곡면의 넓이

학습목표

- 곡면의 넓이

이변수함수 $z=f(x,y)$를 갖는 곡면의 넓이

곡면 S : $z=f(x,y)$, f는 연속인 편도함수를 갖는다.

가정

$f(x,y) \geq 0$

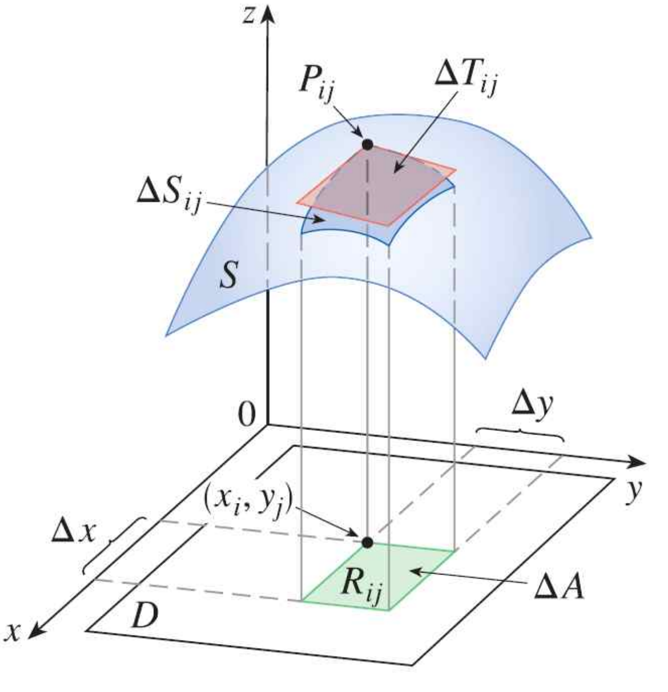
D : 직사각형 모양의 f의 정의역

분할 R_{ij}의 넓이 : $\Delta A = \Delta x \Delta y$

$(x_i,\ y_j)$: R_{ij}의 한 점

점 $P_{ij}\,(x_i,\ y_i,\ f(x_i,\ y_i))$

: 점 $(x_i,\ y_j)$ 위의 S 상의 점

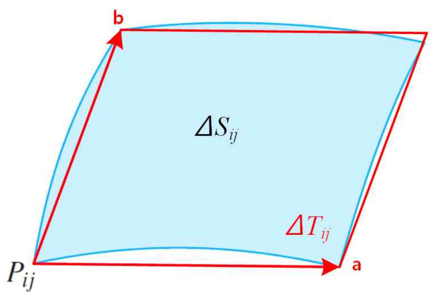


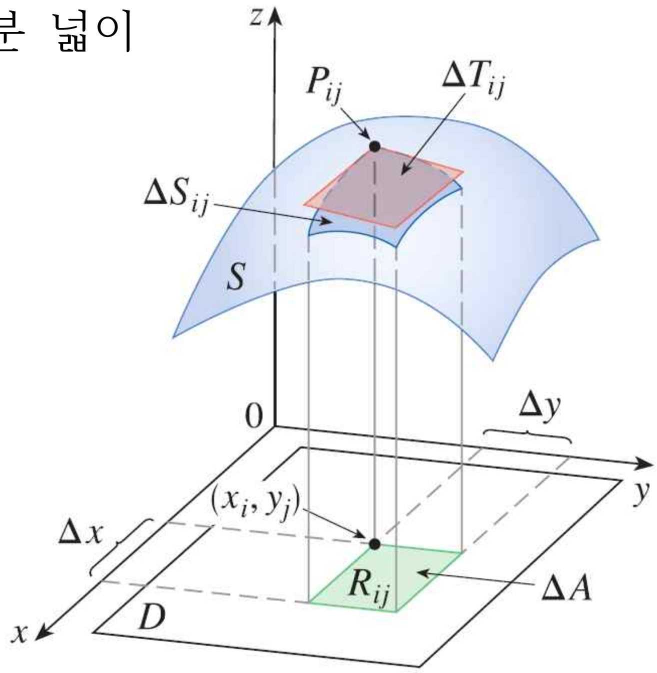
점 P_{ij}에서 S의 접평면은 P_{ij} 근방에서 S에 근사적으로 접근한다.

$\varDelta T_{ij}$: R_{ij} 바로 위에 있는 이 접평면의 부분(평행사변형)의 넓이

$\varDelta S_{ij}$: R_{ij} 바로 위에 있는 S의 부분 넓이

$\varDelta S_{ij} \approx \varDelta T_{ij}$

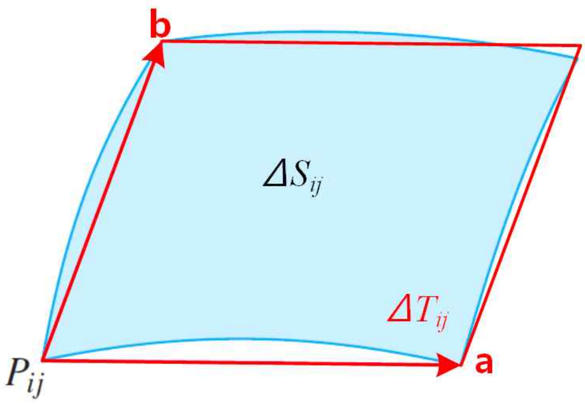


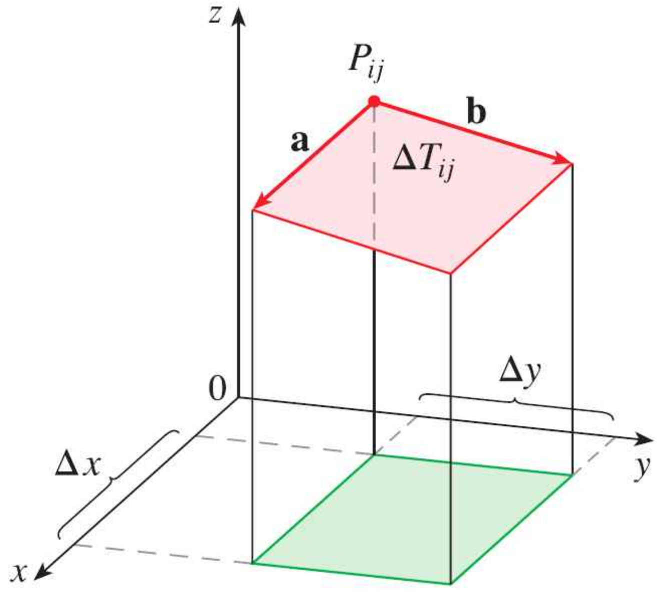


곡면 S의 넓이(surface area) : $A(S)$

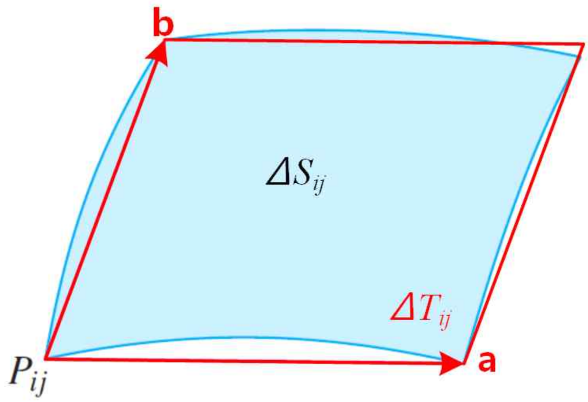
$$A(S) = \sum_{i=1}^{m}\sum_{j=1}^{n} \varDelta S_{ij} \approx \sum_{i=1}^{m}\sum_{j=1}^{n} \varDelta T_{ij}$$

$$= \lim_{m,n \to \infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \varDelta T_{ij}$$





$$\mathbf{a} = \Delta x\ \mathbf{i} + f_x(x_i, y_j)\ \Delta x\ \mathbf{k}$$
$$\mathbf{b} = \Delta y\ \mathbf{j} + f_y(x_i, y_j)\ \Delta y\ \mathbf{k}$$



$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\ \Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\ \Delta y \end{vmatrix}$$

$$= -f_x(x_i, y_j)\ \Delta x\ \Delta y\, \mathbf{i} - f_y(x_i, y_j)\ \Delta x\ \Delta y\, \mathbf{j} + \Delta x\ \Delta y\, \mathbf{k}$$

$$= \left[\ -f_x(x_i, y_j)\ \mathbf{i} - f_y(x_i, y_j)\ \mathbf{j} + \mathbf{k}\ \right] \Delta A$$

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}| = \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\ \ \Delta A$$

$$A(S) = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$
$$= \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\ dA$$

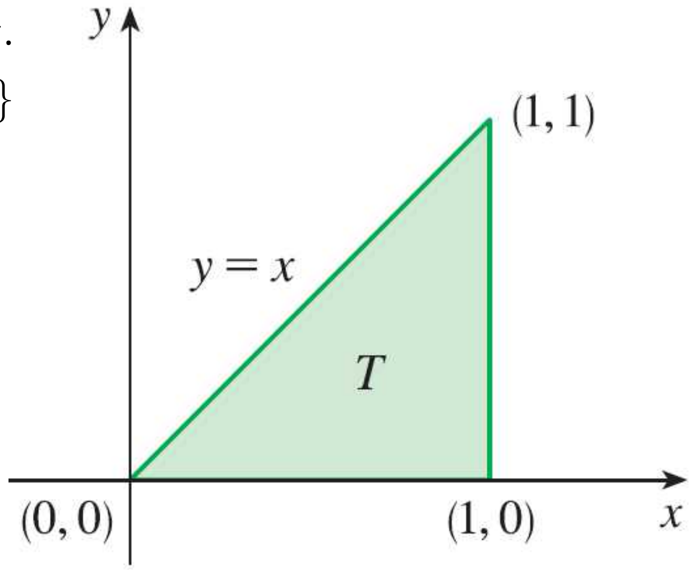
곡면의 넓이

f_x, f_y가 연속일 때, $z = f(x, y)$, $(x, y) \in D$로 표현되는 곡면의 넓이는

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\ dA$$

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2}\ dA$$

예제. 세 꼭짓점이 (0,0), (1,0), (1,1)인 삼각형 영역 T 위의 곡면 $z = x^2 + 2y + 2$의 넓이를 구하여라.



$$T = \{ (x,y) | 0 \le x \le 1,\ 0 \le y \le x \}$$

$$A(S) = \iint_T \sqrt{(2x)^2 + (2)^2 + 1}\ dA$$

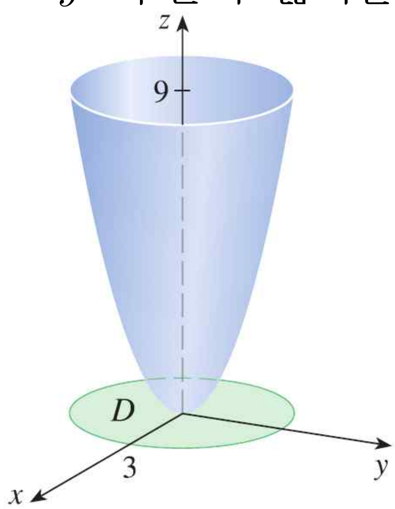
$$= \int_0^1 \int_0^x \sqrt{4x^2 + 5}\ dy\, dx$$

$$= \int_0^1 \left[\sqrt{4x^2 + 5}\, y \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 x\sqrt{4x^2 + 5}\ dx$$

$$= \left[\frac{1}{8} \cdot \frac{2}{3} (4x^2 + 5)^{3/2} \right]_0^1 = \frac{1}{12}(27 - 5\sqrt{5})$$

예제. 평면 $z = 9$ 아래에 놓여 있는 포물면 $z = x^2 + y^2$ 부분의 넓이를 구하여라.



$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2}\ dA$$

$$= \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\ dA$$

$$= \iint_D \sqrt{1 + 4x^2 + 4y^2}\ dA$$

$$= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\ r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 \frac{1}{8} \sqrt{1 + 4r^2}\ (8r\, dr)$$

$$= [\theta]_0^{2\pi} \left[\frac{1}{8}\frac{2}{3}(1 + 4r^2)^{3/2} \right]_0^3 = 2\pi \frac{1}{12}(37\sqrt{37} - 1)$$