Chapter 3. Classical Demand Theory (Propositions)

Proposition 3.C.1. *Suppose that the preference relation $\succsim$ on X is continuous. Then there exists continuous utility function $u(x)$ that represents $\succsim$, i.e., $u(x) \geq u(y)$ if and only if $x \succsim y$.*

Proposition 3.D.1. *If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.*

Proposition 3.D.2. *Suppose that $u(x)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:*

(i) Homogeneity of degree zero in $(p, w) : x(\alpha p, \alpha w) = x(p, w)$ for any p, w and scalar $\alpha > 0$.

(ii) Walras' Law: $p \cdot x = w$ for all $x \in x(p, w)$.

(iii) Convexity/uniqueness: If $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set. Moreover, if $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ consists of a single element.

Proposition 3.D.3. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The indirect utility function $v(p, w)$ is*

(i) Homogeneous of degree zero.

(ii) Strictly increasing in w and nonincreasing in p_l for any l.

(iii) Quansiconvex; that is, the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.

(iv) Continuous in $p \gg 0$ and w.

Proposition 3.E.1. *Suppose $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p \gg 0$. We have* *(i)* *If x^* is optimal in the UMP when wealth is $w > 0$, i.e., $x^* = x(p, w)$, then x^* is optimal in the EMP when the required utility is $u(x^*)$. Moreover, the minimized expenditure in the EMP is w.*

(ii) *If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximized utility in the UMP is u. (*No excess utility)*

Proposition 3.E.2. *Suppose that $u(\cdot)$ is a continuous utility representing a locally non-satiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The expenditure function $e(p, u)$ is*

(i) *Homogeneous of degree one in p.*

(ii) *Strictly increasing in u and nondecreasing in p_l for all l.*

(iii) *Concave in p, i.e., $\alpha e(p, u) + (1 - \alpha)e(p', u) \leq e(ap + (1 - \alpha)p', u)$.*

(iv) *Continuous in $p \gg 0$ and u.*

Using Proposition 3.E.1, we can connect the expenditure function $e(p, u)$ and the indirect utility function $v(p, w)$:

$$e(p, v(p, w)) = w \quad \text{and} \quad v(p, e(p, u)) = u. \tag{3.E.1}$$

Proposition 3.E.3. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on $X = \mathbb{R}^L_+$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ (i.e., expenditure minimizing demand) possesses the following properties:*

1. *Homogeneity of degree zero in p: $h(\alpha p, u) = h(p, u)$ for all p, u and $\alpha > 0$.*

2. *No excess utility: For any $x \in h(p, u)$, $u(x) = u$.*

3. *Convexity/uniqueness: If $\succsim$ is convex, then $h(p, u)$ is a convex set; and if $\succsim$ is strictly convex, then there is a unique element in $h(p, u)$.*

Using Proposition 3.E.1, we can relate the Hicksian and Walrasian demand correspondences as follows: (assuming single-value demand)

$$h(p,u) = x(p,e(p,u)) \quad \text{and} \quad x(p,w) = h(p,v(p,w)). \tag{3.E.4}$$

Proposition 3.E.4. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ and that $h(p,u)$ consists of a single element for all $p \gg 0$. Then the Hicksian demand function $h(p,u)$ satisfies the compensated law of demand: for all p' and p'',*

$$(p'' - p') \cdot [h(p'',u) - h(p'.u)] \leq 0. \tag{3.E.5}$$

Proposition 3.G.1. *Suppose that $u(\cdot)$ is continuous, representing locally nonsatiated and strictly convex preference relation $\succsim$ defined on $X = \mathbb{R}^L_+$. For all p and u,*

$$h(p,u) = \nabla_p e(p,u).$$

Proposition 3.G.2. *Suppose $u(\cdot)$ is continuous utility function representing a locally nonsatiated and strictly convex $\succsim$ on $X = \mathbb{R}^L_+$. Suppose $h(p,u)$ is continuously differentiable at (p,u), and denote the $L \times L$ derivative matrix by $D_p h(p,u)$. Then*

(i) $D_p h(p,u) = D_p^2 e(p,u)$.

(ii) $D_p h(p,u)$ *is negative semidefinite.*

(iii) $D_p h(p,u)$ *is symmetric.*

(iv) $D_p h(p,u) p = 0$.

Proposition 3.G.3. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex $\succsim$ on $X = \mathbb{R}^L_+$. Then for all (p,w), and $u = v(p,w)$, we have*

For all l, k,

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

or

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T$$

Proposition 3.G.4. *Suppose that $u(\cdot)$ is A continuous utility function representing a locally nonsatiated and strictly convex $\succsim$ on $X = \mathbb{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(\bar{p}, \bar{w}) \gg 0$.*

Then

$$x(\bar{p}, \bar{w}) = -\frac{1}{\nabla_w v(\bar{p}, \bar{w})} \nabla_p v(\bar{p}, \bar{w})$$

i.e., for every $l = 1, ..., L$:

$$x_l(\bar{p}, \bar{w}) = \frac{-\partial v(\bar{p}, \bar{w})/\partial p_l}{\partial v(\bar{p}, \bar{w})/\partial w}.$$