# 연쇄법칙

## 학습목표

- 연쇄법칙
- 음함수 미분





$$y = f(x),\ x = g(t)$$
$$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$z = f(x,y),\ x = g(t),\ y = h(t)$인 경우에 대해 고려

**연쇄법칙 : 경우 1**

$z = f(x,y)$가 $x$와 $y$에 관하여 미분가능 함수이고,
$x = g(t),\ y = h(t)$가 모두 $t$의 미분가능 함수라 하자.
그러면 $z$는 $t$의 미분가능 함수이고 다음과 같다.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

**예제**. $x=\sin 2t$, $y=\cos t$일 때, $z=x^2y+3xy^4$이면 $t=0$일 때 $\dfrac{dz}{dt}$를 구하여라.

$$z=(\sin 2t)^2(\cos t)+3(\sin 2t)(\cos t)^4$$

$$\begin{aligned}\frac{dz}{dt} &= 2(\sin 2t)\,(2\cos 2t)\cos t+(\sin 2t)^2\,(-\sin t)\\ &\quad +6(\cos 2t)\,(\cos t)^4+12(\sin 2t)\,(\cos t)^3\,(-\sin t)\\ &= 2xy\,(2\cos 2t)+x^2\,(-\sin t)+3y^4\,(2\cos 2t)+12xy^3\,(-\sin t)\\ &= (2xy+3y^4)\,(2\cos 2t)+(x^2+12xy^3)\,(-\sin t)\end{aligned}$$





$$z=x^2y+3xy^4,\ x=\sin 2t,\ y=\cos t$$

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}\\ &=(2xy+3y^4)\,(2\cos 2t)+(x^2+12xy^3)(-\sin t)\end{aligned}$$

$$t=0 \quad \Rightarrow \quad \begin{cases} x=\sin 0=0 \\ y=\cos 0=1\end{cases}$$

$$\begin{aligned}\left.\frac{dz}{dt}\right|_{t=0} &=(2\cdot 0\cdot 1+3\cdot 1^4)\,(2\cos 0)+(1^2+12\cdot 0\cdot 1^3)(-\sin 0)\\ &=3\cdot 2\cdot 1=6\end{aligned}$$

**예** $x = \sin t,\ y = e^t,\ z = x^2 y - y^2$일 때, $\dfrac{dz}{dt}$를 구하여라.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2xy)\ (\cos t) + (x^2 - 2y)(e^t) \\ &= (2e^t \sin t)\ (\cos t) + (\sin^2 t - 2e^t)(e^t) \\ &= e^t \sin 2t + (\sin^2 t - 2e^t)e^t\end{aligned}$$





**예** $x = \ln t,\ y = t^2,\ z = ye^x$일 때, $\dfrac{dz}{dt}$를 구하여라.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (ye^x)\ (\frac{1}{t}) + (e^x)(2t) \\ &= (t^2\, e^{\ln t})\ (\frac{1}{t}) + (e^{\ln t})(2t) \\ &= t^2 + 2t^2 = 3t^2\end{aligned}$$

**예제**. 이상기체 1몰의 압력 $P$(단위는 킬로파스칼), 부피(단위는 리터), 온도(단위는 켈빈) 사이의 관계식이 방정식 $PV=8.31T$로 주어졌다. 온도가 300K이고 0.1K/$s$의 비율로 증가하며, 부피가 100L이고 0.2L/$s$의 비율로 증가할 때 압력 $P$의 변화율을 구하여라.

$t$ : 시간(초, $s$), $T=300$, $\dfrac{dT}{dt}=0.1$, $V=100$, $\dfrac{dV}{dt}=0.2$

$$PV=8.31T \quad \Rightarrow \quad P=8.31\frac{T}{V}$$

$$\begin{aligned}\frac{dP}{dt} &= \frac{\partial P}{\partial T}\frac{dT}{dt}+\frac{\partial P}{\partial V}\frac{dV}{dt}\\ &= \left(\frac{8.31}{V}\right)\frac{dT}{dt}+\left(-\frac{8.31T}{V^2}\right)\frac{dV}{dt}\\ &= \left(\frac{8.31}{100}\right)(0.1)+\left(-\frac{8.31\times 300}{100^2}\right)(0.2)=-0.4155\end{aligned}$$

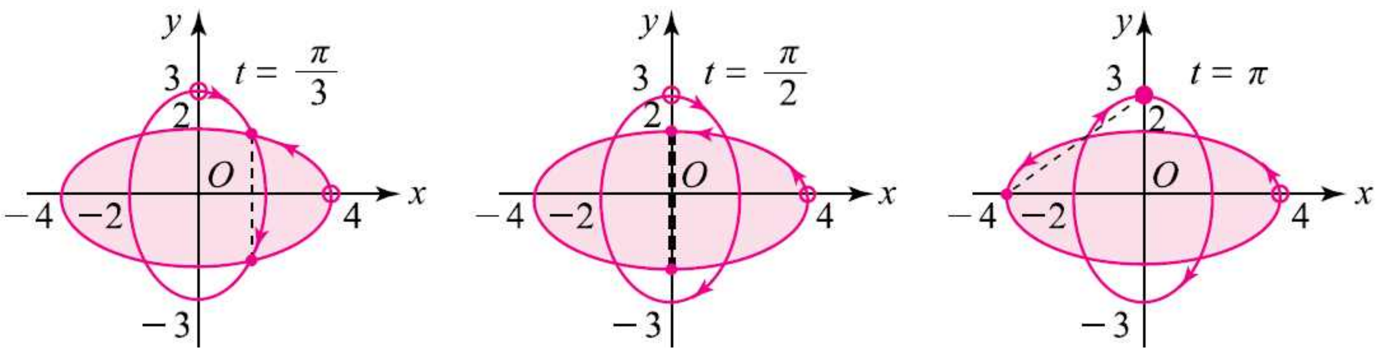




**예** 평면에서 타원을 따라 움직이는 두 물체의 좌표 $P(x_1,y_1)$과 $Q(x_2,y_2)$의 성분 함수가 각각 다음과 같이 매개방정식으로 주어진다고 하자.

$$C_1 : \begin{cases} x_1 = 4\cos t \\ y_1 = 2\sin t \end{cases} \qquad C_2 : \begin{cases} x_2 = 2\sin 2t \\ y_2 = 3\cos 2t \end{cases}$$

시각 $t=\pi$일 때, 두 물체 사이의 거리의 순간 변화율을 찾아라.



두 물체 사이의 거리 : $s(t)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$t=\pi$일 때 $P(-4,0)$, $Q(0,3)$

$s(\pi)=\sqrt{(0-(-4))^2+(3-0)^2}=5$

$$\frac{ds}{dt}=\frac{\partial s}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial s}{\partial y_1}\frac{dy_1}{dt}+\frac{\partial s}{\partial x_2}\frac{dx_2}{dt}+\frac{\partial s}{\partial y_2}\frac{dy_2}{dt}$$

$$\frac{\partial s}{\partial x_1}=\frac{-(x_2-x_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}=-\frac{1}{5}(0-(-4))=-\frac{4}{5}$$

$$\frac{\partial s}{\partial x_2}=\frac{(x_2-x_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}=\frac{1}{5}(0-(-4))=\frac{4}{5}$$

$$\frac{\partial s}{\partial y_1}=\frac{-(y_2-y_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}=-\frac{1}{5}(3-0)=-\frac{3}{5}$$

$$\frac{\partial s}{\partial y_2}=\frac{(y_2-y_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}=\frac{1}{5}(3-0)=\frac{3}{5}$$

$$\left.\frac{dx_1}{dt}\right|_{t=\pi}=-4\sin t|_{t=\pi}=0$$

$$\left.\frac{dx_2}{dt}\right|_{t=\pi}=4\cos 2t|_{t=\pi}=4$$

$$\left.\frac{dy_1}{dt}\right|_{t=\pi}=2\cos t|_{t=\pi}=-2$$

$$\left.\frac{dy_2}{dt}\right|_{t=\pi}=-6\sin 2t|_{t=\pi}=0$$

$$\frac{ds}{dt}=\frac{\partial s}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial s}{\partial y_1}\frac{dy_1}{dt}+\frac{\partial s}{\partial x_2}\frac{dx_2}{dt}+\frac{\partial s}{\partial y_2}\frac{dy_2}{dt}=\left(-\frac{4}{5}\right)\cdot 0+\left(-\frac{3}{5}\right)\cdot(-2)+\left(\frac{4}{5}\right)\cdot 4+\left(\frac{3}{5}\right)\cdot 0=\frac{22}{5}$$





$z=f(x,y)$, $\begin{cases} x=g(s,t), \\ y=h(s,t) \end{cases}$인 경우에 대해 고려

**연쇄법칙 : 경우 2**

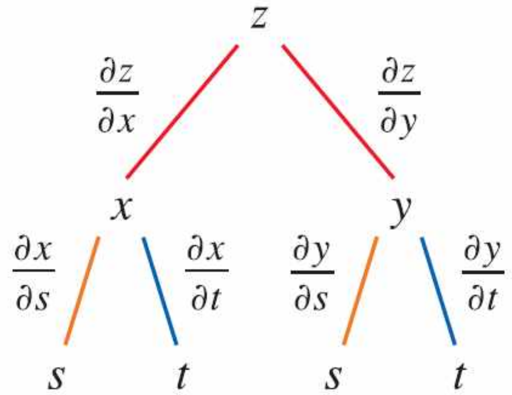
$z=f(x,y)$가 $x$와 $y$에 관하여 미분가능 함수이고,

$x=g(s,t)$, $y=h(s,t)$가 모두 $s,t$의 미분가능한 함수라 하자.

그러면 $z$는 $s$ or $t$의 편미분가능한 함수이고 다음이 성립한다.

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



**예제.** $z = e^x \sin y,\ x = st^2,\ y = s^2 t$일 때,

$\dfrac{\partial z}{\partial s},\ \dfrac{\partial z}{\partial t}$를 구하여라.

$$z = e^{st^2} \sin(s^2 t)$$

$$\frac{\partial z}{\partial s} = \left(t^2 e^{st^2}\right) \sin(s^2 t) + e^{st^2} \left(2st \cos(s^2 t)\right)$$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\ &= t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= (e^x \sin y)(2st) + (e^x \cos y)(s^2) \\ &= 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t)\end{aligned}$$





**예제.** $x = r\cos\theta,\ y = r\sin\theta$일 때, 다음 등식이 성립함을 보여라.

$$\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x} \qquad \frac{\partial x}{\partial \theta} = r^2 \frac{\partial \theta}{\partial x}$$

$$\frac{\partial y}{\partial r} = \frac{\partial r}{\partial y} \qquad \frac{\partial y}{\partial \theta} = r^2 \frac{\partial \theta}{\partial y}$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$r = \sqrt{x^2 + y^2} \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r}$$

$$\theta = \tan^{-1}\frac{y}{x} \quad \Rightarrow \quad \frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} = \frac{-y}{r^2} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{x}{r^2}$$

**예제.** $z=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$일 때, 다음 등식이 성립함을 보여라.

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}(\cos\theta)+\frac{\partial z}{\partial y}(\sin\theta)$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta)$$

$$\begin{aligned}\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2&=\left(\frac{\partial z}{\partial x}(\cos\theta)+\frac{\partial z}{\partial y}(\sin\theta)\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta)\right)^2\\&=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\end{aligned}$$





$u=f(x_1,x_2,\cdots,x_n)$, $x_j=g_j(t_1,t_2,\cdots,t_m)$인 경우에 대해 고려

**연쇄법칙 : 일반적인 경우**

$u$는 $n$개의 변수 $x_1,x_2,\cdots x_n$의 미분가능 함수이고,

각 $x_j$는 $m$개의 변수 $t_1,t_2,\cdots,t_m$의 미분가능 함수라 하자.

그러면 $u$는 $t_1,t_2,\cdots,t_m$의 함수이고,

각 $i=1,2,\cdots,m$에 관하여

$$\frac{\partial u}{\partial t_i}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i}+\cdots+\frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

이다.

**예** $u=f(x,y,z)$, $\begin{cases} x=x(s,t) \\ y=y(s,t) \\ z=z(s,t) \end{cases}$ 인 경우에 대한 연쇄법칙을 찾아라.

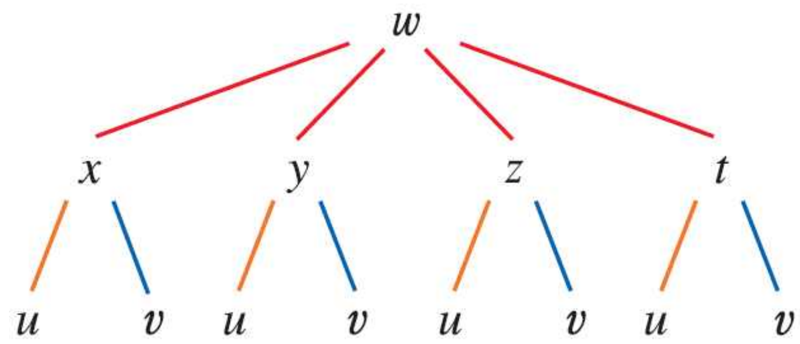
$$\frac{\partial u}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$
$$\frac{\partial u}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$





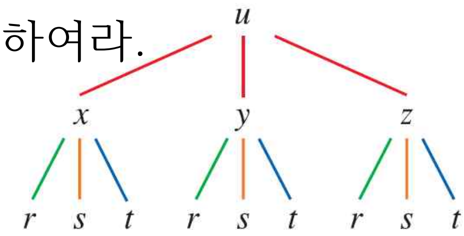
**예제.** $w=f(x,y,z,t)$, $\begin{cases} x=x(u,v) \\ y=y(u,v) \\ z=z(u,v) \\ t=t(u,v) \end{cases}$ 인 경우에 대한 연쇄법칙을 찾아라.

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$
$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$



**예제.** $x = rse^t$, $y = rs^2e^{-t}$, $z = r^2 s\sin t$이고 $u = x^4y + y^2z^3$이면

$r=2$, $s=1$, $t=0$일 때 $\dfrac{\partial u}{\partial s}$의 값을 구하여라.




$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$\begin{aligned}
&= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)\\
&= \left\{4(rse^t)^3(rs^2e^{-t})\right\}(re^t) + \left\{(rse^t)^4 + 2(rs^2e^{-t})(r^2s\sin t)^3\right\}(2rse^{-t}) + \left\{3(rs^2e^{-t})^2(r^2s\sin t)^2\right\}(r^2\sin t)\\
&= 4r^5s^5e^{3t} + \left(r^4s^4e^{4t} + 2r^7s^5e^{-t}\sin^3 t\right)(2rse^{-t}) + 3r^8s^6e^{-2t}\sin^3 t
\end{aligned}$$

$r=2$, $s=1$, $t=0$

$$\begin{aligned}
\frac{\partial u}{\partial s} &= 4r^5s^5e^{3t} + \left(r^4s^4e^{4t} + 2r^7s^5e^{-t}\sin^3 t\right)(2rse^{-t}) + 3r^8s^6e^{-2t}\sin^3 t\\
&= 4\cdot 2^5\cdot 1^5\cdot e^{3\cdot 0} + \left(2^4\cdot 1^4\cdot e^{4\cdot 0} + 2\cdot 2^7\cdot 1^5\cdot e^{-0}\cdot \sin^3 0\right)(2\cdot 2\cdot 1\cdot e^{-0}) + 3\cdot 2^8\cdot 1^6\cdot e^{-2\cdot 0}\cdot \sin^3 0\\
&= 124 + (16+0)(4) + 0 = 192
\end{aligned}$$

$r=2$, $s=1$, $t=0 \quad\Rightarrow\quad x = 2\cdot 1\cdot e^0 = 2$, $y = 2\cdot 1^2\cdot e^{-0} = 2$, $z = 2^2\cdot 1\cdot sin\, 0 = 0$

$$\begin{aligned}
\frac{\partial u}{\partial s} &= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)\\
&= (4\cdot 2^3\cdot 2)(2\cdot e^0) + (2^4 + 2\cdot 2\cdot 0^3)(2\cdot 2\cdot 1\cdot e^{-0}) + (3\cdot 2^2\cdot 0^2)(2^2\cdot \sin 0)\\
&= (64)(2) + (16+0)(4) + (0)(0) = 192
\end{aligned}$$





**예제.** $g(s,t) = f(s^2 - t^2, t^2 - s^2)$이고 $f$가 미분가능일 때,

$g$는 다음 방정식을 만족함을 보여라.

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial s} = 0$$

$x = s^2 - t^2$, $y = t^2 - s^2$

$g(s,t) = f(x,y)$

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

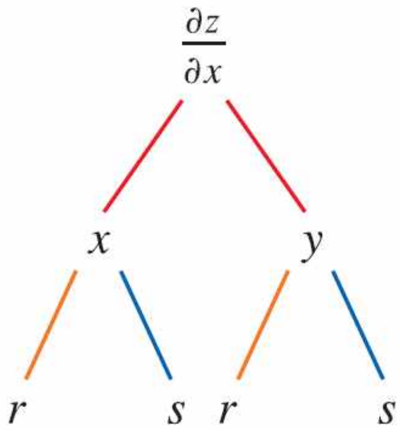
$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial s} = t\left(\frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)\right) + s\left(\frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)\right) = 0$$

**예제.** 만약 $z=f(x,y)$가 연속인 2계 편도함수를 갖고

$x=r^2+s^2,\ y=2rs$이면

(a) $\dfrac{\partial z}{\partial r}$와 (b) $\dfrac{\partial^2 z}{\partial r^2}$를 구하여라.




(a) $\dfrac{\partial z}{\partial r}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r}=\dfrac{\partial z}{\partial x}(2r)+\dfrac{\partial z}{\partial y}(2s)$

(b) $\dfrac{\partial z}{\partial x}=g(x,y),\ \dfrac{\partial z}{\partial y}=h(x,y)$라 놓으면

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial g}{\partial r}=\frac{\partial g}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial g}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}=\frac{\partial^2 z}{\partial x^2}(2r)+\frac{\partial^2 z}{\partial y\,\partial x}(2s)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial h}{\partial r}=\frac{\partial h}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial h}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r}=\frac{\partial^2 z}{\partial x\,\partial y}(2r)+\frac{\partial^2 z}{\partial y^2}(2s)$$





$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}(2r)+\frac{\partial^2 z}{\partial y\,\partial x}(2s)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\,\partial y}(2r)+\frac{\partial^2 z}{\partial y^2}(2s)$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2}&=\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)=\frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x}+2s\frac{\partial z}{\partial y}\right)\\
&=\frac{\partial}{\partial r}(2r)\cdot\frac{\partial z}{\partial x}+2r\cdot\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)+\frac{\partial}{\partial r}(2s)\cdot\frac{\partial z}{\partial y}+2s\cdot\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\\
&=2\cdot\frac{\partial z}{\partial x}+2r\left(2r\frac{\partial^2 z}{\partial x^2}+2s\frac{\partial^2 z}{\partial y\,\partial x}\right)+2s\cdot\left(2r\frac{\partial^2 z}{\partial x\,\partial y}+2s\frac{\partial^2 z}{\partial y^2}\right)\\
&=2\cdot\frac{\partial z}{\partial x}+4r^2\frac{\partial^2 z}{\partial x^2}+4rs\frac{\partial^2 z}{\partial y\,\partial x}+4rs\frac{\partial^2 z}{\partial x\,\partial y}+4s^2\frac{\partial^2 z}{\partial y^2}\\
&=2\cdot\frac{\partial z}{\partial x}+4r^2\frac{\partial^2 z}{\partial x^2}+8rs\frac{\partial^2 z}{\partial x\,\partial y}+4s^2\frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

**예** $z=e^x\sin y,\ x=st^2,\ y=s^2t$일 때, $\dfrac{\partial z}{\partial s},\ \dfrac{\partial^2 z}{\partial s^2}$를 구하여라.

$$\begin{aligned}\frac{\partial z}{\partial s}&=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\\&=(e^x\sin y)(t^2)+(e^x\cos y)(2st)\\&=t^2e^{st^2}\sin(s^2t)+2ste^{st^2}\cos(s^2t)\end{aligned}$$

$$\frac{\partial z}{\partial x}=e^x\sin y=e^{st^2}\sin(s^2t) \qquad \frac{\partial x}{\partial s}=t^2$$

$$\frac{\partial z}{\partial y}=e^x\cos y=e^{st^2}\cos(s^2t) \qquad \frac{\partial y}{\partial s}=2st$$

$$\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial s}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial s}=\frac{\partial^2 z}{\partial x^2}(t^2)+\frac{\partial^2 z}{\partial y\,\partial x}(2st)=(e^x\sin y)(t^2)+(e^x\cos y)(2st)$$

$$\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial s}+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial s}=\frac{\partial^2 z}{\partial x\,\partial y}(t^2)+\frac{\partial^2 z}{\partial y^2}(2st)=(e^x\cos y)(t^2)+(-e^x\sin y)(2st)$$

$$\begin{aligned}\frac{\partial^2 z}{\partial s^2}&=\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}\right)=\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right)\\&=\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial s}\left(\frac{\partial x}{\partial s}\right)+\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial y}\right)\cdot\frac{\partial y}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial s}\left(\frac{\partial y}{\partial s}\right)\\&=\left(\frac{\partial^2 z}{\partial x^2}\cdot(t^2)+\frac{\partial^2 z}{\partial y\,\partial x}\cdot(2st)\right)\cdot(t^2)+\frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial s}(t^2)+\left(\frac{\partial^2 z}{\partial x\,\partial y}\cdot(t^2)+\frac{\partial^2 z}{\partial y^2}\cdot(2st)\right)\cdot(2st)+\frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial s}(2st)\\&=\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\cdot(t^2)+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\cdot(2st)\right]\cdot(t^2)+\frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial s}(t^2)+\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\cdot(t^2)+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\cdot(2st)\right]\cdot(2st)+\frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial s}(2st)\end{aligned}$$





## 음함수 미분

$F(x,y)=0$ 형태의 방정식

즉 $f$의 정의역에 있는 모든 $x$에 관하여 $y=f(x)$이면 $F(x,f(x))=0$

만약 $F$가 미분가능이면, 연쇄법칙의 경우 1을 적용하여 $x$에 관해 식 $F(x,y)=0$의 양변을 미분할 수 있다.

$$\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0, \qquad \frac{dx}{dx}=1$$

$$\boxed{\frac{dy}{dx}=-\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}}=-\frac{F_x}{F_y}, \qquad \frac{\partial F}{\partial y}\neq 0}$$

**예제.** $x^3+y^3=6xy$일 때 $y'$을 구하여라.

$$F(x,y)=x^3+y^2-6xy=0$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{3x^2-6y}{3y^2-6x}=-\frac{x^2-2y}{y^2-2x}$$





함수 $z=f(x,y)$가 방정식 $F(x,y,z)=0$의 형태 : $x,y$의 음함수
즉 $f$의 정의역에 있는 모든 $(x,y)$에 대하여 $F(x,y,f(x,y))=0$
만약 $F,f$가 미분가능이면, 연쇄법칙을 이용하여 다음과 같이 미분

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0, \qquad \begin{cases} \dfrac{\partial}{\partial x}(x)=1, & \dfrac{\partial}{\partial x}(y)=0 \\ \dfrac{\partial}{\partial x}(y)=0, & \dfrac{\partial}{\partial x}(y)=1 \end{cases}$$

$$\boxed{\frac{\partial z}{\partial x}=-\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}=-\frac{F_x}{F_z}, \quad \frac{\partial F}{\partial z}\neq 0 \qquad \frac{\partial z}{\partial y}=-\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}=-\frac{F_y}{F_z}, \quad \frac{\partial F}{\partial z}\neq 0}$$

**예제.** $x^3+y^3+z^3+6xyz+4=0$ 때, $\dfrac{\partial z}{\partial x}$와 $\dfrac{\partial z}{\partial y}$를 구하고

점 $(-1,1,2)$에서의 편도함수의 값을 구하여라.

$$\frac{\partial}{\partial x}\left(x^3+y^3+z^3+6xyz+4\right) = \frac{\partial}{\partial x}(0)$$
$$\frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial x}(y^3)+\frac{\partial}{\partial x}(z^3)+\frac{\partial}{\partial x}(6xyz)+\frac{\partial}{\partial x}(4)=0$$
$$3x^2+0+3z^2\frac{\partial z}{\partial x}+6yz+6xy\frac{\partial z}{\partial x}+0 = 0$$
$$\Rightarrow\quad 3(z^2+2xy)\frac{\partial z}{\partial x}=-3(x^2+2yz)$$

$$\Rightarrow\quad \frac{\partial z}{\partial x}=-\frac{x^2+2yz}{z^2+2xy}$$

$$F(x,y,z)=x^3+y^3+z^3+6xyz+4=0$$
$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2+6yz}{3z^2+6xy}=-\frac{x^2+2yz}{z^2+2xy}$$
$$\left.\frac{\partial z}{\partial x}\right|_{(x,y,z)=(-1,1,2)}=-\frac{(-1)^2+2\cdot1\cdot2}{2^2+2\cdot(-1)\cdot1}=-\frac{5}{2}$$





$$\frac{\partial}{\partial y}\left(x^3+y^3+z^3+6xyz+4\right) = \frac{\partial}{\partial y}(0)$$
$$\frac{\partial}{\partial y}(x^3)+\frac{\partial}{\partial y}(y^3)+\frac{\partial}{\partial y}(z^3)+\frac{\partial}{\partial y}(6xyz)+\frac{\partial}{\partial y}(4)=0$$
$$0+3y^2+3z^2\frac{\partial z}{\partial y}+6xz+6xy\frac{\partial z}{\partial y}+0 = 0$$
$$\Rightarrow\quad 3(z^2+2xy)\frac{\partial z}{\partial x}=-3(y^2+2xz)$$

$$\Rightarrow\quad \frac{\partial z}{\partial x}=-\frac{y^2+2xz}{z^2+2xy}$$

$$F(x,y,z)=x^3+y^3+z^3+6xyz+4=0$$
$$\frac{\partial z}{\partial x}=-\frac{F_y}{F_z}=-\frac{3y^2+6xz}{3z^2+6xy}=-\frac{y^2+2xz}{z^2+2xy}$$
$$\left.\frac{\partial z}{\partial x}\right|_{(x,y,z)=(-1,1,2)}=-\frac{1^2+2\cdot(-1)\cdot2}{2^2+2\cdot(-1)\cdot1}=\frac{3}{2}$$

**예** 방정식 $x^3e^{y+z}-y\sin(x-z)=0$에 의해 $x$와 $y$의 음함수로 정의될 때, $\dfrac{\partial z}{\partial x}$와 $\dfrac{\partial z}{\partial y}$를 찾아라.

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2e^{y+z}\cdot(1)-y\cos(x-z)\cdot(1)}{x^3e^{y+z}\cdot(1)-y\cos(x-z)\cdot(-1)}$$
$$=-\frac{3x^2e^{y+z}-y\cos(x-z)}{x^3e^{y+z}+y\cos(x-z)}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{x^3e^{y+z}\cdot(1)-\sin(x-z)}{x^3e^{y+z}\cdot(1)-y\cos(x-z)\cdot(-1)}$$
$$=-\frac{x^3e^{y+z}-\sin(x-z)}{x^3e^{y+z}+y\cos(x-z)}$$





**예** 변수 $z$가 방정식 $x^3e^{\sin(y-3z)}+y(x+z^2)=0$에 의하여 $x$와 $y$의 음함수로 정의될 때, $\dfrac{\partial z}{\partial x}$와 $\dfrac{\partial z}{\partial y}$를 찾아라.

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2e^{\sin(y-3z)}+y\cdot(1)}{x^3e^{\sin(y-3z)}\cdot(\cos(y-3z)\cdot(-3))+y\cdot(2z)}$$
$$=-\frac{3x^2e^{\sin(y-3z)}+y}{-3x^3\cos(y-3z)e^{\sin(y-3z)}+2yz}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{x^3e^{\sin(y-3z)}\cdot(\cos(y-3z)\cdot(1))+(1)\cdot(x+z^2)}{x^3e^{\sin(y-3z)}\cdot(\cos(y-3z)\cdot(-3))+y\cdot(2z)}$$
$$=-\frac{x^3\cos(y-3z)e^{\sin(y-3z)}+x+z^2}{-3x^3\cos(y-3z)e^{\sin(y-3z)}+2yz}$$