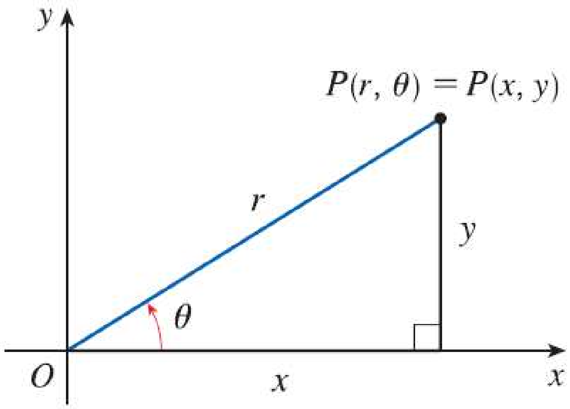
극좌표에서의 이중적분

학습목표

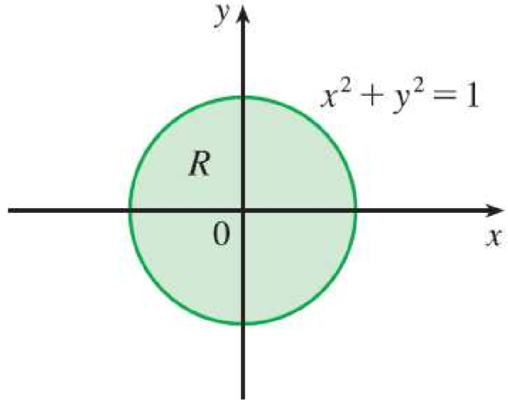
- 극좌표
- 극좌표에서의 이중적분

극좌표

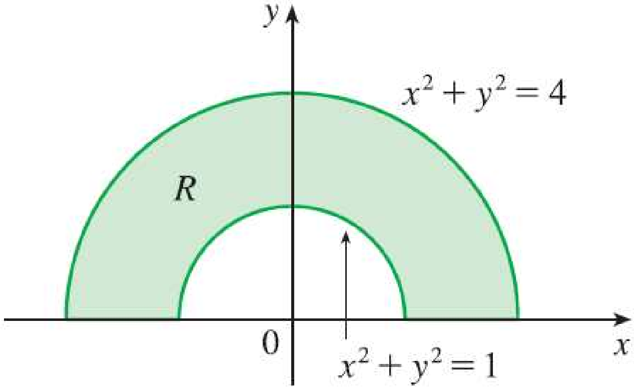


$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

$$x = r\cos\theta, \qquad y = r\sin\theta$$



(a)



(b)

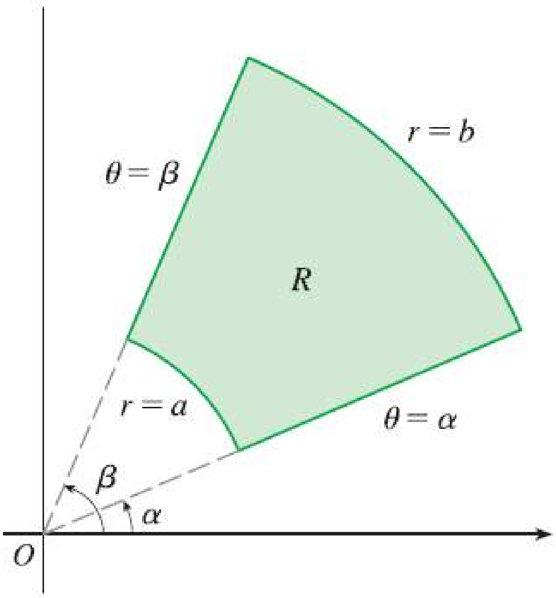
$$\begin{aligned}
\text{(a)}\ \ R &= \left\{ (x,y) \,|\, -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \right\} \\
&= \left\{ (r,\theta) \,|\, 0 \le r \le 1,\ 0 \le \theta \le 2\pi \right\}
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\ \ R = &\left\{ (x,y) \,|\, -2 \le x \le -1,\ \ 0 \le y \le \sqrt{4-x^2} \right\} \\
&\cup \left\{ (x,y) \,|\, -1 \le x \le 1,\ \ \sqrt{1-x^2} \le y \le \sqrt{4-x^2} \right\} \\
&\cup \left\{ (x,y) \,|\, 1 \le x \le 2,\ \ 0 \le y \le \sqrt{4-x^2} \right\} \\
= &\left\{ (r,\theta) \,|\, 0 \le r \le 1,\ 0 \le \theta \le 2\pi \right\}
\end{aligned}$$

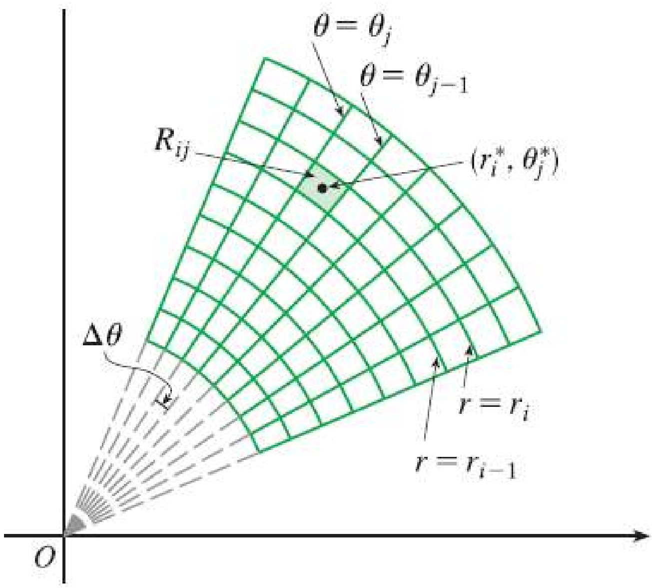
극좌표에서의 이중적분

극장방형(polar rectangle, 극직사각형)

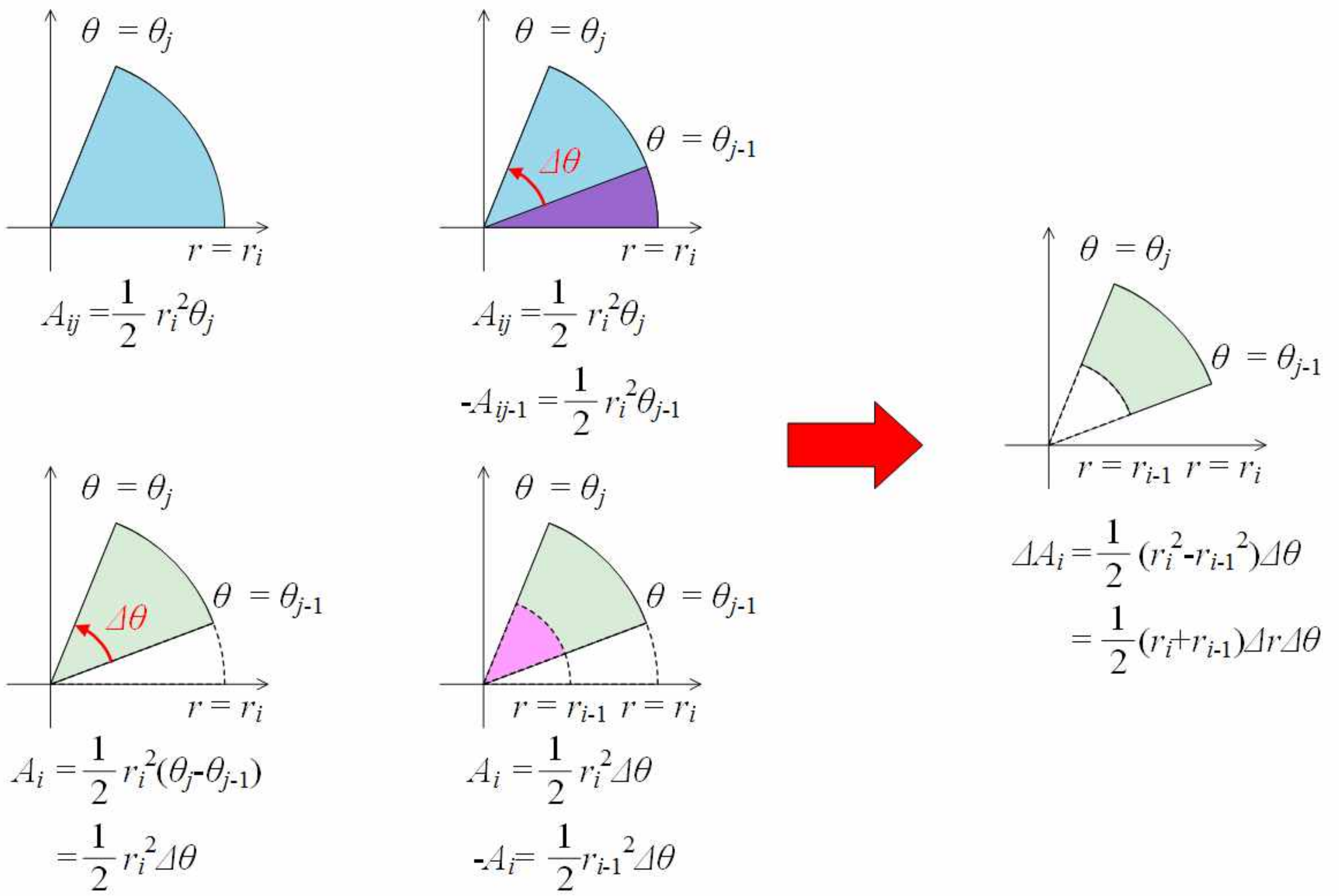
$$R = \{ (r,\theta) \,|\, a \le r \le b,\ \alpha \le \theta \le \beta \}$$



극장방형



R의 부분 극장방형의 분할



부분 극장방형

$$R_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$$

R_{ij}의 중심의 극좌표

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i), \quad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$$

$$\begin{aligned}
\Delta A_i &= \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta, \qquad \Delta\theta = \theta_j - \theta_{j-1} \\
&= \frac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\Delta\theta \\
&= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta \\
&= r_i^* \Delta r\, \Delta\theta
\end{aligned}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(r_i^{*}\cos\theta_j^{*},\, r_i^{*}\sin\theta_j^{*}\right)\Delta A_i$$
$$=\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(r_i^{*}\cos\theta_j^{*},\, r_i^{*}\sin\theta_j^{*}\right) r_i^{*}\,\Delta r\,\Delta\theta$$

$g(r,\theta)=rf(r\cos\theta,\, r\sin\theta)$라 하면

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g\left(r_i^{*},\, \theta_j^{*}\right)\Delta r\,\Delta\theta=\int_{\alpha}^{\beta}\int_{a}^{b} g(r,\theta)\,dr\,d\theta$$

$$\iint_R f(x,y)\,dA=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(r_i^{*}\cos\theta_j^{*},\, r_i^{*}\sin\theta_j^{*}\right)\Delta A_i$$
$$=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} g\left(r_i^{*},\, \theta_j^{*}\right)\Delta r\,\Delta\theta$$
$$=\int_{\alpha}^{\beta}\int_{a}^{b} g(r,\theta)\,dr\,d\theta$$
$$=\int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta,\, r\sin\theta)\,r\,dr\,d\theta$$

이중적분에서 극좌표로의 변환

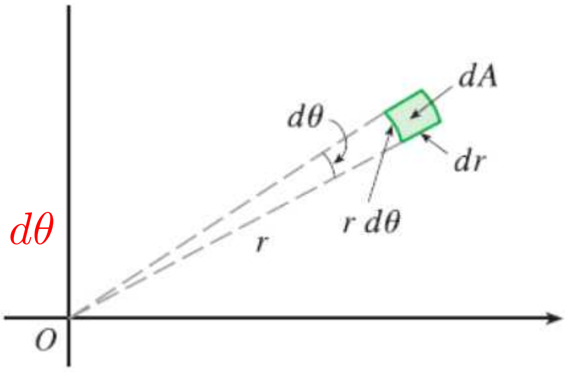
f가 $0 \le a \le r \le b,\ \alpha \le \theta \le \beta$ (단, $0 \le \beta - \alpha \le 2\pi$)인 극직사각형 R에서 연속이면

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta$$

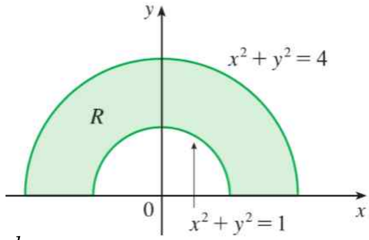
$x = r\cos\theta,\quad y = r\sin\theta$

$dA = dx\,dy$ or $dy\,dx \Rightarrow dA = r\,dr\,d\theta$

(x,y)의 영역 $\Rightarrow$ (r,θ)의 영역



예제. R이 두 개의 원 $x^2+y^2=1$과 $x^2+y^2=4$로 둘러싸인 위쪽 반평면의 영역인 경우에 $\iint_R (3x+4y^2)dA$의 값을 구하여라.



$$\iint_R (3x+4y^2)dA$$

$$= \int_{-2}^{-1}\int_0^{\sqrt{4-x^2}} (3x+4y^2)\,dy\,dx + \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} (3x+4y^2)\,dy\,dx + \int_1^2\int_0^{\sqrt{4-x^2}} (3x+4y^2)\,dy\,dx$$

$$= \int_{-2}^{-1}\left[3xy+\frac{4}{3}y^3\right]_{y=0}^{y=\sqrt{4-x^2}} dx + \int_{-1}^{1}\left[3xy+\frac{4}{3}y^3\right]_{y=\sqrt{1-x^2}}^{y=\sqrt{4-x^2}} dx + \int_1^2\left[3xy+\frac{4}{3}y^3\right]_{y=0}^{y=\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{-1}\left[\left(3x\sqrt{4-x^2}+\frac{4}{3}\sqrt{(4-x^2)^3}\right)-0\right] dx$$

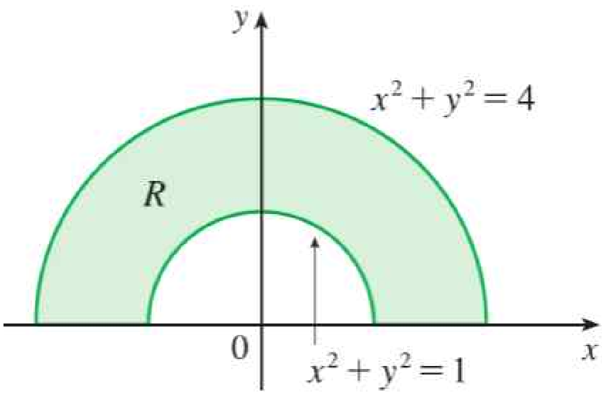
$$+ \int_{-1}^{1}\left[\left(3x\sqrt{4-x^2}+\frac{4}{3}\sqrt{(4-x^2)^3}\right)-\left(3x\sqrt{1-x^2}+\frac{4}{3}\sqrt{(1-x^2)^3}\right)\right] dx$$

$$+ \int_1^2\left[\left(3x\sqrt{4-x^2}+\frac{4}{3}\sqrt{(4-x^2)^3}\right)-0\right] dx$$

$$= (-3\sqrt{3})+\left(-3\sqrt{3}+\frac{8}{3}\pi\right)+(0)+\left(6\sqrt{3}+\frac{8}{3}\pi\right)-(0)-\left(\frac{\pi}{2}\right)+(3\sqrt{3})+\left(-3\sqrt{3}+\frac{8}{3}\pi\right)$$

$$= \frac{15}{2}\pi$$

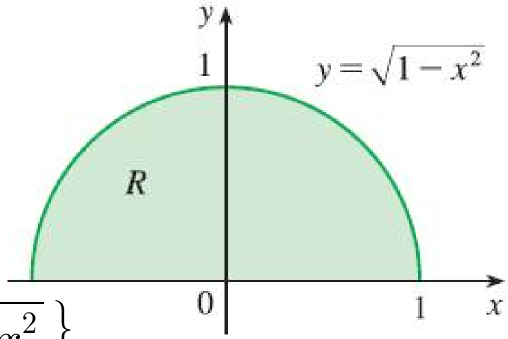
$$\iint_R (3x+4y^2)dA$$
$$= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$
$$= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta$$
$$= \int_0^\pi \left[r^3\cos\theta + r^4\sin^2\theta \right]_{r=1}^{r=2} d\theta$$
$$= \int_0^\pi \left[(2^3-1^3)\cos\theta + (2^4-1^3)\sin^2\theta \right] d\theta$$
$$= \int_0^\pi (7\cos\theta + 15\sin^2\theta)\, d\theta$$
$$= \int_0^\pi \left(7\cos\theta + \frac{15}{2}(1-\cos 2\theta) \right) d\theta$$
$$= \left[7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta \right]_0^\pi$$
$$= \frac{15}{2}\pi$$



$$\iint_R (3x+4y^2)dA$$
$$= \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$
$$= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta$$
$$= \int_1^2 3r^2\, dr \int_0^\pi \cos\theta\, d\theta + \int_1^2 4r^3\, dr \int_0^\pi \sin^2\theta\, d\theta$$
$$= \int_1^2 3r^2\, dr \int_0^\pi \cos\theta\, d\theta + \int_1^2 4r^3\, dr \int_0^\pi \frac{1}{2}(1-\cos 2\theta)\, d\theta$$
$$= \left[r^3 \right]_1^2 \left[\sin\theta \right]_0^\pi + \left[r^4 \right]_1^2 \left[\frac{1}{2}(\theta - \frac{1}{2}\sin 2\theta) \right]_0^\pi$$
$$= (2^3-1^3)(\sin\pi - \sin 0) + (2^4-1^4)\left(\frac{1}{2}(\pi - \frac{1}{2}\sin 2\pi) - \frac{1}{2}(0 - \frac{1}{2}\sin 0) \right)$$
$$= \frac{15}{2}\pi$$

예제. 다음 이중적분을 계산하여라.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$$



$$R=\{(x,y)\mid -1 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}$$
$$=\{(r,\theta)\mid 0 \le \theta \le \pi,\ 0 \le r \le 1\}$$

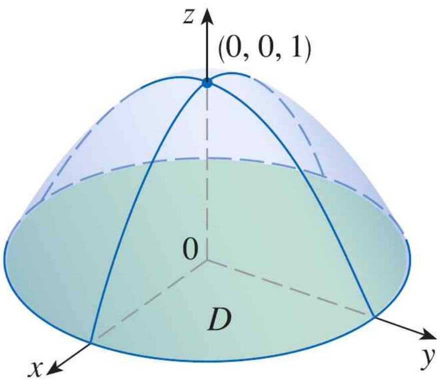
$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx=\int_0^{\pi}\int_0^{1}(r^2)\,r\,dr\,d\theta$$
$$=\int_0^1 r^3\,dr\times\int_0^{\pi}1\,d\theta$$
$$=\left[\frac{r^4}{4}\right]_0^1[\theta]_0^{\pi}=\left[\frac{1^4-0^4}{4}\right][\pi-0]=\frac{1}{4}\pi$$

(극좌표에서의) 이중적분 **다중적분**

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$$
$$=\int_{-1}^{1}\left[x^2y+\frac{y^3}{3}\right]_{y=0}^{y=\sqrt{1-x^2}}dx$$
$$=\int_{-1}^{1}\left(x^2\sqrt{1-x^2}+\frac{\sqrt{(1-x^2)^3}}{3}\right)dx$$
$$=\left[\frac{x\sqrt{1-x^2}(-1+2x^2)+\sin^{-1}x}{8}-\frac{\sqrt{(1-x^2)^3}\left(x\sqrt{1-x^2}(-5+2x^2)-3\sin^{-1}x\right)}{8(1-x^2)^{3/2}}\right]_{-1}^{1}$$
$$=\frac{1}{8}[(\sin^{-1}(1)+3\sin^{-1}(1))-(\sin^{-1}(-1)+3\sin^{-1}(-1)])$$
$$=\frac{4}{8}\left(\frac{\pi}{4}-(-\frac{\pi}{4})\right)=\frac{1}{2}\left(\frac{2\pi}{4}\right)=\frac{\pi}{4}$$

$$\int x^2\sqrt{1-x^2}\,dx \qquad x=\sin\theta,\ dx=\cos\theta\,d\theta$$
$$=\int \sin^2\theta\sqrt{1-\sin^2\theta}\ \cos\theta\,d\theta$$
$$=\int \sin^2\theta\cos\theta\cos\theta\,d\theta$$
$$=\int \sin^2\theta\cos^2\theta\,d\theta$$
$$=\int \frac{1-\cos 2\theta}{2}\,\frac{1+\cos 2\theta}{2}\,d\theta$$
$$=\frac{1}{4}\int (1-\cos^2 2\theta)\,d\theta$$
$$=\frac{1}{4}\int (1-\frac{1+\cos 4\theta}{2})\,d\theta$$
$$=\frac{1}{8}\int (1-\cos 4\theta)\,d\theta$$
$$=\frac{1}{8}(\theta-\frac{1}{4}\sin 4\theta)+C$$
$$=\frac{1}{8}\left[\theta-\frac{1}{4}(4\sin\theta\cos\theta-8\sin^3\theta\cos\theta)\right]+C$$
$$=\frac{1}{8}\left[\sin^{-1}x-x\ \sqrt{1-x^2}+2x^3\ \sqrt{1-x^2}\right]+C$$
$$=\frac{1}{8}\left[\sin^{-1}x+x\ \sqrt{1-x^2}(-1+2x^2\ )\right]+C$$

예제. 평면 $z=0$과 포물면 $z=1-x^2-y^2$으로 둘러싸인 입체의 부피를 구하여라.



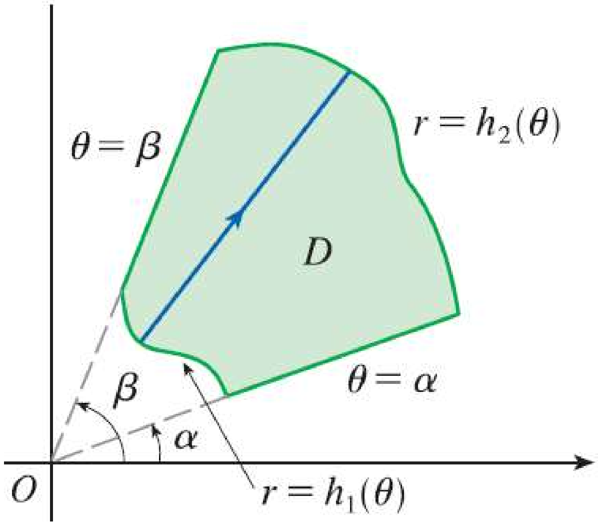
$$D=\{\,(x,y)\,|\,x^2+y^2\ \le\ 1\,\}$$
$$=\{\,(r,\theta)\,|\,0\ \le\ \theta\ \le\ 2\pi,\ \ 0\ \le\ r\ \le\ 1\,\}$$
$$V=\iint_D f(x,y)\,dA, \qquad z=f(x,y)=1-x^2-y^2$$
$$=\iint_D (1-x^2-y^2)\,dA=\int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$
$$=\int_0^{2\pi} 1\,d\theta\int_0^1 (r-r^3)\,dr=[\,\theta\,]_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1$$
$$=2\pi\cdot\frac{1}{4}=\frac{\pi}{2}$$

이중적분에서 극좌표로의 변환

f가 극영역(극좌표 영역)

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

상에서 연속이면



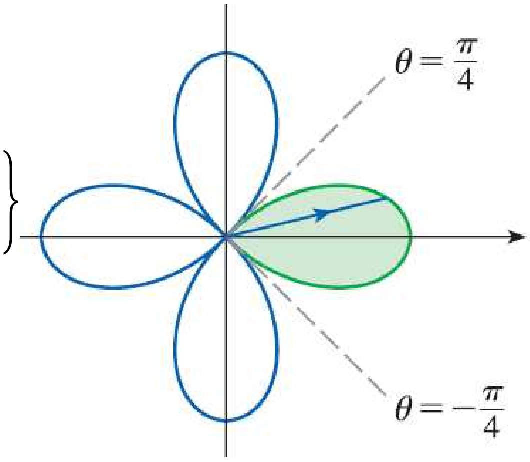
$$\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta$$

극좌표에서의 넓이

$f(x,y) = 1,\ h_1(\theta) = 0,\ h_2(\theta) = h(\theta),\ \alpha \le \theta \le \beta,\ 0 \le r \le h(\theta)$에 의해 유계된 영역 D의 넓이

$$\begin{aligned} A(D) &= \iint_D 1\, dA \\ &= \int_\alpha^\beta \int_0^{h(\theta)} 1\, r\, dr\, d\theta \\ &= \int_\alpha^\beta \left[\frac{r^2}{2} \right]_0^{h(\theta)} d\theta \\ &= \int_\alpha^\beta \frac{1}{2} [\, h(\theta)\,]^2\, d\theta \end{aligned}$$

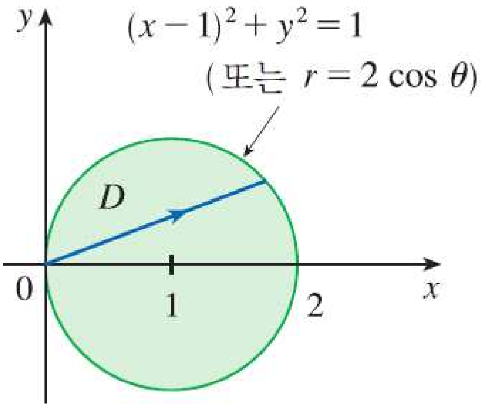
예제. 4엽장미 $r=\cos 2\theta$의 한 개의 닫힌곡선이 둘러싼 내부의 넓이를 이중적분을 이용하여 구하여라.



$$D=\left\{(r,\theta)\mid -\frac{\pi}{4}\le\theta\le\frac{\pi}{4},\ 0\le r\le \cos 2\theta\right\}$$

$$\begin{aligned}A(D)&=\iint_D dA=\iint_D 1\,dA\\&=\int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} r\,dr\,d\theta\\&=\int_{-\pi/4}^{\pi/4}\left[\frac{r^2}{2}\right]_0^{\cos 2\theta}d\theta=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\cos^2 2\theta\,d\theta\\&=\frac{1}{4}\int_{-\pi/4}^{\pi/4}(1+\cos 4\theta)\,d\theta=\frac{1}{4}\left[\theta+\frac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4}=\frac{\pi}{8}\end{aligned}$$

예제. xy-평면 위와 포물면 $z=x^2+y^2$ 아래, 그리고 원주면(원기둥) $x^2+y^2=2x$ 내부에 있는 입체의 부피를 구하여라.



$$x^2+y^2=2x \Rightarrow (x-1)^2+y^2=1$$

$$x^2+y^2=2x \Rightarrow r^2=2r\cos\theta \Rightarrow r=2\cos\theta$$

$$\begin{aligned}D&=\left\{(x,y)\mid 0\le x\le 2,\ -\sqrt{2x-x^2}\le y\le\sqrt{2x-x^2}\right\}\\&=\{(r,\theta)\mid 0\le\theta\le\pi,\ 0\le r\le 2\cos\theta\}\\&=\left\{(r,\theta)\mid -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ 0\le r\le 2\cos\theta\right\}\end{aligned}$$

$$V = \iint_D (x^2+y^2)\, dA = \int_0^{\pi} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \left[\frac{r^4}{4} \right]_0^{2\cos\theta} d\theta = 4\int_0^{\pi} \cos^4\theta\, d\theta$$

$$= 4\int_0^{\pi} \left(\frac{1+\cos 2\theta}{2} \right)^2 d\theta$$

$$= \int_0^{\pi} (1+2\cos 2\theta + \cos^2 2\theta)\, d\theta = \int_0^{\pi} \left(1+2\cos 2\theta + \frac{1+\cos 4\theta}{2} \right) d\theta$$

$$= \left[\frac{3}{2}\theta + 2\cdot\frac{1}{2}\sin 2\theta + \frac{1}{2}\cdot\frac{1}{4}\sin 4\theta \right]_0^{\pi}$$

$$= \frac{3}{2}\pi$$

$$V = \iint_D (x^2+y^2)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \right]_0^{2\cos\theta} d\theta = 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = 8\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= 8\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2} \right)^2 d\theta$$

$$= 2\int_0^{\pi/2} (1+2\cos 2\theta + \cos^2 2\theta)\, d\theta = 2\int_0^{\pi/2} \left(1+2\cos 2\theta + \frac{1+\cos 4\theta}{2} \right) d\theta$$

$$= 2\left[\frac{3}{2}\theta + 2\cdot\frac{1}{2}\sin 2\theta + \frac{1}{2}\cdot\frac{1}{4}\sin 4\theta \right]_0^{\pi/2}$$

$$= 2\cdot\frac{3}{2}\left(\frac{\pi}{2}\right) = \frac{3}{2}\pi$$