Moments :

Let x be the rv with mean μ $(E(x) = \mu)$

$E(x^r)$ is called r^{th} moment of x (r^{th} moment abt origin)

$E(x-\mu)^r$ is called r^{th} central moment of x.

(r^{th} moment abt mean)

$r = 1$ $E(x) \Rightarrow$ the 1^{st} moment is the mean

$r = 2$ $E(x-\mu)^2 \Rightarrow$ the 2^{nd} central moment is variance

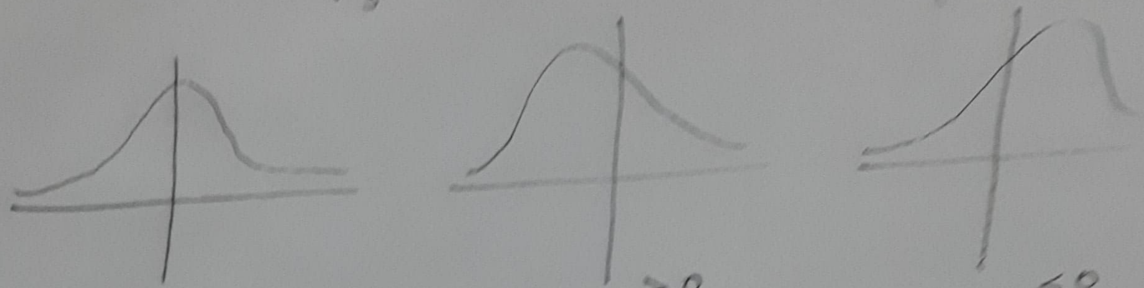
$$E(x-\mu)^2 = E(x^2) - E(x)^2$$

For other values of r also, $E(x^r)$ is a [illegible] measure

$E(x) = \mu$?

$E(x-\mu)^r = \mu_r$ $\quad\quad$ $E(x^r) = \mu_r'$

* The quantity $\frac{\mu_3}{\sigma_3}$ is a measure of symmetry (skewness)



	0	>0	<0
Measure of skewness } :	symmetric dist	skewed to right	skewed to left

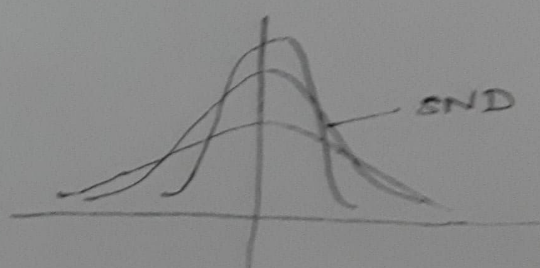
$$\frac{E(x-\mu)^3}{\left[E(x-\mu)^2\right]^{3/2}} \longrightarrow \frac{\mu_3}{\sigma 3}$$

* The quantity $\frac{\mu_4}{\sigma^4}$ is a measure of kurtosis (peakedness)

= 3 for SND

> 3 for a dist which is more peak than SND

< 3 for a dist which is less peak than SND



* Moment means importance. With the knowledge, we know the centre (mean), the spread (var), the symmetry, the kurtosis (peakedness) and so on – a complete pic abt the shape of dist.

generating function : MGF

For a rv X - $\boxed{M_x(t) = E(e^{tx})}$ (mean above; has finite values below)

If $M_x(t)$ is defined for all values of t in an interval $(-\delta, \delta)$ for $\delta > 0$, then $M_x(t)$ is called the moment generating function of X.

neighbourhood points (δ units away from 0)

Note: $M_x(t)$ is finite in some neighbourhood of 0. $(-\delta, \delta)$

If this condition is not satisfied, some moments may not exist.

Moments of rv can be obtained from MGF.

Generating moments from MGF

* Method 1: $\frac{d^r}{dt^r}\left[M_x(t)\right]$ at $t=0$ gives $E(x^r)$

the r^{th} derivative of MGF at $t=0$ is r^{th} moment

Proof:

$$M_x'(t) = \frac{d}{dt}\left[E(e^{tx})\right]$$

$$= E\left[\frac{d}{dt}(e^{tx})\right]$$

$$= E(xe^{tx})$$

$$\left[M_x'(t)\right]_{t=0} = E(x)$$

1^{st} derivative of MGF at $t=0$ is $E(x)$

$$\left[M_x''(t)\right]_{t=0} = E(x^2)$$

Hence proved.

* Method 2: $M_x(t) = E(e^{tx})$

$$= E\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \dots \frac{t^r x^r}{r!} + \dots\right]$$

$$= 1 + \frac{t}{1!}E(x) + \frac{t^2}{2!}E(x^2) + \dots \frac{t^r}{r!}E(x^r) + \dots$$

The coefficient of $\frac{t^r}{r!}$ in the expansion of Mgf is the r^{th} moment.

8)

* Find the mgf of binomial dist with parameters n and p. Also find its mean and variance.

Soln: pmf of BD $= f(x) = nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2 \dots n$

mgf $M_x(t) = E(e^{tx})$

$$= \Sigma\, e^{tx}\, nC_x\, p^x q^{n-x} \qquad x = 0, \dots n$$

$$= \Sigma\, nC_x\, (pe^t)^x q^{n-x} \qquad (p+q)^n = \Sigma\, nC_x\, p^x q^{n-x}$$

$$= (pe^t + q)^n$$

$$[M_x'(t)]_{t=0} = \left[n(pe^t+q)^{n-1} pe^t\right]_{t=0} \qquad [p+q=1]$$

$$\boxed{\text{Mean} = np} = E(x)$$

$$[M_x''(t)]_{t=0} = \left[n(n-1)(pe^t+q)^{n-2} pe^t\, pe^t + n(pe^t+q)^{n-1} pe^t\right]_{t=0}$$

$$= np + p^2(n^2 - n)$$

$$= np[1 + np - p]$$

$$= np(q + np)$$

$$E(x^2) = npq + n^2p^2$$

Variance $= E(x^2) - E(x)^2 = npq + n^2p^2 - n^2p^2 = \boxed{npq}$

...d the mgf of exponential distribution with ...ameter λ. Also find mean and variance

...n: pdf of ED : $\lambda e^{-\lambda x}$

mgf of ED $M_x(t) = E(e^{tx})$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-x(\lambda - t)} dx$$

$$= \lambda \left[\frac{e^{-x(\lambda - t)}}{t - \lambda} \right]_0^\infty$$

$$= \lambda \left[0 - \frac{1}{t - \lambda} \right]$$

$$M_x(t) = \frac{\lambda}{\lambda - t}$$

$$[M_x'(t)]_{t=0} = \lambda \frac{-1}{(\lambda - t)^2} (-1) \Big|_{t=0}$$

$$\boxed{E(x) = \frac{1}{\lambda} = \text{Mean}}$$

$$[M_x''(t)]_{t=0} = \lambda \frac{-2}{(\lambda - t)^3} (-1) \Big|_{t=0} = \frac{2}{\lambda^2}$$

$$Var(x) = E(x^2) - E(x)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

* Find mgf of poisson distribution. Find its mean & variance.

Soln: pmf of PD : $f(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

mgf of PD $M_x(t) = E(e^{tx})$

$$= \sum e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} \qquad x = 0, 1, \ldots$$

$$= e^{-\lambda} \sum \frac{e^{tx} \lambda^x}{x!} = e^{-\lambda} \sum \frac{(e^t \lambda)^x}{x!}$$

$$= e^{-\lambda} e^{e^t \lambda}$$

$$\boxed{e^u = \sum_{x=0}^{\infty} \frac{u^x}{x!}}$$

$$M_x(t) = e^{\lambda(e^t - 1)}$$

$$[M_x'(t)]_{t=0} = e^{\lambda(e^t-1)} (\lambda e^t) \Big/_{t=0}$$

$$\boxed{E(x) = \lambda = \text{Mean}}$$

$$[M_x''(t)]_{t=0} = e^{\lambda e^t - \lambda} \lambda e^t \lambda e^t + e^{\lambda(e^t-1)} \lambda e^t \Big/_{t=0}$$

$$= \lambda^2 + \lambda$$

Variance : $E(x^2) - E(x)^2 = \lambda^2 + \lambda - \lambda^2 = \boxed{\lambda}$

* Find the mgf of SND.

pdf of ~~S~~ND : $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

pdf of SND : $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $\qquad \sigma = 1 ; \mu = 0$

$M_x(t) = E(e^{tx})$

mgf of SND $= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - \frac{x^2}{2}} dx$$

$$\cancel{= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - \frac{x^2}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2tx)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\,(x^2 - 2tx + t^2 - t^2)}\, dx$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\,(x-t)^2}\, dx$$

Let $u = x - t \quad du = dt$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\, du \qquad \because \int_{-\infty}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \text{ is pdf of SND}$$

$$M_x(t) = \frac{e^{t^2/2}}{\cancel{\sqrt{2\pi}}}\,(1) = \boxed{e^{t^2/2}}$$

$$M_x'(t)\Big|_{t=0} = \frac{\cancel{1}}{\cancel{\sqrt{2\pi}}}\, e^{t^2/2} \left(\frac{2t}{2}\right)\Bigg|_{t=0} = \boxed{0 = \text{Mean}}$$

$$M_x''(t)\Big|_{t=0} = \frac{\cancel{1}}{\cancel{\sqrt{2\pi}}}\left[e^{t^2/2}\left(\frac{2t}{2}\right)\left(\frac{2t}{2}\right) + e^{t^2/2}\,\frac{2}{2}\right]\Bigg|_{t=0}$$

$$= \frac{\cancel{1}}{\cancel{\sqrt{2\pi}}}\,[1]$$

Variance : $E(x^2) - E(x)^2 = 1 - 0^2 = \boxed{1}$,

PROPERTIES OF MGF:

* Two rv x and Y with same mgf have the same prob dist

ie) mgf of a rv is unique

* $M_x(0) = 1$

c is constant

* If $M_x(t)$ is known, $M_{cx}(t) = M_x(ct)$

?

(In the mgf of x, replace t by ct

* $M_{c+x}(t) = e^{ct}\, M_x(t)$

$$\left[\begin{array}{l} M_x(t) = E(e^{tx}) \\ M_{c+x}(t) = E\left(e^{t(c+x)}\right) \\ = E(e^{tc}\, e^{tx}) = e^{ct} M_x(t) \end{array}\right.$$

* mgf of sum of 2 independant rv

If x and y are independant $\rightarrow M_{x+y}(t)$

Proof: $M_{x+y}(t) = E(e^{t(x+y)})$

$= E(e^{tx}e^{ty})$

$= E(e^{tx})\,E(e^{ty})$ $\because$ x & y are

$= M_x(t) \cdot M_y(t)$

* Use the mgf of SND, find the mgf of ND with mean μ and SD σ.

Soln: WKT mgf of SND $= e^{t^2/2}$ $\qquad Z = \frac{X-\mu}{\sigma}$

$M_Z(t) = e^{t^2/2}$ $\qquad X = \mu + \sigma Z$

mgf of ND: $M_X(t) = M_{\mu+\sigma Z}(t)$

By property $M_{c+X}(t) = e^{ct} M_X(t)$,

$M_{\mu+\sigma Z}(t) = e^{\mu t} M_{\sigma Z}(t)$

By property $M_{cX}(t) = M_X(ct)$

$= e^{\mu t} M(\sigma t)$

$= e^{\mu t} e^{\sigma^2 t^2/2}$

$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$

* Use mgf of ND to find its mean & variance

Soln: $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$E(x) = M_X'(t)\Big|_{t=0} = e^{\mu t + \frac{\sigma^2 t^2}{2}}(\mu + \sigma^2 t)\Big|_{t=0}$

$\boxed{\text{Mean} = \mu}$

$E(x^2) = M_X''(t)\Big|_{t=0} = e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot (\sigma^2) + (\mu + \sigma^2 t)\, e^{\mu t + \frac{\sigma^2 t^2}{2}} (\mu + \sigma^2 t)\Big|_{t=0}$

$E(x^2) = \sigma^2 + \mu^2$

Variance $E(x^2) - E(x)^2$

$= \sigma^2 + \mu^2 - \mu^2$

$\boxed{var(x) = \sigma^2}$

Let x be a rv with mgf $M_x(t) = e^{2t^2}$

Find $P[0 < x < 1]$

<u>Soln:</u> $M_x(t)$ of ND $= e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$e^{2t^2} = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$\therefore \boxed{\mu = 0}$ $\frac{\sigma^2}{2} = 2$ $\sigma^2 = 4$ $\boxed{\sigma = 2}$

$X \sim ND(0, 2)$ $Z = \frac{X - \mu}{\sigma} = \frac{X}{2}$

$P[0 < X < 1] = P[0 < Z < 1/2]$

$= \Phi(0.5) - \Phi(0)$

$= 0.6915 - 0.5000$

$P[0 < X < 1] = 0.1915$

* Let X be a rv with pmf $P(x) = 2\left(\frac{1}{3}\right)^x$, $x = 1, 2 \ldots$

Find mgf of X and E(x).

<u>Soln:</u> mgf of $M_x(t)$ $M_x(t) = E(e^{tx})$

$\cancel{E(x)} = \sum e^{tx}\, 2\left(\frac{1}{3}\right)^x$ $x = 1, 2, 3 \ldots$

$\cancel{= \left[\frac{2}{3} + \frac{4}{9} + \frac{6}{27}\right.}$

$= 2 \sum e^{tx} \left(\frac{1}{3}\right)^x = 2 \sum \left(\frac{e^t}{3}\right)^x$

$= 2\left[\frac{e^t}{3} + \left(\frac{e^t}{3}\right)^2 + \left(\frac{e^t}{3}\right)^3 + \ldots\right]$

$= 2\left[\frac{1}{1 - \frac{e^t}{3}} - 1\right]$ $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$

$= 2\left[\frac{3}{3 - e^t} - 1\right]$

$$mgf = \frac{6}{3-e^t} - 2$$

$$E(x) = M_x'(t)\Big|_{t=0} = \frac{6(-1)}{(3-e^t)^2}(-e^t)\Big|_{t=0} = \frac{6e^t}{(3-e^t)^2}\Big|_{t=0}$$

$$E(x) = \frac{6}{4} = \frac{3}{2}$$

Sum of independent rv

If $x_1, x_2 \ldots x_n$ are independant rv with mgf's $M_{x_1}(t), M_{x_2}(t) \ldots M_{x_n}(t)$, then mgf of $X = x_1 + x_2 + \ldots x_n$ is

$$M_X(t) = M_{x_1}(t) \times M_{x_2}(t) \times \ldots M_{x_n}(t)$$

① Sum of 2 ind normal rv is a normal rv

Proof: let $x_1 \sim N(\mu_1, \sigma_1)$ & $x_2 \sim N(\mu_2, \sigma_2)$

let $X = x_1 + x_2$

$$M_X(t) = M_{x_1}(t) \times M_{x_2}(t)$$

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}}$$

$$= e^{(\mu_1 + \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}$$

X is normal rv with mean $\mu_1 + \mu_2$ & var $\sigma_1^2 + \sigma_2^2$

② If $x_1, x_2 \ldots x_K$ are independant binomial rv with parameter $(n_1, p), (n_2, p) \ldots (n_K, p)$ then $x_1 + x_2 + \ldots x_n$ is binomial with parameter $(n_1 + n_2 + \ldots n_K, p)$

Proof: mgf of $X = x_1 + x_2 + \ldots x_K$

$$M_X(t) = (q+pe^t)^n$$

$$M_X(t) = (q+pe^t)^{n_1} \times (q+pe^t)^{n_2} \times \ldots (q+pe^t)^{n_K}$$

$$M_X(t) = (q+pe^t)^{n_1+n_2+\ldots n_K}$$

If $X_1, X_2, \dots X_k$ are independant poisson rv parameter $\lambda_1, \lambda_2, \dots \lambda_k$ then $X_1 + X_2 + \dots X_k$ a poisson rv with mean $\lambda_1 + \lambda_2 + \dots \lambda_k$

$$M_x(t) = e^{\lambda_1(e^t-1)} \times e^{\lambda_2(e^t-1)} \times \dots e^{\lambda_k(e^t-1)} \qquad \boxed{M_K(t) = e^{\lambda(e^t-1)}}$$

$$= e^{(e^t-1)(\lambda_1+\lambda_2+\dots\lambda_k)}$$

Erlang & Gamma distribution

* Erlang distribution

Generalisation of exponential dist

While exponential dist describes the time btw two successive events, the k-Erlang rv describes the time btw any event & the k^{th} occurrence

eg: Exp time for next arrival

Erl time for 4^{th} arrival

k is always +ve

Pdf: $\boxed{f(x) = \dfrac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{\Gamma(k)}}$, $x \geq 0$ $\qquad \Gamma(k) = (k-1)!$

parameters : λ and k

Note: If $k=1$, Erlong dist reduces to Exp dis with parameter λ

Mean $E(x) = \dfrac{k}{\lambda}$ $\qquad$ Variance $= \dfrac{k}{\lambda^2}$

* Gamma distribution

Generalisation of Erlang distribution

For any real no r, a continuous r.v. x follows Gamma dist if its pdf is given by

$$f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{r-1}}{\Gamma r}, \quad x \geq 0$$

Mean $E(x) = \frac{r}{\lambda}$ Variance $\frac{r}{\lambda^2}$

(4) * Find mgf of r - Erlang dist

pdf of r Erlang $= \frac{\lambda e^{-\lambda x} (\lambda x)^{r-1}}{\Gamma r}$

mgf

$$M_X(t) = \int_0^\infty e^{tx} \frac{\lambda e^{-\lambda x} (\lambda x)^{r-1}}{\Gamma r} dx$$

$$= \frac{\lambda^r}{\Gamma r} \int_0^\infty e^{tx} e^{-\lambda x} x^{r-1} dx$$

$$= \frac{\lambda^r}{\Gamma r} \int_0^\infty e^{(t-\lambda)x} x^{r-1} dx = \frac{\lambda^r}{\Gamma r} \int_0^\infty e^{-(\lambda-t)x} x^{r-1} dx$$

$$= \frac{\lambda^r}{\Gamma r} \frac{\Gamma r}{(\lambda-t)^r} \qquad \left[\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}\right]$$

$$M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^r$$

(4) Sum of r iid exponential rv with parameter λ is r Erlang rv with parameter r and λ.

Proof: $X = X_1 + X_2 + \ldots X_n$

$$M_X(t) = M_{X_1}(t) * M_{X_2}(t) \times \ldots M_{X_r}(t)$$

$$= \left(\frac{\lambda}{\lambda-t}\right)\left(\frac{\lambda}{\lambda-t}\right) \cdots \left(\frac{\lambda}{\lambda-t}\right).$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^r$$