Advanced Microeconomics

Assignment 3

3.E.9 Use the relations in (3.E.1) to show that the properties of the indirect utility function identified in Proposition 3.D.3 imply Proposition 3.E.2. Likewise, use the relations in (3.E.1) to prove that Proposition 3.E.2 implies Proposition 3.D.3.

Solution. We first show that by relations in (3.E.1), the properties of the indirect utility function in Proposition 3.D.3 imply the properties of the expenditure function in Proposition 3.E.2.
Let $p \gg 0, p' \gg 0, u \in \mathbb{R}, u' \in \mathbb{R}$, and $\alpha \geq 0$.

(i) **Homogeneity:** Let $\alpha > 0$. Define $w = e(p, u)$, then $u = v(p, w)$ by the second relation in (3.E.1). Hence,

$$e(\alpha p, u) = e(\alpha p, v(p, w)) = e(\alpha p, v(\alpha p, \alpha w)) = \alpha w = \alpha e(p, u),$$

where the second equality follows from the homogeneity of $v(p, w)$ and the third from the first relation in (3.E.1). Therefore, the expenditure function is homogeneous of degree one in p.

(ii) **Monotonicity:** Let $u' > u$. Define $w = e(p, u)$ and $w' = e(p, u')$, then $u = v(p, w)$ and $u' = v(p, w')$. Since $v(p, w)$ is strictly increasing in w, we must have $w' > w$, that is, $e(p, u') > e(p, u)$.

Next let $p' \geq p$. Define $w = e(p, u)$ and $w' = e(p', u)$, then by the second relation in (3.E.1), $u = v(p, w) = v(p', w')$. Since $v(p, w)$ is strictly increasing in w and nonincreasing in p_l for any l, we must have $w' \geq w$, that is, $e(p', u) \geq e(p, u)$. Therefore, the expenditure function $e(p, u)$ is strictly increasing in u and nondecreasing in p_l for any l.

(iii) **Concavity:** Let $\alpha \in [0, 1]$. Define $w = e(p, u)$ and $w' = e(p', u)$, then $u = v(p, w) = v(p', w')$. Define $p'' = \alpha p + (1 - \alpha)p'$ and $w'' = \alpha w + (1 - \alpha)w'$. Then, by the quasiconvexity of $v(p, w)$, $v(p'', w'') \leq u$. Hence, since $v(p, w)$ is strictly increasing in w and $v(p'', e(p'', u)) = u$, we must have $e(p'', u) \geq w''$, that is,

$$e(\alpha p + (1 - \alpha)p', u) \geq \alpha e(p, u) + (1 - \alpha)e(p', u).$$

Therefore, the expenditure function $e(p, u)$ is concave in p.

(iv) **Continuity:** Suppose the sequence $\{(p^n, u^n)\}_{n=1}^{\infty}$ converges to (p, u), we show that $\lim_{n\to\infty} e(p^n, u^n) = e(p, u)$. Suppose to the contrary that $\lim_{n\to\infty} e(p^n, u^n) = w \neq$

$e(p,u)$ for some $w \in \mathbb{R}$. On the one hand, by the second relation in (3.E.1), $v(p^n, e(p^n, u^n)) = u^n$, which converges to u by assumption. On the other hand, since $v(p,w)$ is continuous in (p,w) and is strictly increasing in w, $\lim_{n\to\infty} e(p^n, u^n) = w \neq e(p,u)$ implies that $\lim\limits_{n\to\infty} v(p^n, e(p^n, u^n)) = v(p,w) \neq v(p, e(p,u)) = u$. Therefore, $\lim_{n\to\infty} v(p^n, e(p^n, u^n)) = u$ and $\lim_{n\to\infty} v(p^n, e(p^n, u^n)) \neq u$, which constitutes a contradiction. Hence, we must have $\lim_{n\to\infty} e(p^n, u^n) = e(p,u)$, i.e., $e(p,u)$ is continuous in (p,u).

Now we show that by relations in (3.E.1), the properties of the expenditure function in Proposition 3.E.2 imply the properties of the indirect utility function in Proposition 3.D.3.
Let $p \gg 0, p' \gg 0, w \in \mathbb{R}, w' \in \mathbb{R}$, and $\alpha \geq 0$.

(i) **Homogeneity:** Let $\alpha > 0$. Define $u = v(p,w)$, then by the first relation in (3.E.1), $e(p,u) = w$. Hence,

$$v(\alpha p, \alpha w) = v(\alpha p, \alpha e(p,u)) = v(\alpha p, e(\alpha p, u)) = u = v(p,w),$$

where the second equality follows from the homogeneity of $e(p,u)$ and the third from the second relation in (3.E.1). Therefore, the indirect utility function $v(p,w)$ is homogeneous of degree zero.

(ii) **Monotonicity:** Let $w' > w$. Define $u = v(p,w)$ and $u' = v(p,w')$, then $e(p,u) = w$ and $e(p,u') = w'$. Since $e(p,u)$ is strictly increasing in u, we must have $u' > u$, that is, $v(p,w') > v(p,w)$. Therefore, the indirect utility function is strictly increasing in w.

Next let $p' \geq p$. Define $u = v(p,w)$ and $u' = v(p',w)$, then $e(p,u) = e(p',u') = w$. Since $e(p,u)$ is strictly increasing in u and nondecreasing in p_l for any l, we must have $u' \leq u$, that is $v(p',w) \leq v(p,w)$. Therefore, the indirect utility function is nonincreasing in p.

(iii) **Quasiconvexity:** Let $\alpha \in [0,1]$. Define $u = v(p,w)$ and $u' = v(p',w')$, then $e(p,u) = w$ and $e(p,u') = w'$. Without loss of generality, assume that $u' \geq u$. Define $p'' = \alpha p + (1-\alpha)p'$ and $w'' = \alpha w + (1-\alpha)w'$, and we show that $v(p'', w'') \leq u'$. Since $u' = v(p'', e(p'', u'))$ and $v(p,w)$ is strictly increasing in w, it suffices to show that $e(p'', u') \geq w''$. This is proved as follows:

$$\begin{aligned} e(p'', u') &\geq \alpha e(p, u') + (1-\alpha)e(p', u') \\ &\geq \alpha e(p, u) + (1-\alpha)e(p', u) \\ &= \alpha w + (1-\alpha)w' = w'', \end{aligned}$$

where the first inequality follows from the concavity of $e(p, u)$ in p, and the second from the monotonicity of $e(p, u)$ in u. Therefore, the indirect utility function $v(p, w)$ is quasiconvex.

(iv) **Continuity:** Suppose the sequence $\{(p^n, w^n)\}_{n=1}^{\infty}$ converges to (p, w), we show that $\lim_{n\to\infty} v(p^n, w^n) = v(p, w)$. Suppose to the contrary that $\lim_{n\to\infty} v(p^n, w^n) = u \neq v(p, w)$ for some $u \in \mathbb{R}$. On the one hand, $e(p^n, v(p^n, w^n)) = w^n$, which converges to w by assumption. On the other hand, as $e(p, u)$ is continuous in (p, u) and is strictly increasing in u, $e(p^n, v(p^n, w^n))$ converges to $e(p, u) \neq e(p, v(p, w)) = w$. Therefore, $\lim_{n\to\infty} e(p^n, v(p^n, w^n)) = w$ and $\lim_{n\to\infty} e(p^n, v(p^n, w^n)) \neq w$, which constitutes a contradiction. Hence, we must have $\lim_{n\to\infty} v(p^n, w^n) = v(p, w)$, i.e., $v(p, w)$ is continuous in (p, w).