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    - Idea: Study groups that act "nicely" on rooted regular n-ary trees.
- (3) *Computational questions*
  - Idea: There are many fundamental computational questions for finitely generated groups. They are not decidable in general, but can be solved when you restrict to nice groups.
  - Def 0.1: Let G be a group with finite generating set A. The **Word Problem (WP)** for (G,A) asks if there exists an algorithm that, upon input of any word $w \in (A \cup A^{-1})^*$, can determine whether $w =_G 1$.
  - Rmk: The WP does not ask if there is an algorithm to find the algorithm!
  - Thm 0.2: (Boone; Novikov; 1955) There is a finitely presented group G with no WP solution.
  - Def 0.4: The **Uniform Word Problem (UWP)** asks if there exists an algorithm that, upon input of any finite presentation $\langle A \mid R \rangle$ and word $w \in (A \cup A^{-1})^*$, can determine whether $w =_{\langle A|R \rangle} 1$.
  - Def 0.7: The **Isomorphism Problem (IP)** asks if there exists an algorithm that, upon input of any two finite presentations $\langle A \mid R \rangle$ and $\langle B \mid S \rangle$, can determine whether $\langle A \mid R \rangle \cong \langle B \mid S \rangle$.
  - Def 0.10: Let G be a group with a finite generating set A and let H be a subgroup of G. The **Subgroup Membership Problem (SMP)** for (G,A,H) asks if there exists an algorithm that,

upon input of any word $w \in (A \cup A^{-1})^*$, can determine whether w represents an element of H.

- Def 0.12: Let G be a group with a finite generating set A. The **Uniform Subgroup Membership Problem (USMP)** for (G,A) asks if there exists an algorithm that, upon input of any finite set of words $x_1,...,x_n,w \in (A \cup A^{-1})^*$, can determine whether w represents an element of the subgroup $< x_1,...,x_n >$.

Chapter 1: Fundamental concepts of infinite group theory

- *Section A: A short (and incomplete) review*
  - (i) *Groups, Isomorphism Problem and invariants, and actions*
    - Motivation: Studying symmetries of objects
    - Def 1.1: A **binary operation** on a set G is a function $G \times G \to G$.
    - Def 1.2: A **monoid** is a set M with a binary operation (called **monoid multiplication**; the operation maps an ordered pair (a,b) to an element of M denoted ab) satisfying the following:
      (1) For all a,b,c in M, (ab)c = a(bc). (**associative**)
      (2) There is an element 1 such that a1 = a = 1a for all a in M. (**identity**)
    - Def 1.3: A **group** is a set G with a binary operation (called **group multiplication**; the operation maps an ordered pair (a,b) to an element of G denoted ab) satisfying the following:
      (1) For all a,b,c in G, (ab)c = a(bc). (**associative**)
      (2) There is an element 1 such that a1 = a = 1a for all a in G. (**identity**)
      (3) For each element a in G, there is an element b in G such that ab = 1 = ba. (**inverse**)
    - *Examples*
      - Def 1.5: Let X be a set. The **permutation group of X** is **Perm(X)** = $\{f:X \to X \mid f$ is a bijection$\}$ with the group operation of function composition.
      - Def 1.6: Let (X,d) be a metric space. The **isometry group of X** is **Isom(X)** = $\{f:X \to X \mid f$ is a bijection and $d(p,q)=d(f(p),f(q))$ for all $p,q \in X\}$ with the group operation of function composition.
    - Def 1.8: A **homomorphism** from a group G to a group H is a function $f:G \to H$ satisfying $f(gg') = f(g)f(g')$ for all $g,g' \in G$.
    - Def 1.9: An **isomorphism** from G to H is a homomorphism from G to H that is a bijection. Two groups G and H are **isomorphic**, written $G \cong H$, if there is an isomorphism from G to H.
    - Def 1.10: The **Isomorphism Problem** asks whether there exists an algorithm that can determine, upon input of two groups G and H, whether or not $G \cong H$. The **Classification**

Problem asks whether there exists an algorithm that can enumerate (list) all of the groups up to isomorphism.

- Rmk: There cannot be an IP or CP algorithm for *all* groups, but there are IP and CP algorithms for classes of "nice" groups.
- *Isomorphism invariants*
  - Def 1.12: An **isomorphism invariant** is a property P of groups such that whenever G ≅ H and G has P then H has P.
  - Lemma 1.13: If P is an isomorphism invariant, G is a group that has P, and H is a group that does not have P, then G is not isomorphic to H.
  - Def 1.14: A group G is **abelian** (also called **commutative**) if for every a,b ∈ G, ab = ba.
  - Def 1.15: The **abelianization** of a group G is the quotient group $\mathbf{G_{ab}}$:= G/[G,G] where [G,G] is the **commutator subgroup [G,G]** := $\{aba^{-1}b^{-1} \mid a,b \in G\}$. The element $aba^{-1}b^{-1}$ of G is denoted **[a,b]** and called the **commutator of a with b**.
  - Lemma 1.16: If G and H are groups and $G_{ab} \not\cong H_{ab}$, then G ≇ H.
  - Def 1.18: A group G is a **finite** group if the set G is finite. The **order of a group** G (denoted **|G|**) is the number of elements in the set G.
  - Def 1.19: The **order of an element** g of G (denoted **|g|**) is the smallest positive integer n such that $g^n = 1$; if there is no such integer, then the order of g is infinite.
  - Def 1.20: A group G is **torsion-free** if all of the nonidentity elements of G have infinite order. A group G is **torsion** if all of the elements of G have finite order.
  - Thm 1.22: The following are isomorphism invariants: (a) "abelian". (b) The abelianization of the group, up to isomorphism. (c) The order of the group. (d) The set of orders of elements in the group. (e) "torsion-free". (f) "torsion".
- *Group actions*
  - Def 1.25: A **group action** of a group G on a set X is a function G × X → X (written (g,x) → gx) satisfying:
    (1) g(g'x) = (gg')x for all g,g' ∈ G and x ∈ X, and
    (2) 1x = x for all x ∈ X.
  - Lemma 1.26: Let G be a group and let X be a set.
    (a) If G acts on the set X (with action denoted by ·), then the function f:G → Perm(X) defined by (f(g))(x) := g · x (for all g in G and x in X) is a well-defined group homomorphism.
    (b) If f:G → Perm(X) is a homomorphism, then the function :G × X → X defined by g · x := (f(g))(x) (for all g in G and x in X) is a group action.
  - Def 1.27: Let G be a group acting on a set X. The **equivalence relation on X induced by the action of G**, written $\mathbf{\sim_G}$, is defined by p $\sim_G$ q if and only if there is a g ∈ G such that p = gq. The set of equivalence classes X/$\sim_G$ is written **X/G**.
  - Def 1.28: Let G be a group acting on a set X, let p ∈ X, and let Y ⊆ X. The **orbit** of p is the equivalence class of p; that is, $\mathbf{Orbit_G(p)}$:= [p] = {gp | g ∈ G}.
    The **stabilizer of p** is $\mathbf{Stab_G(p)}$:= {g ∈ G | gp = p}.

The **pointwise stabilizer of Y** is **$PtStab_G$(Y)** := {g ∈ G | gy = y for all y ∈ Y}.
The **setwise stabilizer of Y** is **$SetStab_G$(Y)** := {g ∈ G | gy ∈ Y for all y ∈ Y}.

- (ii) *Presentations*
  - Def 1.50: Let B be a set. The **free monoid on B**, denoted **B^***, is the set of all (finite) strings written in the alphabet B, including the empty word, denoted **1**. An element of B^* is called a **word over B**.
  - Def 1.51: Let A be a set, let $A^{-1} := \{a^{-1} \mid a \in A\}$ be a set that bijects to A, and let ~ be the smallest equivalence relation on $(A \cup A^{-1})^*$ such that $xaa^{-1}y \sim xy \sim xa^{-1}ay$ for all a ∈ A and $x,y \in (A \cup A^{-1})^*$. The **free group on A**, denoted **F(A)**, is the quotient set $(A \cup A^{-1})^*/\sim$ with the group operation [v][w] := [vw] where vw is the concatenation of the words v and w. In the case that |A| = n, this group is also denoted **F_n** and called the **free group of rank n**.
  - Def 1.52: A **reduced word** over a set A is a word $w \in (A \cup A^{-1})^*$ that does not contain a subword of the form aa^{-1} or $a^{-1}a$ for any a ∈ A.
  - Lemma 1.53: The function f: {reduced words over A} → F(A) defined by f(w) := [w] is a bijection.
  - Def 1.60: Let A be a set and let R be a subset of F(A). The **normal subgroup of F(A) generated by R** is **$< R >^N$** := $\{u_1r_1^{e_1}u_1^{-1} \cdots u_kr_k^{e_k}u_k^{-1} \mid k \geq 0$, and $r_i \in R$, $e_i \in \{1,-1\}$, and $u_i \in F(A)$ for each $1 \leq i \leq k\}$.
  - Def 1.61: Let A be a set and let R be a subset of F(A). The group **presented by** the **presentation < A | R >** is the quotient group F(A)/<lt; R $>^N$. The set A is the set of **generators**, the set R is the set of **defining relators**, and the set of equations {r = 1 | r ∈ R} is the set of **defining relations** of the presentation. The elements of $< R >^N$ are the **relators** of the group presented by < A | R >.
  - Lemma 1.62: The group < A | R > is the largest group generated by A satisfying $r =_G 1$ for all r ∈ R.
  - Def 1.63: For a set A, R ⊆ F(A), and words $v,w \in (A \cup A')^*$, the equation **v = w** means that v and w are the same word, **$v =_{F(A)} w$** means that [v] = [w] in the group F(A), and **$v =_G w$** means that $[v]< R >^N = [w]< R >^N$ in the group G := < A | R >.
  - Lemma 1.64: If G is a group, then G has a presentation; moreover, G is presented by G = < G | ab = (ab) for all a,b ∈ G >.
  - Prop 1.166: If G = < A | R >, then the abelianization of G is presented by $G_{ab} = < A \mid R \cup \{aba^{-1}b^{-1} \mid a,b \in A\} >$.
  - Thm 1.70: (**HBT** = "Homomorphism Building Theorem for presentations"): Let G = < A | R >, let H be a group, and let f:A → H be a function satisfying the property that for all words $b_1^{e_1} \cdots b_m^{e_m} \in R$ (with each $b_i \in A$ and $e_i \in \{1,-1\}$), $f(b_1)^{e_1} \cdots f(b_m)^{e_m} =_H 1$. Then there is a unique group homomorphism h:G → H satisfying h(a) = f(a) for all a ∈ A.
  - *Isomorphism invariants:*
    - Def 1.72: A subset A of G is a **generating set** for G if every element of G is a (finite) product of elements of A and their inverses. (This is written **G = ⟨ A ⟩**.)
      A group G is **finitely generated (f.g.)** if there is a finite subset A of G that generates G.

A group G is **cyclic** if there is an element a of G satisfying $G = \langle \{a\} \rangle$.
A group G is **finitely presented (f.p.)** if G = < A | R > for some finite sets A and R.

- Lemma 1.73: A group G is finitely generated if and only if G is (isomorphic to) a quotient of F(A) for some finite set A.
- Thm 1.75: The following are isomorphism invariants: (g) "finitely generated". (h) "cyclic". (i) "finitely presented".

- Def 1.77: Let A be a set, let $R \subseteq F(A)$, let b be a letter not in A, let $w \in F(A)$, and let $r \in \langle R \rangle^N$. The operations $\langle A \mid R \rangle \leftrightarrow \langle A \cup \{b\} \mid R \cup \{b = w\} \rangle$ and $\langle A \mid R \rangle \leftrightarrow \langle A \mid R \cup \{r\} \rangle$ are **Tietze transformations**.
- Thm 1.78: (**Tietze's Theorem**) If $\langle A \mid R \rangle \cong \langle B \mid S \rangle$, then there is a finite sequence of Tietze transformations from < A | R > to < B | S >.

- (iii) *Cayley graphs and Cayley complexes*
  - Def 1.80: Let G be a group with a generating set A. The **Cayley graph** for G with respect to A is the 1-dimensional CW complex **Γ = Γ(G,A)** with vertex set G and for each $g \in G$ and $a \in A$, a directed edge from g to ga labeled a.
  - Thm 1.82: Let G = < A > and let Γ be the Cayley graph. Then:
    (a) Γ is a path-connected CW complex.
    (b) For each vertex v of Γ and for each $a \in A$, there is exactly 1 edge out of v labeled a and exactly 1 edge into v labeled a.
    (c) G acts on Γ by $g \cdot h := (gh)$ and $g \cdot e_{h,a} := e_{(gh),a}$ (for all $h \in G$, $a \in A$), and this action satisfies:
    (c-i) (**free**): whenever $g \in G$ and $p \in \Gamma$ with gp = p, then g = 1;
    (c-ii) (**vertex-transitive**): whenever $v,w \in \Gamma^{(0)}$, there is a $g \in G$ with gv = w; and
    (c-iii) (**isomorphisms of directed labeled graph**): for each $g \in G$, the action of g is a bijection $:\Gamma \to \Gamma$ that maps vertices to vertices, and maps edges to edges preserving both directions and labels.
  - *Connections to group actions and covering space theory:*
    - Def 1.85: The **Cayley complex** associated to a presentation < A | R > of a group G is a 2-dimensional CW complex $\mathcal{C}$ **=** $\mathcal{C}$**(G,A,R)** with 1-skeleton Γ(G,A). The set of faces is in bijection with G × R; for each $g \in G$ and $r \in R$, the attaching map $\varphi_{g,r}: S^1 \to \Gamma$ of the face $f_{g,r}$ satisfies $\varphi_{g,r} \circ \omega$:= edge path in Γ starting at g labeled by r.
    - Def 1.87: The **presentation complex** associated to a presentation < A | R > of a group G is a CW complex with one vertex v, an edge e_a for each $a \in A$ (with attaching maps gluing both endpoints of e_a to v), and a face f_r for each $r \in R$ with attaching map determined by following the edges according to the word r.
    - Thm 1.88 (**"2-Way Street Thm"**): For every group G, there is a 2-dimensional PC CW complex X with $\pi_1(X) \cong G$. Moreover, if < A | R > is a presentation of G and Y is the associated presentation complex, then $\pi_1(Y) \cong G$.
    - Thm 1.89: Let < A | R > be a presentation of a group G, let $\mathcal{C}$ be the Cayley complex, and let X be the presentation complex. Then

(a) the action of G on $\mathcal{C}$, given by $g \cdot h := (gh)$, $g \cdot e_{h,a} := e_{(gh),a}$, and $g \cdot f_{h,r} := f_{(gh),r}$ for all $h \in G$, $a \in A$, and $r \in R$, is a covering space action;
(b) $\mathcal{C}/G \cong X$; and
(c) $\mathcal{C}$ is a simply-connected CW complex, and hence the composition $\mathcal{C} \to \mathcal{C}/G \to X$; is the universal covering space of X.

- *Section B: Normal forms, rewriting systems, and the Word Problem*

  - Def 1.100: Let A be a finite generating set for a group G, let π: F(A) → G be the corresponding surjective group homomorphism, and let ρ: $(A \cup A^{-1})^* \to F(A)$ be defined by $\rho(w) := [w]$ for all words w over $A \cup A^{-1}$.
    If $g \in G$, $w \in (A \cup A^{-1})^*$, and $g = \pi \circ \rho(w)$, then the word w **represents** g; in symbols, $w =_G g$.
    A **set of normal forms for G over A** is a subset $N \subset (A \cup A^{-1})^*$ satisfying the property that the restriction $\pi \circ \rho(w)|_N$: N → G is a bijection.
    If $w \in N$ and $g = \pi \circ \rho(w)$, then w is **the normal form of g**.
  - Examples
  - Def 1.105: Let A be a finite set. A **finite convergent rewriting system (CRS) over A** is a finite subset $R \subset A^* \times A^*$ such that the rewritings $xuy \to xvy$ for all $x,y \in A^*$ and $(u,v) \in R$ satisfy:
    (a) (**Termination**:) There is no infinite sequence of rewritings $x_1 \to x_2 \to x_3 \to \cdots$.
    (b) (**Confluence**:)
    (b-i) Whenever $(rs,v),(st,w) \in R$ with $s \neq 1$, then there exist a word $z \in A^*$ and finite sequences of rewritings $vt \to \cdots \to z$ and $rw \to \cdots \to z$.
    (b-ii) Whenever $(s,v),(rst,w) \in R$ (with $s \neq 1$), then there exist a word $z \in A^*$ and finite sequences of rewritings $rvt \to \cdots \to z$ and $w \to \cdots \to z$.
  - Def 1.106: For a rewriting system R over A, the set A is the **alphabet** and the elements of R are the **(rewriting) rules** of the rewriting system.
    A pair of rules of the form [$(rs,v),(st,w) \in R$ with $s \neq 1$] or [$(s,v),(rst,w) \in R$ (with $s \neq 1$)] is called a **critical pair** of R, and in conditions (b-i) and (b-ii) of Def 1.105, the word z and the rewritings to z are called a **resolution** of the critical pair.
    An **irreducible word** (for R) is a word that does not contain a subword u for any $(u,v) \in R$.
    The symbol $w \to^* x$ denotes that there is a finite sequence of rewritings from w to x.
  - Def 1.108: A **finite convergent rewriting system (CRS) for a group G** is a finite CRS such that G is presented as a monoid by G = Mon < A | {u=v | (u,v) ∈ R} >.
  - Thm 1.110: Let (A,R) be a finite CRS for a group G. Then:
    (a) The set **Irr(R)** := {irreducible words for R} is a set of normal forms for G.
    (b) Given any word $w \in A^*$, there is a finite sequence of rewritings from w to the normal form

representing the same element of G.
(c) The group G has a decidable Word Problem.

- *Termination and partial orders:*
  - Def 1.120: Let S be a set. A strict partial order > on S is **well-founded** if there is no infinite sequence of elements of S satisfying $x_1 > x_2 > x_3 > \cdots$.
  - Def 1.121: Let A be a set. A relation > on A^* is **compatible with concatenation** if whenever w,x,y,z ∈ A^* and w > x then ywz > yxz.
  - Def 1.22: Let A be a set. A **termination order** on A^* is a well-founded strict partial order that is compatible with concatenation.
  - Prop 1.123: Let A be a set and let $R \subseteq A^* \times A^*$. Let > be a termination order on A^*. If u > v for all (u,v) ∈ R, then the rewritings xuy → xvy for all x,y ∈ A^* and (u,v) ∈ R satisfy the termination property.
  - Def 1.125: Let A be a finite set with a total order >. The **shortlex** order $>_{sl}$ on A^* induced by > is defined by: For each $x = a_1a_2\cdots a_m$ and $y = b_1b_2\cdots b_n$ in A^*, where each $a_i,b_j \in A$, then $x >_{sl} y$ iff either (i) m > n, or (ii) m=n and there is an index i such that $a_j = b_j$ for all $1 \le j < i$ and $a_i > b_i$.
  - Def 1.126: Let A be a finite set with a total order >, and let w: A → $\mathbb{N}$ be a function. The **weightlex** order $>_{wl}$ on A^* induced by > and w is defined by: For each $x = a_1a_2\cdots a_m$ and $y = b_1b_2\cdots b_n$ in A^*, where each $a_i,b_j \in A$, then $x >_{wl} y$ iff either (i) $w(a_1) + w(a_2) + \cdots + w(a_m) > w(b_1) + w(b_2) + \cdots + w(b_n)$, or (ii) $w(a_1) + w(a_2) + \cdots + w(a_m) > w(b_1) + w(b_2) + \cdots + w(b_n)$ and there is an index i such that $a_j = b_j$ for all $1 \le j < i$ and $a_i > b_i$.
  - Prop 1.127: Let A be a finite set with a total order >. (a) The shortlex order on A^* induced by > is a well-founded strict partial order that is compatible with concatenation.
    (b) If w: A → $\mathbb{N}$, then the weightlex order on A^* induced by > and w is a well-founded strict partial order that is compatible with concatenation.
- Examples

* *Section C: Examples of f.p. groups, van Kampen diagrams, and graph products*

- Prop 1.40: The free group on a set A is presented by F(A) = < A | >. The set of reduced words over A is a set of normal forms for F(A).
- Prop 1.42: The free abelian group on a set A is presented by $\mathbb{Z}^A$ = < A | {ab=ba | a,b ∈ A} >. Given a total order < on A, the set of words $\{a_1^{j_1} \cdots a_k^{j_k} \mid k \ge 0, a_i \in A, j_i \in \mathbb{Z}, \text{ and } a_1^{j_1} < \cdots < a_k^{j_k}\}$ is a set of normal forms for $\mathbb{Z}^A$.
- *Van Kampen diagrams*
  - Def 1.144: Let G be a group with a presentation $\mathcal{P}$ = <A | R> and let w be a word over A ∪ A^{-1} satisfying $w =_G 1$. A **van Kampen diagram** for w with respect to $\mathcal{P}$ is a planar, simply-connected, finite 2-dimensional CW complex Δ with a basepoint vertex $*$ satisfying: (1)

Each edge of Δ is directed and labeled by an element of A. (2) Each 2-cell of Δ is oriented and has boundary labeled by a word in R. (3) The directed counterclockwise loop based at $*$ around the boundary of Δ is labeled by w.

  - Examples
  - Thm 1.145: Let G be a group with a presentation $\mathcal{P}$ = <A | R>.
    (a) For any word w over $A \cup A^{-1}$, $w =_G 1$ if and only if there is a van Kampen diagram for w w.r.t. $\mathcal{P}$.
    (b) Let $\mathcal{C}$ be the Cayley complex $\mathcal{C}(G,\mathcal{P})$, and let Δ be any van Kampen diagram w.r.t. $\mathcal{P}$. There is a unique continuous function f: $\Delta \to \mathcal{C}$ such that $f(*) = 1$ and f maps vertices to vertices, maps edges to edges preserving direction and labels, and maps faces to faces preserving orientation and boundary labels.
- Def 1.150: A **Coxeter group** is a group with a presentation of the form $\langle s_1, s_2, \ldots, s_n \mid (s_is_j)^{m_{ij}} = 1$ for all $i < j$, and $s_i^2 = 1$ for all $i\rangle$, where $2 \le m_{ij} \le \infty$ for all i,j.
- *Graph products*
  - Def 1.155: Let Λ be a finite simple graph such that each vertex v in $V(\Lambda)$ is labeled by a group G_v with a presentation $\langle A_v \mid R_v\rangle$. The **graph product** induced by the labeled graph Λ is the group
    GΛ $:= \langle \cup_{v \in V(\Lambda)} A_v \mid \cup_{v \in V(\Lambda)} R_v \cup \{ab = ba \mid a \in A_u, b \in A_v$, and u,v are adjacent vertices in $\Lambda\} \rangle$.
  - Prop 1.58: There are 4 equivalent views of G × H:
    (a) Presentation view: If $G = \langle A \mid R \rangle$ and $H = \langle B \mid S \rangle$, then $G \times H = \langle A \cup B \mid R \cup S \cup \{ab = ba \mid a \in A, b \in B\} \rangle$.
    (b) Graph product view: G × H is the graph product induced by a graph Λ with vertices labeled G and H and an edge between them.
    (c) Element view: G × H is the Cartesian product set with componentwise multiplication.
    (d) Superlative view: G × H is the largest group generated by G and H such that the subgroups G and H commute.
  - Def 1.160: Let G_α be a group, and write $G_\alpha = \langle A_\alpha \mid R_\alpha \rangle$, for each α. The **free product** of the G_α is the group $*_\alpha$ $\mathbf{G_\alpha} := \langle \cup_\alpha A_\alpha \mid \cup_\alpha R_\alpha \rangle$.
  - Def 1.161: Let G_α be a group for all α. A **reduced sequence** for the collection of groups G_α is a sequence of group elements (or word) of the form $g_1 \cdots g_k$ such that $k \ge 0$, for each $i \in \{1,\ldots,k\}$ there is an index α_i such that $g_i \in G_{\alpha_i} - \{1_{G_{\alpha_i}}\}$, and for each $i \in \{1,\ldots,k-1\}$, $\alpha_i \neq \alpha_{i+1}$. In the case of two groups G and H, a **reduced sequence** for G,H is a word of one of the forms $g_1h_1 \cdots g_kh_k$, $g_1h_1 \cdots h_{k-1}g_k$, $h_1g_2 \cdots g_kh_k$, or $h_1g_2 \cdots h_{k-1}g_k$, such that $k \ge 0$, and each $g_i \in G - \{1_G\}$ and $h_i \in H - \{1_H\}$.
  - Prop 1.162: There are 4 equivalent views of $G * H$:
    (a) Presentation view: If $G = \langle A \mid R \rangle$ and $H = \langle B \mid S \rangle$, then $G * H = \langle A \cup B \mid R \cup S \rangle$.
    (b) Graph product view: $G * H$ is the graph product induced by a graph Λ with vertices labeled G and H and no edges.
    (c) Element view: $G * H$ is the set of reduced sequences for G,H with group operation

given by concatenation and reduction (in the groups G and H) to a reduced sequence.
(d) Superlative view: $G * H$ is the largest group generated by G and H.

- Def 1.65: Let Λ be a finite simple graph.
The **right-angled Artin group (raag)** induced by Λ is the induced graph product in which each vertex is labeled by $\mathbb{Z}$.
The **right-angled Coxeter group (racg)** induced by Λ is the induced graph product in which each vertex is labeled by $\mathbb{Z}/2$.

- Prop 1.70: (a) The fundamental group of a compact connected orientable surface of genus g is $\pi_1(S_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1,b_1] \cdots [a_g,b_g] = 1 \rangle$.
(b) The fundamental group of the Klein bottle is presented by $\pi_1(K^2) = \langle a,b \mid bab^{-1} = a^{-1} \rangle$.
- Prop 1.72: (a) The **special linear group $SL_2(\mathbb{Z})$** of 2 × 2 integer matrices with integer entries and determinant 1, with group operation given by matrix multiplication, is presented by $SL_2(\mathbb{Z}) = \langle s,t \mid s^2 = t^3, s^4 = 1 \rangle$.
(b) The **projective special linear group $PSL_2(\mathbb{Z})$** := $SL_2(\mathbb{Z}) / \langle -I \rangle^N$ is presented by $PSL_2(\mathbb{Z}) = \langle s,t \mid s^2 = t^3 = 1 \rangle$.
- Def 1.74: Let $p,q \in \mathbb{Z}$. The **Baumslag-Solitar group BS(p,q)** is the group presented by $BS(p,q) := \langle a,t \mid ta^pt^{-1} = a^q \rangle$.

Chapter 2: Groups and trees

- *Section A: Fundamental groups of graphs of groups*

  - Def 2.1: A **graph of spaces** is a tuple $\mathcal{S} = (\Lambda, \{X_v\}, \{Y_e\}, \{f_e,f_e'\})$ where Λ is a connected directed graph, with each vertex v labeled by a space X_v and each edge e labeled by a space Y_e, and for each edge e there are continuous functions $f_e: Y_e \to X_{t(e)}$ and $f_e': Y_e \to X_{i(e)}$ (where t(e) and i(e) are the terminal and initial vertices of the edge e).
  - Def 2.2: A **graph of groups** is a tuple $\mathcal{G} = (\Lambda, \{G_v\}, \{H_e\}, \{h_e,h_e'\})$ where Λ is a connected directed graph, such that each vertex v labeled by a group G_v and each edge e labeled by a group H_e, and for each edge e there are *injective* group homomorphisms $h_e: H_e \to G_{t(e)}$ and $h_e': H_e \to G_{i(e)}$ (where t(e) and i(e) are the terminal and initial vertices of the edge e).
  - Prop 2.5: Let Λ be a connected directed graph. Let u be a vertex of Λ and let T be a maximal tree of Λ. For each edge e of Λ, let $s_e: I \to \Lambda$ be the loop at u that follows the path in T from u to the initial vertex i(e), traverses e, and then follows the path in T from the terminal vertex t(e) back to u. Then $\pi_1(\Lambda)$ is the free group generated by the set
$\{s_e \mid e$ is an edge of Λ that is not in T$\}$.
  - Def 2.6: Let $\mathcal{G} = (\Lambda, \{G_v\}, \{H_e\}, \{h_e,h_e'\})$ be a graph of groups. Suppose that $G_v = \langle A_v \mid R_v \rangle$ for each vertex v, and $H_e = \langle B_e \rangle$ for each edge e. Let u be a vertex of Λ, let T be a maximal tree

in Λ, and for each edge e let s_e be the loop in Λ defined in Prop 2.5. The **fundamental group** of this graph of groups with respect to u and T is $\pi_1(\mathcal{G})$:= < A | R > where
$A := (\cup_{v \in Vert(\Lambda)} A_v) \cup \{s_e \mid e \in Edge(\Lambda)\}$, and
$R := (\cup_{v \in Vert(\Lambda)} R_v) \cup \{s_e = 1 \mid e \text{ is an edge in } T\} \cup \{s_e h_e'(b) s_e^{-1} = h_e(b) \mid e \in Edge(\Lambda) \text{ and } b \in B_e\}$.

- Prop 2.8: For a graph of groups $\mathcal{G}$, the fundamental group of $\mathcal{G}$ is independent of the choice of basepoint or maximal tree, up to isomorphism.
- (i) *Free products with amalgamation:*:
  - Def 2.10: Let H = < A | R > and K = < B | S > be groups with presentations, let L = < C >, and let i:L → H and j:L → K be *injective* homomorphisms. The **free product with amalgamation (FPWA)** (or **amalgamated product**) $H *_L K$ is the group
    $H *_L K := \langle A \cup B \mid R \cup S \cup \{i(c) = j(c) \mid c \in C\} \rangle$.
  - Prop 2.11: A FPWA $H *_L K$ is the fundamental group of a graph of groups with one edge (labeled L) and two vertices (labeled H and K). Moreover, H, K, and i(L)=j(L) are all (isomorphic to) subgroups of $H *_L K$.
  - Def 2.13: Let G be a group and let H be a subgroup of G. A **left transversal** for H in G is a subset T ⊆ G satisfying the property that for each left coset in G/H, there is exactly one element t ∈ T contained in that coset.
    Similarly, a **right transversal** for H in G is a subset of G containing exactly one element from each right coset of H in G.
  - Thm 2.15: (**Normal forms for FPWA**): Let H = < A >, K = < B >, and L = < C > be groups with inverse-closed generating sets, and let i:L → H and j:L → K be injective homomorphisms.
    (a) Let M_H be a subset of A^* satisfying the property that for each right coset in i(L)\H other than $i(L)1_H$, there is exactly one word in M_H representing an element of the coset. Similarly let M_K be a subset of B^* that contains exactly one word representing an element of each right coset of j(L)\K other than $j(L)1_K$. Let M_L be a set of normal forms for the elements of $i(L) - \{1_{i(L)}\}$ over A. Then D := A ∪ B is an inverse-closed generating set for $H *_L K$, and the set
    $N := \{\ell p_1 q_1 \cdots p_n q_n \mid n \geq 0, \ell \in M_L \cup \{1\}, p_1 \in M_H \cup \{1\}, q_n \in M_K \cup \{1\}, p_i \in M_H \text{ for all } i > 1, \text{ and } q_i \in M_K \text{ for all } i < n\}$
    is a set of (**right greedy**) normal forms for the FPWA $H *_L K$.
    (b) Let M_H' be a subset of A^* satisfying the property that for each left coset in H/i(L) other than $1_H i(L)$, there is exactly one word in M_H' representing an element of the coset. Similarly let M_K' be a subset of B^* that contains exactly one word representing an element of each left coset of K/j(L) other than $1_K j(L)$. Let M_L be a set of normal forms for the elements of $i(L) - \{1_{i(L)}\}$ over A. Then D := A ∪ B is an inverse-closed generating set for $H *_L K$, and the set
    $N' := \{p_1 q_1 \cdots p_n q_n \ell \mid n \geq 0, \ell \in M_L \cup \{1\}, p_1 \in M_H' \cup \{1\}, q_n \in M_K' \cup \{1\}, p_i \in M_H' \text{ for all } i > 1,$

and $q_i \in M_K'$ for all $i < n$}
is a set of (**left greedy**) normal forms for the FPWA $H *_L K$.

- Def 2.17: Let H, K, and L be groups, and let $i:L \to H$ and $j:L \to K$ be injective homomorphisms. The **Bass-Serre tree** for the FPWA $G := H *_L K$ is the graph T with vertex set $V(T) := \{gH \mid g \in G\} \cup \{gK \mid g \in G\}$, and edge set $E(T) :=$ {edge labeled gL between vertex gH and vertex gK | $g \in G$}.
- Thm 2.18: The Bass-Serre tree T for a FPWA $G = H *_L K$ is a tree.
- Def 2.20: Let G be a group acting on a set X. The action of G on X is **free** if whenever $g \in G$, $p \in X$, and $gp = p$, then $g = 1_G$.
- Def 2.21: Let G be a group acting on a graph Ω.
  The action of G on Ω is **simplicial** if the action by each element of G maps vertices to vertices and edges to edges and preserves adjacency (that is, commutes with attaching maps).
  The action of G on Ω is **without inversion** if for every $g \in G$ and edge e of Ω, either g maps e to e preserving direction, or g doesn't map e to e.
- Thm 2.22: A FPWA $G = H *_L K$ acts (on the left) on the Bass-Serre tree T, by $g(g'H) := (gg')H$ and $g(g'K) := (gg')K$ for the vertices, and $g(g'L) := (gg')L$ on the edges. Moreover, this action is simplicial and without inversion.
- Thm 2.24: If $H *_L K$ is an amalgamated product of finitely presented groups H and K, then $H *_L K$ is finitely presented iff L is finitely generated.
- Thm 2.26: Let Λ be a finite connected simple graph and let GΛ be the corresponding right-angled Artin group. Let v be a vertex of Λ and let Ψ be the subgraph of Λ whose vertex set is $\{w \in V(\Lambda) \mid \exists$ an edge from v to w in Λ$\}$ and whose edge set is all of the edges between these vertices in Λ; that is, Ψ is an *induced* subgraph. Let Φ be the induced subgraph with vertex set $V(\Lambda) - \{v\}$. Show that GΛ is isomorphic to an amalgamated product $(Z \times G\Psi) *_{G\Psi} G\Phi$.

- (ii) *HNN extensions*:
  - Def 2.30: (a) Let $H = < A \mid R >$ be a group with presentation, let $L = < D >$, and let $i:L \to H$ and $j:L \to H$ be *injective* homomorphisms. The **HNN extension** $H *_L$ is the group
    $H *_L := < A \cup \{t\} \mid R \cup \{ti(d)t^{-1} = j(d) \mid d \in D\} >$.
    (b) Let $H = < A \mid R >$ be a group, let $B = < D >$ and C be subgroups of H, and let $\varphi:B \to C$ be an isomorphism. The **HNN extension** $H *_\varphi$ is the group
    $H *_\varphi := < A \cup \{t\} \mid R \cup \{tdt^{-1} = \varphi(d) \mid d \in D\} >$.
  - Prop 2.31: An HNN extension $H *_L$ (or $H *_\varphi$) is the fundamental group of a graph of groups with one edge (labeled L) and one vertex (labeled H). Moreover, H, $B = i(L)$, and $C = j(L)$ are all (isomorphic to) subgroups of $H *_L$.
  - Thm 2.35: (**Normal forms for HNN extensions**): Let $H = < A >$ and $L = < C >$ be groups with inverse-closed generating sets, and let $i:L \to H$ and $j:L \to H$ be injective homomorphisms.
    (a) Let M_i be a subset of A^* satisfying the property that for each right coset in $i(L)\backslash H$ other

than $i(L)1_H$, there is exactly one word in M_i representing an element of the coset. Similarly let M_j be a subset of A^* that contains exactly one word representing an element of each right coset in $j(L)\backslash H$ other than $j(L)1_H$. Let M_L be a set of normal forms for the elements of $i(L) - \{1_{i(L)}\}$ over A. Then $D := A \cup \{t,t^{-1}\}$ is an inverse-closed generating set for $H *_L$, and the set
$N := \{\ell p_0 t^{e_1} p_1 \cdots t^{e_n} p_n \mid n \geq 0, \ell \in M_L \cup \{1\}, p_0 \in M_i \cup \{1\}$, for all $k > 0$ either [$e_k = 1$ and $p_k \in M_i \cup \{1\}$] or [$e_k = -1$ and $p_k \in M_j \cup \{1\}$], and for all $0 < k < n$, if $p_k = 1$ then $e_k \neq -e_{k+1}\}$
is a set of (**right greedy**) normal forms over D for the HNN extension $H *_L$.
(b) Let M_i' be a subset of A^* satisfying the property that for each left coset in $H/i(L)$ other than $1_H i(L)$, there is exactly one word in M_i representing an element of the coset. Similarly let M_j' be a subset of A^* that contains exactly one word representing an element of each left coset in $H/j(L)$ other than $1_H j(L)$. Let M_L be a set of normal forms for the elements of $i(L) - \{1_{i(L)}\}$ over A. Then $D := A \cup \{t,t^{-1}\}$ is an inverse-closed generating set for $H *_L$, and the set
$N := \{p_0 t^{e_1} p_1 \cdots t^{e_n} p_n \ell \mid n \geq 0, \ell \in M_L \cup \{1\}, p_n \in M_i' \cup \{1\}$, for all $k < n$ either [$e_{k+1} = 1$ and $p_k \in M_j' \cup \{1\}$] or [$e_{k+1} = -1$ and $p_k \in M_i' \cup \{1\}$], and for all $0 < k < n$, if $p_k = 1$ then $e_k \neq -e_{k+1}\}$
is a set of (**left greedy**) normal forms over D for the HNN extension $H *_L$.

- Def 2.37: Let H and L be groups, and let $i:L \to H$ and $j:L \to H$ be injective homomorphisms. The **Bass-Serre tree** for the HNN extension $G := H *_L$ is the graph T with vertex set $V(T) := \{gH \mid g \in G\}$, and edge set $E(T) :=$ {edge labeled $gj(L)$ between vertex gH and vertex $gtH \mid g \in G$}.
- Thm 2.38: The Bass-Serre tree T for an HNN extension $G = H *_L$ is a tree.
- Thm 2.42: An HNN extension $G = H *_L$ acts (on the left) on the Bass-Serre tree T, by $g(g'H) := (gg')H$ for the vertices, and $g(g'j(L)) := (gg')j(L)$ on the edges. Moreover, this action is simplicial and without inversion.
- Thm 2.44: If $H *_L$ is an HNN extension of a finitely presented group H, then $H *_L$ is finitely presented iff L is finitely generated.

- Thm 2.50: Let $\mathcal{G} = (\Lambda, \{G_v\}, \{H_e\}, \{h_e,h_e'\})$ be a graph of groups with at least one edge, and let $G := \pi_1(\mathcal{G})$.
(a) If Λ is a tree, then there is a vertex u adjacent to only one edge f; let Λ' be the subgraph of Λ induced by the set of vertices $\mathrm{Vert}(\Lambda) - \{u\}$ (with the same vertex and edge groups). Then G splits as an amalgamated product $G \cong \pi_1(\Lambda') *_{H_f} G_u$.
(b) If Λ is not a tree, then there is an edge f satisfying the property that the graph Λ'' obtained from Λ by removing f is connected. Then G splits as an HNN extension $G \cong \pi_1(\Lambda) *_{H_f}$.
(c) G acts simplicially and without inversion on a tree.
- Def 2.52: Let $\mathcal{G} = (\Lambda, \{G_v\}, \{H_e\}, \{h_e,h_e'\})$ be a graph of groups and let $G := \pi_1(\mathcal{G})$. The **Bass-Serre tree** for the graph of groups $\mathcal{G}$ is the graph T with vertex set $V(T) := \{gG_v \mid g \in G, v \in \mathrm{Vert}(\Lambda)\}$, and edge set $E(T) :=$ {edge labeled $gh_e'(H_e)$ between vertex $gG_{i(e)}$ and vertex $gs_eG_{t(e)} \mid g \in G, e \in \mathrm{Edge}(\Lambda)$}.

- Thm 2.53: (**Bass-Serre Theorem, part I**) Let $\mathcal{G} = (\Lambda, \{G_v\}, \{H_e\}, \{h_e,h_e'\})$ be a graph of groups and let $G := \pi_1(\mathcal{G})$. Let T be the associated Bass-Serre tree.
  (a) The Bass-Serre tree T is a tree.
  (b) The group G acts (on the left) on T, by $g(g'G_v) := (gg')G_v$ for the vertices, and $g(g'h_e'(H_e) := (gg')h_e'(H_e)$ on the edges. Moreover, this action is without inversion.
  (c) The graphs T/G and Λ are isomorphic.
- Def 2.58: The action of a group G on a topological space X is **cocompact** if the quotient space X/G is compact.
- Prop 2.59: Let G be a group that acts on set T. (a) If $p \in T$ and $g \in G$, then $Stab_G(g \cdot p) = g\ Stab_G(p)\ g^{-1}$.
  (b) Suppose further that T is a tree and the action of G is simplicial and without inversion. If e is an edge of T, then $PtStab_G(e) = SetStab_G(e)$ is a subgroup of the groups $Stab_G(t(e))$ and $Stab_G(i(e))$. (This group is also written $\mathbf{Stab_G(e)}$.)
- Thm 2.60: (**Bass-Serre Theorem, part II**) Let G be a group that acts on a tree T simplicially, cocompactly, and without inversion. Let $q: T \to T/G$ be the quotient map. For each vertex v of Λ, let v' be a vertex of T such that $v = q(v')$, and let $G_v := Stab_G(v')$. For each edge e of Λ, let e' be an edge of T such that $e = q(e')$, and let $H_e := Stab_G(e')$. Define $h_e: H_e \to G_{t(e)}$ to be the composition of the inclusion $H_e \to Stab_G(t(e'))$ with the conjugation homomorphism $Stab_G(t(e')) \to G_{t(e)}$. Similarly define $h_e': H_e \to G_{i(e)}$ to be the composition of the inclusion $H_e \to Stab_G(i(e'))$ with the conjugation homomorphism $Stab_G(i(e')) \to G_{i(e)}$. Then $\mathcal{G} := (\Lambda, \{G_v\}, \{H_e\}, \{h_e,h_e'\})$ is a graph of groups, and $G \cong \pi_1(\mathcal{G})$.
- Thm 2.61: (**Bass-Serre Theorem, part III**) Let $R := \{(G,T) \mid T$ is a tree and G is a group acting simplicially, cocompactly, and without inversion on T$\}$. Let $S := \{(\mathcal{G},G) \mid \mathcal{G}$ is a graph of groups and $G = \pi_1(\mathcal{G})\}$. Let $f: R \to S$ map the pair (G,T) to the graph of groups defined in Thm 2.60 (with graph T/G and vertex,edge groups obtained as stabilizers), and let $g: S \to R$ map the pair $(\mathcal{G},G)$ to the pair (G,T) where T is the Bass-Serre tree of the graph of groups. Then f and g are inverse functions.

- *Section B: Groups of piecewise linear homeomorphisms*

  - (i)*$PL_+(I)$*:
    - Def 2.100: $\mathbf{PL_+(I)}$ is the group of orientation-preserving piecewise linear homemorphisms of I = [0,1], with the operation of composition.
    - Lemma 2.101: $PL_+(I)$ is (uncountable and hence is) not finitely generated.
    - Def 2.102: For each $g \in PL_+(I)$, the **breakpoints** of g are the elements $t \in I$ such that the slopes of g to the left and right of t are not equal; that is, $g'_-(t) \neq g'_+(t)$.
    - Lemma 2.103: For each $g \in PL_+(I)$, the number of breakpoints of g is finite.
    - Def 2.110: For each $g \in PL_+(I)$, the **support** of g is the set $\mathbf{Supp(g)} := \{t \in I \mid g(t) \neq t\}$.

- Lemma 2.111: Let $f,g \in PL_+(I)$.
  (a) If $Supp(f) \cap Supp(g) = \emptyset$, then $fg = gf$.
  (b) $Supp(gfg^{-1}) = g(Supp(f))$.
  (c) For each $g \in PL_+(I)$, the support of g is a disjoint union of finitely many open intervals in I.
- Def 2.112: Let $g \in PL_+(I)$, and let $A_1,...,A_k$ be the disjoint open intervals whose union is Supp(g). For each $1 \le i \le k$, define $g_i:I \to I$ by $g_i(t) := g(t)$ for all $t \in A_i$ and $g_i(t) := t$ for all $t \in I - A_i$. The functions g_i are the **one-bump factors** of g.

◦ (ii) *Thompson's group F*:

- Def 2.120: **Thompson's group F** is the subgroup of $PL_+(I)$ of all $g \in PL_+(I)$ satisfying the property that all of the slopes of the linear pieces of g lie in $\{2^k \mid k \in \mathbb{Z}\}$ and all of the breakpoints of g lie in $\mathbb{Z}[1/2]$.
- Thm 2.121: Thompson's group F is presented by
  $F = \langle x,y \mid [y,xyx^{-2}], [y,x^2yx^{-3}] \rangle$
  where x is the PL homeomorphism of I defined by $x(t) = 2t$ for all $t \in [0,1/4]$, $x(t) = t + 1/4$ for all $t \in [1/4,1/2]$, and $x(t) = (1/2)t + 1/2$ for all $t \in [1/2,1]$, and y is the PL homeomorphism of I defined by $y(t) = t$ for all $t \in [0,1/2]$, $y(t) = 2t - 1/2$ for all $t \in [1/2,5/8]$, $y(t) = t + 1/8$ for all $t \in [5/8,3/4]$, and $y(t) = (1/2)t + 1/2$ for all $t \in [3/4,1]$.
- Thm 2.122: (Guba, Sapir) Let x,y be the generators of Thompson's group F from Thm 2.121 and let $A = \{x,x^{-1},y,y^{-1}\}$. The rewriting system defined by
  $R := \{aa^{-1} \to 1 \mid a \in A\} \cup \{y^e x^i y \to x^i y x^{-(i+1)} y^e x^{i+1} \mid i \ge 1, e = \pm 1\} \cup \{y^e x^j y^{-1} \to x^{j+1} y^{-1} x^{-j} y^e x^j \mid j \ge 2, e = \pm 1\}$
  is a convergent rewriting system for F over A. Consequently the set N of reduced words over A that also do not contain any subwords of the form $y^{\pm 1} xx^* y$ or $y^{\pm 1} x^2 x^* y^{-1}$ is a set of normal forms for F over A.

◦ (iii) *Van Kampen diagrams and the seashell method for finitely presented groups*:

- Def 2.125: Let G be a group with a finite presentation $\mathcal{P} = \langle A \mid R \rangle$ and let N be a set of normal forms for G over $A^{\pm 1}$ satisfying the property that $1 \in N$ and whenever s is a proper subword of a word in N then $s \neq_G 1$. Given any word $u \in N$ and letter $b \in A$, if v is the normal form in N for the group element represented by ua, then a van Kampen diagram for the word uav^{-1} is called an **icicle**.
  For any word $w = a_1 \cdots a_k$ with each $a_i \in A^{\pm 1}$ and $w =_G 1$, a **seashell diagram** Δ for w is constructed from icicles as follows: Let $v_0 := 1$, and for all $1 \le i \le k$, let $v_i :=$ the normal form of $a_1 \cdots a_i$ and let Δ_i be an icicle (i.e., van Kampen diagram) for the word $v_{i-1}a_iv_i^{-1}$. Then Δ is the CW-complex obtained from the disjoint union of the Δ_i complexes by gluing Δ_i and Δ_{i+1} along the common boundary word a_i for all $1 \le i \le k-1$.
- Thm 2.126: Let G be a group with a finite presentation $\mathcal{P} = \langle A \mid R \rangle$, let N be a set of normal forms for G over $A^{\pm 1}$ satisfying the property that $1 \in N$ and whenever s is a proper subword of a word in N then $s \neq_G 1$, and let w be a word over $A^{\pm 1}$ such that $w =_G 1$. Then

any seashell diagram for w constructed from icicles (with respect to $\mathcal{P}$ and N) is a van Kampen diagram for w.

- (iv) *Van Kampen diagrams for Thompson's group F*:
  - Prop 2.130: Let x,y be the generators of Thompson's group F from Thm 2.121, and for each natural number i, let $r_i := [y,x^iyx^{-(i+1)}]$. Then $r_i =_F 1$. Moreover, for all i ≥ 3, there is a van Kampen diagram for the word r_i over the presentation F = < x,y | r_1, r_2> that is built inductively from two 2-cells labeled r_1, two copies of the van Kampen diagram for r_{i-2}, and one copy of the van Kampen diagram for r_{i-1}.
  - Thm 2.133: Let $\mathcal{P}$ be the presentation of Thompson's group F from Thm 2.121, let A = $\{x^{\pm 1}, y^{\pm 1}\}$, and let N be the set of normal forms for F over A from Thm 2.122. Given any word u ∈ N and letter b ∈ A, if v is the normal form in N for the group element represented by ua, then there is a van Kampen diagram (an icicle) for the word uav^{-1} consisting of a line segment and a finite stack of van Kampen diagrams for words of the form r_i from Prop 2.130.
- (v) *Trees and Thompson's group F*:
  - Def 2.140: A **rooted binary tree** T is a tree with a distinguished vertex r (the **root**) satisfying the property that for every vertex v of T other than r, the number of edges adjacent to T is either 1 or 3.
    In the case that the number of edges adjacent to v is 1, the vertex v is a **leaf** of T.
    A **caret** of T consists of a non-leaf vertex v of T together with the two edges adjacent to v that do not lie on a geodesic (i.e. shortest length) path from v to the root r.
  - Def 2.142: A **tree pair diagram** for Thompson's group F is an ordered pair, written (T → U), of two rooted binary trees T and U such that T and U have the same number of leaves. A tree pair diagram (T → U) is **reduced** if, when the leaves of the trees T and U are numbered (starting with 1 for each tree) from left to right, there is no pair of carets from T,U labeled by the same two leaf numbers.
  - Thm 2.143: (Cannon, Floyd, Parry) Thompson's group F is isomorphic to the group $\mathscr{F}$ of reduced tree pair diagrams, in which multiplication (T → U) · (V → W) is accomplished by adding carets in corresponding positions in T,U (resulting in a tree pair diagram (T' → U')) and adding carets in corresponding positions V,W (to get (V' → W')) so that W' = T'; then (T → U) · (V → W) is defined to be the reduced tree pair diagram obtained from (V' → U') by removing corresponding carets until the diagram is reduced.
  - Lemma 2.144: For each f ∈ $\mathscr{F}$ (the group in Thm 3.130), if f = (T → U), then f^{-1} = (U → T).
  - Thm 2.145: The word problem is solvable for Thompson's group F.
  - Def 2.147: For any integer n ≥ 2, the **Higman-Thompson group F_n** is the subgroup of $PL_+(I)$ of all g ∈ $PL_+(I)$ satisfying the property that all of the slopes of the linear pieces of g lie in $\{n^k \mid k \in \mathbb{Z}\}$ and all of the breakpoints of g lie in $\mathbb{Z}[1/n]$.
  - Prop 2.148: The Higman-Thompson group F_3 is isomorphic to a subgroup of Thompson's group F.
- (vi) *Algebraic properties of $PL_+(I)$ and Thompson's group F*:

- Thm 2.150: $PL_+(I)$ is torsion-free.
- Thm 2.154: For Thompson's group F:
  (a) The abelianization of F is $F_{ab} = F/[F,F] \cong \mathbb{Z}^2$, generated by the images of the generators x,y of F.
  (b) The commutator subgroup [F,F] is simple.
  (c) Every nontrivial normal subgroup of F contains [F,F].
  (d) (Brin, Squier) F does not contain a subgroup isomorphic to the free group of rank 2.

- (vii) *Solvable groups and wreath products of groups*:
  - Def 2.160: Let A and H be groups. The **restricted wreath product** of A by H is the semidirect product $\mathbf{A} \wr \mathbf{H} := (\bigoplus_{h \in H} A) \rtimes H$ where for each $h,k \in H$ and for each a in the copy of A in the h position, kak^{-1} := a in the copy of A in the kh position.
  - Def 2.161: The **lamplighter group** is the wreath product group $(\mathbb{Z}/2) \wr \mathbb{Z}$.
  - Prop 2.162: The lamplighter group G is finitely generated, with the presentation
    $G = \langle a,t \mid a^2 = 1, \{[t^i a t^{-i}, t^j a t^{-j}] \mid i,j \in \mathbb{Z}\}\rangle$.
  - Def 2.165: A group G is **solvable** if there is a finite sequence $1 = G_0 \rhd G_1 \rhd \cdots \rhd G_n = G$ such that each G_{i+1}/G_i is abelian.
    The minimum possible number n for such a sequence is the **derived length** of G, denoted **DerLen(G)**.
  - Thm 2.166: (a) If H is a subgroup of a solvable group G, then H is solvable and DerLen(H) ≤ DerLen(G).
    (b) If Q is a quotient of a solvable group G, then Q is solvable and DerLen(Q) ≤ DerLen(G).
    (c) If N is a normal subgroup of a group G and both N and G/N are solvable, then G is solvable, and DerLen(G) ≤ DerLen(N) + DerLen(G/N).
  - Def 2.167: Let G be a group with a finite inverse-closed generating set A. The **Solvable Subgroup Recognition Problem (SSRP)** for G over A is decidable if there is an algorithm that upon input of a finite list of words $w_1,\dots,w_k$ over A can determine whether the subgroup $\langle w_1,\dots,w_k\rangle$ of G generated by those words is a solvable group.

- (viii) *Solvable and computable subgroups of $PL_+(I)$ and the Subgroup Membership Problem*:
  - *Iterated wreath products of $\mathbb{Z}$*:
    - Def 2.170: Let $W_1 := \mathbb{Z}$, and for all n ≥ 2, let $W_n := W_{n-1} \wr \mathbb{Z}$.
    - Prop 2.171: For each n, the group W_n is solvable with derived length n.
    - Thm 2.172: For all n ≥ 1, Thompson's group F contains a subgroup isomorphic to W_n. Consequently, F is not solvable.
    - Thm 2.173: (Bleak) Let $W := \bigoplus_{n \in \mathbb{N}} W_n$. Let H be any subgroup of $PL_+(I)$.
      (a) If H is solvable, then H is isomorphic to a subgroup of W.
      (b) If H is not solvable, then H contains a subgroup isomorphic to W.
  - *Solvability of groups generated by two 1-bump functions*:
    - Thm 2.175: Let $f,g \in PL_+(I)$ be 1-bump functions with Supp(f) = (a,b) and Supp(g) = (c,d), and let $H = \langle f,g\rangle$.

(a) If Supp(f) ∩ Supp(g) = ∅, then H is abelian (and hence H is solvable).
(b) If Supp(f) ⊊ Supp(g) and either a = c or b = d, then for all n ≥ 1 the group W_n is isomorphic to a subgroup of H, and hence H is not solvable.
(c) If Supp(f) ∩ Supp(g) ≠ ∅ but Supp(f) ⊄ Supp(g) and Supp(g) ⊄ Supp(f) [that is, either a < c < b < d or c < a < d < b], then for all n ≥ 1 the group W_n is isomorphic to a subgroup of H, and hence H is not solvable.
(d) If Supp(f) ⊊ Supp(g) and Supp(f) ∩ Supp(gfg^{-1}) = ∅, then H ≅ $W_2 = \mathbb{Z} \wr \mathbb{Z}$, and hence H is solvable.
(d') If Supp(f) ⊊ Supp(g) and Supp(f) ∩ Supp(gfg^{-1}) ≠ ∅, then H is not solvable.
(e) If Supp(f) = Supp(g), then further subcases are needed to determine whether H is solvable or not solvable.

- *The split group and bounding the derived length of solvable subgroups*:
  - Def 2.180: Let H be a subgroup of $PL_+(I)$. The **split group of H** is the group **Spl(H)** generated by the set of 1-bump factors of the elements of H.
  - Thm 2.181: (Golan) Let H be a subgroup of $PL_+(I)$. Then Spl(Spl(H)) = Spl(H).
  - Thm 2.182: (Bleak) Let H be a subgroup of $PL_+(I)$. Then H is solvable if and only if Spl(H) is solvable. Moreover, when these groups are solvable, then DerLen(H) = DerLen(Spl(H)).
  - Thm 2.185: Let G be a subgroup of $PL_+(I)$ with finite generating set A. If G is solvable, then DerLen(G) is at most the number of breakpoints of the elements of A.
- *Computable subgroups and the SSRP*:
  - Meta-def 2.188: Let C be a subgroup of $PL_+(I)$. Then C is **computable** if a computer can: (a) Multiply and invert elements in Spl(C). (b) Determine breakpoints and endpoints of components of support of elements of Spl(C), and compute the action of Spl(C) on these points. (c) Compute slopes at support endpoints and compute the 1-bump factors of elements of Spl(C). (d) Multiply and invert elements of the group of slopes of a finite set of 1-bump functions in Spl(C) at a common endpoint, determine whether this is a discrete subgroup of $\mathbb{R}_{>0}$, and if so, compute the least slope greater than 1 in this group.
  - Prop 2.189: Thompson's group F and the Higman-Thompson groups F_n are computable subgroups of $PL_+(I)$.
  - Thm 2.190 (Bleak, Brough, Hermiller) The Solvable Subgroup Recognition Problem is decidable for all finitely generated computable subgroups of $PL_+(I)$. Moreover, if the input group is solvable, the algorithm can also determine the derived length.
- *The Subgroup Membership Problem*:
  - Def 2.193: A finite set A of 1-bump functions in $PL_+(I)$ is **in general position** if for all f,g,h ∈ A: (a) If f ≠ g, then Supp(f) ≠ Supp(g). (b) If Supp(f) ∩ Supp(g) ≠ ∅ and Supp(f) ⊄ Supp(g), then Supp(g) ⊊ Supp(f), and Supp(f) and Supp(g) do not share a common endpoint. (c) If Supp(f) ⊂ Supp(h) and Supp(g) ⊂ Supp(h), then for all nonzero integers i, Supp(h^ifh^{-i}) ∩ Supp(f) = ∅ and Supp(h^ifh^{-i}) ∩ Supp(g) = ∅.

- Thm 2.194: Let G be a finitely generated subgroup of $PL_+(I)$. Then G is split and solvable if and only if G has a finite generating set of 1-bump functions in general position.
- Thm 2.195: (B,B,H) Let G be a finitely generated computable subgroup of $PL_+(I)$, and let H be a finitely generated, split, solvable subgroup of G. Then the Subgroup Membership Problem for H in G is decidable.
- Thm 2.196: If H is a finitely generated computable subgroup of $PL_+(I)$ that is split and solvable, then there is an algorithm that upon input of a finite generating set A for H, outputs a finite generating set B for H of 1-bump functions in general position.
- Rmk 2.197: In general for finitely generated subgroups H of Thompson's group F, decidability of the SMP for H in F is an open question.

- (ix) *PL($\mathbb{R}$)*:
  - Def 2.200: **PL($\mathbb{R}$)** is the group of piecewise linear homemorphisms of $\mathbb{R}$, with the operation of composition. **$PL_+(\mathbb{R})$** is the subgroup of orientation-preserving PL homemorphisms of $\mathbb{R}$.
  - Def 2.202: A homeomorphism f: $\mathbb{R} \to \mathbb{R}$ is **eventually a translation** if there is some real number r > 0 and M,N ∈ $\mathbb{R}$ such that whenever t > r then f(t) = t + M and whenever t < -r then f(t) = t + N.
  - Def 2.203: Let **$PL_2(\mathbb{R})$** denote the group of all g ∈ PL($\mathbb{R}$) satisfying the properties that all of the slopes of the linear pieces of g lie in $\{2^k \mid k \in \mathbb{Z}\}$, all of the breakpoints of g lie in $\mathbb{Z}[1/2]$, and the set of breakpoints is discrete.
    Let **$PL_{2,ev}(\mathbb{R})$** be the subgroup of $PL_2(\mathbb{R})$ consisting of the elements of $PL_2(\mathbb{R})$ that are eventually a translation.
  - Lemma 2.204: If $g \in PL_{2,ev}(\mathbb{R})$, then g has finitely many breakpoints.
  - Lemma 2.205: (a) Define f: $\mathbb{R} \to (0,1)$ by: For every integer j ≥ 0 and t ∈ [j,j+1], let f(t) := $2^{-j-2}(t-j-2) + 1$, and for every integer j ≤ -1 and t ∈ [j,j+1], let f(t) := $2^{j-1}(t-j+1)$. Then f is a piecewise linear homeomorphism.
    (b) Define $\tilde{f}$: $(0,1) \to \mathbb{R}$ by: For every integer j ≥ 0 and $\tilde{t} \in [1-2^{-j-1},1-2^{-j-2}]$, let $\tilde{f}(\tilde{t}) := 2^{j+2}(\tilde{t}-1) + j + 2$, and for every integer j ≤ -1 and $\tilde{t} \in [2^{j-1},2^j]$, let $\tilde{f}(\tilde{t}) := 2^{1-j}\tilde{t} + j - 1$. Then $\tilde{f}$ is a piecewise linear homeomorphism, and $\tilde{f} = f^{-1}$.
  - Thm 2.206: Let F be Thompson's group F. Then $F \cong PL_{2,ev}(\mathbb{R})$.
    Moreover, if f: $\mathbb{R} \to (0,1)$ is the homeomorphism defined in Lemma 2.205, and h: F → $PL_{2,ev}(\mathbb{R})$ is defined by h(g)(t) := $f^{-1} \circ g \circ f$ for all g ∈ F and t ∈ $\mathbb{R}$, then h is well-defined and an isomorphism.
  - *Normalizers, automorphism groups, and centers*:
    - Def 2.210: Let G be a group and let H be a subgroup of G. The **normalizer of H in G** is the subgroup of G given by **$N_G(H)$** := $\{g \in G \mid gHg^{-1} = H\}$.
    - Def 2.211: Let G be a group. An **automorphism** of G is an isomorphism :G → G.
      The **automorphism group** of G is the group **Aut(G)** of all automorphisms of G, with group operation given by composition. For each g ∈ G, the **inner automorphism induced by g** is the automorphism φ_g:G → G defined by $\varphi_g(h) := ghg^{-1}$ for all g ∈ G.

The **inner automorphism group** of G is the subgroup **Inn(G)** of Aut(G) consisting of the inner automorphisms.

- Def 2.212: Let G be a group. The **center** of the group G is the subgroup **Z(G)** := {g ∈ G | gh=hg for all h ∈ G}.
- Prop 2.213: If G is a group, then:
  (a) Z(G) is a normal subgroup of G.
  (b) For any element g ∈ G, $\varphi_g = Id_G$ if and only if g ∈ Z(G).
  (c) Inn(G) ≅ G/Z(G).

- Thm 2.216: (Brin) Let F be Thompson's group F.
  (a) Aut(F) ≅ $N_{PL_2(\mathbb{R})}(PL_{2,ev}(\mathbb{R}))$.
  (b) Aut([F,F]) ≅ $PL_2(\mathbb{R})$.

- *Section C: Self-similar groups and more computational questions*

  - (i) *Finite state automata and finite state transducers*:
    - Def 2.250: A **finite state automaton (FSA)** is a tuple M = (A,Q,q_0,P,δ) where: A is a finite set (the **alphabet**), Q is a finite set (the **set of states**), q_0 ∈ Q (the **initial state**), P ⊆ Q (the **set of accept states**), and δ: Q × A → Q (the **transition function**).
    - Def 2.251: Let M = (A,Q,q_0,P,δ) be an FSA. The **FSA graph** for M is a directed graph Γ(M) with vertex set V(Γ(M)) := Q, and directed edge set
      E(Γ(M)) := {edge from q to q' labeled by a | q ∈ Q, a ∈ A, and δ(q,a) = q'}.
    - Def 2.253: Let A be a set. A **language over A** is a subset of A^*.
    - Def 2.254: Given an FSA M = (A,Q,q_0,P,δ), the **language L(M) accepted by M** is the set of all words over A that can be read in the FSA graph Γ(M) along a path starting at q_0 and ending at a state in P.
    - Def 2.255: Let A be a set and let K,L ⊆ A^*. The **concatenation K · L** is the language K · L := {uv | u ∈ K, v ∈ L}.
      The **Kleene star of L** is the language $\mathbf{L^*}$:= {1} ∪ $(\cup_{j \geq 1} L^j)$, where L^1 := L and for all j ≥ 2, L^j := L^{j-1} · L.
    - Def 2.256: A language L over a finite set A is called **regular** if L is an element of $\mathscr{R}$, where $\mathscr{R}$ is the smallest subset of the set $\mathcal{P}(A^*)$ of languages over A containing the set of all finite languages and closed under the operations ∪, ∩ ·, *, and A^* - ( ).
    - Thm 2.257: Let A be a finite set. A language L ⊆ A^* is regular if and only if L is the language accepted by an FSA.
    - Def 2.260: A **finite state transducer (FST)** is a tuple M = $(A,Q,\delta,\varepsilon,q_0,)$ where: A is a finite set (the **alphabet**), Q is a finite set (the **set of states**), q_0 ∈ Q (the **initial state**), δ: Q × A → Q (the **transition function**), and ε: Q × A → A (the **output function**).

- Def 2.261: Let M = $(A,Q,\delta,\varepsilon,q_0)$ be an FST. The **FST graph** for M is a directed graph $\Gamma(M)$ with vertex set $V(\Gamma(M)) := Q$, and directed edge set
  $E(\Gamma(M)) :=$ {edge from q to q' labeled by "a|b" | $q \in Q$, $a \in A$, $\delta(q,a) = q'$, and $\varepsilon(q,a) = b$}.
- Def 2.264: Let M = $(A,Q,\delta,\varepsilon,q_0)$ be an FST and let $w \in A^*$. The **output word** of M when w is input is computed by: Let p be the directed edge path starting at q_0 whose edge labels' left entries are labeled by the word w; then the output word is the word labeling the right entries of the edges along p.
- Rmk 2.266: We will often consider FST's without an initial state, in which we will consider using each state in turn as an initial state.
- Def 2.267: The **binary adding machine** is the FST M = $(\{0,1\},\{s,t\},\delta,\varepsilon,s)$ where $\delta(s,0) = t$, $\delta(s,1) = s$, $\varepsilon(s,0) = 1$, $\varepsilon(s,1) = 0$, and $\delta(t,0) = t$, $\delta(t,1) = t$, $\varepsilon(t,0) = 0$, $\varepsilon(t,1) = 1$.

- (ii) *Automorphism groups of regular rooted trees*:
  - Def 2.270: Let A be a finite set. The **regular rooted tree T_A** induced by A is a tree with vertex set $V(T_A) := A^*$, directed edge set
    $E(T_A) :=$ {edge from w to wa labeled a | $w \in A^*$ and $a \in A$},
    and root λ, where **λ** := the empty word.
  - Def 2.271: Let A be a finite set. An **automorphism of T_A** is a graph isomorphism $:T_A \to T_A$ mapping the root to the root.
    The **automorphism group** of T_A is the group **Aut(T_A)** of automorphisms of T_A with the composition operation.
  - Lemma 2.272: (1) If $g \in \mathrm{Aut}(T_A)$, then the restriction of g to the vertices of T_A is a bijection $g:A^* \to A^*$.
    (2) Suppose that $h:A^* \to A^*$ is a bijection satisfying $h(\lambda) = \lambda$ and whenever $w \in A^*$ and $a \in A$ then $h(wa) = h(w)b$ for some $b \in A$ (that is, for each $w \in A^*$ there is a permutation $\pi_{h,w}:A \to A$ such that $h(wa) = h(w)\pi_{h,w}(a)$ for all $a \in A$). Then h induces an automorphism $:T_A \to T_A$ whose restriction to the vertices A^* of T_A is h.
  - Def 2.74: Let A be a finite set, let $g \in \mathrm{Aut}(T_A)$, and let $v \in A^*$. The **restriction of g to v** is the automorphism $\mathbf{g|_v} \in \mathrm{Aut}(T_A)$ defined by $g(vw) = g(v)\, g|_v(w)$ for all $w \in A^*$; that is, $g|_v(w)$:= the suffix of g(vw) after g(v).
  - Prop 2.75: Let $g,h \in \mathrm{Aut}(T_A)$ and $v,w \in A^*$.
    (a) $g|_{vw} = (g|_v)|_w$.
    (b) $(gh)|_v = g|_{h(v)}\, h|_v$.
  - Def 2.77: Let $g \in \mathrm{Aut}(T_A)$ and $v \in A^*$. The **permutation of A induced by g at v** is the permutation $\boldsymbol{\pi_{g,v}}:A \to A$ defined by $g(va) = g(v)\, \pi_{g,v}(a)$ for all $a \in A$ (that is, $\pi_{g,v}$ is the restriction of the automorphism $g|_v$ to A).
  - Def 2.78: Let $g \in \mathrm{Aut}(T_A)$. The **portrait of g** is the tree T_A with the permutation $\pi_{g,v}$ labeled at each vertex v.

- (iii) *Automaton groups*:
  - Def 2.275: An FST M = (A,Q,δ,ε) is **invertible** if for all $q \in Q$, the function $\varepsilon_q:A \to A$ defined by $\varepsilon_q(a) := \varepsilon(q,a)$ is a bijection (that is, ε_q is a permutation of A).

- Thm 2.277: Let M = (A,Q,δ,ε) be an invertible FST. If q ∈ Q, then q induces a bijection q':$A^* \to A^*$ by q(w) := output word of M when q is the start state and the word w is input. The state q also induces an automorphism of T_A whose restriction to the vertices of T_A is q'.
- Def 2.280: A group G is an **automaton group** if there is an invertible FST M = (A,Q,δ,ε) such that G is the subgroup of Aut(T_A) generated by Q.
- Prop 2.282: Let M be the binary adding machine FST. (a) The automorphism of T_A induced by t is the identity. (b) The automaton group of M is the infinite cyclic group $\mathbb{Z}$.
- More examples

- (iv) *The Word Problem for automaton groups*:
  - Def 2.300: Let M = (A,Q,δ,ε) and M' = (A,Q',δ',ε') be FST's. The **product FST M · M'** is M · M' = (A,Q × Q',δ'', ε'') where ε''((g,g'),a) := &epsilon(g,ε'(g',a)) and δ''((g,g'),a) := (δ(g,ε'(g',a)),δ(g',a)) for all g ∈ Q, g' ∈ Q', and a ∈ A.
    (In shorthand: (gg')(a) := g(g'(a)), and $(gg')|_a = g|_{g'(a)}g'|_a$.)
  - Def 2.302: Let M = (A,Q,δ,ε) be an invertible FST. The **inverse FST M^{-1}** is M^{-1} = (A,Q^{-1},δ',ε') where ε'(g^{-1},a) := the letter y ∈ A such that ε(g,y) = a, and δ'(g^{-1},a) := $(\delta(g,\varepsilon'(g^{-1},a)))^{-1}$.
  - Examples
  - Thm 2.305: (a) If M = (A,Q,δ,ε) and M' = (A,Q',δ',ε') are FST's and g ∈ Q and g' ∈ Q', then the automorphism gg' of T_A is the state gg' of the product automaton M · M'.
    (b) If M = (A,Q,δ,ε) is an FST and g ∈ Q, then the automorphism g^{-1} of T_A is the state g^{-1} of the product automaton M · M'.
  - Cor 2.307: If G is an automaton group, then the Word Problem for G is decidable.
  - Rmk 2.308: Let G be an automaton group with FST M = (A,Q,δ,ε). The Word Problem solution for G is decided as follows: Upon input of a word w ∈ $(Q \cup Q^{-1})^*$, build the FST M_w containing w as a state using the constructions in definitions 2.300 and 2.302 (which can be done in finitely many steps). Then w $=_G$ 1 if and only if for every state g in M_w that can be reached from w (by a directed path), the output function ε_w of M_w at g induces the identity permuation on A.

- (v) *The Finiteness Problem and the Conjugacy Problem for automaton groups*:
  - *The Grigorchuk group*
    - Def 2.310: The **Grigorchuk group** is the automaton group of the FST M = ({0,1}, {**I**,a,b,c,d},δ,ε) where δ(a,0) = **I**, δ(a,1) = **I**, δ(b,0) = a, δ(b,1) = c, δ(c,0) = a, δ(c,1) = d, δ(d,0) = **I**, δ(d,1) = b, δ(**I**,0) = **I**, δ(**I**,1) = **I**, ε(a,0) = 1, ε(a,1) = 0, ε(b,0) = 0, ε(b,1) = 1, ε(c,0) = 0, ε(c,1) = 1, ε(d,0) = 0, ε(d,1) = 1, ε(**I**,0) = 0, ε(**I**,1) = 1.
    - Thm 2.311: Let G be the Grigorchuk group. Then:
      (a) G is finitely generated by {a,b,c,d} and each of these elements has order 2.
      (b) G is not finitely presented.
      (c) G is infinite.
      (d) Every element of G has finite order. Moreover, for each g ∈ G, the order of G is a power of 2.

(e) G is a subgroup of a finitely presented group H generated by two elements of infinite order.
(f) G is **just infinite**: Every proper quotient of G is finite.
(g) G has decidable Conjugacy Problem.

- *The Conjugacy Problem and automaton groups*
  - Def 2.320: Let G be a group with a finite inverse-closed generating set A. The **Conjugacy Problem (CP)** for (G,A) asks if there exists an algorithm that, upon input of any pair of words $v,w \in A^*$ can determine whether there exists a $g \in G$ such that $w =_G gvg^{-1}$.
  - Examples of (automaton) groups with decidable CP
  - Thm 2.322: (Sunic, Ventura) There exists an automaton group with undecidable Conjugacy Problem.
  - Prop 2.335: Let **FAut(T_A)** be the subset of Aut(T_A) that is the union of all of the automaton groups. Then FAut(T_A) is a countable subgroup of Aut(T_A).
  - Rmk 2.337: Decidability of the Conjugacy Problem is an open question in FAut(T_A). Elements of this group can be input using FST's, or using words over a 2-element generating set of a group H containing FAut(T_A) as a subgroup (from Ex2.A.7).
- *The Finiteness Problem, self-similar groups, and permutational wreath products*
  - Def 2.340: The **Finiteness Problem** for finitely presented groups asks if there exists an algorithm that, upon input of any finite presentation < A | R > can determine whether the group presented by < A | R > is finite.
  - Def 2.341: The **Finiteness Problem** for automaton groups asks if there exists an algorithm that, upon input of any FST M can determine whether the automaton group defined by M is finite.
  - Rmk 2.342: In infinite group theory, computation is often (usually) a matter of art; that is, collecting a lot of tools, techniques, etc. In general there is no algorithm to find WP, CP, FinP, etc. algorithms. Instead there are procedures to find those algorithms - procedures may stop an give the algorithm, or may run forever and never succeed.
  - *The Finiteness Problem and rewriting systems:*
    - Def 2.344: Let G be a group with a finite convergent rewriting system (CRS) R over an alphabet A. The associated **irreducible word automaton** is the FSA M(A,R) := (A,Q,1,Q-{F},δ) where the state set is Q := S ∪ {F} such that S := {proper prefixes of left hand sides of rules of R}, and the transition function $\delta:Q \times A \to Q$ is given by δ(w,a) := F if wa can be rewritten using R, and δ(w,a) := the maximal suffix of wa that is in S otherwise.
    - Thm 2.345: If G is a group with a finite convergent rewriting system (CRS) R over an alphabet A and M(A,R) is the associated irreducible word automaton, then the language of M(A,R) is the set of irreducible words (that is, normal forms) for the rewriting system R.

- Thm 2.346: The Finiteness Problem is decidable for groups with finite convergent rewriting systems. Moreover, given (input) a finite CRS (A,R) for a group G, the group G is finite if and only if there are no directed circuits in the FSA M(A,R).
- Examples of Finiteness Problem solutions for classes of groups
- Def 2.350: Let H be a group acting on a set X and let G be a group. The **permutational wreath product** of G by H is the semidirect product $\mathbf{G \wr^X H} := (\bigoplus_{p \in X} G) \rtimes H$ where for each $h \in H$ and for each g in the copy of G in the p position, $hgh^{-1} :=$ g in the copy of G in the h(p) position.
- Prop 2.351: Let G and H be groups, and let X be the set H. Let H act on X by $h \cdot x := hx$ for all $h \in H$ and $x \in X$ (where $h \cdot x$ denotes the action, and hx denotes multiplication in H). Then $G \wr^X H = G \wr H$.
- Thm 2.353: Let $A = \{a_1,...,a_n\}$ be a finite set. Let Perm(A) be the group of permutations of A, acting on the set A by permutation. Define the function $\psi:\mathrm{Aut}(T_A) \to \mathrm{Aut}(T_A) \wr^A \mathrm{Perm}(A)$ by $\psi(g) := (g|_{a_1},...,g|_{a_n})\, \pi_{g,\lambda}$ for each $g \in \mathrm{Aut}(T_A)$. Then ψ is an isomorphism. The function ψ is called the **wreath recursion** map.
- Def 2.355: A group G is **self-similar** if G is a subgroup of $\mathrm{Aut}(T_A)$ for some finite set A and satisfies the property that for all $g \in G$ and $v \in A^*$, the automorphism $g|_v$ is also in G.
- Prop 2.356: If G is an automaton group, then G is a self-similar group.
- Cor 2.357: Let $G < \mathrm{Aut}(T_A)$ be a self-similar group. Then restriction of the wreath recursion map gives a well-defined monomorphism $\psi:G \to G \wr^A \mathrm{Perm}(A)$.
- Def 2.360: Let A be a finite set and let $n \in \mathbb{N}_0$. The **n-th level** of the regular rooted tree T_A is the set A^n of vertices labeled by words of length n over A.
- Def 2.361: Let G be a subgroup of $\mathrm{Aut}(T_A)$ and let $n \in \mathbb{N}_0$. The **stabilizer of level n in G** is the subgroup of G given by
  $\mathbf{Stab_G(n)} := \mathrm{PtStab}_G(A^n)$.
- Thm 2.362: Let G be the Grigorchuk group and let $f: \mathrm{Stab}_G(1) \to G$ be defined by $f(h) := h|_0$ for all $h \in \mathrm{Stab}_G(1)$. Then f is a surjective homomorphism. Moreover, $\mathrm{Stab}_G(1) \lneq G$, and hence G is an infinite group.