

Chapter 3. Classical Demand Theory

3.A. Introduction: Take \succsim as the primitive

- (1) What assumption(s) on \succsim do we need to represent \succsim with a utility function?
- (2) How to perform utility maximization and derive the demand function?
- (3) How to derive the utility as a function of prices and wealth, or the indirect utility function?
- (4) How to perform expenditure minimization and derive the expenditure function?
- (5) How are demand function, indirect utility function, and expenditure function related?

3.B. Preference Relations: Basic Properties/Assumptions

In the classical approach to consumer demand, the analysis of consumer behavior begins by specifying the consumer's preferences over the commodity bundles in the consumption set. Consumption set is defined as $X \subset \mathbb{R}_+^L$. Consumer's preferences are captured by the preference relation \succsim .

Rationality We would assume *Rationality* (*Completeness and Transitivity*) throughout the chapter. Definition 3.B.1 below repeats the formal definition of *Rationality*.

Definition 3.B.1. The preference relation \succsim on X is rational if it possesses the following two properties:

- (i) **Completeness:** For all $x, y \in X$, we have $x \succsim y$ or $y \succsim x$ (or both).
- (ii) **Transitivity:** For all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Desirability Assumptions The first *Desirability* Assumption we consider is *Monotonicity*: larger amounts of commodities are preferred to smaller ones.

For Definition 3.B.2, we assume that the consumption of a larger amounts of goods is always feasible in principle; that is, if $x \in X$ and $y \geq x$, then $y \in X$.

Definition 3.B.2. The preference relation \succsim on X is *monotone* if $x, y \in X$ and $y \gg x$ implies $y \succ x$. It is *strongly monotone* if $y \geq x$ & $y \neq x$ implies $y \succ x$.

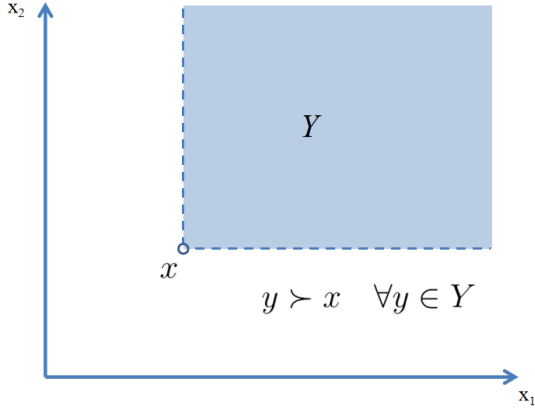


Figure 1: Monotonicity

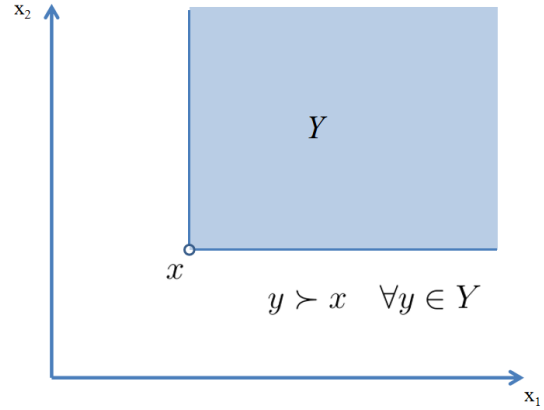


Figure 2: Strong Monotonicity

Remark.

- If \succsim is monotone, we may have indifference with respect to an increase in the amount of some but not all commodities.
- If \succsim is strongly monotone, y is strictly preferred to x if y is larger than x for some commodity and is no less for any other commodities.

Claim. If \succsim is strongly monotone, then it is monotone.

Proof. For $x, y \in X$, if $y \gg x$, then $y \geq x$ and $y \neq x$. Since \succsim is strongly monotone, we have $y \succ x$. Thus, \succsim is monotone. \square

Example. Here is an example of a preference that is monotone, but not strongly monotone:

$$u(x_1, x_2) = x_1 \text{ in } \mathbb{R}_+^2.$$

1. The preference is monotone: if $y \gg x$, then $y_1 > x_1$. So, we must have $u(y_1, y_2) > u(x_1, x_2)$, which implies $y \succ x$.

2. The preference is not strongly monotone: For $x = (1, 2)$ and $y = (1, 3)$, we have $y \geq x$, but $u(y) = u(x)$, which implies $y \sim x$.

For much of the theory, a weaker desirability assumption, *local nonsatiation*, suffices.

Definition 3.B.3. The preference relation \succsim on X is *locally nonsatiated* if for every $x \in X$ and every $\varepsilon > 0$, $\exists y \in X$ such that $\|y - x\| \leq \varepsilon$ and $y \succ x$.

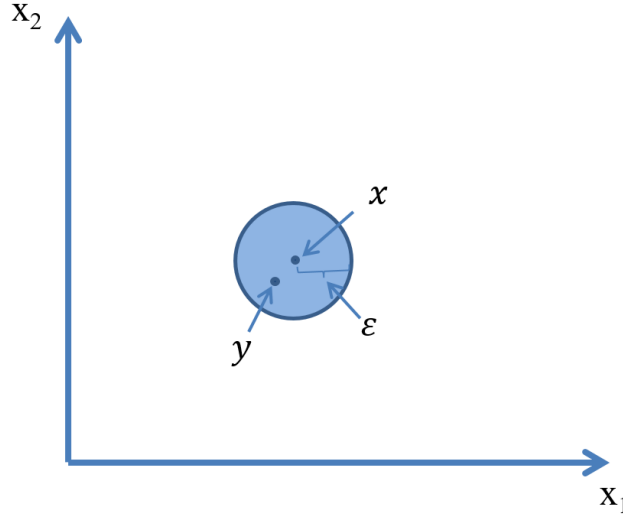


Figure 3: Test for Local Nonsatiation

Claim. *Local nonsatiation* is a weaker desirability assumption compared to *monotonicity*.

If \succsim is monotone, then it is locally nonsatiated.

Proof. Fix some $\varepsilon > 0$. Let there be an arbitrary $x \in X$ and $e = (1, \dots, 1)$. For any $\lambda > 0$, we also have $y = x + \lambda e \in X$. Since clearly $y = x + \lambda e \gg x$, by monotonicity, $y \succ x$. On the other hand, $\|y - x\| = \sqrt{L\lambda^2} = \lambda\sqrt{L}$. Thus, for $\lambda < \frac{\varepsilon}{\sqrt{L}}$, $\|y - x\| \leq \varepsilon$. Since x is arbitrary, the existence of the point $y = x + \lambda e$ where $\lambda < \frac{\varepsilon}{\sqrt{L}}$ implies that \succsim is locally nonsatiated. \square

Example. Here is an example of a preference that is locally nonsatiated, but not monotone:

$$u(x_1, x_2) = x_1 - |1 - x_2| \text{ in } \mathbb{R}_+^2.$$

1. The preference is locally nonsatiated: Fix an $\varepsilon > 0$. We can find $\lambda > 0$ such that $\lambda < \varepsilon$. Denote $y = (y_1, y_2) = (x_1 + \lambda, x_2)$. Then $u(y) - u(x) = \lambda > 0$, which

implies $y \succ x$. On the other hand, $\|y - x\| = \sqrt{\lambda^2} = \lambda < \varepsilon$. Therefore, \succsim is locally nonsatiated.

2. The preference is not monotone: For $x = (1, 1)$ and $y = (1.5, 2)$, we have $y \gg x$, but $u(x) = 1 > u(y) = 0.5$, which implies $x \succ y$.

Indifference sets Given \succsim and x , we can define 3 related sets of consumption bundles.

1. The indifferent set is $\{y \in X : y \sim x\}$.
2. The upper contour set is $\{y \in X : y \succsim x\}$.
3. The lower contour set is $\{y \in X : x \succsim y\}$.

Implication of local nonsatiation One implication of local nonsatiation (and, hence, of monotonicity) is that it rules out “thick” indifference sets.

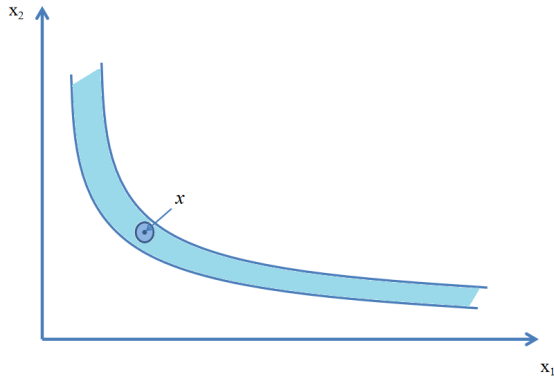


Figure 4: Violation of local nonsatiation

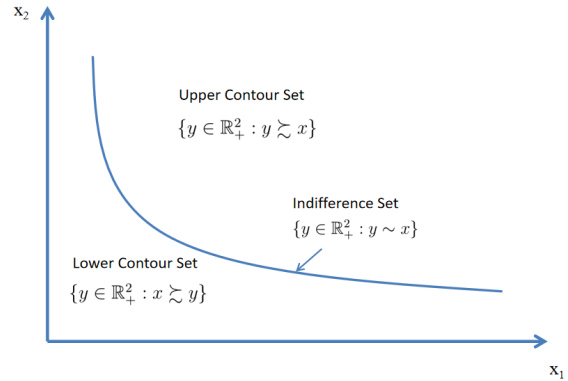


Figure 5: Compatible with local nonsatiation

Exercise 3.B.2

The preference relation \succsim defined on the consumption set $X = \mathbb{R}_+^L$ is said to be *weakly monotone* if and only if $x \geq y$ implies that $x \succsim y$. Show that if \succsim is transitive, locally nonsatiated, and weakly monotone, then it is monotone.

Convexity Assumptions

Definition 3.B.4. The preference relation \succsim on X is *convex* if for every $x \in X$, the upper contour set of x , $\{y \in X : y \succsim x\}$ is convex; that is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1 - \alpha)z \succsim x$ for any $\alpha \in [0, 1]$.

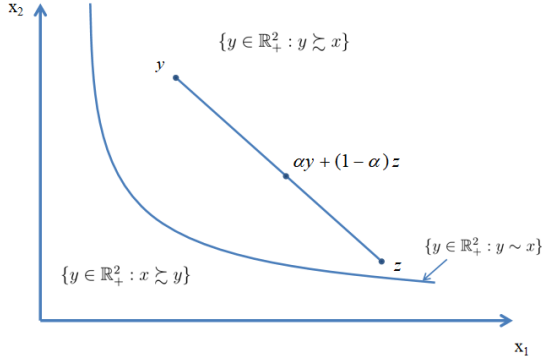


Figure 6: Convex Preference

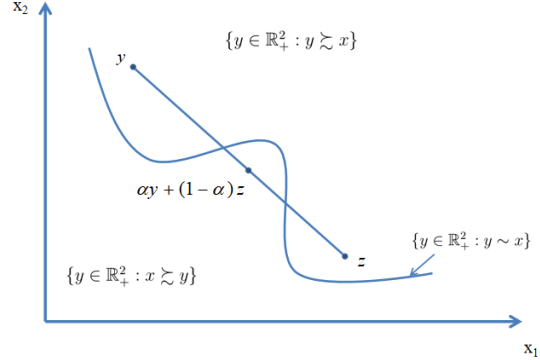


Figure 7: Nonconvex Preference

Properties associated with convexity

- (i) *Diminishing marginal rates of substitution*: with convex preferences, from any initial consumption X , and for any two commodities, it takes an increasingly larger amounts of one commodity to compensate for successive unit losses of the other.
- (ii) Preference for diversity (implied by (i)): under convexity, if x is indifferent to y , then $\frac{1}{2}x + \frac{1}{2}y$ cannot be worse than x or y .

Definition 3.B.5. The preference relation \succsim on X is *strictly convex* if for every $x \in X$, we have that $y \succsim x$ and $z \succsim x$, and $y \neq z$ implies $\alpha y + (1 - \alpha)z \succ x$ for all $\alpha \in (0, 1)$.

Homothetic and Quasilinear Preference In applications (particularly those of an econometric nature), it is common to focus on preferences for which it is possible to deduce the consumer's entire preference relation from a single indifference set. Two examples are the classes of *homothetic* and *quasilinear* preferences.

Definition 3.B.6. A monotone preference relation \succsim on $X = \mathbb{R}_+^L$ is *homothetic* if all indifference sets are related by proportional expansion along rays; that is, if $x \sim y$, then $\alpha x \sim \alpha y$ for any $\alpha \geq 0$.

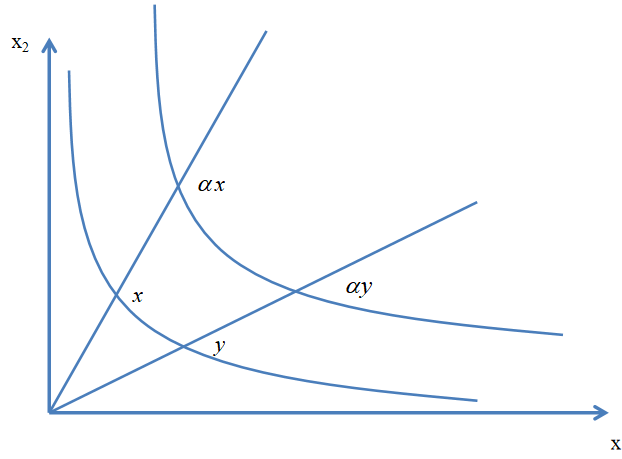


Figure 8: Homothetic Preference

Definition 3.B.7. The preference relation \succsim on $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is *quasilinear* with respect to commodity 1 (the *numeraire* commodity) if

- (i) All the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, 0, \dots, 0)$ and any $\alpha \in \mathbb{R}$.
- (ii) Good 1 is desirable; that is $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

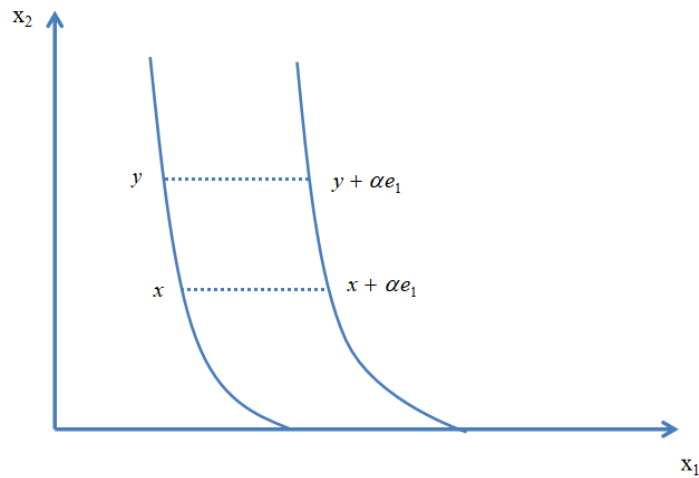


Figure 9: Quasilinear Preference

3.C. Preference and Utility

Key Question. When can a rational preference relation be represented by a utility function?

Answer: If the preference relation is continuous.

Definition 3.C.1. The preference relation \succsim on X is *continuous* if it is preserved in the limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for all n , $x = \lim_{n \rightarrow \infty} x^n$, $y = \lim_{n \rightarrow \infty} y^n$, we have $x \succsim y$.

Interpretation: Consumer's preferences cannot exhibit jumps. For example, the consumer preferring each elements in the sequence x^n to the corresponding element in the sequence y^n but suddenly reversing her preference at the limiting points of these sequences x and y .

Claim. \succsim is continuous if and only if for all x , the upper contour set $\{y \in X : y \succsim x\}$ and the lower contour set $\{y \in X : x \succsim y\}$ are both closed.

Proof. We only provide the proof for “only if” part. “if” part is more advanced and not required by this course.

Definition 3.C.1 implies that for any sequence of points $\{y^n\}_{n=1}^{\infty}$ with $x \succsim y^n$ for all n and $y = \lim_{n \rightarrow \infty} y^n$, we have $x \succsim y$ (just let $x^n = x$ for all n). Hence the closedness of lower contour set is implied. Similarly, we could show the closedness of upper contour set. \square

Example 3.C.1. Lexicographic Preference Relation on \mathbb{R}^2

$X = \mathbb{R}^2$; $x \succ y$ if either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$. $x \sim y$ if $x_1 = y_1$ and $x_2 = y_2$.

Claim. Lexicographic Preference Relation on \mathbb{R}^2 is not continuous.

Proof. Consider sequence of bundles (x^n, y^n) where $x^n = (1 + \frac{1}{n}, 1)$ and $y^n = (1, 2)$. $x = \lim_{n \rightarrow \infty} x^n = (1, 1)$, $y = \lim_{n \rightarrow \infty} y^n = (1, 2)$. $x^n \succ y^n$ for all n but $x \prec y$. \square

Claim. Lexicographic Preference Relation on \mathbb{R}^2 cannot be represented by $u(\cdot)$.

Proof. Here, we only provide a sketch of proof with the help of Figure 10. The contradiction comes from the one-to-one mapping $r(\cdot) : \mathbb{R} \rightarrow \mathbb{Q}$. For detailed proof, please refer to Chapter 1 Lecture Notes.

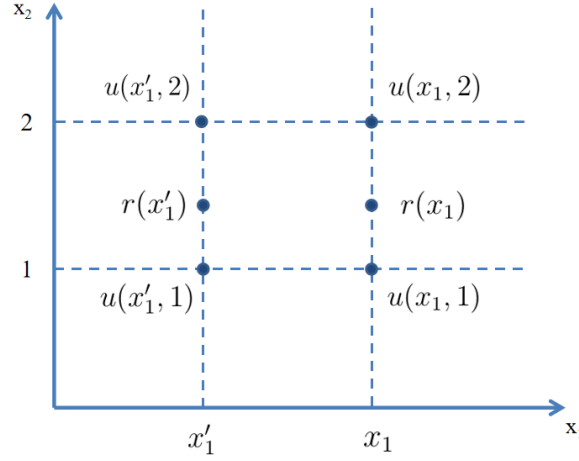


Figure 10: Lexicographic Preference

Alternatively, we could use the fact that upper and lower contour sets of a continuous preference must be closed. It is shown in Figure 12 that the upper and lower contour set of Lexicographic preference are not closed.

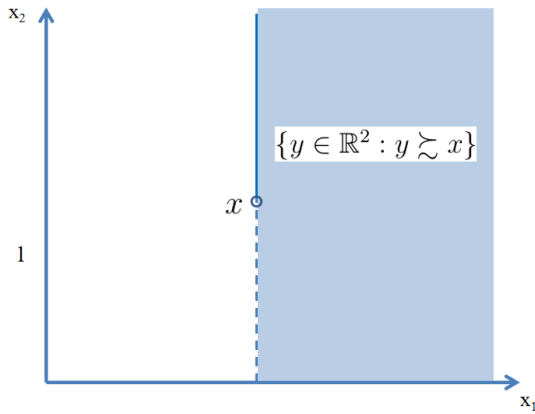


Figure 11: Upper Contour Set

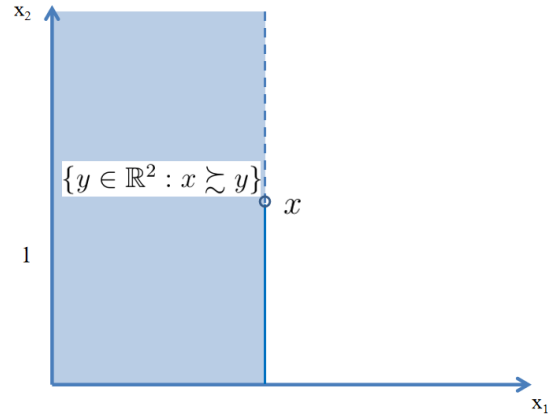


Figure 12: Lower Contour Set

□

Proposition 3.C.1. Suppose that the preference relation \succsim on X is continuous. Then there exists continuous utility function $u(x)$ that represents \succsim , i.e., $u(x) \geq u(y)$ if and only if $x \succsim y$.

For this course, we will only prove a simplified version of Proposition 3.C.1, which assumes that the preference relation \succsim is also monotone.

Proof. Define $Z := \{x \in \mathbb{R}_+^L : x_l = x_k \text{ for all } k, l = 1, \dots, L\}$. Define $e := (1, \dots, 1)$ with L elements. Then, $\alpha e \in Z$ for $\alpha \geq 0$.

Step 1: Construction of a utility function

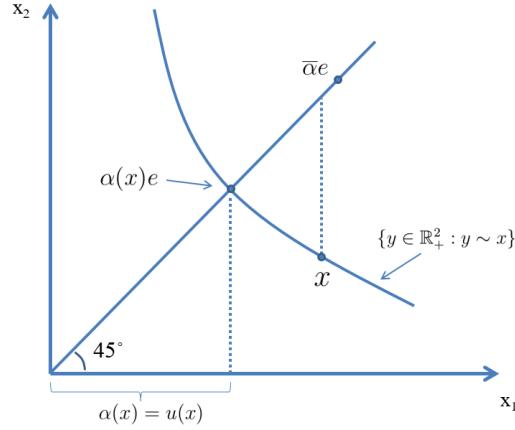


Figure 13: Construction of Utility Function

We want to show that for all $x \in \mathbb{R}_+^L$, \exists unique $\alpha \geq 0$ such that $\alpha e \sim x$, $\alpha_1 e \succ x$ if $\alpha_1 > \alpha$, $x \succ \alpha_2 e$ if $\alpha_2 < \alpha$.

Monotonicity implies $x \succsim 0$. Also, for $\bar{\alpha}$ such that $\bar{\alpha}e \gg x$, we have $\bar{\alpha}e \succ x$. Monotonicity and continuity of preference can be used to show that there exists a unique $\alpha(x) \in [0, \bar{\alpha}]$, such that $\alpha(x)e \sim x$.

Formally, continuity of preference implies that $A^+ = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$ and $A^- = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$ are closed.

Define $\underline{\alpha}^+ := \min\{A^+\}$, and $\bar{\alpha}^- := \max\{A^-\}$.

Then by monotonicity, $\alpha e \succ x$, $\forall \alpha > \underline{\alpha}^+$ and $\alpha e \prec x$, $\forall \alpha < \bar{\alpha}^-$.

Now we show that $\underline{\alpha}^+ = \bar{\alpha}^-$.

Suppose $\underline{\alpha}^+ > \bar{\alpha}^-$. Then $\exists \alpha \in (\bar{\alpha}^-, \underline{\alpha}^+)$ such that $\alpha e \not\succsim x$ and $x \not\succsim \alpha e \implies$ incomplete preference;

Suppose $\underline{\alpha}^+ < \bar{\alpha}^-$. Then $\exists \alpha \in (\bar{\alpha}^-, \underline{\alpha}^+)$ such that

$$\alpha e \succ x (\because \alpha > \underline{\alpha}^+) \quad \& \quad \alpha e \prec x (\because \alpha < \bar{\alpha}^-) \implies \text{contradiction}$$

Thus, there exists a scalar α such that $\alpha e \sim x$.

Furthermore, by monotonicity, $\alpha_1 e \succ \alpha_2 e$ whenever $\alpha_1 > \alpha_2$. Hence, there can be at most one scalar satisfying $\alpha e \sim x$. This scalar is $\alpha(x)$.

Set $u(x) = \alpha(x)$. We have shown that $u(x)$ exists and is unique.

Step 2: $u(x)$ represents \succsim , i.e., $\alpha(x) \geq \alpha(y)$ iff $x \succsim y$.

Note that $\alpha(x)e \sim x$ (Step 1).

If $\alpha(x) \geq \alpha(y)$, then $x \sim \alpha(x)e \succsim \alpha(y)e \succsim y$ (\because preference is monotone) $\implies x \succsim y$.

If $x \succsim y$, then $\alpha(x)e \succsim \alpha(y)e$, monotonicity implies $\alpha(x) \geq \alpha(y)$.

Since $u(x) = \alpha(x)$, $u(x)$ represents \succsim .

Step 3: $u(x) = \alpha(x)$ is continuous, i.e., take any sequence $\{x^n\}_{n=1}^{\infty}$ with $x = \lim_{n \rightarrow \infty} x^n$, we have $\lim_{n \rightarrow \infty} \alpha(x^n) = \alpha(x)$.

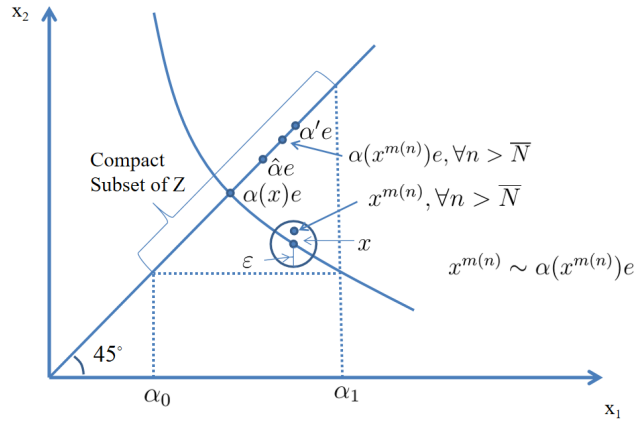


Figure 14: Proof of continuity

Note that $x^n \geq 0$ for all n . Also, there exists $\alpha_1 > 0$ such that $\alpha_1 e \succ x$. For any sufficiently small $\varepsilon > 0$, for all $x' \in X$ such that $\|x' - x\| \leq \varepsilon$, we have $\alpha_1 e \succ x'$ and $\alpha(x') < \alpha_1$. In other words, $\exists N$ s.t. $\forall n > N$, $0 \leq \alpha(x^n) \leq \alpha_1$. Since $\alpha(x^1), \dots, \alpha(x^N)$ are also finite, every element in the sequence $\{\alpha(x^n)\}_{n=1}^{\infty}$ is finite and belongs to some compact space $[0, \alpha_2]$ for some sufficiently large α_2 . By Bolzano-Weierstrass Theorem, the infinite sequence $\{\alpha(x^n)\}_{n=1}^{\infty}$ must have a convergent subsequence.

Now we show that all converging subsequences of $\{\alpha(x^n)\}_{n=1}^{\infty}$ converges to $\alpha(x)$.

Suppose otherwise and that $\lim_{n \rightarrow \infty} \alpha(x^{m(n)}) = \alpha' > \alpha(x)$. [The proof for the case $\alpha(x) > \alpha'$ is similar.] Let $\hat{\alpha} = \frac{1}{2}(\alpha' + \alpha(x))$, so $\alpha(x) < \hat{\alpha} < \alpha'$, $x \sim \alpha(x)e \prec \hat{\alpha}e \prec \alpha'e$.

Recall that $x^{m(n)} \sim \alpha(x^{m(n)})e$ for all n by construction.

(a) Since $\lim_{n \rightarrow \infty} \alpha(x^{m(n)}) = \alpha'$, $\exists \bar{N}$ s.t. $\forall n \geq \bar{N}$, $x^{m(n)} \sim \alpha(x^{m(n)})e > \hat{a}e$.

(b) On the other hand, since $\lim_{n \rightarrow \infty} x^{(n)} = x \prec \hat{a}e$, continuity of preference implies $\exists \tilde{N}$ s.t. $\forall n \geq \tilde{N}$, $x^{m(n)} \prec \hat{a}e$.

Combining (a) and (b), $\forall n \geq \max\{\bar{N}, \tilde{N}\}$, $\hat{a}e \succ x^{m(n)} \sim \alpha(x^{m(n)})e \succ \hat{a}e \implies$ Contradiction.

Note that all the convergent subsequences should cover all infinite subsequences. Suppose not. Then by the Bolzano-Weierstrass Theorem, these infinite sequences should have convergent subsequence(s), contradicting that we have already considered all convergent subsequences. Therefore, we can conclude that $\{\alpha(x^n)\}_{n=1}^\infty$ converges to αx . \square

Remark. $u(x)$ is not unique, any increasing transformation $v(x) = f(u(x))$ will represent \succsim . We can also introduce countably many jumps in $f(\cdot)$.

Claim. If $u(\cdot)$ is a continuous utility function representing \succsim , then \succsim is continuous.

Proof. Take a sequence of pairs $\{(x^n, y^n)\}_{n=1}^\infty$ such that $x^n \succsim y^n$ for all n . Then, $u(x^n) \geq u(y^n)$ for all n . Denote $x = \lim_{n \rightarrow \infty} x^n$ and $y = \lim_{n \rightarrow \infty} y^n$. The continuity of $u(\cdot)$ implies that $\lim_{n \rightarrow \infty} u(x^n) = u(x)$, $\lim_{n \rightarrow \infty} u(y^n) = u(y)$ and thus $u(x) \geq u(y)$. Hence, $x \succsim y$. Therefore, \succsim is continuous.

Assumptions of differentiability of $u(x)$ The assumption of differentiability is commonly adopted for technical convenience, but is not applicable to all useful models.

Here is an example of preference that is not differentiable.

Example (Leontief Preference). $x \succsim y$ if and only if $\min\{x_1, x_2\} \geq \min\{y_1, y_2\}$.

$f(\min\{x_1, x_2\})$ is not differentiable because the left and right derivatives are different:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^-} \frac{f(\min\{x + \varepsilon, x\}) - f(\min\{x, x\})}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0^+} \frac{f(\min\{x, x\}) - f(\min\{x - \varepsilon, x\})}{\varepsilon} = f'(x) \\ \lim_{\varepsilon \rightarrow 0^+} \frac{f(\min\{x + \varepsilon, x\}) - f(\min\{x, x\})}{\varepsilon} &= 0. \end{aligned}$$

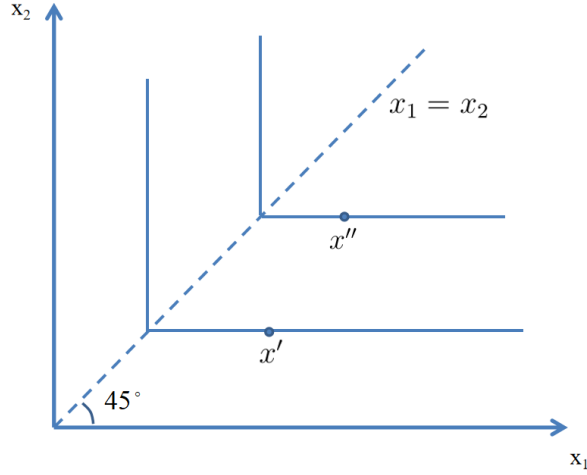


Figure 15: Leontief Preference

Implications of \succsim and u

- (i) \succsim is convex $\iff u : X \rightarrow \mathbb{R}$ is quasi-concave.
- (ii) continuous \succsim on \mathbb{R}_+^L is homothetic $\iff u(x)$ is H.D.1.
- (iii) continuous \succsim on $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is quasilinear with respect to Good 1 \iff

$$u(x) = x_1 + \phi(x_2, \dots, x_L)$$

Definition. The utility function $u(\cdot)$ is *quasiconcave* if the set $\{y \in \mathbb{R}_+^L : u(y) \geq u(x)\}$ is convex for all x or, equivalently, if $u(\alpha x + (1 - \alpha)y) \geq \min\{u(x), u(y)\}$ for all x, y and all $\alpha \in [0, 1]$. If $u(\alpha x + (1 - \alpha)y) > \min\{u(x), u(y)\}$ for $x \neq y$ and $\alpha \in (0, 1)$, then $u(\cdot)$ is *strictly quasiconcave*.

Proof.

- (i) \succsim is convex $\iff y \succsim x$ and $z \succsim x$, then $\alpha y + (1 - \alpha)z \succsim x$ for any $\alpha \in [0, 1]$ \iff
 $u(y) \geq u(x)$ and $u(z) \geq u(x)$, then $u(\alpha y + (1 - \alpha)z) \geq u(x)$ for any $\alpha \in [0, 1]$ \iff
 $\{y \in \mathbb{R}_+^L : u(y) \geq u(x)\}$ is convex $\iff u : X \rightarrow \mathbb{R}$ is quasi-concave.
- (ii) (a) “ \Leftarrow ” Suppose $u(x)$ is H.D.1., i.e., $u(\alpha x) = \alpha u(x)$ and $u(\alpha y) = \alpha u(y)$.

Also suppose $x \sim y \iff u(x) = u(y)$. Then $u(\alpha y) = \alpha u(y) = \alpha u(x) = u(\alpha x) \implies \alpha x \sim \alpha y \implies \succsim$ is homothetic.

- (b) “ \Rightarrow ” Suppose \succsim is homothetic. Suppose $u(x)e \sim x$. It’s W.L.O.G to take $u(x)$ as a utility function.

Homothetic \succsim implies $\alpha u(x)e \sim \alpha x$. We also have $u(\alpha x)e \sim \alpha x$. Then $\alpha u(x)e \sim u(\alpha x)e \Rightarrow u(\alpha x) = \alpha u(x) \Rightarrow u(x)$ is H.D.1.

- (iii) a) “ \Leftarrow ” Suppose $u(x) = x_1 + \phi(x_2, \dots, x_L)$. Then, $u(x + \alpha e_1) = x_1 + \alpha + \phi(x_2, \dots, x_L) = \alpha + u(x)$. Similarly, $u(y + \alpha e_1) = \alpha + u(y)$. Therefore, $x \sim y \Rightarrow u(x) = u(y) \Rightarrow u(x + \alpha e_1) = u(y + \alpha e_1) \Rightarrow (x + \alpha e_1) \sim (y + \alpha e_1) \Rightarrow \succsim$ is quasilinear.

- b) In general cases, for some consumption bundle $(0, x_2, \dots, x_L)$, there exist a consumption bundle $(x_1^*, 0, \dots, 0)$, such that $(0, x_2, \dots, x_L) \sim (x_1^*, 0, \dots, 0)$. We therefore define the mapping from (x_2, \dots, x_L) to x_1^* by a function $x_1^* = \phi(x_2, \dots, x_L)$. From \succsim being quasilinear, we have

$$(x_1, x_2, \dots, x_L) \sim (x_1 + \phi(x_2, \dots, x_L), 0, \dots, 0).$$

Therefore, \succsim admits a utility function of the form $u(x) = x_1 + \phi(x_2, \dots, x_L)$. \square

Exercise 3.C.6

Suppose that in a two-commodity world, the consumer’s utility function takes the form $u(x) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho}$. This utility function is known as the *constant elasticity of substitution* (or *CES*) utility function.

- (a) Show that when $\rho = 1$, indifference curves become linear.
- (b) Show that as $\rho \rightarrow 0$, this utility function comes to represent the same preferences as the (generalized) Cobb-Douglas utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$.
- (c) Show that as $\rho \rightarrow -\infty$, indifference curves become “right angles”; that is, this utility function has in the limit the indifference map of the Leontief utility function $u(x_1, x_2) = \min \{x_1, x_2\}$.

3.D. Utility Maximization Problem (UMP)

We assume throughout that preference is rational, continuous, and locally nonsatiated, and we take $u(x)$ to be a continuous utility function representing these preferences. We also assume that the consumption set is $X = \mathbb{R}_+^L$.

The consumer's problem of choosing her most preferred consumption bundle $x(p, w)$ can be stated as a *Utility Maximization Problem (UMP)*:

$$\begin{aligned} \max_{x \in \mathbb{R}_+^L} \quad & u(x) \\ \text{s.t.} \quad & p \cdot x \leq w \end{aligned}$$

Proposition 3.D.1. *If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.*

Proof. $B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$ is compact, i.e.,

(i) bounded: $0 \leq x_l \leq w/p_l$, for $p_l > 0$

(ii) closed (it contains all the limit points): Consider a sequence $\{x^n\}_{n=1}^\infty$ where $x^n \in B_{p,w}$, or $p \cdot x^n \leq w$ for all n and $x = \lim_{n \rightarrow \infty} x^n \notin B_{p,w}$ or $p \cdot x > w$. There exists $\varepsilon > 0$ such that for all y satisfying $\|y - x\| < \varepsilon$, $p \cdot y > w$. Therefore, $\exists N > 0$, s.t. $\forall n \geq N$, $p \cdot x^n > w$. This contradicts $p \cdot x^n \leq w$ for all n .

By *Extreme Value Theorem*, a continuous function always has a maximum value on any compact set. □

Here, we provide two counter examples where the solution of UMP does not exist.

Counter Examples.

(i) $B_{p,w}$ is not closed: $p \cdot x < w$

(ii) $u(x)$ is not continuous: $u(x) = \begin{cases} p \cdot x & \text{for } p \cdot x < w \\ 0 & \text{for } p \cdot x = w \end{cases}$

Properties of the Walrasian demand correspondence/functions The solution of UMP, denoted by $x(p, w)$, is called *Walrasian* (or *ordinary* or *market*) *demand correspondence*. When $x(p, w)$ is single valued for all (p, w) , we refer to it as *Walrasian* (or *ordinary* or *market*) *demand function*.

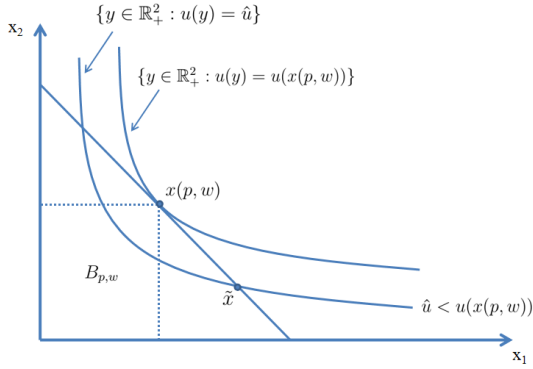


Figure 16: Single solution

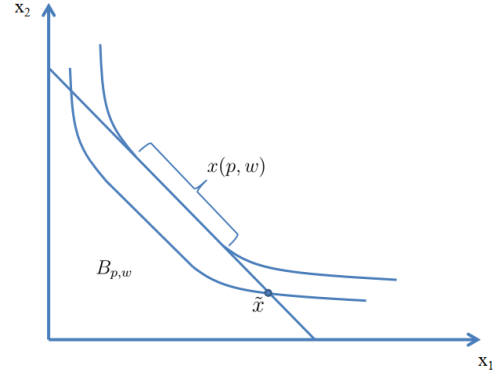


Figure 17: Multiple solutions

Proposition 3.D.2. Suppose that $u(x)$ is a continuous utility function representing a locally nonsatiated preference relation \succsim defined on the consumption set $X = \mathbb{R}_+^L$. Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:

- (i) Homogeneity of degree zero in (p, w) : $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and scalar $\alpha > 0$.
- (ii) Walras' Law: $p \cdot x = w$ for all $x \in x(p, w)$.
- (iii) Convexity/uniqueness: If \succsim is convex, so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set. Moreover, if \succsim is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ consists of a single element.

Recall,

Definition. The preference relation \succsim on X is *convex* if for every $x \in X$, the upper contour set of x , $\{y \in X : y \succsim x\}$ is convex; that is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1 - \alpha)z \succsim x$ for any $\alpha \in [0, 1]$.

Definition. The utility function $u(\cdot)$ is *quasiconcave* if the set $\{y \in \mathbb{R}_+^L : u(y) \geq u(x)\}$ is convex for all x or, equivalently, if $u(\alpha x + (1 - \alpha)y) \geq \min\{u(x), u(y)\}$ for all x, y and all $\alpha \in [0, 1]$.

Proof.

$$(i) \text{ H.D.}\emptyset : \{x \in \mathbb{R}_+^L : p \cdot x \leq w\} = \{x \in \mathbb{R}_+^L : \alpha p \cdot x \leq \alpha w\}.$$

The set of feasible consumption bundles in the UMP is unaffected by α . Therefore,
 $x(p, w) = x(\alpha p, \alpha w)$.

(ii) Walras' Law: Suppose $p \cdot x(p, w) < w$. Then $\exists \varepsilon > 0$ such that $\forall y$ such that $\|y - x(p, w)\| < \varepsilon$, $p \cdot y < w$. Local nonsatiation implies $\exists y$ with $y \in X$ and $\|y - x(p, w)\| < \varepsilon$ such that $y \succ x(p, w) \implies$ contradiction with $x(p, w)$ being optimal.

(iii) Suppose $x, x' \in x(p, w)$ and $x \neq x'$. Then $u(x) = u(x')$. Quasiconcavity of u implies
 $u(\alpha x + (1 - \alpha)x') \geq \min\{u(x), u(x')\} \implies \alpha x + (1 - \alpha)x' \in x(p, w)$

Now suppose $u(x)$ is strictly quasiconcave. Suppose $x, x' \in x(p, w)$ and $x \neq x'$. Then $u(x) = u(x')$. Strict quasiconcavity implies $u(\alpha x + (1 - \alpha)x') > u(x) = u(x')$. This contradicts that $x, x' \in x(p, w)$. Therefore, $x(p, w)$ is single valued.

Alternative proof for (iii): Suppose $x, x' \in x(p, w)$. Then $x \sim x' \succsim y, \forall y \in B_{p, w}$.
 $\bar{x} = \alpha x + (1 - \alpha)x' \succsim x \sim x' \implies \bar{x} \succsim y, \forall y \in B_{p, w}$. Strict convexity $\bar{x} \succ x \sim x'$
 which contradicts $x, x' \in x(p, w)$. \square

We will take a break to review some mathematical results before proceeding with this Chapter. Read “Math Review: Maximization Problem” for details.