Joint moments, covariance & corelation

Joint moments of 2 rv's x & y give the info about the joint behaviour

$E[x^J y^k]$ - jk^{th} joint moment of x & y

moments
$E(x^r) = r^{th}$ moment of x
$E(x-\mu)^r \to r^{th}$ central.

$j=0 \Rightarrow k^{th}$ moment of y
$k=0 \to j^{th}$ moment of x.

For $j=k=1$ $E(xy)$ is the corelation of x & y

If $\boxed{E(xy)=0}$, we say x & y are orthogonal.

$E[(x-\bar{x})^j (y-\bar{y})^k] \to jk^{th}$ joint central moment of x & y

WKT
$E(x) \to$ mean
$E[(x-\mu)^2] \to$ variance
$E(x^2) - E(x)^2$

→ for $j=0$, $k=2$

$$E[(y-\bar{y})^2] = \text{variance of } y \,(\text{var}(y))$$

→ for $j=2$, $k=0$

$$E[(x-\bar{x})^2] = \text{variance of } x \,(\text{var}(x))$$

→ for $j=k=1$

$$E[(x-\bar{x})(y-\bar{y})] \Rightarrow \text{covariance of } x \,\&\, y$$

$$\boxed{= E(xy) - E(x)E(y)}$$

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Covariance of independent r.v's

If x & y are independent then :

WKT

$f(x,y) = f_x(x) f_y(y)$

$E(XY) = E(x) E(y)$

$$E(xy) = E(x)\,E(y)$$

(ie) $\boxed{cov(x,y) = 0}$

* Covariance ~~6~~ is not true

$$cov(x,y) = 0 \not\Rightarrow x, y \text{ are independent.}$$

Eg.

Q) Let x be a uniform rv over $(-1,1)$ and $y = x^2$

$$E(x) = \frac{-1+1}{2} = 0$$ // since x is a uniform rv

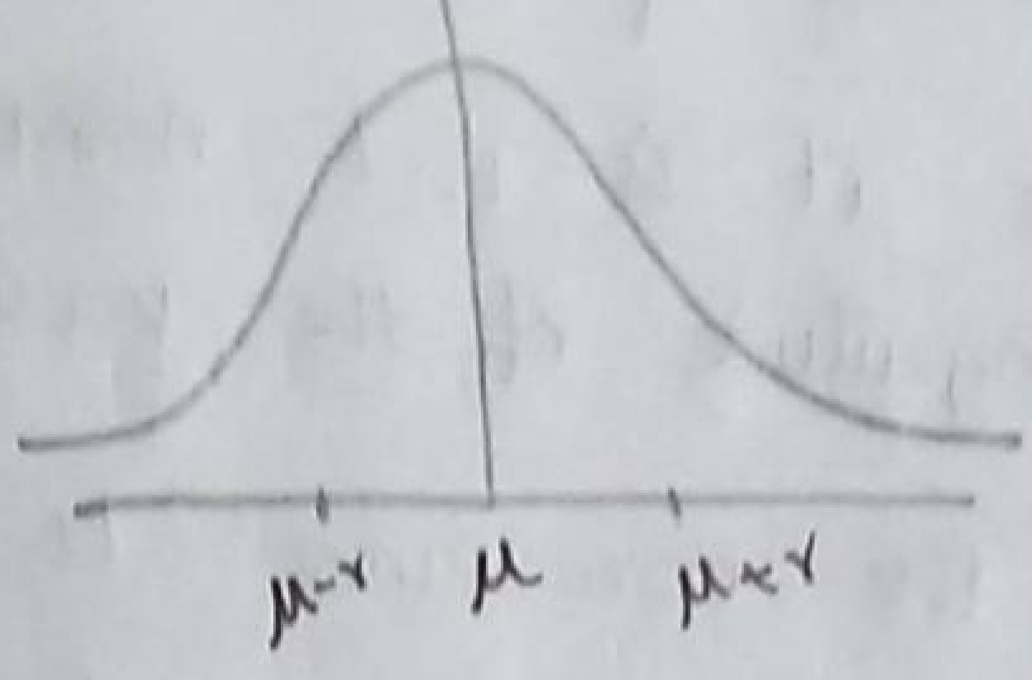
$$E(x^3) = 0$$ // Becos in the interval a to b if the fun is odd $\int f(x)\,dx$ will always be zero

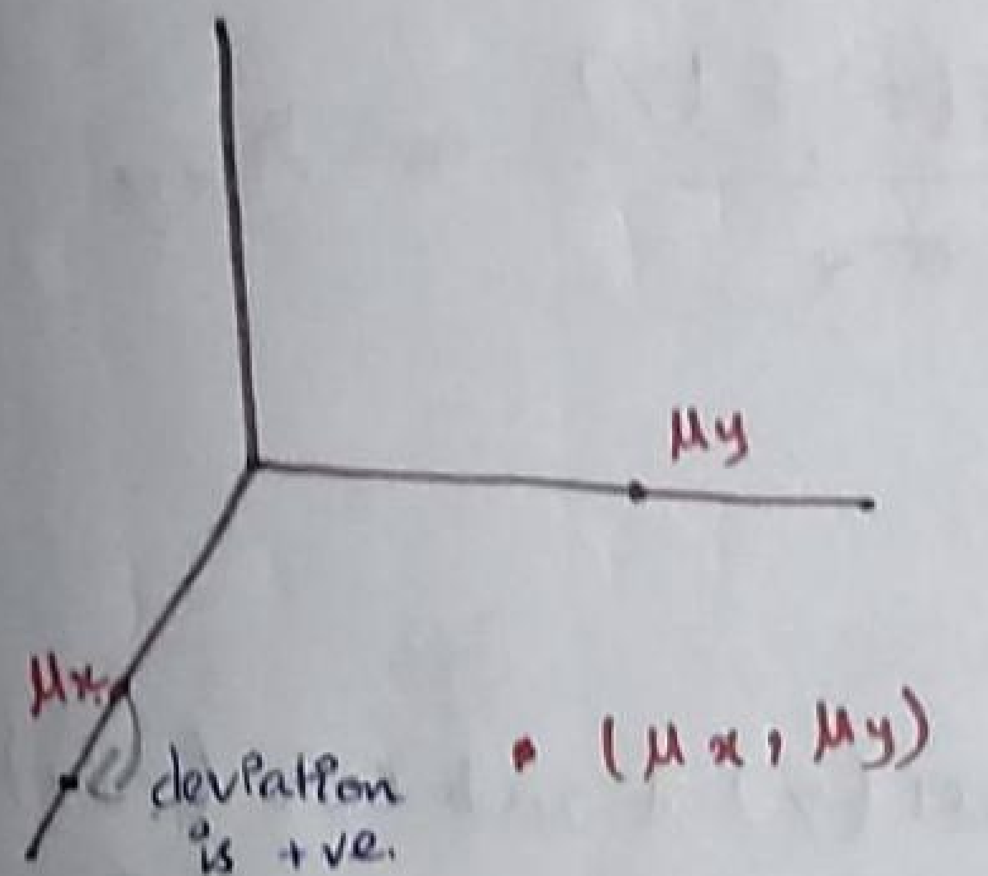
$$cov(x,y) = E(xy) - E(x)\,E(y)$$
$$= E(x^3) - E(x)\,E(y)$$
$$= 0$$

$f(x) \to$ curve or a line

$f(xy) \to$ surface

Covariance measures the deviations of the points (x,y) from $E(x) = \mu_x$ and $E y = \mu_y$





The both deviation is +ve or -ve, the both will increase or decrease.

→ If $x - \mu_x$ and $y - \mu_y$ tend to have the same signs (both are +ve or both are -ve), then

$$cov(x, y) > 0$$

→ If $x - \mu_x$ and $y - \mu_y$ both have opp signs then

$$cov(x, y) < 0$$

→ $cov(x, y)$ is not independent of the units in which x and y are measured

Q] suppose x and y are rv's measured in cm having $cov(x, y) = 0.15$

If we change the units of x and y to mm (i.e)

$x_1 = 10x \quad y_1 = 10y$ as new rv's

$$cov(x_1, y_1) = cov(10x, 10y)$$
$$= 15$$

Hence the magnitude of covariance is not easy to interpret because it is not normalised and hence depends on the magnitude of the variables.

So, we normalise the covariance to measure the ~~coralation~~ correlation in absolute scale

$$\rho_{xy} \xrightarrow{\text{strength}} = \frac{cov(x,y)}{\sigma_x \sigma_y} = \frac{E(xy) - E(x)E(y)}{\sigma_x \sigma_y} = r_{xy}$$

Result.

$$var(ax+by) = a^2 var(x) + b^2 var(y) + 2ab\, cov(x,y)$$

Proof

$$var(ax+by) = E\left[(ax+by) - (a\mu_x + b\mu_y)\right]^2$$

$$= E\left[a(x-\mu_x) + b(y-\mu_y)\right]^2$$

$$= E\left[a^2(x-\mu_x)^2 + b^2(y-\mu_y)^2 + 2ab(x-\mu_x)(y-\mu_y)\right]$$

$$= a^2 E(x-\mu_x)^2 + b^2 E(y-\mu_y)^2 + 2ab\, E\left[(x-\mu_x)(y-\mu_y)\right]$$

$$= a^2 var(x) + b^2 var(y) + 2ab\, cov(x,y)$$

Cor:

$$var\left(\frac{x}{\sigma_x} + \frac{y}{\sigma_y}\right) = 2 + 2\rho_{xy}$$

$$var\left(\frac{x}{\sigma_x} - \frac{y}{\sigma_y}\right) = 2 - 2\rho_{xy}$$

from $var(ax+by) = a^2 var(x) + b^2 var(y) + 2ab\sigma_x\sigma_y\rho_{xy}$

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Properties

i) $\rho_{xy} = r$ $\boxed{-1 \le r \le 1}$

Proof :

$$\text{var}\left(\frac{x}{\sigma_x} + \frac{y}{\sigma_y}\right) \geq 0$$

$$2 + 2\rho_{xy} \geq 0$$

$$2 + 2r \geq 0$$

$$r \geq -1$$

$$\text{var}\left(\frac{x}{\sigma_x} - \frac{y}{\sigma_y}\right) \geq 0$$

$$2 - 2\rho_{xy} \geq 0$$

$$2 - 2r \geq 0$$

$$r \leq 1$$

(ii) $r_{xy} = 1$ iff $y = ax + b$ for constants a, b and $a > 0$

Proof

suppose $r_{xy} = 1$

$$\text{var}\left(\frac{x}{\sigma_x} - \frac{y}{\sigma_y}\right) = 2 - 2(1) = 0$$

$$\frac{x}{\sigma_x} - \frac{y}{\sigma_y} = c \text{ , constant rv}$$

$$\frac{y}{\sigma_y} = \frac{x}{\sigma_x} - c.$$

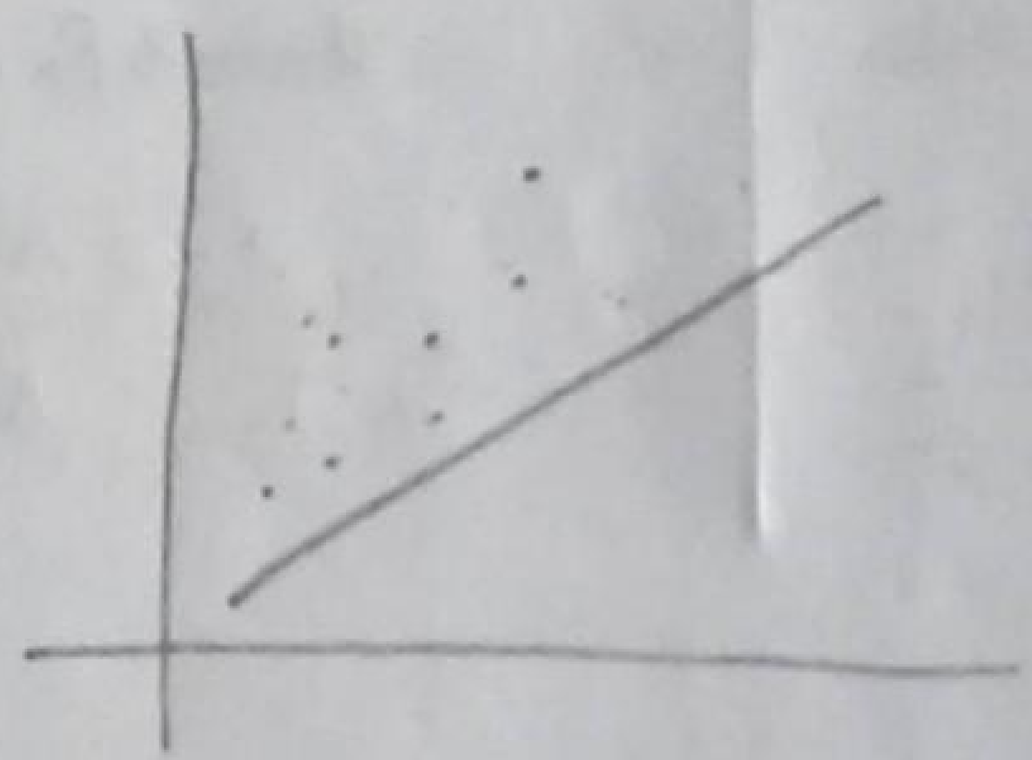
$$y = \underbrace{\boxed{\frac{\sigma_y}{\sigma_x}}}_{a} x \underbrace{\boxed{- c\sigma_y}}_{b}$$

Q] If x & y are r.v's with joint pdf

$$f(x, y) = x + y, \quad 0 < x, y < 1$$

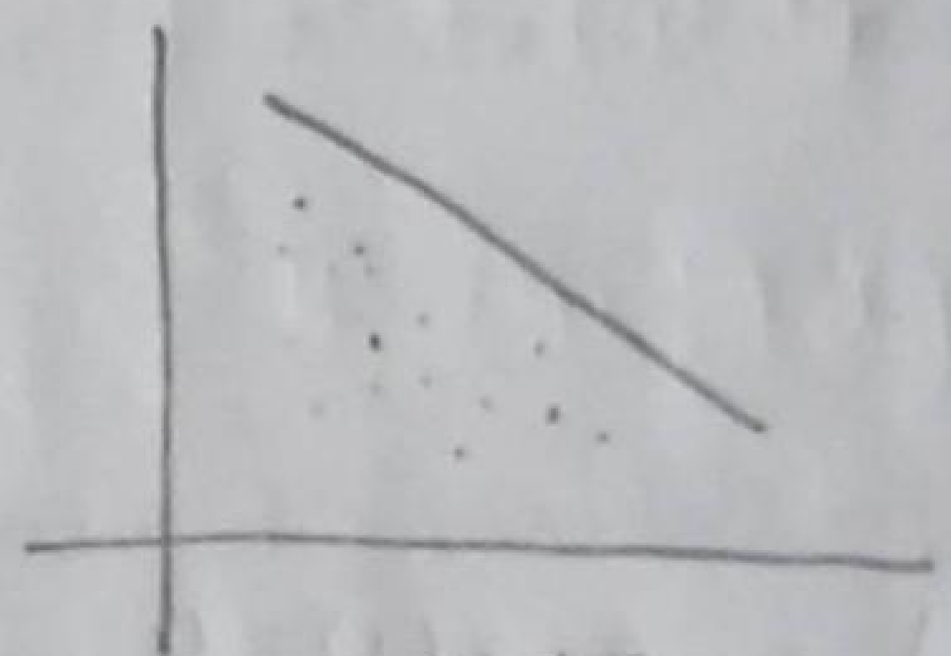
Show that x & y are not linearly related.

has no perfect relation → The pts do not lie on a line



+ve

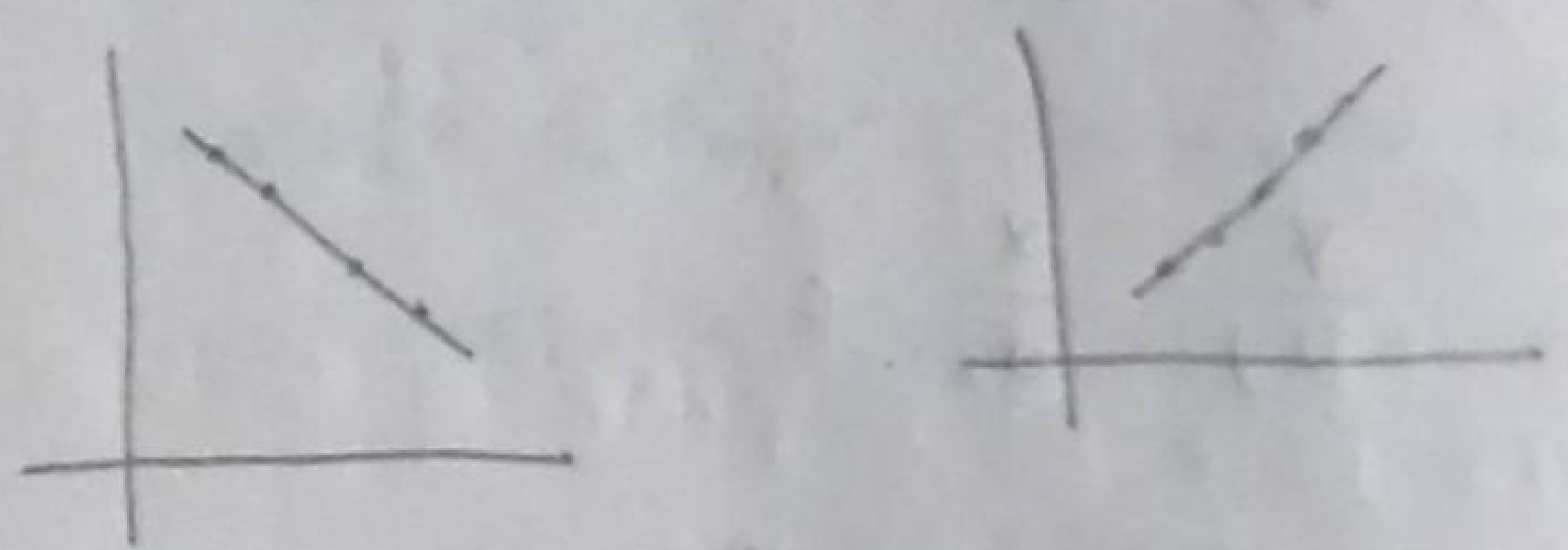
$\gamma > 0$



-ve

$\gamma < 0$

Perfect relation



First, we find marginal densities of x and y

$$f_x(x) = \int f(x, y)\, dy$$

$$= \int_0^1 (x + y)\, dy$$

$$= \left[xy + \frac{y^2}{2} \right]_0^1$$

$$\boxed{f_x(x) = x + \frac{1}{2} \qquad 0 < x < 1}$$

// lly

$$\boxed{f_Y(y) = y + \frac{1}{2}, \quad 0 < y < 1}$$

$$E(x) = \int x \, f_x(x) \, dx$$

$$= \int_0^1 x \left(x + \frac{1}{2} \right) dx$$

$$= \int_0^1 x^2 + \frac{x}{2} \, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4} \right]_0^1 = \left[\frac{1}{3} + \frac{1}{4} \right] = \frac{7}{12}$$

// lly

$$E(y) = \int y \, f_y(y) \, dy$$

$$= \int_0^1 y \left(y + \frac{1}{2} \right) dy$$

$$= \frac{7}{12}$$

$$E(xy) = \iint (xy) \, f(x, y) \, dy \, dx$$

$$= \int_0^1 \int_0^1 xy \, (x + y) \, dy \, dx$$

$$= \int_0^1 \int_0^1 x^2 y + xy^2 \, dy \, dx$$

$$= \int_0^1 \left[x^2 \left(\frac{y^2}{2} \right) + x \left(\frac{y^3}{3} \right) \right]_0^1 dx$$

$$= \int_0^1 \left(\frac{1}{2} x^2 + \frac{1}{3} x \right) dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} \right]_0^1 + \frac{1}{3} \left[\frac{x^2}{2} \right]_0^1$$

$$= 1/3$$

$E(x^2) = \int x^2 f_x(x)\,dx$

$= \int_0^1 x^2 \left(x+\frac{1}{2}\right) dx$

$= \int_0^1 x^3 + \frac{x^2}{2}\, dx.$

$= \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1$

$= \frac{1}{4} + \frac{1}{6} = 5/12$

$E(y^2) = \int y^2 f_y(y)\,dy$

$= \int_0^1 y^2 \left(y+\frac{1}{2}\right) dy$

$= 5/12.$

$var(x) = E(x^2) - E(x)^2$

$= \frac{5}{12} - \frac{49}{12^2} = \frac{11}{144}$

WKT

$\sigma_x = \sqrt{var(x)}$

$var(y) = E(y^2) - E(y)^2$

$= \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$

WKT

$\sigma_y = \sqrt{var(y)}$

correlation coefficient $\Big\}$ $\rho_{xy} = \dfrac{E(xy) - E(x)E(y)}{\sigma_x \sigma_y}$

$= \dfrac{\frac{1}{3} - \frac{49}{144}}{\frac{\sqrt{11}}{12} \cdot \frac{\sqrt{11}}{12}}$

WKT

$cov(x,y) = E(xy) - E(x)E(y)$

$\rho_{xy} = \dfrac{cov(x,y)}{\sigma_x \sigma_y}$

$$= \frac{\frac{48-49}{144}}{\frac{11}{144}}$$

$$= -\frac{1}{11} \neq \pm 1$$

$\therefore$ x and y are not linearly related.

WKT

$E(x+y) = E(x) + E(y)$

$Var(x+y) = var(x) + var(y) + 2cov(xy)$

Mean and variance of sum of r.v's

for any list of rv's $X_1, \ldots X_n$

let $W_n = X_1 + \cdots + X_n$

$$\boxed{\begin{aligned} E(W_n) &= \sum_i E(X_i) \\ Var(W_n) &= \sum_i var(X_i) + 2\sum_{i<j} cov(X_i, X_j) \end{aligned}}$$

If $X_1, \ldots X_n$ are pairwise independent or pairwise uncorrelated, then

$$\boxed{Var(W_n) = \sum_i var(X_i)}$$

Result ~ Sum of iid random variables

If $X_1, \ldots X_n$ are iid rv's each with mean μ & variance σ^2, then

$\boxed{E(W_n) = n\mu}$ $\boxed{var(W_n) = n\sigma^2}$

$X_1, X_2 \ldots$ be a sequence of random variables, with $E(X_1)=0$ and $\text{cov}(X_i - X_j) = 0.8^{|i-j|}$. Find the and variance of $Y_i = X_i + X_{i-1} + X_{i-2}$

$E(X_i) = 0$

$Var(Y_i) = Var(X_i + X_{i-1} + X_{i-2})$

$= var(X_i) + var(X_{i-1}) + var(X_{i-2}) + 2\,cov(X_i, X_{i-1})$

$+ 2\,cov(X_{i-1}, X_{i-2}) + 2\,cov(X_i, X_{i-2})$

WKT

$cov(X,Y) = E(XY) - E(X)E(Y)$

$cov(X,X) = E(X^2) - E(X)^2$

$= 1 + 1 + 1 + 2(0.8) + 2(0.8) + 2(0.8)^2$

$= 7.48$

At a party of $n \geq 2$ people each person throws his/her hat in a common box. They are shuffled and each person draws ~~the~~ a hat without replacement. If a person draws his/her hat it is success. What are mean & variance of the no of success

let $X_i = \begin{cases} 1 & \text{if person } i \text{ takes his / her own hat} \\ 0 & o/w \end{cases}$

Binomial

$\frac{1}{n} = p$

$q = 1-p$

$P[X_i = 1] = \frac{1}{n}$ $\quad$ $P[X_i = 0] = 1 - \frac{1}{n}$

No. of success $S_n = X_i + \ldots + X_n$

$E(X_i) = \frac{1}{n}$ & $var(X_i) = \frac{1}{n}\left(1 - \frac{1}{n}\right)$ $\quad$ var $= pq$

$E(S_n) = n \cdot \frac{1}{n} = 1$

$E(S_n) = n\mu$

$Var(S_n) = n\sigma^2$

$cov(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)$

to find $E(X_i X_j)$

$$X_i X_j = \begin{cases} 1 & \text{iff } x_i = x_j = 1 \\ 0 & o/w \end{cases}$$

$E(X_i X_j) = 1 \; P_{X_i X_j}(1,1) + 0$

$\quad = 1 \; P_{X_i|X_j}(1|1) \; P_{X_j}(1)$

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$

$f(x,y) = f_{X|Y}(x|y) f(y)$

Given $X_j = 1$, the j^{th} person draws his hat; then $X_i = 1$, iff the i^{th} person draws his hat from $n-1$ other hats

$P_{X_i|X_j}(1|1) = \frac{1}{n-1}$

$E(X_i X_j) = \frac{1}{n-1} \cdot \frac{1}{n}$

$cov(X_i, X_j) = \frac{1}{n(n-1)} - \frac{1}{n} \cdot \frac{1}{n}$

$cov(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)$

$\quad = \frac{1}{n(n-1)} - \frac{1}{n^2}$

$var(S_n) = Var(X_1) + \dots Var(X_n) + 2\,{}^nC_2 \; cov(X_i, X_j)$

$\quad = n\left(\frac{1}{n} - \frac{1}{n^2}\right) + \cancel{2}\,\frac{n(n-1)}{\cancel{2}}\left\{\frac{1}{n(n-1)} - \frac{1}{n^2}\right\}$

$\quad = \left(1 - \frac{1}{n}\right) + \left(1 - \frac{n-1}{n}\right) = 1$

Note

suppose each person immediately returns the hat into the box. What are the mean & variance?

$$E(S_n) = 1$$

$$Var(S_n) = n\left(\frac{1}{n} - \frac{1}{n^2}\right)$$

$$= 1 - \frac{1}{n}$$

cov is 0 becau[se] they are indepen[dent] & identical (i) the rv's are II[D]

If 2 rv's are ind[ependent] cov is zero

Markov inequality

If x is a rv that takes non-negative values then for any $a > 0$.

upper bound for $P[x \geq a]$

Proof:

$$E(x) = \int_0^\infty x f(x)\,dx$$

$$= \int_0^a x f(x)\,dx + \int_a^\infty x f(x)\,dx$$

$$\geq \int_a^\infty x f(x)\,dx$$

$$\geq \int_a^\infty a f(x)\,dx$$

$$E(x) \geq a\,P[x \geq a]$$

$$P[x \geq a] \leq \frac{E(x)}{a}$$

Chebyshev's inequality

If x is a rv with mean μ and variance σ^2 the for any value of $k > 0$

$$P[|x-\mu| \geq k] \leq \frac{\sigma^2}{k^2}$$

Proof:

$(x-\mu)^2$ is a non-negative rv

By markov's inequality

WKT

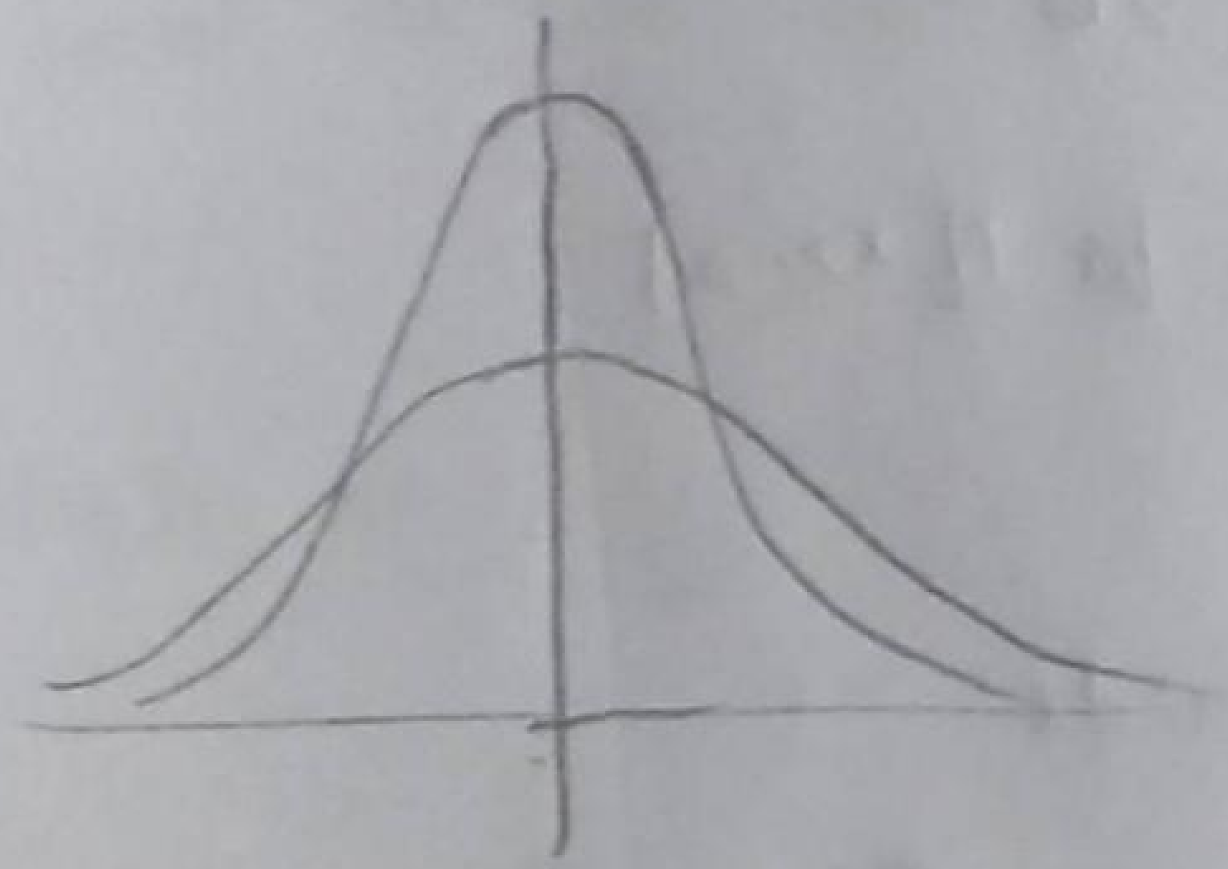
$$P[x \geq a] \leq \frac{E(x)}{a}$$

$$P[(x-\mu)^2 \geq k^2] \leq \frac{E(x-\mu)^2}{k^2}$$

variance

$$= \frac{\sigma^2}{k^2}$$

$P[|x-\mu| \geq k] \leq \dfrac{\sigma^2}{k^2}$. $P[x \geq a] \leq \dfrac{E(x)}{a}$



→ If a rv has small variance then the probability that it takes values far away from the mean is small.

→ If $k = n\sigma$, then Chebyshev's inequality becomes

$$P[|x-\mu| \geq n\sigma] \leq \frac{1}{n^2}$$

$\frac{\sigma^2}{k^2}$ $k = n\sigma$

$\frac{\sigma^2}{n^2\sigma^2} = \frac{1}{n^2}$

The probability that x deviates from its mean by atleast n SD's is $\leq \frac{1}{n^2}$

Suppose the no. of items produced in a factory in a week is a RV with a mean of 500

i) what can be said about the prob that this week's production will be atleast 10000.

ii) If the variance is known to be 100, what can be said about the probability that this weeks production will be b/w 400 and 600?

i) x : no. of items produced in a week

$x \geq 0$ and $E(x) = 500$

$$P[x \geq 1000] \leq \frac{500}{1000} = 0.5$$

Markov inequality

$$P[x \geq a] \leq \frac{E(x)}{a}$$

ii) Given $\sigma^2 = 100$

By Chebyshev's inequality

$$P[|x - \mu| \geq k] \leq \frac{\sigma^2}{k^2}$$

$$P[|x - 500| \geq 100] \leq \frac{100}{100^2} = 0.01$$

$\Rightarrow \geq 100$ upper bound

$500 - 100 < x < 500 + 100$.

$$\Leftarrow P[|x - 500| < 100] \geq 0.99.$$

$\Rightarrow < 100$ lower bound

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