

## Chapter 2. Consumer Choice

### 2.A. Introduction

In this chapter, we perform analysis of choice structure in the context of consumption. In other words, we analyze consumer demand for commodities.

### 2.B. Commodities

The decision problem faced by the consumer is to choose the consumption levels of various goods or services. We call the goods and services *commodities*. A *commodity vector* (or commodity *bundle*) is a point

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix} \in \mathbb{R}^L$$

- Number of commodities  $L$ , indexed by  $l = 1, 2, \dots, L$ .
- $\mathbb{R}^L$  is the commodity *space*.
- $x_l$  is the amount of commodity  $l$  consumed.

*Remark.* Time (see the example below) and location (see Figure 3), could be built into the definition of a commodity.

For example,  $x_1$  could be bread today, and  $x_2$  could be bread tomorrow. (In this example, we ignore other commodities.) Alice who plans to consume 5 slices of bread today and 6 slices of bread tomorrow would have a commodity vector

$$x = \begin{bmatrix} x_1 = 5 \\ x_2 = 6 \end{bmatrix} \in \mathbb{R}^2.$$

### 2.C. Consumption Set

The *consumption set* is a subset of the commodity space  $\mathbb{R}^L$ , denoted by  $X \subset \mathbb{R}^L$ , whose elements are the consumption bundles that the individual can conceivably consume given the physical and institutional constraints imposed by his environment.

Below are some examples of 2 commodities, i.e.,  $L = 2$ , with *Physical Constraints*:

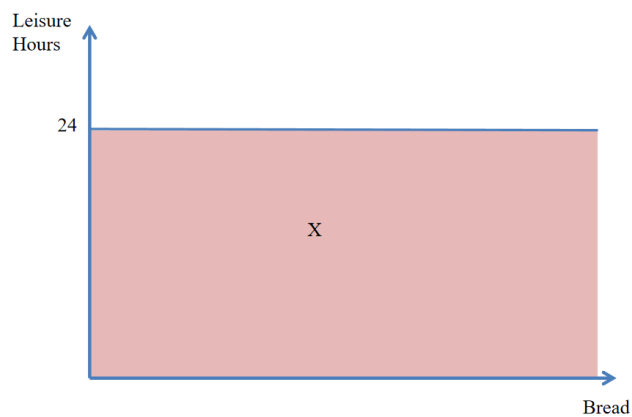


Figure 1: Possible consumption levels of bread and leisure in a day

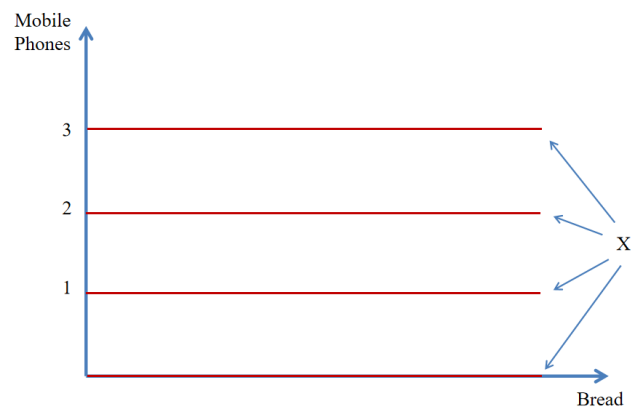


Figure 2: Possible consumption levels of bread and mobile phones

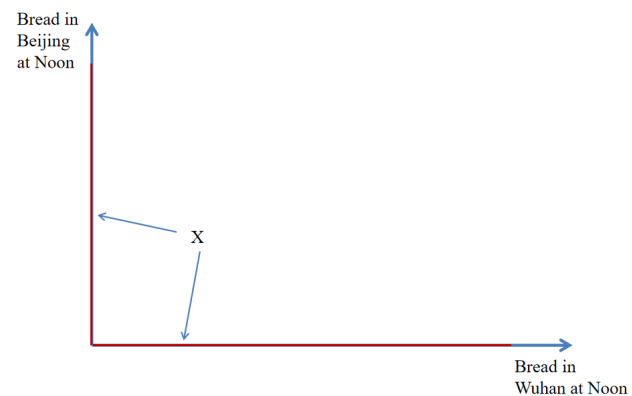


Figure 3: Possible consumption levels of bread in Beijing and Wuhan at noon

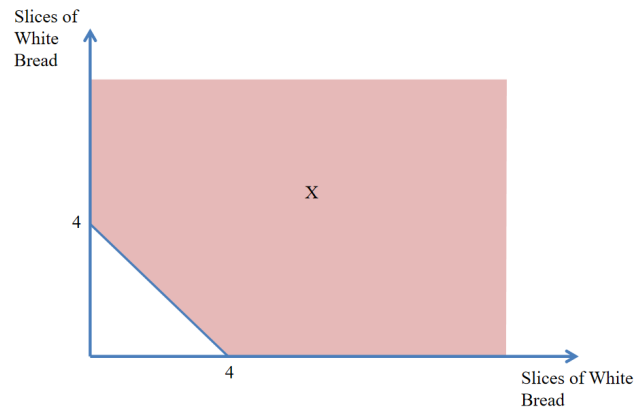


Figure 4: Possible consumption levels of bread where  
the minimum survival amount is 4 slices and only 2 types of bread are available

There could also be *Institutional Constraints*.

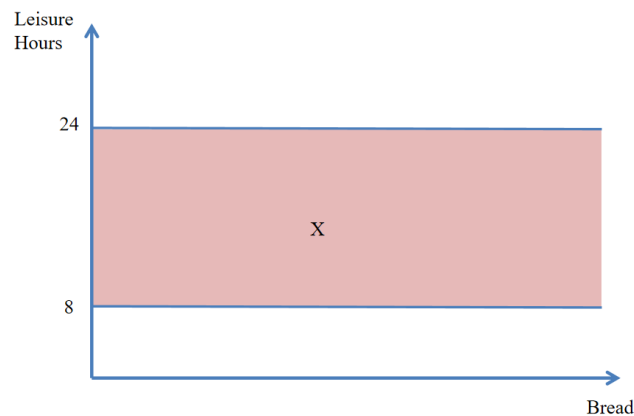
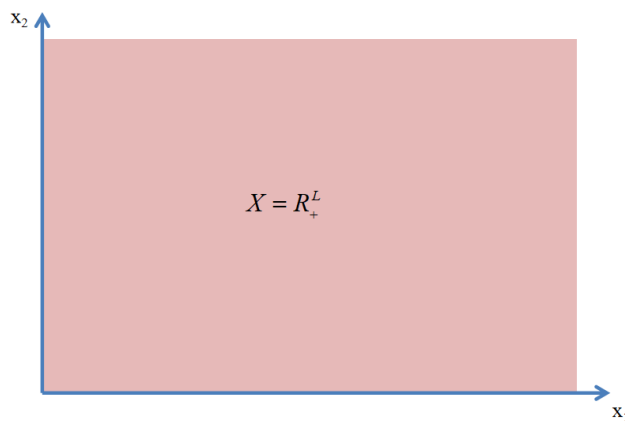


Figure 5: Possible consumption levels of bread and leisure in a day with a law requiring that  
no one work more than 16 hours a day

Practically, to keep our discussion in this section as straightforward as possible, we adopt the simplest consumption set:

$$X = \mathbb{R}_+^L = \{x \in \mathbb{R}^L : x_l \geq 0 \text{ for } l = 1, 2, \dots, L\}.$$

Below is an illustration of the consumption set  $\mathbb{R}_+^L$  in 2 dimensions, i.e.,  $\mathbb{R}_+^2$ .


 Figure 6: The consumption set  $\mathbb{R}_+^L$ 

*Remark.*  $X$  is convex:  $x \in X, x' \in X \implies \alpha x + (1 - \alpha)x' \in X$ .

**Proof.** Given any two commodities  $l, k = 1, \dots, L$ .

$$x_l \geq 0, x_k \geq 0 \implies \alpha x_l + (1 - \alpha)x_k \geq 0$$

□

Much of the theory to be developed applies also for more general convex consumption sets (for example, the consumption sets illustrated in Figures 1, 4, 5).<sup>1</sup>

## 2.D. Competitive Budgets (Affordability)

In addition to the physical and institutional constraints, the consumer also faces *economic* constraint: affordability.

To formalize the economics constraint, we assume that  $L$  commodities are all traded at public dollar prices and that consumers are *price takers*. Formally, prices are represented by the *price vector*:

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L$$

**Assumption.**  $p \gg 0$ , i.e.,  $p_l > 0, \forall l$ .

Throughout the course, we make the above assumption, even though the assumption may not be reasonable. There exist scenarios in real life that  $p_l = 0$ , or even  $p_l < 0$ . We provide two counter examples below.

<sup>1</sup>You should check by yourselves that the consumption sets in Figures 1, 4, 5 are convex.

### Counter Examples.

1. Someone invites you: for you,  $p_l = 0$ .
2. Sometimes parents pay kid to read books: for the kid,  $p_l < 0$ .

**Economic-Affordability Constraint** The affordability of a consumption bundle depends on

1. market prices:  $p = (p_1, \dots, p_L)$
2. consumer's wealth level (in dollars):  $w$

The consumption bundle  $x \in \mathbb{R}_+^L$  is affordable if

$$p \cdot x = p_1x_1 + \dots + p_Lx_L \leq w.$$

### Walrasian budget set

**Definition 2.D.1.** The Walrasian, or competitive budget set  $B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$  is the set of all feasible consumption bundles for the consumer who faces market prices  $p$  and has wealth  $w$ .

The consumer's problem is to choose *consumption bundle*  $x$  from  $B_{p,w}$ .

The set  $\{x \in \mathbb{R}_+^L : p \cdot x = w\}$  is called the *budget hyperplane*.

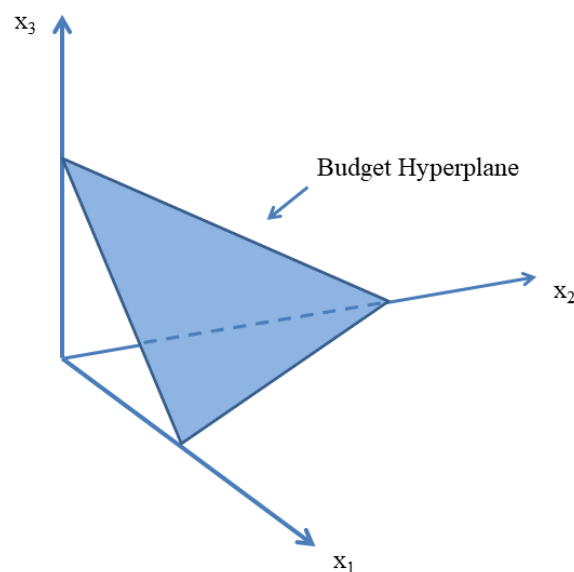


Figure 7: Budget Hyperplane (3 commodities)

When  $L = 2$ , Budget Hyperplane is Budget Line. The slope  $-\frac{p_1}{p_2}$  captures the rate of exchange between the two commodities.

- $\frac{p_1}{p_2}$  describes the units of  $x_2$  the consumer can obtain by giving up one unit of  $x_1$ :  
 one unit of  $x_1 \implies p_1$  of money  $\implies \frac{p_1}{p_2}$  units of  $x_2$

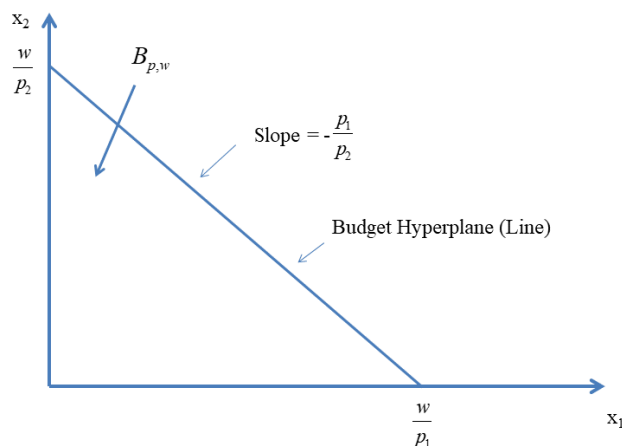


Figure 8: Budget hyperplane (line) for two commodities

The price vector  $p$ , drawn from any point  $\bar{x}$  on the budget hyperplane, must be orthogonal to any vector starting at  $\bar{x}$  and lying on the budget hyperplane.

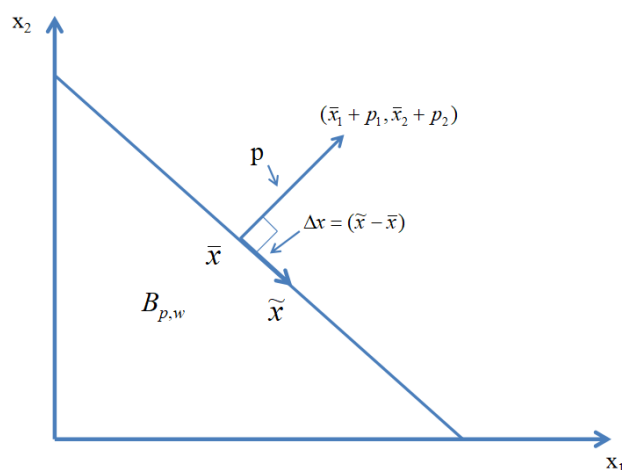


Figure 9: The geometric relationship between  $p$  and the budget hyperplane

To check the orthogonality, we need to check whether  $p \cdot \Delta x = 0$ , where  $\Delta x = \tilde{x} - \bar{x}$  and  $\tilde{x}, \bar{x}$  are on the budget hyperplane. This is true because  $p \cdot \tilde{x} = p \cdot \bar{x} = w$ .

**Walrasian budget set  $B_{p,w}$  is convex.**

**Proof.** We need to show that for all  $x, x' \in B_{p,w}$ ,  $x'' = \alpha x + (1 - \alpha)x' \in B_{p,w}$ .

First,  $x, x' \in \mathbb{R}_+^L \implies x'' \in \mathbb{R}_+^L$ . Second, since  $p \cdot x \leq w$  and  $p \cdot x' \leq w$ , we have  $p \cdot x'' = p \cdot [\alpha x + (1 - \alpha)x'] = \alpha(p \cdot x) + (1 - \alpha)(p \cdot x') \leq w$ .

Thus,  $x'' \in B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$ .  $\square$

### Exercise 2.D.1

A consumer lives for two periods, denoted 1 and 2, and consumes a single consumption good in each period. His wealth when born is  $w > 0$ . What is his (lifetime) Walrasian budget set?

### Exercise 2.D.2

A consumer consumes one consumption good  $x$  and hours of leisure  $h$ . The price of the consumption good is  $p$ , and the consumer can work at a wage rate of  $s = 1$ . What is the consumer's Walrasian budget set?

*Remark.* The convexity of  $B_{p,w}$  depends on the convexity of the consumption set.  $B_{p,w}$  will be convex as long as  $X$  is.

**Proof.** We need to show that for all  $x, x' \in B_{p,w}$ ,  $x'' = \alpha x + (1 - \alpha)x' \in B_{p,w}$ .

First,  $x, x' \in X \implies x'' \in X$  since  $X$  is convex. Second, since  $p \cdot x \leq w$  and  $p \cdot x' \leq w$ , we have  $p \cdot x'' = p \cdot \alpha x + (1 - \alpha)x' = \alpha(p \cdot x) + (1 - \alpha)(p \cdot x') \leq w$ .

Thus,  $x'' \in B_{p,w} = \{x \in X : p \cdot x \leq w\}$ .  $\square$

## 2.E. Demand Functions and Comparative Statics

The consumer's *Walrasian* (or *market*, or *ordinary*) *demand correspondence*  $x(p, w)$  assigns a set of chosen consumption bundles for each  $(p, w)$ .

When  $x(p, w)$  is single-valued, we refer to it as a *demand function*.

**Assumption.**

1.  $x(p, w)$  is homogeneous of degree zero.
2.  $x(p, w)$  satisfies Walras' law.

General definition of Homogeneous Functions:

**Definition.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is Homogeneous of Degree  $k$  if for any  $\alpha > 0$ ,

$$f(\alpha x_1, \alpha x_2, \dots, \alpha x_n) = \alpha^k f(x_1, x_2, \dots, x_n).$$

**Example.** 1.  $f(x, y) = xy$  is Homogeneous of Degree 2.

2.  $f(x, y, z) = \frac{x}{y} + \frac{2z}{x}$  is Homogeneous of Degree 0.

3.  $f(x_1, x_2) = Ax_1^a x_2^b$  is Homogeneous of Degree  $a + b$ .

4.  $f(x_1, x_2) = x_1 + x_2^2$  is not a Homogeneous Function.

Homogeneity in Example 1 to 3 could be easily checked. For Example 4, we provide a proof.

**Proof.** We prove by contradiction.

Suppose  $f(x_1, x_2) = x_1 + x_2^2$  is not Homogeneous of Degree  $k$ . We must have

$$\begin{aligned} f(\alpha x_1, \alpha x_2) &= \alpha^k f(x_1, x_2) \\ \implies \alpha x_1 + (\alpha x_2)^2 &= \alpha^k (x_1 + x_2^2) \quad \forall \alpha > 0, x_1, x_2 \in \mathbb{R} \end{aligned}$$

In particular, taking  $\alpha = 2$  gives

$$2x_1 + 4x_2^2 = 2^k x_1 + 2^k x_2^2.$$

Then, for  $(x_1, x_2) = (1, 0)$  and  $(x_1, x_2) = (0, 1)$ , we have  $k = 1$  and  $k = 2$  respectively, which constitutes a contradiction.

**Definition 2.E.1.** The Walrasian demand correspondence  $x(p, w)$  is homogeneous of degree zero (H.D. $\emptyset$ ) if  $x(\alpha p, \alpha w) = x(p, w)$  for any  $p, w$  and  $\alpha > 0$ .

*Remark.* A change from  $(p, w)$  to  $(\alpha p, \alpha w)$  does not change the consumer's set of feasible consumption bundles, i.e.,  $B_{p, w} = B_{\alpha p, \alpha w}$ . H.D. $\emptyset$  means that individual's choice depends only on the set of feasible points.



*Remark.* Implication of H.D.Ø: it is without loss to *normalize* the level of one of the  $L + 1$  independent variables at an arbitrary level. One common normalization is  $p_l = 1$  for some  $L$ . Another is  $w = 1$ .

**Definition 2.E.2.** The Walrasian demand correspondence  $x(p, w)$  satisfies Walras' law if for every  $p \gg 0$  and  $w > 0$ , we have  $p \cdot x = w$  for all  $x \in x(p, w)$ .

*Remark.* Walras' law says that the consumer fully expends his wealth.

**Question:** Is Walras' law reasonable?

**Answer:** It's more reasonable if  $w$  refers the life-time income and  $x$  refers to life-time demands. Even then, it's still controversial.

### Exercise 2.E.1

Suppose  $L = 3$ , and consider the demand function  $x(p, w)$  defined by

$$\begin{aligned} x_1(p, w) &= \frac{p_2}{p_1 + p_2 + p_3} \frac{w}{p_1} \\ x_2(p, w) &= \frac{p_3}{p_1 + p_2 + p_3} \frac{w}{p_2} \\ x_3(p, w) &= \frac{\beta p_1}{p_1 + p_2 + p_3} \frac{w}{p_3} \end{aligned}$$

Does this demand function satisfy homogeneity of degree zero and Walras' law when  $\beta = 1$ ? What about when  $\beta \in (0, 1)$ ?

For the remainder of the section, we assume that  $x(p, w)$  is single-valued, continuous, and differentiable.

### 2.E.1. Comparative statics (with respect to $p$ and $w$ )

The examination of a change in outcome in response to a change in underlying economic parameters is known as *comparative statics* analysis.

This section examines how the consumer's choice would vary with changes in his wealth and in prices.

**Wealth Effects** For fixed prices  $\bar{p}$ ,  $x(\bar{p}, w)$  is called the consumer's *Engel function*.

Its image in  $\mathbb{R}_+^L$ ,  $E_{\bar{p}} = \{x(\bar{p}, w) : w > 0\}$  is the *wealth expansion path*.

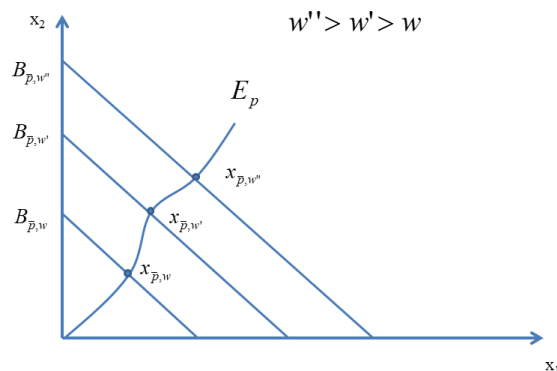


Figure 10: Wealth expansion path at  $\bar{p}$

The derivative  $\frac{\partial x_l(p, w)}{\partial w}$  is the *wealth effect* for the  $l^{th}$  good.

- A commodity  $l$  is *normal* at  $(p, w)$  if  $\frac{\partial x_l(p, w)}{\partial w} \geq 0$ .
- A commodity  $l$  is *inferior* at  $(p, w)$  if  $\frac{\partial x_l(p, w)}{\partial w} < 0$ .

In matrix notation, the wealth effects are  $D_w x(p, w) = \begin{bmatrix} \frac{\partial x_1(p, w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(p, w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L$ .

**Price Effects** The demand function for good  $l$  could be represented as a function of  $p_l$ , keeping other things equal, i.e.,  $x(p_l, \bar{p}_{-l}, \bar{w})$ .

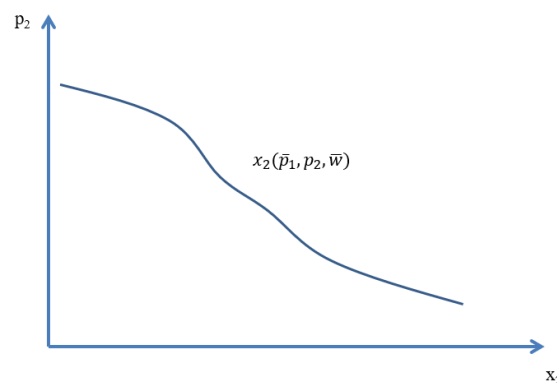


Figure 11: Demand for good 2 as a function of its price

Another useful representation of the consumers' demand at different prices  $p_l$  is the locus of points demanded in  $\mathbb{R}_+^L$ , for fixed  $p_{-l}$  and  $w$ . This is known as an *offer curve*.

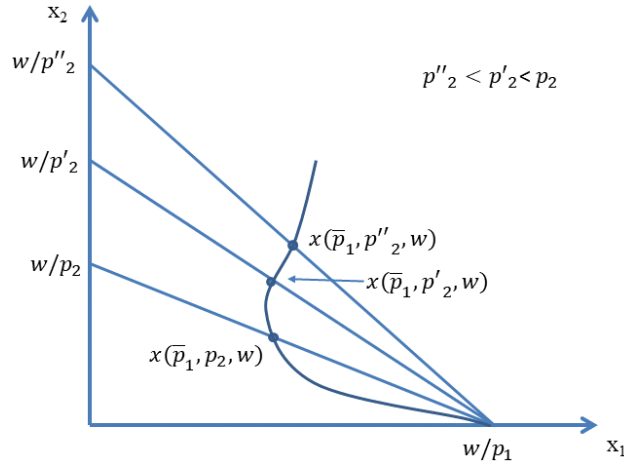


Figure 12: Offer Curve

The derivative  $\frac{\partial x_l(p, w)}{\partial p_k}$  is the *price effect* of  $p_k$  on the demand for good  $l$ .

- Good  $l$  is a *Giffen good* if  $\frac{\partial x_l(p, w)}{\partial p_l} > 0$ . (Example: potatoes at low wealth level)

In matrix notation, the price effects are  $D_p x(p, w) = \begin{bmatrix} \frac{\partial x_1(p, w)}{\partial p_1} & \dots & \frac{\partial x_1(p, w)}{\partial p_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_L(p, w)}{\partial p_1} & \dots & \frac{\partial x_L(p, w)}{\partial p_L} \end{bmatrix}$ .

## 2.E.2. Implications of homogeneity and Walras' law for price and wealth effects

### Implication of H.D. $\emptyset$

**Proposition 2.E.1.** *If the Walrasian demand function  $x(p, w)$  is homogeneous of degree zero, then for all  $p$  and  $w$ :*

$$\sum_{k=1}^L \frac{\partial x_l(p, w)}{\partial p_k} p_k + \frac{\partial x_l(p, w)}{\partial w} w = 0, \text{ for } l = 1, \dots, L. \quad (2.E.1)$$

*In matrix notation, it is expressed as*

$$D_p x(p, w) p + D_w x(p, w) w = 0. \quad (2.E.2)$$

**Proof.**

$$x(p, w) \text{ is H.D.}\emptyset \implies x_l(\alpha p, \alpha w) = x_l(p, w), \text{ for } l = 1, \dots, L$$

Differentiating both sides of the equation with respect to  $\alpha$  gives

$$\frac{\partial x_l(\alpha p, \alpha w)}{\partial \alpha} = 0 \implies \sum_{k=1}^L \frac{\partial x_l(\alpha p, \alpha w)}{\partial (\alpha p_k)} p_k + \frac{\partial x_l(\alpha p, \alpha w)}{\partial (\alpha w)} w = 0.$$

Setting  $\alpha = 1$  implies the result.  $\square$

Dividing the expression by  $x_l$ :

$$\sum_{k=1}^L \frac{\partial x_l(p, w)}{\partial p_k} \frac{p_k}{x_l(p, w)} + \frac{\partial x_l(p, w)}{\partial w} \frac{w}{x_l(p, w)} = 0, \text{ for } l = 1, \dots, L.$$

i.e.,

$$\sum_{k=1}^L \varepsilon_{lk}(p, w) + \varepsilon_{lw}(p, w) = 0, \text{ for } l = 1, \dots, L. \quad (2.E.3)$$

*Intuition:* The above equation describes the percentage change in  $x_l$  if all prices and wealth changes 1%. Basically, the equation captures the definition of H.D. $\emptyset$ .

**TWO implications of Walras' Law** ( $p \cdot x(p, w) = w$ )

**Proposition 2.E.2** (Cournot Aggregation). *If the Walrasian demand function  $x(p, w)$  satisfies the Walras' Law, then for all  $p$  and  $w$ :*

$$\sum_{l=1}^L p_l \frac{\partial x_l(p, w)}{\partial p_k} + x_k(p, w) = 0, \text{ for } k = 1, 2, \dots, L, \quad (2.E.4)$$

or written in matrix notation,

$$p \cdot D_p x(p, w) + x(p, w)^T = 0^T. \quad (2.E.5)$$

**Proof.**

$$\begin{aligned} p \cdot x(p, w) = w &\implies \frac{\partial}{\partial p_k} (p \cdot x(p, w)) = 0 \implies p \cdot \frac{\partial x(p, w)}{\partial p_k} + x_k(p, w) = 0 \\ &\implies \sum_{l=1}^L p_l \frac{\partial x_l(p, w)}{\partial p_k} + x_k(p, w) = 0. \quad \square \end{aligned}$$

*Intuition:* Total expenditure cannot change in response to a change in prices.

**Proposition 2.E.3** (Eagel Aggregation). *If the Walrasian demand function  $x(p, w)$  satisfies Walras' Law, then for ALL  $p$  and  $w$ :*

$$\sum_{l=1}^L p_l \frac{\partial x_l(p, w)}{\partial w} = 1, \quad (2.E.6)$$

or, written in matrix notation,

$$p \cdot D_w x(p, w) = 1. \quad (2.E.7)$$

**Proof.**

$$p \cdot x(p, w) = w \implies \frac{\partial}{\partial w}(p \cdot x(p, w)) = 1 \implies p \cdot \frac{\partial x(p, w)}{\partial w} = 1 \implies \sum_{l=1}^L p_l \frac{\partial x_l(p, w)}{\partial w} = 1. \quad \square$$

*Intuition:* Total expenditure must change by an amount equal to any wealth change.

#### Exercise 2.E.3

Use Proposition 2.E.1 to 2.E.3 to show that  $p \cdot D_p x(p, w) p = -w$ .

#### Exercise 2.E.5

Suppose that  $x(p, w)$  is a demand function which is homogeneous of degree one with respect to  $w$  and satisfies Walras' law and homogeneity of degree zero. Suppose also that all the cross-price effects are zero, that is  $\partial x_l(p, w) / \partial p_k = 0$  whenever  $k \neq l$ . Show that this implies that for every  $l$ ,  $x_l(p, w) = \alpha_l w / p_l$ , where  $\alpha_l > 0$  is a constant independent of  $(p, w)$ .

#### Exercise 2.E.7

A consumer in a two-good economy has a demand function  $x(p, w)$  that satisfies Walras' law. His demand function for the first good is  $x_1(p, w) = \alpha w / p_1$ . Derive his demand function for the second good. Is his demand function homogeneous of degree zero?

### Exercise 2.E.8

Show that the elasticity of demand for good  $l$  with respect to price  $p_k$ ,  $\varepsilon_{lk}(p, w)$ , can be written as  $\varepsilon_{lk}(p, w) = d \ln(x_l(p, w)) / d \ln(p_k)$ , where  $\ln(\cdot)$  is the natural logarithm function. Derive a similar expression for  $\varepsilon_{lw}(p, w)$ . Conclude that if we estimate the parameters  $(\alpha_0, \alpha_1, \alpha_2, \gamma)$  of the equation  $\ln(x_l(p, w)) = \alpha_0 + \alpha_1 \ln p_1 + \alpha_2 \ln p_2 + \gamma \ln w$ , these parameter estimates provide us with estimates of the elasticities  $\varepsilon_{l1}(p, w)$ ,  $\varepsilon_{l2}(p, w)$ , and  $\varepsilon_{lw}(p, w)$ .

## 2.F. Weak Axiom of Revealed Preference and Law of Demand

Implicit assumptions:  $x(p, w)$  is single-valued, homogeneous of degree zero, and satisfies Walras' Law.

**Definition 2.F.1.** The Walrasian demand function  $x(p, w)$  satisfies the weak axiom of revealed preference (W.A.R.P) if the following holds for any two price-wealth situations  $(p, w)$  and  $(p', w')$ :

If

$$p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w)^2$$

then

$$p' \cdot x(p, w) > w'$$

**Definition stated using language in Chapter 1** Let  $B_{p,w}$  denote the budget set given  $p$  and  $w$ ; and  $B_{p',w'}$  denote the budget set given  $p'$  and  $w'$ .  $p \cdot x(p', w') \leq w$  means that  $x(p', w')$  is also affordable under  $B_{p,w}$ . Through the choice given  $B_{p,w}$ ,  $x(p, w)$  is revealed preferred to  $x(p', w')$ . Therefore, by W.A.R.P, it must not be revealed that  $x(p', w')$  is preferred to  $x(p, w)$ . In other words, if  $x(p, w)$  is not chosen given the budget  $B_{p',w'}$ , it must be that it is not affordable, i.e.,  $p' \cdot x(p, w) > w'$ , or  $x(p, w) \notin B_{p',w'}$ .

The below figure illustrates an example of demand function  $x(p, w)$  that satisfies W.A.R.P.

<sup>2</sup>Note that  $x(p, w)$  is the demand given  $(p, w)$  and  $x(p', w')$  is the demand given  $(p', w')$ .

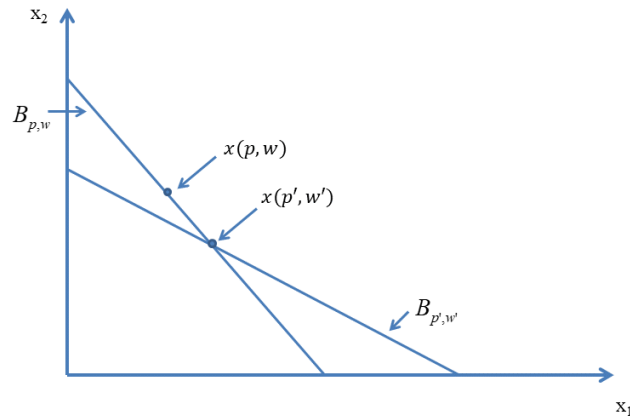


Figure 13: Demand satisfying W.A.R.P

**Violation of W.A.R.P** W.A.R.P may be violated only if both  $x(p, w)$  and  $x(p', w')$  belong to both  $B_{p,w}$  and  $B_{p',w'}$ .

The below figure illustrates an example of demand function  $x(p, w)$  that violates W.A.R.P.

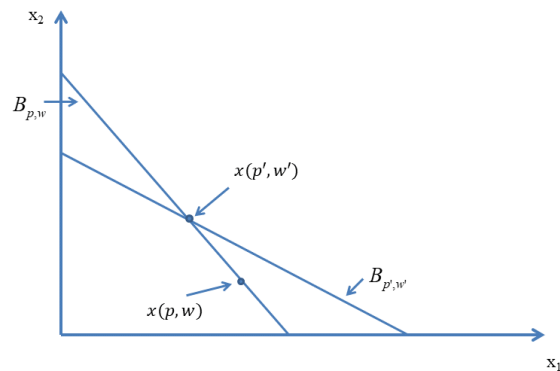


Figure 14: Demand violating W.A.R.P

### Implications of W.A.R.P

**Uncompensated price change:  $p_1$  to  $p'_1$**  An uncompensated price change is a change in price without a corresponding change in wealth. Such a price change would affect the consumer in two ways:

- change the relative cost of commodities;
- change the consumer's real wealth.

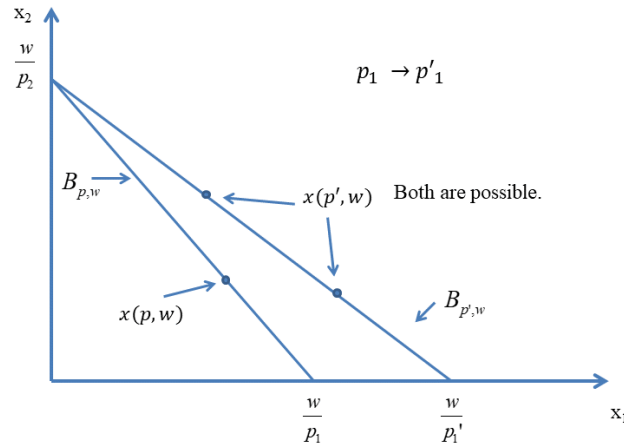


Figure 15: Uncompensated price change

No prediction on change in demand can be drawn.

**Compensated price change** Imagine a situation in which a change in prices is accompanied by a change in the consumer's wealth that makes her initial consumption bundle just affordable at the new prices. That is,  $w' = p' \cdot x(p, w)$ . The wealth adjustment is  $\Delta w = \Delta p \cdot x(p, w)$ . This kind of wealth adjustment is called *Slutsky wealth compensation*. The price changes that are accompanied by compensating wealth changes are called (*Slutsky*) *compensated price changes*.

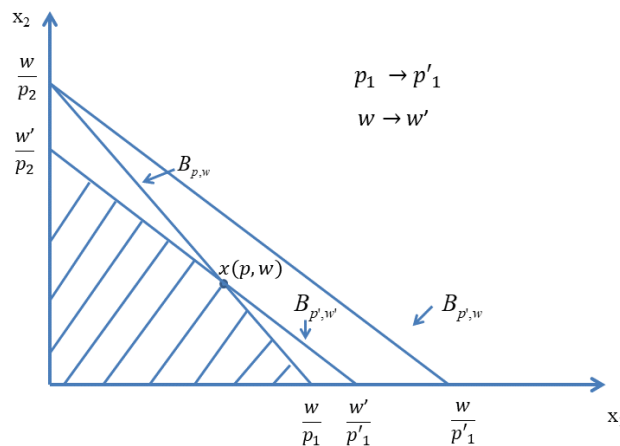


Figure 16: Compensated price change

The shaded area is revealed not as good as  $x(p, w)$ . So, the bundles in the area won't be picked after price change. *This implies  $x_1$  must increase after compensated price change.*



Another perspective is that  $x(p', w') \succsim x(p, w)$ , so it cannot be  $x(p, w) \succ x(p', w')$ .

**How to check whether W.A.R.P. is Satisfied.** It would be easier to check whether W.A.R.P. is satisfied for compensated price changes.

*Question.* What does it mean that W.A.R.P. is satisfied for all compensated price changes?

Recall,

1. W.A.R.P: If  $p \cdot x(p', w') \leq w$  and  $x(p', w') \neq x(p, w)$ , then  $p' \cdot x(p, w) > w'$ .
2. Compensated price change means  $p' \cdot x(p, w) = w'$ .

**W.A.R.P is satisfied for all compensated price change means:** For any price change from  $(p, w)$  to  $(p', w')$  such that  $p' \cdot x(p, w) = w'$ , if  $x(p', w') \neq x(p, w)$ , then  $p \cdot x(p', w') > w$ . [New bundle chosen after compensated price change is unaffordable under original price and wealth.]

**W.A.R.P is violated for some compensated price change means:** There exists a price change from  $(p, w)$  to  $(p', w')$  such that  $p' \cdot x(p, w) = w'$ ,  $x(p', w') \neq x(p, w)$  and  $p \cdot x(p', w') \leq w$ .

Figure 17 below depicts an example compensated price change. Changes in demand to  $x(p', w')$  in the red area constitutes violation of W.A.R.P.

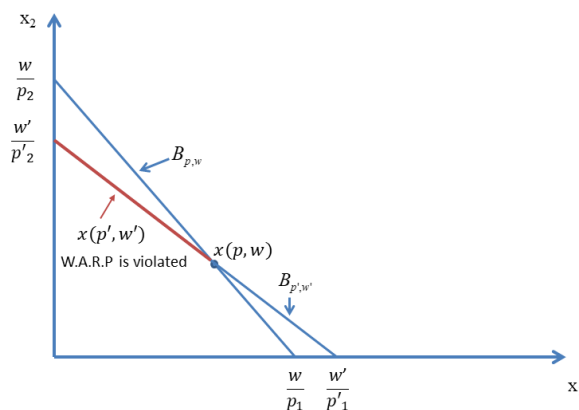


Figure 17: Compensated price change

Next, we present a useful lemma which makes it easier to check whether a demand function satisfies W.A.R.P.

**Lemma 1.** *W.A.R.P holds for all price changes if and only if it holds for all compensated price changes.*

**Proof.** *Necessary* (only if) part is obvious.

*Sufficiency* (if): Suppose that W.A.R.P is violated for some uncompensated price change. We'll show that it must also be violated for some compensated price change. [In other words, if it holds for all compensated price changes, then it also holds for all uncompensated price changes.]

Consider two price-wealth pairs  $(p', w')$  and  $(p'', w'')$  such that  $x(p', w') \neq x(p'', w'')$ ,  $p' \cdot x(p'', w'') \leq w'$  and  $p'' \cdot x(p', w') \leq w''$ .

If one of the weak inequalities holds in equality, then either the change from  $(p', w')$  to  $(p'', w'')$  or the change from  $(p'', w'')$  to  $(p', w')$  is a compensated price change.

Therefore, we restrict attention to the case of  $p' \cdot x(p'', w'') < w' = p' \cdot x(p', w')$  and  $p'' \cdot x(p', w') < w'' = p'' \cdot x(p'', w'')$ , as shown in the following figure:

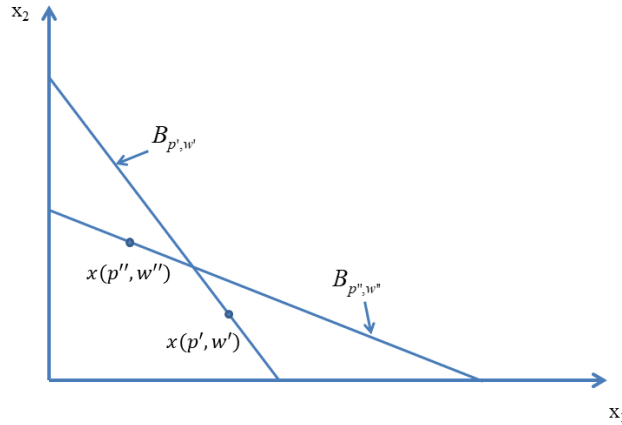


Figure 18: Uncompensated price change

Note that there exists  $\alpha \in (0, 1)$  such that

$$(\alpha p' + (1 - \alpha)p'') \cdot x(p', w') = (\alpha p' + (1 - \alpha)p'') \cdot x(p'', w'').^3$$

<sup>3</sup>You can verify this by showing that when  $\alpha = 0$ , LHS < RHS and when  $\alpha = 1$ , LHS > RHS.



In Proposition 2.F.1 below, we show that the weak axiom can be equivalently stated in terms of the demand response to compensated price changes.

**Proposition 2.F.1.** *Suppose that the Walrasian demand function  $x(p, w)$  is homogeneous of degree zero and satisfies Walras' Law, Then  $x(p, w)$  satisfies W.A.R.P if and only if the following property holds:*

*For ANY compensated price change from an initial situation  $(p, w)$  to a new price-wealth pair  $(p', w') = (p', p \cdot x(p, w))$ , we have*

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0, \quad (2.F.1)$$

*with strict inequality whenever  $x(p, w) \neq x(p', w')$ .*

Before proving the result, let's rewrite (2.F.1).

$$(p' - p) \cdot [x(p', w') - x(p, w)] = p' \cdot x(p', w') - p' \cdot x(p, w) - p \cdot [x(p', w') - x(p, w)].$$

Note that  $p' \cdot x(p', w') - p' \cdot x(p, w) = 0$  because we consider compensated price changes.

Therefore, (2.F.1) is equivalent to

$$p \cdot [x(p', w') - x(p, w)] \geq 0 \quad (> 0 \text{ if } x(p', w') \neq x(p, w)). \quad (*)$$

Below, we provide a formal proof of Proposition 2.F.1.

**Proof.**

(i) *W.A.R.P implies Equation (\*).*

If  $x(p, w) = x(p', w')$ , LHS of (\*) is 0 and the inequality holds obviously.

Suppose  $x(p, w) \neq x(p', w')$ . Since  $p' \cdot x(p, w) = w'$ ,  $x(p, w)$  is affordable under  $(p', w')$ , yet it is not chosen. W.A.R.P implies that  $x(p', w')$  is not affordable under  $(p, w)$ , i.e.,  $p \cdot x(p', w') > p \cdot x(p, w)$ . This is (\*).

(ii) *If Equation (\*) holds for all compensated price changes, then W.A.R.P holds for all compensated price changes. (And by means of Lemma 1, W.A.R.P also holds for all uncompensated price changes.)*

Equivalently, we prove that if W.A.R.P is violated for some compensated price changes, then (\*) is also violated.

Suppose W.A.R.P is violated for some compensated price changes, then there exists a compensated price change from  $(p, w)$  to  $(p', w')$ ,  $p' \cdot x(p, w) = w'$ , such that  $x(p', w') \neq x(p, w)$  and  $p \cdot x(p', w') \leq w = p \cdot x(p, w)$ . Then,  $p \cdot [x(p', w') - x(p, w)] \leq 0$ , and (\*) is violated.  $\square$

*Remark.* The inequality (2.F.1) can be interpreted as a form of the Law of Demand: Demand and price move in opposite directions. Since it only holds for compensated price changes, it is called the *Compensated Law of Demand*.

- As illustrated in Figure 15, W.A.R.P does not generate definitive prediction on the demand changes in response to *uncompensated* price changes.

**Weak Axiom and Differentiable Demand** Consider a differentiable change in price  $dp$ , compensated by the change in wealth

$$dw = x(p, w) \cdot dp.$$

Proposition 2.F.1 implies

$$dp \cdot dx \leq 0. \quad (2.F.5)$$

By chain rule, the differential change in demand induced by this compensated price change is

$$\begin{aligned} dx &= D_p x(p, w) dp + D_w x(p, w) dw \\ \implies dx &= D_p x(p, w) dp + D_w x(p, w) dw \\ \implies dx &= D_p x(p, w) dp + D_w x(p, w) [x(p, w) \cdot dp] \\ \implies dx &= [D_p x(p, w) + D_w x(p, w) x(p, w)^T] dp \end{aligned} \quad (2.F.8)$$

Define

$$S(p, w) = D_p x(p, w) + D_w x(p, w) x(p, w)^T$$

as the *substitution matrix* or *Slutsky matrix*. In matrix notation, it is

$$S(p, w) = \begin{bmatrix} s_{11}(p, w) & \cdots & s_{1L}(p, w) \\ & \ddots & \\ s_{L1}(p, w) & \cdots & s_{LL}(p, w) \end{bmatrix},$$

where the  $(l, k)^{th}$  entry is

$$s_{l,k}(p, w) = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w).^5$$

$s_{l,k}(p, w)$  are known as *substitution effects*.

**Implications of the substitution effects**  $s_{l,k}(p, w)$  is the change in demand for good  $l$  given a change in  $p_k$  and a compensating change in  $w$ .

Wealth Change:

$$\begin{aligned} w &= \sum_{l=1}^L x_l(p, w) p_l \text{ \& } w' = \sum_{l \neq k} x_l(p, w) p_l + x_k p'_k \\ \implies w' - w &= (p'_k - p_k) x_k(p, w) \\ \implies \Delta w &= \Delta p_k x_k(p, w) \end{aligned}$$

Suppose  $\Delta p_k = 1$ , then  $\Delta w = x_k(p, w)$ . The effect of the change of  $\Delta p_k = 1$  and the compensating change in wealth gives  $s_{lk}(p, w)$ :

$$\begin{aligned} &\underbrace{\frac{\partial x_l(p, w)}{\partial p_k} \Delta p_k}_{\text{price effect due to } \Delta p_k} + \underbrace{\frac{\partial x_l(p, w)}{\partial w} \Delta w}_{\text{wealth effect due to compensating change in wealth}} \\ &= \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w) \\ &= s_{lk}(p, w). \end{aligned}$$

**Negative semidefiniteness of Slutsky matrix** (2.F.8) and (2.F.5) gives

$$dp^T S(p, w) dp \leq 0, \forall p.$$

The result is summarized in Proposition 2.F.2 below.

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<sup>5</sup> $S_{lk}(p, w)$  is not directly observable, but can be inferred if we can estimate  $x(p, w)$ .

**Proposition 2.F.2.** *If a differentiable Walrasian demand function  $x(p, w)$  satisfies Walras' Law, homogeneous of degree zero, and W.A.R.P, then at any  $(p, w)$ , the Slutsky matrix  $S(p, w)$  satisfies  $v \cdot S(p, w)v \leq 0$  for any  $v \in \mathbb{R}^L$ . i.e.  $S(p, w)$  is negative semidefinite.*

*Remark.* Proposition 2.F.2 does not imply, in general, that the matrix  $S(p, w)$  is symmetric.

- For  $L = 2$ ,  $S(p, w)$  is necessarily symmetric. (Exercise 2.F.11)

#### Exercise 2.F.11

Show that for  $L = 2$ ,  $S(p, w)$  is always symmetric. [Hint: Use Proposition 2.F.3.]

- When  $L > 2$ ,  $S(p, w)$  is not necessarily symmetric, under the assumptions so far (H.D.Ø, Walras' Law, and W.A.R.P).
- Symmetry of  $S(p, w)$  is connected with maximization of rational preferences. (It will be introduced in Chapter 3.)

**Corollary.** *The substitution effect of good  $l$  with respect to its own price is always non-positive, i.e.,  $s_{ll}(p, w) \leq 0$ .*

**Proof.** Since  $S(p, w)$  is negative semidefinite, i.e.,  $v \cdot S(p, w)v \leq 0$  for any  $v \in \mathbb{R}^L$ .

Pick  $v^T = \begin{bmatrix} v_1 & \cdots & v_{l-1} & v_l & v_{l+1} & \cdots & x_L \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$ .

Then,  $v \cdot S(p, w)v = s_{ll}(p, w) \leq 0$ . □

*Remark.* An implication of  $s_{ll}(p, w) \leq 0$  is that a good can be a **Giffen good** at  $(p, w)$  only if it is **inferior**.

**Proof.**  $s_{ll}(p, w) = \frac{\partial x_l(p, w)}{\partial p_l} + \frac{\partial x_l(p, w)}{\partial w} x_l(p, w) \leq 0$ .

Then, if  $\frac{\partial x_l(p, w)}{\partial p_l} > 0$  (Giffen good), we must have  $\frac{\partial x_l(p, w)}{\partial w} < 0$  (inferior). □

*Remark.* Suppose a differentiable Walrasian demand function  $x(p, w)$  satisfies Walras' law, homogeneous of degree zero, and the Slutsky matrix  $S(p, w)$  is negative semidefinite. It is not necessarily true that  $x(p, w)$  satisfies W.A.R.P. That is, *Negative semidefiniteness of  $S(p, w)$  is not sufficient for W.A.R.P.*

Below we provide a counter example. (Exercise 2.F.16 in the book)

**Example.** Consider a setting where  $L = 3$  and a consumer whose consumption set is  $\mathbb{R}$ . Suppose that his demand function  $x(p, w)$  is

$$\begin{aligned}x_1(p, w) &= \frac{p_2}{p_3} \\x_2(p, w) &= -\frac{p_1}{p_3} \\x_3(p, w) &= \frac{w}{p_3}.\end{aligned}$$

The demand satisfies

- (a)  $x(p, w)$  is H.D. $\emptyset$  and satisfies Walras' law.
- (b)  $x(p, w)$  violates W.A.R.P.
- (c)  $v \cdot S(p, w)v = 0$  for all  $v \in \mathbb{R}^3$ .

**Solution.**

- (a) H.D. $\emptyset$  can be checked as follows:

$$\begin{aligned}x_1(\alpha, p, \alpha w) &= \alpha p_2 / \alpha p_3 = p_2 / p_3 = x_1(p, w), \\x_2(\alpha, p, \alpha w) &= -\alpha p_1 / \alpha p_3 = -p_1 / p_3 = x_2(p, w), \\x_3(\alpha, p, \alpha w) &= \alpha w / \alpha p_3 = w / p_3 = x_3(p, w).\end{aligned}$$

As for Walras' law,

$$p_1 x_1(p, w) + p_2 x_2(p, w) + p_3 x_3(p, w) = p_1 p_2 - p_2 p_1 + p_3 w / p_3 = w.$$

- (b) Let  $p = (1, 2, 1)$ ,  $w = 1$ ,  $p' = (1, 1, 1)$ , and  $w' = 2$ , then  $x(p, w) = (2, -1, 1)$  and  $x(p', w') = (1, -1, 2)$ . Thus,  $p' \cdot x(p, w) = 2 \leq w'$  and  $p \cdot x(p', w') = 1 \leq w$ . Hence, W.A.R.P is violated.



(c) First, we compute  $S(p, w)$ .

$$S(p, w) = \begin{bmatrix} 0 & 1/p_3 & -p_2/p_3^2 \\ -1/p_3 & 0 & p_1/p_3^2 \\ p_2/p_3^2 & -p_1/p_3^2 & 0 \end{bmatrix}$$

Then,

$$\begin{aligned} v \cdot S(p, w) &= \left[ -\frac{v_2}{p_3} + \frac{p_2 v_3}{p_3^2} \quad \frac{v_1}{p_3} - \frac{p_1 v_3}{p_3^2} \quad -\frac{p_2 v_1}{p_3^2} + \frac{p_1 v_2}{p_3^2} \right] \\ v \cdot S(p, w) v &= -\frac{v_2 v_1}{p_3} + \frac{p_2 v_3 v_1}{p_3^2} + \frac{v_1 v_2}{p_3} - \frac{p_1 v_3 v_2}{p_3^2} - \frac{p_2 v_1 v_3}{p_3^2} + \frac{p_1 v_2 v_3}{p_3^2} = 0. \end{aligned}$$

*Remark.* The sufficient condition is  $v \cdot S(p, w) v < 0$  whenever  $v \neq \alpha p$  for any scalar  $\alpha$ . That is,  $S(p, w)$  must be negative definite for all vectors other than those that are proportional to  $p$ .

The proof is out of the scope of this course. See [Samuelson \(1947\)](#) or [Kihlstrom, Mas-Colell, and Sonnenschein \(1976\)](#) for an advanced treatment.

### More properties on Slutsky matrix

**Proposition 2.F.3.** *Suppose that the Walrasian demand function  $x(p, w)$  is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then,  $p \cdot S(p, w) = 0$  and  $S(p, w)p = 0$  for any  $(p, w)$ .*

**Proof.**

$$\begin{aligned} p \cdot S(p, w) &= p \cdot D_p x(p, w) + p \cdot D_w x(p, w) x(p, w)^T \\ &= p \cdot D_p x(p, w) + x(p, w)^T \quad (\text{by Proposition 2.E.3}) \\ &= 0^T \quad (\text{by Proposition 2.E.2}) \end{aligned}$$

$$\begin{aligned} S(p, w)p &= D_p x(p, w)p + D_w x(p, w)x(p, w)^T p \\ &= -D_w x(p, w)w + D_w x(p, w)w \quad (\text{by Proposition 2.E.1 and Walras' law}) \\ &= 0 \quad \square \end{aligned}$$

It follows from Proposition 2.F.3 that the negative semidefiniteness of  $S(p, w)$  established in Proposition 2.F.2 cannot be extended to negative definiteness. As an example, see Exercise 2.F.17.

### Exercise 2.F.17

In an  $L$ -commodity world, a consumer's Walrasian demand function is

$$x_k(p, w) = \frac{w}{\sum_{l=1}^L p_l} \text{ for } k = 1, \dots, L.$$

- (a) In this demand function homogeneous of degree zero in  $(p, w)$ ?
- (b) Does it satisfy Walras' law?
- (c) Does it satisfy the weak axiom?
- (d) Compute the Slutsky substitution matrix for this demand function. Is it negative semidefinite? Negative definite? Symmetric?

### Choice-based Approach and Preference-based Approach

*Remark.*  $\mathcal{B}^W = \{B_{p,w} : p \gg 0, w > 0\}$  does NOT contain all two- and three-element subsets of  $X$ . Therefore, choice-based approach  $\neq$  preference-based approach.

**Example 2.F.1.** In a three-commodity world, consider the three budget sets determined by the price vectors  $p^1 = (2, 1, 2)$ ,  $p^2 = (2, 2, 1)$ ,  $p^3 = (1, 2, 2)$  and wealth = 8 (the same for the three budgets). Suppose that the respective (unique) choices are  $x^1 = (1, 2, 2)$ ,  $x^2 = (2, 1, 2)$ ,  $x^3 = (2, 1, 1)$ . For these three budgets, any two pairs of choices satisfy W.A.R.P but  $x^3$  is revealed preferred to  $x^2$ ,  $x^2$  is revealed preferred to  $x^1$ , and  $x^1$  is revealed preferred to  $x^3$ .

- We check W.A.R.P for budget 1 and 2, the satisfaction of W.A.R.P for the rest of the pairs could be shown similarly. W.A.R.P is satisfied for budget 1 and 2 since we have  $p^2 \cdot x^1 = 8$ ,  $x^1 \neq x^2$  and  $p^1 \cdot x^2 = 9 > 8$ .
- For revealed preference,  $x^2$  is revealed preferred to  $x^1$  since  $p^2 \cdot x^1 = 8$ , implying that  $x^1$  is affordable under budget 2 but not chosen. Other pairs could be similarly checked.

**Summary of Chapter 2** Taking choice as the primitive, we look at the implications of these assumptions:

- (i)  $x(p, w)$  is homogeneous of degree zero
- (ii)  $x(p, w)$  satisfies Walras' Law
- (iii)  $x(p, w)$  satisfies the W.A.R.P  $\implies$  Compensated Law of Demand
- (iv)  $x(p, w)$  is differentiable  $\implies$  negative semidefinite Slutsky matrix

## References

Kihlstrom, Richard, Andreu Mas-Colell, and Hugo Sonnenschein. 1976. "The Demand Theory of the Weak Axiom of Revealed Preference." *Econometrica* :971–978.

Samuelson, Paul Anthony. 1947. *Foundations of Economic Analysis*.