Math 911 Fall 2023 Exercises

Problem set 1:

- **Ex1.1.** For each of the following presentations of a group G, find (and prove) a sequence of Tietze transformations to show that the group is isomorphic to a familiar group with a simpler presentation.
  (a) $G = < a, b \mid bab^{-1} = a^2, aba^{-1} = b >$.
  (b) $G = < x, y, z \mid [x,z] = z^{-1} >$.
- **Ex1.2.** For each of the groups below:
  (a) Find a CRS and a set of normal forms.
  (b) Draw the Cayley graph Γ (and Cayley complex $\mathcal{C}$) with respect to the given presentation, and in another color trace all of the paths in Γ that start at the identity vertex and are labeled by normal forms from your rewriting system.
  (c) When a word w over the generating set is given, use your rewriting system to determine whether $w =_G 1$ (where G is the respective group).
  - Raag examples:
    - $F_2 \times \mathbf{Z} = < x, y, z \mid [x,z] = 1, [y,z] = 1 >$
    - $\mathbf{Z}^2 * \mathbf{Z} = < x, y, z \mid [x,y] = 1 >$
  - Coxeter example:
    - $Cox_{3,3,3} = < a, b, c \mid a^2 = b^2 = c^2 = 1, (ab)^3 = 1, (ac)^3 = 1, (bc)^3 = 1>$
      Word: $w = babcacabab$.
  - 3-generator/3-relator examples, **with alphabet** $A = \{x^\pm, y^\pm, z^\pm\}$:
    - The Heisenberg group $H = < x,y,z \mid [x,y] = z, [x,z] = 1, [y,z] = 1 >$
      Word: $w = x^n y x^{-n} y^{-1} z^{-n}$.
    - $G_1 = < x, y, z \mid [x,y] = 1, zxz^{-1} = xy^2, zyz^{-1} = y >$
    - $G_2 = < x, y, z \mid [x,y] = 1, zxz^{-1} = x^{-1}, zyz^{-1} = y^{-1} >$
    - $G_3 = < x, y, z \mid [x,y] = 1, zxz^{-1} = x^2, zyz^{-1} = y >$
    - $G_4 = < x, y, z \mid [x,y] = 1, z^2 = 1, zxz^{-1} = y >$
      Word: $w = x^{-2}zyzx$.
  - $BS(1,4) = < a,t \mid tat^{-1} = a^4 >$
- **Ex1.3.** For each of the groups H, G_1, G_2, G_3, G_4 in Ex1.2., determine whether or not the group is a semidirect product of < x,y > by < z >.
- **Ex1.4.** *Closure properties*:
  (a) Show that the class of finitely presented groups *is* closed under taking extensions. (Hint: It may be helpful to think about this using the concept of normal forms.)
  (b) Prove that the class of finitely generated groups *is not* closed under taking subgroups, by using covering space theory to show that if $G = F_2$ is the free group of rank 2 and $N = [G,G]$ is the

commutator subgroup of G, then N is not finitely generated. (Hint: This is frequently a covering space theory homework from Math 872 - in fact it's part of problem 6 in Section 1.A of Hatcher's book. Consider the covering space of the graph $S^1 \vee S^1$ corresponding to N.)
(c) Prove that the class of finitely presented groups *is* closed under taking finite index subgroups, by using covering space theory to show that if G is finitely presented and H is a finite index subgroup of G, then H is finitely presented. (Hint: This also is often a homework problem in Math 872: Start with a finite presentation complex X for G, and find the covering space $p:Y \to X$ corresponding to the subgroup H. What do you know about the complex Y?)
(d) Determine whether the class of groups with finite convergent rewriting systems *is or is not* closed under taking quotient groups.

- **Ex1.5.** For each of the following classes (sets) of groups, determine whether the class is closed under taking extensions, semidirect products, direct products, graph products, and/or abelianization.
  - (a) Finite groups
  - (b) Infinite groups
  - (c) Abelian groups
  - (d) Finitely generated groups
  - (e) Finitely presented groups
  - (f) Pick your own favorite class of groups!
- **Ex1.6.** Show that a group G is a semidirect product of N by H if and only if G is a split extension of N by H.

Problem Set 1 is due by Tuesday September 19 at 10:00pm.
Subassignment to be handed in for grading: Ex1.1(a), Ex1.2 for the group G_1 (using the alphabet specified for the G_i groups), Ex1.3 for the group G_1, Ex1.4(d), Ex 1.5(a,b,c,f) for "extensions", Ex1.6.

Problem set 2:

- **Ex1.7.** Let P be an isomorphism invariant group property. For each of the following properties P, decide whether (i) virtually P = P, (ii) residually P = P, (iii) locally P = P, and/or (iv) poly P = P.
  (a) Finitely generated
  (b) Torsion-free
- **Ex1.8.** Let P be an isomorphism invariant group property. Prove the following.
  (a) If P is a group property that is inherited by subgroups, then every subgroup of a residually-P group is residually-P.

(b) If P is a group property that is inherited by subgroups, then a group G is locally-P if and only if every finitely generated subgroup of G is P.

- **Ex1.9.** For each of the following, prove or disprove that the property is preserved by taking direct products:
  (a) Virtually abelian
  (b) Residually finite
  (c) Locally finite
  (d) Polycyclic
- **Ex2.1.**
  (a) Show that for all integers m,n,p,q, the group $G = \langle x,y,z \mid [x,y] = 1, zxz^{-1} = x^m y^n, zyz^{-1} = x^p y^q \rangle$ is solvable.
  (b) For each of the groups G_1, G_2, G_3 in Ex1.2, determine whether or not the group is also polycyclic or nilpotent. (Note: You must prove your answer directly, and may not use the theorems from class on ascending HNN extensions.)
  (c) Using the software package GAP (Groups, Algorithms, Programming), check your answers in parts (a-b) for the groups in (b) using the commands IsSolvableGroup, IsPolycyclicGroup, and IsNilpotentGroup, (or other related commands); see the GAP reference manual at **https://docs.gap-system.org/doc/ref/chap0_mj.html** **(https://docs.gap-system.org/doc/ref/chap0_mj.html)** and the GAP tutorial at **https://docs.gap-system.org/doc/tut/chap0_mj.html** **(https://docs.gap-system.org/doc/tut/chap0_mj.html)** for details.
- **Ex2.2.** A group G is **supersolvable** if G is poly-normal-cyclic; that is, if there is a sequence $1 = G_0 \lhd G_1 \lhd \dots \lhd G_n = G$ in which each quotient G_{i+1} / G_i is cyclic and each group G_i is normal in G. [This rather unfortunate name is used throughout the literature!]
  (a) Show that the class of supersolvable groups is closed under taking subgroups.
  (b) Show that the class of supersolvable groups is not closed under taking extensions.
  (Hint for counterexample for (b): Let $X := \langle a,b \mid a^2 = b^2 = (ab)^4 = 1 \rangle = D_8$, $A := \langle c,d \mid cd = dc, c^3 = d^3 = 1 \rangle = \mathbf{Z}_3 \times \mathbf{Z}_3$, and let $\varphi : X \to \mathrm{Aut}(A)$ be the homomorphism defined by $\varphi(a)(c) = d$, $\varphi(a)(d) = c$, $\varphi(b)(c)=c^{-1}$, and $\varphi(b)(d) = d$. Let G be the semidirect product $A \rtimes_\varphi X$. Show that A and X are supersolvable. To show that G is not supersolvable, since G is finite, there are only finitely many possible cyclic series to consider.)

Problem Set 2 is due by Friday October 6 at 10:00pm.
Subassignment to be handed in for grading: Ex1.7(b), Ex1.9(b,d) [only a brief justification, not necessarily a full proof, is needed in this problem], Ex2.1(a).
(Ex2.1(b) for both "polycyclic" and "nilpotent" will either be due at a later date or done in class.)

Problem set 3:

- **Ex2.3.** Show that if $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ is a central-normal series for a group G, then $\gamma_{n-i+1}(G) \leq G_i \leq \zeta_i(G)$ for all $0 \leq i \leq n$.
- **Ex2.4.** Prove that the class of nilpotent groups is closed under subgroups, quotients, and finite direct products.
- **Ex2.5.** Suppose that G is a finitely generated group, and for each natural number *i* let γ_i be the *i*th group in the lower central series for G.
  (a) Show that for all natural numbers *n* and for all $0 \leq i \leq n$ the subgroup γ_i/γ_n of G/γ_n is finitely generated.
  (Hint: Use the commutator identities [a,bc]=[a,b][a,c][[c,a],b] and [ab,c]=[b,c][[c,b],a][a,c] (or any other commutator identities you might find!).)
  (b) Show that if G is a finitely generated nilpotent group, then γ_i is a finitely generated subgroup for all *i*.
- **Ex2.6.** Let G be a group. Show that G has the property max iff there is no infinite strictly ascending chain $G_1 \lneqq G_2 \lneqq \ldots$ of subgroups of G.
- **Ex2.7.** See Ex2.2 for the definition of supersolvable group.
  (a) Find (and prove) an example of a finitely generated group that is supersolvable but not nilpotent.
  (b) Find (and prove) an example of a polycyclic group that is not supersolvable.
- **Ex2.8.** Let $H = F_2$ be the free group on 2 generators, let $\varphi : H \to H$ be a monomorphism, and let $G := H *_\varphi = \langle a, b, t \mid tat^{-1} = \varphi(a), tbt^{-1} = \varphi(b) \rangle$ be the associated ascending HNN extension.
  (a) Show that if φ is the identity map, then the distortion function δ_H^G is linear.
  (b) Show that if φ is an isomorphism, then $H \lhd G$ and G is a semidirect product of H with $\mathbb{Z}$.
  (c) Show that φ induces a homomorphism $\varphi' : \mathbb{Z}^2 \to \mathbb{Z}^2$.
  (d) Suppose that φ is an isomorphism and the induced map φ' has an eigenvalue with absolute value > 1. Determine whether or not the distortion function δ_H^G must be at least exponential.
- **Ex2.9.** For each of the groups G_1, G_2, G_3 in Ex1.2, answer the following.
  (a) Find as many lower and upper bounds as you can for the distortion of the subgroup $\langle x,y \rangle$ in G_i.
  (b) Find as many lower and upper bounds as you can for the growth function of G_i with respect to the generating set $\{x,y,z\}^{\pm 1}$.
  (Note: In both parts you must prove your answer directly, and may not use the theorems from class on ascending HNN extensions.)

Problem Set 3 is due by: Saturday November 6 at 10:00pm.
Subassignment to be handed in for grading: Ex2.1(b) for G_1, Ex2.6, Ex2.7(a), Ex2.9 for G_1

Problem set 4:

- **Ex3.1.** Find a K(G,1) space and a free resolution of $\mathbb{Z}$ over $\mathbb{Z}$G, and use these to compute the homology groups $H_i(G)$ for all indices i, for the following groups:
  (a) The braid group B_3 on 3 strands G = < a,b | aba = bab >.
  (b) The Baumslag-Solitar BS(1,2) group G = < a,t | $tat^{-1} = a^2$ >.
  (c) The Heisenberg group G = < x, y, z | $[x,y] = z$, $[x,z] = 1$, $[y,z] = 1$ >.
  (d) The ascending HNN extension of $\mathbb{Z}^2$ defined by G_2 = < x, y, z | $[x,y] = 1$, $zxz^{-1} = x^{-1}$, $zyz^{-1} = y^{-1}$ >
- **Ex3.2.** Let G be a group. The **augmentation map** e:**Z**G → **Z** is the group homomorphism (uniquely) defined by e(g) = 1 for all g in G. Let I be the kernel of e and let S be any set of elements of G. Show that S generates G if and only if the T := {s-1 | s ∈ S} generates I as a left ideal of $\mathbb{Z}$G (i.e., as a left $\mathbb{Z}$G-module).
- **Ex3.3. Homology and subgroups:** Let G be a group and let H be a subgroup of G.
  (a) Let Y be a K(G,1), let $\tilde{Y}$ be the universal covering space of Y, and let Z := $\tilde{Y}$/H be the quotient of $\tilde{Y}$ by the action of H. Show that Z is a K(H,1).
  (b) Show that if $F_\bullet \to \mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}$G, then this sequence is also a free resolution of $\mathbb{Z}$ over $\mathbb{Z}$H.
  (c) Use the results of parts (a) and (b) to show that if G contains a torsion element g, then every K(G,1) space is infinite dimensional, and every free resolution $F_\bullet \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ over $\mathbb{Z}$G satisfies $F_n \neq 0$ for all n ≥ 0.
- **Ex3.4. Homology and free products:** Let G_1 and G_2 be groups, and let G := $G_1 * G_2$ be their free product. For each i ∈ {1,2}, let K_i be a $K(G_i)$ and let v_i be a vertex in K_i. Let Z := $(K_1 \coprod K_2)/\sim$, where ~ is the smallest equivalence relation satisfying $v_1 \sim v_2$.
  (a) Show that Z is a K(G,1).
  (b) Use the Mayer-Vietoris Theorem (from Math 872) and the result of part (a) to show that $H_0(G) = \mathbb{Z}$ and $H_n(G) \cong H_n(G_1) \oplus H_n(G_2)$ for all n ≥ 1.
  (c) Use the result in (b) to compute the homology groups of the infinite dihedral group D_∞ = < a,b | $a^2 = 1 = b^2$ >.
- **Ex3.5. Homology and amalgamated products:** Let G_1,G_2, and J be groups, and let i:J → G_1 and k:J → G_2 be injective group homomorphisms. Let G := $G_1 *_J G_2$ be the associated amalgamated product.
  (a) For each i ∈ {1,2}, let K_i be a $K(G_i)$ and let $p_i:\tilde{K}_i \to K_i$ be its universal covering space. Let K_J be a K(J,1). Let f_i: $\tilde{K}_i/J \to \tilde{K}_J$ be a homotopy equivalence. (See Hatcher p. 90 Thm 1.B.8 for a proof that f_i exists.) Let Z := $(K_1 \coprod (K_J \times I) \coprod K_2)/\sim$, where ~ is the smallest equivalence relation satisfying $(x,0) \sim p_1(f_1(x))$ and $(x,1) \sim p_2(f_2(x))$ for all x ∈ K_J. Use the Seifert-van Kampen Theorem (from Math 872) to show that $\pi_1(Z) \cong G$.
  (b) The space Z in part (a) is a K(G,1); use the Mayer-Vietoris Theorem with this space to show that the homology of the amalgamated product satisfies a long exact sequence

$\cdots \to H_n(J) \to H_n(G_1) \oplus H_n(G_2) \to H_n(G) \to H_{n-1}(J) \cdots$.

(c) Use the result in part (b) to compute the homology groups of the group $G = SL_2(\mathbb{Z}) = (\mathbb{Z}/4) *_{\mathbb{Z}/2} (\mathbb{Z}/6)$.

- **Ex3.6.** For each of the following groups, compute the geometric and cohomological dimensions. Also determine the maximal number n such that the group has type F_n, the maximal n such that the group has type FP_n, and whether or not the group has type F or FF.
  (a) The Baumslag-Solitar group BS(1,2).
  (b) The Baumslag-Solitar group BS(2,2).
  (c) The group $\oplus_{i \in \mathbb{Z}} \mathbb{Z}$.
  (d) The infinite dihedral group $D_\infty = < a,b \mid a^2 = 1 = b^2 >$.
  (e) The ascending HNN extension of $\mathbb{Z}^2$ defined by $G_2 = < x, y, z \mid [x,y] = 1, zxz^{-1} = x^{-1}, zyz^{-1} = y^{-1} >$.
- **Ex3.7. Homology and rewriting systems:**
  (a) Find the shortlex minimal convergent rewriting system for the symmetric group $S_3 = < a,b \mid a^2 = 1, b^2 = 1, (ab)^3 = 1>$. Using **Squier's paper (https://canvas.unl.edu/courses/160712/files/16893576?wrap=1) (https://canvas.unl.edu/courses/160712/files/16893576/download?download_frd=1)** , compute a partial free resolution of $\mathbb{Z}$ over $\mathbb{Z}S_3$ out to dimension 3 (including the boundary map d_3), and use this to compute $H_2(S_3)$.
  (b) For the infinite dihedral group G = Mon$< a,b \mid a^2 = 1, b^2 = 1 >$, find a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ using **Groves's paper (https://canvas.unl.edu/courses/160712/files/16893575?wrap=1) (https://canvas.unl.edu/courses/160712/files/16893575/download?download_frd=1)** .

Problem Set 4 is due by: Friday 12/8/2023 at 10:00pm.
(Ex3.1(a-b) was done in class on 11/16/2023.)
Subassignment to be handed in for grading: Ex3.1(d), Ex 3.2 (in the S gen G implies T gen I direction), Ex3.3(c), Ex3.4(c), Ex3.6(a,d)
[Added after class on 12/7: For Ex3.6 on the graded subassignment, you can either hand in Ex3.6(a) or Ex3.6(a,d) for the points on Ex3.6, since we discussed the solution to Ex3.6(d) in class today.]