직사각형 위에서의 이중적분

정적분

부피와 이중적분

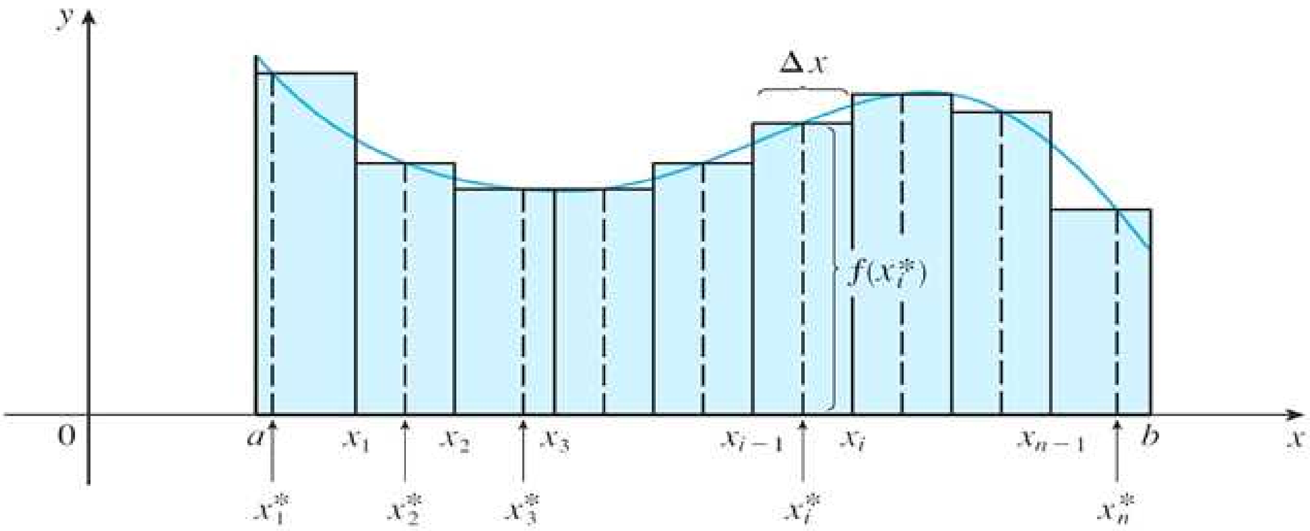
중점법칙

반복적분

푸비니의 정리

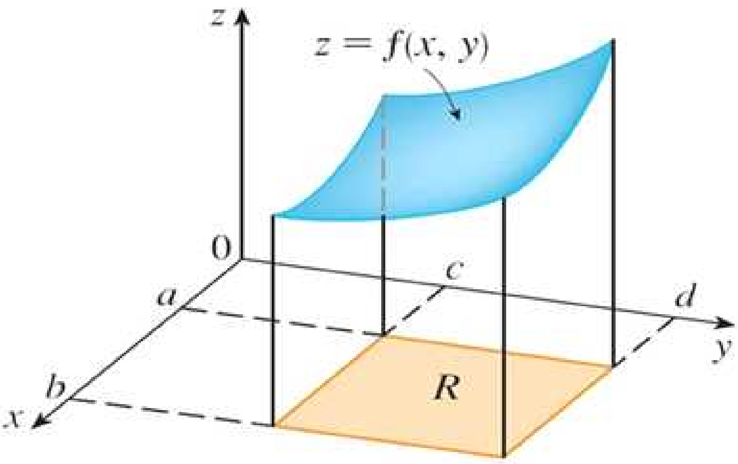
평균값

정적분



$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i{}^*)\,\Delta x$$

부피와 이중적분



폐직사각형 영역

$R = [a,\ b] \times [c,\ d]$
$\quad = \{(x,y) \in \mathrm{R}^3 \mid a \le x \le b,\ c \le y \le d\}$

에서 정의된 이변수함수 f에 대하여, $f(x,y) \ge 0$을 가정

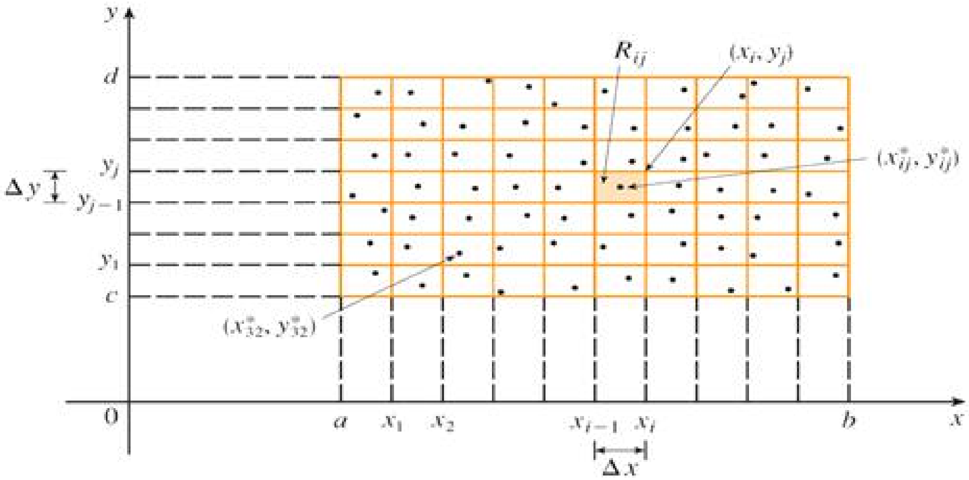
f의 그래프 : 방정식 $z = f(x,y)$를 갖는 한 곡면

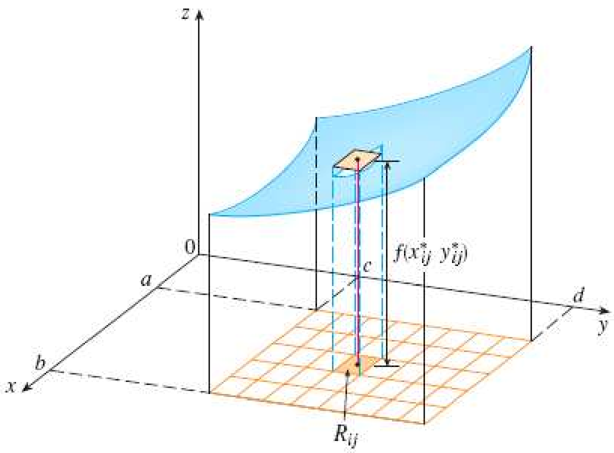
S : R 위와 f의 그래프 아래에 있는 입체

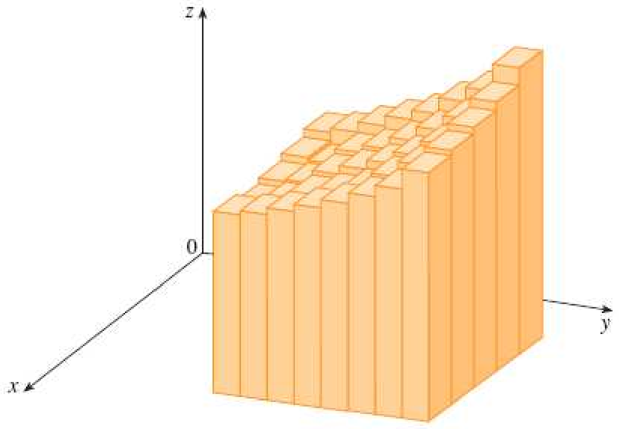
$S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le f(x,y),\ (x,y) \in R\}$

$$\begin{cases} \Delta x = \dfrac{b-a}{m} \\ \Delta y = \dfrac{d-c}{n} \end{cases}, \qquad \Delta A = \Delta x \Delta y$$

$R_{ij} = [x_{i-1},\ x_i] \times [y_{j-1},\ y_j]$
$\quad = \{(x,y) \mid x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_i\}$







$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

정의

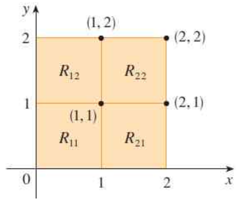
직사각형 R 위의 f에 대한 **이중적분(double integral)**은 다음 식의 극한이 존재하는 경우이다.

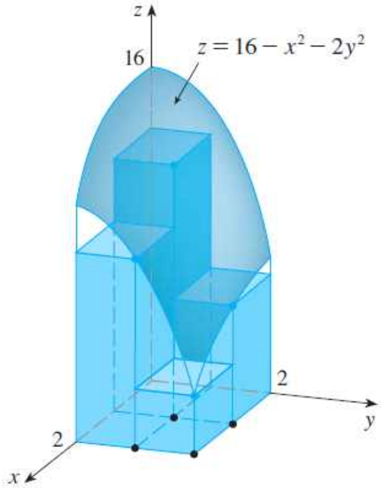
$$\iint_R f(x,y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

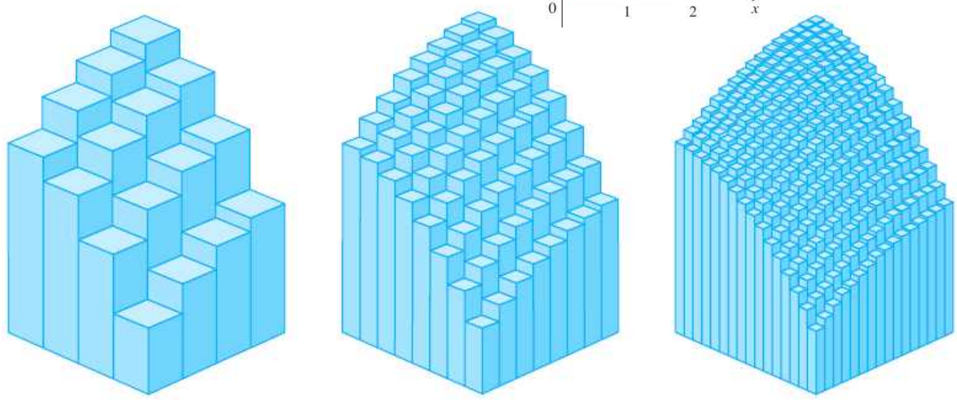
$f(x,y) \geq 0$이면 직사각형 R 위에 있고, 곡면 $z = f(x,y)$ 아래에 놓인 입체의 부피 V는 다음과 같다.

$$V = \iint_R f(x,y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

예제 타원포물면 $z=16-x^2-2y^2$ 아래, $R=[0,2]\times[0,2]$ 위에 놓인 입체의 부피







(a) $m=n=4,\ V\approx 41.5$ (b) $m=n=8,\ V\approx 44.875$ (c) $m=n=16,\ V\approx 46.46875$

$m=n=2$일 때,
$\Delta x=1,\ \Delta y=1 \quad \Rightarrow \quad \Delta A=1$

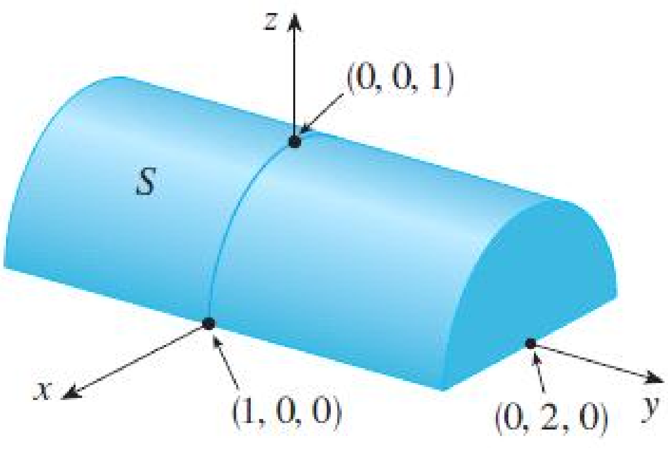
$$V=\sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i,\,y_j)\,\Delta A$$
$$=f(1,1)\,\Delta A+f(1,2)\,\Delta A+f(2,1)\,\Delta A+f(2,2)\,\Delta A$$
$$=13\cdot 1+7\cdot 1+10\cdot 1+4\cdot 1=34$$

$$V=\iint_R (16-x^2-2y^2)\,dA \qquad =\int_0^2\int_0^2 (16-x^2-2y^2)\,dy\,dx$$
$$=\int_0^2\left(\left[16y-x^2y-\frac{2}{3}y^3\right]_{y=0}^{y=2}\right)dx=\int_0^2\left(32-2x^2-\frac{16}{3}\right)dx$$
$$=\left[\frac{80}{3}x-\frac{2}{3}x^3\right]_0^2 \qquad =\frac{160}{3}-\frac{16}{3}=\frac{144}{3}=48$$

예제 $R=\{(x,y)\mid -1\le x\le 1,\ -2\le y\le 2\}$일 때

다음 적분을 구하여라.

$$\iint_R \sqrt{1-x^2}\ dA$$



S의 부피

: 반지름이 1이고 높이가 4인 원기둥의 부피의 반

$$\iint_R \sqrt{1-x^2}\ dA=\frac{1}{2}\pi(1)^2\times 4$$
$$=2\pi$$

중점 법칙

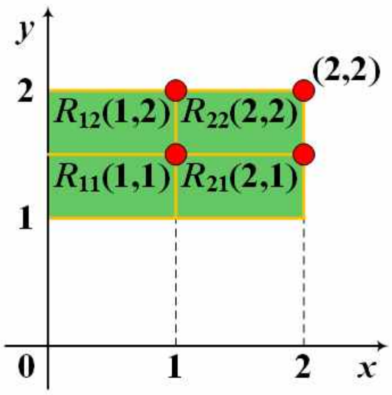
이중적분에 대한 중점 법칙

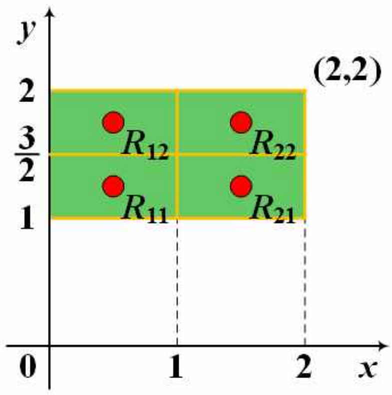
직사각형 R 위의 f에 대한 **이중적분(double integral)**은 다음 식의 극한이 존재하는 경우이다.

$$\iint_R f(x,y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x_i}, \overline{y_i})\, \Delta A$$

단, $\overline{x_i}$: $[x_{i-1}, x_i]$의 중점, $\overline{y_j}$: $y_{j-1}, y_j]$의 중점

예제 $R=\{(x,y)\,|\,0 \le x \le 2,\ 1 \le y \le 2\}$이고 $m=n=2$일 때 중점 법칙을 이용하여 $\iint_R (x-3y^2)\, dA$를 추정하여라. (정답 : -12)





$$\begin{aligned} V &= \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\, \Delta A \\ &= f(1,1)\, \Delta A + f(1,2)\, \Delta A + f(2,1)\, \Delta A + f(2,2)\, \Delta A \\ &= (-2)\cdot\frac{1}{2} + (-11)\cdot\frac{1}{2} + (-1)\cdot\frac{1}{2} + (-10)\cdot\frac{1}{2} \\ &= -\frac{24}{2} = -12 \end{aligned}$$

$$\begin{aligned} V &= \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\, \Delta A \\ &= f(\tfrac{1}{2},\tfrac{5}{4})\, \Delta A + f(\tfrac{1}{2},\tfrac{7}{4})\, \Delta A + f(\tfrac{3}{2},\tfrac{5}{4})\, \Delta A + f(\tfrac{3}{2},\tfrac{7}{4})\, \Delta A \\ &= (-\frac{76}{16})\cdot\frac{1}{2} + (-\frac{139}{16})\cdot\frac{1}{2} + (-\frac{51}{16})\cdot\frac{1}{2} + (-\frac{123}{16})\cdot\frac{1}{2} \\ &= -\frac{95}{4}\cdot\frac{1}{2} = -\frac{23.75}{2} = -11.875 \end{aligned}$$

반복적분

f : 직사각형 $R=\{(x,y)|a\le x\le b,\ c\le y\le d\}$ 상에서 적분가능한 이변수함수

$$\iint_R f(x,y)\,dA=\begin{cases}\displaystyle\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_a^b\underbrace{\left[\int_c^d f(x,y)\,dy\right]}_{y\text{에 관해 적분}}dx\\[2ex]\displaystyle\int_c^d\int_a^b f(x,y)\,dx\,dy=\int_c^d\underbrace{\left[\int_a^b f(x,y)\,dx\right]}_{x\text{에 관해 적분}}dy\end{cases}$$

예제 반복적분을 계산하여라.

(a) $\displaystyle\int_0^3\int_1^2 x^2y\,dy\,dx$

$$\begin{aligned}&\int_0^3\int_1^2 x^2y\,dy\,dx\\&=\int_0^3\left[\int_1^2 x^2y\,dy\right]dx\\&=\int_0^3\left(\left[x^2\frac{y^2}{2}\right]_{y=1}^{y=2}\right)dx\\&=\int_0^3\left(\frac{x^2}{2}\left[2^2-1^2\right]\right)dx\\&=\int_0^3\frac{3}{2}x^2\,dx=\left[\frac{1}{2}x^3\right]_0^3\\&=\frac{1}{2}\left[3^3-0^3\right]=\frac{27}{2}\end{aligned}$$

(b) $\displaystyle\int_1^2\int_0^3 x^2y\,dx\,dy$

$$\begin{aligned}&\int_1^2\int_0^3 x^2y\,dx\,dy\\&=\int_1^2\left[\int_0^3 x^2y\,dx\right]dy\\&=\int_1^2\left(\left[\frac{x^3}{3}y\right]_{x=0}^{x=3}\right)dy\\&=\int_1^2\left(\left[3^3-0^3\right]\frac{y}{3}\right)dy\\&=\int_1^2 9y\,dy=\left[\frac{9}{2}y^2\right]_1^2\\&=\frac{9}{2}\left[2^2-1^2\right]=\frac{27}{2}\end{aligned}$$

푸비니 정리

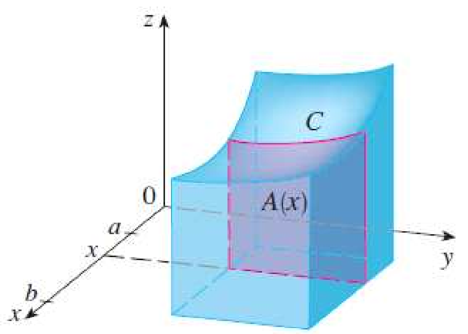
푸비니 정리(Fubini's Theorem)

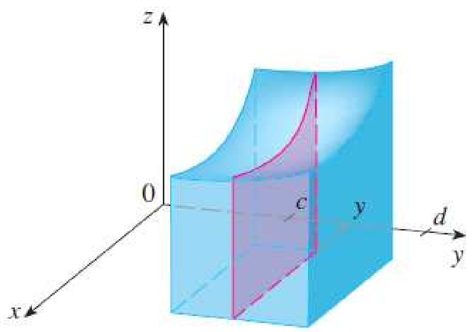
f가 직사각형 $R = \{(x,y) | a \le x \le b,\ c \le y \le d\}$ 상에서 연속이면 다음 식이 성립한다.

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

더 일반적으로 만약 f가 R상에서 유계이고 f가 오직 유한개의 매끄러운 곡선상에서 불연속이며, 반복적분이 존재한다면 이 식이 성립한다.

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$





$$\iint_R f(x,y)\, dA = V = \int_a^b A(x)\, dx,$$

$$A(x) = \int_c^d f(x,y)\, dy$$

$$\iint_R f(x,y)\, dA = V = \int_c^d A(y)\, dy,$$

$$A(y) = \int_a^b f(x,y)\, dx$$

예제 $R=\{(x,y)\,|\,0\le x\le 2,\ 1\le y\le 2\}$일 때 이중적분 $\iint_R (x-3y^2)\,dA$를 계산하여라.

$$\begin{aligned}\iint_R (x-3y^2)\,dA &= \int_0^2\int_1^2 (x-3y^2)\,dy\,dx\\ &= \int_0^2 \left[xy-y^3\right]_{y=1}^{y=2} dx \quad = \int_0^2 [(2x-8)-(x-1)]\,dx\\ &= \int_0^2 x-7\,dx \quad = \left[\frac{x^2}{2}-7x\right]_0^2\\ &= \frac{2^2}{2}-7\cdot 2=-12\end{aligned}$$

$$\begin{aligned}\iint_R (x-3y^2)\,dA &= \int_1^2\int_0^2 (x-3y^2)\,dx\,dy\\ &= \int_1^2 \left[\frac{x^2}{2}-3xy^2\right]_{x=0}^{x=2} dy \quad = \int_1^2 \left[\left(\frac{2^2}{2}-3\cdot 2y^2\right)-0\right] dy\\ &= \int_1^2 2-6y^2\,dy \quad = \left[2y-2y^3\right]_1^2\\ &= (2\cdot 2-2\cdot 2^3)-(2\cdot 1-2\cdot 1^3)=-12\end{aligned}$$

예제 $R=\{(x,y)\,|\,1\le x\le 2,\ 0\le y\le \pi\}$일 때 $\iint_R y\sin(xy)\,dA$를 계산하여라.

$$\begin{aligned}\iint_R y\sin(xy)\,dA &= \int_0^\pi\int_1^2 y\sin(xy)\,dx\,dy\\ &= \int_0^\pi y\left[-\frac{1}{y}\cos(xy)\right]_{x=1}^{x=2} dy\\ &= \int_0^\pi y\cdot\frac{1}{y}(-\cos 2y+\cos y)\,dy\\ &= \left[-\frac{1}{2}\sin 2y+\sin y\right]_0^\pi=0\end{aligned}$$

$$\begin{aligned}\iint_R y\sin(xy)\,dA &= \int_1^2\int_0^\pi y\sin(xy)\,dy\,dx\\ &= \int_1^2\left(-\frac{\pi}{x}\cos(\pi x)+\frac{1}{x^2}\sin(\pi x)\right)dx\end{aligned}$$

$$\begin{aligned}\int_0^\pi y\sin(xy)\,dy &= \left[-\frac{y}{x}\cos(xy)\right]_{y=0}^{y=\pi}-\int_0^\pi\left(-\frac{1}{x}\cos(xy)\right)dy\\ &= \left[\left(-\frac{\pi}{x}\cdot\cos(\pi x)\right)-\left(-\frac{0}{x}\cdot 1\right)\right]+\left[\frac{1}{x^2}\sin(xy)\right]_{y=0}^{y=\pi}\\ &=-\frac{\pi}{x}\cos(\pi x)+\left[\frac{1}{x^2}(\sin(\pi x)-\sin 0)\right]\\ &=-\frac{\pi}{x}\cos(\pi x)+\frac{1}{x^2}\sin(\pi x)\end{aligned}$$

예제 타원포물면 $x^2+2y^2+z=16$과 평면 $x=2,\ y=2$, 그리고 세 좌표평면으로 둘러싸인 입체 S의 부피를 구하여라.

$z=f(x,y)=16-x^2-2y^2$

$R=[0,2]\times[0,2]$

$$\begin{aligned} V&=\iint_R(16-x^2-2y^2)\,dA\\ &=\int_0^2\int_0^2(16-x^2-2y^2)\,dx\,dy\\ &=\int_0^2\left[16x-\frac{x^3}{3}-2xy^2\right]_{x=0}^{x=2}dy\\ &=\int_0^2\left[\left(32-\frac{8}{3}-4y^2\right)-0\right]dy\\ &=\int_0^2\left(\frac{88}{3}-4y^2\right)dy\\ &=\left[\frac{88}{3}y-\frac{4}{3}y^3\right]_0^2\\ &=\left(\frac{88}{3}\cdot 2-\frac{4}{3}\cdot 2^3\right)-0=\frac{144}{3}=48 \end{aligned}$$

$$\begin{aligned} V&=\iint_R(16-x^2-2y^2)\,dA\\ &=\int_0^2\int_0^2(16-x^2-2y^2)\,dy\,dx\\ &=\int_0^2\left[16y-x^2y-\frac{2}{3}y^3\right]_{y=0}^{y=2}dx\\ &=\int_0^2\left[\left(32-2x^2-\frac{16}{3}\right)-0\right]dx\\ &=\int_0^2\left(\frac{80}{3}-2x^2\right)dx\\ &=\left[\frac{80}{3}x-\frac{2}{3}x^3\right]_0^2\\ &=\left(\frac{80}{3}\cdot 2-\frac{2}{3}\cdot 2^3\right)-0=\frac{144}{3}=48 \end{aligned}$$

$f(x,y)$가 오직 x만의 함수와 오직 y만의 함수의 곱으로 인수분해될 수 있는 특별한 경우

즉,

$f(x,y)=g(x)\cdot h(y)$이고 $R=[a,b]\times[c,d]$라 하면 푸비니 정리에 의해

$$\iint_R f(x,y)\,dA=\iint_R[g(x)\cdot h(y)]\,dA=\left[\int_a^b g(x)\,dx\right]\times\left[\int_c^d h(y)\,dy\right]$$

$$\boxed{\iint_R[g(x)h(y)]\,dA=\int_a^b g(x)\,dx\int_c^d h(y)\,dy,\quad R=[a,b]\times[c,d]}$$

예제 $R=[0,\pi/2]\times[0,\pi/2]$일 때,

$\iint_R \sin x \cos y \, dA$를 계산하여라.

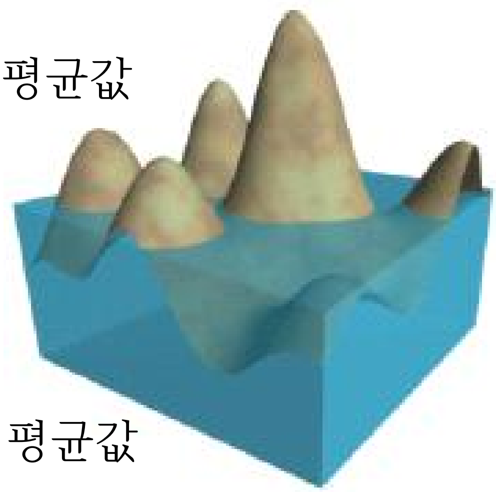
$$\begin{aligned}
&\iint_R \sin x \cos y \, dA \\
&= \int_0^{\pi/2}\int_0^{\pi/2} \sin x \cos y \, dy \, dx \\
&= \int_0^{\pi/2} [\sin x \sin y]_{y=0}^{y=\pi/2} \, dx \\
&= \int_0^{\pi/2} \left(\sin x \sin\left(\frac{\pi}{2}\right) - \sin x \sin(0)\right) dx \\
&= \int_0^{\pi/2} \sin x \, dx \\
&= [-\cos x]_0^{\pi/2} \\
&= \left(-\cos\left(\frac{\pi}{2}\right) + \cos 0\right) \\
&= 0+1=1
\end{aligned}$$

$$\begin{aligned}
&\iint_R \sin x \cos y \, dA \\
&= \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy \\
&= [-\cos x]_0^{\pi/2} \, [\sin y]_0^{\pi/2} \\
&= \left(-\cos\left(\frac{\pi}{2}\right) + \cos 0\right) \times \left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) \\
&= (0+1)\times(1-0) \\
&= 1 \cdot 1 \\
&= 1
\end{aligned}$$

평균값

구간 $[a,b]$상에서 정의된 일변수함수 f의 평균값

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$



직사각형 R상에서 정의된 이변수함수 f의 평균값

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x,y)\,dA, \quad A(R)\text{은 } R\text{의 넓이}$$