Chapter 1. Preference and Choice

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1.A. Introduction

Two approaches to modeling individual choice behavior:

1. Preference-based Approach: preference as primative (rationality axioms) $\Longrightarrow$ consequences on choices

2. Choice-based Approach: choice behavior as primative (axioms on behavior)

1.B. Preference Relations

X: Set of Alternatives.

- For example, if Alice just graduated from Wuhan University majoring in economics, then her set of alternatives is: $X = \{$go to graduate school and study economics, go to a Big-4 firm, go to work for the government, ..., run a small business$\}$.

We use capital letters (like X and B) for a set of alternatives, small letters (like x and y) for a specific choice alternative.

Defining Preference Relations

Denote by $\succsim$ *the preference relation* defined on the set X, allowing the comparison of any x and y in X.

- $x \succsim y$: pronounced as "x is preferred to y" or "x is at least as good as y." The first usage is more common.

- *Strict preference* $\succ$: $x \succ y \iff x \succsim y$ but not $y \succsim x$ (i.e., $y \not\succsim x$) ("x is strictly preferred to y.")

- *Indifference* $\sim$: $x \sim y \iff x \succsim y$ and $y \succsim x$ ("x is indifferent to y.")

Rational Preference

Not all preference relations make sense.

For example, consider Alice's preference:

- "Hot and Dry Noodles" $\succ$ "Doupi" (dòu pí)
- "Doupi" $\succ$ "Xiaolongbao" (xiǎo lóng bāo)
- "Xiaolongbao" $\succsim$ "Hot and Dry Noodles"

Alice must have a hard time choosing her breakfast from

$X = \{$Hot and Dry Noodles, Doupi, Xiaolongbao$\}$.

Rational Preference

Definition 1.B.1 (Rational preference)**.** The preference relation $\succsim$ is **rational** if it possesses these two properties:

(i) Completeness: $\forall x, y \in X$, $x \succsim y$ or $y \succsim x$. (rules out $x \not\succsim y$ and $y \not\succsim x$)

(ii) Transitivity: $\forall x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Rational Preference

> *Question.* In the example above, which property does Alice's preference relation violate?

Answer: Transitivity.

The first two bullet points

- "Hot and Dry Noodles" $\succ$ "Doupi" (dòu pí)
- "Doupi" $\succ$ "Xiaolongbao" (xiǎo lóng bāo)

implies "Hot and Dry Noodles" $\succ$ "Xiaolongbao"

which contradicts "Xiaolongbao" $\succsim$ "Hot and Dry Noodles"

Implications of Rational Preference on $\succ$ and $\sim$

The following propositions follow from the definition of *rational preference*.

Proposition 1.B.1. *If $\succsim$ is rational, then:*

(i) $\succ$ *is both irreflexive ($x \succ x$ never holds) and transitive.*

(ii) $\sim$ *is reflexive ($x \sim x$), transitive and symmetric (if $x \sim y$, then $y \sim x$).*

(iii) if $x \succ y \succsim z$, then $x \succ z$. (slightly stronger than transitivity in (i))

Utility Functions

Definition 1.B.2. A function $u : X \to \mathbb{R}$ is a utility function representing preference relation $\succsim$ if

$$x \succsim y \iff u(x) \geq u(y) \text{ for all } x, y \in X. \tag{1}$$

The utility function is nothing but assigning each choice x with a number $u(x)$. Obviously, the function u satisfying Condition (1) is not unique.

Example. $u(x) \geq u(y) \iff \alpha u(x) \geq \alpha u(y)$ for all $\alpha > 0$.

Utility Functions

Exercise. *Show that if* $f : \mathbb{R} \to \mathbb{R}$ *is a* ***strictly increasing*** *function and* $u : X \to \mathbb{R}$ *is a utility function representing preference relation* $\succsim$*, then the function* $v : X \to \mathbb{R}$ *defined by* $v(x) = f(u(x))$ *is also a utility function representing preference relation* $\succsim$*.*

Utility Functions

Question. When can a preference relation be represented by a utility function?

Answer: Only if the preference relation is rational. See the next proposition.

Utility Functions

Proposition 1.B.2. *If the preference relation* $\succsim$ *can be represented by a utility function (i.e.* $\exists u(\cdot)$ *s.t.* $u(x) \geq u(y)$ *iff* $x \succsim y$*), then* $\succsim$ *is rational (i.e. complete & transitive).*

Utility Functions

Question. If $\succsim$ is rational, does there exist a utility function u representing $\succsim$?

Answer: Not always. Rationality is just a necessary condition for the existence of a utility representation, but not sufficient. See the counterexample below.

Lexicographic Preference

Definition (Lexicographic Preference)**.** Let $X = \mathbb{R}^2$. The preference relation $\succsim$ is a *lexicographic preference* if for all $x, y \in X$, $x \succsim y$ whenever (i) $x_1 > y_1$ or (ii) $x_1 = y_1$ and $x_2 \geq y_2$.

Claim. The lexicographic preference on $\mathbb{R}^2$ do *not* have a utility representation.

Example of Lexicographic Preference

Alice is considering buying a new phone. The relevant attributes include brand name, price, CPU, and so on. For simplicity, suppose Alice only cares about the brand (Apple or Huawei) and price. Alice is a Apple fan and strictly prefers an iPhone to a Huawei Phone regardless of the price. For Alice,

$$(\text{Apple}, 5000) \succ (\text{Apple}, 8000) \succ (\text{Huawei}, 5000).$$

Utility Function

Remark. If X is **finite** and $\succsim$ is a rational preference relation on X, then there is a utility function $u : X \to R$ that represents $\succsim$.

1.C. Choice Rules

A *choice structure* $(\mathscr{B}, C(\cdot))$ consists of two ingredients:

(i) $\mathscr{B}$ is a family (a set) of nonempty subsets of X: that is, every $B \in \mathscr{B}$ is a set $B \in X$.

- In consumer theory, B are budget sets.
- $\mathscr{B}$ needs NOT to include all possible subsets of X.

(ii) $C(\cdot)$ is a choice rule that assigns a nonempty subset of chosen elements $C(B) \subset B$ for every $B \in \mathscr{B}$.

- $C(B)$ is a set of *acceptable alternatives*.

Choice Rules

Example 1.C.1. $X = \{x, y, z\}, \mathscr{B} = \{\{x, y\}, \{x, y, z\}\}$

Choice Structure 1 $(\mathscr{B}, C_1(\cdot))$:

$$C_1(\{x, y\}) = \{x\}, C_1(\{x, y, z\}) = \{x\}$$

Choice Structure 2 $(\mathscr{B}, C_2(\cdot))$:

$$C_2(\{x, y\}) = \{x\}, C_2(\{x, y, z\}) = \{x, y\}$$

Under $(\mathscr{B}, C_2(\cdot))$, y is acceptable only if z is available.

Go to Example 1.C.2

Choice Rules

You might find the choice structure 2 unreasonable.

Consider the following conversation.

Waiter:	Coffee or Tea?
Customer:	Coffee, please.
Waiter:	Sure. Oh sorry, actually we also serve coke. Do you want some coke?
Customer:	*Since coke is available, I'd prefer tea rather than coffee.*

Weak Axiom of Revealed Preference (W.A.R.P)

Definition 1.C.1. The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the weak axiom of revealed preference (W.A.R.P) if the following property holds:

If for some $B \in \mathscr{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathscr{B}$ with $x, y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$.

Weak Axiom of Revealed Preference (W.A.R.P)

In the last example, $(\mathscr{B}, C_2(\cdot))$ violates W.A.R.P since

$y \in C_2(\{x, y, z\})$, $x, y \in \{x, y\}$, $x \in C_2(\{x, y\})$ but

$y \notin C_2(\{x, y\})$.

[Think of $\{x, y, z\}$ as B and $\{x, y\}$ as B' in Definition 1.C.1.]

IDEA: Agent's choice between x and y should not be affected

by irrelevant options/alternatives.

Revealed Preference: Preference inferred from/ revealed through Choice

Definition 1.C.2. Given a choice structure ($\mathscr{B}$, $C(\cdot)$), the **revealed preference relation** $\succsim^*$ is defined by

$$x \succsim^* y \iff \exists B \in \mathscr{B} \text{ s.t. } x, y \in B \text{ and } x \in C(B).$$

$x \succsim^* y$ reads "x is revealed at least as good as y"

Revealed Preference

- $x \succ^* y$:

  $\exists B \in \mathscr{B}$ s.t. $x, y \in B$ and $x \in C(B),$ and $y \notin C(B)$. (

  "x is revealed preferred to y")

- $\succsim^*$ needs not to be complete or transitive.

- "Revealed preference" is defined reference to B.

  (Compare with "preference")

- **Restatement of W.A.R.P**: If $x \succsim^* y$, then $y \not\succ^* x$.

  (only imposed on $B \in \mathscr{B}$)

Revealed Preference

Example 1.C.2. Recall Example 1.C.1.

$(\mathscr{B}, C_1(\cdot))$: $x \succ^* y$ and $x \succ^* y$, $x \succ^* z$

$(\mathscr{B}, C_2(\cdot))$: $x \succ^* y$ and $y \succsim^* x \implies$ contradicts W.A.R.P

Useful alternative statements of W.A.R.P

Restatement of W.A.R.P 1. *$x, y \in B$, $x \in C(B)$, $y \in C(B')$ & $x \notin C(B')$, then $x \notin B'$.*

Proof. Proof by contradiction. If $x \in B'$ & $y \in C(B')$, W.A.R.P $\implies x \in C(B')$. $\square$

Restatement of W.A.R.P 2. *Suppose that $B, B' \in \mathscr{B}$, that $x, y \in B$, and that $x, y \in B'$. Then if $x \in C(B)$ and $y \in C(B')$, we must have $\{x, y\} \subset C(B)$ and $\{x, y\} \in C(B')$.*

The proof is left as an exercise.

1.D. Relationship between Preference Relations & Choice Rules

More precisely, we want to know the relationship between *rational preference* and *W.A.R.P*.

(i) Does Rational Preference imply W.A.R.P? (Yes)

(ii) Does W.A.R.P imply Rational Preference? (Maybe)

Preference Generated Choice Structure

Consider rational preference $\succsim$ on X.

Define: $C^*(B, \succsim) = \{x \in B\colon x \succsim y$ for every $y \in B\}$

- Elements of $C^*(B, \succsim)$ are DM's most preferred alternatives in B.

- Assumption: $C^*(B, \succsim)$ is nonempty for all $B \in \mathscr{B}$.

Remark. If X is **finite**, then any rational preference relation generates a nonempty choice rule.

The proof is left as an exercise.

Preference Generated Choice Structure

We say that the preference $\succsim$ *generates* the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$.

Proposition 1.D.1. *Suppose* $\succsim$ *is a rational preference relation. Then the choice structure generated by* $\succsim$*,* $(\mathscr{B}, C^*(\cdot, \succsim))$ *satisfies W.A.R.P.*

Preference Generated Choice Structure

Definition 1.D.1. Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ rationalizes $C(\cdot)$ relative to $\mathscr{B}$ if $C(B) = (C^*(B, \succsim))$ for all $B \in \mathscr{B}$, that is, if $\succsim$ generates the choice structure $(\mathscr{B}, C(\cdot))$.

Preference Generated Choice Structure

1. If a rational preference relation rationalizes the choice rule, we can interpret the DM's choices as if she were a preference maximizer.

2. In general, there may be more than one rationalizing preference relation $\succsim$ for a given choice structure $(\mathscr{B}, C(\cdot))$.

   Example.

   $X = \{x, y\}, \mathscr{B} = \{\{x\}, \{y\}\}, C(\{x\}) = x, C(\{y\}) = y$.

Preference Generated Choice Structure

Example 1.D.1. $X = \{x, y, z\}$,

$\mathscr{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$[1],

$C(\{x, y\}) = \{x\}, C(\{y, z\}) = \{y\}, C(\{x, z\}) = \{z\}$.

This choice structure satisfies the W.A.R.P.

However, it cannot be rationalized by a rational preference.

Remark. W.A.R.P is defined by $\mathscr{B}$. And the choice is not challenged by having to choose from $\{x, y, z\}$.

[1] $\{x, y, z\}$ is not empirically relevant.

Preference Generated Choice Structure

Proposition 1.D.2. *If* $(\mathscr{B}, C(\cdot))$ *is a choice structure such that*

(i) the W.A.R.P is satisfied, [$x \succsim^* y$*, then* $y \not\succ^* x$*]*

(ii) $\mathscr{B}$ *includes all subsets of* X *of up to three elements,*

then $\exists$ *rational* $\succsim$ *that rationalizes* $C(\cdot)$ *relative to* $\mathscr{B}$*, i.e.,*

$$C(B) = (C^*(B, \succsim)), \forall B \in \mathscr{B}.$$

Furthermore, this rational preference relation is unique.

Summary of Chapter 1

- Preference relation $\succsim$ is binary relation on choice set X.

- $\succsim$ is rational if Completeness & Transitivity.

- Choice function $C(\cdot)$ is defined on $\mathscr{B}$, NOT on X.

  (Assumptions: W.A.R.P & $C(\cdot) \neq \varnothing$)

- Rational Preference implies W.A.R.P.

  But for W.A.R.P to imply Rational Preference, it requires $C(\cdot) \neq \varnothing$ and that $\mathscr{B}$ includes all 2 & 3-element subsets of X.