Random Variables

A random variable is a function that assigns to each O/C $w \in S$ a real number

event $A \subseteq S$

A real valued function $X: \Omega \to R$ is a r.v.

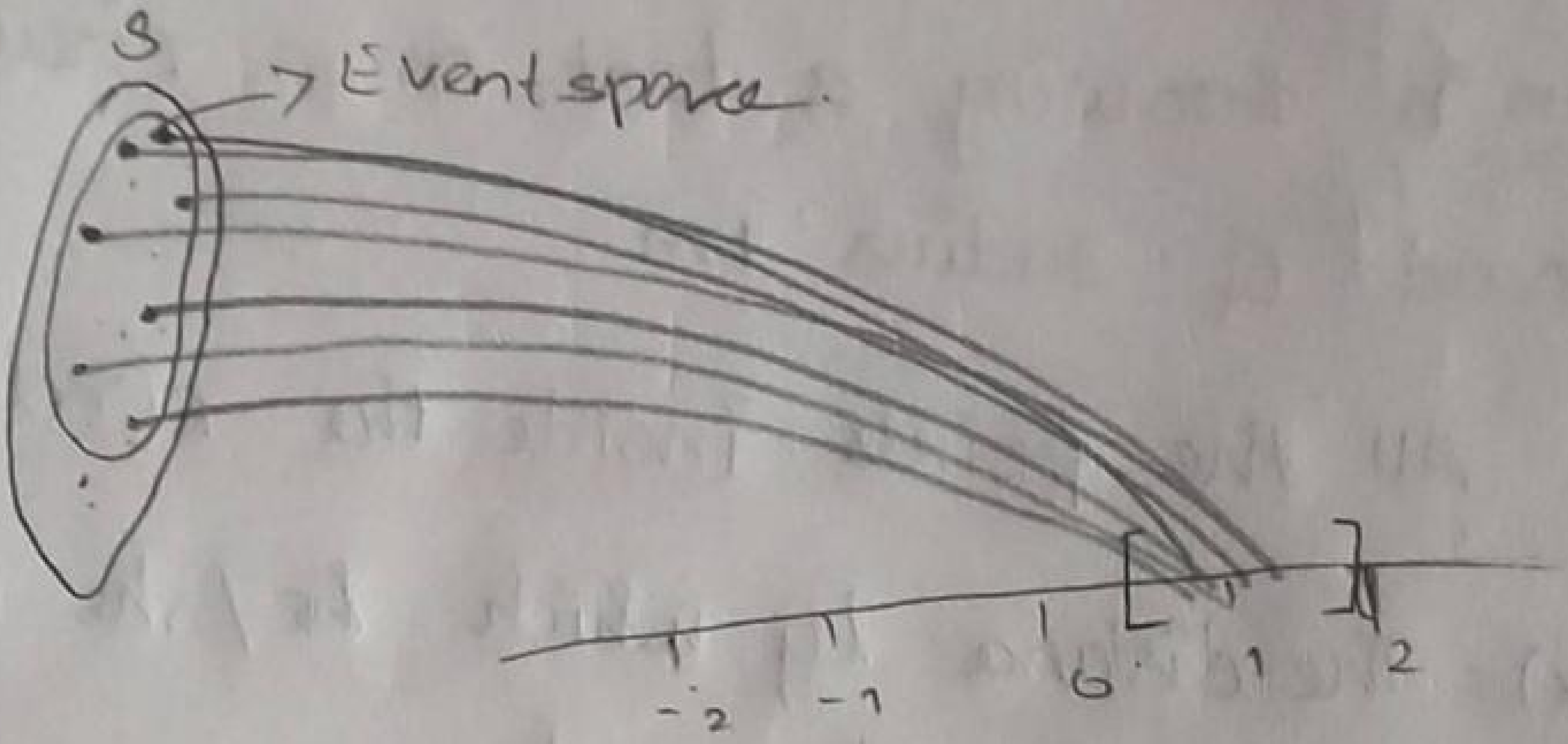
Ω = Event space

if for each interval $I \subseteq R$,

$\{ w : X(w) \in I \}$ is an event.

outcomes

numbers in this Interval



A real valued function

Example: 3 satellites are launched in space

O/c : FFF FFS FSF FSS SFF SFS SSF SSS

R.V. X # ~~of satellites sent to orbit~~

space (X) = { 0, 1, 2, 3 } | set of values taken by random variables

A person tosses a coin until head appears

O/c = H, TH, TTH, TTTH . . .

Random variable : x no of tails for getting head

space (x) : { 1, 2, 3 . . . }

3) A person is waiting for a call from his / her friend starting from 6AM. 3/8/2021

O/c All time instances from 6:00 AM of 3/8/2021

RV x: The amount of time he/she waits for the call

space (x) = $[0, \infty)$

4) A person is throwing a dart on a circular dart board of radius 1m

O/c : All the points inside the circle

Space(x) : The distance by which he/she miss the center of board.

$\therefore [0, 1]$

Random variable

Discrete

It has discrete values. Takes countable no of values

Eg: 1 and 2

Continous

has continous values Takes any value in the interval of ranges.

Eg: 3 and 4

Example 5

tossing 2 dice

X : sum of 2 dice

Y : Max of the 2dice

X_i : o/c of the i^{th} die

$X = X_1 + X_2$

$Y = e^X$

An example of event space

$X = \{0, 1, 2, 3\}$

$\{ [X=0], [X=1] [X=2] [X=3] \}$

partition of X

For a random variable X,

Let $A_x = \{ w \in S \ \ X(w) = x \}$

then,

$A_x \cap A_y = \varphi$ for $x \neq y$

$\bigcup_x A_x = S$

The collection of events $\{A_x\}$ for all x defines an event space.

Distribution function (df) (or) cumulative density fn

For a r.v X, the function $F : R \to [0, 1]$ defined by $F(x) = P[X \leq x]$ for $x \in R$ is call df or cdf.

Example:

$x = \{0, 1, 2, 3\}$

$P[x=0] = 1/8$

$P[x=1] = 3/8$

$P[x=2] = 3/8$

$P[x=3] = 1/8$

$F(1) = 4/8$

$F(1.2) = 4/8$

$F(2.7) = 7/8$

$F(5) = 1$

$F(-3) = 0$

$F(3) = 1$

sine by df (or) cdf

WKT

$F(x) = P[X \leq x]$

<u>Properties</u>:

1) F is non-decreasing function

$x_1 < x_2 \Rightarrow F(x_1) \leq F(x_2)$

F(x) can either increase or stay level.

It cannot decrease

<u>Proof</u>:

$x_1 < x_2$

$[x \leq x_1] < [x \leq x_2]$

If $A \leq B$

$P(A) \leq P(B)$

$P[x \leq x_1] \leq P[x \leq x_2]$

$F(x_1) \leq F(x_2)$

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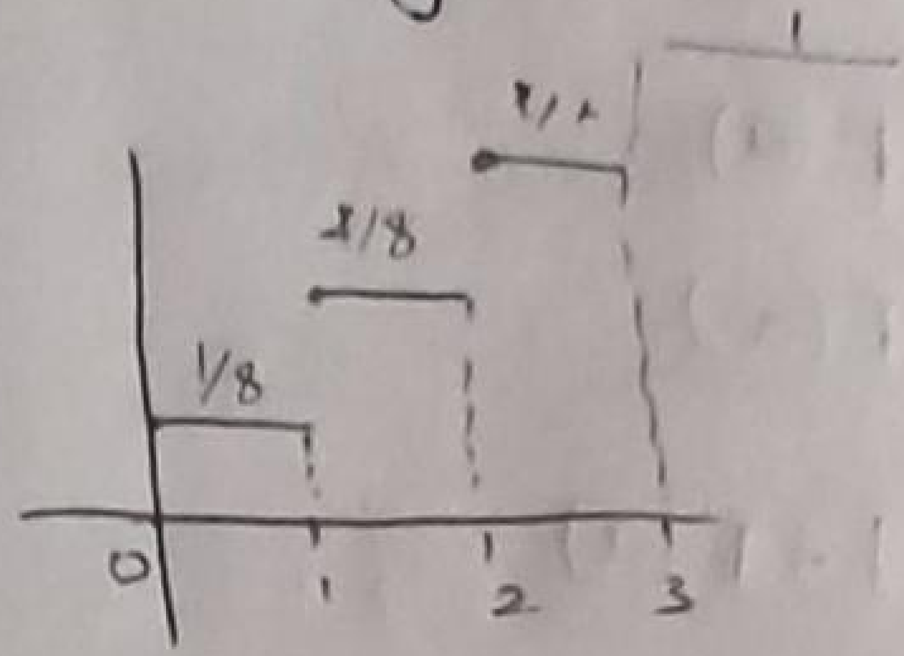
2) Function is right continuous



3) $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

4) $F(\infty) = \lim_{x \to \infty} F(x) = 1$

If $F(a^+) = \lim_{x \to a^+} f(a)$

$F(a^-) = \lim_{x \to a^-} f(x)$

When debaling with a random variable (i.e) X. for any a, b where $a < b$, we want to final the probabilities for one or more of the following events

NOTE:

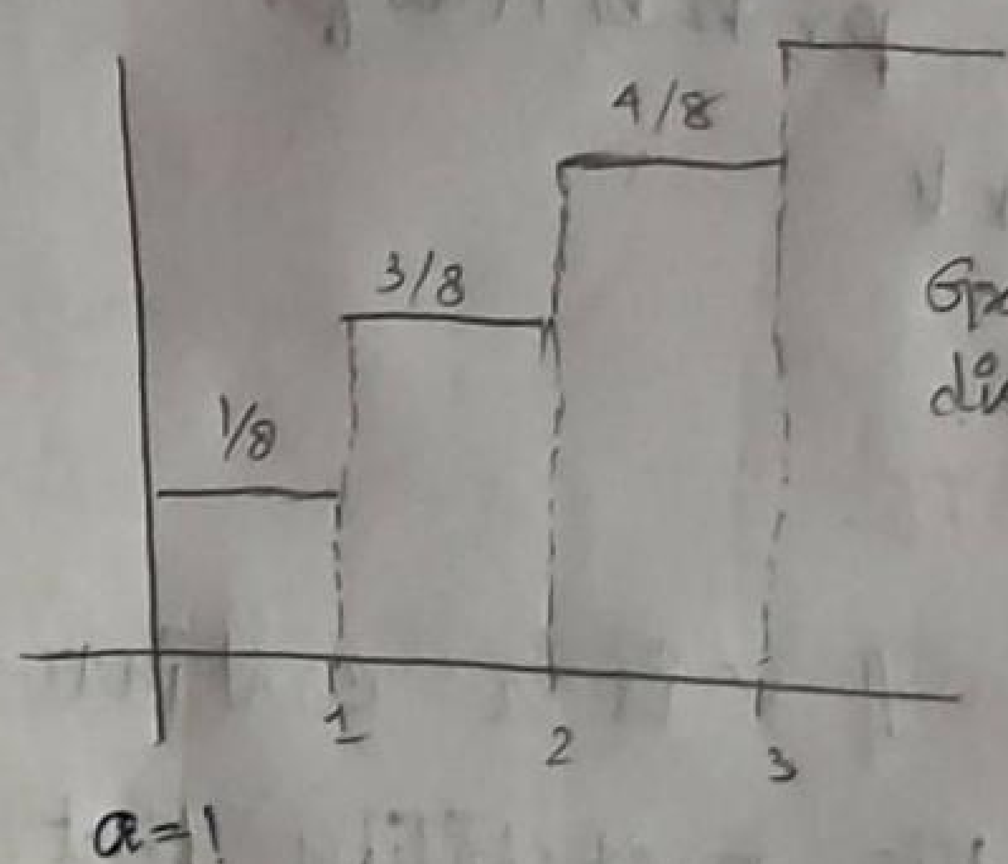
If X is a continuous random variable from $P[X=a] = 0$ the probability that X takes a particular value is zero.

Event	Prob
$[x \le a]$	$F(a)$
$[x < a]$	$F(a^-)$
$[x > a]$	$1 - F(a)$
$[x = a]$	$P[x \le a] - P[x < a] = F(a) - F(a^-)$
$[x \ge a]$	$1 - F(a^-)$
$[x \ne a]$	$1 - [F(a) - F(a^-)]$
$[a < x < b]$	$F(b^-) - F(a)$
$[a < x \le b]$	$F(b) - F(a)$
$[a \le x \le b]$	$F(b) - F(a^-)$
$[a \le x < b]$	$F(b^-) - F(a^-)$

$[x \le a] = F(a) = \frac{4}{8}$

$[x < a] = F[1^-] = \frac{1}{8}$

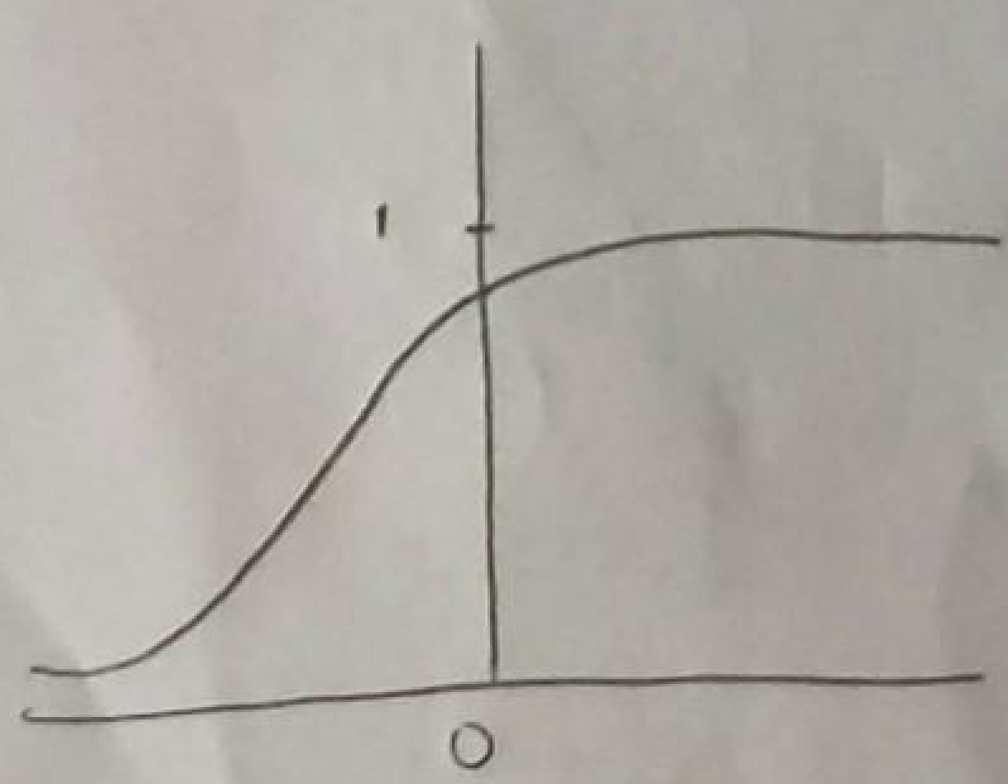
$[x > a] = 1 - \frac{4}{8} = \frac{4}{8}$



Graph distribution

$a = 1$

Graph of continous R.V



$P(x = 1.5) = F(1.5) - F(1.5)$

$P(x = 1) = \frac{4}{8} - \frac{1}{8} = \frac{3}{8}$

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Probability distribution

probability mass function - if x is discrete.

probability density function - if x is continuous

Proof for PMF

f is the prob of a discrete random variable x, if

$$f(x_i) = P[x = x_i] \quad \forall x_i \in \text{space}(x)$$

Note

i) $f(x_i) \geq 0$

(ii) $\sum_{x_i} f(x_i) = 1$

If prob f is known, F can be found

$$F(x) = P[X \leq x] = \sum_{x_i < x} f(x_i)$$

Proof for pdf

For a continuous random variable X,

We talk about the probability that x takes a value in an interval the function f is called the pdf of a continuous r.v x if

$$\int_a^b f(x)\,dx = P[a \leq X \leq b] \qquad \sum_{x_i} f(x_i) = 1$$

$$F(a) = \int_{-\infty}^{a} f(x)\,dx$$

$$F(x) = P(X \leq x)$$

Note:

i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ii) $F(x)$ is a prob measure But $f(x)$ is not a prob measure unless it is multiplied by an infinitesimal Δx to yield

$$f(x)\,\Delta x = P[x < X < x + \Delta x]$$

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Define f as follows

$$f(x) = \lim_{\Delta x \to 0^+} \frac{P[x < X \leq x + \Delta x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0^+} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

$$= F'(x)$$

1) The pdf of a rv X is given by

$$f(x) = \begin{cases} k(2x - x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find k

(ii) Find $P[X > 1]$

(i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_0^2 k(2x - x^2)\,dx = 1$

$\boxed{k = 3/4}$

(ii) $P[X > 1] = \int_1^2 f(x)\,dx = \int_1^2 \frac{3}{4}(2x - x^2)\,dx = \frac{1}{2}$

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2) The cdf of a r.v X is given by

$$F(x) = \begin{cases} 0 & ; x < 2 \\ k(x-2) & , 2 < x \leq 2 \\ 1 & x \geq 6 \end{cases}$$

(i) Find k

(ii) Find $P[x > 4]$

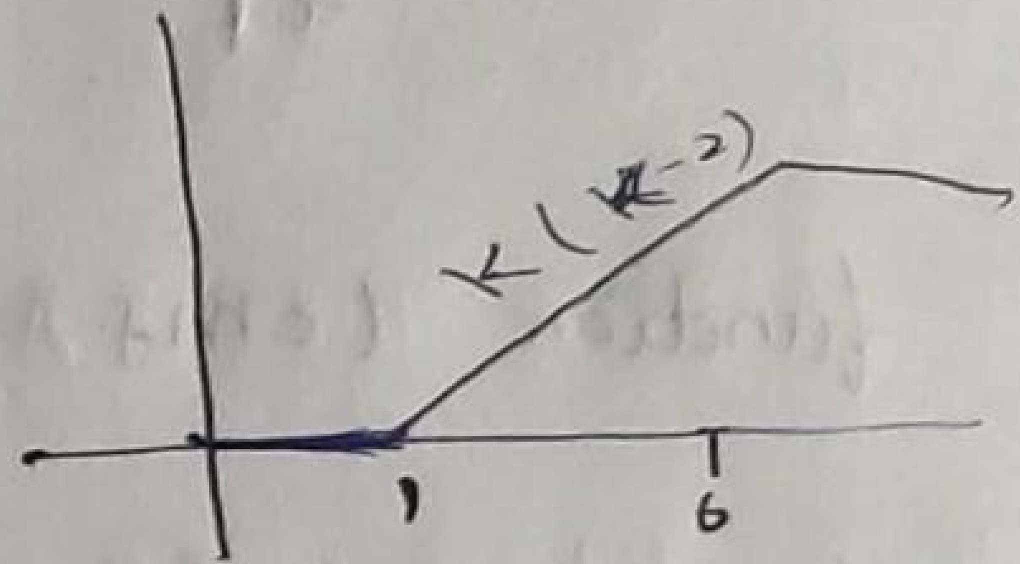
(iii) find $P[3 \leq X \leq 5]$

(i) $F(6) = 1$

$K(6-2) = 1$

$4K = 1$

$k = 1/4$



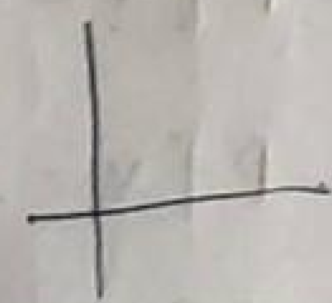
(ii) $P[x \geq 4] = 1 - F(4)$

$= 1 - k(4-2)$

$= 1 - 2k$

$= 1/2$

$x = [2, 6]$



(iii) $P[3 \leq x \leq 5] = F(5) - F(3^-)$

Two device are rolled ~~twice~~ let x denote the absolute value of the difference of the o/e's obtain the pmf of x.

Space (x) : { 0, 1, 2, 3, 4, 5 }

$X = x$	0	1	2	3	4	5
$P(x)$	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	

$$\sum_{x_i} P(x_i) = 1$$

The function (cmf) of a rate x is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ x/4 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ \frac{x}{12} + \frac{1}{2} & 2 \le x < 3 \\ 1 & x \geq 3 \end{cases}$$

(i) Sketch the graph of f.

(ii) Find $P[x < 2]$

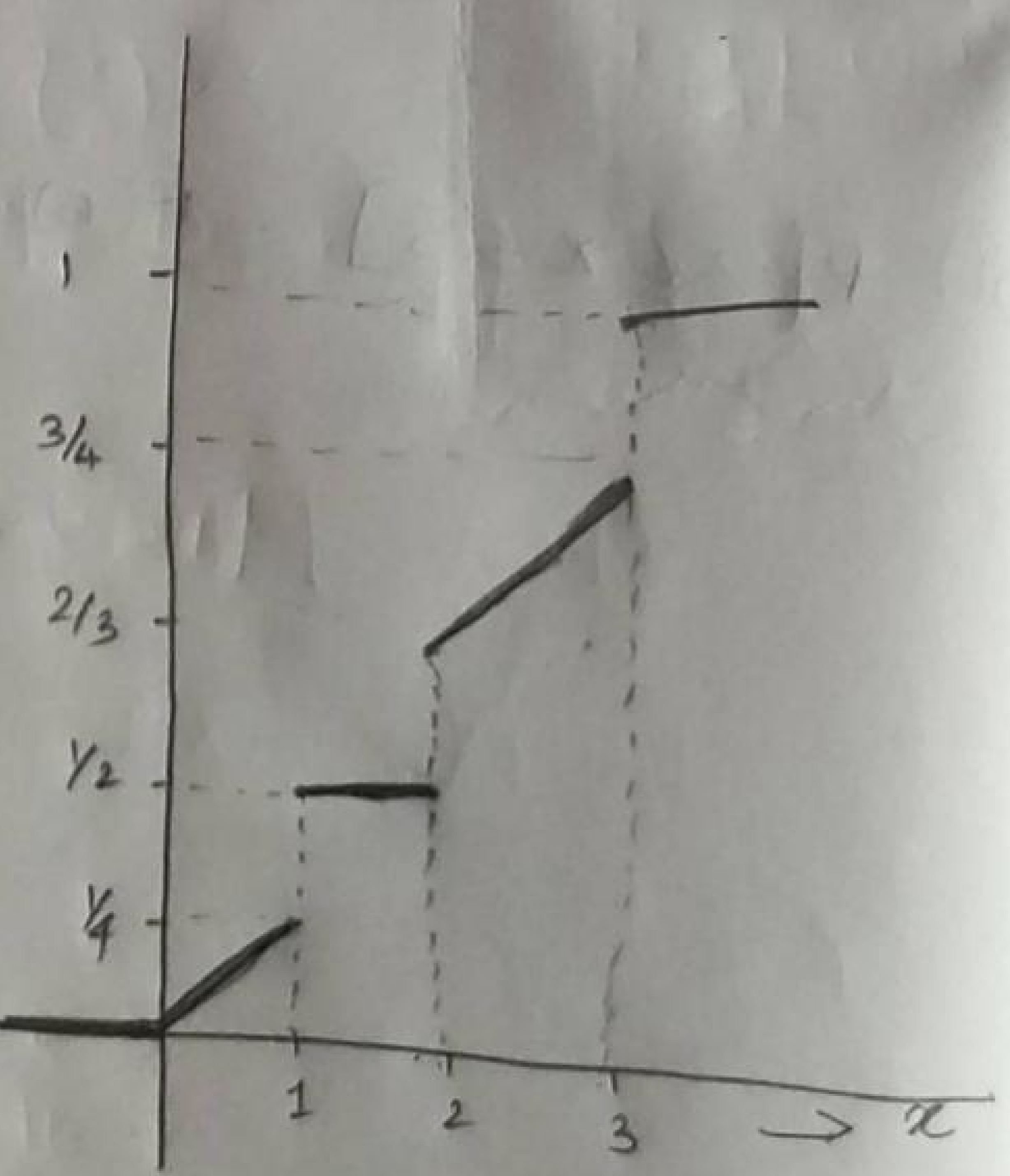
$P[x = 2]$

$P[1 \le x < 3]$

$P[x > 3/2]$

$P[x = 5/2]$

$P[2 < x \le 7]$



i) $P[x < 2]$

$= F(2^-) = 1/2$

(ii) $P[x = 2]$

$= F(2) - F(2^-)$

$= \left(\frac{2}{12} + \frac{1}{2}\right) - \frac{1}{2} = 1/6.$

(iii) $P[1 \leq x < 3]$

$= F(3^-) - F(1)$

$= \left(\frac{3}{12} + \frac{1}{2}\right) - \frac{1}{4} = 1/2.$

(iv) $P[x > 3/2]$

$= 1 - P(3/2)$

$= 1 - \frac{1}{2} = 1/2.$

(v) $P[x = 5/2]$

$= F(5/2) - F(5/2)$

$= 0.$

(vi) $P[2 < x \leq 7]$

$= F(7) - F(2)$

$= 1 - \left(\frac{2}{12} + \frac{1}{2}\right) = 1/3$

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The sales (in \$) of a convenience store on a randomly selected day is a RV with CDF

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2/2 & 0 \le x < 1 \\ k(4x - x^2) & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

a) Suppose this store's total sales on any given day is less than \$ 2000 find the value of 'k'

b) Let A : Tmrw's sales is btw \$ 500 & \$ 1500

B : Tmrw's sales is over \$ 1000

Find P(A) & P(B). Are A & B independent.

a) Given $P[\text{sales} < \$ 2000] = P[x < 2] = 1$

$F(2^-) = 1$

$k(4(2) - 2^2) = 1$

$\boxed{k = 1/4}$

b) $P(A) = P[0.5 \le x < 1.5)$

$= F(1.5^-) - F(0.5^-)$

$= \frac{1}{4}\left[4\left(\frac{3}{2}\right) - \left(\frac{3}{2}\right)^2\right] - \frac{1}{2}\left(\frac{1}{2}\right)^2$

$= \frac{1}{4}\left[\frac{15}{\cancel{4}} - \frac{1}{2}\right] = \frac{13}{16}$

$= 1 - F(1)$

$= 1 - \frac{1}{4}\left[4(1) - 1^2\right]$

$= 1 - \frac{3}{4}$

$= 1/4$

$P(A \cap B) = P[1/2 \le x < 3/2 \; , x \ge 1]$

$= P[1 < x < 3/2]$

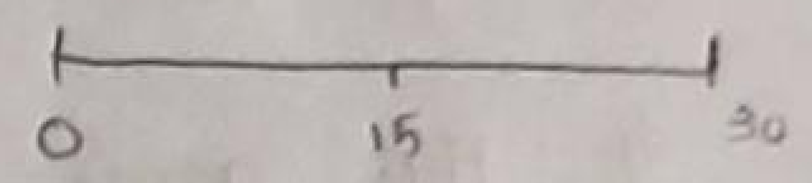
$= P\left(\frac{3}{2}\right)^- - F(1)$

$= \frac{1}{4}\left(\frac{1.5}{\cancel{4}}\right) - \frac{3}{4} = \frac{9}{16}$

$P(A) \cdot P(B) = \frac{13}{16} \cdot \frac{1}{4}$ Non independent

Suppose that a bus arrives at a bus stop every day b/w 10:00 and 10:30 at a random. let 'X' denote the waiting time. find cdf of F

$F : \mathbb{R} \to [0, 1]$



$x \to$ waiting time

$x = [0, 30)$

$F(15) = P[x \leq 15]$

$= 15/30$

[uniform random variable]

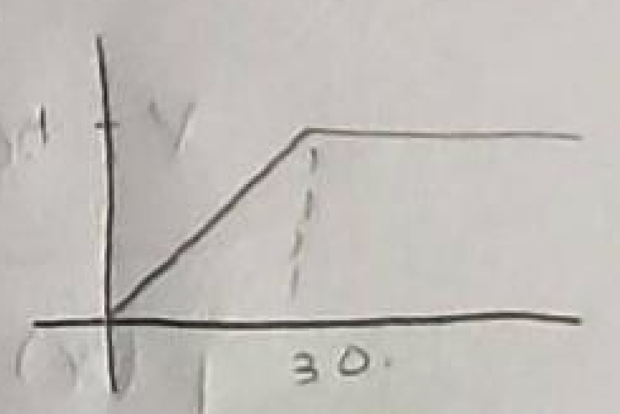
<u>To find $F(t)$</u>

For $t < 0$ $\quad F(t) = 0$

For $0 \leq t < 30$, $\quad F(t) = \dfrac{t - 0}{30 - 0} = \dfrac{t}{30}$

For $t \geq 30$; $\quad F(t) = 1$

$$F(t) = \begin{cases} 0 & t < 30 \\ t/30 & 0 \leq t < 30 \end{cases}$$



Let x be a point selected at random from $(0, 1)$. Find the distribution function (CDF) of $y = \dfrac{x}{1+x}$

Refer prev prob

$$\text{CDF of } x = F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

To find CDF F_Y of $y = \dfrac{x}{1+x}$

$$F_Y(y) = P[Y \leq y]$$

$$= P\left[\frac{X}{1+X} \leq y\right]$$

$$= F_X\left(\frac{y}{1-y}\right) \quad 0 \leq y \leq 1/2$$

$\dfrac{x}{1+x} \leq y$

$x \leq y + xy$

$x(1-y) \leq y$

Mean and Variance.

Q) In a casino game, the probability of loosing $1 is 0.6 and the probability of winning 1$, 2$ and 3$ are 0.3, 0.08, 0.02

If this game played n times, find the gain.

$(-1\$)(0.6)n + (1)(0.3)n + 2(0.08)n + 3(0.02)n = -0.08n$

In an average there is a loss of 0.08 $ per game

let x : Gain (a r.v)

$E(x) = -0.08$ is a expected value of x

$X = x$	-1	1	2	3
$p(x)$	0.6	0.3	0.08	0.02

Then,

$$E(x) = (-1)(0.6) + 1(0.3) + 2(-0.08) + 3(0.02)$$
$$= -0.08$$

$$\boxed{E(x) = \sum_x x \cdot P(x)}$$

For a random variable x, the exp. value of x is defined by

$$E(x) = \begin{cases} \sum_x x \cdot P(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x \cdot f(x)\,dx & \text{if } x \text{ is con.} \end{cases}$$

$E(x)$ is weighted average of all values of x in which the weight assigned to each $x \in X$ is $p(x)$

Mean $E(X)$ is a measure of the centre of probability distribution.

Properties.

$E(K) = K$ where K is constant

$E(Kx) = K \cdot E(X)$

$E(X \pm Y) = E(X) \pm E(Y)$

Variance : -> spread or how the values of random variable are spread or dispersed.

Suppose we measure a of contanin quantity

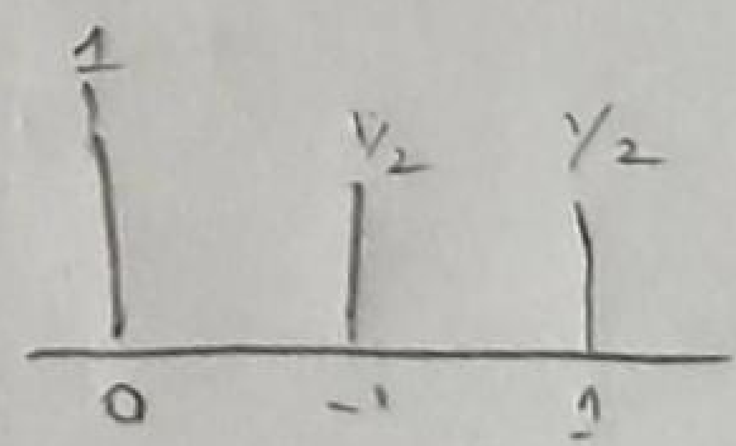
Let X Error (~~true~~ value - value obtained in measurement)

X is a randon variable with $E(X) = 0$.

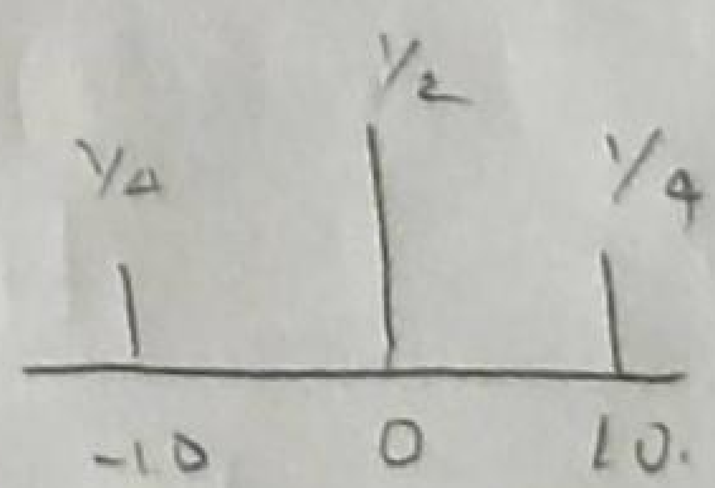
Consider the random variable s

$x = 0$ with prob 1

$$y = \begin{cases} -1 & \text{with prob } 1/2 \\ 1 & \text{with prob } 1/2 \end{cases}$$



$$z = \begin{cases} -10 & \text{with prob } 1/4 \\ 0 & \text{with prob } 1/2 \\ 10 & \text{with prob } 1/4 \end{cases}$$



$E(X) = \mu$