Memoryless property of Exp distⁿ

$$P[X > t + s \mid X > s] = P[X > t]$$

If x life time of an instrument, the probability that the instrument lives for more than t+s years given it has already lasted more than s years is the same as initial prob that it lives for more t yrs.

Proof

$$\text{RHS} = P[X > t] = \int_t^\infty f(x)\, dx$$

$$= \int_t^\infty \lambda e^{-\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_t^\infty$$

$$= -\left[0 - e^{-\lambda t} \right]$$

$$\boxed{P[X > t] = e^{-\lambda t}}$$

LHS

$$P[X > t+s \mid X > s] = \frac{P[X > t+s,\ X > s]}{P[X > s]}$$

$$= \frac{P[X > t+s]}{P[X > s]}$$

$$= \frac{e^{-(t+s)\lambda}}{e^{-\lambda s}}$$

$$\boxed{P[X > t+s \mid X > s] = e^{-\lambda t}}$$

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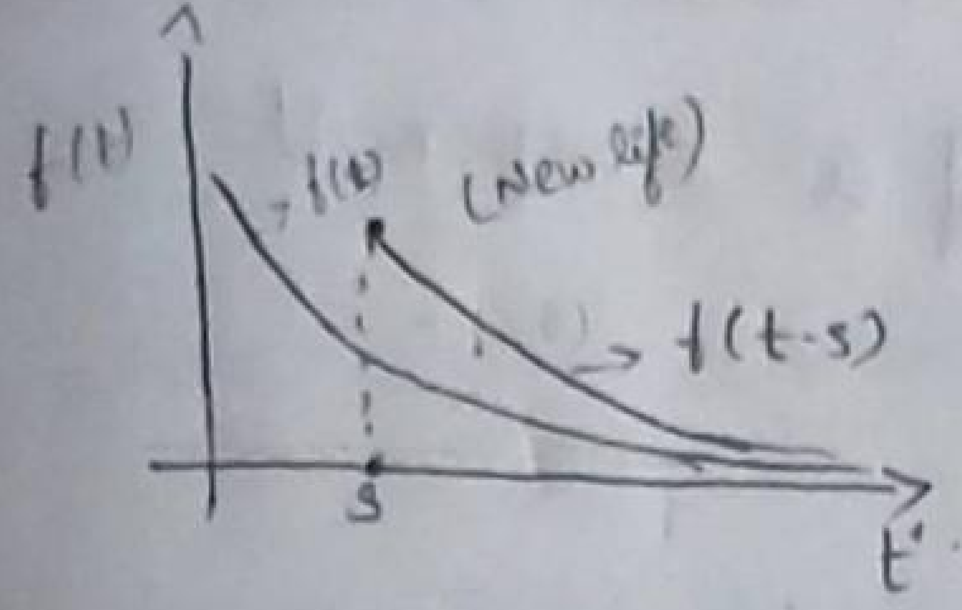


$$f(x) = \lambda e^{-\lambda t}, \quad x \geq 0$$

Suppose the ~~amt~~ of time one spends in a bank follows ED with a mean of 10 mins. What is the probability that a customer will spend more than 15 mins in the bank?

What is the prob that she will spend more than 15 mins given she is still in the bank after 10 mins.

x ~ time spent in the bank (in mins) mean $\frac{1}{\lambda} = 10$

$x \sim Expo\left(\lambda = \frac{1}{10}\right)$

WKT

$P[x > t] = e^{-\lambda t}$

$$P[x > 15] = e^{-1/10(15)}$$
$$= e^{-1.5}$$

$$P[x > 10+5 \mid x > 10] = P[x > 5] \quad \text{(Memoryless property)}$$
$$= e^{\frac{-1}{10}(5)}$$
$$= e^{-0.5}$$

\# The question before special continous dists: guest arrives

$\lambda = 5$ per hour

$\lambda = \frac{5}{60}$ per minutes

x - Time for the arrival of next guest

$x \sim Expo\left(\lambda = \frac{1}{12}\right)$

→ guest not arriving till 10 mins

→ guest not arriving till 10 mins and arriving after 2 mins

$$\boxed{1 - P[x > 10+2 \mid x > 10]} = 1 - P[x > 2]$$
$$= 1 - e^{-1/12(2)} = 1 - e^{-1/6}$$

↓

P[Next guest arriving after 2 mins]

Weibull distribution:

x: time to failure of a component

If pdf of x is

$$\boxed{f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}} \; ; x \geq 0$$

then x follows weibull distribution

parameters α and β

α ↙ scale parameter β ↘ shape parameter

Mean & variance:

$$E[x^{r}] = \int_0^{b} x^{r} f(x)\, dx$$

du

$$= \int_0^{\infty} x^{r} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} dx$$

Take $u = \alpha x^{\beta} \rightarrow x^{\beta} = \frac{u}{\alpha} \;\; = \;\; x = \left(\frac{u}{\alpha}\right)^{1/\beta}$

$du = \alpha\beta x^{\beta-1} dx$

$$= \int_0^{\infty} e^{-u} \left(\frac{u}{\alpha}\right)^{r/\beta} du$$

$$= \frac{1}{\alpha^{r/\beta}} \int_0^{\infty} e^{-u} u^{r/\beta} du$$

WKT

$n-1 = \frac{r}{\beta}$

$n = \frac{r}{\beta}+1$

$$= \frac{1}{\alpha^{r/\beta}} \Gamma\left(\frac{r}{\beta}+1\right)$$

$$\boxed{r=1} \Rightarrow E(x) = \frac{1}{\alpha^{1/\beta}} \Gamma\left(\frac{1}{\beta}+1\right)$$

$$\boxed{r=2} \Rightarrow E(x^2) = \frac{1}{\alpha^{2/\beta}} \Gamma\left(\frac{2}{\beta}+1\right)$$

Failure rate | Hazard rate $z(t)$

$$z(t) = \alpha\beta\, t^{\beta-1}$$

where α is constant

Weibull

$x \Rightarrow$ time to failure

$$\boxed{f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}}$$

$\alpha \Rightarrow$ scale

$\beta \Rightarrow$ shape

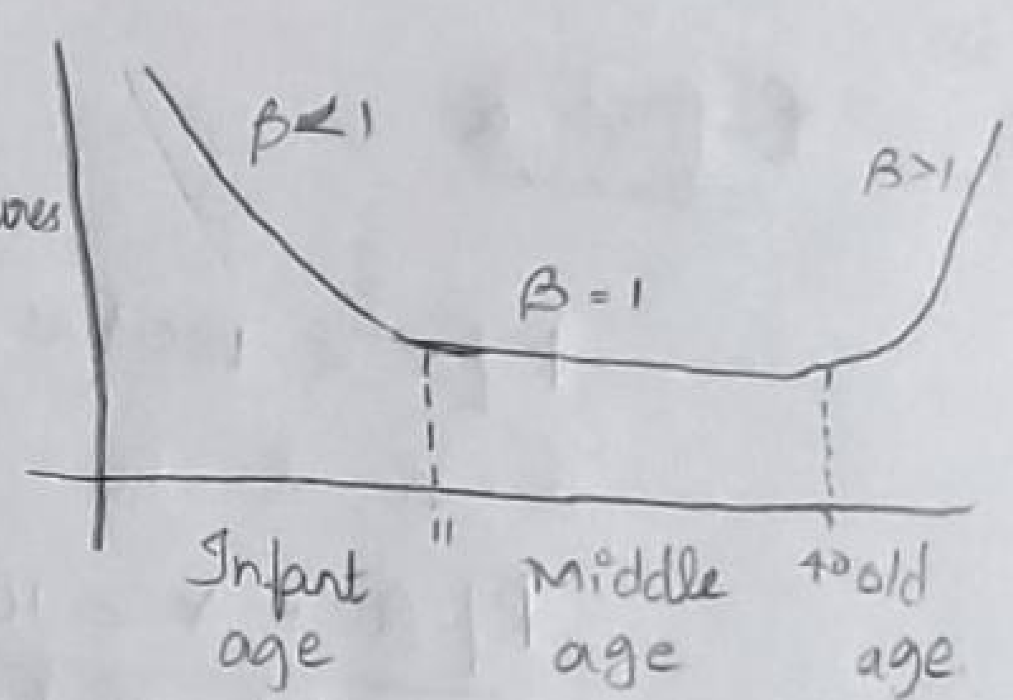
$$E(X^{r}) = \frac{1}{\alpha^{r/\beta}} \Gamma\left(\frac{r}{\beta}+1\right)$$

$\beta < 1 \;\rightarrow\; z(t)$ is decreasing

the equipment strengthen over time.

$\beta = 1 \;\rightarrow\; z(t) = \alpha$, constant

Becomes exponential distribution in which the memeoryless property prevails



$\beta > 1 \;\Rightarrow\; z(t)$ is increasing,

components wear out over time.

Suppose the life time (in hrs) of a certain weibull. with $\alpha = 0.1$ & $\beta = 0.3$

X : life time (in hrs)

$X \sim$ Weibull $(\alpha = 0.1, \beta = 0.5)$

(i) $P[X > 300] = e^{-(0.1)(300)^{0.5}}$ $\Rightarrow P[X>t] = e^{-\alpha t^{\beta}}$

$= e^{-1.732} = 0.176$

(ii) Mean life time $E(X) = \dfrac{1}{(0.1)^{\frac{1}{(0.5)}}} \Gamma\left(\dfrac{1}{0.5}+1\right)$

$$E(X^{r}) = \frac{1}{\alpha^{r/\beta}} \Gamma\left(\frac{r}{\beta}+1\right)$$

$= 100 \times \Gamma(2+1)$

$= 100 \times \Gamma(3) = 100 \times 2!$

$= 200!$

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Suppose the life time (in hrs) of a component weibull with $\beta = 2$ from past experience it is known that 15% of the components that have lasted 90 hrs. Find before 100 hrs. Determine the value of α.

x : lifetime (in hrs)

$x \sim$ weibull ($\alpha = ?$, $\beta = 2$)

$P(x > t) = e^{-\alpha t^{\beta}}$

$F(t) = 1 - e^{-\alpha t^{\beta}}$

$\Downarrow$

$P[x \leq t]$

To find α

$$P[x < 100 \text{ hrs} \mid x > 90 \text{ hrs}] = 15\% = 0.15$$

$$\frac{P[90 < x < 100]}{P[x > 90]} = 0.15$$

$$\frac{F(100) - F(90)}{P[x > 90]} = 0.15$$

$$\frac{(1 - e^{-\alpha(100)^2}) - (1 - e^{-\alpha(90)^2})}{e^{-\alpha(90)^2}} = 0.15$$

$$\frac{e^{-8100\alpha} - e^{-10000\alpha}}{e^{-8100\alpha}} = 0.15$$

$$1 - e^{-1900\alpha} = 0.15$$

$$e^{-1900\alpha} = 0.85$$

Taking ln on both sides,

$$-1900\alpha = \ln(0.85)$$

$$\alpha = \frac{\ln(0.85)}{-1900}$$

$$= 0.00009$$

The life time of a certain components from weibull with $\beta = 2$. Find the value of α given that the prob that the component, life exceeds 5 yrs is $e^{-0.25}$. Find the mean & variance.

X : life time (in yrs)

$X \sim$ weibull ($\alpha = ?$, $\beta = 2$)

<u>To find α</u>

given $P[X > 5] = e^{-0.25}$ $\qquad$ $P[X > t] = e^{-\alpha t^{\beta}}$

$$e^{-\alpha (5)^2} = e^{-0.25}$$

$$25\alpha = 0.25$$

$$\boxed{\alpha = 0.01}$$

$$E(X) = \frac{1}{\alpha^{\gamma/\beta}} \Gamma\left(\frac{\gamma}{\beta} + 1\right)$$

$$= \frac{1}{(0.01)^{1/2}} \Gamma\left(\frac{1}{2} + 1\right)$$

$$= \frac{1}{0.1} \Gamma\left(\frac{1}{2} + 1\right)$$

$\Gamma(n+1) = n\Gamma(n!)$

$$= 10 \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$\Gamma(1/2) = \sqrt{\pi}$

$$= 5\sqrt{\pi}$$

$$E(X^2) = \frac{1}{(0.01)^{2/2}} \Gamma\left[\frac{2}{2} + 1\right]$$

$$= 100\, \Gamma(2)$$

$$= 100 \times 1! = 100$$

$$\text{variance}(X) = E(X^2) - E(X)^2$$

$$= 100 - 25\pi$$

Normal / Gaussian Distribution. (Z)

pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

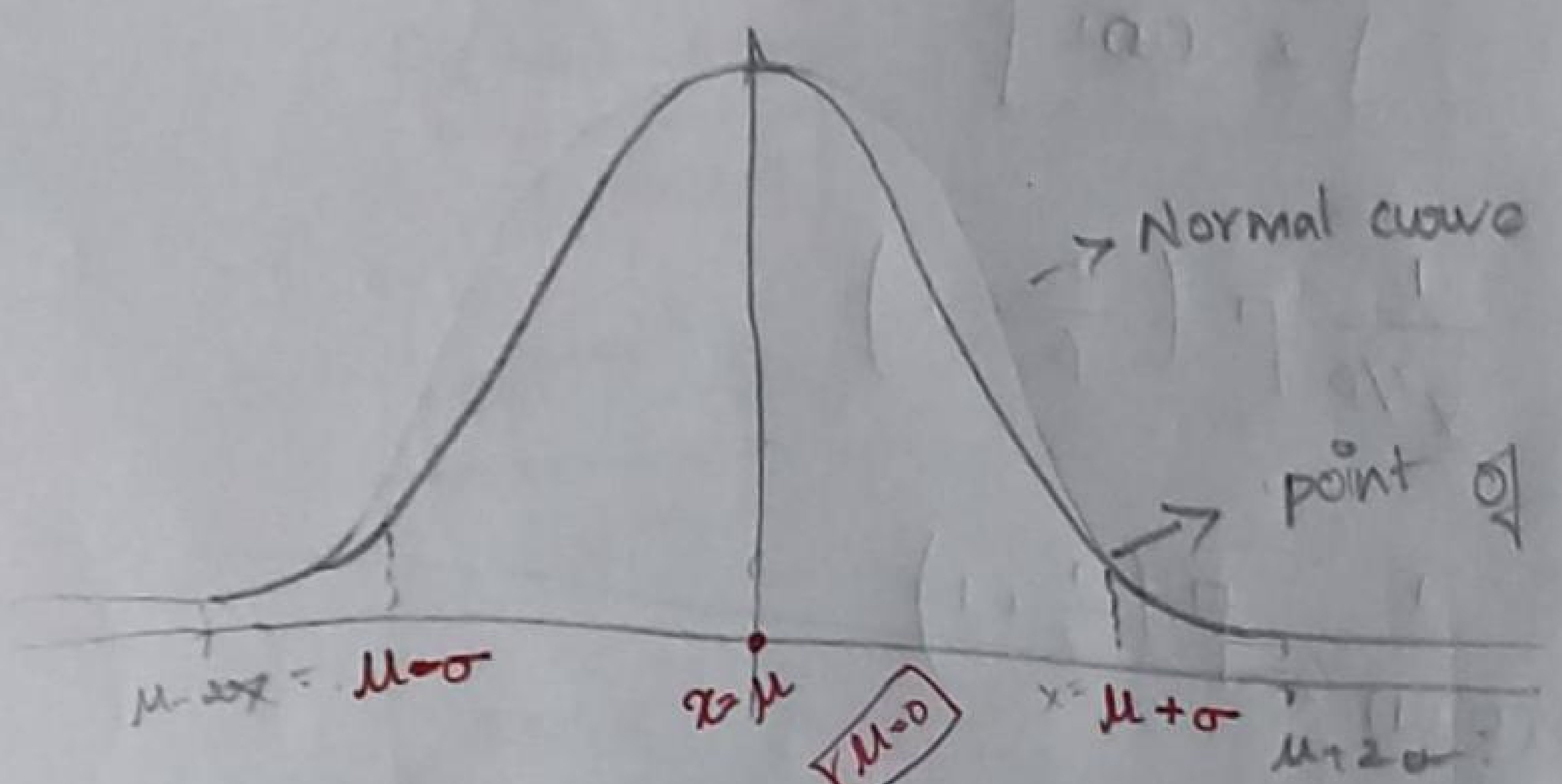
parameters μ & σ.

μ → mean

σ → std deviation

Special case : Standard Normal distribution

It is a normal distribution with $\mu = 0$ & $\sigma = 1$.



Normal curve :→ graph of the pdf of normal distribution.

Properties of normal

→ Symmetric about the mean $x = \mu$

→ Mean = Median = Mode

→ $x = \mu - \sigma$, $x = \mu + \sigma$ are points of inflexion

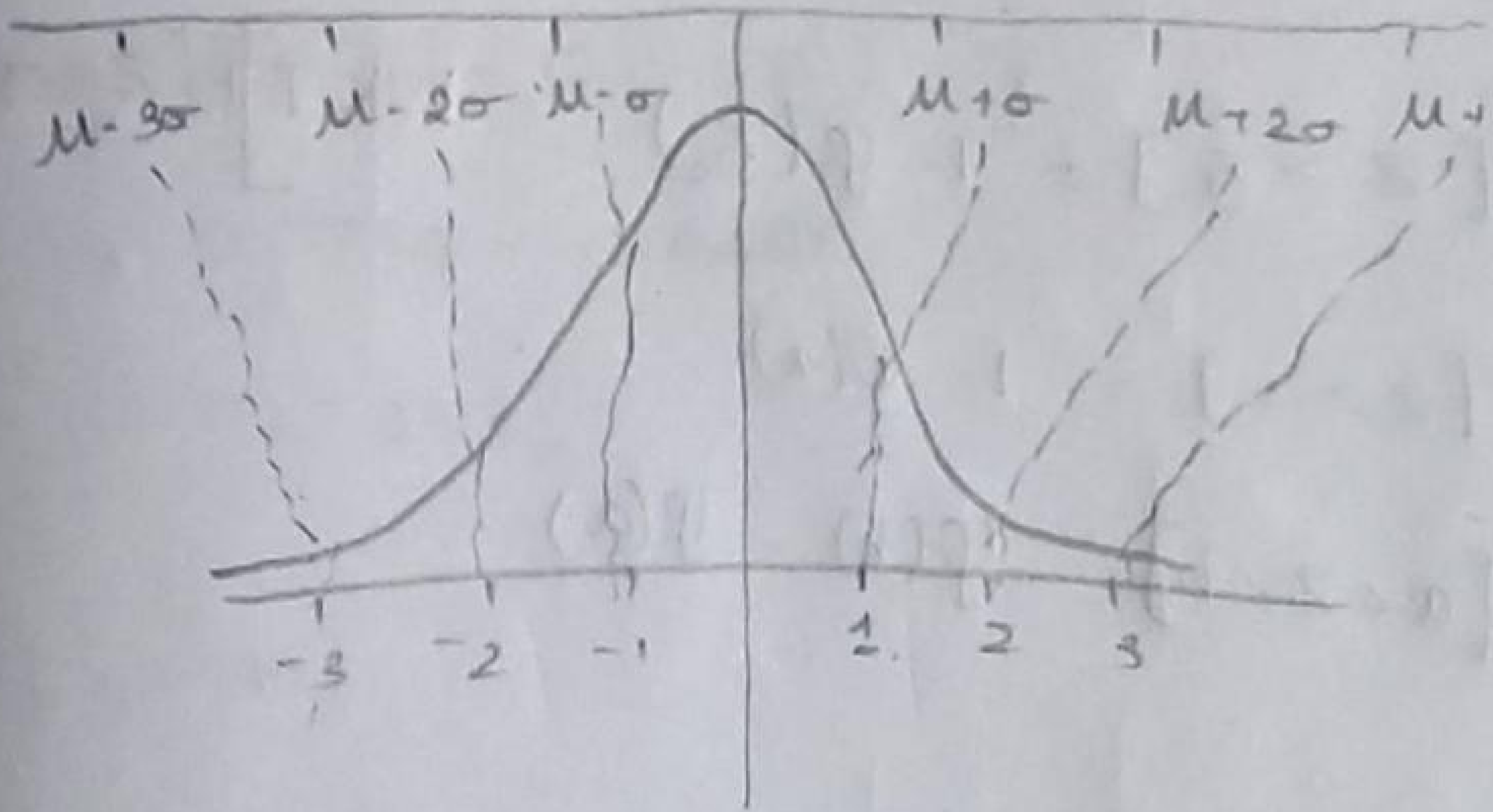
→ x axis is an asymptote

→ $\int_a^b f(x)\,dx$ = Area under normal curve $x = a$, $x = b$

⇒ Normal table contain probability for standard normal distribution (SND) · $\mu = 0$, $\sigma = 1$.

⇒ An arbitrary normal distribution with any mean μ and and SD σ can be converted into a SND by the transformation = $Z = \frac{x-\mu}{\sigma}$

where Z is the SN variable.



$\mu - 3\sigma = -3$ $\qquad$ $\mu + 3\sigma = 3$

$\mu - 2\sigma = -2$ $\qquad$ $\mu = 0$ $\qquad$ $\mu + 2\sigma = 2$

$\mu - \sigma = -1$ $\qquad$ $\mu + \sigma = 1$.

∴ $\boxed{\text{Any ND} \rightarrow \text{SND}}$

Emperical rules:

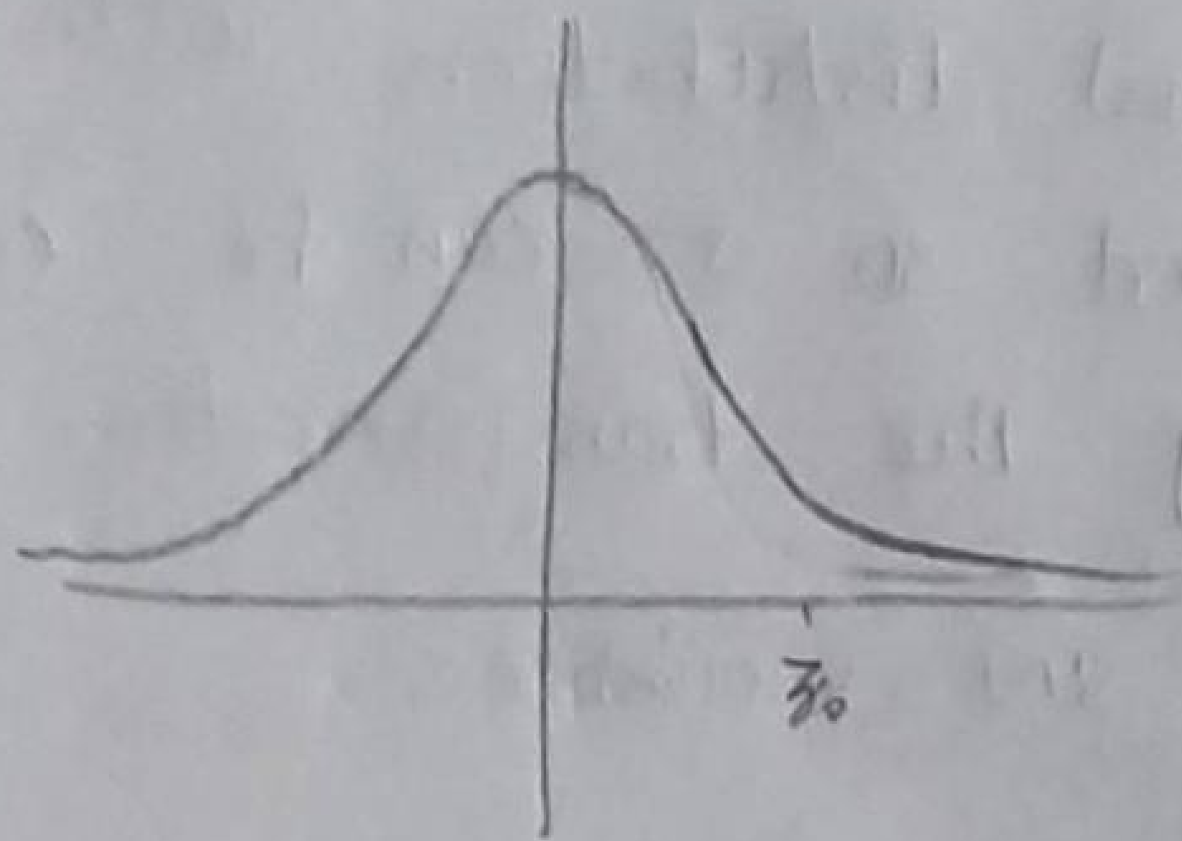
$P[\mu - \sigma < x < \mu + \sigma] = 68\%$

$P[\mu - 2\sigma < x < \mu + 2\sigma] = 95\%$

$P[\mu - 3\sigma < x < \mu + 3\sigma] = 99.7\%$

68 95 99.7 rule

From normal table one can find $\boxed{P[Z \le z_0] = \varphi(z_0)}$
for any $z_0 \in \mathbb{R}$



$$P[Z > z_0] = 1 - \varphi(z_0)$$

WKT $P[Z \le z_0] = \varphi(z_0)$

$$P[Z > a] = 1 - \varphi(a)$$

$$P[a < Z < b] = \varphi(b) - \varphi(a)$$

A survey indicates that for each trip to the super market, a customer spends on average of 45 mins with a SD of 12 mins. The length of time spent follows ND. Wt is the prob a randomly selected customer will in the store.

(i) b/w 24 and 54 min

(ii) more than 39 mim

(iii) b/w 33 and 60 min

X : Time spent in the store (in mins

$X \sim N(\mu = 45, \sigma = 12)$

$$Z = \frac{X - \mu}{\sigma}$$

(i) $P[24 < x < 54]$

$P[Z < Z_0] = \varphi(z_0)$

$$= P\left[\left(\frac{24-45}{12}\right) < Z < \left(\frac{54-45}{12}\right)\right]$$

$$= P[-1.75 < Z < 0.75]$$

$$= \varphi(0.75) - \varphi(-1.75)$$

$$= 0.7734 - 0.0401$$

$$= 0.7333$$

(ii) $P[x > 39]$

$$= P\left[Z > \frac{39-45}{12}\right]$$

$$= P[Z > -0.5]$$

$$= 1 - \varphi(-0.5) = 1 - 0.3085 = 0.6915$$

(iii) $P[33 < x < 60]$

$$= P\left[\left(\frac{33-45}{12}\right) < Z < \left(\frac{60-45}{12}\right)\right]$$

$$= P[-1 < Z < 1.25]$$

$$= \varphi(1.25) - \varphi(-1)$$

$$= 0.8944 - 0.1587$$

$$= 0.7357$$

The scores on a test given to 1lkh students are normally distributed with mean 500 & SD 100
wt should be the score of student be to place them among the top 10% ?

x = Scores

$x \sim ND(\mu = 500, \sigma = 100)$

To find x such that

$P[X \geq x] = 10\%$

$P\left[Z \boxed{\geq} \frac{x-500}{100}\right] = \boxed{0.1}$

$(1-x)$

$P\left[Z < \frac{x-500}{100}\right] = \boxed{0.9}$

$P(Z-z_0) = \varphi(z_0)$

$\varphi\left(\frac{x-500}{100}\right) = 0.9$

(Finding in the reverse in ~~the~~ table)

From Normal table for prob = 0.9 (0.887), the Z score is 1.28

1.28 => 0.8997 Approx to 0.9

$\frac{x-500}{100} = 1.28$

$\boxed{x = 628}$

In a ND 31 % of the items are under 45 & 8% of the items are over 64. Find the mean & variance

$x \sim N(\mu, \sigma)$

given $P[x < 45] = 31\% = 0.31$

$P\left[Z < \frac{45-\mu}{\sigma}\right] = 0.31$

$\varphi\left[\frac{45-\mu}{\sigma}\right] = 0.31$

$0.3085 = -0.5$

// From table $\frac{45-\mu}{\sigma} = -0.5$

$\mu - 0.5\sigma = 45 \rightarrow ①$

Also,

$P[x > 64] = 8\% = 0.08$

$P\left[Z \boxed{>} \frac{64-\mu}{\sigma}\right] = \boxed{0.08}$

$P\left[Z \boxed{<} \frac{64-\mu}{\sigma}\right] = \boxed{0.92}$

$\varphi\left(\frac{64-\mu}{\sigma}\right) = 0.92$

// From table for prob 0.92, Z score = 1.41

$\frac{64-\mu}{\sigma} = 1.41$

$64 - \mu = 1.41\sigma$

$\mu + 1.41\sigma = 64 \rightarrow ②$

solving ① & ②

$\mu - 0.5\sigma = 45$

$-\mu - 1.41\sigma = -64$

$-1.91\sigma = -19$

$\boxed{\sigma = 9.948}$

$\mu - 0.5(9.948) = 45$

$\boxed{\mu = 47.974}$

Approx values

$\sigma = 10$

$\mu = 50.$

Moments

Let x be a r.v with mean $E(x) = \mu$.

$E(x^r)$ is called r^{th} moment of x (moment abt origin)

$E(x-\mu)^r$ is called r^{th} central moment of x (moment about mean)

Any central moment can be expressed as

n^{th} moment of origin

$$\boxed{E(x-\mu)^2 = E(x^2) - E(x)^2}$$

$-$

1^{st} moment of origin to the n power.

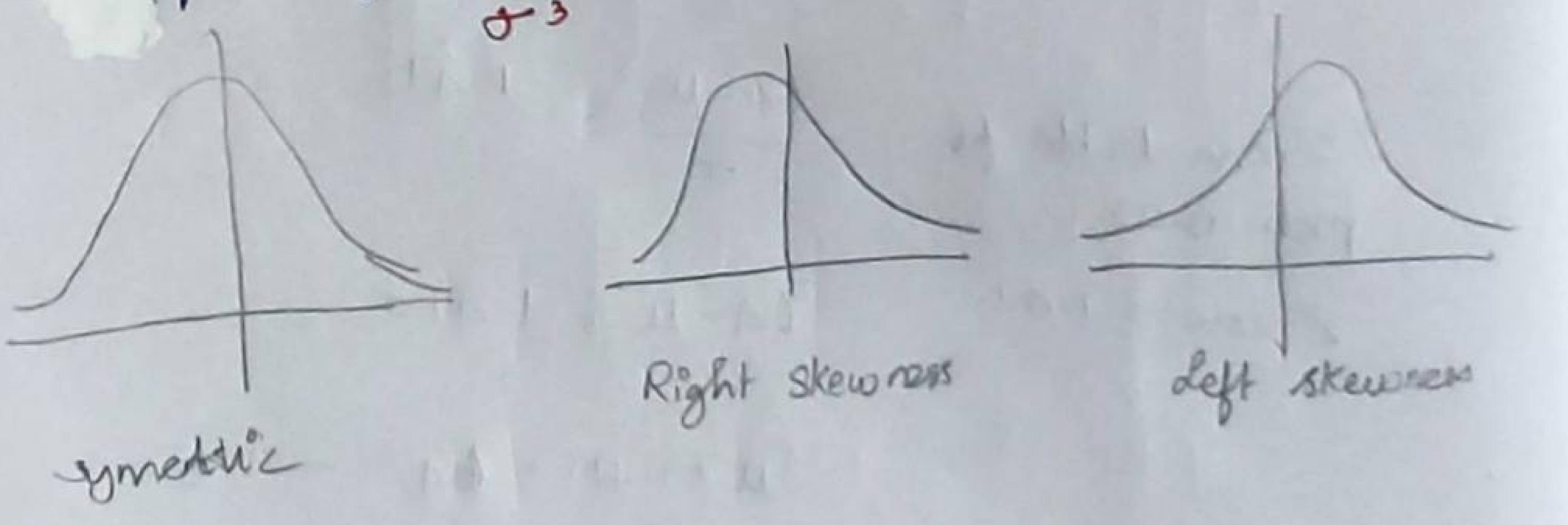
$r=1$ · $E(x)$ - the 1^{st} moment is the mean.

$r=2$ $E(x-\mu)^2$ - the 2^{nd} central moment is the variance

For other ~~variance~~ values of r also, $E(x^r)$ is a valuable measure

$$\boxed{\begin{array}{ll} E(x) = \mu & E(x^r) = \mu'_r \\ E(x-\mu)^r = \mu_r & \end{array}}$$

The quantity $\frac{\mu_3}{\sigma^3}$ is a measure of symmetry (skew)



It is called skewness

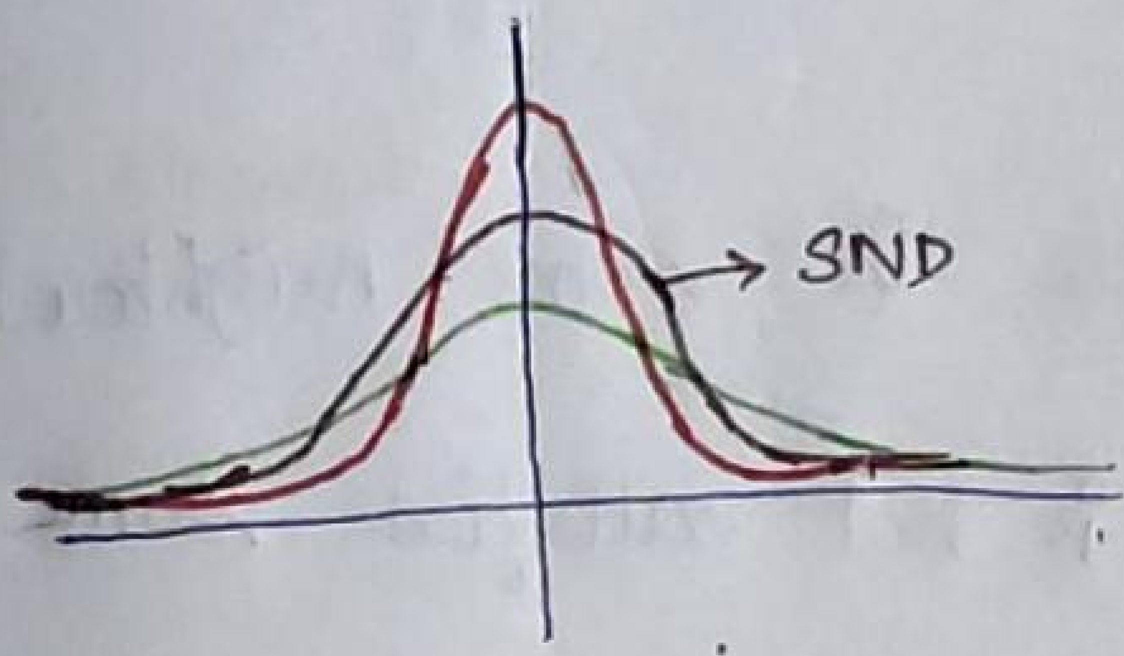
$= 0$, then the distribution is symmetric

> 0 , then the distribution is skewed on the right

< 0 , then the distribution is skewed on the left

The quantity $\frac{\mu_4}{\sigma_4}$ is a measure of Kurtosis

(peakness)



$= 3$ for SND

> 3 for a distribution which is more peaked than SND

< 3 for a distribution which is less peaked than SND

Moment means importance

With the knowledge of moments we know

Moment generating function (MGF)

For a r.v X,

$$M_X(t) = E(e^{tX})$$ e^{tX} is another r.v

If $M_X(t)$ is defined for all values of t in an interval $(-\delta, \delta)$ for $\delta > 0$, then $M_X(t)$ is called the moment generating function of X

Note:

$M_X(t)$ is finite in some neighbourhood of 0 $(-\delta, \delta)$

If this condition is not satisfied some moments may not exist.

Generating moments from mgf

method 1

$$\frac{d^r}{dt^r}\left[M_X(t)\right]_{t=0} = E(X^r)$$

the r^{th} derivative of mgf about $t=0$ is the r^{th} moment

Proof:

$$M_X'(t) = \frac{d}{dt}\{E(e^{tX})\}$$

$$= E\left\{\frac{d}{dt}(e^{tX})\right\}$$

$$= E\left[X e^{tX}\right]$$

$$M_X(t)\Big|_{t=0} = E(X)$$

1^{st} derivative of mgf @ $t=0$ is $E(X)$

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Method 2.

$$M_x(t) = E(e^{tx})$$

$$= E\left(1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots \frac{t^r x^r}{r!} + \cdots\right)$$

$$= 1 + \frac{t}{1!}E(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \cdots \frac{t^r}{r!}(Ex^r)$$

the coefficient of $\frac{t^r}{r!}$ in the expansion of mgf is

the r^{th} moment.