Sum of independent random variables

If $x_1 \ldots x_n$ are independent rvs with mgfs $M_{x_1}(t) \ldots M_{x_n}(t)$, then the mgf of $X = x_1 + \cdots + x_n$ is

$$M_X(t) = M_{x_1}(t) + \ldots M_{x_n}(t)$$

1) Sum of 2 independent ~~random~~ normal random variable is a normal random variable

Proof

let $X_1 \sim N(\mu_1, \sigma_1)$

$X_2 \sim N(\mu_2, \sigma_2)$

Normal of mgf

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

let $X = X_1 + X_2$

$$M_X(t) = M_{X_1}(t) \cdot M_{X_2}(t)$$

$$= e^{\mu_1 t + \frac{\sigma_1^2 t}{2}} \cdot e^{\mu_2 t + \frac{\sigma_2^2 t}{2}}$$

$$= \boxed{e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2)^2 t}{2}}}$$

This is in the form of Normal of mgf

X is normal with mean $\mu_1 + \mu_2$

& variance $\sigma_1^2 + \sigma_2^2$

2) If $x_1 \dots x_n$ are independent binomial random variables with parameters $(n_1, p) \dots\dots (n_k, p)$

then $x_1 + \dots x_n$ is binomial with parameters $(n_1 + \dots n_k, p)$

(Becos p is a common parameter for all x's)

Proof:

mgf of $x = x_1 + \dots + x_n$

mgf of Binomial $(q + pe^t)^n$

$$M_x(t) = (q + pe^t)^{n_1} \times \dots \times (q + pe^t)^{n_k}$$

$$= (q + pe^t)^{n_1 + \dots n_k}$$

3) If $x_1 \dots x_k$ are independent poisson random variables with parameters $\lambda_1 \dots \lambda_k$ then $x_1 + \dots x_k$ is a poisson random variable with mean $\lambda_1 + \dots + \lambda_k$

Proof:

mgf of poisson $e^{\lambda(e^t - 1)}$

$$M_x(t) = e^{\lambda_1 (e^t - 1)} \times \dots e^{\lambda_k (e^t - 1)}$$

$$= e^{\underbrace{(\lambda_1 + \dots \lambda_k)}_{\lambda \text{ vals}} (e^t - 1)}$$

ERLONG | GAMMA distribution.

ERLONG distribution

exponential distribution describes the time b/w successive events

Generalization of expo distribution

While expo distribution describes the time b/w successive events, the k Erlong random variable describes the time b/w any and the kth occurence of the event.

exponential [X : Time for the next arrival

(Time b/w one arrival & next arrival)

Erlong Y : Time for the K^{th} arrival

eg:

Time for 4^{th} arrival.

Y : 4 Erlong.

Pdf

$$\boxed{f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{k-1}}{\Gamma(k)} , x \geq 0}$$

parameters λ & K

when K = 1 -> erlong distribution reduces to exponential distribution with parameter λ

Be as,

Y : 1 Erlong = exponential distribution

Mean & variance.

Mean $E(x) = \frac{K}{\lambda}$

$var(x) = K/\lambda^2$

Erlong is for K^{th} arrival.

expo is for 1^{st} arrival.

For expo distribution

$E(x) = 1/\lambda$

$var(x) = 1/\lambda^2$

The generalized form of Erlang dis.. is called gamma distribution

For any real number r, a continuous r.v, X follow gamma ~~differentiation~~ distribution is its pdf is given by

pdf

$$\boxed{f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{r-1}}{\Gamma(r)}, \quad x \geq 0}$$

$X \sim \text{Gamma}(r, \lambda)$

Mean $E(x) = r/\lambda$

variance $= r/\lambda^2$

sum of ~~all~~ exponential r.v is Erlang r.v

mgf of Exponential r.v $= \frac{\lambda}{\lambda - t}$

mgf of r - Erlang $= \left(\frac{\lambda}{\lambda - t}\right)^r$

To find mgf of r-erlang distribution.

pdf of r- Erlang

$$f(x) = \frac{\lambda e^{-\lambda x} \cdot (\lambda x)^{r-1}}{\Gamma(r)}, \quad x \geq 0$$

mgf ~~of~~ $M_x(t) = E(e^{tx})$

$$= \int_0^\infty e^{tx} \frac{\cdot \lambda e^{-\lambda x} (\lambda x)^{r-1}}{\Gamma(r)} dx.$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty e^{-(\lambda-t)x} x^{r-1} dx$$

WKT

$$\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(n)}{a^n}$$

$$= \frac{\lambda^r}{\Gamma(r)} \cdot \frac{\Gamma(r)}{(\lambda-t)^r}$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^r$$

Sum of r exponential r.v with parameter λ is r- Erlong r.v ~~is the~~ with parameter r and λ

$$X = X_1 + \cdots + X_r$$

Proof :

$$M_X(t) = M_{X_1}(t) + \cdots M_{X_r}(t)$$

$$= \left(\frac{\lambda}{\lambda-t}\right) \times \cdots \times \left(\frac{\lambda}{\lambda-t}\right)$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^r .$$

Mgf of r Erlong $= \left(\frac{\lambda}{\lambda-t}\right)^r$

Joint probability distribution

When ~~on~~ 2 or more random variable defined on the ~~joint~~ same sample space.

consider,

$x \to$ height of a student

$y \to$ weight of the same student

$f_{xy}(x, y) \to$ joint pdf

WKT

$F(x) = P[X \leq x]$

$P_{xy}(x, y) = P[X = x ; Y = y] \to$ joint pmf

$F_{xy}(x, y) = P[X \leq x, Y \leq y] \to$ joint cdf.

~~the individual random variable~~.

The individual prob distri of rv x & y are called marginal distributions denoted by $f_x(x), f_y(y)$

Marginal distributions of x and y

Let $P_{xy}(x, y)$ be the joint prob of x and y.

then marginal distribution of x

$$P_x(x) = \sum_{y \in \gamma} P_{xy}(x, y)$$

$$P_y(y) = \sum_{x \in \gamma} P_{xy}(x, y)$$

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Eg:

2 dice are rolled. Let x denotes max of the 2 throws, y denotes the no. of times an even no appears. Find the joint pmf of x and y. Also find the marginal distribution of x & y.

$y = \{0, 1, 2\}$

$x = \{1, 2, 3, 4, 5, 6\}$

(x, y)

$x \Rightarrow$ max of 2 throws

$y \Rightarrow$ no. of times an even no. appears

	x \ y	0	1	2	Pdf	
(1,1)	1	1/36	0	0	1/36	$P_x(1)$
(1,2) (2,1) (2,2)	2	0	2/36	1/36	3/36	$P_x(2)$
(3,1) (3,3) (1,3) (3,2) (2,3)	3	3/36	2/36	0	5/36	$P_x(3)$
(4,1) (1,4) (4,3) (3,4) (4,2) (4,4) (2,4)	4	0	4/36	3/36	7/36	$P_x(4)$
	5	5/36	4/36	0	9/36	$P_x(5)$
1,	6	0	6/36	5/36	11/36	$P_x(6)$ (3,6) (5,6) (6,1) (6,3) (6,5) (1,6) (6,6) (2,6) (6,2) (4,6) (6,4)
		9/36	18/36	9/36	$\boxed{1}$	
		$P_y(0)$	$P_y(1)$	$P_y(2)$		

This is ~~Jo~~ Jmf of x, y.

pdf here is the marginal probability distribution of x

$P_y(0)$, $P_y(1)$, $P_y(2)$ is the marginal distribution of y

Marginal distribution of x

x	1	2	3	4	5	6
$P_x(x)$	1/36	3/36	5/36	7/36	9/36	11/3

Marginal distribution of y

y	0	1	2
$P_y(y)$	9/36	18/36	9/36

Properties of joit pmf / pdf.

Pmf

→ $P_{xy}(x,y) \geq 0$

→ $\sum_{x \in X} \sum_{y \in Y} p_{xy}(x,y) = 1$

Pdf

→ $f_{xy}(x,y) \geq 0$

→ $\int\int f_{xy}(x,y)\, dy\, dx = 1$

If $f_{xy}(x,y)$ is the joint pdf of x & y then

marginal distribution of x : $f_x(x) = \int f(x,y)\, dy$

marginal distribution of y : $f_y(y) = \int f(x,y)\, dx$

Note

If x & y are independent rv

$f(x,y) = f_x(x)\, f_y(y)$

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Conditional probability :

Conditional distribution of Y given $X = x$

For 2 events A, B

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad , \quad P(B) \neq 0$$

For 2 random variable X & Y with joint pmf / pdf $f(x,y)$, the conditional distribution of Y given $X=x$ is given by

$$\boxed{f_{Y|X=x}(y|x) = \frac{f(x,y)}{f_X(x)} \quad , \quad f_x(x) \neq 0}$$

function of y

Conditional expectation & variance

the conditional expectation of Y given $X=x$ is given by

// If X & Y are discrete

$$E[Y|X=x] = \sum_{y \in Y} y \cdot f_{Y|x}(y|x)$$

~~sum~~ value of the random variable and conditional probability

// If X & Y are continuous

$$E[Y|X=x] = \int y \, f_{y|x}(y|x) \, dy$$