A toy problem with internal data (A)

(1) Take the diffusion equation

$$(\star)\begin{cases} -\nabla\cdot\gamma\nabla u+\sigma u=0 & \text{in } \Omega \\ u=g & \text{on } \partial\Omega \end{cases}$$

with Ω a nice domain in $\mathbb{R}^2$. Say the unit square $\Omega=[0,1]^2$

(a) take $\gamma=0.02$ for simplicity

(b) take $\sigma(x)=\begin{cases} 0.2 & \text{in } [0.5,0.7]^2 \\ 0.1 & \text{otherwise} \end{cases}$

(c) Solve $(\star)$ and compute (with a finite element method).

$$H(x)=\sigma(x).u(x).$$

(d) This H is you datum.

(e) You can add noise to H, for instance $H^*=(1+\text{rand}).H$

random field between $[-0.05, 0.05]$

(2) To reconstruct σ ~~xxx~~ from H^*

You can do the following steps:

(a) Solve $\begin{cases} -\nabla\cdot\gamma\nabla u^* + H = 0 \\ u^* = g \end{cases}$

to find u^*.

(b) reconstruct σ as $\sigma^* = \dfrac{H^*}{u^*}$.

Depending on the noise you added to H, you can get different reconstructions.

(3) You can choose a different γ to see its impact on the reconstruction of σ.

for instance, take

$$\gamma(x) = \begin{cases} 0.03 & \text{in } [0.2, 0.4]^2 \\ 0.01 & \text{otherwise.} \end{cases}$$

(4) Now, assume that you have $J \geq 2$ data sets

(a) For each g_j $j=1,2,\ldots,J$, Solve

$$(*)\quad \begin{cases} -\nabla\cdot\gamma\nabla u_j + \sigma u_j = 0 \\ u_j = g_j \end{cases}$$

(b) Compute $H_j = \sigma u_j$ $\quad j=1,\ldots,J$.

(c) You can add some noise to H_j again.

(d) To reconstruct σ, you can do an average like this:

- Solve $\begin{cases} -\nabla\cdot\gamma\nabla u_j^* + H_j = 0 \\ u_j^* = g_j \end{cases}$ $\quad j=1,\ldots,J$.

- reconstruct σ as

$$\sigma^* = \frac{1}{J}\sum \frac{H_j}{u_j^*}.$$

(e) An alternative way of reconstruction is this:

- Solve $\begin{cases} -\nabla \cdot \gamma \nabla u^* + \frac{1}{J}\sum_{j=1}^{J} H_j = 0 \\ u^* = \frac{1}{J}\sum_{j=1}^{J} g_j \end{cases}$

- reconstruct σ as

$$\sigma^* = \left(\frac{1}{J}\sum_{j=1}^{J} H_j\right) \Big/ u^*$$

$\boxed{\text{A toy Problem with internal data (B)}}$

(1) Take the diffusion equations

$$\begin{cases} -\nabla\cdot\gamma\nabla u_j + \sigma u_j = 0 & \text{in } \Omega \\ u = g_j & \text{on } \partial\Omega. \end{cases}$$

Take again $\Omega = [0,1]^2$

(a) take $\gamma(x) = \begin{cases} 0.03 & x \in [0.4, 0.6]^2 \\ 0.01 & \text{otherwise} \end{cases}$

$$\sigma(x) = \begin{cases} 0.2 & x \in [0.2, 0.8] \times [0.2, 0.3] \\ 0.1 & \text{otherwise} \end{cases}$$

(b) Solve (*) to get data

$$H_j(x) = \sigma(x) \cdot u_j(x). \quad j = 1, 2, \dots, J.$$

(2) We can add noise to the data as usual.

For instance, $H_j^* = (1 + \underbrace{\text{random}}_{\in [-0.05,\, 0.05]}) \cdot H_j$

(3) Assume now that both γ and σ are not known.

The strategy of reconstructions in the previous case does not work anymore.

(4) We can use numerical optimization tools for the reconstruction.

We search for (γ, σ) that minimize

$$\Phi(\gamma,\sigma) = \frac{1}{2}\sum_{j=1}^{J}\int_{\Omega}(\sigma u_j - H_j^*)^2\,dx + \frac{\beta}{2}\int_{\Omega}\left(|\nabla\gamma|^2 + |\nabla\sigma|^2\right)dx$$

with β a small regularization parameter, say $\beta = 10^{-5}$.

(5) Newton type of algorithms can be used to solve this minimization problem.

(6) A critical issue will be how to compute the gradient of Φ w.r.t γ and σ.

This is usually handled with the adjoint state method.