Advanced Microeconomics

Assignment 3

3.D.5 Consider again the CES utility function of Exercise 3.C.6, and assume that $\alpha_1 = \alpha_2 = 1$.

(b) Verify that these functions satisfy all the properties of Propositions 3.D.2 and 3.D.3.

Solution.

(b) **Proposition 3.D.2**

(i) **Homogeneity of degree zero** of the demand function. For any p, w and $\alpha > 0$, we have

$$x_1(\alpha p, \alpha w) = \frac{(\alpha p_1)^{\frac{1}{\rho-1}}}{(\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}} \alpha w = \frac{p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w = x_1(p, w),$$

$$x_2(\alpha p, \alpha w) = \frac{(\alpha p_2)^{\frac{1}{\rho-1}}}{(\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}} \alpha w = \frac{p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w = x_2(p, w).$$

(ii) **Walras' law**. Direct calculation gives

$$\begin{aligned} p_1 x_1 + p_2 x_2 &= p_1 \frac{p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w + p_2 \frac{p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w \\ &= \frac{p_1^{\frac{\rho}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w + \frac{p_2^{\frac{\rho}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w \\ &= w, \end{aligned}$$

(iii) The **uniqueness** is trivial: $x(p, w)$ is unique given the explicit expression.

Proposition 3.D.3

(i) **Homogeneity of degree zero** in price of the indirect utility function. For any $\alpha > 0$, we have

$$\begin{aligned} v(\alpha p, w) &= \left((\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} \alpha w \\ &= \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w \\ &= v(p, w). \end{aligned}$$

(ii) **Monotonicity** of the indirect utility function. Since for any $p \gg 0, w > 0$,

$$\begin{aligned}
\frac{\partial v(p,w)}{\partial w} &= \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} > 0,\\
\frac{\partial v(p,w)}{\partial p_l} &= \frac{1-\rho}{\rho}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-2\rho}{\rho}} w \left(\frac{\rho}{\rho-1} p_l^{\frac{1}{\rho-1}}\right)\\
&= -\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-2\rho}{\rho}} w p_l^{\frac{1}{\rho-1}} < 0 \text{ for } l = 1, 2.
\end{aligned}$$

Hence, the indirect function is strictly increasing in w and strictly decreasing in p_l for all l.

(iii) **Quasiconvexity**. To prove quasiconvexity, we claim that, by homogeneity of degree zero, it suffices to prove that for any $\bar{v} \in \mathbb{R}$ and $w > 0$, the set $\{p \in \mathbb{R}^2_{++} : v(p,w) \leq \bar{v}\}$ is convex.

For $\rho \to 0$, the utility function is Cobb-Douglas, and the indirect utility function is given by $v(p,w) = \frac{w^2}{4p_1p_2}$ which is convex in p [1], and hence quasiconvex.

For $\rho < 0$, since $p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is concave in p [2], the set $\{p : v(p,w) \leq \bar{v}\} = \{p : p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} \geq \bar{v}^{\frac{\rho}{1-\rho}}\}$ is convex.

For $\rho \in (0,1)$, since $p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is convex in p [3], the set $\{p : v(p,w) \leq \bar{v}\} = \{p : p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} \leq \bar{v}^{\frac{\rho}{1-\rho}}\}$ is convex.

To justify the claim, note that quasiconvexity states that

$$v(p,w) \leq \bar{v}, v(p',w') \leq \bar{v} \implies v(\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w') \leq \bar{v}.$$

And homogeneity of degree zero implies $v(p,w) = v\left(\frac{p}{w}, 1\right)$. Hence, quasiconvexity is equivalent to

$$v\left(\frac{p}{w}, 1\right) \leq \bar{v}, v\left(\frac{p'}{w'}, 1\right) \leq \bar{v} \implies v\left(\frac{\alpha p + (1-\alpha)p'}{\alpha w + (1-\alpha)w'}, 1\right) \leq \bar{v}. \tag{1}$$

[1]The Hessian matrix of the function $v(p,w) = \frac{w^2}{4p_1p_2}$ is $w^2 \begin{bmatrix} \frac{1}{2p_1^3p_2} & \frac{1}{4p_1^2p_2^2} \\ \frac{1}{4p_1^2p_2^2} & \frac{1}{2p_1p_2^3} \end{bmatrix}$, which is positive definite when $p \gg 0$.

[2]The Hessian matrix of the function $g(p) = p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is $\begin{bmatrix} \frac{\rho}{(\rho-1)^2} p_1^{\frac{2-\rho}{\rho-1}} & 0 \\ 0 & \frac{\rho}{(\rho-1)^2} p_2^{\frac{2-\rho}{\rho-1}} \end{bmatrix}$, which is negative definite when $\rho < 0$ and $p \gg 0$.

[3]The Hessian matrix of the function $g(p) = p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is $\begin{bmatrix} \frac{\rho}{(\rho-1)^2} p_1^{\frac{2-\rho}{\rho-1}} & 0 \\ 0 & \frac{\rho}{(\rho-1)^2} p_2^{\frac{2-\rho}{\rho-1}} \end{bmatrix}$, which is positive definite when $\rho \in (0,1)$ and $p \gg 0$.

Let $p_1 = \frac{p}{w}$, $p_2 = \frac{p'}{w'}$, then

$$\frac{\alpha p + (1-\alpha)p'}{\alpha w + (1-\alpha)w'} = \beta p_1 + (1-\beta)p_2,$$

where

$$\beta = \frac{\alpha w}{\alpha w + (1-\alpha)w'} \in (0,1).$$

Now if the set $\{p \in \mathbb{R}^L_{++} : v(p,w) \leq \bar{v}\}$ is convex, we have

$$v(p_1,1) \leq \bar{v}, v(p_2,1) \leq \bar{v} \implies v(\beta p_1 + (1-\beta)p_2, 1) \leq \bar{v}.$$

Therefore, we have established (1).

(iv) **Continuity** follows from the functional form of $v(p,w)$.