

Chapter 3. Classical Demand Theory

(Part 1)

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3.A. Introduction: Take \succsim as the primitive

- (1) Assumption(s) on \succsim so that \succsim can be represented with a utility function
- (2) Utility maximization and demand function
- (3) Utility as a function of prices and wealth (indirect utility)
- (4) Expenditure minimization and expenditure function
- (5) Relationship among demand function, indirect utility function, and expenditure function

3.B. Preference Relations: Basic Properties

Rationality We would assume *Rationality* (*Completeness and Transitivity*) throughout the chapter.

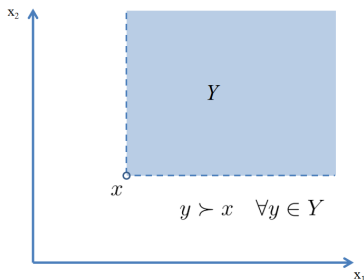
Definition 3.B.1. The preference relation \succsim on X is rational if it possesses the following two properties:

(i) **Completeness:** For all $x, y \in X$, we have $x \succsim y$ or $y \succsim x$ (or both).

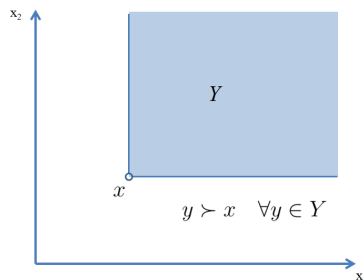
(ii) **Transitivity:** For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Monotonicity

Definition 3.B.2. The preference relation \succsim on X is *monotone* if $x, y \in X$ and $y \gg x$ implies $y \succ x$. It is *strongly monotone* if $y \geq x$ & $y \neq x$ implies $y \succ x$.



Monotonicity



Strong Monotonicity

Monotonicity

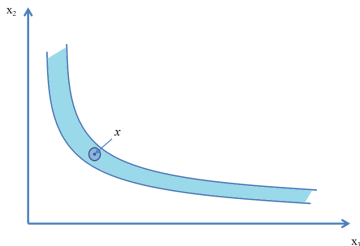
Claim. If \succsim is strongly monotone, then it is monotone.

Example. Here is an example of a preference that is monotone, but not strongly monotone:

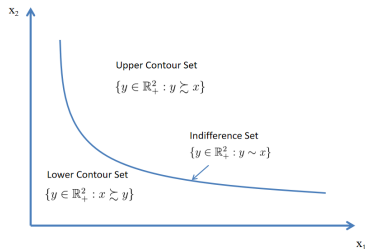
$$u(x_1, x_2) = x_1 \text{ in } \mathbb{R}_+^2.$$

Local Nonsatiation

Definition 3.B.3. The preference relation \succsim on X is *locally nonsatiated* if for every $x \in X$ and every $\varepsilon > 0$, $\exists y \in X$ such that $\|y - x\| \leq \varepsilon$ and $y \succ x$.



Violation



Compatible

Local Nonsatiation

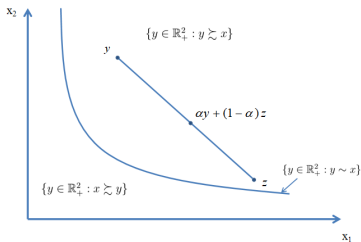
Claim. *Local nonsatiation* is a weaker desirability assumption compared to *monotonicity*. If \succsim is monotone, then it is locally nonsatiated.

Example. Here is an example of a preference that is locally nonsatiated, but not monotone:

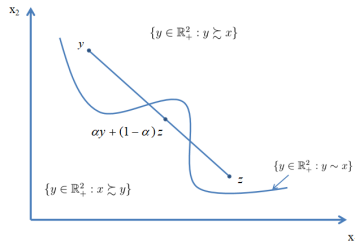
$$u(x_1, x_2) = x_1 - |1 - x_2| \text{ in } \mathbb{R}_+^2.$$

Convexity Assumptions

Definition 3.B.4. The preference relation \succsim on X is *convex* if for every $x \in X$, the upper contour set of x , $\{y \in X : y \succsim x\}$ is convex; that is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1 - \alpha)z \succsim x$ for any $\alpha \in [0, 1]$.



Convex



Nonconvex

Properties associated with convexity

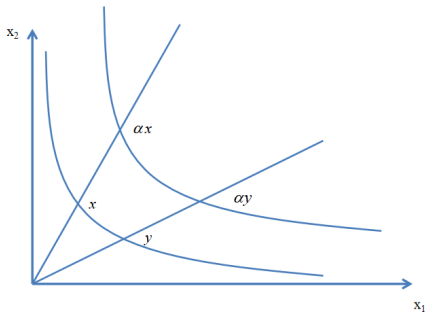
- (i) Diminishing marginal rates of substitution
- (ii) Preference for diversity (implied by (i))

Strict Convexity

Definition 3.B.5. The preference relation \succsim on X is *strictly convex* if for every $x \in X$, we have that $y \succsim x$ and $z \succsim x$, and $y \neq z$ implies $\alpha y + (1 - \alpha)z \succ x$ for all $\alpha \in (0, 1)$.

Homothetic Preference

Definition 3.B.6. A monotone preference relation \succsim on $X = \mathbb{R}_+^L$ is *homothetic* if all indifference sets are related by proportional expansion along rays; that is, if $x \sim y$, then $\alpha x \sim \alpha y$ for any $\alpha \geq 0$.



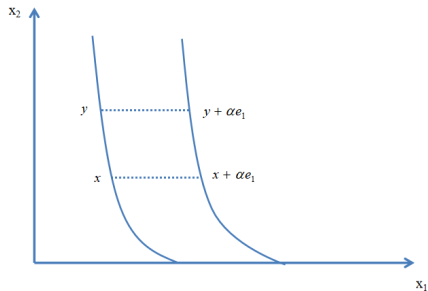
Homothetic Preference

Quasilinear Preference

Definition 3.B.7. \succsim on $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is *quasilinear* with respect to commodity 1 (*numeraire* commodity) if

- (i) All the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, 0, \dots, 0)$ and any $\alpha \in \mathbb{R}$.
- (ii) Good 1 is desirable; that is $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

Quasilinear Preference



Quasilinear Preference

3.C. Preference and Utility

Key Question. When can a rational preference relation be represented by a utility function?

Answer: If the preference relation is continuous.

Continuous Preference

Definition 3.C.1. The preference relation \succsim on X is *continuous* if it is preserved in the limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for all n , $x = \lim_{n \rightarrow \infty} x^n$, $y = \lim_{n \rightarrow \infty} y^n$, we have $x \succsim y$.

Remark. \succsim is continuous if and only if for all x , the upper contour set $\{y \in X : y \succsim x\}$ and the lower contour set $\{y \in X : x \succsim y\}$ are both closed.

Continuous Preference

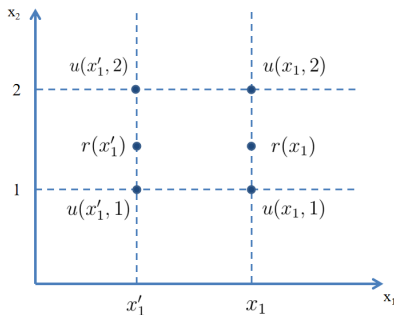
Example 3.C.1. Lexicographic Preference Relation on \mathbb{R}^2

$X = \mathbb{R}^2$; $x \succ y$ if either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$. $x \sim y$ if $x_1 = y_1$ and $x_2 = y_2$.

Claim. Lexicographic Preference Relation on \mathbb{R}^2 is not continuous.

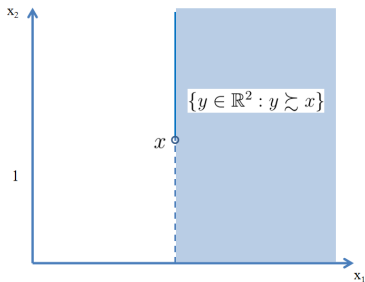
Continuous Preference

Claim. Lexicographic Preference Relation on \mathbb{R}^2 cannot be represented by $u(\cdot)$.

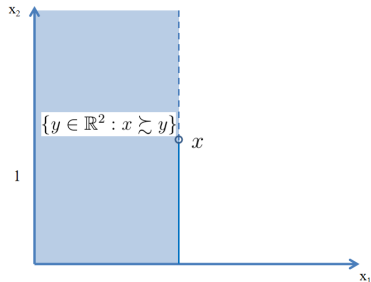


Lexicographic Preference

Continuous Preference Alternatively, we could use the fact that upper and lower contour sets of a continuous preference must be closed.



Upper Contour Set



Lower Contour Set

Continuous Preference

Proposition 3.C.1. *Suppose that the preference relation \succsim on X is continuous. Then there exists continuous utility function $u(x)$ that represents \succsim , i.e., $u(x) \geq u(y)$ if and only if $x \succsim y$.*

For this course, we will only prove a simplified version of Proposition 3.C.1, which assumes that the preference relation \succsim is also monotone.

Continuous Preference

Remark. $u(x)$ is not unique, any increasing transformation $v(x) = f(u(x))$ will represent \succsim . We can also introduce countably many jumps in $f(\cdot)$.

Claim. If $u(\cdot)$ is a continuous utility function representing \succsim , then \succsim is continuous.

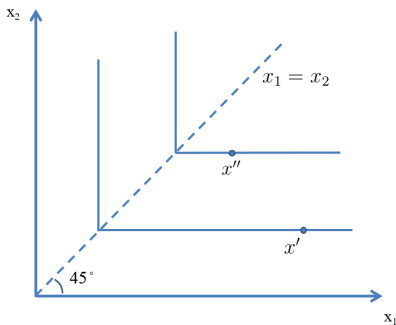
Assumptions of differentiability of $u(x)$

The assumption of differentiability is commonly adopted for technical convenience, but is not applicable to all useful models.

Assumptions of differentiability of $u(x)$

Here is an example of preference that is not differentiable.

Example (Leontief Preference). $x \succsim y$ if and only if $\min\{x_1, x_2\} \geq \min\{y_1, y_2\}$.



Leontief Preference

Implications of \succsim and u

- (i) \succsim is convex $\iff u : X \rightarrow \mathbb{R}$ is quasi-concave.
- (ii) continuous \succsim on \mathbb{R}_+^L is homothetic $\iff u(x)$ is H.D.1.
- (iii) continuous \succsim on $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is quasilinear with respect to Good 1 $\iff u(x) = x_1 + \phi(x_2, \dots, x_L)$

Quasiconcave Utility

Definition. The utility function $u(\cdot)$ is *quasiconcave* if the set $\{y \in \mathbb{R}_+^L : u(y) \geq u(x)\}$ is convex for all x or, equivalently, if $u(\alpha x + (1 - \alpha)y) \geq \min\{u(x), u(y)\}$ for all x, y and all $\alpha \in [0, 1]$. If $u(\alpha x + (1 - \alpha)y) > \min\{u(x), u(y)\}$ for $x \neq y$ and $\alpha \in (0, 1)$, then $u(\cdot)$ is *strictly quasiconcave*.

3.D. Utility Maximization Problem (UMP)

Assume throughout that preference is *rational*, *continuous*, *locally nonsatiated*, and $u(x)$ continuous.

Consumer's *Utility Maximization Problem (UMP)*:

$$\max_{x \in \mathbb{R}_+^L} u(x)$$

$$\text{s.t. } p \cdot x \leq w$$

Existence of Solution

Proposition 3.D.1. *If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.*

Existence of Solution

Here, we provide two counter examples where the solution of UMP does not exist.

Counter Examples.

(i) $B_{p,w}$ is not closed: $p \cdot x < w$

(ii) $u(x)$ is not continuous:

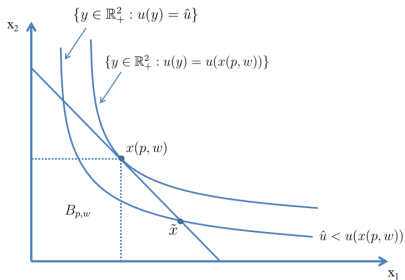
$$u(x) = \begin{cases} p \cdot x & \text{for } p \cdot x < w \\ 0 & \text{for } p \cdot x = w \end{cases}$$

Walrasian demand correspondence/functions

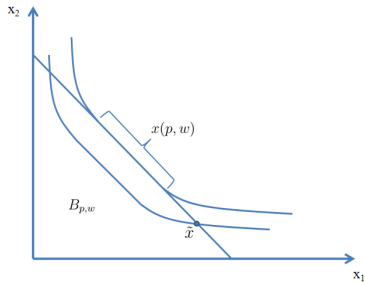
The solution of UMP, denoted by $x(p, w)$, is called *Walrasian* (or *ordinary* or *market*) *demand correspondence*.

When $x(p, w)$ is single valued for all (p, w) , we refer to it as *Walrasian* (or *ordinary* or *market*) *demand function*.

Walrasian demand correspondence/functions



Single solution



Multiple solutions

Properties of Walrasian demand correspondence

Proposition 3.D.2. *Suppose that $u(x)$ is a continuous utility function representing a locally nonsatiated preference relation \succsim defined on the consumption set $X = \mathbb{R}_+^L$. Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:*

- (i) *Homogeneity of degree zero in (p, w) : $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and scalar $\alpha > 0$.*
- (ii) *Walras' Law: $p \cdot x = w$ for all $x \in x(p, w)$.*

Properties of Walrasian demand correspondence

Proposition 3.D.2 (continued).

(iii) *Convexity/uniqueness: If \succsim is convex, so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set. Moreover, if \succsim is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ consists of a single element.*