Reachability of Fair Allocations via Sequential Exchanges

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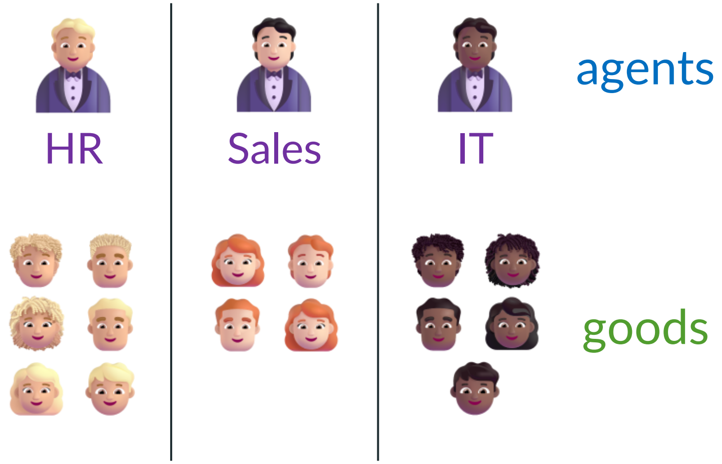
Fair Division of Indivisible Goods

- **Fair Division of Indivisible Goods**
  - The study of allocating goods fairly among competing agents.

- **Example.** A company wishes to allocate its employees to different departments in a fair manner.
  - agent → head of department
  - good → employee



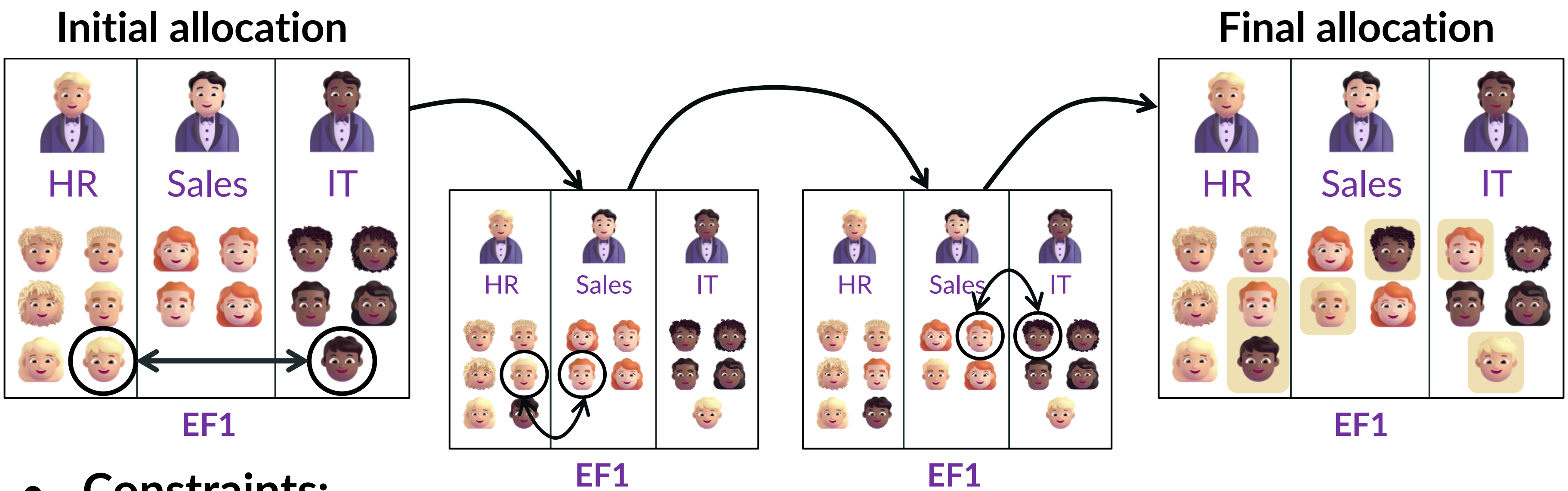
- **Fairness notion**
  - Envy-freeness up to one good (EF1)

- An EF1 allocation always exists[†]

[†] On Approximately Fair Allocations of Indivisible Goods (Lipton, Markakis, Mossel, and Saberi, 2004)

Reachability of Fair Allocations

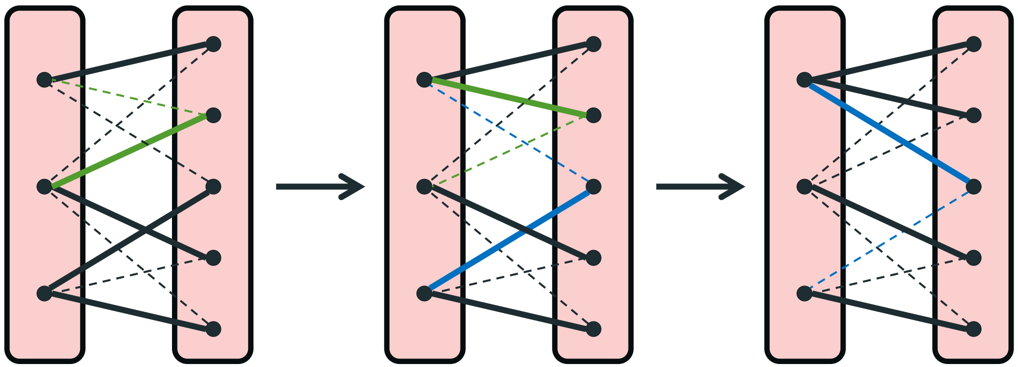
- We take a "dynamic" perspective.



- **Constraints:**
    - Every month, two employees from different departments will be selected to exchange positions.
    - EF1 must be maintained throughout the whole process.
- **Question:** Is it always possible to reach the desired final allocation?

Reconfiguration

- Reachability problems are also known as reconfiguration problems
- Examples:
  - Minimum spanning tree[1]
  - Graph coloring[2]
  - Perfect matching[3]



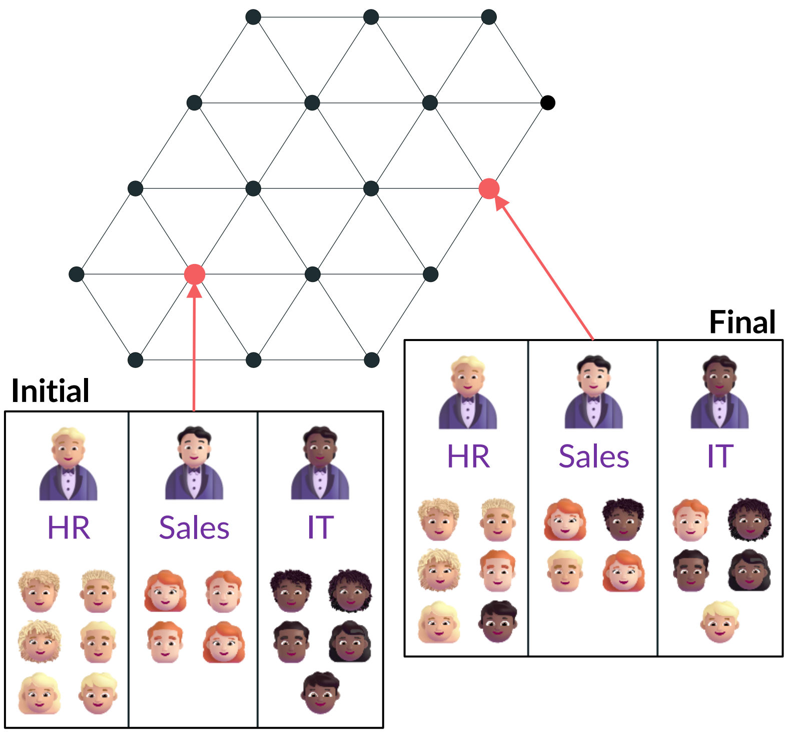
- Reachability has also been studied in voting[4 5]

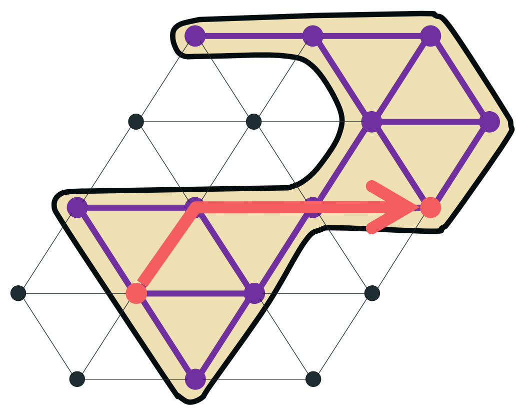
[1] On the Complexity of Reconfiguration Problems (Ito, Demaine, Harvey, Papadimitriou, Sideri, Uehara, and Uno, 2011)
[2] Finding Shortest Paths between Graph Colorings (Johnson, Kratsch, Kratsch, Patel, and Paulusma, 2016)
[3] The Perfect Matching Reconfiguration Problem (Bonamy, Bousquet, Heinrich, Ito, Kobayashi, Mary, Mühlenthaler, and Wasa, 2019)
[4] On Swap-Distance Geometry of Voting Rules (Obraztsova, Elkind, Faliszewski, and Slinko, 2013)
[5] On Swap Convexity of Voting Rules (Obraztsova, Elkind, and Faliszewski, 2020)

Exchange Graph

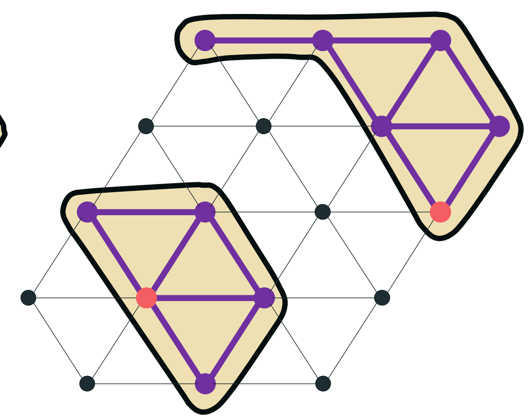
- **Vertices**
  - All allocations with the same size vector.

- **Edges**
  - Two allocations are adjacent iff they can be reached via an exchange.





EF1 exchange graph is connected



EF1 exchange graph is not connected

- **Question:** Is the EF1 exchange graph always connected?

Two Agents, General Utilities

- **Theorem.** ✗ There exists an instance for **two agents** such that the EF1 exchange graph is **not connected**.

agent 1	3	3	0	0	2	2	2	2
agent 2	3	3	0	0	1	1	1	1

agent 1	3	3	0	0	2	2	2	2

agent 1 envies agent 2 by more than one good!

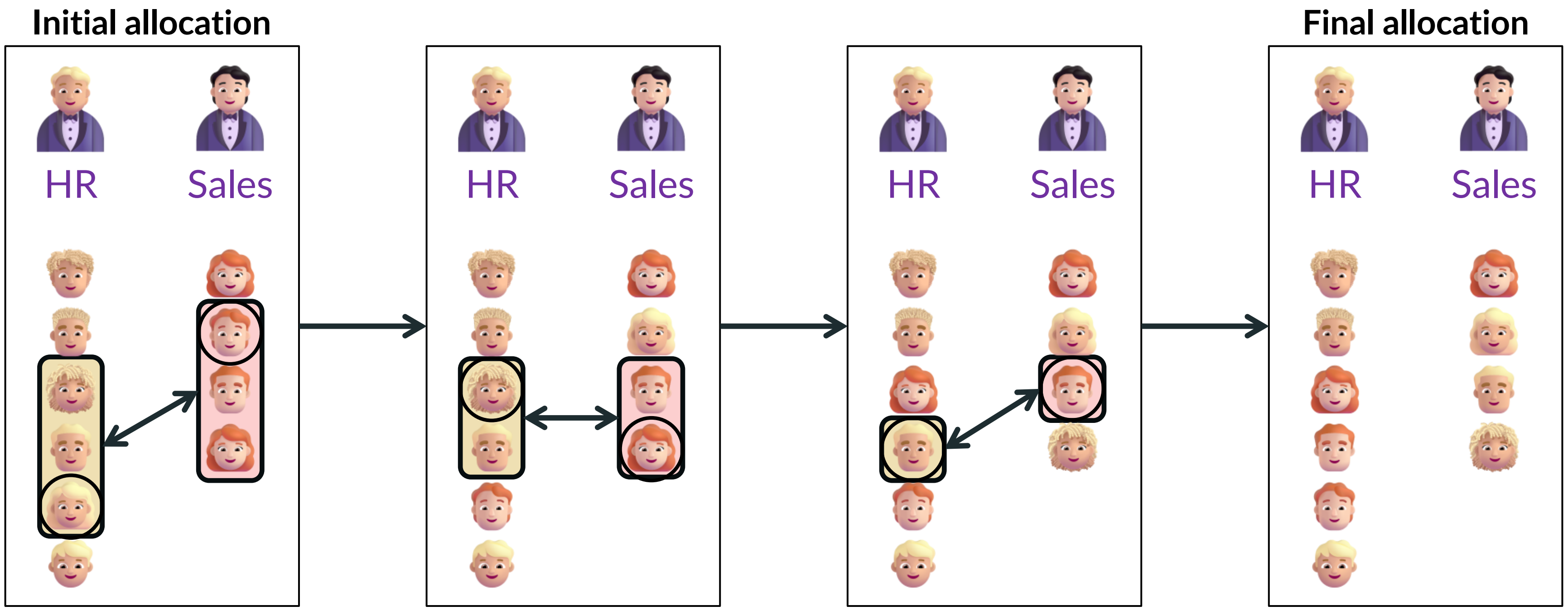
agent 2	3	3	0	0	1	1	1	1

agent 2 envies agent 1 by more than one good!

Two Agents, Identical / Binary Utilities

- **Theorem.** ✓ The EF1 exchange graph is **connected** for any instance with **two agents** and **identical utilities**.

- **Theorem.** ✓ The EF1 exchange graph is **connected** for any instance with **two agents** and **binary utilities**.



Three or More Agents

- **Theorem.** ✗ For every $n \geq 3$, there exists an instance for ***n* agents with identical utilities** such that the EF1 exchange graph is **not connected**.

agent 1	4	3	1				
agent 2				4	2	2	
agent 3							4

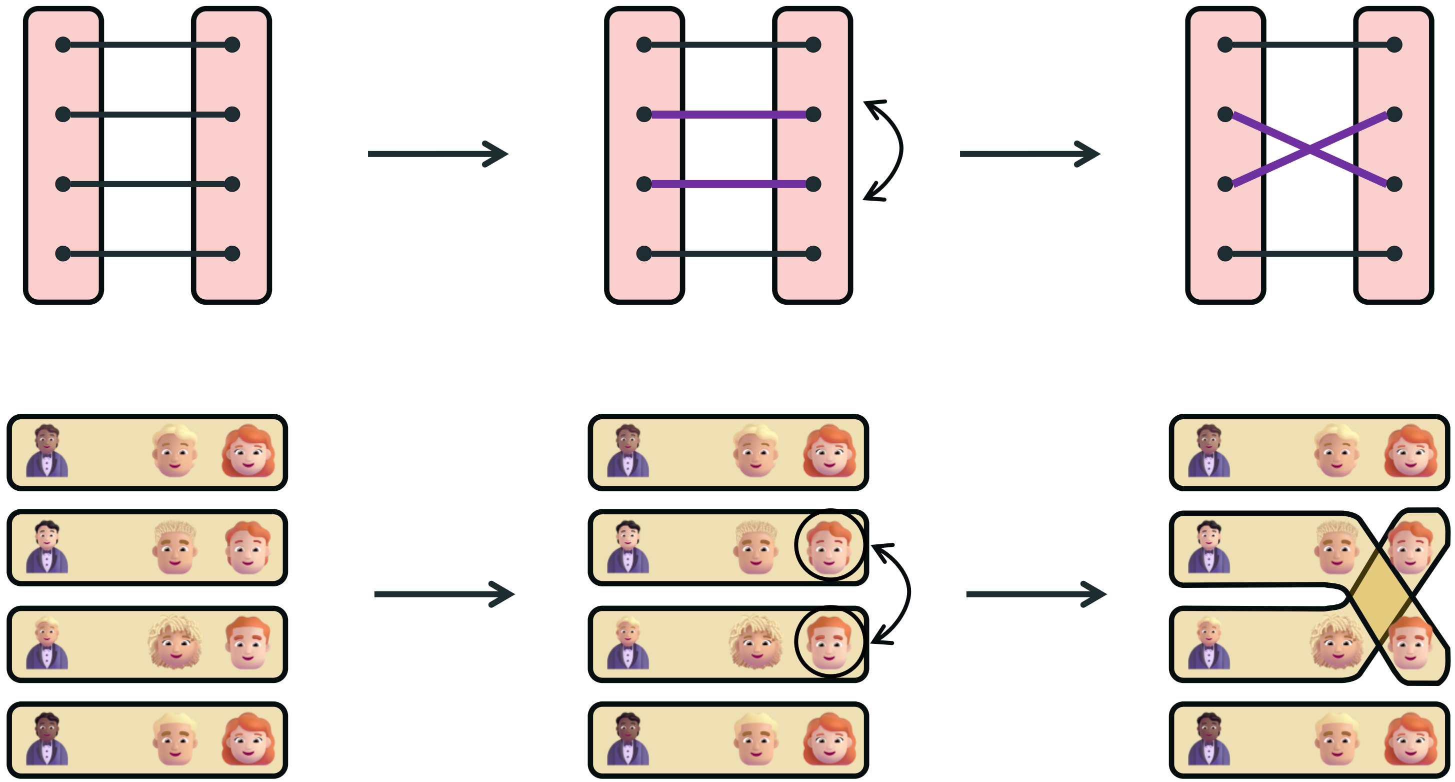
- **Theorem.** ✗ For every $n \geq 3$, there exists an instance for ***n* agents with binary utilities** such that the EF1 exchange graph is **not connected**.

- **Theorem.** ✓ The EF1 exchange graph is **connected** for any instance with **identical binary utilities**.

Connectivity is PSPACE-Complete

- **Theorem.** Determining the existence of an EF1 exchange path between two EF1 allocations is PSPACE-complete.

- *Proof.* Use a reduction from Perfect Matching Reconfiguration.†



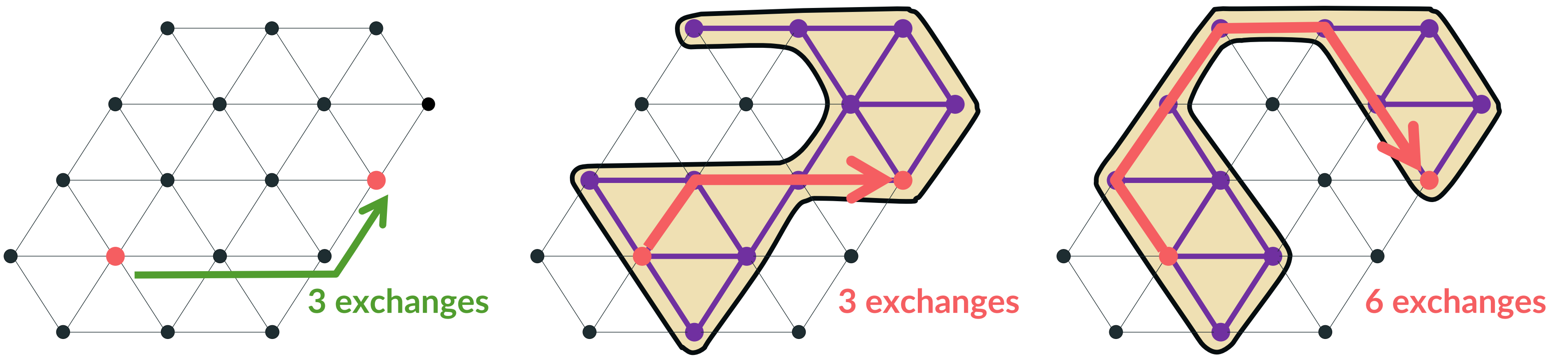
† The Perfect Matching Reconfiguration Problem (Bonamy, Bousquet, Heinrich, Ito, Kobayashi, Mary, Mühlenthaler, and Wasa, 2019)

Summary for Connectivity

utilities	general	identical	binary	identical binary
two agents	✗	✓	✓	✓
≥ three agents	✗ PSPACE-complete	✗	✗	✓

Optimality of EF1 Exchange Path

- We only consider instances where the EF1 exchange graph is connected.



Exchange graph

Optimal EF1 exchange path exists

Optimal EF1 exchange path does not exist

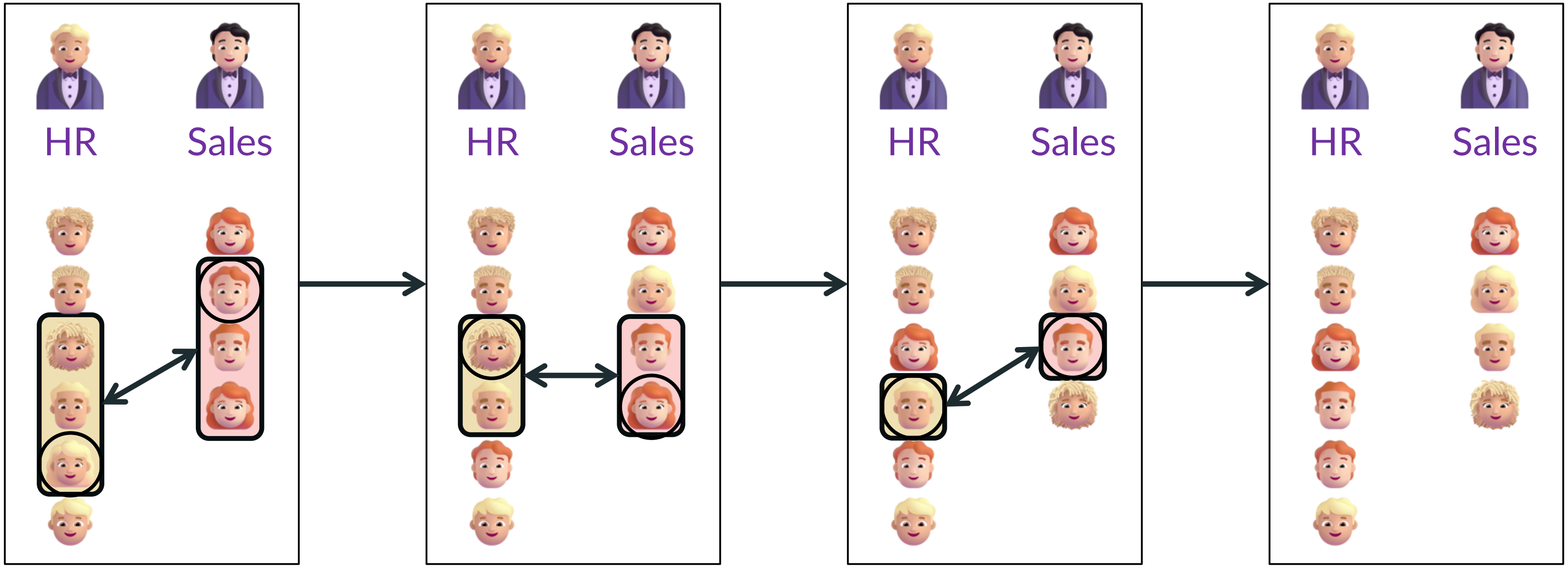
- **Question:** Is there always an optimal EF1 exchange path?

- **Theorem.** ✗ There exists an instance for **two agents** such that the EF1 exchange graph is connected, but for some pair of EF1 allocations, **no optimal EF1 exchange path exists** between them.

utilities		general	identical	binary	identical binary
two agents	connected? optimal?	✗ ✗	✓	✓	✓
≥ three agents	connected? optimal?	✗	✗	✗	✓

Two Agents, Identical / Binary Utilities

- **Theorem.** ✓ An **optimal EF1 exchange path exists** between any two allocations in any instance with **two agents** and **identical or binary utilities**.



utilities		general	identical	binary	identical binary
two agents	connected?	✗	✓	✓	✓
	optimal?	✗	✓	✓	✓
≥ three agents	connected?	✗	✗	✗	✓
	optimal?				

Three or More Agents

- **Theorem.** ✗ For every $n \geq 3$, there exists an instance for ***n* agents with identical binary utilities** such that the EF1 exchange graph is connected, but for some pair of EF1 allocations, **no optimal EF1 exchange path exists** between them.

- **Theorem.** Determining the **existence of an optimal EF1 exchange path** between two EF1 allocations is **NP-hard**, even for four agents with identical utilities.

- **Theorem.** Finding the **optimal number of exchanges** from the initial allocation to the final allocation on the exchange graph is **NP-hard** (disregarding the EF1 property).

utilities		general	identical	binary	identical binary
two agents	connected?	✗	✓	✓	✓
	optimal?	✗	✓	✓	✓
≥ three agents	connected?	✗	✗	✗	✓
	optimal?	✗	✗	✗	✗

Conclusion

- **Summary**
  -

utilities		general	identical	binary	identical binary
two agents	connected?	✗	✓	✓	✓
	optimal?	✗	✓	✓	✓
≥ three agents	connected?	✗	✗	✗	✓
	optimal?	✗	✗	✗	✗

  - Connectivity of EF1 exchange graph is PSPACE-complete.
  - Optimality of EF1 exchange path is NP-hard.
  - Finding optimal number of exchanges (disregarding EF1) is NP-hard.

- **Future work**
  - Transfers, instead of exchanges
  - Other fairness notions besides EF1