Math 911 Spring 2023 Exercises

Problem set 1:

- **Ex1.1.** For each of the following presentations of a group G, find (and prove) a sequence of Tietze transformations to showing that the group is isomorphic to a familiar group with a simpler presentation. Then determine whether your answer can be used to decide whether the group G is an infinite group.
  (a) G = $< a, b \mid bab^{-1} = a^2, aba = b^2 >$.
  (b) G = $< x, y \mid [y,xyx^{-2}] = 1, [y,x^2yx^{-3}] = 1, [x,y] = 1 >$. (This group G is the abelianization of "Thompson's group F", which we'll discuss more in Chapter 2.)
- **Ex1.2.** In each of the following, find the shortlex CRS for the group with the generating set and the total order given. Then use your CRS to determine whether the group is in finite. (Hint: How many irreducible words are there?)
  (a) $G := < a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = 1, (ab)^4 = 1, (ac)^3 = 1, (ad)^4 = 1 >$, with the generating set A = {a,b,c}, under each of the following orders on A:
  (a-i) a < b < c < d.
  (a-ii) d < a < b < c.
  (a-iii) d < b < c < a.
  (b) $H := < a,b,c \mid a^2 = b^2 = c^2 = 1, (ab)^2 = 1, (ac)^3 = 1, (bc)^5 = 1 >$, with the generating set B = {a,b,c} and order a < b < c on B.
- **Ex1.3.** For each of the groups below:
  (a) Find a CRS and a set of normal forms.
  (b) Draw the Cayley graph Γ (and Cayley complex $\mathcal{C}$) with respect to the given presentation, and in another color trace all of the paths in Γ that start at the identity vertex and are labeled by normal forms from your rewriting system.
  (c) When a word *w* over the generating set is given, use your rewriting system to determine whether $w =_G 1$ (where G is the respective group). If $w =_G 1$, draw a van Kampen diagram for *w*.
  - Raag examples:
    - $F_2 \times \mathbf{Z} = < x, y, z \mid [x,z] = 1, [y,z] = 1 >$
    - $\mathbf{Z}^2 * \mathbf{Z} = < x, y, z \mid [x,y] = 1 >$
  - Coxeter examples:
    - $Cox_{2,4,4} = < a, b, c \mid a^2 = b^2 = c^2 = 1, (ab)^2 = 1, (ac)^4 = 1, (bc)^4 = 1>$
      Word: $w = babcacabab$.
    - $Cox_{2,3,\infty} = < a, b, c \mid a^2 = b^2 = c^2 = 1, (ab)^2 = 1, (ac)^3 = 1 >$
  - The Heisenberg group $< x,y,x \mid [x,y] = z, [x,z] = 1 = [y,z] >$
    Word: $w = x^n y x^{-n} y^{-1} z^{-n}$.
  - $BS(1,4) = < a,t \mid tat^{-1} = a^4 >$

- 3-generator/3-relator examples, **with alphabet** $A = \{x^{\pm}, y^{\pm}, z^{\pm}\}$:
  - $G_1 = < x, y, z \mid [x,y] = 1, zxz^{-1} = xy^2, zyz^{-1} = y >$
  - $G_2 = < x, y, z \mid [x,y] = 1, zxz^{-1} = x^{-1}, zyz^{-1} = y^{-1} >$
  - $G_3 = < x, y, z \mid [x,y] = 1, zxz^{-1} = x^2, zyz^{-1} = y >$
  - $G_4 = < x, y, z \mid [x,y] = 1, z^2 = 1, zxz^{-1} = y >$
    Word: $w = x^{-2}zyzx$.

- **Ex1.4.** *Closure properties for the class (set) of finitely presented groups*: Let G be a group with a normal subgroup N and quotient Q := G/N. The group G is called an **extension of N by Q**.
  (a) Show that the class of f.p. groups *is* closed under taking extensions, by proving that if N and Q are finitely presented, then so is G. (Hint: It may be helpful to think about this using the concept of normal forms.)
  (b) Prove that the class of finitely presented groups *is not* closed under taking supergroups, by giving an example of a group G that is not finitely generated that contains a normal subgroup N that is finitely presented.
  (c) Prove that the class of finitely presented groups *is not* closed under taking subgroups, by using covering space theory to show that if $G = F_2$ is the free group of rank 2 and N = [G,G] is the commutator subgroup of G, then N is not finitely generated. (Hint: This is frequently a covering space theory homework from Math 872 - in fact it's part of problem 6 in Section 1.A of Hatcher's book. Consider the covering space of the graph $S^1 \vee S^1$ corresponding to N.)
  (d) Prove that the class of finitely presented groups *is* closed under taking finite index subgroups, by using covering space theory to show that if G is finitely presented and H is a finite index subgroup of G, then H is finitely presented. (Hint: This also is often a homework problem in Math 872: Start with a finite presentation complex X for G, and find the covering space $p: Y \to X$ corresponding to the subgroup H. What do you know about the complex Y?)
- **Ex1.5.** For each of the following classes (sets) of groups, determine whether the class is closed under taking direct products, free products, graph products, and/or abelianization.
  - Finite groups
  - Infinite groups
  - Finitely generated groups
  - Finitely presented groups
  - Free groups
  - Abelian groups
  - Right-angled Artin groups
  - Pick your own favorite class of groups!
- **Ex1.6.** Use the presentation of the Heisenberg group H in Ex1.3, and let $A = \{x,y,z\}^{\pm 1}$.
  - Write the word $w_n = x^n y^n x^{-n} y^{-2n} x^{-n} y^n x^n$ in the free group F(A) as a product of conjugates of relators in the presentation of H for $n \leq 2$. Use this to draw van Kampen diagrams for w_1 and w_2.
  - Create a conjecture (and prove it if you can) for (an upper bound on) the area of the word w_n for all n.

Problem Set 1 is due by Monday, February 20 at 10:00pm. (Ex1.2(a) was done in class on February 7.)
Subassignment to be handed in for grading: Ex1.1(a), Ex1.3 for the group G_4 (using the alphabet specified for the G_i groups), Ex1.4(a) and (b), Ex 1.5 for "infinite groups".

Problem set 2:

- **Ex2.A.1.** For each of the following groups and FPWA decompositions $G = H *_L K$, (a) draw the Bass-Serre tree for the FPWA, (b) draw the Cayley graph for (G,A), (c) discuss any connections (or lack of relationship) between the drawings in (a-b), and (d) write out a set of left greedy normal forms and a set of right greedy normal forms for G over $A^{\pm 1}$.
  (a) $G = SL_2(\mathbb{Z})$, $H = \mathbb{Z}/4 = < a \mid a^4 = 1 >$, $K = \mathbb{Z}/6 = < b \mid b^6 = 1 >$, $L = \mathbb{Z}/2 = < c \mid c^2 = 1 >$, $i:L \to H$ is defined by $i(c) = a^2$, $j:L \to K$ is defined by $j(c) = b^3$. A = {a,b}.
  (b) $G = \mathbb{Z}^2$, $H = \mathbb{Z}^2 = < a,b \mid ab = ba >$, $K = \mathbb{Z} = < c \mid \ >$, $L = \mathbb{Z} = < d \mid \ >$, $i:L \to H$ is defined by $i(d) = a$, $j:L \to K$ is defined by $j(d) = c$. A = {a,b,c}.
- **Ex2.A.2.** For each of the following groups and HNN extension decompositions $G = H *_L$, (a) draw the Bass-Serre tree for the HNN extension, (b) draw the Cayley graph for (G,A), (c) discuss any connections (or lack of relationship) between the drawings in (a-b), and (d) write out a set of left greedy normal forms and a set of right greedy normal forms for G over $A^{\pm 1}$.
  (a) G = BS(1,2), $H = \mathbb{Z} = < a \mid \ >$, $L = \mathbb{Z} = < c \mid \ >$, $i:L \to H$ is defined by $i(c) = a$, $j:L \to H$ is defined by $j(c) = a^2$. A = {a,t}.
  (b) $G = \mathbb{Z}^2$, $H = \mathbb{Z} = < a \mid \ >$, $L = \mathbb{Z} = < c \mid \ >$, $i:L \to H$ is defined by $i(c) = a$, $j:L \to H$ is defined by $j(c) = a^2$. A = {a,t}.
- **Ex2.A.3.** Prove the following closure results for finite presentability.
  (a) Prove that if $H *_L K$ is an amalgamated product of finitely presented groups H and K, then $H *_L K$ is finitely presented iff L is finitely generated.
  (b) Prove that if $H *_L$ is an HNN extension of a finitely presented group H, then $H *_L$ is finitely presented iff L is finitely generated.
- **Ex2.A.4.** Let Λ be a finite connected simple graph and let GΛ be the corresponding right-angled Artin group. Let v be a vertex of Λ and let Ψ be the subgraph of Λ whose vertex set is $\{w \in V(\Lambda) \mid \exists$ an edge from v to w in $\Lambda\}$ and whose edge set is all of the edges between these vertices in Λ; that is, Ψ is an *induced* subgraph. Let Φ be the induced subgraph with vertex set V(Λ) - {v}. Show that GΛ is isomorphic to an amalgamated product $(Z \times G\Psi) *_{G\Psi} G\Phi$.
- **Ex2.A.5. Degrees of vertices in Bass-Serre trees:**
  Let H and L be groups, and let $i:L \to H$ and $j:L \to H$ be injective homomorphisms. Let T be the **directed Bass-Serre tree** for the HNN extension $G := H *_L$; that is, T is the directed graph with vertex set $V(T) := \{gH \mid g \in G\}$, and directed edge set E(T) := {directed edge labeled gj(L) from the vertex gH to the vertex gtH | $g \in G$}.
  (a-i) Show that whenever $g \in G$ and $m \in L$, then $gH = gj(m)H$, $gj(L) = gj(m)j(L)$, and $gtH = gj(m)tH$,

and hence the directed edge labeled gj(L) and the directed edge labeled gj(m)j(L) are the same edge.

(a-ii) Show that whenever $g \in G$ and $h \in H$ with $h \notin j(L)$, then $gH = ghH$, $gj(L) \neq ghj(L)$, and $gtH \neq ghtH$, and hence the directed edges labeled gj(L) and ghj(L) have the same initial vertex gH but distinct terminal vertices, and are not the same edge. (*Hint:* Using proof by contradiction, suppose that $gtH = ghtH$, and hence $ht =_G th'$ for some h' in H. Write out left greedy normal forms for both ht and th' to get a contradiction.)

(a-iii) Show that there is a bijection from the set of directed edges in T with initial vertex gH to the set of cosets H/j(L) (that is, cosets of the form hj(L) for $h \in H$). (*Hint:* Use a left greedy normal form for g in your proof.)

(b-i) Show that whenever $g \in G$ and $m \in L$, then $gt^{-1}H = gi(m)t^{-1}H$, $gt^{-1}j(L) = gi(m)t^{-1}j(L)$, and $gH = gi(m)t^{-1}tH$, and hence the directed edge labeled $gt^{-1}j(L)$ and the directed edge labeled $gi(m)t^{-1}j(L)$ are the same edge.

(b-ii) Show that whenever $g \in G$ and $h \in H$ with $h \notin i(L)$, then $gt^{-1}H \neq ght^{-1}H$, $gt^{-1}j(L) \neq ght^{-1}j(L)$, and $gH = gtt^{-1}H = ghtt^{-1}H$, and hence the directed edges labeled $gt^{-1}j(L)$ and $ght^{-1}j(L)$ have the same terminal vertex gH but distinct initial vertices, and are not the same edge. (*Hint:* Using proof by contradiction, suppose that $gt^{-1}H = ght^{-1}H$, and hence $ht^{-1} =_G t^{-1}h'$ for some h' in H. Write out left greedy normal forms for both ht^{-1} and $t^{-1}h'$ to get a contradiction.)

(b-iii) Show that there is a bijection from the set of directed edges in T with terminal vertex gH to the set of cosets H/i(L) (that is, cosets of the form hi(L) for $h \in H$). (*Hint:* Use a left greedy normal form for g in your proof.)

(c) Use the information in parts (a-iii) and (b-iii) to draw the directed Bass-Serre tree for the HNN extension $G = BS(2,3) = \langle a,t \mid ta^2t^{-1} = a^3 \rangle$ where $H = \mathbb{Z} = \langle a \mid \;\rangle$, $L = \mathbb{Z} = \langle c \mid \;\rangle$, and $i,j: L \to H$ are the monomorphisms defined by $i(c) = a^2$ and $j(c) = a^3$.

(d) Let T' be the directed graph with vertex set $V(T') := \{gH \mid g \in G\}$, and directed edge set $E(T') := \{$directed edge labeled gi(L) from vertex $gt^{-1}H$ to vertex $gH \mid g \in G\}$. Show that there is a bijection $f:T \to T'$ that is the identity on vertices and maps edges to edges (preserving adjacency and direction).

- **Ex2.A.6. Hopfian groups**:
  A group G is called **Hopfian** if every surjective homomorphism $f: G \to G$ is an isomorphism.
  (a) Show that the group $G = \mathbb{Z} = \langle a \mid \;\rangle$ is Hopfian.
  (b) Let $G = BS(2,3) = \langle a,t \mid ta^2t^{-1} = a^3 \rangle$. Show that G is not Hopfian, by proving the following steps:
  (b-i) Show that there is a (well-defined) homomorphism $h:G \to G$ satisfying $h(a) = a^2$ and $h(t) = t$.
  (b-ii) Show that h is surjective by finding an element $g \in G$ satisfying $h(g) = a$.
  (b-iii) Write out the left greedy normal forms for the HNN extension G.
  (b-iv) Show that h is not an isomorphism by finding a nonempty left greedy normal form w representing a nonidentity element of G that is in the kernel of h.
- **Ex2.A.7. Embedding countable groups into 2-generated groups using HNN extensions:**
  Let H be any countable group. Prove that H is a subgroup of a group G generated by two elements of infinite order, by proving the following steps:

(a) Write H = < A | R > where A = {a_1, a_2, ...} is countable. Let J := H ∗ < c,d | > be the free product of H with a free group of rank 2. Let S be the subset of J defined by S := {c} ∪ {d^ncd^{-n}} and similarly let T := {d} ∪ {$a_nd^ncd^{-n}$} (where in each case the values of n are the indices of the generators of A). Let S' be the subgroup of J generated by S and let T' be the subgroup of J generated by T. Show that there is an isomorphism φ: S' → T' such that φ(c) = d and φ(d^ncd^{-n}) = $a_nd^ncd^{-n}$ for all n ≥ 1. (*Hint:* Use the normal forms for a free product to show that S' is isomorphic to the free group of S and T' is isomorphic to the free group on T.)

(b) Let G := J $*_{\varphi}$ be the HNN extension of J with respect to φ. Show that G is generated by c and t. (*Hint:* Use ToC Def 2.30 to write down a presentation for G. Then explain how to use (countably many) Tietze transformations to find a presentation for G with only the generators c and t.)

(c) Explain how you know that H is isomorphic to a subgroup of G, and that c and t are elements of infinite order in G. (You may reference any result from the Table of Contents in your explanation.)

(d) Show that if H is finitely presented, then G is finitely presented.

- **Ex2.A.8.** Prove Thm 2.50: Let $\mathcal{G}$ = (Λ, {G_v}, {H_e}, {h_e,h_e'}) be a graph of groups with at least one edge, and let G := $\pi_1(\mathcal{G})$.
  (a) If Λ is a tree, then there is a vertex u adjacent to only one edge f; let Λ' be the subgraph of Λ induced by the set of vertices Vert(Λ) - {u} (with the same vertex and edge groups). Show that G splits as an amalgamated product G ≅ π_1(Λ') $*_{H_f}$ G_u.
  (b) If Λ is not a tree, then there is an edge f satisfying the property that the graph Λ" obtained from Λ by removing f is connected. Show that G splits as an HNN extension G ≅ π_1(Λ) $*_{H_f}$.
  (c) Explain how (a) and (b) show that G acts simplicially and without inversion on a tree.

Problem Set 2 is due by Monday April 3 at 10:00pm. (Ex2.A.1, Ex2.A.2, and Ex2.A.4 were discussed in class on March 2 and 7.)
Subassignment to be handed in for grading: Ex2.A.5(c), Ex2.A.6, Ex2.A.7(b)-(d).

Problem set 3:

- **Ex2.B.1.** Let x and y be the generators of Thompson's group F from ToC Thm 2.121.
  (a) Compute Supp(xyx^{-1}).
  (b) Define z ∈ F by: z(t) = 2t for all t ∈ [0,1/8], z(t) = t + 1/8 for all t ∈ [1/8,1/4], z(t) = (1/2)t + 1/4 for all t ∈ [1/4,1/2], and z(t) = t for all t ∈ [1/2,1]. Compute Supp(zyz^{-1}).
  (c) Define w ∈ F by: w(t) = t for all t ∈ [0,1/2], w(t) = 2t -1/2 for all t ∈ [1/2,5/8], w(t) = (1/2)t + 7/16 for all t ∈ [5/8,7/8], and w(t) = t for all t ∈ [7/8,1]. Compute Supp(xwx^{-1}).
- **Ex2.B.2.** Let G be a group with a finite presentation $\mathcal{P}$ = < A | R >. The **area** of a van Kampen diagram Δ is **Area(Δ)** := the number of 2-cells in Δ. For any word w over $A^{\pm 1}$ with w $=_G$ 1, the **area** of the word w is **Area(w)** := min{Area(Δ) | Δ is a vKd for w}.
  In this problem, you are asked to find an upper bound for the areas of the words r_i in Thompson's group F. More precisely, let $\mathcal{P}$ be the presentation of Thompson's group F from ToC Thm 2.121;

that is, $\mathcal{P}$ = < x,y | $[y,xyx^{-2}]$, $[y,x^2yx^{-3}]$ >.

(a) Using the construction of ToC Prop 2.130, drawn a van Kampen diagram for the words $r_3 := y\, x^3yx^{-4}\, y^{-1}\, x^4y^{-1}x^{-3}$ and $r_4 := y\, x^4yx^{-5}\, y^{-1}\, x^5y^{-1}x^{-4}$.

(b) Define the sequence C by C(1) = 1, C(2) = 1, and C(i) = C(i-1) + 2C(i-2) + 2 for all i ≥ 3. Compute the first 7 numbers in this sequence. (Unfortunately, when you enter this list into the Online Encyclopedia of Integer Sequences, you'll find it's not there.) Show that $C(i) = (2/3)2^i - (2/3)(-1)^i - 1$ for all i.

(c) For each i ≥ 1 let r_i be the word $r_i := [y,x^iyx^{-(i+1)}]$. Use ToC Prop 2.130 (and/or the **pictorial proof of Prop 2.130 (https://canvas.unl.edu/courses/147968/files/14953925?wrap=1)** **(https://canvas.unl.edu/courses/147968/files/14953925/download?download_frd=1)** from class on 3/28/2023) and the result in (b) to explain how you know that Area(r_i) < 3^i for all i.

- **Ex2.B.3.** (a) Using the view of Thompson's group F as the group of reduced tree pair diagrams, use the group operation on tree pair diagrams to compute the reduced tree pair diagram for the following elements of Thompson's group F:
  (a-i) f := $xyx^{-1}y^{-1}$.
  (a-ii) g := x^4.
  (a-iii) gfg^{-1}.
  (b) Recall from a Desmos computation shown in class that Supp(f) and Supp(gfg^{-1}) are disjoint. Can you see that in your tree pair diagrams from parts (a-i) and (a-iii)?
- **Ex2.B.4.** Using the generators x,y of Thompson's group F from ToC Thm 2.121 and the elements z,w of Thompson's group F from Ex2.B.1, determine whether the following subgroups of F are solvable:
  (a) < y, xyx^{-1} >.
  (b) < z, xwx^{-1} >.
  (c) < x, w >.
- **Ex2.B.5.** Prove ToC Thm 2.166:
  (a) If H is a subgroup of a solvable group G, then H is solvable and DerLen(H) ≤ DerLen(G).
  (b) If Q is a quotient of a solvable group G, then Q is solvable and DerLen(Q) ≤ DerLen(G).
  (c) If N is a normal subgroup of a group G and both N and G/N are solvable, then G is solvable, and DerLen(G) ≤ DerLen(N) + DerLen(G/N).
- **Ex2.B.6.** Let A = <B | S> and H = <C | T> be groups, and let G := A ≀ H be the wreath product of A by H.
  (a) Write out a presentation for the group G.
  (b) If A = <a | a^2 = 1> ≅ $\mathbb{Z}/2$ and H = <c | c^2 = 1> ≅ $\mathbb{Z}/2$, write out a presentation for the wreath product G = A ≀ H, and use your presentation to show that G is isomorphic to a familiar group.
- **Ex2.B.7.** For each of the classes of groups in Ex1.5, determine whether the class is closed under taking wreath products.
- **Ex2.B.8.** Let x,y be the generators of Thompson's group F from ToC Thm 2.121 and let z,w,f,g be the elements of Thompson's group F from Ex2.B.1 and Ex2.B.3. In each of the following, determine whether the word v lies in the subgroup H of F.

(a) v = w and H = < y, z >.
(b) v = z and H = < f,g >.

- **Ex2.B.9.** Let G be a group. See ToC Defs 2.211,2.212 for the definitions of inner automorphism and center. In the following, prove ToC Prop 2.213:
  (a) Show that Z(G) is a normal subgroup of G.
  (b) Show that for any element $g \in G$, $\varphi_g = Id_G$ if and only if $g \in Z(G)$.
  (c) Show that $Inn(G) \cong G/Z(G)$.
- **Ex2.B.10.** Show that the center of Thompson's group F is the trivial group.
  (Hint: Suppose to the contrary that f is in Z(F) and $f \neq_F 1$. What does the fact that f commutes with the generator x tell you about the support of f? Using that information, what does the fact that f commutes with y tell you further?)
- **Ex2.B.11.** Let x be the generator of Thompson's group F in ToC Thm 2.121 and let $h:F \to PL_{2,ev}(\mathbb{R})$ be the isomorphism from Thm 2.206. For all t in each of the following intervals, compute (h(x))(t).
  (a) $(-\infty,-1)$.
  (b) $(-1,0)$.
  (c) $(0,\infty)$.

Problem Set 3 is due by Monday May 1 at 10:00pm. (Ex2.B.1 was discussed in class in March; Ex2.B.8 and Ex2.B.11 were discussed in class in April 13-18.)
Subassignment to be handed in for grading: Ex2.B.2(c), Ex2.B.3(a-i),, Ex2.B.4(c), Ex2.B.7(for each of the properties infinite and abelian), Ex2.B.10

Problem set 4:

- **Ex2.C.1.** Let M be the finite state transducer (FST) defined by $M=(\{0,1\},\{a,b\},\delta,\varepsilon)$ where $\delta(a,0) = b$, $\delta(a,1) = b$, $\delta(b,0) = b$, $\delta(b,1) = a$, and $\varepsilon(a,0) = 1$, $\varepsilon(a,1) = 0$, $\varepsilon(b,0) = 0$, $\varepsilon(b,1) = 1$. Let G be the automaton group generated by M.
  (a) Determine what word is output by this FST when the start state is a and the word 00101110 is input.
  (b) Compute the product FST $M \cdot M$.
  (c) Compute the inverse FST M^{-1}.
  (d) Draw a portrait for the element $aba^{-1}b^{-1}$ of G.
  (e) Find a presentation for G, and find a familiar group that is isomorphic to G. (Prove your answers.)
- **Ex2.C.2.** Let M be the finite state transducer (FST) defined by $M=(\{0,1\},\{a,b\},\delta,\varepsilon)$ where $\delta(a,0) = b$, $\delta(a,1) = a$, $\delta(b,0) = b$, $\delta(b,1) = a$, and $\varepsilon(a,0) = 1$, $\varepsilon(a,1) = 0$, $\varepsilon(b,0) = 0$, $\varepsilon(b,1) = 1$. Let G be the automaton group generated by M.
  (a) Determine what word is output by this FST when the start state is a and the word 00101110 is input.

(b) Compute the product FST $M \cdot M$ and the inverse FST M^{-1}.

(c) Compute an FST that has a state that gives the automorphism of the tree T_A represented by the word aba^2. Then use your FST to determine whether $aba^2 =_G 1$.

- **Ex2.C.3.** Let A be a finite set and let G be a subgroup of $Aut(T_A)$ generated by a finite set Q.
  The **Schreier graph of level n** for G with respect to Q is the directed graph $\Sigma_n = \Sigma_n(G,Q)$ with vertex set $V(\Sigma_n) := A^n$ (that is, the set of words over A of length n), and directed edge set $E(\Sigma_n) :=$ {directed edge from w to q(w) | $w \in A^n$ and $q \in M$}.
  A **simple circuit** in a directed graph is an edge path following arrow directions starting and ending at the same vertex, but that otherwise does not repeat any vertices or edges.
  (a) Show that if $g \in Q$ and there is a simple circuit labeled by g^m in Σ_n for some $m,n \in \mathbb{N}$, then the order of the element g of G is at least m.
  (b) Show that the group $Stab_G(n)$ is represented by the set of all words x over $Q^{\pm 1}$ satisfying the property that for all $w \in A^n$, the word x labels a circuit in Σ_n at w.
  (c) For the FST M, group G, and generating set Q of Ex2.C.2:
  (c-i) Draw the Schreier graph Σ_1, and use this graph to determine representatives for all of the elements of the subgroup $Stab_G(1)$ of G.
  (c-ii) Draw the Schreier graphs Σ_n for n = 2,3,4. Use these to find lower bounds for the orders of the elements a and b of G.
  (c-iii) Challenge problem: Determine the order of the element a of G. (Hint: Drawing more Schreier graphs can be helpful. Note: This problem is worth trying for a short while, but not for more than an hour.)
- **Ex2.C.4.** Let A = {0,1,2}, let H := Perm(A) act on A by permutations, and let $G = \mathbb{Z}/2$.
  (a) What is the order of the permutational wreath product $J := G \wr^A H$?
  (b) Write out a generating set and a set of normal forms for J.
- **Ex2.C.5.** Give a solution to the Conjugacy Problem for the group G_4 of Ex1.3.
- **Ex2.C.6.** Give a solution to the Finiteness Problem for the following classes of groups. (In each case, part of your solution should be an explanation of how the group is input to the algorithm.)
  (a) The class of right-angled Coxeter groups.
  (b) The class of finitely generated subgroups of $PL_+(\mathbf{I})$.
  (c) The class of Baumslag-Solitar groups $BS(p,q) = \langle a,t \mid ta^pt^{-1} = a^q \rangle$.
- **Ex2.C.7.** Find a finite convergent rewriting system for the group $G = \langle a,b \mid aba = bab, bab^{-1} = a^2 \rangle$. Construct the irreducible word automaton (ToC Def 2.344) from your rewriting system, and use your automaton to determine whether the group G is finite.

Problem Set 4 is due by is due by Friday May 19 at 4:00pm. (Ex2.C.1(b,c,e) was discussed in class in April 25 - May 4; Ex2.C.6(a,c) was discussed in class May 9.)
There is no subassignment to be handed in for grading.