Approximate Envy-Freeness in Graphical Cake Cutting

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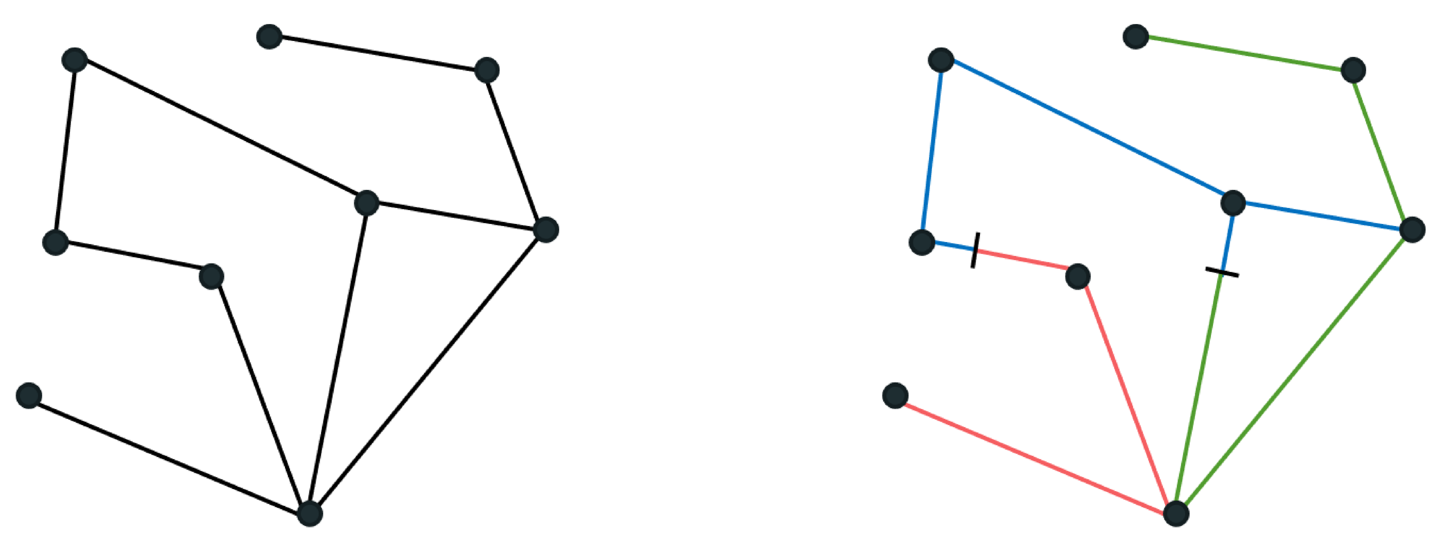
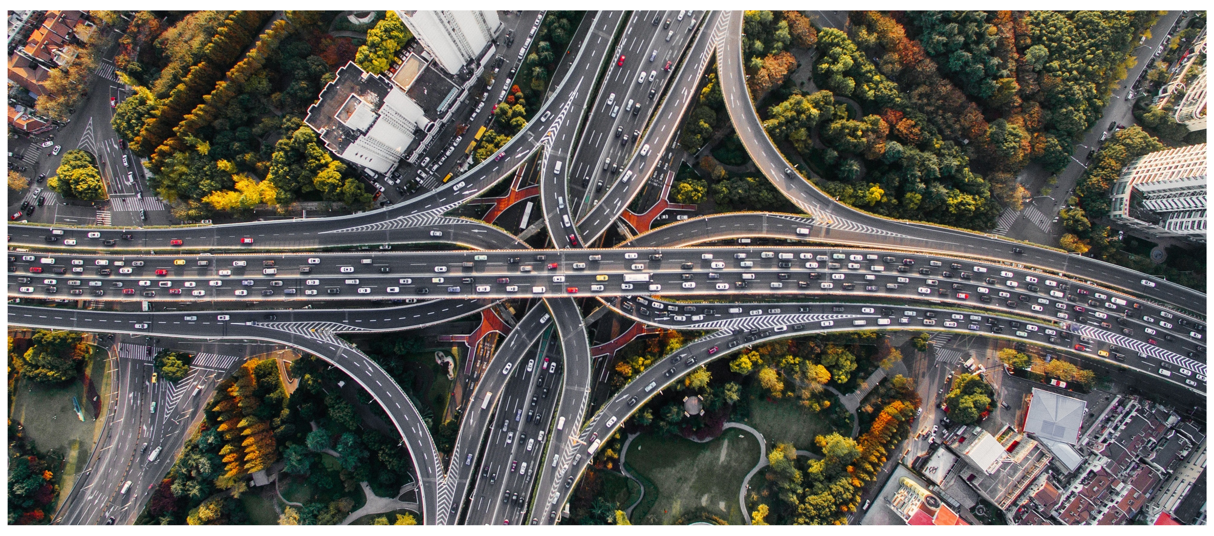
Graphical Cake Cutting



- We consider the problem of fairly allocating a divisible resource among agents who may have different values for different parts of the cake.
- The cake is represented by the edges of a connected graph; each edge can be subdivided into segments to be allocated to different agents.
- Appropriate when the resource corresponds to a network, such as a road network, a railway network, or a power cable network, to be divided among different companies for the purpose of construction or maintenance.



Fairness Notions

- Envy-freeness: agents rather have their own pieces of cake than others'.
  - An envy-free allocation may not exist in graphical cake cutting.
  - We consider approximations of envy-freeness.
    - α-EF: agents do not envy others by a factor of $> \alpha$ ($\alpha \geq 1$).
    - α-additive-EF: agents do not envy others by an amount $> \alpha$ ($\alpha \in [0, 1]$).
- Other fairness notions studied are proportionality [Bei & Suksompong '21] and maximin share [Elkind, Segal-Halevi & Suksompong '21].
  - These only consider each agent's piece of cake relative to the whole cake, rather than relative to other agents' pieces of cake.

Our Results

	General graphs	Star graphs
Non-identical valuations	1/2-additive-EF	$(3+\epsilon)$-EF
Identical valuations	$(2+\epsilon)$-EF	2-EF

- All of our results come with algorithms with time polynomial in the number of agents, the size of the graph, and, if applicable, $1/\epsilon$.

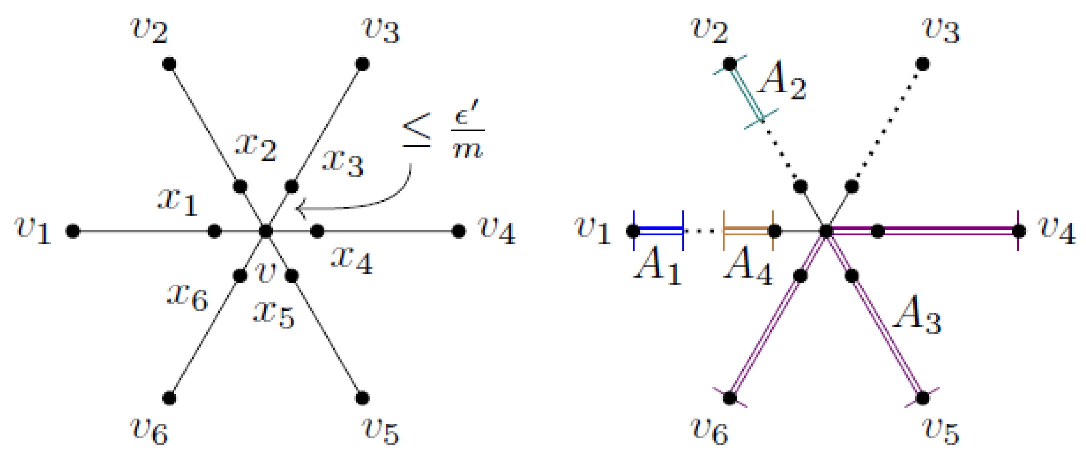
Non-Identical Valuations

Theorem 1. There exists a $1/2$-additive-EF allocation for a general graph.

- Generalize ideas from interval cake cutting [Goldberg, Hollender & Suksompong '20].
- Find a bundle worth less than $1/2$ to every agent and at least $1/4$ to some agent; allocate it to one such agent who values it at least $1/4$.
- We can find such a bundle using the DIVIDE algorithm.
- Remove this agent along with her bundle; repeat the process with the remaining graph and remaining agents until the whole graph is allocated.

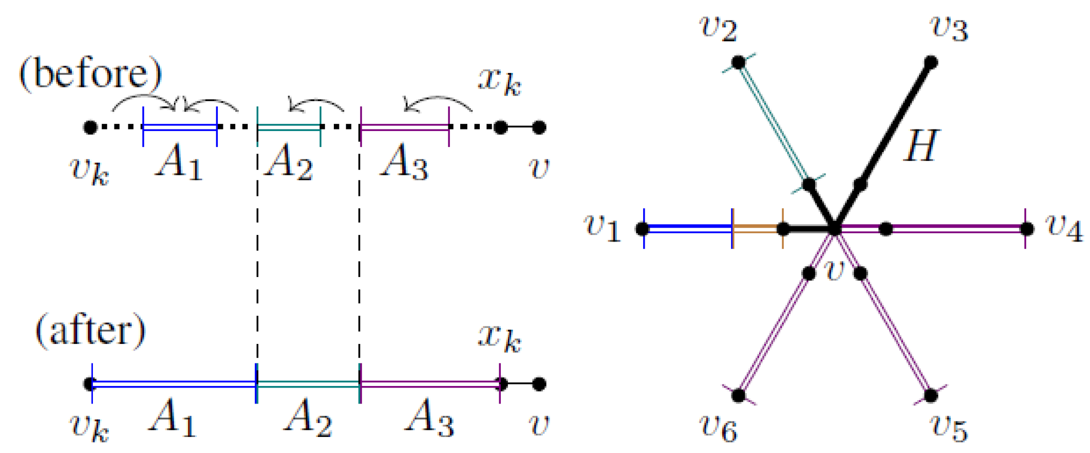
Theorem 2. There exists a $(3+\epsilon)$-EF allocation for a star graph.

- Generalize ideas from interval cake cutting [Arunachaleswaran, Barman, Kumar & Rathi '19].



(a) Phase 1 (b) Phase 2

- **Phase 1.** Find a point on each edge close to the center vertex.
- **Phase 2.** Repeatedly find an unallocated bundle worth slightly more than some agent's current bundle; allow that agent to relinquish her existing bundle for this new one. This new bundle could be a segment of an edge (**Phase 2a**) or a union of multiple complete edges (**Phase 2b**).



(c) Phase 3 (d) Phase 4

- **Phase 3 & 4.** Append the remaining graph to the agents.

Identical Valuations

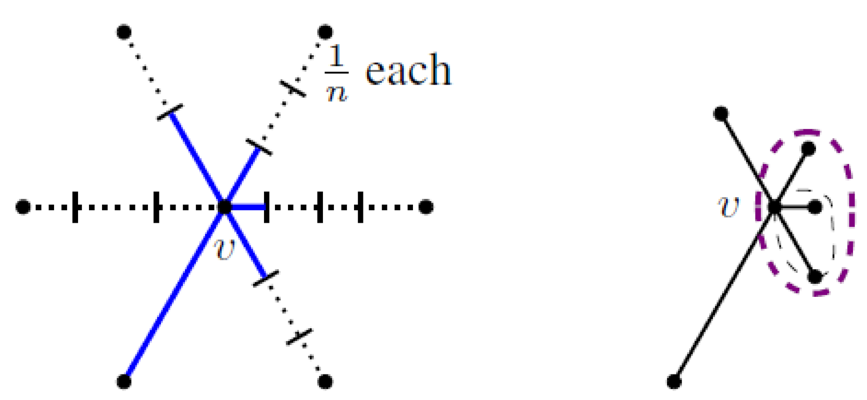
Theorem 3. There exists a 4-EF allocation for a general graph and agents with identical valuations.

- Generalize ideas from Theorem 1 which had used a threshold of $1/4$.
- Use an adaptive threshold of $\frac{1}{2}\left(\frac{2i}{2n-1} - \sum_{j=1}^{i-1}\mu(A_j)\right)$ for agent i, which takes into account the bundles received by the previous agents.

Theorem 4. There exists a $(2+\epsilon)$-EF allocation for a general graph and agents with identical valuations.

- Generalize ideas from partitioning indivisible edges of a graph [Chu, Wu, Wang & Chao '10].
- Starting with a 4-EF allocation, find a minimum-maximum path, and adjust the bundles along this path accordingly.
- Repeat this process until the allocation is $(2+\epsilon)$-EF.

Theorem 5. There exists a 2-EF allocation for a star graph and agents with identical valuations.



- Find segments worth $1/n$ and allocate them to the agents.
- There will be a star of stubs left; group these stubs repeatedly until the number of groups is equal to the number of agents remaining.
- Assign each group to each of the remaining agents.

Beyond One Connected Piece

- Allow agents to receive a small number of connected pieces using the notion of path similarity number.

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