On Connected Strongly-Proportional Cake-Cutting

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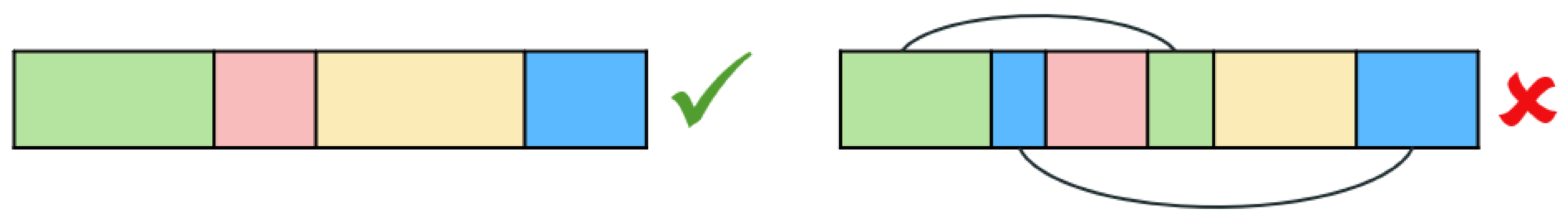
Introduction

- **Cake-cutting.** The problem of fairly dividing a resource, also known as a cake, among n agents.
  - **Cake** is represented by a unit interval $[0, 1]$.
  - **Divisible.** The cake can be cut into arbitrarily small parts.
  - **Heterogenous.** Each agent may have different valuations over different parts of the cake.
  - **Connectedness.** Each agent receives a connected piece of cake.



- **Applications.** Dividing land between owners; or dividing the time slots of a meeting room between different teams.



- **Strong-proportionality.** Each agent i receives a piece of cake worth more than their entitlement w_i.

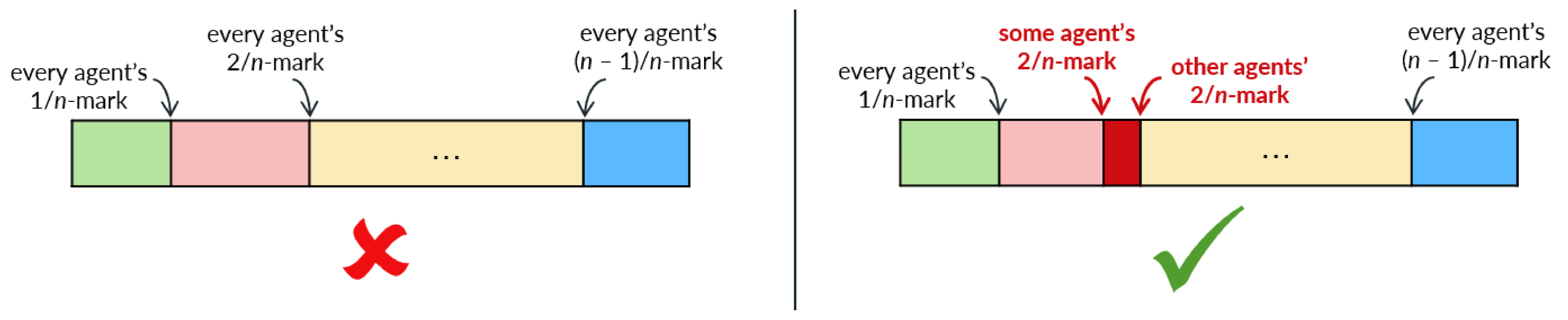
Questions

- What are the **conditions** for the existence of a connected strongly-proportional allocation?
- What are the **query complexities** to determine these conditions?

Hungry Agents with Equal Entitlements

- **Hungry.** An agent is hungry if every interval of the cake has positive value to them.
- **Equal entitlements.** Every agent's entitlement is exactly $1/n$; therefore, a strongly-proportional allocation gives every agent more than $1/n$.

Theorem 1. For n hungry agents with equal entitlements, a connected strongly-proportional allocation exists if and only if some t/n-mark of two agents are different.



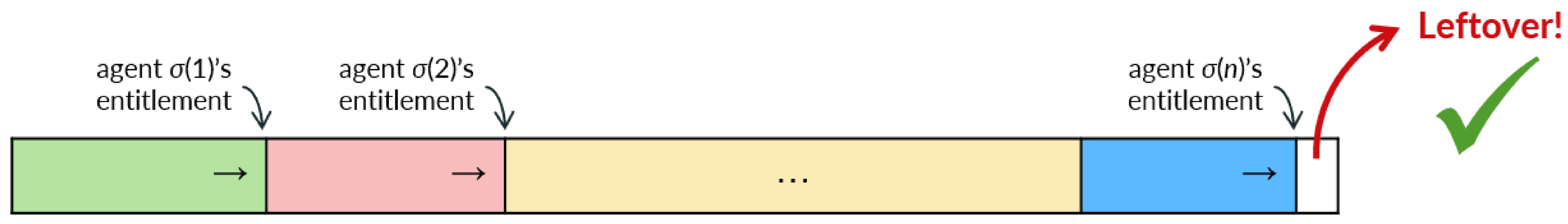
- If every agent's t/n-marks are the same, then giving some agent more than $1/n$ will sacrifice some other agent's piece. ✗
- If two agents' t/n-marks are different, then we can first assign every agent $1/n$ of the cake, then slightly adjust the boundaries for all the pieces. ✓

Theorem 2. For n hungry agents with equal entitlements, the existence of a connected strongly-proportional allocation can be determined in $\Theta(n^2)$ queries.

- **Upper bound.** Verify the condition in **Theorem 1** by asking every agent for their t/n-marks. This requires $n(n-1) \in O(n^2)$ queries.
- **Lower bound.** If some agent's t/n-mark is not known, then the condition in **Theorem 1** cannot be verified. This requires $n(n-1)/2 \in \Omega(n^2)$ queries.

Upper Bound for General Case

Theorem 3. A connected strongly-proportional allocation exists if and only if there exists a permutation σ of the agents such that $\text{MARK}_\sigma(0, \mathbf{w}) < 1$.



- **Meaning of $\text{MARK}_\sigma(0, \mathbf{w})$:** Agents go in sequence of σ and make their rightmost marks worth their entitlements on the cake one after another.
- If $\text{MARK}_\sigma(0, \mathbf{w}) < 1$ for some permutation σ, then each agent shall simply encroach on the piece to their right to get more than their entitlement.
- If a connected strongly-proportional allocation exists, then we define σ as the order based on that allocation, and we guarantee $\text{MARK}_\sigma(0, \mathbf{w}) < 1$.

Theorem 4. The existence of a connected strongly-proportional allocation can be determined in $O(n \cdot 2^n)$ queries.

- Verify the condition in **Theorem 3** by finding the best $\text{MARK}_\sigma(0, \mathbf{w})$.
- Use dynamic programming to reduce the number of queries to $O(n \cdot 2^n)$.

Lower Bound for Hungry Agents with Generic Entitlements

Theorem 5. Any algorithm that decides the existence of a connected strongly-proportional allocation for n hungry agents with generic entitlements requires $\Omega(n \cdot 2^n)$ queries.

- Construct two contrasting instances such that when fewer than $n \cdot 2^{n-2} \in \Omega(n \cdot 2^n)$ queries are made, the algorithm cannot differentiate between them.
- **Instance 1:** Every agent has uniform and identical distribution on the cake such that a connected strongly-proportional allocation does not exist.
- **Instance 2:** Some agent's marks are not fully known, so we adjust that agent's distribution from **Instance 1** such that a connected strongly-proportional allocation exists.

Lower Bound for Agents with Equal Entitlements

Theorem 6. Any algorithm that decides the existence of a connected strongly-proportional allocation for n agents with equal entitlements requires $\Omega(n \cdot 2^n)$ queries.

Agent 1	0	$a_1/(n-2)$		0	$a_1/(n-2)$	0	$1-a_1$	0
$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Agent $n-1$	0	$a_{n-1}/(n-2)$	(total: $n-2$ identical copies)	0	$a_{n-1}/(n-2)$	0	$1-a_{n-1}$	0
Agent n	$1/n$	0		$1/n$	0	$1/n$	0	$1/n$

- Construct the instance above and choose $a_1, \dots, a_{n-1}$ carefully.
- In a connected strongly-proportional allocation, agent n is forced to receive the two rightmost $1/n$ pieces.
- The remaining pieces are worth a_i to agent $i \in \{1, \dots, n-1\}$.
- After removing "0" pieces and normalizing, the reduced cake is for $n-1$ hungry agents with generic entitlements—we make use of **Theorem 5**.

Conclusion

	hungry agents	**general agents**
equal entitlements	$\Theta(n^2)$	$\Theta(n \cdot 2^n)$
unequal entitlements	$\Theta(n \cdot 2^n)$	$\Theta(n \cdot 2^n)$

- **Full version of our paper:** Stronger than strongly-proportional; Pies.
- **Future work:** Chores; Other fairness notions; More complex topologies beyond the unit interval.

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