Math 911 Fall 2023 Table Of Contents

Chapter 0: Overview of the course

- (1) *Introduction, and combinatorial and geometric group theory*
  - *Basic constructions of new groups, homomorphisms and group properties from old ones:*
    - Idea: Decompose groups into simpler groups
    - The four standard constructions of groups and homomorphisms: Extensions, graph products, amalgamated products, HNN extensions
    - The four standard constructions of properties: Virtually, poly, residually, locally
  - *Basic questions for finitely generated groups:*
    - Idea: There are many fundamental computational questions for finitely generated groups. They are not decidable in general, but can be solved when you restrict to nice groups.
    - Def 0.1: Let G be a group with finite generating set A. The **Word Problem (WP)** for (G,A) asks if there exists an algorithm that, upon input of any word $w \in (A \cup A^{-1})^*$, can determine whether $w =_G 1$.
    - Rmk: The WP does not ask if there is an algorithm to find the algorithm!
    - Thm 0.2: (Boone; Novikov; 1955) There is a finitely presented group G with no WP solution.
    - Def 0.4: The **Isomorphism Problem (IP)** asks if there exists an algorithm that, upon input of any two finite presentations $< A \mid R >$ and $< B \mid S >$, can determine whether $< A \mid R > \cong < B \mid S >$.
  - Idea: View a group as a graph or as a metric space, or study the group using its actions on graphs and metric spaces
- (2) *Algebraic group theory*
  - Idea: Measure how close a group is to being abelian
  - Poly properties
  - {Abelian groups} $\subsetneq$ {nilpotent groups} $\subseteq$ {soluble groups}
  - {F.g. abelian groups} $\subsetneq$ {f.g. nilpotent groups} $\subsetneq$ {polycyclic groups} $\subsetneq$ {f.g. soluble groups}
- (3) *Topological and algebraic group theory: Homology of groups*
  - Idea: Measure how close a group is to being finite
  - Topological view: Build a space from a group, then compute its homology
  - Algebraic view: Build a chain complex of modules and module homomorphisms, then compute its homology
  - Idea: Get much more information if both views are used together

Chapter 1: Fundamental concepts of infinite group theory

- *Section A: A short (and incomplete) review of topics in Math 817 and Math 872*

- (i) *Groups, Isomorphism Problem and invariants, and actions*
  - Motivation: Studying symmetries of objects
  - Def 1.1: A **binary operation** on a set G is a function G × G → G.
  - Def 1.2: A **monoid** is a set M with a binary operation (called **monoid multiplication**; the operation maps an ordered pair (a,b) to an element of M denoted ab) satisfying the following:
    (1) For all a,b,c in M, (ab)c = a(bc). (**associative**)
    (2) There is an element 1 such that a1 = a = 1a for all a in M. (**identity**)
  - Def 1.3: A **group** is a set G with a binary operation (called **group multiplication**; the operation maps an ordered pair (a,b) to an element of G denoted ab) satisfying the following:
    (1) For all a,b,c in G, (ab)c = a(bc). (**associative**)
    (2) There is an element 1 such that a1 = a = 1a for all a in G. (**identity**)
    (3) For each element a in G, there is an element b in G such that ab = 1 = ba. (**inverse**)
  - *Examples*
    - Def 1.5: Let X be a set. The **permutation group of X** is **Perm(X)** = {f:X → X | f is a bijection} with the group operation of function composition.
    - Def 1.6: Let (X,d) be a metric space. The **isometry group of X** is **Isom(X)** = {f:X → X | f is a bijection and d(p,q)=d(f(p),f(q)) for all p,q ∈ X} with the group operation of function composition.
  - Def 1.8: A **homomorphism** from a group G to a group H is a function f:G → H satisfying f(gg') = f(g)f(g') for all g,g' ∈ G.
  - Def 1.9: An **isomorphism** from G to H is a homomorphism from G to H that is a bijection. Two groups G and H are **isomorphic**, written G ≅ H, if there is an isomorphism from G to H.
  - Def 1.10: The **Isomorphism Problem** asks whether there exists an algorithm that can determine, upon input of two groups G and H, whether or not G ≅ H. The **Classification Problem** asks whether there exists an algorithm that can enumerate (list) all of the groups up to isomorphism.
  - Rmk: There cannot be an IP or CP algorithm for *all* groups, but there are IP and CP algorithms for classes of "nice" groups.
  - *Isomorphism invariants*
    - Def 1.12: An **isomorphism invariant** is a property P of groups such that whenever G ≅ H and G has P then H has P.
    - Lemma 1.13: If P is an isomorphism invariant, G is a group that has P, and H is a group that does not have P, then G is not isomorphic to H.
    - Def 1.14: A group G is **abelian** (also called **commutative**) if for every a,b ∈ G, ab = ba.
    - Def 1.15: The **abelianization** of a group G is the quotient group $\mathbf{G_{ab}}$:= G/[G,G] where [G,G] is the **commutator subgroup [G,G]** := < {$aba^{-1}b^{-1}$ | a,b ∈ G} >. The element $aba^{-1}b^{-1}$ of G is denoted **[a,b]** and called the **commutator of a with b**.
    - Lemma 1.16: If G and H are groups and $G_{ab} \ncong H_{ab}$, then G ≇ H.
    - Def 1.18: A group G is a **finite** group if the set G is finite. The **order of a group** G (denoted **|G|**) is the number of elements in the set G.

        - Def 1.19: The **order of an element** g of G (denoted **|g|**) is the smallest positive integer n such that $g^n = 1$; if there is no such integer, then the order of g is infinite.
        - Def 1.20: A group G is **torsion-free** if all of the nonidentity elements of G have infinite order. A group G is **torsion** if all of the elements of G have finite order.
        - Thm 1.22: The following are isomorphism invariants: (a) "abelian". (b) The abelianization of the group, up to isomorphism. (c) The order of the group. (d) The set of orders of elements in the group. (e) "torsion-free". (f) "torsion".
    - *Group actions*
        - Def 1.25: A **group action** of a group G on a set X is a function G × X → X (written (g,x) → gx) satisfying:
          (1) g(g'x) = (gg')x for all g,g' ∈ G and x ∈ X, and
          (2) 1x = x for all x ∈ X.
        - Lemma 1.26: Let G be a group and let X be a set.
          (a) If G acts on the set X (with action denoted by ·), then the function f:G → Perm(X) defined by (f(g))(x) := g · x (for all g in G and x in X) is a well-defined group homomorphism.
          (b) If f:G → Perm(X) is a homomorphism, then the function :G × X → X defined by g · x := (f(g))(x) (for all g in G and x in X) is a group action.
        - Def 1.27: Let G be a group acting on a set X. The **equivalence relation on X induced by the action of G**, written $\sim_G$, is defined by p $\sim_G$ q if and only if there is a g ∈ G such that p = gq. The set of equivalence classes $X/\sim_G$ is written **X/G**.
        - Def 1.28: Let G be a group acting on a set X and let p ∈ X. The **orbit** of p is the equivalence class of p; that is, **$\mathrm{Orbit}_G(p)$** := [p] = {gp | g ∈ G}. The **stabilizer of p** is **$\mathrm{Stab}_G(p)$** := {g ∈ G | gp = p}.
  - (ii) *Presentations*
    - Def 1.50: Let B be a set. The **free monoid on B**, denoted **B^***, is the set of all (finite) strings written in the alphabet B, including the empty word, denoted **1**. An element of B^* is called a **word over B**.
    - Def 1.51: Let A be a set, let $A^{-1} := \{a^{-1} \mid a \in A\}$ be a set that bijects to A, and let ~ be the smallest equivalence relation on $(A \cup A^{-1})^*$ such that $xaa^{-1}y \sim xy \sim xa^{-1}ay$ for all a ∈ A and x,y ∈ $(A \cup A^{-1})^*$. The **free group on A**, denoted **F(A)**, is the quotient set $(A \cup A^{-1})^*/\sim$ with the group operation [v][w] := [vw] where vw is the concatenation of the words v and w. In the case that |A| = n, this group is also denoted **F_n** and called the **free group of rank n**.
    - Def 1.52: A **reduced word** over a set A is a word w ∈ $(A \cup A^{-1})^*$ that does not contain a subword of the form aa^{-1} or $a^{-1}a$ for any a ∈ A.
    - Lemma 1.53: The function f: {reduced words over A} → F(A) defined by f(w) := [w] is a bijection.
    - Prop 1.58: Let G = < A > be generated by A.
      (a) Let S be a subset of G, and let N = < S > be the subgroup of G generated by S. The subgroup N is normal in G if and only if for all b ∈ $A \cup A^{-1}$ and s ∈ S, bsb^{-1} ∈ N.
      (b) Let T be a subset of $(A \cup A^{-1})^*$, and let N be the subgroup of G generated by the elements of G represented by the words in T. The subgroup N is normal in G if and only if for all b ∈ $A \cup A^{-1}$ and t ∈ T, the word btb^{-1} represents an element of N.

- Def 1.60: Let A be a set and let R be a subset of F(A). The **normal subgroup of F(A) generated by R** is **$< R >^N$** := $\{u_1 r_1^{e_1} u_1^{-1} \cdots u_k r_k^{e_k} u_k^{-1} \mid k \geq 0$, and $r_i \in R$, $e_i \in \{1,-1\}$, and $u_i \in F(A)$ for each $1 \leq i \leq k\}$.
- Def 1.61: Let A be a set and let R be a subset of F(A). The group **presented by** the **presentation < A | R >** is the quotient group F(A)/<lt; R >^{N}. The set A is the set of **generators**, the set R is the set of **defining relators**, and the set of equations $\{r = 1 \mid r \in R\}$ is the set of **defining relations** of the presentation. The elements of $< R >^N$ are the **relators** of the group presented by < A | R >.
- Lemma 1.62: The group < A | R > is the largest group generated by A satisfying $r =_G 1$ for all $r \in R$.
- Def 1.63: For a set A, $R \subseteq F(A)$, and words $v,w \in (A \cup A')^*$, the equation **v = w** means that v and w are the same word, **$v =_{F(A)} w$** means that [v] = [w] in the group F(A), and **$v =_G w$** means that $[v]< R >^N = [w]< R >^N$ in the group G := < A | R >.
- Lemma 1.64: If G is a group, then G has a presentation; moreover, G is presented by G = < G | ab = (ab) for all $a,b \in G$ >.
- Prop 1.66: If G = < A | R >, then the abelianization of G is presented by $G_{ab} = < A \mid R \cup \{aba^{-1}b^{-1} \mid a,b \in A\} >$.
- Thm 1.70: (**HBT** = "Homomorphism Building Theorem for presentations"): Let G = < A | R >, let H be a group, and let f:A → H be a function satisfying the property that for all words $b_1^{e_1} \cdots b_m^{e_m} \in R$ (with each $b_i \in A$ and $e_i \in \{1,-1\}$), $f(b_1)^{e_1} \cdots f(b_m)^{e_m} =_H 1$. Then there is a unique group homomorphism h:G → H satisfying h(a) = f(a) for all $a \in A$.
- *Isomorphism invariants:*
  - Def 1.72: A subset A of G is a **generating set** for G if every element of G is a (finite) product of elements of A and their inverses. (This is written **G = ⟨ A ⟩**.)
    A group G is **finitely generated (f.g.)** if there is a finite subset A of G that generates G.
    A group G is **cyclic** if there is an element a of G satisfying G = ⟨ {a} ⟩.
    A group G is **finitely presented (f.p.)** if G = < A | R > for some finite sets A and R.
  - Lemma 1.73: A group G is finitely generated if and only if G is (isomorphic to) a quotient of F(A) for some finite set A.
  - Thm 1.75: The following are isomorphism invariants: (g) "finitely generated". (h) "cyclic". (i) "finitely presented".
- Def 1.77: Let A be a set, let $R \subseteq F(A)$, let b be a letter not in A, let $w \in F(A)$, and let $r \in < R >^N$. The operations < A | R > ↔ < A ∪ {b} | R ∪ {b = w} > and < A | R > ↔ < A | R ∪ {r} > are **Tietze transformations**.
- Thm 1.78: (**Tietze's Theorem**) If < A | R > and < B | S > are finite presentations, then < A | R > ≅ < B | S > if and only if there is a finite sequence of Tietze transformations from < A | R > to < B | S >.

◦ (iii) *Cayley graphs and Cayley complexes*

- Def 1.80: Let G be a group with a generating set A. The **Cayley graph** for G with respect to A is the 1-dimensional CW complex **Γ = Γ(G,A)** with vertex set G and for each $g \in G$ and $a \in A$, a directed edge $e_{g,a}$from g to ga labeled a.
- Thm 1.82: Let G = < A > and let Γ = Γ(G,A) be the Cayley graph. Then:
  (a) Γ is a path-connected CW complex.

(b) For each vertex v of Γ and for each $a \in A$, there is exactly 1 edge out of v labeled a and exactly 1 edge into v labeled a.
(c) G acts on Γ by $g \cdot h := (gh)$ and $g \cdot e_{h,a} := e_{(gh),a}$ (for all $h \in G$, $a \in A$), and this action satisfies:
(c-i) (**free**): whenever $g \in G$ and $p \in \Gamma$ with $gp = p$, then $g = 1$;
(c-ii) (**vertex-transitive**): whenever $v,w \in \Gamma^{(0)}$, there is a $g \in G$ with $gv = w$; and
(c-iii) (**isomorphisms of directed labeled graph**): for each $g \in G$, the action of g is a bijection $:\Gamma \to \Gamma$ that maps vertices to vertices, and maps edges to edges preserving both directions and labels.

- *Connections to group actions and covering space theory:*
  - Def 1.85: The **Cayley complex** associated to a presentation < A | R > of a group G is a 2-dimensional CW complex $\mathcal{C}$ **=** $\mathcal{C}$**(G,A,R)** with 1-skeleton Γ(G,A). The set of faces is in bijection with G × R; for each $g \in G$ and $r \in R$, the attaching map $\varphi_{g,r}: S^1 \to \Gamma$ of the face $f_{g,r}$ satisfies $\varphi_{g,r} \circ \omega$:= edge path in Γ starting at g labeled by r.
  - Def 1.87: The **presentation complex** associated to a presentation < A | R > of a group G is a CW complex with one vertex v, an edge e_a for each $a \in A$ (with attaching maps gluing both endpoints of e_a to v), and a face f_r for each $r \in R$ with attaching map determined by following the edges according to the word r.
  - Thm 1.88 (**"2-Way Street Thm"**): For every group G, there is a 2-dimensional PC CW complex X with $\pi_1(X) \cong G$. Moreover, if < A | R > is a presentation of G and Y is the associated presentation complex, then $\pi_1(Y) \cong G$.
  - Thm 1.89: Let < A | R > be a presentation of a group G, let $\mathcal{C}$ be the Cayley complex, and let X be the presentation complex. Then
(a) the action of G on $\mathcal{C}$, given by $g \cdot h := (gh)$, $g \cdot e_{h,a} := e_{(gh),a}$, and $g \cdot f_{h,r} := f_{(gh),r}$ for all $h \in G$, $a \in A$, and $r \in R$, is a covering space action;
(b) $\mathcal{C}/G \cong X$; and
(c) $\mathcal{C}$ is a simply-connected CW complex, and hence the composition $\mathcal{C} \to \mathcal{C}/G \to X$; is the universal covering space of X.

- *Section B: Normal forms, rewriting systems, and the Word Problem*
  - Idea: In order to build the Cayley graph of a finitely generated group, first find unique representatives for each of the group elements.
  - Def 1.100: Let A be a finite generating set for a group G, let $\pi: F(A) \to G$ be the corresponding surjective group homomorphism, and let $\rho: (A \cup A^{-1})^* \to F(A)$ be defined by $\rho(w) := [w]$ for all words w over $A \cup A^{-1}$.
If $g \in G$, $w \in (A \cup A^{-1})^*$, and $g = \pi \circ \rho(w)$, then the word w **represents** g; in symbols, $w =_G g$.
A **set of normal forms for G over A** is a subset $N \subset (A \cup A^{-1})^*$ satisfying the property that the restriction $\pi \circ \rho(w)|_N: N \to G$ is a bijection.
If $w \in N$ and $g = \pi \circ \rho(w)$, then w is **the normal form of g**.
  - Examples

- Def 1.105: Let A be a finite set. A **finite convergent rewriting system (CRS) over A** is a finite subset $R \subset A^* \times A^*$ such that the rewritings $xuy \to xvy$ for all $x,y \in A^*$ and $(u,v) \in R$ satisfy:
  (a) (**Termination**:) There is no infinite sequence of rewritings $x_1 \to x_2 \to x_3 \to \cdots$.
  (b) (**Confluence**:)
  (b-i) Whenever $(rs,v),(st,w) \in R$ with $s \neq 1$, then there exist a word $z \in A^*$ and finite sequences of rewritings $vt \to \cdots \to z$ and $rw \to \cdots \to z$.
  (b-ii) Whenever $(s,v),(rst,w) \in R$ (with $s \neq 1$), then there exist a word $z \in A^*$ and finite sequences of rewritings $rvt \to \cdots \to z$ and $w \to \cdots \to z$.
- Def 1.106: For a rewriting system R over A, the set A is the **alphabet** and the elements of R are the **(rewriting) rules** of the rewriting system.
  A pair of rules of the form [$(rs,v),(st,w) \in R$ with $s \neq 1$] or [$(s,v),(rst,w) \in R$ (with $s \neq 1$)] is called a **critical pair** of R, and in conditions (b-i) and (b-ii) of Def 1.105, the word z and the rewritings to z are called a **resolution** of the critical pair.
  An **irreducible word** (for R) is a word that does not contain a subword u for any $(u,v) \in R$.
  The symbol $w \to^* x$ denotes that there is a finite sequence of rewritings from w to x.
- Def 1.108: A **finite convergent rewriting system (CRS) for a group G** is a finite CRS such that G is presented as a monoid by $G = \mathrm{Mon} < A \mid \{u=v \mid (u,v) \in R\} >$.
- Thm 1.110: Let (A,R) be a finite CRS for a group G. Then:
  (a) The set **Irr(R)** := {irreducible words for R} is a set of normal forms for G.
  (b) Given any word $w \in A^*$, there is a finite sequence of rewritings from w to the normal form representing the same element of G.
  (c) The group G has a decidable Word Problem.
- *Proving termination with partial orders:*
  - Def 1.120: Let S be a set. A strict partial order > on S is **well-founded** if there is no infinite sequence of elements of S satisfying $x_1 > x_2 > x_3 > \cdots$.
  - Def 1.121: Let A be a set. A relation > on A^* is **compatible with concatenation** if whenever $w,x,y,z \in A^*$ and $w > x$ then $ywz > yxz$.
  - Def 1.122: Let A be a set. A **termination order** on A^* is a well-founded strict partial order that is compatible with concatenation.
  - Thm 1.123: Let A be a set and let $R \subseteq A^* \times A^*$. Let > be a termination order on A^*. If $u > v$ for all $(u,v) \in R$, then the rewritings $xuy \to xvy$ for all $x,y \in A^*$ and $(u,v) \in R$ satisfy the termination property.
- *Examples of termination orders:*
  - Def 1.125: Let A be a finite set with a total order >. The **shortlex** order $>_{sl}$ on A^* induced by > is defined by: For each $x = a_1a_2\cdots a_m$ and $y = b_1b_2\cdots b_n$ in A^*, where each $a_i,b_j \in A$, then $x >_{sl} y$ iff either (i) $m > n$, or (ii) $m=n$ and there is an index i such that $a_j = b_j$ for all $1 \le j < i$ and $a_i > b_i$.
  - Def 1.126: Let A be a finite set with a total order >, and let $w: A \to \mathbb{N}$ be a function. The **weightlex** order $>_{wl}$ on A^* induced by > and w is defined by: For each $x = a_1a_2\cdots a_m$ and $y = b_1b_2\cdots b_n$ in A^*, where each $a_i,b_j \in A$, then $x >_{wl} y$ iff either (i) $w(a_1) + w(a_2) + \cdots + w(a_m) > w(b_1) + w(b_2) + \cdots + w(b_n)$, or (ii) $w(a_1) + w(a_2) + \cdots + w(a_m) > w(b_1) + w(b_2) + \cdots + w(b_n)$ and there is an index i such that $a_j = b_j$ for all $1 \le j < i$ and $a_i > b_i$.
  - Prop 1.127: Let A be a finite set with a total order >. (a) The shortlex order on A^* induced by > is a well-founded strict partial order that is compatible with concatenation.

(b) If w: A → $\mathbb{N}$, then the weightlex order on A^* induced by > and w is a well-founded strict partial order that is compatible with concatenation.

- Examples

- *Section C: An initial catalog of examples of groups*

  - *Free and free abelian groups:*
    - Prop 1.140: The free group on a set A is presented by F(A) = < A | >. The set of reduced words over A is a set of normal forms for F(A).
    - Def 1.141: The **free abelian group** on a set A is presented by $\mathbb{Z}^A$ = < A | {ab=ba | a,b ∈ A} >.
    - Prop 1.142: Let G be the free abelian group on a set A. Given a total order < on A, the set of words $\{a_1^{j_1} \cdots a_k^{j_k} \mid k \geq 0$, each $a_i \in A$, each $j_i \in \mathbb{Z}$, and $a_1^{j_1} < \cdots < a_k^{j_k}\}$ is a set of normal forms for $\mathbb{Z}^A$.
  - *Coxeter and Artin groups*
    - Def 1.144: A **Coxeter group** is a group with a presentation of the form $\langle s_1, s_2, \ldots, s_n \mid (s_is_j)^{m_{ij}} = 1$ for all i < j, and $s_i^2 = 1$ for all i$\rangle$, where $2 \leq m_{ij} \leq \infty$ for all i,j.
    - Rmk: Every Coxeter group is generated by a set of reflections of a space.
    - Def 1.145: The **dihedral group D_n** is the Coxeter group generated by two Coxeter generators s_1 and s_2 with m_{12} = n/2.
    - Def 1.147: An **Artin group** is a group with a presentation of the form $\langle s_1, s_2, \ldots, s_n \mid [s_i,s_j,m_{ij}] = [s_j,s_i,m_{ij}]$ for all i < j $\rangle$, where $2 \leq m_{ij} \leq \infty$ for all i,j, and the symbol [s,t,m] denotes the alternating word ststs... of length m.
    - Def 1.149: The **braid group on n strands B_n** is the Artin group $\langle s_1, s_2, \ldots, s_{n-1} \mid [s_i,s_j] = 1$ for all |i-j|>1, and $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for all i < n-1 $\rangle$.
    - Def 1.150: An **right-angled Artin group**, or **raag**, is an group satisfying the property that $m_{ij} \in \{2,\infty\}$ for all i,j.
  - *Fundamental groups of manifolds*
    - Def 1.152: A **surface group** is the fundamental group of a compact connected surface.
    - Prop 1.153: (a) The fundamental group of a compact connected orientable surface of genus g is presented by $\pi_1(S_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1,b_1] \cdots [a_g,b_g] = 1 \rangle$.
      (b) The fundamental group of the Klein bottle is presented by $\pi_1(K^2) = \langle a,b \mid bab^{-1} = a^{-1} \rangle$.
  - *Matrix groups*
    - Def 1.155: The **special linear group $SL_n(\mathbb{Z})$** is the group of n × n integer matrices with integer entries and determinant 1, with group operation given by matrix multiplication.
      The **projective special linear group $PSL_2(\mathbb{Z})$** is the quotient **$PSL_2(\mathbb{Z})$** := $SL_2(\mathbb{Z}) / \langle -I \rangle^N$.
      The **Heisenberg group H** is the group of upper triangular matrices with integer entries and all 1's on the diagonal.
    - Prop 1.156: (a) The special linear group is presented by $SL_2(\mathbb{Z}) = \langle s,t \mid s^2 = t^3, s^4 = 1 \rangle$.
      (b) The projective special linear group is presented by $PSL_2(\mathbb{Z}) = \langle s,t \mid s^2 = t^3 = 1 \rangle$.
      (c) The Heisenberg group is presented by < x,y,z | [x,z] = [y,z] = 1, [x,y] = z >.

- Def 1.158: Let p,q $\in \mathbb{Z}$. The **Baumslag-Solitar group BS(p,q)** is the group presented by BS(p,q) := $< a,t \mid ta^pt^{-1} = a^q >$.

- *Section D: Constructions of groups, group homomorphisms, and group properties*

  - Rmk: The following three basic constructions of new groups from old can also be viewed as the three basic ways to decompose a group into "simpler" subgroups: Extensions, graph products, and fundamental groups of graphs of groups.
  - (i) *Extensions of groups*
    - Def 1.170: If N is a normal subgroup of a group G, and Q := G/N, then G is an **extension of N by Q**.
    - Thm 1.171: Let G be an extension of N by Q, and let $i:N \to G$ be the inclusion map and let $q:G \to Q$ be the quotient map. (a) If N and Q are finitely generated, then so is G. Moreover, if N is finitely generated by the set A and Q is finitely generated by the set B, and $B' \subseteq G$ satisfies the property that the restriction $q|_{B'}:B' \to Q$ gives a bijection onto B, then G is finitely generated by $i(A) \cup B'$. (b) If N and Q are finitely presented, then G is finitely presented.
    - Def 1.173: An extension G of N by Q is **split** if there is a group homomorphism $s:Q \to G$ such that $q \circ s = 1_Q$, where $q:G \to Q$ is the quotient map.
    - Def 1.174: Let G be a group. An **automorphism** of G is an isomorphism $:G \to G$.
      The **automorphism group Aut(G)** is the group of automorphisms of G, with group operation given by composition.
    - Def 1.175: Let N = < A | R > and H = < B | S > be groups with presentations, and let $\varphi: H \to \mathrm{Aut}(N)$ be a homomorphism. The **semidirect product** of N by H with respect to φ is the group $\mathbf{N \rtimes_\varphi H} := < A \cup B \mid R \cup S \cup \{bab^{-1} = (\varphi(b))(a) \mid a \in A^{\pm 1}, b \in B^{\pm 1}\} >$.
    - Thm 1.176: Let N and H be groups, let $\varphi: H \to \mathrm{Aut}(N)$ be a homomorphism, let $A := N - \{1_N\}$, let $B := H - \{1_H\}$, and let $C := A \cup B$. The set $\{nh \mid n \in A \cup \{1\}, h \in B \cup \{1\}\}$ is a set of normal forms for the semidirect product group $N \rtimes_\varphi H$ over the generating set C.
    - Thm 1.177: Let G, N, and H be groups. The group G is a semidirect product of N by H if and only if G is a split extension of N by H.
  - (ii) *Graph products of groups*
    - Def 1.180: Let H = < A | R > and J = < B | S > be groups with presentations.
      The **free product** of H and J is the group $\mathbf{H * J} := < A \cup B \mid R \cup S >$.
      The **direct product** of H and J is the group $\mathbf{H \times J} := < A \cup B \mid R \cup S \cup \{ba = ab \mid a \in A, b \in B\} >$.
    - Thm 1.182: Let H and J be groups, let $A := H - \{1_H\}$, let $B := J - \{1_J\}$, and let $C := A \cup B$.
      (a) A **reduced sequence** over H and J is a word in C^* of the form $h_1j_1h_2j_2 \cdots h_nj_n$ such that $n \geq 0$, $h_1 \in A \cup \{1\}$, each $h_i \in A$ for $i \geq 2$, each $j_k \in B$ for $k \leq n-1$, and $j_n \in B \cup \{1\}$.
      (b) The set of reduced sequences over H and J is a set of normal forms for the free product group $H * J$ over the generating set C.

(c) The set $\{hj \mid h \in A \cup \{1\}, j \in B \cup \{1\}\}$ is a set of normal forms for the direct product group $H \times J$ over the generating set C.

- Def 1.185: Let Λ be a finite simple graph with vertex set V and edge set E, and for each vertex $v \in V$, let $G_v = < A_v \mid R_v >$ be a group with a presentation. The associated **graph product** is the group presented by $\mathbf{G\Lambda} := < \cup_{v \in V} A_v \mid \cup_v R_{v \in V} \cup \{ab=ba \mid a \in A_u, b \in A_v, \text{and } (u,v) \in E\} >$.
- Def 1.188: A **right-angled Artin group (raag)** is a graph product in which each vertex group is an infinite cyclic group.
  A **right-angled Coxeter group (raag)** is a graph product in which each vertex group has order 2.

- (iii) *Amalgamated products and HNN extensions*
  - Rmk: Amalgamated products and HNN extensions are two special cases of the third basic decomposition of a group into subgroups (or construction of new groups from old), called fundamental groups of graphs of groups.
  - Def 1.190: Let $G = < A \mid R >$ and $H = < B \mid S >$ be groups with presentations, let $J = < C >$ be a group with generating set, and let $i:J \to G$ and $k:J \to H$ be injective group homomorphisms. The **amalgamated product** of G and H along J with respect to i and k is the group $\mathbf{G *_J H} := < A \cup B \mid R \cup S \cup \{i(c) = k(c) \mid c \in C\}>$.
  - Prop 1.191: Let $G *_J H$ be an amalgamated product of G and H along J with respect to homomorphisms $i:J \to G$ and $k:J \to H$. Let $f:G \to G * H$ be the homomorphism mapping $f(g) := g$ (a reduced sequence of length one) for all $g \in G$, and let $q:G * H \to G *_J H$ be the quotient map. Then $q \circ f:G \to G *_J H$ is a monomorphism.
  - Def 1.194: (a) Let $G = < A \mid R >$ be a group with presentation, let $J = < C >$ be a group with generating set, and let $i:J \to G$ and $k:J \to G$ be injective group homomorphisms.
    The **HNN extension** of G over J with respect to i and k is the group $\mathbf{G *_J} := < A \cup \{t\} \mid R \cup \{ti(c)t^{-1} = k(c) \mid c \in C\}>$.
    (b) Let $G = < A \mid R >$ be a group with presentation, let H and K be subgroups of G, let C be a generating set of H, and let $\varphi:H \to K$ be an isomorphism. The **HNN extension** of G over φ is the group $\mathbf{G *_\varphi} := < A \cup \{t\} \mid R \cup \{tct^{-1} = \varphi(c) \mid c \in C\}>$.
  - Prop 1.195: Let $G *_J$ be an HNN extension of G over J with respect to homomorphisms $i,k:J \to G$. Let $f:G \to G * < t \mid \ >$ be the homomorphism mapping $f(g) := g$ (a reduced sequence of length one) for all $g \in G$, and let $q:G * < t \mid \ > \to G *_J$ be the quotient map. Then $q \circ f:G \to G *_J$ is a monomorphism.
    Similarly, if $G *_\varphi$ is an HNN extension of G over φ, then the map $q \circ f:G \to G *_\varphi$, defined by $q \circ f(g) :=_{G *_\varphi} g$ for all $g \in G$, is a monomorphism.
  - Lemma 1.196: $BS(1,2) \cong \mathbb{Z} *_\mathbb{Z}$.
- (iv) *Abelianization*
  - Def 1.197: Let $G = < A \mid R >$ be a group with presentation. The **abelianization** of G is the group $\mathbf{G^{ab}} := < A \mid R \cup \{ab=ba \mid a,b \in A\} >$.
    The **commutator subgroup** of G is the group $\mathbf{[G,G]} := < \{[a,b] \mid a,b \in A\} >$.
  - Prop 1.198: For any group G, $[G,G] \triangleleft G$ and $G^{ab} \cong G/[G,G]$ is the largest abelian quotient of G.
  - Thm 1.199: If G and H are groups and $G^{ab} \ncong H^{ab}$, then $G \ncong H$.
- (v) *Constructing properties*

- Rmk: Examples of isomorphism invariant group properties include infinite, finite, finitely generated, finitely presented, torsion, torsion-free, abelian, nilpotent, and solvable.
- Def 1.200: Let P be a group property. A group G is **virtually** P if there is a finite index subgroup H of G such that H has P.
- Thm 1.202: (a) Virtually finite = virtually trivial = finite.
  (b) Virtually abelian ≠ abelian. Moreover, the dihedral group D_6 is virtually abelian but not abelian.
- Def 1.205: Let P be a group property. A group G is **residually** P if for all $g \in G$ with $g \neq 1_G$, there is a P group H and homomorphism $\varphi: G \to H$ such that $\varphi(g) \neq 1_H$.
- Thm 1.206: (a) Residually finite ≠ finite. Moreover, $\mathbb{Z}$ is residually finite but not finite.
  (b) Residually abelian = abelian.
- Def 1.210: Let P be a group property. A group G is **locally** P if every finite subset $S \subseteq G$ is contained in a P subgroup of G.
- Thm 1.211: (a) Locally finite ≠ finite. Moreover, $\bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is locally finite but not finite.
  (b) Locally abelian = abelian.
- Def 1.220: Let P be a group property. A group G is **poly**-P if there is a finite sequence of subgroups
  $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ in which each G_{i+1}/G_i has the property P.
  Each G_i is a **term**, each G_{i+1}/G_i is a **factor**, and if $G_{i+1} \neq G_i$ for all i, then n is the **length**, of the sequence.
  The sequence is called a **subnormal series** for G.
- Thm 1.224: (a) Polyfinite = finite.
  (b) Polyabelian ≠ abelian. Moreover, the dihedral group D_6 is polyabelian but not abelian.
- Rmk: Poly properties are often proved by induction.

Chapter 2: Abelian and geometric group theory: Poly properties and "abelian-ness"

- *Section A: Closure properties and examples*

  - Thm 2.1: The finitely generated groups $\mathbb{Z}^n$ for all n and the Heisenberg group H are polycyclic.
  - Thm 2.2: Braid groups and raags are polyfree.
  - Thm 2.4: Let P be an isomorphism invariant group property.
    (a) If P is inherited by subgroups, then every subgroup of a poly-P group is poly-P.
    (b) If P is inherited by quotient groups, then every quotient of a poly-P group is poly-P.
    (c) P is preserved when taking extensions if and only if poly-P = P.
  - Thm 2.5: Let P be an isomorphism invariant group property. Then poly-(poly-P) = poly-P.
  - Thm 2.7: (a) Poly-finitely-presented = finitely presented. (b) Poly-finitely-generated = finitely generated. (c) Polycyclic ≠ cyclic. (Example: $\mathbb{Z}^2$.) (d) Polyfree ≠ free. (Example: $\mathbb{Z}^2$.) (e) Polytrivial = trivial. (f) Polytorsion = torsion. (g) Polyabelian ≠ abelian.

- *Section B: Measuring "abelian-ness" of groups*

  - (i) *Solvable groups*
    - Def 2.10: A group G is **solvable**, or **soluble**, if G is polyabelian.
    - Def 2.11: The **derived length** of a solvable group G, denoted **dl(G)**, is the length of the shortest abelian subnormal series for G.
    - Def 2.12: A group G is **metabelian** if G is solvable with derived length 2.
    - Cor 2.13: The class of solvable groups is closed under taking subgroups, quotients, and extensions.
    - Thm 2.15: The Heisenberg group H and the Baumslag-Solitar groups BS(1,n) are metabelian.
    - Def 2.17: Let G be a group. The **derived series** of G is the sequence $G \triangleright G^{(1)} \triangleright G^{(2)} \triangleright G^{(3)} \cdots$ where $\mathbf{G^{(1)}}$:= [G,G] is the **derived group** of G, and for all n > 1, $\mathbf{G^{(n)}} := [G^{(n-1)},G^{(n-1)}]$ is the **n-th derived group** of G.
    - Prop 2.18: Let G be a group. (a) For all n, the group $G^{(n)}$ is normal in G. (b) The derived series is a (possibly infinite) abelian normal series.
    - Thm 2.20: Let G be a group. The group G is solvable if and only if $G^{(n)} = 1$ for some natural number n.
      Moreover, if G is solvable, then $dl(G) = \min\{n \mid G^{(n)} = 1\}$ and the derived series (ending at $G^{(dl(G))}$) is a minimal length abelian normal series for G.
    - Thm 2.22: (**Witt-Hall identities** or **commutator calculus**): Let G be a group and let a,b,c,g ∈ G. (i) $[a,b][b,a] =_G 1$. (ii) $g[a,b]g^{-1} =_G [gag^{-1},gbg^{-1}]$. (iii) $[a,bc] =_G [a,b][a,c][[c,a],b]$. (iv) $[ab,c] =_G [b,c][[c,b],a][a,c]$.
    - Def 2.24: Let G be a group and let $a_1,...,a_n \in G$. Then $\mathbf{[a_1,...,a_n]} := [[\cdots[[a_1,a_2],a_3], \cdots],a_n]$ is an **n-fold commutator** in G.
  - (ii) *Nilpotent groups*
    - Def 2.30: The **center** of a group G is the subgroup **Z(G)** := {g ∈ G | gh=hg for all h ∈ G}.
    - Def 2.31: A group G is **nilpotent** if G is "poly-central-normal"; that is, if there is a sequence $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ in which each G_i is a normal subgroup of G, and each G_{i+1}/G_i is in the center of G/G_i.
    - Def 2.32: Let G be a nilpotent group. The **nilpotence class** of G, denoted **c(g)**, is the length of a shortest central normal series for G.
    - Thm 2.33: Every nilpotent group is solvable.
    - Rmk: Nilpotence class, derived length, and nilpotence versus solvability all measure how close a group is to being abelian.
    - Prop 2.35: The Heisenberg group H is nilpotent but not abelian. Moreover, H is the free nilpotent group of class 2.
    - Def 2.39: Let G be a group. The **lower central series** for G is the sequence $G = \gamma_1(G) \triangleright \gamma_2(G) \triangleright \gamma_3(G) \triangleright \cdots$ where $\boldsymbol{\gamma_1}(G) := G$, and for each n ≥ 1, $\boldsymbol{\gamma_{n+1}}(G) := [\gamma n(G),G]$.
    - Thm 2.40: The lower central series is a (possibly infinite) central normal series.

- Def 2.42: Let G be a group. The **upper central series** for G is the sequence $1 = \zeta_0(G) \triangleleft \zeta_1(G) \triangleleft \zeta_2(G) \triangleleft \cdots$ such that $\boldsymbol{\zeta_0(G)} := 1$, $\boldsymbol{\zeta_1(G)} := Z(G)$, and for all $n \geq 1$, $\boldsymbol{\zeta_{n+1}(G)} := \pi_n^{-1}(Z(G/\zeta_n(G)))$ where $\pi_n\colon G \to G/\zeta_n(G)$ is the quotient map.
- Thm 2.43: The upper central series is a (possibly infinite) central normal series.
- Thm 2.44: Let G be a group. The following are equivalent: (a) G is nilpotent. (b) $\gamma_n(G) = 1$ for some n. (c) $\zeta_n(G) = G$ for some n.
  Moreover, if G is nilpotent, then $c(G) = \min\{n \mid \gamma_n(G) = 1\} - 1 = \min\{n \mid \zeta_n(G) = G\}$, and both the lower central series (ending at $\gamma_{c(G)+1}(G)$) and the upper central series (ending at $\zeta_{c(G)}(G)$) are minimal length central normal series for G.
- Prop 2.46: The Baumslag-Solitar group BS(1,2) is solvable but not nilpotent.
- Thm 2.48: The class of nilpotent groups is closed under subgroups, quotients, and finite direct products.
- Thm 2.50: If G is a nontrivial nilpotent group, then $dl(G) \leq \log_2(c(G)) + 1$.

◦ (iii) *Poly-finitely-generated-abelian groups*

- Def 2.60: A group G has the property **max** if every subgroup of G is finitely generated.
- Thm 2.62: The class of max groups is closed with respect to subgroups, quotients, and extensions.
- Cor 2.63: Poly-max = max.
- Cor 2.64: Every polycyclic group has max.
- Thm 2.66: Let G be a group. The following are equivalent:
  (a) G is polycyclic.
  (b) G is finitely generated and solvable and G has max.
  (c) G is solvable and G has max.
  (d) G is poly-(finitely-generated-abelian).
- Prop 2.68: If G is a finitely generated nilpotent group, then each group $\gamma_i(G)$ in the lower central series for G is finitely generated.
- Thm 2.69: Every finitely generated nilpotent group is polycyclic.
- Prop 2.71: The Baumslag-Solitar group BS(1,2) is finitely generated and solvable but not polycyclic.
- Def 2.73: Let H be a group and let $\alpha\colon H \to H$ be an injective homomorphism. Let $\varphi\colon H \to \alpha(H)$ be the isomorphism $\varphi := \alpha|^{\alpha(H)}$. The **ascending HNN extension** induced by α is $\mathbf{H *_\alpha} := H *_\varphi$.
- Prop 2.74: The Baumslag-Solitar group BS(1,n) is the ascending HNN extension $BS(1,n) = \mathbb{Z} *_\alpha$ associated to the monomorphism $\alpha\colon\mathbb{Z} \to \mathbb{Z}$ given by multiplication by n.
- Thm 2.75: Let $\alpha\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be an injective homomorphism with associated matrix M, and let $G_\alpha := \mathbb{Z}^n *_\alpha$ be the associated ascending HNN extension. Then:
  (a) G_α is solvable.
  (b) G_α is polycyclic if and only if $\det(M) = \pm 1$.
  (c) G_α is nilpotent-by-finite if and only if every eigenvalue λ of M satisfies $|\lambda| = 1$.
- Cor 2.76: The group $G = \langle a,b,t \mid [a,b]=1, tat^{-1} = a^4b, tbt^{-1} = a \rangle$ is polycyclic and finitely generated but not nilpotent.
- Cor 2.77: {f.g. abelian groups} ⊊ {f.g. nilpotent groups} ⊊ {polycyclic groups} ⊊ {f.g. solvable groups} ⊊ {f.g. groups}.

- *Section C: Geometry and algorithms for poly-abelian groups*

  - Def 2.100: Let G be a group with a finite inverse-closed generating set A. The **word metric $d_{G,A}$** on G with respect to A is defined by $d_{G,A}(g,h) := \min\{n \mid h = ga_1 \cdots a_n$ for some $a_1,...,a_n \in A\}$ for all $g,h \in G$.
  - Def 2.101: Let G be a group with a finite inverse-closed generating set A, and let $\Gamma = \Gamma(G,A)$ be the associated Cayley graph. The **path metric d_{path}** on Γ is defined by making edge isometric to I = [0,1] (with the Euclidean metric); then $d_{path}(x,y) := \min\{\text{length}(p) \mid p$ is a path in Γ from x to y$\}$ for all $x,y \in \Gamma$.
  - Prop 2.102: Let G be a group with a finite inverse-closed generating set A, and let $\Gamma = \Gamma(G,A)$ be the associated Cayley graph. For each $g,h \in G$, $d_{G,A}(g,h) = d_{path}(g,h)$.
  - Def 2.104: Let G = < A > be a group with a finite generating set and let $\Gamma = \Gamma(G,A)$ be the associated Cayley graph. For any $x,y \in \Gamma$, a **geodesic** from x to y is a path p from x to y satisfying $\text{length}(p) = d_{path}(x,y)$.
  - (i) *Subgroup distortion*:
    - Def 2.110: Let G = < A > and H = < B > be groups with finite inverse-closed generating sets with H < G. The **distortion** of H is G is the function $\delta_H^G: \mathbb{N} \to \mathbb{N}$ defined by $\delta_H^G(n) := \max\{d_{H,B}(1,h) \mid h \in H$ and $d_{G,A}(1,h) \le n\}$.
    - Def 2.111: Let $f,g: \mathbb{N} \to \mathbb{N}$. Then **$f \preccurlyeq g$** if there is a constant C such that $f(n) \le Cg(Cn + C) + C$ (+ Cn) for all $n \in \mathbb{N}$.
      If $f \preccurlyeq g$ and $f \succcurlyeq g$, then we write **$f \approx g$** and say that f and g are **equivalent**.
    - Prop 2.112: Let $f,g: \mathbb{N} \to \mathbb{N}$.
      (a) If f and g are polynomials, then $f \approx g$ if and only if f and g have the same degree.
      (b) If a > 1 and b > 1 are real numbers and $f(n) = a^n$ and $g(n) = b^n$ for all n, then $f \approx g$.
    - Thm 2.115: Let G = < A > = < A' > and H = < B > = < B' > be groups with finite inverse-closed generating sets with H < G. Then $\delta_{H,B}^{G,A} \approx \delta_{H,B'}^{G,A'}$.
    - Prop 2.121: Let H be the subgroup generated by $B = \{a^{\pm 1}\}$ in the group $G = BS(1,2) = < a,t \mid tat^{-1} = a^2 >$. The distortion function δ_H^G grows at least exponentially.
    - Thm 2.122: Let $H \cong \mathbb{Z}^n$ be a free abelian group on the inverse-closed generating set B and let $\alpha: H \to H$ be an injective homomorphism with associated matrix M. Let $G_\alpha := \mathbb{Z}^n *_\alpha$ be the associated ascending HNN extension with generating set $B \cup \{t^{\pm 1}\}$. The following are equivalent:
      (a) G_α is not nilpotent-by-finite.
      (b) Some eigenvalue λ of M satisfies $|\lambda| \neq 1$.
      (c) The distortion function δ_H^G grows at least exponentially.
  - (ii) *Growth functions*:
    - Def 2.130: Let G = < A > be a group with a finite inverse-closed generating set. The **cumulative growth function** of G with respect to A is the function $\beta_{G,A}: \mathbb{N} \to \mathbb{N}$ defined by $\beta_{G,A}(n) := |\{g \in G \mid d_{word}(1,g) \le n\}|$.
    - Examples

- Prop 2.132: If G = < A > = < A' > is a group with a finite inverse-closed generating sets, then $\beta_{G,A} \approx \beta_{G,A'}$.
- Thm 2.135: (Gromov) A finitely generated group G has polynomial growth if any only if G is virtually nilpotent.

- (iii) *Word problems and rewriting systems*:
  - Thm 2.140: The class of groups with finite convergent rewriting systems is closed under taking extensions.
  - Cor 2.141: If G is a polycyclic group, then G has a finite convergent rewriting system.
  - Thm 2.144: There are finitely presented solvable groups that do not have solvable word problem.
  - Def 2.145: The class of **constructible groups** is the smallest class of groups that contains the trivial group and that is closed under forming extensions in which the normal subgroup is constructible and the quotient is finite, amalgamated free products in which both factors as well as the amalgamated subgroup are constructible, and HNN extensions in which the base group and associated subgroups are constructible.
  - Thm 2.146: (Groves, Smith) A finitely generated solvable group G has a finite convergent rewriting system if and only if G is constructible.

Chapter 3: Topological and algebraic group theory: Homology and "finiteness" -- A practical user's guide

- *Section A: Topological view*

  - (i) *Homology of presentation and Cayley complexes*
    - Def 3.1: Let Y be a CW complex. For each n ≥ 0, the **group of n-chains** of Y is the direct sum $\mathbf{C_n(Y)} := \bigoplus_{\sigma \in J_n} \mathbb{Z}[\sigma]$, where J_n is the set of n-cells of Y.
      The **chain complex** associated to Y is the sequence
      $\cdots \to C_{n+1}(Y) \to C_n(Y) \to C_{n-1}(Y) \to \cdots \to C_1(Y) \to C_0(Y) \to 0$,
      where the **boundary maps** $\boldsymbol{\partial_n}: C_n(Y) \to C_{n-1}(Y)$ are the cellular boundary maps defined in Section 2.2 of Hatcher's text.
      The **n-th homology group of Y** is the group $\mathbf{H_n(Y)} := \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$.
    - Thm 3.3: Let G = < A | R > be a group with a presentation, and let X be the presentation complex of G. The chain groups for X satisfy $C_n(X) = 0$ for all n ≥ 3, $C_2(X) = \bigoplus_{r \in R} \mathbb{Z}[r]$, $C_1(X) = \bigoplus_{a \in A} \mathbb{Z}[a]$, $C_0(X) = \mathbb{Z}[v]$, and the boundary maps are $\mathbb{Z}$-module homomorphisms given by $\partial_n = 0$ for all n ≠ 2, and $\partial_2(1[r]) = e_1[b_1] + \cdots + e_k[b_k]$ for each relator $r = b_1^{e_1} \cdots b_k^{e_k}$ in R, where each $b_i \in A$ and each $e_i \in \{\pm 1\}$.
      The homology groups of X satisfy $H_0(X) \cong \mathbb{Z}$, $H_1(X) \cong G_{ab}$, and $H_n(X) \cong 0$ for all n ≥ 3. However, $H_2(X)$ depends on the presentation.
    - Examples

- Def 3.5: Let G be a group. The **integral group ring** for G is the ring $\mathbb{Z}$**G** := $\bigoplus_{g \in G} \mathbb{Z}g$ (as an abelian group) with multiplication defined by $(\Sigma\, m_g g)(\Sigma\, n_h h) = \Sigma\, \Sigma\, (m_g n_h)(gh)$, for all $g,h \in G$ and $m_g,n_h \in \mathbb{Z}$.
- Thm 3.6: Let G = < A | R > be a group with a presentation, and let $\mathcal{C}$ be the presentation complex of G. The chain groups for X satisfy $C_n(\mathcal{C}) = 0$ for all $n \geq 3$, $C_2(\mathcal{C}) = \bigoplus_{r \in R} \mathbb{Z}G[r]$, $C_1(\mathcal{C}) = \bigoplus_{a \in A} \mathbb{Z}G[a]$, $C_0(\mathcal{C}) = \mathbb{Z}G$, The boundary maps are $\mathbb{Z}G$-module homomorphisms given by $\partial_n = 0$ for all $n \geq 2$, $\partial_0 = 0$; for each generator $a \in A$, $\partial_1(1[a]) = a-1$; and for each relator $r = b_1^{e_1} \cdots b_k^{e_k}$ in R, where each $b_i \in A$ and each $e_i \in \{\pm 1\}$, $\partial_2(1[r]) = \Sigma_{i=1}^k e_i p_i[b_i]$, where $p_i := b_1^{e_1} \cdots b_{i-1}^{e_{i-1}}$ if $e_i = 1$ and $p_i := b_1^{e_1} \cdots b_i^{e_i}$ if $e_i = -1$.
  The homology groups of $\mathcal{C}$ satisfy $H_0(\mathcal{C}) \cong \mathbb{Z}$, $H_1(\mathcal{C}) \cong 0$, and $H_n(\mathcal{C}) \cong 0$ for all $n \geq 3$.
  However, $H_2(\mathcal{C})$ depends on the presentation.
- Examples

◦ (ii) *The topological definition of $H_n(G)$, and K(G,1) spaces*

- Def 3.20: A space Y is **contractible** if Y is homotopy equivalent to a space with exactly one point.
- Prop 3.21: A space Y is contractible if and only if the identity map Id_Y is homotopic to a constant function.
- Def 3.30: For a group G, a **K(G,1)** (also called an **Eilenberg-MacLane space** for G) is a path-connected CW complex Y with $\pi_1(Y) = G$ satisfying the property that the universal covering space of Y is contractible.
- Thm 3.33: For a PC CW complex Y with $\pi_1(Y) = G$ and universal covering space $p:\tilde{Y} \to Y$, the following are equivalent: (a) Y is a K(G,1). (b) $\tilde{Y}$ is contractible. (c) $H_n(\tilde{Y}) = 0$ for all $n > 1$.
- Def 3.35: Let G be a group, and let Y be a K(G,1) space. The **n-th homology group of G** is **H_n(G)** := $H_n(Y)$.
- Thm 3.36: If G is a group and Y and Z are both K(G,1) spaces, then $H_n(Y) \cong H_n(Z)$. Hence $H_n(G)$ is well-defined.
- Thm 3.37: If G is a group, then $H_0(G) = \mathbb{Z}$ and $H_1(G) = G_{ab}$.
- *Examples of computing $H_n(G)$:*
  - Prop 3.40: For the infinite cyclic group $\mathbb{Z}$, S^1 and $S^1 \times I$ are both K($\mathbb{Z}$,1)s. Moreover, $H_n(\mathbb{Z}) = \mathbb{Z}$ for n=0,1, and $H_n(\mathbb{Z}) = 0$ for $n \geq 2$.
  - Prop 3.41: For the free group F_n, the bouquet of n circles is a K(F_n,1). Moreover, $H_0(F_n) = \mathbb{Z}$, $H_1(F_n) = \mathbb{Z}^n$, and $H_n(F_n) = 0$ for $n \geq 2$.
  - Prop 3.44: For the group $\mathbb{Z}/2$, the presentation complex for the presentation < a | $a^2 = 1$ > is not a K($\mathbb{Z}/2$,1).
- *Zero divisors in group rings:*
  - Def 3.50: A **zero divisor** in a ring R is an element $r \in R$ with $r \neq 0$ such that there is an $s \in R$ with $s \neq 0$ and $rs = 0$.
  - Prop 3.52: If g is an element of a group G with finite order $n > 1$, then g is a zero divisor in $\mathbb{Z}G$.
  - Def 3.53: Let G be a group and let K be a field The **group ring KG** := $\bigoplus_{g \in G} Kg$ (as an abelian group) with multiplication defined by $(\Sigma\, m_g g)(\Sigma\, n_h h) = \Sigma\, \Sigma\, (m_g n_h)(gh)$, for all $g,h \in G$ and $m_g,n_h \in K$.

    - Conj 3.55: (**Kaplansky's zero divisor conjecture (ZDC)**): If G is a torsion-free group and K is a field, then the group ring KG has no zero divisors.
    - Prop 3.56: If G satisfies the ZDC for K = $\mathbb{Q}$, then the integral group ring $\mathbb{Z}$G has no zero divisors.
    - Def 3.60: A group G is a **1-relator group** if G has a presentation of the form G = < A | r=1 > (with one relator).
    - Def 3.61: A group G is **indicable** if either G is the trivial group or there exists a surjective homomorphism :G → $\mathbb{Z}$.
    - Prop 3.62: A group G is **locally indicable** if every finitely generated subgroup of G is indicable.
    - Def 3.63: A group G is **left-orderable** if there is a total order < on G satisfying gh < gj whenever g,h,j ∈ G and h < j.
    - Thm 3.64: Let G be a finitely generated group. Then G is left-orderable if and only if G is (isomorphic to) a subgroup of **$\mathrm{Homeo}_+(\mathbb{R})$**, the group of orientation preserving homeomorphisms of $\mathbb{R}$.
    - Thm 3.66: The ZDC holds for the following groups:
      t.f. virtually solvable groups,
      t.f. 1-relator groups,
      t.f. free-by-cyclic groups,
      (t.f.) left orderable groups (including Thompson's group F), and
      (t.f.) locally indicable groups (Higman's Thm).
    - Thm 3.68: (Howie's Thm): If G = < A | R > and H = < B | S > are locally indicable groups, and if r is a cyclically reduced word in elements of G and H of length at least 2 (that is, r = $u_1v_1u_2v_2 \cdots u_nv_n$ where n ≥ 2 and each $u_i \in (A \cup A^{-1})^*$ satisfies $u_i \neq_G 1_G$ and each $v_i \in (B \cup B^{-1})^*$ satisfies $v_i \neq_H 1_H$) that is not a proper power in the group G ∗ H = < A ∪ B | R ∪ S > (that is, $r \neq_{G*H} j^n$ for any j ∈ G ∗ H and n ≥ 2), then the group $(G * H)/\langle r \rangle^N$ = < A ∪ B | R ∪ S ∪ {r} > is locally indicable.
    - Examples
  - *More examples of computing $H_n(G)$:*
    - Prop 3.70: For the group $\mathbb{Z}^2$, the torus T^2 is a K($\mathbb{Z}^2$,1). Moreover, $H_0(\mathbb{Z}^2) = \mathbb{Z}$, $H_1(\mathbb{Z}^2) = \mathbb{Z}^2$, $H_2(\mathbb{Z}^2) = \mathbb{Z}$, and $H_n(\mathbb{Z}^2) = 0$ for n ≥ 3.
    - Prop 3.71: For the group BS(1,2), the presentation complex for the presentation < a,t | $tat^{-1} = a^2$ > is a K(BS(1,2),1). Moreover, $H_n(BS(1,2)) = \mathbb{Z}$ for n=0,1, and $H_n(BS(1,2)) = 0$ for n ≥ 2.
  - Thm 3.80: Let G and H be groups.
    (a) If G ≅ H, then $H_n(G) \cong H_n(H)$ for all n ≥ 0.
    (b) If there is an $n \in \mathbb{N}_0$ with $H_n(G) \ncong H_n(H)$, then G ≇ H.
  - Thm 3.82: For every group G and presentation $\mathcal{P}$ of G, the corresponding presentation complex X is the 2-skeleton of a CW complex X that is a K(G,1).
  - *Yet more examples of computing $H_n(G)$:*
    - Rmk: Let Y be a CW complex Y, let n be an index n ≥ 0, $\partial_n:C_n(Y) \to C_{n-1}(Y)$ be the boundary map, and let σ be an n-cell σ. In order to compute $\partial_n(1[\sigma])$, the n-cell σ is viewed

as an n-cell with an orientation (or direction) on the boundary of σ inherited from an orientation on the boundary sphere S^{n-1} used in gluing σ into Y.

- Prop 3.85: For the group $\mathbb{Z}^3$, there is a $K(\mathbb{Z}^3,1)$ consisting of the presentation complex for the presentation < a,b,c | [a,b] = [a,c] = [b,c] = 1 >, with one additional 3-cell. Moreover, $H_0(\mathbb{Z}^3) = \mathbb{Z}$, $H_1(\mathbb{Z}^3) = \mathbb{Z}^3$, $H_2(\mathbb{Z}^3) = \mathbb{Z}^3$, $H_3(\mathbb{Z}^3) = \mathbb{Z}$, and $H_n(\mathbb{Z}^3) = 0$ for n ≥ 4.
- Prop 3.86: Examples from Ex3.1. (To be filled in later!)

- Prop 3.90: If X is a contractible space, then $H_0(X) = \mathbb{Z}$ and $H_n(X) = 0$ for all n ≥ 1.
- Def 3.92: Let Y be a CW complex, and let J_0 be the set of 0-cells of Y. The **augmentation map** e: $C_0(Y) = \bigoplus_{\sigma \in J_0} \mathbb{Z}[\sigma] \to \mathbb{Z}$ is defined by e(1[σ]) := 1 for all σ ∈ J_0.
- Def 3.93: Let Y be a CW complex. The **augmented chain complex** associated to Y is the sequence
  $\cdots \to \tilde{C}_{n+1}(Y) \to \tilde{C}_n(Y) \to \tilde{C}_{n-1}(Y) \to \cdots \to \tilde{C}_1(Y) \to \tilde{C}_0(Y) \to \mathbb{Z} \to 0$,
  where the reduced chain groups and reduced boundary maps satisfy $\tilde{C}_n(Y) = C_n(Y)$ for all n ≥ 0 and $\tilde{\partial}_n = \partial_n$ for all n ≥ 1, and the reduced 0-th boundary map is the augmentation map $\tilde{\partial}_0$ = e: $C_0(Y) \to \mathbb{Z}$.
  The **n-th reduced homology group of Y** is the group $\mathbf{\tilde{H}_n(Y)} := \mathrm{Ker}(\tilde{\partial}_n)/\mathrm{Im}(\tilde{\partial}_{n+1})$.
- Prop 3.94: If X is a contractible space, then $\tilde{H}_n(X) = 0$ for all n ≥ 0.
- Thm 3.96: Let G = < A | R > be a group with a presentation, and let $\mathcal{C}$ be the presentation complex of G.
  The augmentation map $\tilde{\partial}_0$ = e: $C_0(\mathcal{C}) \to \mathbb{Z}G$ satisfies e(g) = 1 for all g ∈ G.
  The reduced homology groups of $\mathcal{C}$ satisfy $\tilde{H}_n(\mathcal{C}) \cong 0$ for all n ≠ 2.

- *Section B: Algebraic view*

  - (i) *Resolutions*
    - Rmk: All modules in this chapter are left modules unless otherwise noted.
    - Def 3.100: Let R be a ring. A **free R-module** is a direct sum of copies of R; that is, ⊕ R.
    - Def 3.102: Let R be a ring. A sequence $\cdots \to M_{n+1} \to M_n \to M_{n-1} \to \cdots$ of R-modules M_n and R-module homomorphisms $d_n:M_n \to M_{n-1}$ is a (algebraic) **chain complex** if $d_n \circ d_{n+1} = 0$ [equivalently, $\mathrm{Im}(d_n) \subseteq \mathrm{Ker}(d_n)$] for all n.
    - Def 3.104: Let R be a ring, let M,N,P be R-modules, and let h:M → N and j:N → P be R-module homomorphisms. Then M → N → P is **exact at N** if Ker(j) = Im(h).
      A sequence $\cdots \to M_{n+1} \to M_n \to M_{n-1} \to \cdots$ of R-modules M_n and R-module homomorphisms $d_n:M_n \to M_{n-1}$ is an **exact sequence** if it is exact at M_n for all n.
    - Def 3.105: Let R be a ring and let M be an R-module. A **free resolution** of M over R is an exact sequence $\cdots \to F_{n+1} \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$, where each F_n is a free R-module.
      This free resolution is also denoted **F. → M → 0**.
    - Thm 3.106: If R is a ring and M is an R-module, then
      (a) M is a quotient of a free R-module.
      (b) There exists a free resolution of M over R.

(c) If $F_n \to \cdots \to F_0 \to M \to 0$ is an exact sequence of free R-modules and R-module homomorphisms, then this sequence extends to a free resolution of M over R.

- Def 3.108: If G is a group, then the **trivial action** of $\mathbb{Z}G$ on $\mathbb{Z}$ is defined by $(\Sigma\, n_g g)\, m := (\Sigma\, n_g)\, m$ whenever each $g \in G$ and each $n_g, m \in \mathbb{Z}$.
- Lemma 3.109: If G is a group, then $M = \mathbb{Z}$ with the trivial $\mathbb{Z}G$-action is a $\mathbb{Z}G$-module.
- Cor 3.111: Every group G has a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.
- Thm 3.112: Let G be a group, let Y be a K(G,1) space, and let $\tilde{Y}$ be its universal covering space.
  (a) The augmented (CW, simplicial, or singular) chain complex for $\tilde{Y}$ is a free resolution of $\mathbb{Z}$ (with the trivial G action) over $\mathbb{Z}G$.
  (b) The (CW, simplicial, or singular) chain complex for Y is a (algebraic) chain complex.
- *Coinvariants:*
  - Def 3.115: Let G be a group and let M be a $\mathbb{Z}G$-module. Then M is also called a **G-module**. The **action of G on M** is the action $g \cdot m := gm$ for all $g \in G$ and $m \in M$.
    The group G **acts trivially** on M if $gm = m$ for all $g \in G$ and $m \in M$.
  - Def 3.117: Let M be a $\mathbb{Z}G$-module. The **group of coinvariants** of M is the $\mathbb{Z}$-module (that is, the abelian quotient group)
    $\mathbf{M_G} := M / < \{gm-m \mid g \in G, m \in M\} >$.
  - Prop 3.118: Let M be a $\mathbb{Z}G$-module.
    (a) M_G is the largest quotient of M on which G acts trivially.
    (b) $M_G \cong M/IM$, where $I := \mathrm{Ker}(e{:}\mathbb{Z}G \to \mathbb{Z})$ is the kernel of the augmentation map.
    [(c) $M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$, where $\mathbb{Z}$ is a right $\mathbb{Z}G$-module with the trivial action.]
  - Prop 3.120: Let $h{:}M \to N$ be a homomorphism of $\mathbb{Z}G$-modules. Then h induces a $\mathbb{Z}$-module homomorphism $\overline{\mathbf{h}}{:}M_G \to N_G$ defined by $\overline{h}(m + IM) := h(m) + IN$ for all $m \in M$.
    That is, $(\ )_G$: ZG-Mod → AbGp is a functor.
  - Def 3.121: Let G be a group. If $F_\bullet \to \mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then the **induced coinvariant complex** is the sequence $\cdots \to (F_{n+1})_G \to (F_n)_G \to (F_{n-1})_G \to \cdots \to (F_1)_G \to (F_0)_G \to 0$ of coinvariant groups and induced homomorphisms $\overline{\boldsymbol{\partial}}_{\mathbf{n}}: (F_n)_G \to (F_{n-1})_G$, also denoted $\mathbf{(F_\bullet)_G \to 0}$.
  - Prop 3.123: If $F_\bullet \to \mathbb{Z} \to 0$ is a (free) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then the induced coinvariant complex is a complex of (free) $\mathbb{Z}$-modules; that is, $\overline{\partial}_n \circ \overline{\partial}_{n+1} = 0$ for all n.
- Thm 3.128: Let G be a group, let Y be a K(G,1) space, and let $\tilde{Y}$ be its universal covering space. The (nonaugmented) chain complex for Y is the complex induced by the coinvariants of the augmented chain complex for $\tilde{Y}$.

◦ (ii) *The algebraic definition of $H_n(G)$*
  - Def 3.130: Let G be a group, let $F_\bullet \to \mathbb{Z} \to 0$ be any free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, and let $(F_\bullet)_G \to 0$ be the induced coinvariant complex. The **n-th homology group of G** is $\mathbf{H_n(G)} := H_n((F_\bullet)_G) = \mathrm{Ker}(\overline{\partial}_n)/\mathrm{Im}(\overline{\partial}_{n+1})$.
  - Thm 3.131: Let G be a group. Then $H_n(G)$ is well-defined for all $n \geq 0$.
  - Thm 3.133: Let G be a group and let $n \geq 0$. The topological and algebraic definitions of $H_n(G)$ define the same group.
  - **Procedure to compute $H_n(G)$ with the algebraic view:**

- **Step 1:** Build a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ using the algorithm in the proof of Thm 3.106. In particular, start with the augmented chain complex
  $\bigoplus_{r \in R} \mathbb{Z}G[r] \to \bigoplus_{a \in A} \mathbb{Z}G[a] \to \mathbb{Z}G \to \mathbb{Z} \to 0$
  of the Cayley complex $\mathcal{C}$ for a presentation < A | R > of G, and extend this to a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.
- **Step 2:** Compute the induced coinvariant complex $(F_G)_{\bullet} \to 0$. In particular, if the free resolution in dimensions up to 2 is the chain complex of the Cayley complex, then the induced coinvariant complex in these dimensions is the complex
  $\bigoplus_{r \in R} \mathbb{Z}[r] \to \bigoplus_{a \in A} \mathbb{Z}[a] \to \mathbb{Z} \to 0$
  of the presentation complex X for the presentation < A | R > of G.
- **Step 3:** Compute the homology groups of the coinvariant complex $(F_G)_{\bullet} \to 0$; these are the homology groups of G.

- *Examples*:
  - Prop 3.140: Let G be a group with a presentation < A | R >. Let $F_0 := \mathbb{Z}G$ and let $\partial_0: F_0 \to \mathbb{Z}$ be the augmentation map. Let $F_1 := \bigoplus_{a \in A} \mathbb{Z}G[a]$ and let $\partial_1: F_1 \to F_0$ be the $\mathbb{Z}G$-module homomorphism defined by $\partial_1(1[a]) := a-1$ for all $a \in A$. Let $F_2 := \bigoplus_{r \in R} \mathbb{Z}G[r]$ and let $\partial_2: F_2 \to F_1$ be the $\mathbb{Z}G$-module homomorphism defined by $\partial_2(1[r]) := \Sigma_{i=1}^{k} e_i p_i[b_i]$, where $p_i =_G b_1^{e_1} \cdots b_i^{e_i}$ if $e_i = 1$ and $p_i =_G b_1^{e_1} \cdots b_{i+1}^{e_{i+1}}$ if $e_i = -1$, for all $r = b_1^{e_1} \cdots b_k^{e_k} \in R$ (where each $b_i \in A$ and each $e_i \in \{\pm 1\}$). Then the sequence $F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$ with these boundary maps is the beginning of a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.
  - Thm 3.142: Let G be a group. Then $H_0(G) = \mathbb{Z}$ and $H_1(G) = G_{ab}$.
  - Thm 3.144: Let $G = \mathbb{Z}/n\mathbb{Z}$ for any $n \geq 2$. Then $H_0(G) = \mathbb{Z}$, $H_i(G) = \mathbb{Z}/n\mathbb{Z}$ for all odd i, and $H_i(G) = 0$ for all even $i \geq 2$.

- *Section C: Measuring "finiteness" of groups*

  - (i) *Definitions*:
    - Def 3.150: The **geometric dimension** of a group G, denoted **gd(G)**, is the minimal dimension of a K(G,1).
    - Def 3.152: Let $F_{\bullet} \to \mathbb{Z} \to 0$ be a free resolution. If $F_n \neq 0$ but $F_k = 0$ for all $k \geq n+1$, then the **length** of the resolution is n. If $F_n \neq 0$ for arbitrarily large n, then the resolution has infinite length.
    - Def 3.153: The **cohomological dimension** of a group G, denoted **cd(G)**, is the minimal length of a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.
    - Thm 3.155: The geometric dimension and cohomological dimension are isomorphism invariants.
    - Def 3.160: Let G be a group.
      The group G satisfies the finiteness condition $\mathbf{F_n}$ if there is a K(G,1) whose n-skeleton has finitely many cells.
      The group G has type $\mathbf{F_\infty}$ if there is a K(G,1) with finitely many cells in every dimension.
      The group G has **type F** if there is a K(G,1) with finitely many cells.

- Def 3.162: Let G be a group.
  The group G satisfies the finiteness condition **FP_n** if there is a free resolution $F_\bullet \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ over $\mathbb{Z}G$ such that F_i is a finitely generated $\mathbb{Z}G$-module for all $i \leq n$.
  The group G has type **FP_∞** if there is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ such that every $\mathbb{Z}G$-module in the resolution is finitely generated.
  The group G has **type FF** if there is a finite length free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ by finitely generated $\mathbb{Z}G$-modules.
- Thm 3.165: F_n, FP_n, F_∞, FP_∞, type F, and type FF are isomorphism invariants.

◦ (ii) *Geometric and cohomological dimension and "freeness"*:
  - Thm 3.170: If G is a group, then gd(G) = 0 if and only if G = 1.
  - Thm 3.171: If G is a group, then TFAE: (a) G is free and nontrivial. (b) gd(G) = 1. (c) cd(G) = 1. (Stallings, Swan)
  - Rmk: The invariants gd and cd measure "freeness" of a group.

◦ (iii) *"Finiteness" properties*:
  - Thm 3.178: (a) For all $n \geq 0$, F_n implies FP_n. (b) F_∞ implies FP_∞. (c) Type F implies type FF.
  - Thm 3.179: Every group has type F_0 and type FP_0.
  - Thm 3.180: If G is a group, then TFAE: (a) G is finitely generated. (b) G has type F_1. (c) G has type FP_1.
  - Thm 3.181: Let G be a group. (a) G has type F_2 if and only if G is finitely presented. (b) If G is finitely presented, then G has type FP_2.
  - Thm 3.182: (Bestvina, Brady) FP_2 does not imply F_2 in general.
  - Thm 3.183 (Stallings, Bieri) Let $n \geq 3$ and let G be the kernel of the homomorphism $h: F_2 \times \cdots \times F_2 \to \mathbb{Z}$ (where there are n factors isomorphic to the free group of rank 2) defined by mapping each generator of each free group to the (same) generator of $\mathbb{Z}$. Then G has type FP_n but not type FP_{n+1}.
  - Rmk: The invariants F_n, FP_n, F_∞, FP_∞, type F, and type FF measure "finiteness" of a group.
  - Def 3.184: (a) The invariants F_n, F_∞, and type F are called **homotopical finiteness conditions**. (b) The invariants FP_n, FP_∞, and type FF are called **homological finiteness conditions**.

◦ (iv) *Computing examples*:
  - Prop 3.190: For every $n \geq 2$, the group $\mathbb{Z}/n\mathbb{Z}$ has type FP_∞ but not type FF.
  - Prop 3.191: Let $G = \langle a,b,c,d \mid [a,b][c,d] = 1 \rangle$ be the fundamental group of a surface of genus 2. Then G has type F_∞, FP_∞, F and FF, and gd(G) = cd(G) = 2. Therefore G is not a free group.
  - Thm 3.193: Let G be a group with a presentation $\langle A \mid R \rangle$ satisfying the property that every relator $r \in R$ has even length. If $w \in (A \cup A^{-1})^*$ and the length of w is odd, then $w \neq_G 1$.

◦ (v) *Finiteness properties for finitely presented groups*:
  - Thm 3.194: For any group G, $cd(G) \leq gd(G)$.
  - Rmk: Thm 3.128 says that every K(G,1) yields a (CW) chain complex that is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. We used this to prove Thms 3.178 and 3.194.
  - Thm 3.196: (Eilenberg, Ganea) If G is a finitely presented group and $F_\bullet \to \mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then this resolution is the (CW) chain complex of the universal cover of a K(G,1).

- Cor 3.197: If G is a finitely presented group, then:
  (a) gd(G) = cd(G).
  (b) For all $n \geq 0$, G is of type F_n iff G is of type FP_n.
  (c) G is F_∞ iff G is FP_∞
  (d) G has type F iff G has type FF.
- (vi) *Algorithms and finiteness properties:*
  - Rmk: Resolutions of critical pairs for a rewriting system define maps $S^2 \to \mathcal{C}$, where $\mathcal{C}$ is the Cayley complex for the associated presentation.
  - Rmk: When extending the chain complex for the Cayley complex of G to a free resolution $F_\bullet \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ over $\mathbb{Z}G$, the module F_2 is generated by the relations among the generators, F_3 is generated by the relations among the relations, F_4 is generated by the relations among the (relations among the relations), etc.
  - Thm 3.200: (Anick, Brown, Farkas, Groves, Squier, ...): Let G be a group. If G has a finite convergent rewriting system, then G has type FP_∞.
  - Cor 3.202: If G has a finite convergent rewriting system, then there is an algorithm to compute $H_n(G)$ for all $n \geq 0$.

- *Section D: Homology and group constructions*

  - Thm 3.210: (**Homomorphisms**:) If $f:G \to J$ is a group homomorphism, then for all $n \geq 0$ there is an induced homomorphism $f_* = H_n(f)$: $H_n(G) \to H_n(J)$. That is, H_n: Gp $\to$ AbGp is a functor.
  - Thm 3.213: (**Subgroups**:) Let G be a group and let H be a subgroup of G.
    (a) If Y is a K(G,1), $\tilde{Y}$ is the universal covering space of Y, and Z := $\tilde{Y}$/H is the quotient of $\tilde{Y}$ by the action of H, then Z is a K(H,1).
    (b) If $F_\bullet \to \mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then this sequence is also a free resolution of $\mathbb{Z}$ over $\mathbb{Z}H$.
    (c) gd(H) ≤ gd(G) and cd(H) ≤ cd(H).
  - Cor 3.214: (**Torsion**:) If G contains a torsion element, then: (a) gd(G) = ∞ and G does not have type F. (b) cd(G) = ∞ and G does not have type FF.
  - Cor 3.215: (**Finite index subgroups**:) Let J be a finite index subgroup of a group G.
    (a) If G has type F_n, then J also has type F_n.
    (b) If G has type FP_n, then J also has type FP_n.
  - Thm 3.220: (**Free products**:) Let G_1 and G_2 be groups, and let G := $G_1 * G_2$ be their free product.
    (a) For each i ∈ {1,2}, let K_i be a $K(G_i)$ and let v_i be a vertex in K_i. Let Z := $(K_1 \coprod K_2)/\sim$, where ~ is the smallest equivalence relation satisfying $v_1 \sim v_2$. Then Z is a K(G,1).
    (b) $H_n(G) \cong H_n(G_1) \oplus H_n(G_2)$ for all $n \geq 1$.
    (c) $gd(G_1 * G_2) \leq \min\{gd(G_1), gd(G_2)\}$.
  - Thm 3.221: (**Amalgamated products**:) Let G_1,G_2, and J be groups, and let $i:J \to G_1$ and $k:J \to G_2$ be injective group homomorphisms. Let G := $G_1 *_J G_2$ be the associated amalgamated product.
    (a) For each i ∈ {1,2}, let K_i be a $K(G_i)$ and let $p_i:\tilde{K}_i \to K_i$ be its universal covering space. Let K_J be

a $K(J,1)$. Then there exists a homotopy equivalence $f_i \colon \tilde{K}_i/J \to \tilde{K}_J$.

(b) Let $Z := (K_1 \coprod (K_J \times I) \coprod K_2)/\sim$, where $\sim$ is the smallest equivalence relation satisfying $(x,0) \sim p_1(f_1(x))$ and $(x,1) \sim p_2(f_2(x))$ for all $x \in K_J$. Then Z is a $K(G,1)$.

(c) The homology of the amalgamated product satisfies a long exact sequence

$\cdots \to H_n(J) \to H_n(G_1) \oplus H_n(G_2) \to H_n(G) \to H_{n-1}(J) \cdots$.