Chapter 5. Production

5.A. Introduction

In this chapter, we study the **supply side** of the economy. In particular, we study how goods and services are produced by "firms". Here, we view firms as "black boxes", transforming inputs into outputs. That is, we ignore the organizational structure within the firms. Please note that this simplification is for the purpose of analyzing market behavior. The study of organizational structure, witch falls outside of the scope of this chapter, is also important and interesting.

5.B. Production Sets

- We consider an economy with L commodities.
- *Production vector* (**including both inputs & outputs**) $y = (y_1, ..., y_L) \in \mathbb{R}^L$ describes the (net) outputs.
  - If $y_l > 0$, l is an output;
  - If $y_l \leq 0$, l is an input.

  Example 5.B.1. Suppose that $L = 5$, Then $y = (-5, 2, -6, 3, 0)$ means that

  (a) 2 amd 3 units of Good 2 and 4 are produced;

  (b) 5 amd 6 units of Good 1 and 3 are used;

  (c) Good 5 is neither produced or used.
- The set of all production vectors that constitute technologically feasible plans is called the *production set* $Y \subset \mathbb{R}^L$.
  - Any $y \in Y$ is feasible;
  - Any $y \notin Y$ is not feasible.
- We can describe the production set Y by a *transformation function* $F(\cdot)$.
  - The production set $Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$.

- $\{y \in \mathbb{R}^L : F(y) = 0\}$ is called the *transformation frontier*.

- Consider changes in y while staying on $F(y) = 0$. For such changes dy along the frontier, we have $dy \cdot \nabla F(y) = 0$.

- Suppose only y_l & y_k change.

$$dF(\bar{y}) = \frac{\partial F(\bar{y})}{\partial y_l} dy_l + \frac{\partial F(\bar{y})}{\partial y_k} dy_k = 0$$
$$\iff \frac{dy_k}{dy_l} = -\frac{\partial F(\bar{y})/\partial y_l}{\partial F(\bar{y})/\partial y_k} = -MRT_{lk}(\bar{y}).$$

  $MRT_{lk}(\bar{y})$ is called the *marginal rate of transformation (MRT) of good l for good k at $\bar{y}$.*

- Figure 1 below presents the production function and transformation frontier for two goods.

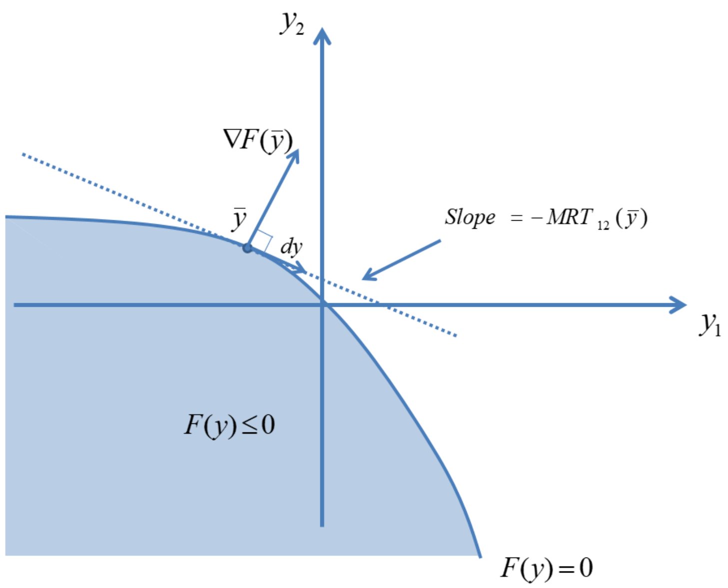


Figure 1: Production Function and Transformation Frontier

Technologies with Distinct Inputs and Outputs

- Suppose there are M outputs and $L - M$ inputs.
  - let $q = (q_1, ..., q_M) \geq 0$ denote the outputs.
  - let $z = (z_1, ..., z_{L-M}) \geq 0$ denote the inputs.

  - e.g. $(y_{L-M+1}, ..., y_L) = (q_1, ..., q_M)$; $(y_1, ..., y_{L-M}) = -(z_1, ..., z_{L-M})$.

- Single-output technology
  - *Production function*: $f(z)$, where $z = (z_1, ..., z_{L-1}) \geq 0$
  - Output: $q \leq f(z)$
  - Production set:

$$Y = \{(-z_1, ..., -z_{L-1}, q) : q - f(z_1, ..., z_{L-1}) \leq 0 \text{ and } (z_1, ..., z_{L-1}) \geq 0\}$$

- Holding the level of output fixed, we define *Marginal rate of technological substitution (MRTS) of input l for input k at $\bar{z}$* as follows:

$$MRTS_{lk}(\bar{z}) = \frac{\partial f(\bar{z})/\partial z_l}{\partial f(\bar{z})/\partial z_k}$$

  - $MRTS_{lk}(\bar{z})$ is the same as $MRT_{lk}(\bar{z}, \bar{q})$, simply a renaming for the substitution between inputs in a single-output case.

Example 5.B.2. Cobb-Douglas Production Function:

$f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$, where $a \geq 0, \beta \geq 0$. ($f(z_1, z_2)$: output, z_1 : input 1, z_2 : input 2.)

MRTS at $z = (z_1, z_2)$ is

$$MRTS_{12}(z) = \frac{\partial f(z_1, z_2)/\partial z_1}{\partial f(z_1, z_2)/\partial z_2} = \frac{\alpha z_1^{\alpha-1} z_2^{\beta}}{\beta z_1^{\alpha} z_2^{\beta-1}} = \frac{\alpha z_2}{\beta z_1}.$$

Remark. In percentage change terms

$$\left[\frac{\partial f(z_1, z_2)}{\partial z_1} \frac{z_1}{f(z_1, z_2)}\right] \Big/ \left[\frac{\partial f(z_1, z_2)}{\partial z_2} \frac{z_2}{f(z_1, z_2)}\right] = \frac{\alpha z_2}{\beta z_1} \frac{z_1}{z_2} = \frac{\alpha}{\beta}.$$

Commonly Assumed Properties of Production Sets

(i) Y is nonempty.

(ii) Y is closed. (technical)

(iii) No free lunch: If $y \geq 0$, then $y = 0$. The idea is that no commodities can be created out of thin air. Production of any commodity requires consumption of some other commodities.

(iv) Possibility of inaction: $0 \in Y$.

(v) Free disposal: If $y \in Y$ and $y' \leq y$, then $y' \in Y$.

- Extra amount of inputs (or outputs) can be disposed at no cost.

(vi) Irreversibility: Suppose $y \in Y$ and $y \neq 0$, then $-y \notin Y$.

- For example, One cannot effortlessly disassemble an iPad and turn it back into its original parts in perfect condition.
- Figure 2 and Figure 3 below depict reversible and irreversible technology respectively.

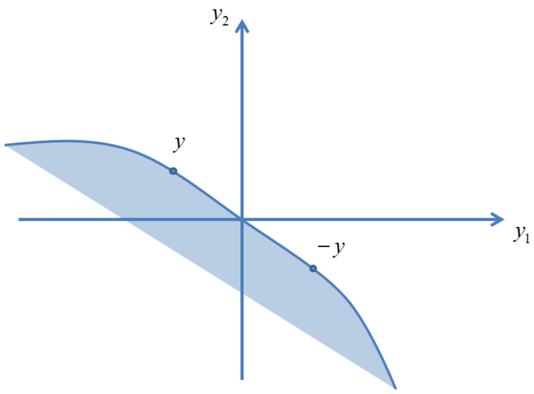


Figure 2: Reversible Technology

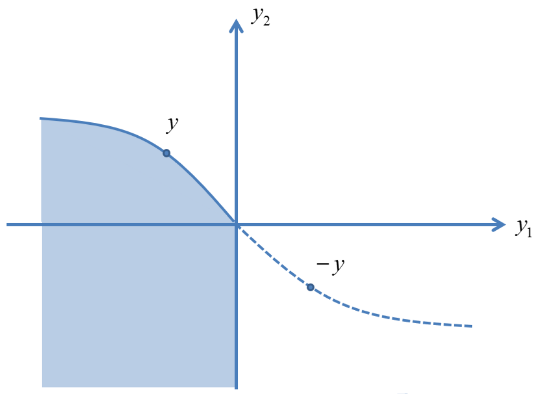


Figure 3: Irreversible Technology

(vii) Nonincreasing returns to scale: $y \in Y$ and $\alpha \in [0, 1] \implies \alpha y \in Y$.

- Smaller scale is more efficient: Half the inputs will get you more than half the outputs.

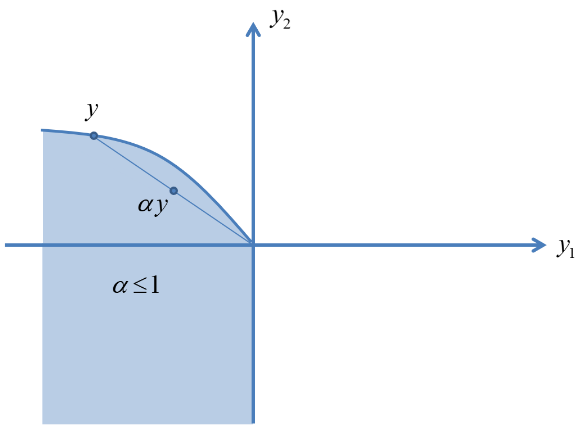


Figure 4: Nonincreasing Returns to Scale Technology

(viii) Nondecreasing returns to scale: $y \in Y$ and $\alpha \geq 1 \implies \alpha y \in Y$.

- Larger scale is more efficient. Double the inputs will get you more than double the outputs.

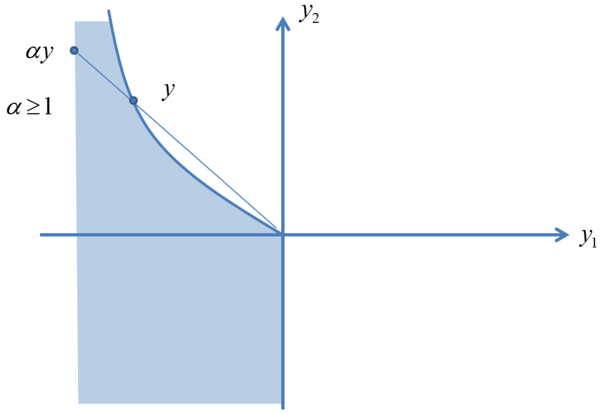
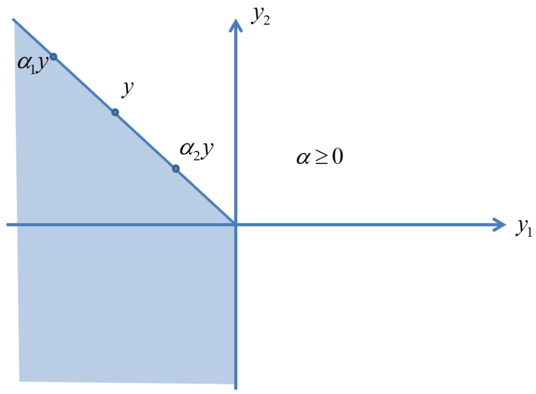


Figure 5: Nondecreasing Returns to Scale Technology

(ix) Constant returns to scale (Cone): $y \in Y$ and $\alpha \geq 0 \implies \alpha y \in Y$.



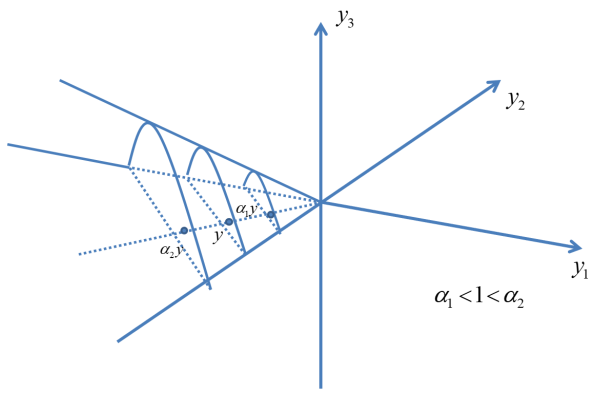


Figure 6: CRS (2 commodities)

Figure 7: CRS (3 commodities)

Exercise 5.B.2

Suppose that $f(\cdot)$ is the production function associated with a single-output technology, and let Y be the production set of this technology. Show that Y satisfies constant returns to scale if and only if $f(\cdot)$ is homogeneous of degree one.

(x) Additivity: Suppose $y \in Y$ and $y' \in Y$. Then $y + y' \in Y$.

- Alternatively, $Y + Y \subset Y$.
- If $y \in Y$, then $ky \in Y$ for all $k \in \mathbb{Z}$.
- This captures an economy with **free entry**: Any existing technology can be added to the existing technologies.

(xi) Convexity: $y, y' \in Y$ and $\alpha \in [0,1] \implies \alpha y + (1-\alpha)y' \in Y$.

- Convexity implies nonincreasing returns to scale: if inaction is possible (i.e., $0 \in Y$), then convexity implies that for any $\alpha \in [0,1]$, $\alpha y = \alpha y + (1-\alpha)0 \in Y$.
- "Balanced" inputs (outputs) are weakly more productive (less costly) than "unbalanced" ones.

Exercise 5.B.3

Show that for a single-output technology, Y is convex if and only if the production function $f(\cdot)$ is concave.

(xii) Convex cone: Y is a convex cone if for any production vector $y, y' \in Y$ and constants $\alpha \geq 0$ & $\beta \geq 0$, we have $\alpha y + \beta y' \in Y$.

- Note that $\alpha y + \beta y'$ can be written as $\gamma[\theta y + (1-\theta)y']$ for some $\gamma \geq 0$ and $\theta \in [0,1]$.[1] The convex combination between y and y' captures the *convex* part of the definition, and $\gamma \geq 0$ captures the *cone* part of the definition.
- The production sets depicted in Figure 6 and Figure 7 are both convex cones.

Proposition 5.B.1. *The production set Y is additive and satisfies the nonincreasing returns condition if and only if it is a convex cone.*

Proof.

1. "$\Longleftarrow$" part: Suppose Y is a convex cone. That is, for any $y, y' \in Y$ and $\alpha \geq 0$ & $\beta \geq 0$, we have $\alpha y + \beta y' \in Y$.

[1] More specifically, we could let $\gamma = \alpha + \beta$ and $\theta = \frac{\alpha}{\alpha+\beta}$, we have $(\alpha + \beta)[\frac{\alpha}{\alpha+\beta} y + \frac{\beta}{\alpha+\beta} y'] \in Y$.

a) Suppose $y,\, y' \in Y$. Let $\alpha = \beta = 1 \implies y + y' \in Y \implies Y$ is Additive.

b) Set $y' = y$ and $\gamma = \alpha + \beta \geq 0$. Then, by Y being a convex cone, we have $y \in Y \implies \gamma y \in Y$ for any $\gamma \geq 0$. Since $y \in Y \implies \gamma y \in Y$ holds for any $\gamma \geq 0$, it must hold for $\gamma \in [0, 1]$, which implies that Y is nonincreasing returns to scale.

2. "$\Longrightarrow$" part: Suppose $y,\, y' \in Y$. By additivity, $k_1 y,\, k_2 y' \in Y$, k_1, for $k_2 \in \mathbb{Z}$. Then by nonincreasing returns to scale, $\theta_1 k_1 y \in Y$ and $\theta_2 k_2 y' \in Y$ for $\theta_1,\, \theta_2 \leq 1$. Again by additivity $\theta_1 k_1 y + \theta_2 k_2 y' \in Y$. Since $\forall\, \alpha, \beta \geq 0$, $\exists\, \theta_1 \leq 1$, $k_1 \in \mathbb{Z}$, $\theta_2 \leq 1$, $k_2 \in \mathbb{Z}$ s.t. $\alpha = \theta_1 k_1$, $\beta = \theta_2 k_2$, then we have that for any $y, y' \in Y$ and $\alpha \geq 0$ & $\beta \geq 0$, $\alpha y + \beta y' \in Y$. That is, Y is a convex cone. $\square$

Remark (on the concept of production set). Production set is a description of techonology. So, if inputs are there, production should be scalable. In other words, decreasing returns to scale observed in real life must be a reflection of scarcity of inputs.

Proposition 5.B.2. *For any convex production set $Y \subset \mathbb{R}^L$ with $0 \in Y$, there is a constant returns, convex production set $Y' \subset \mathbb{R}^{L+1}$ s.t. $Y = \{y \in \mathbb{R}^L : (y, -1) \in Y'\}$.*

Remark. Here "there exists" only means that there exists such an interpretation. It doesn't really mean that the technology necessarily exists.

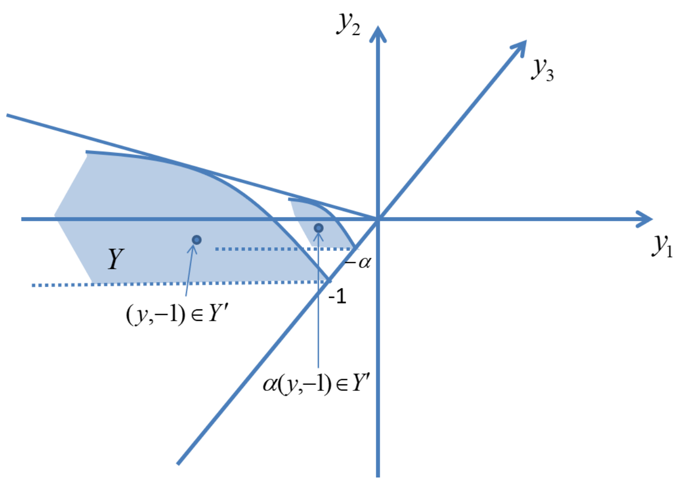


Figure 8: Constant Returns Production Set

Proof. Let $Y' = \{y' \in \mathbb{R}^{L+1} : y' = \alpha(y, -1)$ for some $y \in Y$ & $\alpha \geq 0\}$. See Figure 8.

We now check that Y' is constant returns and convex.

- Constant returns: We need to show that if $y' \in Y'$, then $\gamma y' \in Y'$ for all $\gamma \geq 0$.

  $y' \in Y'$ means $y' = \alpha(y, -1)$ where $y \in Y$ and $\alpha \geq 0$, then for any $\gamma \geq 0$, $\gamma y' = \gamma\alpha(y, -1) \in Y'$, since $y \in Y$ and $\gamma\alpha \geq 0$.

- Convexity: We need to show that if $y'_1, y'_2 \in Y', \beta \in [0, 1]$, then $\beta y'_1 + (1-\beta)y'_2 \in Y'$.

  $y'_1 \in Y'$ and $y'_2 \in Y'$ mean $y'_1 = \alpha_1(y_1, -1)$ where $y_1 \in Y$ and $\alpha_1 \geq 0$ and $y'_2 = \alpha_2(y_2, -1)$ where $y_2 \in Y$ and $\alpha_2 \geq 0$, then for any $\beta \geq 0$,

$$\begin{aligned} &\beta y'_1 + (1-\beta)y'_2 = \beta\alpha_1(y_1, -1) + (1-\beta)\alpha_2(y_2, -1) \\ =&[\alpha_1\beta + \alpha_2(1-\beta)] \left(\frac{\alpha_1\beta}{\alpha_1\beta + \alpha_2(1-\beta)} y_1 + \frac{\alpha_2(1-\beta)}{\alpha_1\beta + \alpha_2(1-\beta)} y_2, -1 \right) \in Y', \end{aligned}$$

  since $\frac{\alpha_1\beta}{\alpha_1\beta+\alpha_2(1-\beta)} y_1 + \frac{\alpha_2(1-\beta)}{\alpha_1\beta+\alpha_2(1-\beta)} y_2 \in Y$ ($\because Y$ is convex) and $\alpha_1\beta + \alpha_2(1-\beta) \geq 0$. □

Remark. Y is not constant returns to scale. But it can be the cross-section of a constant returns to scale $Y' \subset \mathbb{R}^{L+1}$. In essence, the implication is that in a competitive, convex setting, there may be little loss of conceptual generality in limiting to constant returns technologies.

5.C. Profit Maximization and Cost Minimization

- L commodities, priced at $p = (p_1, ...p_L) \gg 0$.
- Firm is *price-taking*.
- Firm's objective is to maximize profit.
- Assume *(i) nonemptiness*, *(ii) closedness*, and *(v) free disposal*.

Profit Maximization Problem

- Profit $= p \cdot y$. This is because y includes both inputs (as negative) and outputs (as positive). Besides, profit can come from multiple products.

- Profit maximization problem

$$\max_{y \in \mathbb{R}^L} p \cdot y$$

$$\text{s.t. } y \in Y(\text{ or } F(y) \leq 0)$$

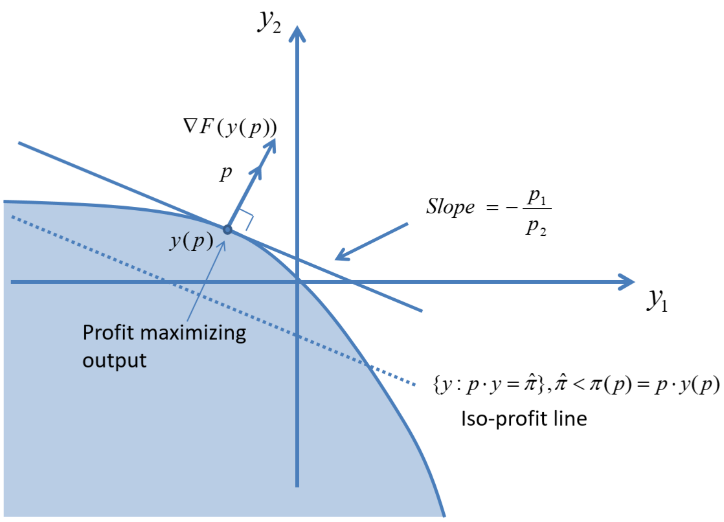


Figure 9: Profit Maximization Problem

- Lagrange Function:

$$\mathcal{L} = p \cdot y - \lambda F(y)$$

- Kuhn-Tucker Conditions:[2]

$$\frac{\partial \mathcal{L}}{\partial y_l} = p_l - \lambda \frac{\partial F(y)}{\partial y_l} = 0 \text{ for } l = 1, .., L, \text{ or } p = \lambda \nabla F(y^*) \tag{1}$$

$$\lambda \geq 0$$

$$\lambda F(y) = 0$$

$$F(y) \leq 0$$

Claim. $F(y) = 0$.

Proof. If $F(y) < 0$, then $\exists y' \gg y$ and $F(y') < 0$. Since y' is feasible and $p \cdot y' > p \cdot y$, it contradicts with the fact that y maximizes profit.

Remark. Equation (1) implies $\frac{p_l}{p_k} = \frac{\partial F(y^*)/\partial y_l}{\partial F(y^*)/\partial y_k} = MRT_{lk}(y^*)$.

[2] Suppose $F(\cdot)$ is differentiable.

- In the case of single-output production, profit $= pf(z) - w \cdot z$ where $p > 0$ is a scalar, and $w = (w_1, ..., w_{L-1}) \gg 0$ is a vector of input prices.

- The profit maximization problem for single-output production is

$$\max_{z \geq 0} pf(z) - w \cdot z$$
$$\text{s.t. } q \leq f(z)$$

Remark. Here $z \geq 0$ is required but not in the previous configuration.

- Lagrange Function:

$$\mathcal{L} = pf(z) - w \cdot z - \lambda(q - f(z))$$

- Kuhn-Tucker Conditions:

$$p\frac{\partial f(z^*)}{\partial z_l} \leq w_l, \text{ with equality if } z_l^* > 0, \text{ for } l = 1, ..., L-1. \tag{2}$$
$$\lambda \geq 0$$
$$\lambda(q - f(z)) = 0$$
$$q \leq f(z)$$
$$z^* \geq 0$$

Equation (2) is equivalent to $p\nabla f(z^*) \leq w$ and $[p\nabla f(z^*) - w] \cdot z^* = 0$.

- Suppose $(z_l^*, z_k^*) \gg 0$. Then,

$$p\frac{\partial f(z^*)}{\partial z_l} = w_l \text{ and } p\frac{\partial f(z^*)}{\partial z_k} = w_k$$
$$\implies \frac{w_l}{w_k} = \frac{\partial f(z^*)/\partial z_l}{\partial f(z^*)/\partial z_k} = MRTS_{lk}(z^*). \tag{3}$$

- Condition (3) can also be rewritten as

$$\frac{1}{w}\frac{\partial f(z^*)}{\partial z_l} = \frac{1}{w}\frac{\partial f(z^*)}{\partial z_k} = \text{ marginal product of \$1.}$$

In other words, when profit is maximized, the marginal product of \$ 1 of production cost spent on each input should be equal. Or else the same production cost should be spent on the input generating a higher marginal product per dollar. It is possible input j generates lower marginal product per dollar than the rest because this input is particularly ineffective. In that case, z_j^* must be zero. Note that the Kuhn-Tucker conditions accommodate this.

- If the production set Y is convex, then the F.O.C in (1) and (2) are not only necessary but also sufficient.

Exercise 5.C.9

Derive the profit function $\pi(p)$ and supply function (or correspondence) $y(p)$ for the single-output technologies whose production functions $f(z)$ are given by

(a) $f(z) = \sqrt{z_1 + z_2}$.

(b) $f(z) = \sqrt{\min\{z_1, z_2\}}$.

(c) $f(z) = (z_1^\rho + z_2^\rho)^{1/\rho}$, for $\rho \leq 1$.

Mathematical Appendix: Separating Hyperplane Theorem Now we need to visit the Mathematical Appendix to retrieve a result that we'll use to prove our next proposition for this chapter.

Theorem M.G.2 (Separating Hyperplane Theorem (Part I))**.** *Suppose that $B \subset \mathbb{R}^N$ is convex and closed, and that $y \notin B$. Then there is a $p \in \mathbb{R}^N$ with $p \neq 0$, and a value $c \in \mathbb{R}$ such that $p \cdot y > c$ and $p \cdot x < c$ for every $y \in B$.*

Proof. For any $z \in \mathbb{R}^N$ and $z \neq y$, define $p = y - z$. First, $p \cdot y > p \cdot z$ because $p \cdot (y - z) = \|y - z\|^2 > 0$. Let $c = p \cdot \left(\frac{y+z}{2}\right)$ so that $p \cdot y > c > p \cdot z$.
Suppose $z = \arg\min_{x \in B} \|y - x\|^2$. (see Figure 10) Consider an arbitrary $x \in B$.

$$
\begin{aligned}
\|z - y\|^2 &\leq \|(1-\lambda)\, z + \lambda x - y\|^2 \\
&= \|(1-\lambda)(z-y) + \lambda (x-y)\|^2 \\
&= (1-\lambda)^2 \|z-y\|^2 + \lambda^2 \|x-y\|^2 + 2(1-\lambda)\lambda (z-y)\cdot(x-y) \\
\implies\ & 0 \leq \lambda(\lambda-2)\|z-y\|^2 + \lambda^2\|x-y\|^2 + 2(1-\lambda)\lambda(z-y)\cdot(x-y) \\
\implies\ & 0 \leq (\lambda-2)\|z-y\|^2 + \lambda\|x-y\|^2 + 2(1-\lambda)(z-y)\cdot(x-y)
\end{aligned}
$$

Taking limit, letting λ go to zero,

$$0 \leq -2(z-y)\cdot(z-y) + 2(z-y)\cdot(x-y)$$

$$0 \leq 2(z-y)\cdot(x-z) = -2p\cdot(x-z)$$

$$p\cdot z \geq p\cdot x.$$

Therefore, $p\cdot y > c > p\cdot z \geq p\cdot x$ for all $x \in B$. $\square$

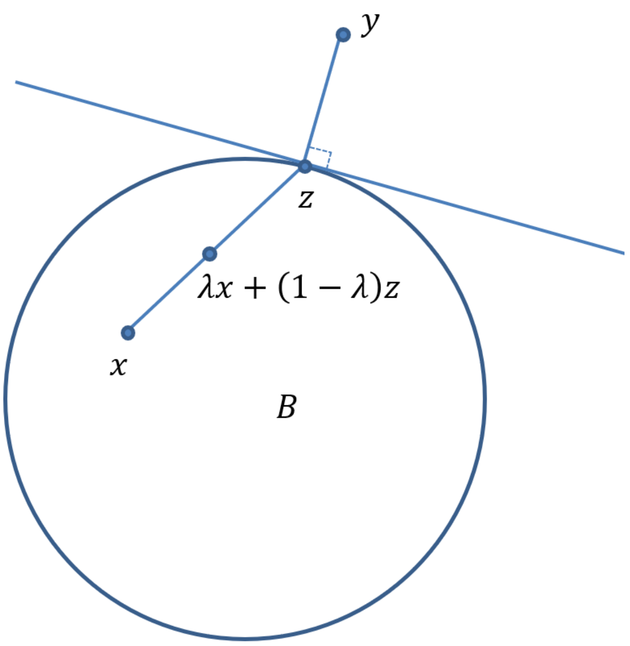


Figure 10: Seperating Hyperplane

Proposition 5.C.1. *Suppose $\pi(\cdot)$ is the profit function of the production set Y and that $y(\cdot)$ is the associated supply correspondence. Assume also that Y is closed and satisfies the free disposal property. Then,*

(i) *$\pi(\cdot)$ is homogeneous of degree one.*

(ii) *$\pi(\cdot)$ is convex.*

(iii) *If Y is convex, then $Y = \{y \in \mathbb{R}^L : p\cdot y \leq \pi(p)$ for all $p \gg 0\}$.*

(iv) *$y(\cdot)$ is homogeneous of degree zero.*

(v) *If Y is convex, then $y(p)$ is a convex set for all p. Moreover, if Y is strictly convex, then $y(p)$ is single-valued (if nonempty).*

(vi) (Hotelling's lemma) If $y(\bar{p})$ consists of a single point, then $\pi(\cdot)$ is differentiable at $\bar{p}$ and $\nabla\pi(\bar{p}) = y(\bar{p})$.

(vii) If $y(\cdot)$ is a function differentiable at $\bar{p}$, then $Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric and positive semidefinite matrix with $Dy(\bar{p})\bar{p} = 0$.

Proof.

(i) & (iv) The solution to

$$\max_{y \in \mathbb{R}^L} \alpha p \cdot y$$
$$\text{s.t. } y \in Y$$

and

$$\max_{y \in \mathbb{R}^L} \alpha p \cdot y$$
$$\text{s.t. } y \in Y$$

are identical. Therefore, $y\left(\alpha p\right) = y\left(p\right)$. This proves (iv).

Next, $\pi(\alpha p) = (\alpha p) \cdot y(p) = \alpha p \cdot y(p) = \alpha\pi(p)$. This proves (i).

(ii) $\pi(p) = p \cdot y(p) \geq p \cdot \tilde{y}$ for any $\tilde{y} \in Y$. $\pi(p') = p' \cdot y(p') \geq p' \cdot \tilde{y}$ for any $\tilde{y} \in Y$.

$$\begin{aligned}
\pi(\alpha p + (1-\alpha)p') &= \alpha p \cdot y(\alpha p + (1-\alpha)p') + (1-\alpha)p' \cdot y(\alpha p + (1-\alpha)p') \\
&\leq \alpha p \cdot y(p) + (1-\alpha)p' \cdot y(p') \\
&= \alpha\pi(p) + (1-\alpha)\pi(p')
\end{aligned}$$

Therefore, $\pi(\cdot)$ is convex.

Intuition: Consider the scenario in which the price is uncertain; with probability α it is p and with probability $(1-\alpha)$ it is p'. If the firm chooses output under this uncertainty, its output is $y\left(\alpha p + (1-\alpha)p'\right)$ and profit is $\pi(\alpha p + (1-\alpha)p')$. The result is same as when the price is fixed at $\alpha p + (1-\alpha)p'$. It is intuitive that the firm's profit would be higher if it gets to know the realization of the price before choosing y. In that case, its expected profit is $\alpha\pi(p) + (1-\alpha)\pi(p')$.

(iii) If Y is convex and closed, then by the Separating Hyperplane Theorem, $\forall x \notin Y$, there exists $p \neq 0$ s.t. $p \cdot x > p \cdot y$ for all $y \in Y$.

Note that $\pi(p) = p \cdot y^* \geq p \cdot y$ for some $y^* \in Y$ and for all $y \in Y$. Therefore, $p \cdot x > \pi(p) \geq p \cdot y$.

Now we establish that $p \geq 0$. Suppose $p_l < 0$ for some l. Then by free disposal, $y - \theta e_l \in Y$ for any $\theta > 0$. This implies that an arbitrarily large profit can be achieved by choosing θ sufficiently large. This contradicts $p \cdot x > p \cdot (y - \theta e_l)$.

Next, we argue that it is without loss of generality to focus on $p \gg 0$. Suppose $p_l = 0$ for some l. Then there exists $\alpha > 0$ sufficiently small such that $(p + \alpha e_l) \cdot x > \pi(p + \alpha e_l) \geq (p + \alpha e_l) \cdot y$ for all $y \in Y$.

Since each $x \notin Y$ (no matter how close to Y) is excluded from the half space $p \cdot y \leq \pi(p)$ for some $p \gg 0$, the intersection of such half spaces for all $p \gg 0$ *excludes all* $x \notin Y$. However, all such half spaces include Y so *their intersection also includes* Y.

Therefore, $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$.

(v) Suppose $y, y' \in y(p) \subset Y$. Then,

$$p \cdot y = p \cdot y' = \pi(p) \text{ and } F(y) \leq 0, F(y') \leq 0$$

$$\Longrightarrow p \cdot (\alpha y + (1-\alpha)y') = \alpha p \cdot y + (1-\alpha) p \cdot y' = \pi(p), \ \forall \alpha \in [0,1]$$

Convexity of Y implies $\alpha y + (1-\alpha)y' \in Y$.

Therefore, $y(p)$ is convex.

Suppose Y is strictly convex. Consider $y, y' \in y(p)$ and $y \neq y'$. Then, $F(\alpha y + (1-\alpha)y') < 0$ for $\alpha \in (0,1)$. Then, $\exists\ y'' \gg \alpha y + (1-\alpha)y'$ s.t. $F(y'') \leq 0$, and $p \cdot y'' > \alpha p \cdot y + (1-\alpha) p \cdot y' = \pi(p)$.

This contradicts the definition of $\pi(p)$.

(vi) Proof of differentiability of $\pi(p)$ is skipped.

Here, we provide some intuition.

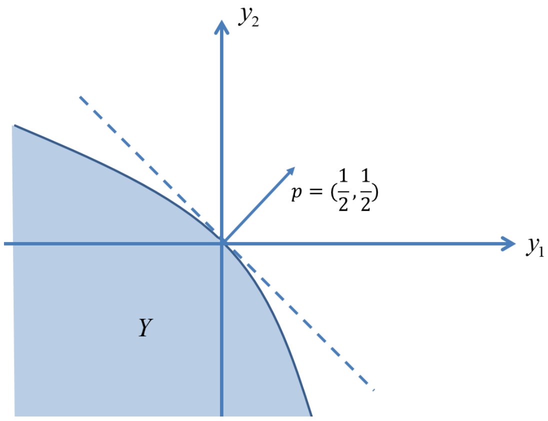


Figure 11: Strictly Convex Production Set (Unique Solution)

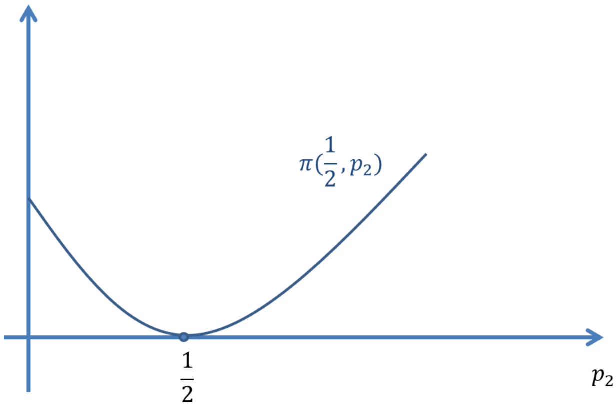


Figure 12: Differentiable Profit Function

Figure 11 depicts the case where $y(\bar{p})$ admits a single solution (i.e, $y(\bar{p}) = (0,0)$) at $\bar{p} = \left(\frac{1}{2}, \frac{1}{2}\right)$. Figure 12 shows the corresponding profit function. As indicated in the figure, $\pi(\frac{1}{2}, \bar{p}_2)$ is a differentiable function of p_2 at $\bar{p}_2 = \frac{1}{2}$.

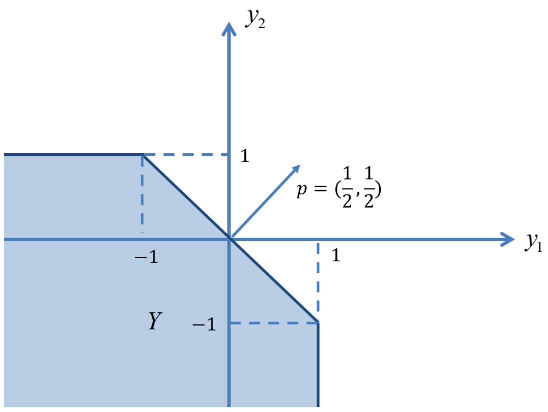


Figure 13: Not Strictly Convex Production Set (Multiple Solutions)

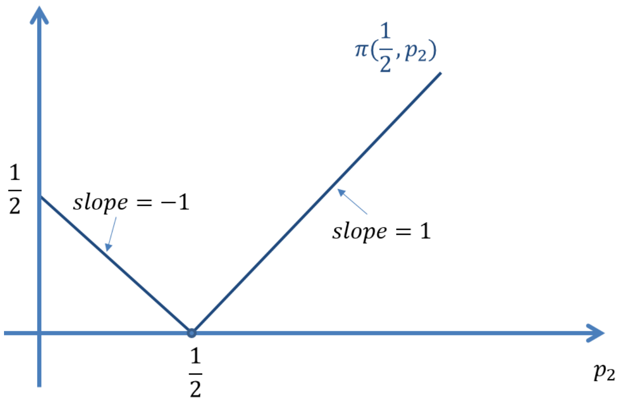


Figure 14: Non-Differentiable Profit Function

Figure 13 depicts the case where $y(\bar{p})$ admits multiple solutions (i.e, $y(\bar{p}) = (y_1, -y_1)$ for $y_1 \in [-1, 1]$) at $\bar{p} = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Figure 14 shows the corresponding profit function. As indicated in the figure, $\pi(\frac{1}{2}, \bar{p}_2)$ is NOT a differentiable function of p_2 at $\bar{p}_2 = \frac{1}{2}$.

Now suppose $\pi(p)$ is differentiable.

The Lagrange Function:

$$\mathcal{L} = p \cdot y - \lambda F(y).$$

$\pi(p)$ could be written as

$$\pi(p) = \mathcal{L}^* = p \cdot y(p) - \lambda F(y(p)).$$

Differentiating $\pi(p)$ with respect to p_l gives

$$\frac{\partial \pi(p)}{\partial p_l} = y_l(p) + \underbrace{\sum_{k=1}^{K} \frac{\partial \mathcal{L}^*}{\partial y_k} \frac{\partial y_k}{\partial p_l}}_{\frac{\partial \mathcal{L}^*}{\partial y_k}=0} = y_l(p).$$

In matrix notation, $\nabla \pi(p) = y(p)$.

(vii)

- $Dy(\bar{p}) = D^2\pi(\bar{p})$ follows (vi) directly.
- Symmetry of $D^2\pi(\bar{p})$ is also standard for $\pi(p)$ being C^2 [Schwarz' theorem].
- Positive semidefiniteness. Taylor expansion:

  $$\pi(p+\alpha z) = \pi(p) + \nabla\pi(p)\cdot(az) + \frac{1}{2}(\alpha z)D^2\pi(p+\beta z)(\alpha z) \text{ for some } \beta \in [0, \alpha].$$

  $$\implies \frac{\alpha^2}{2} z^T D^2\pi(p+\beta z)z = \pi(p+\alpha z) - (\pi(p) + \nabla\pi(p)\cdot(az)) \geq 0 \ (\because \pi \text{ is convex.})$$

  This holds true for α, β arbitrarily small.

  $$\implies z^T D^2\pi(p)z \geq 0.$$

- By (iv), $y(\alpha p) = y(p)$ (H.D.$\emptyset$).

  Differentiating both sides of the equation by α gives: $Dy(\alpha p)p = 0$.

  Setting $\alpha = 1$, we have $Dy(p)p = 0$. □

Remark. $\nexists$ budget constraint, so no "income" effect associated with price change.

Law of Supply

Claim. $(p - p') \cdot (y - y') \geq 0$ [That is, $dp \cdot dy = dp^T Dy dp \geq 0$]

Proof. $(p - p') \cdot (y - y') = (p \cdot y - p \cdot y') + (p' \cdot y' - p' \cdot y) \geq 0.$ □

Cost Minimization

- Cost minimization is necessary (but not sufficient) for profit maximization.
- We focus on single-output production.
- Cost Minimization Problem (CMP):

$$\begin{array}{ccc} \min\limits_{z \geq 0} w \cdot z & \equiv & \max\limits_{z \geq 0} -w \cdot z \\ \text{s.t. } f(z) \geq q & & \text{s.t. } -f(z) \leq -q \end{array}$$

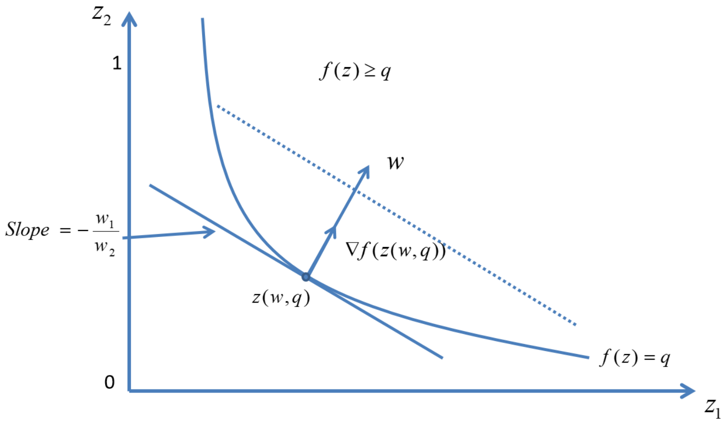


Figure 15: CMP for Single-output Production

- Let $z(w,q)$ denote the solution of CMP, $c(w,q)$ denote the minimized cost, or the cost function. $z(w,q)$ is known as the *conditional factor demand function* or correspondence.
- Lagrange Function:

$$\mathcal{L} = (-w \cdot z) - \lambda(-f(z) + q)$$

- Kuhn-Tucker Conditions:

$$-w_l + \lambda \frac{\partial f(z^*)}{\partial z_l} \leq 0 \Longleftrightarrow w_l \geq \lambda \frac{\partial f(z^*)}{\partial z_l}, \text{with equality if } z_l^* > 0 \tag{4}$$

$$\lambda \geq 0$$

$$\lambda(-f(z) + q) = 0$$

$$-f(z) \leq -q$$

$$z \geq 0$$

- Equation (4) is equivalent to $w \geq \lambda \nabla f(z^*)$ and $[w - \lambda \nabla f(z^*)] \cdot z^* = 0$.

- For any l, k with $(z_l, z_k) \gg 0$, we have
$$\frac{w_l}{w_k} = \frac{\partial f(z^*)/\partial z_l}{\partial f(z^*)/\partial z_k} = MRTS_{lk}$$

- λ measures $\partial c(w,q)/\partial q$, or the marginal cost of production. Here is the explanation:
$$\lambda = w_l \frac{1}{\partial f(z^*)/\partial z_l}.$$
At the margin, it takes $\frac{1}{\partial f(z^*)/\partial z_l}$ units of factor l to produce one more unit of output and each unit of factor l costs w_l.

- As with Profit Maximization Problem, if the production set Y is convex, then F.O.C. (Equation (4)) is not only necessary but also sufficient for z^* to be an optimum in Cost Minimization Problem.

Proposition 5.C.2. *Suppose that $c(w,q)$ is the cost function of a single-output technology Y with production function $f(\cdot)$ and that $z(w,q)$ is the associated conditional factor demand correspondence. Assume also that Y is closed and satisfies the free disposal property. Then,*

(i) *$c(\cdot)$ is homogeneous of degree one in w and nondecreasing in q.*

(ii) *$c(\cdot)$ is a concave function of w.*

(iii) *If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every q, then $Y = \{(-z, q) : w \cdot z \geq c(w,q) \text{ for all } w \gg 0\}$.*

(iv) *$z(\cdot)$ is homogeneous of degree zero in w.*

(v) *If the set $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w,q)$ is a convex set. Moreover, if $\{z \geq 0 : f(z) \geq q\}$ is a strictly convex set, then $z(w,q)$ is single-valued.*

(vi) *(Shepard's lemma) If $z(\bar{w}, q)$ consists of a single point, then $c(\cdot)$ is differentiable with respect to w at $\bar{w}$ and $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$.*

(vii) *If $z(\cdot)$ is differentiable at $\bar{w}$, then $D_w z(\bar{w}, q) = D_w^2 c(\bar{w}, q)$ is symmetric and negative semidefinite matrix with $D_w z(\bar{w}, q)\bar{w} = 0$.*

(viii) *If $f(\cdot)$ is homogeneous of degree one (i.e., exhibits constant returns to scales), then $c(\cdot)$ and $z(\cdot)$ are homogeneous of degree one in q.*

(ix) *If $f(\cdot)$ is concave, then $c(\cdot)$ is a convex function of q (in particular, marginal costs are nondecreasing in q).*

Remark. Note that cost minimization is very similar to expenditure minimization.

Proof.

(i) & (iv) The cost minimization problem

$$\begin{aligned} &\min_{z\geq 0} \alpha w\cdot z \\ &\text{s.t. } f(z)\geq q \end{aligned} \equiv \begin{aligned} &\min_{z\geq 0} w\cdot z \\ &\text{s.t. } f(z)\geq q \end{aligned}$$

Therefore, $z(\alpha w, q) = z(w, q)$, or $z(w, q)$ is H.D.$\emptyset$ in w, which is (iv). For (i),

$$c(\alpha w, q) = aw\cdot z(\alpha w, q) = aw\cdot z(w, q) = \alpha c(w, q).$$

Therefore, $c(w, q)$ is H.D.1 in w.

(ii) To prove concavity in w, we need to show $c(\alpha w + (1-\alpha)w', q) \geq \alpha c(w, q) + (1-\alpha)c(w', q)$ $\forall \alpha \in [0, 1]$. To see this,

$$\begin{aligned} c(\alpha w + (1-\alpha)w', q) &= [\alpha w + (1-\alpha)w']\cdot z(\alpha w + (1-\alpha)w', q) \\ &= \alpha w\cdot z(\alpha w + (1-\alpha)w', q) + (1-\alpha)w'\cdot z(\alpha w + (1-\alpha)w', q) \\ &\geq \alpha w\cdot z(w, q) + (1-\alpha)w'\cdot z(w', q) \\ &= \alpha c(w, q) + (1-\alpha)c(w', q) \\ \Longrightarrow &\ c(\cdot) \text{ is concave in } w. \end{aligned}$$

(iii) Given that $\widetilde{Y}_q = \{z \geq 0 : f(z) \geq q\}$ is convex and closed, by Separating Hyperplane Theorem, $\forall x \notin \widetilde{Y}_q$, there exists $w \neq 0$ s.t. $w\cdot x < w\cdot z$, $\forall z \in \widetilde{Y}_q$.

Note that $c(w, q) = w\cdot z^* \leq w\cdot z$ for some $z^* \in \widetilde{Y}_q$ and for all $z \in \widetilde{Y}_q$. So $w\cdot x < c(w, q) \leq w\cdot z$.

Now, we show that $w \geq 0$. Suppose $w_l < 0$ for some l. In this case $f(z+\theta e_l) \geq q$ so $z + \theta e_l \in \widetilde{Y}_q$ for $\theta > 0$. For all x, there exists $\theta > 0$ sufficiently large such that $w\cdot z > w\cdot x$.

Next, we argue that it is without loss of generality to restrict attention to $w \gg 0$. Suppose $w_l = 0$ for some l. In this case, there exists $\alpha > 0$ sufficiently small such that $(w + \alpha e_l) \cdot x < (w + \alpha e_l) \cdot z$ for all $z \in \widetilde{Y}_q$.

Since every $x \notin Y_q$ is excluded by some half space $w \cdot z \geq c(w, q)$ for some $w \gg 0$, the intersection of all such half spaces for all $w \gg 0$ excludes all $x \notin Y_q$. On the other hand, the intersection still covers $\widetilde{Y}_q$. Therefore, $\widetilde{Y}_q = \{z \in \mathbb{R}^{L-1} : w \cdot z \geq c(w, q)$ for all $w \gg 0\}$. Since $y_L = q$ and $(y_1, ..., y_{L-1}) = -z$, $Y = \{(-z, q) : w \cdot z \geq c(w, q)$ for all $w \gg 0\}$.

(v) Suppose $z_1, z_2 \in z(w, q)$.

Then, $w \cdot z_1 = w \cdot z_2 = c(w, q) \leq w \cdot z$, $\forall z \in Y_q \equiv \{z \geq 0 : f(z) \geq q\}$.

$$w \cdot (\alpha z_1 + (1 - \alpha) z_2) = \alpha w \cdot z_1 + (1 - \alpha) w \cdot z_2 = c(w, q).$$

Since $z_1, z_2 \in Y_q$ and Y_q is convex, $(\alpha z_1 + (1 - \alpha) z_2) \in Y_q$. Therefore, $\alpha z_1 + (1 - \alpha) z_2 \in z(w, q)$.

Suppose $z_1, z_2 \in z(w, q)$ and $z_1 \neq z_2$. If Y_q is strictly convex, then $f(\alpha z_1 + (1 - \alpha) z_2) > q$ and there exists $\theta \in (0, 1)$ such that $f(\theta(\alpha z_1 + (1 - \alpha) z_2)) \geq q$ and $w \cdot \theta(\alpha z_1 + (1 - \alpha) z_2) < c(w, q)$. This contradicts the definition of $c(w, q)$.

(vi) Proof of differentiability of $c(\cdot)$ is skipped.

Here, we provide some intuition. The intuition is similar to the one provided in the proof of Proposition 5.C.1 Part (vi).

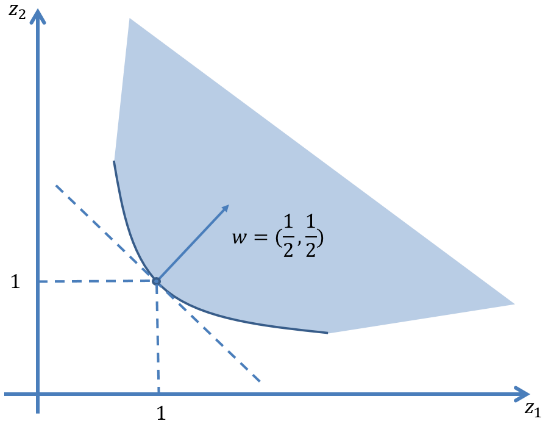


Figure 16: Strictly Convex Production Set (Unique Solution)

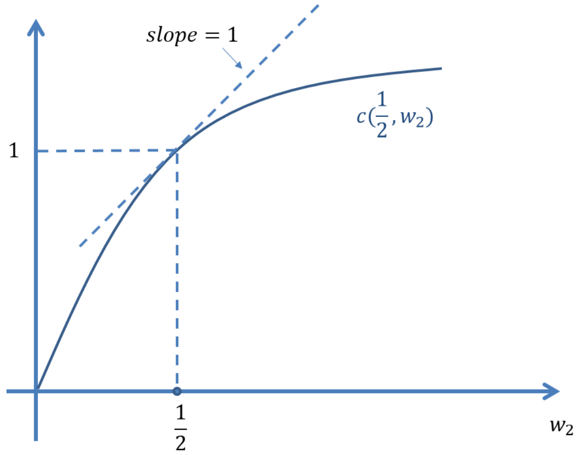


Figure 17: Differentiable Profit Function

Figure 16 depicts the case where $z(\bar{w})$ admits a single solution (i.e, $z(\bar{w}) = (1, 1)$) at $\bar{w} = \left(\frac{1}{2}, \frac{1}{2}\right)$. Figure 17 shows the corresponding profit function. As indicated in the figure, $c(\frac{1}{2}, \bar{w}_2)$ is a differentiable function of w_2 at $\bar{w}_2 = \frac{1}{2}$.

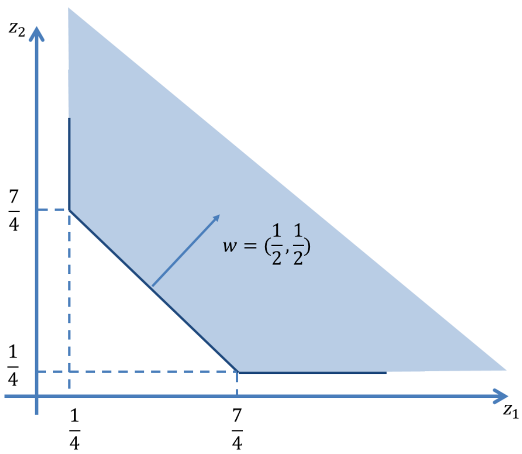


Figure 18: Not Strictly Convex Production Set (Multiple Solutions)

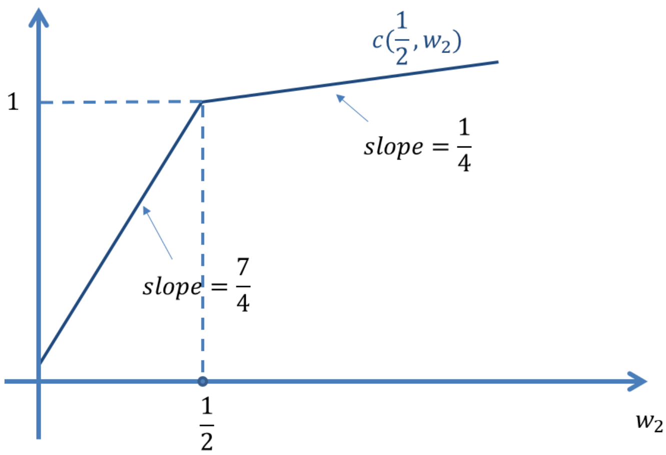


Figure 19: Non-Differentiable Profit Function

Figure 18 depicts the case where $z(\bar{w})$ admits multiple solutions (i.e, $z(\bar{w}) = (z_1, 2-z_1)$ for $z_1 \in [\frac{1}{4}, \frac{7}{4}]$) at $\bar{w} = \left(\frac{1}{2}, \frac{1}{2}\right)$. Figure 19 shows the corresponding profit function. As indicated in the figure, $c(\frac{1}{2}, \bar{p}_2)$ is NOT a differentiable function of w_2 at $\bar{w}_2 = \frac{1}{2}$.

Now suppose $c(\cdot)$ is differentiable.

The Lagrange Function:

$$\mathcal{L} = -w \cdot z - \lambda \left(q - f\left(z\right)\right).$$

$c\left(w, q\right)$ could be written as

$$c\left(w, q\right) = -\mathcal{L}^* = w \cdot z\left(w, q\right) - \lambda \left(f\left(z\left(w, q\right)\right) - q\right).$$

Differentiating $c\left(w, q\right)$ with respect to w_l gives

$$\frac{\partial c\left(w, q\right)}{\partial w_l} = z_l\left(w, q\right) - \underbrace{\sum_{k=1}^{L-1} \frac{\partial \mathcal{L}^*}{\partial z_k}\frac{\partial z_k}{\partial w_l}}_{\frac{\partial \mathcal{L}^*}{\partial z_k}=0} = z_l\left(w, q\right).$$

In matrix notation, $\nabla_w c(w, q) = z(w, q)$.

(vii)
- $D_w z(\bar{w}, q) = D_w^2 c(\bar{w}, q)$ follows differentiability of $z(\cdot)$ and (vi) immediately.
- Symmetry of $D_w^2 c(\bar{w}, q)$ is standard for $c(\bar{w}, q)$ being C^2 [Schwarz' theorem].
- Positive semidefiniteness. Taylor expansion:

$$c(w + \alpha v, q) = c(w, q) + D_w c(w, q)\alpha v + \frac{1}{2}(\alpha v) D_w^2 c(w + \beta v, q)(\alpha v) \text{ for } \beta \in [0, \alpha]$$

$$\implies \frac{\alpha^2}{2} v^T D_w^2 c(w + \beta v, q) v = c(w + \alpha v, q) - c(w, q) - D_w c(w, q)\alpha v \leq 0$$

$\because c$ is concave in w.

This holds true for α, β arbitrarily small. $\implies v^T D_w^2 c(w, q) v \leq 0$.

- by (iv), $z(\alpha w, q) = z(w, q)$ (H.D.$\emptyset$ in w).

Differentiating both sides of the equation by α gives: $D_w z(\alpha w, q) w = 0$.

Setting $\alpha = 1$, we have $D_w z(w, q) w = 0$.

(viii) Note that $z\left(w, \lambda q\right)$ solves CMP$_1$:

$$\min_z w \cdot z$$
$$\text{s.t. } f(z) \geq \lambda q.$$

Since $f\left(z\right)$ is H.D.1,

$$f\left(z\right) \geq \lambda q \iff \frac{f\left(z\right)}{\lambda} \geq q \iff f\left(\frac{z}{\lambda}\right) \geq q.$$

CMP_1 can be stated as CMP_2

$$\min_{\tilde{z}} w \cdot (\tilde{z})$$
$$\text{s.t. } f(\tilde{z}) \geq q,$$

where $\tilde{z} = z/\lambda$. Let $\tilde{z}^*$ be the solution to CMP_2. Then

$$\frac{z(w, \lambda q)}{\lambda} = \tilde{z}^* = z(q, w)$$
$$\Longrightarrow z(w, \lambda q) = \lambda z(q, w).$$

That is, $z(\cdot)$ is H.D.1 in q. For $c(\cdot)$,

$$c(\lambda q, w) = w \cdot z(\lambda q, w) = \lambda w \cdot z(q, w) = \lambda c(q, w).$$

That is, $c(\cdot)$ is also H.D.1 in q.

(ix) Let $c(w, q_1) = w \cdot z_1$ & $f(z_1) \geq q_1$ and $c(w, q_2) = w \cdot z_2$ & $f(z_2) \geq q_2$.

First, we show that because $f(\cdot)$ is concave, to produce $\alpha q_1 + (1-\alpha)q_2$, the firm can use no more than αz_1 and $(1-\alpha)z_2$ to produce it:

$$f(\alpha z + (1-\alpha)z') \geq \alpha f(z) + (1-\alpha)f(z') \geq \alpha q_1 + (1-\alpha)q_2.$$

This implies an upper bound on the total cost of producing $\alpha q_1 + (1-\alpha)q_2$:

$$\begin{aligned} c(w, \alpha q_1 + (1-\alpha)q_2) &\leq w \cdot (\alpha z_1 + (1-\alpha)z_2) \\ &= \alpha w \cdot z_1 + (1-\alpha) w \cdot z_2 \\ &= \alpha c(w, q_1) + (1-\alpha)c(w, q_2) \end{aligned}$$

That is, $c(\cdot)$ is convex in q. □

From Cost Minimization to Profit Maximization We restate Profit Maximization Problem using the cost function:

$$\max_{q \geq 0} \ pq - c(w, q).$$

Kuhn-Tucker Conditions:

$$p - \frac{\partial c(w, q^*)}{\partial q} \leq 0 \text{ with equality if } q^* > 0 \tag{5}$$

$$q \geq 0.$$

Equation (5) indicates that at an interior optimum (i.e., if $q^* > 0$), *price equals marginal cost*. If $c(w, q)$ is convex in q, then the F.O.C (Equation (5)) is not only necessary but also sufficient for q^* to be the optimal production level.

Example 5.C.1. (Building on Example 5.B.2): Derive the cost and profit functions for the Cobb-Douglas production function $f(z_1, z_2) = z_1^\alpha z_2^\beta$.

Remark. Note that $f(\cdot)$ is constant returns to scale if $\alpha + \beta = 1$, increasing returns to scale if $\alpha + \beta > 1$, and decreasing returns to scale if $\alpha + \beta < 1$.

Solution. We first consider the cost minimization problem.

Cost minimization

$$\begin{aligned} \min_{z_1, z_2 \geq 0} \; & w_1 z_1 + w_2 z_2 \\ \text{s.t.} \; & z_1^\alpha z_2^\beta \geq q \end{aligned} \equiv \begin{aligned} \max_{z_1, z_2 \geq 0} \; & - w_1 z_1 - w_2 z_2 \\ \text{s.t.} \; & - z_1^\alpha z_2^\beta \leq -q \end{aligned}$$

Lagrange Function:

$$\mathcal{L} = -w_1 z_1 - w_2 z_2 - \lambda(-z_1^\alpha z_2^\beta + q)$$

Kuhn-Tucker Conditions:

$$\begin{aligned} & - w_1 + \lambda(\alpha z_1^{\alpha-1} z_2^\beta) \leq 0, \text{ with equality if } z_1 > 0 \\ & - w_2 + \lambda(\beta z_1^\alpha z_2^{\beta-1}) \leq 0, \text{ with equality if } z_2 > 0 \\ & \lambda \geq 0 \\ & \lambda(-z_1^\alpha z_2^\beta + q) = 0 \\ & - z_1^\alpha z_2^\beta \leq -q \\ & z_1, z_2 \geq 0 \end{aligned}$$

Note that for any $q > 0$, $z_1^* > 0$ & $z_2^* > 0$ must hold (if not, $z_1^{*\alpha} z_2^{*\beta} = 0 < q$). Therefore,

$$\frac{w_1}{w_2} = \frac{\alpha}{\beta}\frac{z_2}{z_1} \iff z_2 = z_1 \frac{w_1 \beta}{w_2 \alpha}. \tag{6}$$

Also, it must hold that $z_1^\alpha z_2^\beta = q$. If not, less of both inputs can be used and production cost can be lowered. Plugging (6) into this equality gives

$$z_1(w_1, w_2, q) = q^{1/(\alpha+\beta)}(\frac{\alpha w_2}{\beta w_1})^{\beta/(\alpha+\beta)};$$
$$z_2(w_1, w_2, q) = q^{1/(\alpha+\beta)}(\frac{\beta w_1}{\alpha w_2})^{\alpha/(\alpha+\beta)}.$$

It follows immediately that the (conditional) cost function is

$$\begin{aligned} c(w_1, w_2, q) &= w_1 z_1(w_1, w_2, q) + w_2 z_2(w_1, w_2, q) \\ &= q^{1/(\alpha+\beta)}[(\frac{\alpha}{\beta})^{\beta/(\alpha+\beta)} + (\frac{\alpha}{\beta})^{-\alpha/(\alpha+\beta)}]w_1^{\alpha/(\alpha+\beta}w_2^{\beta/(\alpha+\beta)} \\ &= q^{1/(\alpha+\beta)}\theta\phi(w_1, w_2), \end{aligned} \tag{7}$$

where $\theta = (\frac{\alpha}{\beta})^{\beta/(\alpha+\beta)} + (\frac{\alpha}{\beta})^{-\alpha/(\alpha+\beta)}$ and $\phi(w_1, w_2) = w_1^{\alpha/(\alpha+\beta}w_2^{\beta/(\alpha+\beta)}$.

Profit maximization Cost minimization is only a necessary condition for profit maximization. To maximize profit, the firm must choose the optimal quantity:

$$\max_{q\geq 0} pq - c(w, q)$$

Kuhn-Tucker Conditions:

$$p - \frac{\partial c(w, q^*)}{\partial q} \leq 0 \text{ with equality if } q^* > 0 \tag{8}$$
$$q \geq 0.$$

When (8) holds in equality, $p = MC$.

Plugging (7) into (8), we have

$$p \leq \theta\phi(w_1, w_2)(\frac{1}{\alpha+\beta})q^{1/(\alpha+\beta)-1}, \text{ with equality if } q > 0. \tag{9}$$

Case I When $\alpha + \beta < 1$, $f(\cdot)$ is concave and $c(\cdot)$ is convex in q, i.e., MC increases in q. $\implies$ F.O.C. is sufficient.

At $q = 0$, the R.H.S of (9) is zero and $p \leq 0$ must not hold. So (9) must hold in equality. Optimal q is unique:

$$q(w_1, w_2, q) = \left[(\alpha+\beta)\left(\frac{p}{\theta\phi(w_1, w_2)}\right)\right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

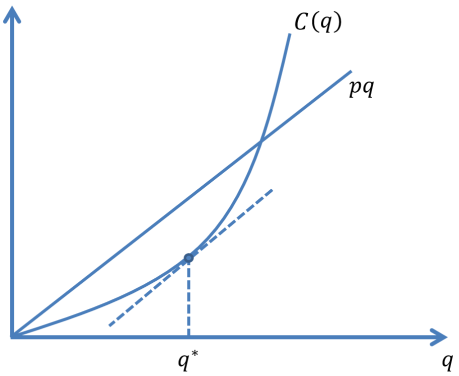


Figure 20: Case I

The factor demands can also be obtained through:

$$z_l(w_1, w_2, q) = z_l(w_1, w_2, q(w_1, w_2, p))$$

So can the profit function:

$$\pi(w_1, w_2, p) = pq(w_1, w_2, p) - w \cdot z(w_1, w_2, q(w_1, w_2, p))$$

Case II When $\alpha + \beta = 1$, (8) $\implies p \leq \theta\phi(w_1, w_2)$

(i) If $\theta\phi(w_1, w_2) > p$, then $q^* = 0$.

(ii) If $\theta\phi(w_1, w_2) < p$, then no solution: higher q the better.

(iii) If $\theta\phi(w_1, w_2) = p$, (knife-edge case): any nonnegative q is a solution.

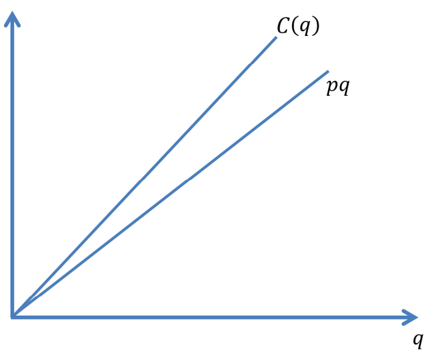


Figure 21: Case II (i)

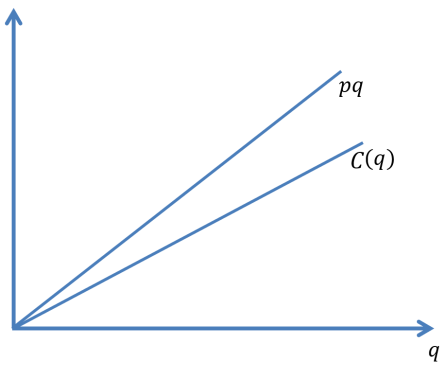


Figure 22: Case II (ii)

Case III When $\alpha + \beta > 1$, then F.O.C only identifies the local minimum.

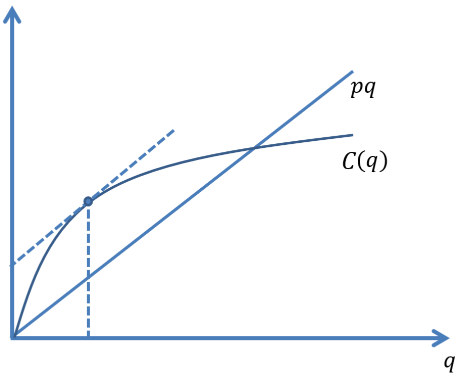


Figure 23: Case III

Exercise 5.C.10

Derive the cost function $c(w, q)$ and conditional factor demand functions (or correspondences) $z(w.q)$ for each of the following single-output constant return technologies with production functions given by

(a) $f(z) = z_1 + z_2$ (perfect substitutable inputs)

(b) $f(z) = \min\{z_1, z_2\}$ (leontief technology)

(c) $f(z) = (z_1^\rho + z_2^\rho)^{1/rho}, \rho \leq 1$ (constant elasticity of substitution technology)

Exercise 5.C.11

Show that $\partial z_l(w, q)/\partial q > 0$ if and only if marginal cost at q is increasing in w_l.

5.D. The Geometry of Cost and Supply on the Single-Output Case

Focusing on the single-output case, we analyze the relationships among: technology, cost function, and supply behavior.

We consider fixed factor prices $\bar{w} \gg 0$, and suppress the dependence on w, defining

$$C(q) = c(\bar{w}, q)$$

$$AC(q) = c(\bar{w}, q)/q$$

$$MC(q) = \partial c(\bar{w}, q)/\partial q$$

Convex Production Set Recall F.O.C for profit maximization: $p \leq C'(q)$ with equality if $q > 0$. If Y is convex, $c(\cdot)$ is convex and F.O.C is sufficient for profit maximization. An example of convex production set is given below:

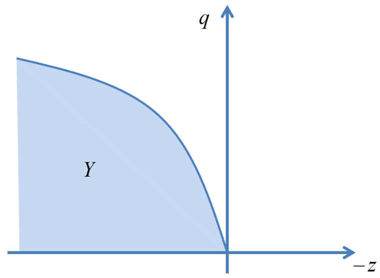


Figure 24: Production Set

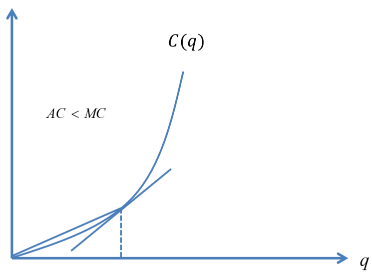


Figure 25: Cost Function

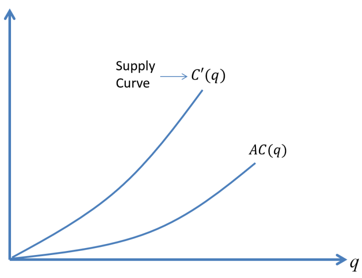


Figure 26: MC and AC

Nonconvex Production Set Y may not be convex. An example of nonconvex production set is given below:

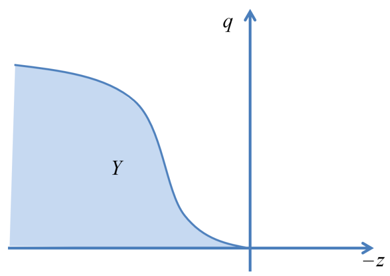


Figure 27: Production Set

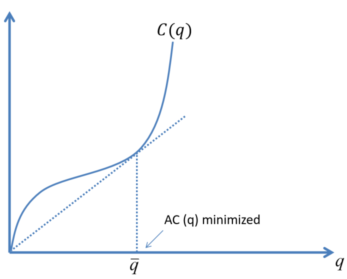


Figure 28: Cost Function

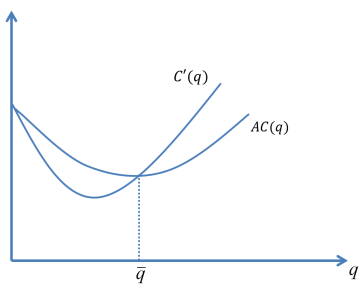


Figure 29: MC and AC

The relationship between Average Cost (AC) and Marginal Cost (MC):

$$AC(q) = c(q)/q$$
$$AC'(q) = \frac{qc'(q) - c(q)}{q^2}$$

The F.O.C. for minimization of AC is $\bar{q}c'(\bar{q}) - c(\bar{q}) = 0$ or $c'(\bar{q}) = \frac{c(\bar{q})}{\bar{q}}$, i.e., AC is minimized when $MC(\bar{q}) = AC(\bar{q})$.

Exercise 5.D.1

Show that $AC(\bar{q}) = C'(\bar{q})$ at any $\bar{q}$ satisfying $AC(\bar{q}) \leq AC(q)$ for all q. Does this result depend on the differentiability of $C(\cdot)$ everywhere?

Fixed cost (but not sunk) Fixed cost arises because some input(s) have to be used before any output can be produced. Since the cost is not sunk, it is still preventable so producing nothing and costing nothing is still an option.

For the firm to be willing to be active in production, the price has to at least cover the average cost of production. Otherwise, the firm will produce nothing.

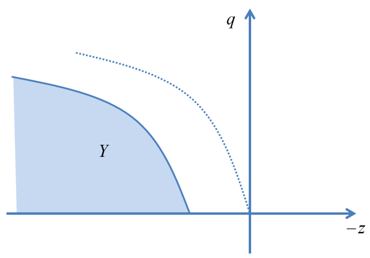


Figure 30: Production Set

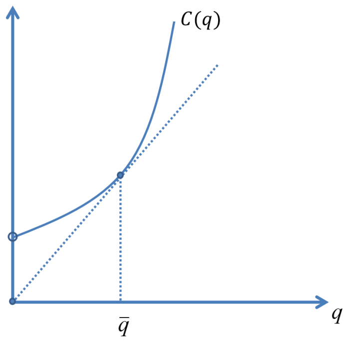


Figure 31: Cost Function

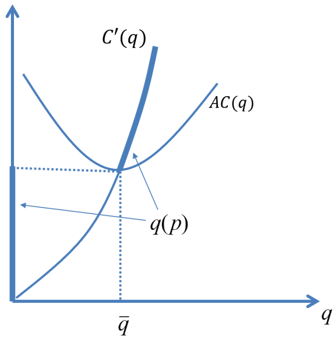


Figure 32: MC and AC

Sunk cost When cost is sunk, it is no longer preventable. So it is not an option to use no inputs and incur no cost.

In deciding whether to be active in production or not, sunk cost should not be part of the consideration because by gone is by gone. Therefore, even if the price falls below the average cost, it may still be economically profitable (without accounting for the sunk cost) to be active in production.

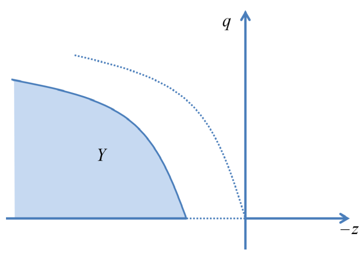


Figure 33: Production Set

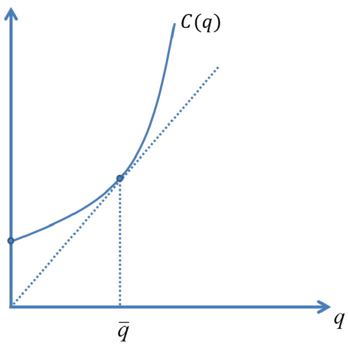


Figure 34: Cost Function

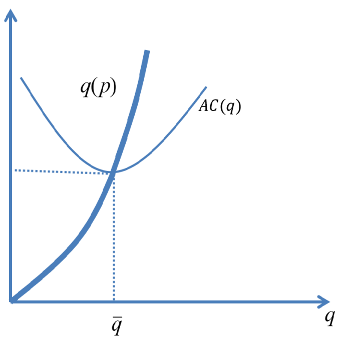


Figure 35: MC and AC

Exercise 5.D.2

Depict the supply locus for a case with partially sunk costs, that is, where $C(q) = K + C_v(q)$ if $q > 0$ and $0 < C(0) < K$.

Long-run and short-run cost functions In Figure 36, the cost function excluding any prior input commitments is depicted by $C(\cdot)$. We call it the *long-run cost function*. If one input, say z_2, is fixed at level $\bar{z}_2$ in the short-run, then the *short-run cost function* of the firm becomes $C(q \mid \bar{z}_2) = \bar{w}_1 z_1 + \bar{w}_2 \bar{z}_2$, where z_1, is chosen so that $f(z_1, \bar{z}_2) = q$.

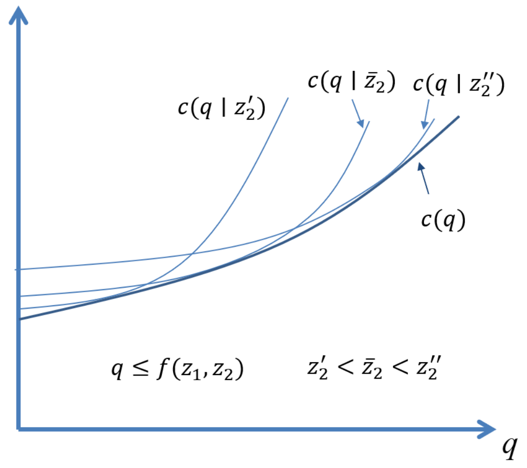


Figure 36: LR and SR Cost Functions

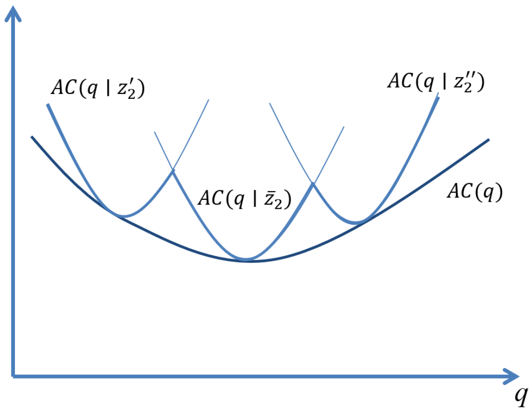


Figure 37: LR and SR AC

Exercise 5.D.3

Suppose that a firm can produce good L from $L-1$ factor inputs ($L > 2$). Factor prices are $w \in \mathbb{R}^{L-1}$ and the price of output is p. The firm's differentiable cost function is $c(w, q)$. Assume that this function is strictly convex in q. However, although $c(w, q)$ is the cost function when all factors can be freely adjusted, factor 1 cannot be adjusted in the short run.

Suppose that the firm is initially at a point where it is producing its long-run profit-maximizing output level of good L given prices w and p, $q(w, p)$ [i.e., the level that is optimal under the long-run cost conditions described by $c(w, q)$], and that all inputs are optimally adjusted [i.e., $z_l = z_l(w, q(w, p))$ for all $l = 1, ..., L-1$, where $z_l(\cdot, \cdot)$ is the long-run input demand function]. Show that the firm's profit-maximizing output response to a marginal increase in the price of good L is larger in the long run than in the short run. [*Hint*: Define a short-run cost function $c_s(w, q|z_1)$ that gives the minimized costs of producing output level q given that input 1 is fixed at level z_1.]

5.E. Aggregation

Question. Would the properties of individual supplies be preserved when they are aggregated to market supply?

Question. Would merger affect supply behavior?

- J production units/plants
- Y_j is nonempty, closed
- $\pi_j(p)$: profit function
- $y_j(p)$: supply correspondence
- Aggregate supply correspondence:

$$y(p) = \sum_{j=1}^{J} y_j(p) = \{y \in \mathbb{R}^L : y = \sum_j y_j \text{ for some } y_j \in y_j(p), j = 1, ..., J\}$$

- Suppose $y_j(p)$ is single-valued & differentiable.
  - From Proposition 5.C.1, $Dy_j(p)$ is symmetric & positive semidefinite.
  - Because these properties are preserved under addition, $Dy(p) = \sum_j Dy_j(p)$ is also symmetric and positive semidefinite.
  - Positive semidefiniteness implies law of supply in aggregate:

$$dp \cdot dy = dp^T Dy(p) dp \geq 0.$$

- Alternatively, based on "revealed preference"-like argument, we obtain

$$(p - p') \cdot [y_j(p) - y_j(p')] \geq 0.$$

  - Summing over j, we have $(p - p') \cdot \left[\sum_j y_j(p) - \sum_j y_j(p')\right] \geq 0$.

- Aggregate production set:

$$Y = Y_1 + ... + Y_J = \{y \in \mathbb{R}^L : y = \sum_j y_j \text{ for some } y_j \in Y_j, j = 1, ..., J\}$$

  - Suppose Y is feasible to a single owner who maximizes total profit from J plants' production.

Proposition 5.E.1. *For all $p \gg 0$, we have*

(i) $\pi^*(p) = \sum_j \pi_j(p)$

(ii) $y^*(p) = \sum_j y_j(p)$

Proof.

(i) First, the owner of all J plants can at least replicate what the J individual owners do. This implies

$$\pi_j^*(p) \geq \pi_j(p) \implies \pi^*(p) = \sum_j \pi_j^*(p) \geq \sum_j \pi_j(p)$$

On the other hand, given the definition of $\pi_j(p)$, $\pi_j^*(p) \leq \pi_j(p)$ for all j.

Therefore, $\pi_j^*(p) = \pi_j(p)$ and $\pi^*(p) = \sum_j \pi_j(p)$.

(ii) We need to show $\sum_j y_j(p) \subseteq y^*(p)$ and $y^*(p) \subseteq \sum_j y_j(p)$. First, we prove the former. Suppose $y \in \sum_j y_j(p)$. Then $p \cdot y = \sum_j \pi_j(p) = \pi^*(p)$ (where the second equality is implied by (i)). This implies $y \in y^*(p)$.

Now, we prove the latter. Suppose $y \in y^*(p)$. Then $p \cdot y = \pi^*(p) = \sum_j \pi_j(p)$ (where the second equality is implied by (i)). This implies $y_j \in y_j(p)$ for all j. Therefore, $y \in \sum_j y_j(p)$. $\square$

Remark. This result that merger does affect supply behavior holds only because the firms are **price takers**. When these firms set prices to compete, the prices they set will have externality on each other's profit. After merger, the owner of all the plants typically will raise the prices to reduce the negative externality of low prices on other plants' profits.

5.F. Efficient Production (Narrow notion of efficiency)

> *Question.* When do we regard production as nonwasteful?

We take the prices as exogenously fixed and do not discuss whether the prices are too high or too low when we discuss the efficiency of a profit maximizing firm.

Definition 5.F.1. A production vector $y \in Y$ is efficient if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

Proposition 5.F.1. *If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient.*

Proof. Suppose y is not efficient. Then $\exists y' \in Y$ s.t. $y' \geq y$ and $y' \neq y$. This implies $p \cdot y' > p \cdot y$ for all $p \gg 0$; so y is not profit maximizing. □

Exercise 5.F.1. Suppose $p_1 = 0$ & $p_2 > 0$. Then for all p_2, both y and y' maximize profit but y' is NOT efficient. This illustrates the importance of $p \gg 0$ in Proposition 5.F.1.

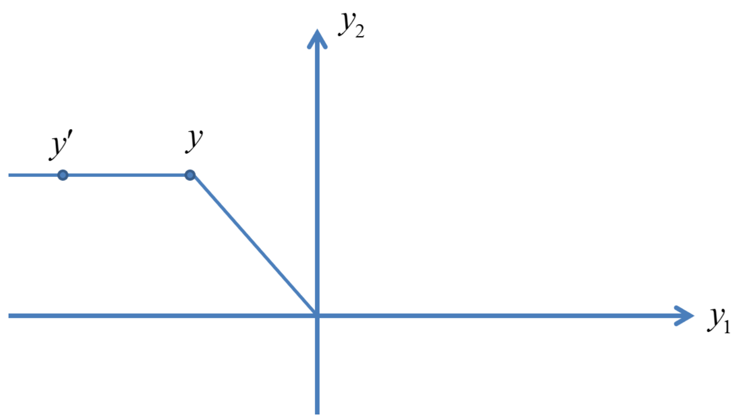


Figure 38: $p_1 = 0$

Now we need to visit the Mathematical Appendix to retrieve a result that we'll use to prove our next proposition for this chapter.

Theorem M.G.2 (Separating Hyperplane Theorem (Part II))**.** *Suppose that the convex sets $A, B \subset \mathbb{R}^N$ are disjoint (i.e., $A \cap B = \varnothing$). Then there is $p \in \mathbb{R}^N$ with $p \neq 0$, and a value $c \in \mathbb{R}$, such that $p \cdot x \geq c$ for every $x \in A$ and $p \cdot y \leq c$ for every $y \in B$. That is, there is a hyperplane that separates A and B, leaving A and B on different sides of it.*

Proof. Consider arbitrary $x \in A$ and $y \in B$ and let $z = x - y$. Since A and B are disjoint, $z \neq 0$. Let $D = \left\{ z \in \mathbb{R}^N : z = x - y \text{ for some } x \in A \text{ and some } y \in B \right\}$.
Now we show that D is convex. Suppose $z_1, z_2 \in D$. Then

$$\alpha z_1 + (1 - \alpha) z_2 = [\alpha x_1 + (1 - \alpha) x_2] - [\alpha y_1 + (1 - \alpha) y_2].$$

Since A and B are convex, $\alpha x_1 + (1 - \alpha) x_2 \in A$ and $\alpha y_1 + (1 - \alpha) y_2 \in B$.

So $\alpha z_1 + (1-\alpha) z_2 \in D$. Since $z \neq 0$ for all $z \in D$, $0 \notin D$. Applying Part I of the theorem, there exists $p \neq 0$ such that $0 \leq p \cdot z$ for all $z \in D$. In other words, $0 \leq p \cdot (x-y)$ or $p \cdot y \leq p \cdot x$ for all $x \in A$ and $y \in B$. To complete the proof, let

$$c = \frac{\inf_{x \in A} p \cdot x + \sup_{y \in B} p \cdot y}{2}.$$

□

Proposition 5.F.2. *Suppose that Y is convex. Then every efficient production $y \in Y$ is a profit-maximizing production for some nonzero price vector $p \geq 0$.*

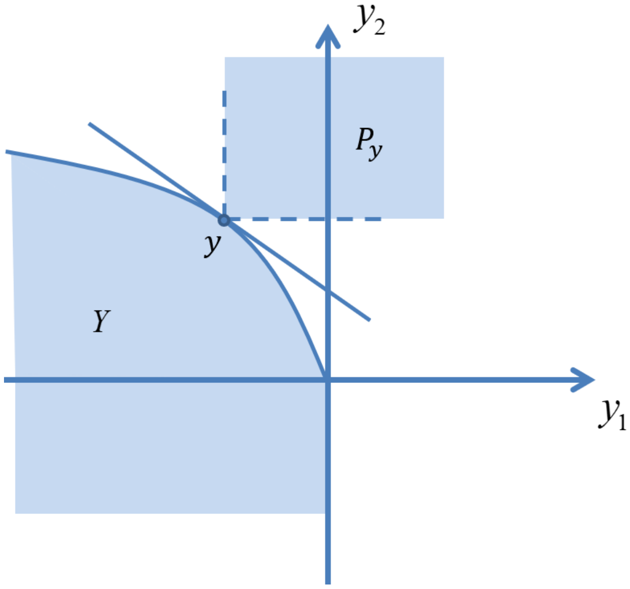


Figure 39: Proposition 5.F.2

Proof. Suppose $y \in Y$ is efficient and define $P_y = \{y' \in \mathbb{R}^L : y' \gg y\}$.

Since y is efficient, $\nexists x \in Y$, s.t. $x \geq y$ & $x \neq y$. Therefore, $Y \cap P_y = \varnothing$.

Since P_y is convex and disjoint from Y, by the separating hyperplane theorem, $\exists p \neq 0$ s.t. $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$ and $y'' \in Y$.

In particular, $p \cdot y' \geq p \cdot y$ for every $y' \gg y$ and for this to hold, it requires that $p \geq 0$. Suppose otherwise and that $p_l < 0$ for some l. Then with y'_l sufficiently large, $p \cdot y' < p \cdot y$ necessarily holds, constituting a contradiction.

Next, we show that $p \cdot y \geq p \cdot y''$ for every $y'' \in Y$. Suppose otherwise and that $p \cdot y < p \cdot y''$ for some $y'' \in Y$. Then there exists $\varepsilon > 0$ sufficiently small such that $p \cdot (y + \varepsilon e) < p \cdot y''$.

However, $(y + \varepsilon e) \in P_y$ and this constitutes a contradiction. Therefore, y maximizes profit given price vector $p \geq 0$. $\square$

The end of the second sentence of Proposition 5.F.2 cannot be read as " $p \gg 0$" . The following example illustrates why:

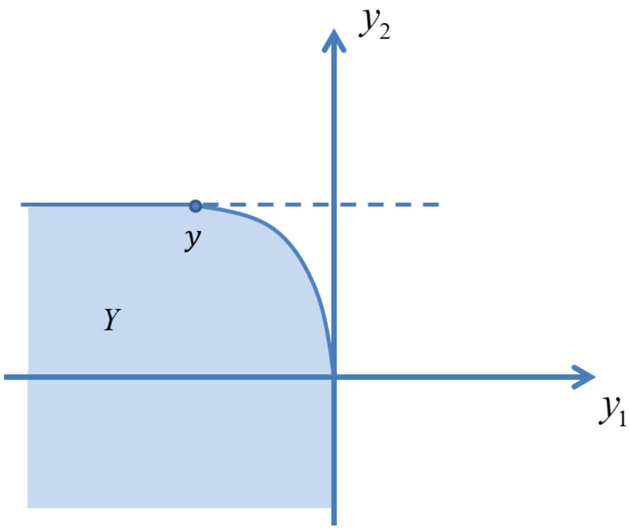


Figure 40: Proposition 5.F.2

The production vector y is efficient but is not profit-maximizing for any $p \gg 0$. It's profit-maximizing for some (p_1, p_2) with $p_1 = 0$.