

# Chapter 2. Consumer Choice

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## 2.A. Introduction

In this chapter, we perform analysis of choice structure in the context of **consumption**. In other words, we analyze consumer demand for commodities.

## 2.B. Commodities

The decision problem faced by the consumer is to choose the consumption levels of commodities (goods and services).

A *commodity vector* (or *commodity bundle*) is a point

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix} \in \mathbb{R}^L$$

- $\mathbb{R}^L$  is the commodity *space*.
- $x_l$  is the amount of commodity  $l$  consumed.

## Commodities

*Remark.* Time (see the example below) and location (see Figure 3), could be built into the definition of a commodity.

For example,  $x_1$  could be bread today, and  $x_2$  could be bread tomorrow. (In this example, we ignore other commodities.)

Alice who plans to consume 5 slices of bread today and 6 slices of bread tomorrow would have a commodity vector

$$x = \begin{bmatrix} x_1 = 5 \\ x_2 = 6 \end{bmatrix} \in \mathbb{R}^2.$$

## 2.C. Consumption Set

The *consumption set* is a subset of the commodity space  $\mathbb{R}^L$ , denoted by  $X \subset \mathbb{R}^L$ , whose elements are the consumption bundles that the individual can conceivably consume given the physical and institutional constraints imposed by his environment.

## *Physical Constraints*

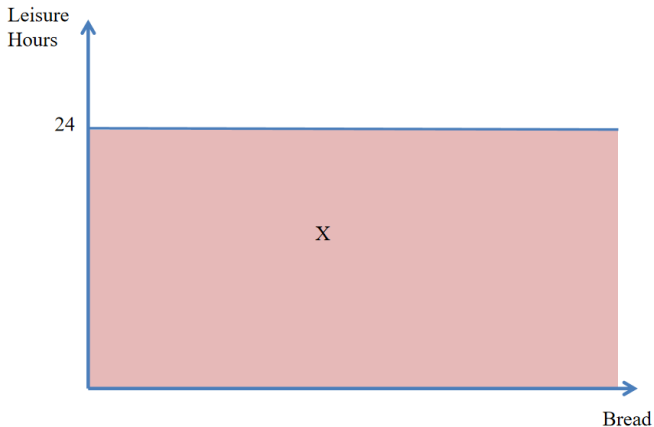


Figure 1: Possible consumption levels of bread and leisure in a day

## Physical Constraints

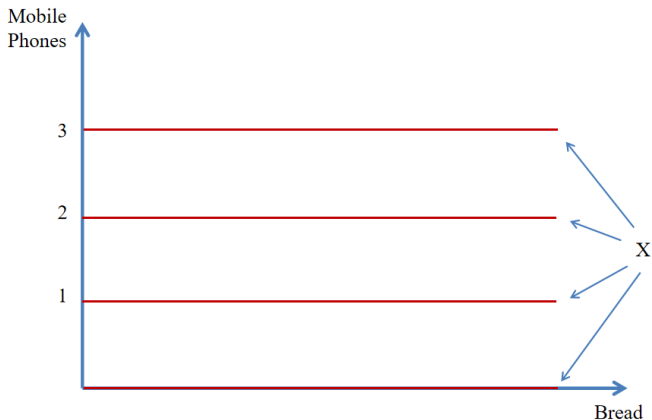


Figure 2: Possible consumption levels of bread and mobile phones

## *Physical Constraints*

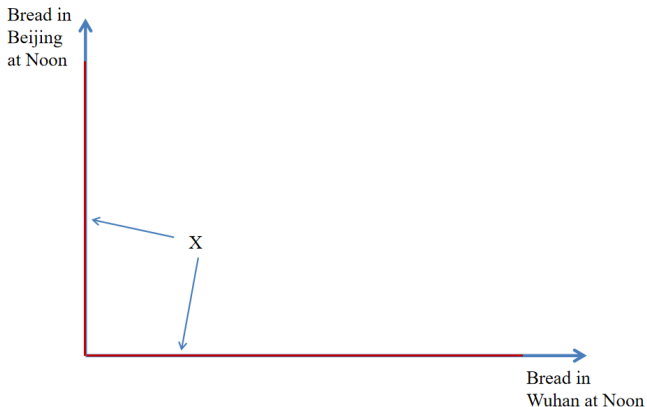


Figure 3: Possible consumption levels of bread in Beijing and Wuhan at noon



## Physical Constraints

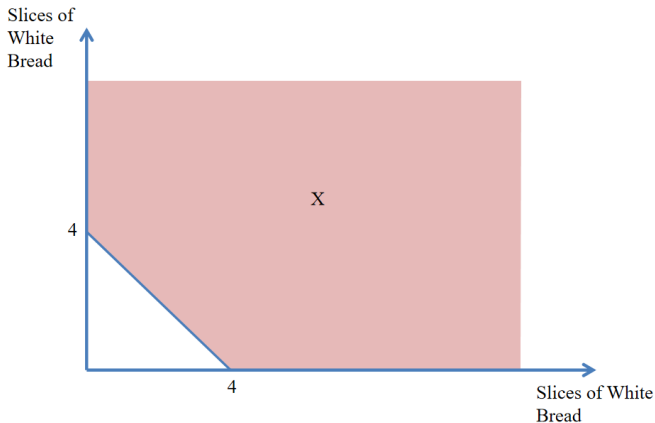


Figure 4: Possible consumption levels of bread where the minimum survival amount is 4 slices and only 2 types of bread are available

There could also be *Institutional Constraints*.

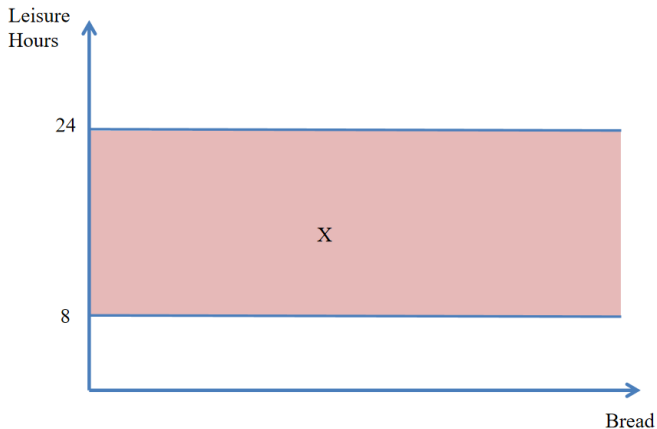


Figure 5: Possible consumption levels of bread and leisure in a day with a law requiring that no one work more than 16 hours a day

Practically, we adopt the simplest consumption set:

$$X = \mathbb{R}_+^L = \{x \in \mathbb{R}^L : x_l \geq 0 \text{ for } l = 1, 2, \dots, L\}.$$

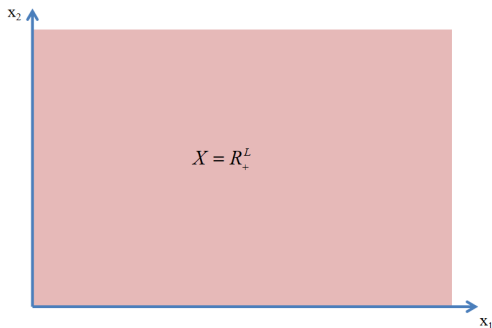


Figure 6: The consumption set  $\mathbb{R}_+^L$

## Consumption Set

*Remark.*  $X$  is convex:  $x \in X, x' \in X \implies \alpha x + (1-\alpha)x' \in X$ .

**Proof.** Given any two commodities  $l, k = 1, \dots, L$ .

$$x_l \geq 0, x_k \geq 0 \implies \alpha x_l + (1 - \alpha)x_k \geq 0 \quad \square$$

Much of the theory to be developed applies also for more general convex consumption sets (for example, the consumption sets illustrated in Figures 1, 4, 5).<sup>1</sup>

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<sup>1</sup>You should check by yourselves that the consumption sets in Figures 1, 4, 5 are convex.

## 2.D. Competitive Budgets (Affordability)

In addition to the physical and institutional constraints, the consumer also faces *economic* constraint: affordability.

Assumptions:

- $L$  commodities are all traded at public dollar prices.
- Consumers are *price takers*.

## Competitive Budgets

Formally, prices are represented by the *price vector*:

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L$$

**Assumption.**  $p \gg 0$ , i.e.,  $p_l > 0, \forall l$ .

## Competitive Budgets

*Question.* Do you think this assumption is reasonable?

## Competitive Budgets

### Counter Examples.

1. Someone invites you: for you,  $p_l = 0$ .
2. Sometimes parents pay kid to read books: for the kids,  
 $p_l < 0$ .



## Economic-Affordability Constraint

The affordability of a consumption bundle depends on

1. market prices:  $p = (p_1, \dots, p_L)$
2. consumer's wealth level (in dollars):  $w$

The consumption bundle  $x \in \mathbb{R}_+^L$  is affordable if

$$p \cdot x = p_1 x_1 + \dots + p_L x_L \leq w.$$

## Walrasian budget set

**Definition 2.D.1.** The Walrasian, or competitive budget set  $B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$  is the set of all feasible consumption bundles for the consumer who faces market prices  $p$  and has wealth  $w$ .

The consumer's problem is to choose *consumption bundle*  $x$  from  $B_{p,w}$ .

## Walrasian budget set

The set  $\{x \in \mathbb{R}_+^L : p \cdot x = w\}$  is called the *budget hyperplane*.

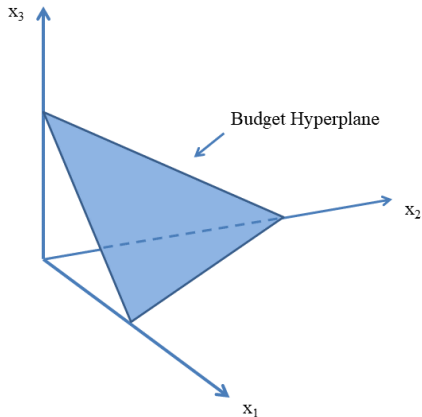


Figure 7: Budget Hyperplane (3 commodities)

## Walrasian budget set

When  $L = 2$ , Budget Hyperplane is Budget Line.

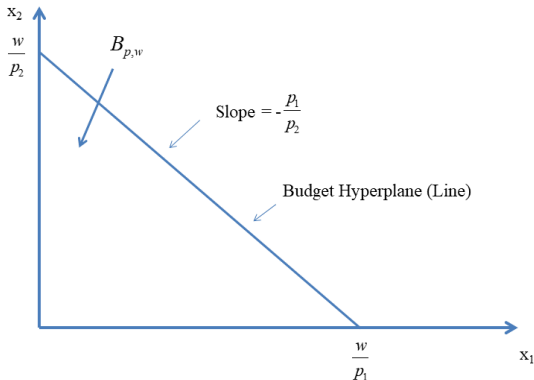


Figure 8: Budget hyperplane (line) for two commodities

## Walrasian budget set

The slope  $-\frac{p_1}{p_2}$  captures the rate of exchange between the two commodities.

- $\frac{p_1}{p_2}$  describes the units of  $x_2$  the consumer can obtain by giving up one unit of  $x_1$ :

$$\text{one unit of } x_1 \implies p_1 \text{ of money} \implies \frac{p_1}{p_2} \text{ units of } x_2$$

## Walrasian budget set

$p$  is orthogonal to any vector starting at  $\bar{x}$  and lying on the budget hyperplane.

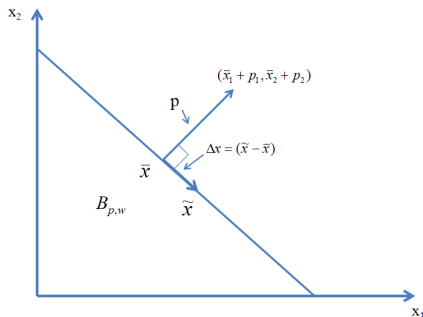


Figure 9: The geometric relationship between  $p$  and the budget hyperplane

**Walrasian budget set  $B_{p,w}$  is convex.**

**Proof.** We need to show that for all  $x, x' \in B_{p,w}$ ,  $x'' = \alpha x + (1 - \alpha)x' \in B_{p,w}$ .

First,  $x, x' \in \mathbb{R}_+^L \implies x'' \in \mathbb{R}_+^L$ . Second, since  $p \cdot x \leq w$  and  $p \cdot x' \leq w$ , we have  $p \cdot x'' = p \cdot \alpha x + (1 - \alpha)x' = \alpha(p \cdot x) + (1 - \alpha)(p \cdot x') \leq w$ .

Thus,  $x'' \in B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$ . □

*Remark.* The convexity of  $B_{p,w}$  depends on the convexity of the consumption set.  $B_{p,w}$  will be convex as long as  $X$  is.

## 2.E. Demand Functions and Comparative Statics

The consumer's *Walrasian* (or *market*, or *ordinary*) *demand correspondence*  $x(p, w)$  assigns a set of chosen consumption bundles for each  $(p, w)$ .

When  $x(p, w)$  is single-valued, we refer to it as a *demand function*.



# Demand Functions

## Assumption.

1.  $x(p, w)$  is homogeneous of degree zero.
2.  $x(p, w)$  satisfies Walras' law.

# Homogeneous Functions

**Definition.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is Homogeneous of Degree  $k$  if for any  $\alpha > 0$ ,

$$f(\alpha x_1, \alpha x_2, \dots, \alpha x_n) = \alpha^k f(x_1, x_2, \dots, x_n).$$

## Homogeneous of Degree Zero

**Definition 2.E.1.** The Walrasian demand correspondence  $x(p, w)$  is homogeneous of degree zero (H.D.0) if  $x(\alpha p, \alpha w) = x(p, w)$  for any  $p, w$  and  $\alpha > 0$ .

*Remark.* Since  $B_{p,w} = B_{\alpha p, \alpha w}$ , H.D.0 means that individual's choice depends only on the set of feasible points.

*Remark.* Implication of H.D.0: it is without loss to *normalize* the level of one of the  $L+1$  independent variables at an arbitrary level.

## Walras' Law

**Definition 2.E.2.** The Walrasian demand correspondence  $x(p, w)$  satisfies Walras' law if for every  $p \gg 0$  and  $w > 0$ , we have  $p \cdot x = w$  for all  $x \in x(p, w)$ .

*Remark.* Walras' law says that the consumer fully expends his wealth.

## Walras' Law

*Question.* Is Walras' law reasonable?

## Walras' Law

*Question.* Is Walras' law reasonable?

It's more reasonable if  $w$  refers the life-time income and  $x$  refers to life-time demands. Even then, it's still controversial.

## Demand Functions

For the remainder of the section, we assume that  $x(p, w)$  is single-valued, continuous, and differentiable.

*Remark.*  $\mathcal{B}^{\mathcal{W}} = \{B_{p,w} : p \gg 0, w > 0\}$  does NOT contain all two- and three-element subsets of  $X$ . Therefore, choice based approach  $\neq$  preference-based approach.

## **Comparative statics (with respect to $p$ and $w$ )**

The examination of a change in outcome in response to a change in underlying economic parameters is known as *comparative statics* analysis.

This section examines how the consumer's choice would vary with changes in his wealth and in prices.



## Wealth Effects

For fixed prices  $\bar{p}$ ,  $x(\bar{p}, w)$  is called the consumer's *Eagel function*. Its image in  $\mathbb{R}_+^L$ ,  $E_{\bar{p}} = \{x(\bar{p}, w) : w > 0\}$  is the *wealth expansion path*.

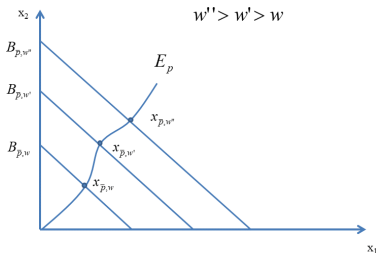


Figure 10: Wealth expansion path at  $\bar{p}$

## Wealth Effects

The derivative  $\frac{\partial x_l(p,w)}{\partial w}$  is the *wealth effect* for the  $l^{th}$  good.

- A commodity  $l$  is *normal* at  $(p, w)$  if  $\frac{\partial x_l(p,w)}{\partial w} \geq 0$ .
- A commodity  $l$  is *inferior* at  $(p, w)$  if  $\frac{\partial x_l(p,w)}{\partial w} < 0$ .

In matrix notation, the wealth effects are

$$D_w x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L.$$

## Price Effects

The demand function for good  $l$  could be represented as a function of  $p_l$ , keeping other things equal, i.e.,  $x(p_l, \bar{p}_{-l}, \bar{w})$ .

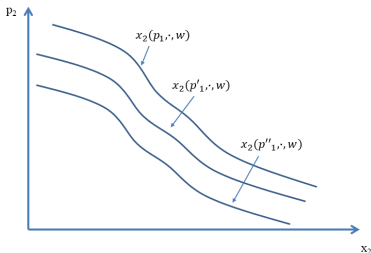


Figure 11: Demand for good 2 as a function of its price

## Price Effects

Another useful representation of the consumers' demand at different prices  $p_l$  is the locus of points demanded in  $\mathbb{R}_+^L$ , for fixed  $p_{-l}$  and  $w$ . This is known as an *offer curve*.

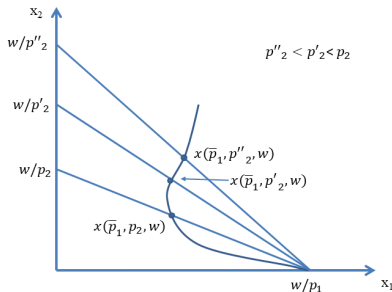


Figure 12: Offer Curve

## Price Effects

The derivative  $\frac{\partial x_l(p,w)}{\partial p_k}$  is the *price effect* of  $p_k$  on the demand for good  $l$ .

- Good  $l$  is a *Giffen good* if  $\frac{\partial x_l(p,w)}{\partial p_l} > 0$ . (Example: potatoes at low wealth level)

In matrix notation, the price effects are

$$D_p x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \dots & \frac{\partial x_1(p,w)}{\partial p_L} \\ & \ddots & \\ \frac{\partial x_L(p,w)}{\partial p_1} & \dots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}.$$

## Implications of homogeneity for price and wealth effects

**Proposition 2.E.1.** *If the Walrasian demand function  $x(p, w)$  is homogeneous of degree zero, then for all  $p$  and  $w$ :*

$$\sum_{k=1}^L \frac{\partial x_l(p, w)}{\partial p_k} p_k + \frac{\partial x_l(p, w)}{\partial w} w = 0, \text{ for } l = 1, \dots, L. \quad (2.E.1)$$

*In matrix notation, it is expressed as*

$$D_p x(p, w)p + D_w x(p, w)w = 0. \quad (2.E.2)$$

## Implication of homogeneity for price and wealth effects

Divide the expression by  $x_l$ :

$$\sum_{k=1}^L \frac{\partial x_l(p, w)}{\partial p_k} \frac{p_k}{x_l(p, w)} + \frac{\partial x_l(p, w)}{\partial w} \frac{w}{x_l(p, w)} = 0, \text{ for } l = 1, \dots, L.$$

i.e.,

$$\sum_{k=1}^L \varepsilon_{lk}(p, w) + \varepsilon_{lw}(p, w) = 0, \text{ for } l = 1, \dots, L. \quad (2.E.3)$$

*Intuition:* The above equation describes the percentage change in  $x_l$  if all prices and wealth changes 1%. Basically, the equation captures the definition of H.D.Ø.

## Implications of Walras' Law for price and wealth effects

**Proposition 2.E.2.** *If the Walrasian demand function  $x(p, w)$  satisfies the Walras' Law, then for all  $p$  and  $w$  :*

$$\sum_{l=1}^L p_l \frac{\partial x_l(p, w)}{\partial p_k} + x_k(p, w) = 0, \text{ for } k = 1, 2, \dots, L, \quad (2.E.4)$$

*or written in matrix notation,*

$$p \cdot D_p x(p, w) + x(p, w)^T = 0^T. \quad (2.E.5)$$

*Intuition:* Total expenditure cannot change in response to a change in prices.



## Implications of Walras' Law for price and wealth effects

**Proposition 2.E.3.** *If the Walrasian demand function  $x(p, w)$  satisfies Walras' Law, then for ALL  $p$  and  $w$ :*

$$\sum_{l=1}^L p_l \frac{\partial x_l(p, w)}{\partial w} = 1, \quad (2.E.6)$$

*or, written in matrix notation,*

$$p \cdot D_w x(p, w) = 1. \quad (2.E.7)$$

*Intuition:* Total expenditure must change by an amount equal to any wealth change.

## **2.F. Weak Axiom of Revealed Preference and Law of Demand**

Implicit assumptions:  $x(p, w)$  is single-valued, homogeneous of degree zero, and satisfies Walras' Law.

## W.A.R.P and Law of Demand

**Definition 2.F.1.** The Walrasian demand function  $x(p, w)$  satisfies the weak axiom of revealed preference (W.A.R.P) if the following holds for any two price-wealth situations  $(p, w)$  and  $(p', w')$ : If  $p \cdot x(p', w') \leq w$  and  $x(p', w') \neq x(p, w)$ ,<sup>2</sup> then  $p' \cdot x(p, w) > w'$ .

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<sup>2</sup>Note that  $x(p, w)$  is the demand given  $(p, w)$  and  $x(p', w')$  is the demand given  $(p', w')$ .

## Definition stated using language in Chapter 1

Let  $B_{p,w}$  denote the budget set given  $p$  and  $w$ ; and  $B_{p',w'}$  denote the budget set given  $p'$  and  $w'$ .  $p \cdot x(p', w') \leq w$  means that  $x(p', w')$  is also affordable under  $B_{p,w}$ . Through the choice given  $B_{p,w}$ ,  $x(p, w)$  is revealed preferred to  $x(p', w')$ . Therefore, by W.A.R.P, it must not be revealed that  $x(p', w')$  is preferred to  $x(p, w)$ . In other words, if  $x(p, w)$  is not chosen given the budget  $B_{p',w'}$ , it must be that it is not affordable, i.e.,  $p' \cdot x(p, w) > w'$ , or  $x(p, w) \notin B_{p',w'}$ .

## Demand Satisfying W.A.R.P

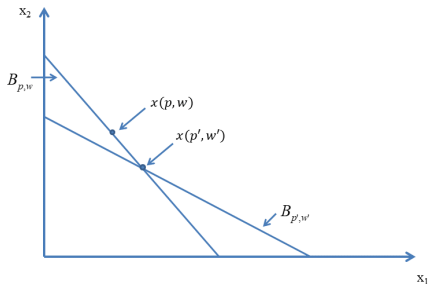


Figure 13: Demand satisfying W.A.R.P

## Violation of W.A.R.P

W.A.R.P may be violated only if both  $x(p, w)$  and  $x(p', w')$  belong to both  $B_{p,w}$  and  $B_{p',w'}$ .

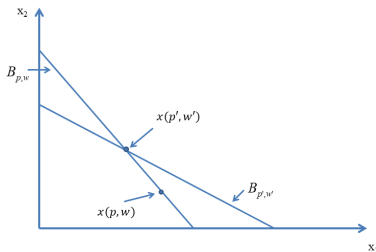


Figure 14: Demand violating W.A.R.P

## **Uncompensated price change**

Such a price change would affect the consumer in two ways:

- change the relative cost of commodities;
- change the consumer's real wealth.

## W.A.R.P and Uncompensated price change

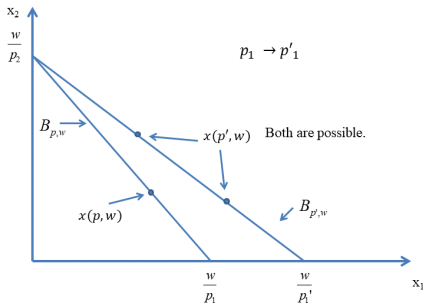


Figure 15: Uncompensated price change

Assuming W.A.R.P, no prediction on change in demand can be drawn.



## Compensated price change

Imagine a situation in which a change in prices is accompanied by a change in the consumer's wealth that makes her initial consumption bundle just affordable at the new prices. That is,  $w' = p' \cdot x(p, w)$ . The wealth adjustment is  $\Delta w = \Delta p \cdot x(p, w)$ . This kind of wealth adjustment is called *Slutsky wealth compensation*. The price changes that are accompanied by compensating wealth changes are called *(Slutsky) compensated price changes*.

## Compensated price change

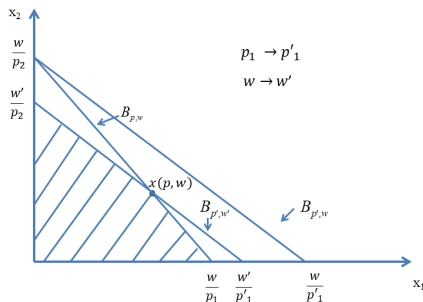


Figure 16: Compensated price change

- $x_1$  must increase after compensated price change.
- $x(p', w') \succsim x(p, w)$ , so it cannot be  $x(p, w) \succ x(p', w')$ .

## **How to check whether W.A.R.P. is Satisfied.**

It would be easier to check whether W.A.R.P is satisfied for compensated price changes.

*Question.* What does it mean that W.A.R.P is satisfied for all compensated price changes?

## **W.A.R.P is satisfied for all compensated price change**

For any price change from  $(p, w)$  to  $(p', w')$  such that  $p' \cdot x(p, w) = w'$ , if  $x(p', w') \neq x(p, w)$ , then  $p \cdot x(p', w') > w$ .

[New bundle chosen after compensated price change is unaffordable under original price and wealth.]

## **W.A.R.P is violated for some compensated price change**

There exists a price change from  $(p, w)$  to  $(p', w')$  such that  $p' \cdot x(p, w) = w'$ ,  $x(p', w') \neq x(p, w)$  and  $p \cdot x(p', w') \leq w$ .

## W.A.R.P and Compensated Price Change

Figure 17 below depicts an example compensated price change.

Changes in demand to  $x(p', w')$  in the red area constitutes violation of W.A.R.P.

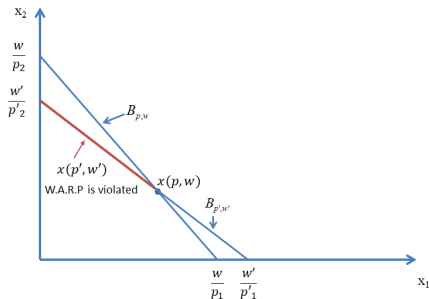


Figure 17: Compensated price change

## **W.A.R.P and Compensated Price Change**

Next, we present a useful lemma which makes it easier to check whether a demand function satisfies W.A.R.P.

**Lemma 1.** *W.A.R.P holds for all price changes if and only if it holds for all compensated price changes.*

## W.A.R.P and Compensated Price Change

**Proposition 2.F.1.** *Suppose that the Walrasian demand function  $x(p, w)$  is homogeneous of degree zero and satisfies Walras' Law, Then  $x(p, w)$  satisfies W.A.R.P if and only if the following property holds: For ANY compensated price change from an initial situation  $(p, w)$  to a new price-wealth pair  $(p', w') = (p', p \cdot x(p, w))$ , we have*

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0, \quad (2.F.1)$$

*with strict inequality whenever  $x(p, w) \neq x(p', w')$ .*

## W.A.R.P and Compensated Price Change

*Remark.* The inequality (2.F.1) can be interpreted as a form of the Law of Demand: Demand and price move in opposite directions. Since it only holds for compensated price changes, it is called the *Compensated Law of Demand*.

- As illustrated in Figure 15, W.A.R.P does not generate definitive prediction on the demand changes in response to *uncompensated* price changes.



## Weak Axiom and Differentiable Demand

**Proposition 2.F.2.** *If a differentiable Walrasian demand function  $x(p, w)$  satisfies Walras' Law, homogeneous of degree zero, and W.A.R.P, then at any  $(p, w)$ , the Slutsky matrix  $S(p, w)$  satisfies  $v \cdot S(p, w)v \leq 0$  for any  $v \in \mathbb{R}^L$ . i.e.  $S(p, w)$  is negative semidefinite.*

$S(p, w)$  is defined as

$$S(p, w) = D_p x(p, w) + D_w x(p, w)x(p, w)^T$$

as the *substitution matrix* or *Slutsky matrix*.

## Slutsky Matrix

In matrix notation, it is

$$S(p, w) = \begin{bmatrix} s_{11}(p, w) & \cdots & s_{1L}(p, w) \\ & \ddots & \\ s_{L1}(p, w) & \cdots & s_{LL}(p, w) \end{bmatrix},$$

where the  $(l, k)^{th}$  entry is

$$s_{l,k}(p, w) = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w).$$

$s_{l,k}(p, w)$  are known as *substitution effects*.

## Slutsky Matrix

*Remark.* Proposition 2.F.2 does not imply, in general, that the matrix  $S(p, w)$  is symmetric.

- For  $L = 2$ ,  $S(p, w)$  is necessarily symmetric.
- When  $L > 2$ ,  $S(p, w)$  is not necessarily symmetric, under the assumptions so far (H.D.Ø, Walras' Law, and W.A.R.P).
- Symmetry of  $S(p, w)$  is connected with maximization of rational preferences. (It will be introduced in Chapter 3.)

## Slutsky Matrix

**Corollary.** *The substitution effect of good  $l$  with respect to its own price is always nonpositive, i.e.,  $s_{ll}(p, w) \leq 0$ .*

*Remark.* An implication of  $s_{ll}(p, w) \leq 0$  is that a good can be a **Giffen good** at  $(p, w)$  only if it is **inferior**.

## More properties on Slutsky matrix

**Proposition 2.F.3.** *Suppose that the Walrasian demand function  $x(p, w)$  is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then,  $p \cdot S(p, w) = 0$  and  $S(p, w)p = 0$  for any  $(p, w)$ .*

## Summary of Chapter 2

Taking choice as the primitive, we look at the implications of these assumptions:

- (i)  $x(p, w)$  is homogeneous of degree zero
- (ii)  $x(p, w)$  satisfies Walras' Law
- (iii)  $x(p, w)$  satisfies the W.A.R.P  $\implies$  Compensated Law of Demand
- (iv)  $x(p, w)$  is differentiable  $\implies$  negative semidefinite Slutsky matrix