On Differential Forms and the Stokes' Theorem

Sherlock Dong

Berkshire School
Euler's Circle Independent Research
June 2024

Acknowledgements

The author would like to thank Ari Krishna for their mentorship and advice on this paper.The author would also like to thank Simon Rubinstein-Salzedo for his lectures on mathematics and his support throughout.

1 Introduction

The study of differential topology started in the 20th century. In the following decades, people succeeded in connecting calculus and topology using a newly developed concept, differential forms. Mathematicians who study this subject generalize a lot of ideas in calculus, in order to make them work in higher-dimensional spaces. Though scientists are still working to try to understand the universe with higher dimensions, theoretical physicists are already able to use this subject to study the universe.

However, when we are learning differentiation and integration in calculus nowadays, we still learn the ideas that Issac Newton and Gottfried Leibniz formalizes in the 17th century first. Though this method can make students easily understand basic calculus, it creates inconvenience at the same time for those who are going to learn differential topology in the future; a lot of notions have been redefined over the last century.

This paper intends to discuss the Stokes' Theorem. To do this, modern ideas in calculus will also be covered. After proving the Stokes' Theorem, some applications of the theorem will also be shown.

2 Differential Form

2.1 Some Linear Algebra

I will start this section by introducing some concepts that will be used in discussing differential forms.

Definition 2.1. A linear functional on $\mathbb{R}^n$ is a linear map $A : \mathbb{R}^n \to \mathbb{R}$ such that $A(tv) = tA(v)$ and $A(v+u) = A(u) + A(v)$ for all $v, u \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

In other words, a linear functional is a linear map from a vector space to its scalar fields.

Definition 2.2. The dual space of a vector space V, written as V^*, is the space of all linear functionals on V.

Example 2.2.1. Let $V = P^n$, the set of all polynomials with rank n. Define a function $f : f(p) = p(1)$ and suppose the polynomial $p_1 = x^2 + 2x + 1$. the map f makes the following change: $f(p_1) = p(1) = 4$, and it indeed is an element in $\mathbb{R}$. Thus, f is a linear functional on V, which is also an element in V^*.

Since we are talking about high-dimensional spaces, here I need to specify the idea of being tangent or cotangent at a point. In particular, while they still refer to the same ideas, there can obviously be multiple tangent or cotangent vectors in space, which forms another space themselves.

Definition 2.3. Let $p \in \mathbb{R}^n$, then the tangent space to $\mathbb{R}^n$ at p is the set of pairs

$$T_p\mathbb{R}^n := (p, v)|v \in \mathbb{R}^n$$

Cotangent space is associated with tangent space at point p. It is essentially the dual space of which is defined above.

Definition 2.4. Let $p \in \mathbb{R}^n$, then the cotangent space to $\mathbb{R}^n$ at p is the space of all linear functionals that take a tangent vector to $\mathbb{R}$:

$$T_p^*\mathbb{R}^n = T_p\mathbb{R} \to \mathbb{R}$$

Thus, an element of $T_p^*\mathbb{R}^n$ is called the cotangent vector to $\mathbb{R}^n$ at p.

Theorem 2.1. The projection principle

Let U be an open subset of the vector space V, and let W be an n-dimensional real vector space with basis $w_1, w_2, ..., w_n$. Then there is a bijection between the function $f : U \to W$ and n continuous linear functionals $f_1, f_2, ..., f_n$ by projection, which means

$$f(v) = f_1(v)w_1 + f_2(v)w_2 + ...f_n(v)w_n$$

Proof. Because $f(v)$ maps V to W, which has basis $w_1, w_2, ..., w_n$, and the $n-tuple f_1, f_2, ..., f_n$ maps the vector v to scalars. When these scalars are multiplied with each w_i for $1 \leq i \leq n$, they are bound to have a vector in W because of the definition of the vector. □

2.2 Wedge product and Forms

To understand the calculation in spaces, we first need to define the wedge product. It is also called the exterior product.

Definition 2.5. Let V be a vector space and m a positive integer, then the space $\wedge^m(V)$ is the wedge product

$$v_1 \wedge v_2 \wedge ... \wedge v_m$$

with the properties of an Abelian group; that is to say,

- $... \wedge (v_1 + v_2) \wedge ... = (... \wedge v_1 ...) + (... \wedge v_2 ...)$
- $... \wedge (cv_1) \wedge v_2 ... = ... \wedge v_1 \wedge (cv_2) ...$
- $... \wedge v \wedge v \wedge ... = 0$
- $... \wedge v \wedge w \wedge ... = -(... \wedge w \wedge v ...)$

Wedge product acts on tangent vectors; what it does is that it generalizes the cross product of 3-vectors. Let's look at an example of the calculation.

Example 2.5.1. Suppose V is a vector space and we have two vectors $v, w \in V$ such that there are $(v_1, v_2, v_3), (w_1, w_2, w_3) \in V \subseteq \mathbb{R}^3$, then the 3-wedge product, written as $\wedge^3(V)$, would be the abelian group generated by elements of the form $v \wedge w$. According to the properties of wedge product stated above, we can make some changes to it so that it is easier to calculate:

$$\begin{aligned} v \wedge w &= (v_1e_1 + v_2e_2 + v_3e_3) \wedge (w_1e_1 + w_2e_2 + w_3e_3) \\ &= (v_1w_1)e_1 \wedge e_1 + (v_1w_2)e_1 \wedge e_2 + (v_1w_2)e_1 \wedge e_3 + (v_2w_1)e_2 \\ &\wedge e_1 + (v_2w_2)e_2 \wedge e_2 + (v_2w_3)e_2 \wedge e_3 + (v_3w_1)e_3 \wedge e_1 + (v_3w_2)e_3 \wedge e_2 + (v_3w_3)e_3 \wedge e_3 \\ &= (v_1w_2 - v_2w_1)e_1 \wedge e_2 + (v_1w_3 - v_3w_1)e_1 \wedge e_3 + (v_2w_3 - v_3w_2)e_2 \wedge e_3 \end{aligned} \tag{1}$$

What I did just now is that I just used three of the only four properties of the wedge product to make the calculation easier.

Now, we can start to think about differential forms. Because it is a relatively complicated notion, I will start from low-dimension spaces. Let's first take a look at $0-form$ and $1-form$

First, consider $U \subseteq V$, where U is an open subset of the vector space V. Then, let us suppose that there is a function $f : U \to \mathbb{R}$, which assigns a certain number to every point $p \in U$, and that is a $0 - form$

Definition 2.6. A $0 - form$ α on U is a smooth function: $\alpha : U \to \mathbb{R}$.

By analogy, a $1 - form$ assigns each point p to a vector in $T_p^*\mathbb{R}^n$. In other words, let V be a vector space of dimension n, then a 1-form is a function $F : \mathbb{R}^n \to V^*$.

We will now go through the formal definition of the forms.

Definition 2.7. Let U be an open subset of $\mathbb{R}^n$. A $k - form$ α on U is a function which assigns to each point p an element $\alpha_p \in \wedge^k(T_p^*\mathbb{R}^n)$.

Example 2.7.1. Let V= $\mathbb{R}^4$ with standard basis e_1, e_2, e_3, e_4. Then a 2-form α at point p is

$$\alpha_p = f_1(p)e_1 \wedge e_2 + f_2(p)e_1 \wedge e_3 + f_3(p)e_1 \wedge e_4 + f_4(p)e_2 \wedge e_3 + f_5(p)e_2 \wedge e_4 + f_6(p)e_3 \wedge e_4$$

Calculating it would be the same as in example 1.

Let's do another example here without specifying the dimension of the form.

Example 2.7.2. Let $\alpha = \sum_I f_I dx_I$ be a k - *form*, and let $\beta = \sum_J g_J dx_J$ be an $l-$ form, Then their wedge product is the $(k + l)$ form $\alpha \wedge \beta$ given by

$$\alpha \wedge \beta = \sum_I \sum_J f_I g_J dx_I dx_J$$

.

From Theorem 2.1, it can be seen that in order to express α, we need to specify a function on U for each basis elements of $\wedge^k(T_p^*\mathbb{R}^n)$. Therefore, by taking any basis $e_i \in U$ and the basis elements would give a more general form of α :

$$\alpha_p = \sum_{1 \leq i_i < i_2 < ... < i_k \leq n} f_{i_1, i_2, ... i_n}(p) e_{i_1}^* \wedge e_{i_2}^* \wedge ... \wedge e_{i_k}^*$$

If we use I to denote $i_1, i_2, ... i_n$, this would become $\alpha = \sum_I f_I de_I$.

Note that the sum works the same way, as well as the basis elements.

Example 2.7.3. Let $\alpha_1, \alpha_2, ..., \alpha_k$ be one-forms. Then $\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_k$ is a k- form whose value at a point $p \in \mathbb{R}$ is the wedge product

$$(\alpha_1)_p \wedge ...(\alpha_k)_p$$

2.3 Exterior Differentiation

Before going forward into exterior differentiation, let's look at the cotangent space again. Let U be an open subset of $\mathbb{R}^n$. Suppose we have a function $f : U \to \mathbb{R}$, then for each $p \in U$ and $q = f(p)$, one has a linear map

$$df_p : T_p\mathbb{R}^n \to T_q\mathbb{R}$$

According to the definition of the cotangent spaces, this is essentially sending $(q, v) \to v$, where v is a vector, an element in $T_q\mathbb{R}$. Therefore, this differential is a linear map from $T_p\mathbb{R}^n$ to $\mathbb{R}$, which is an element of the dual space $T_p^*\mathbb{R}^n$.

Therefore, this way of assigning to p a value obtained from df_p is called a differential 1-form on U, and we denote it as df. It is also called the derivative of f.

We now define the exterior derivative that we always see in calculus.

Definition 2.8. The exterior derivative of a function $f : U \to \mathbb{R}$is

$$df := \sum_i \frac{\partial f}{\partial e_i} e_i^*$$

Here, people in history also decided to make some changes to the notations. Instead of e_i^*, people adapted $(de_i)_p$ to replace it. Similar to df_p in Example 2.4, people omitted the point p to make it more general. Thus, the modern way of writing this exterior derivative becomes

$$df := \sum_i \frac{\partial f}{\partial e_i} de_i$$

Example 2.8.1. Suppose $f(x, y, z) = 2x$. If we want to take the derivative of this function using the definition of the exterior derivatives:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= 2xdx$$

.

Thus, the exterior derivative: $\frac{df}{dx} = 2x$.

In this example, we took partial derivatives of the function f with respect to its three different bases. Note that here, $2x$ is also its total derivative. Now, let's do a more interesting one.

Example 2.8.2. Let $f(x, y, z) = x^2 + xyz + z^3$. The exterior derivative is just $(2x + yz)dx +$

$xzdy + (3z^2 + xy)dz$. Computing $d(df)$ will be

$$d^2 f = ((2dx + ydz)) \wedge dx + (xdz + zdx) \wedge dy + (6zdz + xdy + ydx) \wedge dz$$

$$= ydz \wedge dx + zdy \wedge dx + xdz \wedge dy + zdx \wedge dy + xdy \wedge dz + ydx \wedge dz$$

$$= 0$$

From this example, we see how we can use the properties of the wedge product in the calculation for higher order derivatives. Interestingly, there is a theorem on this type of problem.

Theorem 2.2. Suppose U is an open subset of $\mathbb{R}^n$. Let α be any k-form, then $d^2\alpha = 0$.

Proof. An easy approach is by induction. For the base case, we notice that for a function $f(x)$, if $k = 1$, $d(d\alpha) = f_1(x)dx \wedge dx = 0$ for any $f \in C^\infty(U)$(i.e. smooth functions).

For the induction step, assume we know the result of $k-1$ functions while we now have $f_1, f_2, f_3, ..., f_k \in C^\infty(U)$, and let $\omega = df_2 \wedge df_3 \wedge ... \wedge df_k$. By the induction hypothesis as well as the properties of the wedge product, we see that

$$d(df_1 \wedge df_2 \wedge ... \wedge df_k) = d(df_1 \wedge \omega)$$

$$= d(df_1) \wedge \omega - df_1 \wedge \omega$$

$$= 0$$

. □

For k-forms, the exterior derivatives work the same way. The exterior differentiation of a k-form is called a differential k-form.

Definition 2.9. Let $\alpha = \sum_I f_I de_I$, then we define the exterior derivative as $d\alpha := \sum_I df_I \wedge de_I = \sum_I \sum_j \frac{\partial f_I}{\partial e_j} de_j \wedge de_I$

Definition 2.10. A k-form α is closed if $d\alpha = 0$.

A k-form α is exact if for some $k-1$ -form β, $d\beta = \alpha$. Note that if $k = 0$, α is only exact when $\alpha = 0$.

Theorem 2.3. Exact forms are closed.

Proof. For a k-form α, if it is an exact form, then there is another differential k-1 form β, such that $d\beta = \alpha$. From theorem 2.3, we know that $d(d\beta) = 0$, implying that $d\alpha = 0$. Therefore, Exact forms are closed. □

2.4 Pullback operation

Before going into integration, I will talk about one more important concept regarding the differential forms.

Let U be an open subset of $\mathbb{R}^n$, V be an open subset of $\mathbb{R}^m$, (m and n can be the same) and $f \in C^\infty : U \to V$. For a point in U, the derivative of f at p

$$df_p : T_p\mathbb{R}^n \to T_f(p)\mathbb{R}^m$$

is a linear map. More specifically, this is a map from a set of tangent vectors at point p to the tangent vectors at point $f(p)$. What we get from a pullback map is the opposite

$$df^*p \coloneqq (df_p)^* : \wedge^k(T_f^*(p)\mathbb{R}^m \to \wedge^k(T_p^*\mathbb{R}^n)$$

For differential forms, let α be a k-form on V, then at $f(p) \in V$, the value of α at point $f(p)$ is an element in the set $\wedge^k(T_q^*\mathbb{R}^m)$.

Therefore, an operation for a differential k-form α also gives an element

$$df_p^*\alpha_f(p) \in \wedge^k(T_p^*\mathbb{R}^n)$$

More generally, for every point p, we can assign a value to it in the same method:

$$p \to (df_p)^*\alpha_f(p)$$

which is the very definition for differential k-forms. We denote this operation by $f^*\alpha$, and we call it the pullback along the map f.

Another way of understanding the pullback operation is by projection. Suppose $w_1, w_2, ..., w_n$ is a basis of U and $v_1, v_2, ..., v_m$ is a basis of V. Then, by Theorem 2.1, for a vector $u \in U, f(u) = f_1(u)w_1 + f_2(u)w_2 + ... + f_n(u)w_n$. Then, from the definition of a k-form α, which is $\alpha = \sum_{I \subseteq 1,...,n} f_I w_I$, we define the pullback operation ω as following:

$$f^* \circ \alpha = \sum_{I \subseteq 1,...,m} (f_I\omega)(D\omega_{i_1} \wedge D\omega_{i_2} \wedge ... \wedge D\omega_{i_k})$$

Note the subscripts here, I is the set $i_1, i_2, ..., i_n$

There are some important properties of the pullback operation that can be useful in later calculation.

Let U be an open subset of $\mathbb{R}^n$ and V be an open subset of $\mathbb{R}^m$. Let $f : U \to V$ a C^∞ map.Let $\omega, \alpha, and\beta$ be a differential k-form.

Let ω be a 0 form. Then,

1. $f^*\omega(p) = \omega(f(p))$

2. $f^*(\alpha + \beta) = f^*(\alpha) + f^*(\beta)$

3. $f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta)$

4. Linearity: $f^*(c\alpha + \beta) = cf^*\alpha + f^*\beta$

5. Naturality: $f^*d(\alpha) = d(f^*\alpha)$

Proof. 1. Because ω is a 0 form,

$$\wedge^0(T_p^*) = \wedge^0(T_{f(p)}^*) = \mathbb{R}$$

the pullback map in this case would just be an identity map. Thus for 0 forms, the property stands.

2. Because df is a linear map, we see that $(df_p)^*(\alpha+\beta)_f(p) = (df_p)^*(\alpha)_{f(p)} + (df_p)^*(\beta)_{f(p)}$. From this, we can have property 2.

3. From the properties of the wedge product, we can see that $df_p^*(\alpha)_f(p) \wedge (\beta)_f(p) = df_p^*(\alpha)_f(p) \wedge df_p^*(\beta)_f(p)$

4. Linearity can be proved using property 1.

5. Naturality will be proved in section 4.3, differential forms on manifolds.

□

Example 2.10.1. Let $V = \mathbb{R}^2$ with basis e_1, e_2, $V' = \mathbb{R}^3$ with basis w_1, w_2, w_3, a, b being two constants, and suppose $f : V \to V'$ is

$$f_1(ae_1 + be_2) = (a^2)w_1 + (ab)w_3$$

$$f_2(ae_1 + be_2) = (b^2)w_2$$

If I define a form α at point$p$ $\alpha_p = f_1(p)w_1 \wedge w_3 + f_2(p)w_2$, so the pullback operationω on this form at point is

$$(\omega^*\alpha)_p = f_1(\omega(p))(3ae_1^* + be_2^*) \wedge f_2(\omega(p))(2be_2^*)$$

$$= f_1(\omega(p))f_2(\omega(p))6abe_1^* \wedge e_2^*$$

This falls in the form as we define this operation, which can be further calculated given the functions and the values of a, b.

2.5 Applications of Differential Forms

In physics, differential forms is very useful. For example, it is the language for **Maxwell's equations**, which makes it being written fairly compactly. For energy, differential forms offer a clear view of the energy function. From which, we can see that it does not change along the integral curve of the vector, which means that energy does not change with respect to time [1]. But that is not the focus of this paper. Now, I will move on to talk about how to integrate them, the normal thing to do after learning how to differentiate functions.

3 Integrating Differential Forms

3.1 Integration

In multivariable calculus, line integrals are defined, while we integrate over curves, spaces, etc. Integrating a differential form is the same thing, where we add up all the values of the differential k-form α along a curve c.

Definition 3.1. Suppose c is a curve and α is a differential k-form. An integral of α over $[a, b]$ is

$$\int_c \alpha := \int_{[a,b]} \alpha_{c(t)} c'(t) dt$$

Note that here, $c'(t)$ stands for the tangent vector along the curve at point t. I can also use the pullback operation in exterior differentiation to define this integration

$$\int_c \alpha = \int_{[a,b]} c^* \alpha$$

3.2 Cells

In order to make the discussion of integration more general, I need to define two notions here.

Definition 3.2. A $k-$ cell is a smooth function : $c : [a_1, b_1] \times [a_2, b_2] \times ... \times [a_k, b_k] \to V$

Example 3.2.1. Let's look at some examples on cells.

- A 0 - cell is just a single point that is assigned a value.
- A 1-cell, in analogy, is a curve.
- A 3-cell is a 3-dimensional figure. For example, suppose $c : [0, 2\pi] \times [0, r_1] \times [0, h] \to V$ by $c : (r, \theta, z) \to (r\cos\theta, r\sin\theta, z)$ can be thought to be a cylinder with radius $\frac{r_1}{2}$ and height h.

Now we know how to use cells, and we can generalize the previous definition on integral for differential forms, since we don't have to limit it to curves, but we can use cells to denote higher dimensional functions.

Definition 3.3. Suppose α is a k-form and c is a k-cell $c : [a,b]^k \to V$. Then integrating it over c can be defined as

$$\int_c \alpha := \int_{[a,b]^k} c^* \alpha$$

Of course the cell c does not have to have the same interval for all of its dimensions, but they are dealt with in the same way.

Example 3.3.1. Let us take a look again at the cylinder in example 3.1. Take the three-form α that gives $e_1^* \wedge e_2^* \wedge e_3^*$ at every point on c. Then,

$$c^*\alpha = (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \wedge dz$$

$$= rdrd\theta dz$$

Integrate it,

$$\int_c \alpha = \int_0^h \int_0^{2\pi} \int_0^{r_1} rdr \wedge d\theta \wedge dz$$

In multivariable calculus people omit the wedge product sign, and the result is $\pi h(r_1)^2$, which is the volume for a cylinder.

3.3 Stokes' Theorem on cells

First, I introduce a notion that enables us to consider multiple cells at once.

Definition 3.4. A *k-chain U* is a formal linear combination of k-cells over U. It can be expressed as:

$$c = a_1c_1 + a_2c_2 + ... + a_kc_k$$

where each $a_i \in \mathbb{R}$, and c_i is a *k-cell*. Therefore, the integral can be alternatively defined as: $\int_c \alpha = \sum_i a_i \int c_i$

Second, we need to talk about the intervals, or the boundary. Previously, when I define a cell: $c : [a,b] \to V$, the boundary is actually a 0 chain $c(b) - c(a)$. Let's see the formal definition.

Definition 3.5. Suppose $c : [a_1,b_1] \wedge [a_2,b_2] \wedge ... \wedge [a_k,b_k] \to U$ is a k-cell, then the boundary of c, denoted as $\partial c : [a_1,b_1] \wedge [a_2,b_2] \wedge ... \wedge [a_{k-1},b_{k-1}] \to U$, is the (k-1) chain defined as follow: For each $i = 1,...,k$, define $k-1$- chain by

$$c_i^{start} : (t_1,...,t_{k-1}) \mapsto c(t_1,...ti-1,a_i,t_i,...,t_k)$$

$$c_i^{end} : (t_1, ..., t_{k-1}) \mapsto c(t_1, ...ti - 1, b_i, t_i, ..., t_k)$$

Then,

$$\partial c := \sum_{i=1}^{k} (-1)^{i+1} (c_i^{stop} - c_i^{start})$$

The boundary of a chain, on the other hand, is the sum of the boundaries of each cell: $\partial(\sum a_i c_i) = \sum a_i \partial c_i$

Now we have enough materials to prove the Stokes' Theorem for cells.

Theorem 3.1. Let U be an open subset of the vector space V, let $c : [a_1, b_1] \wedge [a_2, b_2] \wedge ... \wedge [a_k, b_k]$ be a k-cell, and let $\alpha : U \to \wedge^{k-1}(T_p^* \mathbb{R}^n)$ be a (k-1) form. Then,

$$\int_c d\alpha = \int_{\partial c} \alpha$$

The proof will be given later, as this is a preliminary version of the theorem.

Example 3.5.1. Suppose $k = 1$, and c is a cell that has an interval $[a, b]$, so then $\partial c = b - a$. Apply Stokes' Theorem,

$$\int_c d\alpha = \alpha_b - \alpha_a$$

This is the fundamental theorem of calculus.

Though compare to Stokes' theorem in multivariable calculus, this form is more general. But it is not generalized enough. Specifically, this is the Stokes' Theorem for cells.

4 Manifolds and Forms

4.1 Some topology

In order to discuss differential forms on manifolds, we first need to understand manifolds. In fact, the Earth that mankind lives on is a manifold. For a long time in history, people believe that the Earth is flat, as it is observed by everyone on Earth [2], while in fact, the Earth is a 3-dimensional sphere. The space that locally looks like $\mathbb{R}^n$ is a **n-manifold**.

Before giving the formal definition of a n-manifold, I will talk a little about some important topological concepts.

Definition 4.1. Let X be a topological space and $x \in X$, a neighborhood of x is a set U which contains an open set V containing x.

Definition 4.2. Suppose f is a smooth function. If f is a bijection and f and f^{-1} are both smooth maps, then f is a diffeomorphism.

Definition 4.3. Let n, n_1 be two non-negative integers with $n \leq n_2$. An subset U of $\mathbb{R}^{n_1}$ is a n-manifold if for every point $p \in U$, there exists a neighborhood V of p in $\mathbb{R}^{n_1}$, an open subset $X \subseteq \mathbb{R}^n$ and a diffeomorephism $f : X \to U \cap V$.

In other words, U is a n-manifold if near every point p, it looks like an open subset of $\mathbb{R}^n$. Let's look at some examples.

Example 4.3.1. Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$a smooth function. The graph of this function

$$\Gamma_f = (x, f(x)) \in \mathbb{R}^{n+1} | x \in U$$

is an n-manifold in $\mathbb{R}^{n+1}$. The reason is that, the map from U to $\mathbb{R}^{n+1}$, which maps x to $(x, f(x))$ has some hidden properties. It is bijection and the inverse of it is also smooth. Thus, this map is diffeomorphism, indicating that Γ_f is an n-manifold.

Example 4.3.2. Suppose there are two submanifolds X_1, X_2 of$\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$, each has dimension n_1, n_2. Taking the direct product,

$$X_1 \times X_2 = (x_1, x_2) | x_{1,2} \in X_{1,2}$$

is an $(n_1 + n_2)$ dimensional submanifold of $\mathbb{R}^{N_1+N_2} = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

Note that a submanifold is a subset of a manifold while conserving all of the properties and structures,

Why?

For point $p_i \in X_i$ for $i = 1, 2$, there exists a neighborhood at each point in $\mathbb{R}^{N_i}$m an open set $U_i \in \mathbb{R}^{n_i}$. We also note that there should also exist a diffeomorephism $f : U_i \to X_i \cap V_i$. Let $U = U_1 \times U_2, V = V_1 \times V_2, X = X_1 \times X_2$.

Another way of considering this problem is start by analyzing the manifold locally. If we have take the direct product of two manifolds, from the definition and the properties of direct product, the dimensions here are additive, which means the result would have the sum of the dimensions of the previous two manifolds. Therefore, the result makes sense, that $X_1 \times X_2$ is $n_1 + n_2$ dimensional.

Definition 4.4. A topological covering of a $n - manifold$ X is another space X_0 with a continuous map $\pi : X_0 \to X$ such that X_0 is a union of subsets of X whose union is all of X. We call X_0 the covering (cover) to X.

Definition 4.5. A n-dimensional half space $\mathbb{H}^n$ is a portion of a n-dimensional space obtained by removing the part lying on one side of a$n - 1$ dimensional half plane. In other words, it is literally half of a n-dimensional space.

Definition 4.6. Let X be an $n - manifold$. A partition of unity is a set of continuous functions from X to the unit interval $[0, 1]$ such that for every point $p \in X$,

- There exists neiborhoods of X, but a finite number of functions in the set are 0.
- The sum of all functions in the set is 1.

We say this partition of unity subordinates to X.

4.2 Regular Values

Let's turn our attention to yet another way of viewing manifolds: by viewing them as solutions of systems of equations [1]. To do this, we need to discuss **regular values**.

Definition 4.7. A point $a \in \mathbb{R}^n$ is a regular value of a smooth function f if for every point $p \in f^{-1}(a)$, the map f is a submersion at p.

Here, if f is said to be a submersion at p, the differential $Df(p) : \mathbb{R}^N \to \mathbb{R}^n$ is and only is surjective, which means $n \leq N$. There would be no regular values if $n > N$.

Theorem 4.1. Let $k = N - n$. if a is a regular value of f, the set $X := f^{-1}(a)$ is an k manifold.

Proof. This proof was inspired by [3].

□

Example 4.7.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a map such that $x_1, x_2, ..., x_{n+1} \to x_1^2 + x_2^2 + ... + x_{n+1}^2 - 1$. If we take the total derivative of $f(x)$, we get

$$D(f(x)) = 2(x_1, x_2, ..., x_{n+1})$$

the dimensions of the two sides are equal, so it is a submersion at$x$. In fact, it is a submersion at all points on the $n - sphere$. From Theorem 4.1, we see that an $n - sphere$ is an $n - manifold \in \mathbb{R}^{n+1}$.

4.3 Differential Forms on Manifolds

It often occurs to us that some easy ideas can be generalized to become fairly complicated. Now, let's introduce two more of those.

Definition 4.8. Let X be a manifold. A vector field on X is a function v which assigns to each point $p \in U$ an element $v(p)$ of T_pX.

Definition 4.9. Let X be a manifold. A k form is a function w which assigns to each point $p \in X$ an element $w_p \in \wedge^k(T_p^*X)$.

Does it look familiar? Yes, it was a set of elements of the image of a pullback operation, except that here, I changed $\mathbb{R}^n$ to a manifold X.

Definition 4.10. Let X be an $n - manifold$ and U an open subset of X. Then the set U is a parametrizable open set if there exists an open set $U_0 \in \mathbb{R}^n$ and a diffeomorphism $f : U_0 \to U$.

Note that here, the diffeomorphism is a parametrization of U. Therefore, we can also say that U is parametrizable if there exists a parametrization of it having U itself as its image.

Now, I need to restate the smoothness of a $k - form$ and a vector field v on manifolds as well.

Definition 4.11. Let X be a manifold. A $k - form \omega$ is smooth if locally at every point $p \in X$, ω is smooth on a neighborhood of p. Similarly, a vector field v is smooth if for every point $p \in X$, v is smooth on a neighborhood of p.

They are basically the same ideas for smoothness on $\mathbb{R}^n$, except here, we need to consider one more thing: the generalized idea of "neighborhood".

Theorem 4.2. Let X, Y be two $n - manifolds$ and $f : X \to Y$ a smooth map. Then for a differential $k - form\ \alpha$, we have $f^*(d\alpha) = d(f^*\alpha)$

Proof. For every point $p \in X$, we can check how this works in its neighborhood. Let $q = f(p)$ and let U, V be parametrizable neighborhoods of p, q. Let $\phi : U_0 \to U$ and $\psi : V_0 \to V$ be the parametrizations, we obtain a parametrization map $g : U_0 \to V_0, g = \psi^{-1} \circ f \circ \phi$; therefore,

$$\begin{aligned} \phi^* d(f^*\alpha) &= d\phi^* f^* \alpha \\ &= d(f \circ \phi)^* \alpha \\ &= d(\psi \circ g)^* \alpha \\ &= dg^*(\psi^* \alpha) \\ &= g^* d\phi^* \alpha \\ &= g^* \psi^* d\alpha \\ &= \phi^* f^* d\alpha \end{aligned} \tag{2}$$

Thus the equation stands. □

4.4 Orientations

We finally come to the stage where we can talk about integral calculus of forms on manifolds. Before talking about Stokes' Theorem, there is another thing that we need to know.

Definition 4.12. Let X be an $n - manifold$. An orientation of X is a rule for assigning each point $p \in X$ an orientation of T_pX.

In fact, this is putting a label (i.e. a *plus* or a *minus*) on points $p \in X$.

The reason for having an *orientation* is that, in a space, we need to pick a direction for the tangent vectors that we are taking of the points. Let's look at a definite integral for an example.

Example 4.12.1. As we know,

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

We can easily find out the orientation for such integrals as it is two-dimensional. In space, however, it would be much harder and we need orientations to help us to find out. In fact, there are some manifolds that are not "orientable".

Definition 4.13. A smooth $n{-}manifold$ is orientable if there exists a differential $n{-}form\omega$ on X such that for every point $p \in X$, $\omega_p \neq 0$.

In particular, we know that ω_p is an element of $\wedge^n(T_p^*X)$; in this case, ω is called a volume form of M.

Example 4.13.1. Let's take a look at Möbius strip. It is obtained from a rectangle which has its both ends glued together. It is not orientable, because the vector does not switch sides when moving along the strip, while it actually has moved all the way.

Definition 4.14. Let M, N be two manifolds. Suppose $f : M \to N$ is a local diffeomorphism. f is **orientation-preserving** if for each point $p \in M$, df_p takes the oriented bases of T_pM to the oriented bases of $T_{f(p)}N$.

In other words, a diffeomorphism is orientation-preserving if the isomorphism df_p does not change the directions of the two bases. For instance, a parametrization $f : U_0 \to U$ is orientation-preserving.

Definition 4.15. Let D be a smooth domain, and ψ being a parametrization $\psi : U_0 \to U$. We call U a D-adapted parametrizable open set if

$$\psi^{-1}(U \cap D) = \mathbb{H}^n \cap U_0$$

ψ is the a D-adapted parametrization.

5 Stokes' Theorem

5.1 Boundary

Definition 5.1. Let α be a $k - form$ on an oriented $n - manifold$ X. The **support** of α is

$$supp(v) := p \in X | \alpha_p \neq 0$$

We say α is compactly supported if supp(v) is compact as a topological space.

Recall the definition of manifold. It does not include the points on the **edge**. A **manifold with boundary** is a manifold including its boundary points(the edges). Or formally,

Definition 5.2. The boundary of a manifold X is the set of all boundary points and is denoted as ∂X.

Example 5.2.1. Consider a closed unit ball $B^3 := (x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1$. It is a three dimensional **manifold with boundary** and its boundary is the two dimensional unit sphere.

Definition 5.3. An open subset $D \in X$ is a smooth domain if

- The boundary $\partial D$is an $(n-1)$ dimensional submanifold of X;
- The boundary ∂D concides with the boundary of D that is without boundary.

One important reason to generalize the understanding of the boundary here is that, in manifolds the boundary might be oriented. It is oriented on a oriented surface, and the orientation usually remains the same.

Example 5.3.1. Let's look at the three dimensional unit ball: $B^3 := (x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 < 1$. It is not a manifold with boundary, but it has an "edge"(boundary) outside of it, which is still the two dimensional unit sphere. Thus, it is a smooth domain

5.2 Integration on Manifolds

Definition 5.4. Let α be a compactly supported $n - form$ on a smooth domain D, where α can be written as $\alpha = f dx_1 \wedge dx_2 \wedge ... \wedge dx_n$. Let $f_0 : U_0 \to U$ be a parametrization. Since f is oriented, we can define

$$\int_D \alpha = \int_D f_0^* \alpha$$

$$\int_D \alpha = \int_D f dx_1 \wedge dx_2 \wedge ... \wedge dx_n$$

From the additivity properties of Rieaman Integrals, we can legitimately get analogous properties for integrals over manifolds:

$$\int_D (\alpha_1 + \alpha_2) = \int_D \alpha_1 + \int_D \alpha_2$$

Similarly, for a constant c, we also have

$$\int_D c\alpha = c \int_D \alpha$$

However, we need to extend the definition of integral to any compactly supported $n-manifold$. Before that, I will show how I derive the **partition of unity** properties in manifold.

Theorem 5.1. Let $C = \{U_k\}_{k\in I}$ be a covering of X. Let U be an open subset of X. Then there exists a family of smooth functions ρ_i(i.e. partition of unity subordinates to X)such that for $i \geq 1$, it has the following properties:

1. $\rho_i \geq 0$;
2. For every compact set $S \subset X$, there exists a postiive integer N such that if $i > N$, $\text{supp}(\rho_i) \cap S = \varnothing$
3. $\sum_{i=1}^{\infty} \rho_i = 1$
4. For every $i \geq 1$ there exists an index $k \in I$ such that $\text{supp}(\rho_i) \subset U_k$

Proof. For each point p in X, and for some U_k containing this point, select an open set $O_p \in \mathbb{R}^N$ with $p \in O_p$ and $\overline{O_p \cap X} \in U_k$ being true. Let $O := \bigcup_{p\in X} O_p$ and let $\widetilde{\rho_i}$ be a smooth form defined on O. For $i \geq 1$, let $\widetilde{\rho_i}$ be a partition of unity subordinate to O by the $O_p's$. From the second condition of selecting O_p, we see that $U \subseteq O$, so ρ_i is a restriction of $\widetilde{\rho_i}$, and thus it inherits the properties of $\widetilde{\rho_i}$, which are those of a partition of unity(property 1 -3), and being subordinate to the covering(property 4). □

Now, let the covering in theorem5.2 be any covering of an open subset in X, and let $\rho_i$be a family of smooth functions such that for $i \geq 1$, it is a partition of unity subordinate to the cover. Let α be a compact $n - manifond$ on X, we define the integral of α over an open subset X to be

$$\sum_{i=1}^{\infty} \int_X \rho_i \alpha$$

Note that in definition 5.4, α has to be compactly supported. But here, we allow it to be an arbitrary smooth $n - form$ on manifolds.

Theorem 5.2. change of variable formula

Let $X_0$and X be oriented $n - manifolds$ and $f : X_0 \to X$an orientation-preserving diffeomorephism. If W is an open subset of X and $W' := f^{-1}(W)$, then

$$\int_{W'} f^*\alpha = \int_W \alpha$$

where α is a compact $n - form$.

Proof. From above, we see that the integral $\int_W \alpha$ is a sum of smooth forms while each form is supported on a parametrizable open subset, and thus we can assume that α has these properties. Let V be a parametrizable open set containing supp (α) and let $\phi_0 : U \to V$ be an oriented parametrization of V. Because f is a diffeomorphism, it has an inverse f^{-1} which is a diffeomorphism of X onto X_1. Let $V' = f^{-1}(V)$ and let $\phi' = f^{-1} \circ \phi_0$. Note that ϕ' is also an oriented parametrization. Now, because $f \circ \phi'_0 = \phi_0$ and we know from the condition that $W_0 = \phi_0^{-1}(W)$, we have

$$W_0 = (\phi'_0)^{-1}(f^{-1}(W)) = (\phi'_0)^{-1}(W')$$

By the chain rule we have

$$\phi_0^*\alpha = (f \circ \phi'_0)^*\alpha = (\phi'_0)^* f^*\alpha$$

Thus

$$\int_W \alpha = \int_{W_0} \phi_0^*\alpha = \int_{W_0} (\phi'_0)^*(f^*\alpha) = \int_{W'} f^*\alpha$$

□

5.3 Generalized Stokes' Theorem

Let's start with a preliminary version of Stokes' Theorem.

Lemma 5.3. If μ is a compact $n-1-form$ on X, then

$$\int_X d\mu = 0$$

Proof. Let $\rho_i(i \geq 1$ be a partition of unity such that each ρ_i is supported in a parametrizable open set $W_i = W$. We can prove the theorem using the change of variable formula.

Suppose $f : W_0 \to W$is an oriented parametrization of U. Then from Theorem 5.2 and Theorem 4.2, we have

$$\int_W d\mu = \int_{U_0} f^* d\mu = \int_{U_0} d(f^*\mu)$$

□

Let X be an oriented n-dimensional manifold and $D \subset X$ a smooth domain. Because ∂D obtains a natural orientation from D, so if we define $\iota : \partial D \to X$ to be an inclusion map and α a $n-1-form$ on X, $\int_{\partial D} \iota^*\alpha$ is a well-defined integral. Now let's turn our attention to prove Stokes' Theorem.

Theorem 5.4. (Stokes' Theorem) For a $n-1-form\alpha$, we have

$$\int_{\partial D} \iota^*\alpha = \int_D d\alpha$$

Proof. The proof is inspired from [4] Let $\rho_i (i \geq 1)$ be a partition of unity such that for each i, the support of ρ_i is contained in a parametrizable open set $U_i = U$ of one of the following three types:

1. U is a subset of the interior of D.
2. U is a subset of the exterior of D.
3. There exists an open$U_0 \subset \mathbb{R}^n$ and an oriented D-adapted parametrization $\phi : U_0 \widetilde{\to} U$.

If we replace α by the finite sum $\sum_i^\infty \rho_i \alpha$, we will be able to prove the theorem by proving that it works for each $\rho_i \alpha$. Because ρ_i is a partition of unity, we see from Theorem 5.1 property 4 that the support of α is contained in a parametrizable open set U of one of the three types above.

1. If U is of type 1, because we know that ι is an inclusion map, and
$$\int_D d\alpha = \int_U d\alpha = \int_X d\alpha$$
so $\iota^* \alpha = 0$. In this case the left hand side of the Stokes' Theorem equation is 0. From Theorem 5.4, we see that the right hand side would also be 0. The equation stands.
2. If U is of type 2, $\iota^* \alpha = 0$ for the same reason. The restriction of α to D is 0, so both sides of the Stokes' Theorem equation is 0. The equation stands.

Thus, to prove the Stokes' Theorem it is sufficient to prove that U is an oriented D-adapted parametrization open set.

From the properties of such open sets, we can see that the restriction of the map to $U_0 \cap \partial \mathbb{H}^n$ is the diffeomorphism
$$\psi : U_0 \cap \partial \mathbb{H}^n \to U \cap D$$
Another restriction is on ι. If we define $\iota_{\mathbb{R}^{n-1}} : \mathbb{R}^{n-1} \to \mathbb{R}^n$, then
$$\iota_D \circ \psi = \phi \circ \iota_{\mathbb{R}^{n-1}}$$
Note that it is an inclusion map from ∂D to $\mathbb{R}^n$, we can thus rewrite the right hand side of 5.4 to
$$\int_D d\alpha = \int_{\mathbb{H}^n} \phi^* d\alpha = \int_d \mathbb{H}^n \alpha$$

Because we also have 5.3, the left hand side of 5.4 to

$$\begin{aligned}\int_{\partial D} \iota^* \alpha &= \int_{\mathbb{R}^{n-1}} \psi^* \iota^* \alpha \\ &= \int_{\mathbb{R}^{n-1}} \iota^*_{\mathbb{R}^{n-1}} \phi^* \alpha \\ &= \int_{\partial \mathbb{H}^n} \iota^*_{\mathbb{R}^{n-1}} \phi^* \alpha\end{aligned} \tag{3}$$

Here we are able to identify $\mathbb{R}^{n-1}$ with $\partial\mathbb{H}^n$ is because $\mathbb{H}$ is a half space. Consequently, what is left to do now is to prove the Stokes' Theorem on $\mathbb{H}^n$, which is sufficient to prove 5.4.

Let

$$\alpha = \sum_{i=1}^{n} (-1)^{i-1} f_i dx_1 \wedge ... \wedge dx_n$$

Then

$$d\alpha = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge ... \wedge dx_n$$

The integral it on $\mathbb{H}^n$ is

$$\int_{\mathbb{H}^n} d\alpha = \sum_{i=1}^{n} \int_{\mathbb{H}^n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge ... \wedge dx_n$$

We can rearrange the order of integration in each term. For $i > 1$, from the fundamental theorem of calculus, we have

$$\begin{aligned}&\sum_{i=2}^{n} (-1)^{i-1} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i}(x) dx_2 \wedge ... \wedge dx_n \\ &\qquad = \sum_{i=2}^{n} (-1)^{i-1} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f_i(x)|_{x_i=-\infty}^{x_i=\infty} dx_1 \wedge ... \wedge dx_n \\ &\qquad = 0\end{aligned} \tag{4}$$

because $\alpha$is compactly supported. The only thing that is left is dx_1. Because $\mathbb{H}^n$ is a half space, the range of interation is the interval $(-\infty, 0)$

$$\int_{-\infty}^{0} \frac{\partial f_1}{x_1} dx_1 = f(0, x_2, ..., x_n)$$

Integrate it with respect to the remaining variables,

$$\int_{\mathbb{H}^n} = \int_{\mathbb{R}^{n-1}} f(0, x_2, ..., x_n) dx_2 \wedge ... \wedge dx_n$$

For the same reason, $\iota^*_{\mathbb{R}^{n-1}} x_1 = 0$ while $\iota^*_{\mathbb{R}^{n-1}} x_i = x_i$ thus

$$\int_{\mathbb{R}^{n-1}} \iota^*_{\mathbb{R}^{n-1}} \alpha = \int_{\mathbb{R}^{n-1}} f(0, x_2, ..., x_n) dx_2 \wedge ... \wedge dx_n$$

which is the same with 5.3. Hence the two sides of 5.4 are equal, hence the Stokes' Theorem stands. □

There is another way of writing the Stokes' Theorem: Suppose X is an oriented $n-manifold$ with boundary, and α being a compactly supported $(n-1)-form$ on X. Then the Stokes' Theorem states,

$$\int_X d\alpha = \int_{\partial X} \alpha$$

This is the same statement as 5.4, but this one is more commonly seen as it looks more straightforward.

5.4 Application

I will start by talking about its applications in Mathematics. It is widely used in multiple fields, including geometry, topology, and calculus. In fact, the Stokes' Theorem can be used to conduct to two other important theorems in multivariable calculus: the Green's Theorem and the Divergence Theorem. Let's take a look at Green's Theorem together.

Theorem 5.5. Suppose D is a smooth domain in $\mathbb{R}^2$, and P,Q are smooth real-valued functions on D. Then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} PdxQdy$$

Proof. Apply the Stokes' Theorem to the $1-form$ $Pdx + Qdy$, we will have the Green's Theorem. □

While there are multiple applications of Stokes' Theorem in physics, it is comparatively harder to apply the theorem to physics, as manifolds are highly abstract concepts and physicists are still exploring the space and time so that they can use the math that we have for higher-dimensional spaces. However, we can still use Stokes' Theorem a lot in lower dimensional spaces. Let's look at Maxwell-Faraday equation for an example.
The Maxwell-Faraday equation predicts how a magnetic field will interact with an electric circuit to provide an electromotive force(EMF).

Lemma 5.6. (Lenz's Law)
$EMF = -\frac{d\Phi}{dt}$

The proof for this law will not be given here as it is an experimental result. If you are interested, you can give it a try.

Theorem 5.7. Maxwell-Faraday equation

$$\nabla \times E = -\frac{\partial B(t)}{\partial t}$$

Proof. In order to derive the equation, let's imagine a loop, with B being the time-varying electric field. Because the rate of change of the total magnetic flux is equal to the opposite of the EMF, so taking the sum of the magnetic flux (Φ) is the integral of B over the area enclosed by the circuit

$$\Phi(t) = \int_S B(t) dS$$

while the total EMF is the line integral

$$EMF_{total} = \oint_{circuit} d(EMF)$$

From the definition of Voltage(V) and the relationship between V and the electric field E, we have

$$V = \int E dl E = \frac{dV}{dl}$$

rewrite 5.4with the two equations above, we have

$$EMF_{total} = \oint_{circuit} E dl$$

If we apply the Stokes' Theorem on one form(i.e. Green's Theorem)to the right hand side of 5.4 we have

$$\oint_{circuit} E dl = \int_S \nabla \times E dS$$

From the equation above and the Lenz's theorem, we have

$$\int_S \nabla \times E dS = -\frac{d}{dt} \int_S B(t) dS = \int_S \frac{-dB(t)}{dt} dS$$

Integrate both sides,

$$\nabla \times E = -\frac{\partial B(t)}{\partial t}$$

which is the Maxwell-Faraday equation, the third equation in the Maxwell's equations. The other three equations can also be proved □

6 Conclusion

In this paper, the idea of differential forms was introduced and discussed to prove the Stokes' Theorem. The theorem is already used in physics, and it will be more widely applied once humankind succeeds in understanding higher dimensional spaces. The future studies on the Stokes' Theorem should also be how it can be applied to theoretical physics to interpret the universe.

References

[1] H. Flanders, *Differential forms with applications to the physical sciences.* Dover Publications, 1963.

[2] T. F. Glick and S. J. Livesey, *Medieval science, technology, and medicine : an encyclopedia.* New York: Routledge, 2005.

[3] J. Milnor, *Topology from the Differentiable Viewpoint.* Princeton University Press, 1965.

[4] J. M. Lee, *Introduction to Smooth Manifolds.* Springer, 2012.