

# Chapter 3. Classical Demand Theory

## (Part 1)

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### 3.A. Introduction: Take $\succsim$ as the primitive

- (1) Assumption(s) on  $\succsim$  so that  $\succsim$  can be represented with a utility function
- (2) Utility maximization and demand function
- (3) Utility as a function of prices and wealth (indirect utility)
- (4) Expenditure minimization and expenditure function
- (5) Relationship among demand function, indirect utility function, and expenditure function

## 3.B. Preference Relations: Basic Properties

**Rationality** We would assume *Rationality* (*Completeness and Transitivity*) throughout the chapter.

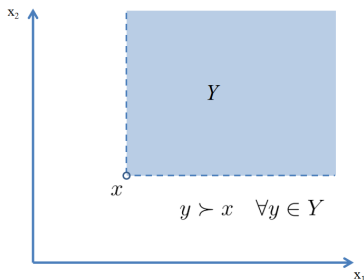
**Definition 3.B.1.** The preference relation *succsim* on  $X$  is rational if it possesses the following two properties:

(i) Completeness: For all  $x, y \in X$ , we have  $x \succsim y$  or  $y \succsim x$  (or both).

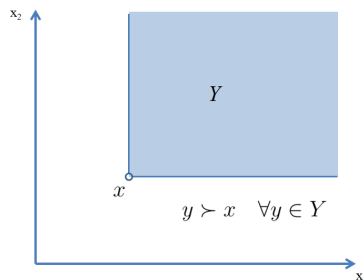
(ii) Transitivity: For all  $x, y, z \in X$ , if  $x \succsim y$  and  $y \succsim z$ , then  $x \succsim z$ .

# Monotonicity

**Definition 3.B.2.** The preference relation  $\succsim$  on  $X$  is *monotone* if  $x, y \in X$  and  $y \gg x$  implies  $y \succ x$ . It is *strongly monotone* if  $y \geq x$  &  $y \neq x$  implies  $y \succ x$ .



Monotonicity



Strong Monotonicity

## Monotonicity

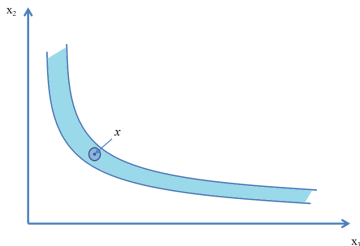
**Claim.** If  $\succsim$  is strongly monotone, then it is monotone.

**Example.** Here is an example of a preference that is monotone, but not strongly monotone:

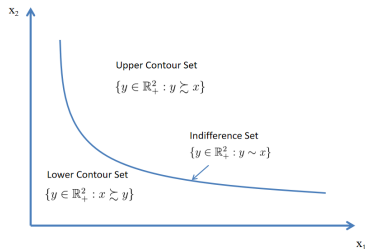
$$u(x_1, x_2) = x_1 \text{ in } \mathbb{R}_+^2.$$

## Local Nonsatiation

**Definition 3.B.3.** The preference relation  $\succsim$  on  $X$  is *locally nonsatiated* if for every  $x \in X$  and every  $\varepsilon > 0$ ,  $\exists y \in X$  such that  $\|y - x\| \leq \varepsilon$  and  $y \succ x$ .



Violation



Compatible

## Local Nonsatiation

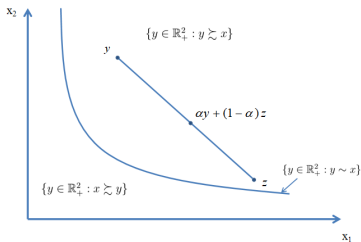
**Claim.** *Local nonsatiation* is a weaker desirability assumption compared to *monotonicity*. If  $\succsim$  is monotone, then it is locally nonsatiated.

**Example.** Here is an example of a preference that is locally nonsatiated, but not monotone:

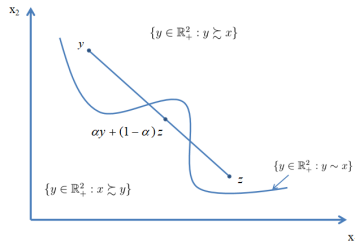
$$u(x_1, x_2) = x_1 - |1 - x_2| \text{ in } \mathbb{R}_+^2.$$

# Convexity Assumptions

**Definition 3.B.4.** The preference relation  $\succsim$  on  $X$  is *convex* if for every  $x \in X$ , the upper contour set of  $x$ ,  $\{y \in X : y \succsim x\}$  is convex; that is, if  $y \succsim x$  and  $z \succsim x$ , then  $\alpha y + (1 - \alpha)z \succsim x$  for any  $\alpha \in [0, 1]$ .



Convex



Nonconvex



## **Properties associated with convexity**

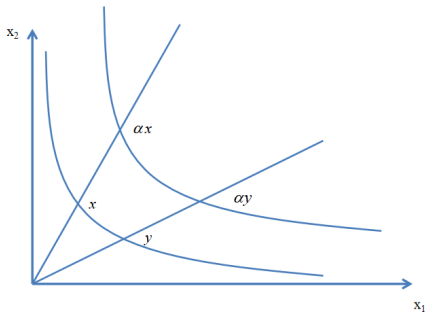
- (i) Diminishing marginal rates of substitution
- (ii) Preference for diversity (implied by (i))

## Strict Convexity

**Definition 3.B.5.** The preference relation  $\succsim$  on  $X$  is *strictly convex* if for every  $x \in X$ , we have that  $y \succsim x$  and  $z \succsim x$ , and  $y \neq z$  implies  $\alpha y + (1 - \alpha)z \succ x$  for all  $\alpha \in (0, 1)$ .

## Homothetic Preference

**Definition 3.B.6.** A monotone preference relation  $\succsim$  on  $X = \mathbb{R}_+^L$  is *homothetic* if all indifference sets are related by proportional expansion along rays; that is, if  $x \sim y$ , then  $\alpha x \sim \alpha y$  for any  $\alpha \geq 0$ .



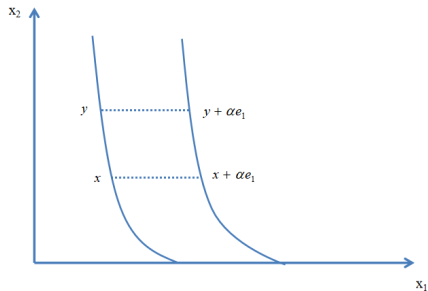
Homothetic Preference

## Quasilinear Preference

**Definition 3.B.7.**  $\succsim$  on  $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$  is *quasilinear* with respect to commodity 1 (*numeraire* commodity) if

- (i) All the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if  $x \sim y$ , then  $(x + \alpha e_1) \sim (y + \alpha e_1)$  for  $e_1 = (1, 0, 0, \dots, 0)$  and any  $\alpha \in \mathbb{R}$ .
- (ii) Good 1 is desirable; that is  $x + \alpha e_1 \succ x$  for all  $x$  and  $\alpha > 0$ .

# Quasilinear Preference



Quasilinear Preference

### 3.C. Preference and Utility

*Key Question.* When can a rational preference relation be represented by a utility function?

*Answer:* If the preference relation is continuous.

## Continuous Preference

**Definition 3.C.1.** The preference relation  $\succsim$  on  $X$  is *continuous* if it is preserved in the limits. That is, for any sequence of pairs  $\{(x^n, y^n)\}_{n=1}^{\infty}$  with  $x^n \succsim y^n$  for all  $n$ ,  $x = \lim_{n \rightarrow \infty} x^n$ ,  $y = \lim_{n \rightarrow \infty} y^n$ , we have  $x \succsim y$ .

*Remark.*  $\succsim$  is continuous if and only if for all  $x$ , the upper contour set  $\{y \in X : y \succsim x\}$  and the lower contour set  $\{y \in X : x \succsim y\}$  are both closed.

## Continuous Preference

**Example 3.C.1.** Lexicographic Preference Relation on  $\mathbb{R}^2$

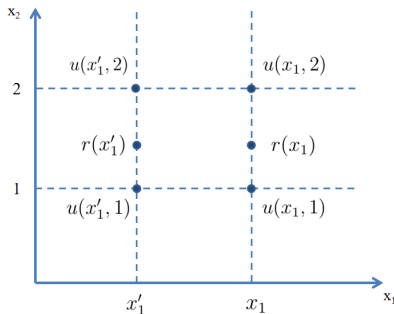
$X = \mathbb{R}^2$ ;  $x \succ y$  if either  $x_1 > y_1$  or  $x_1 = y_1$  and  $x_2 > y_2$ .  $x \sim y$  if  $x_1 = y_1$  and  $x_2 = y_2$ .

**Claim.** Lexicographic Preference Relation on  $\mathbb{R}^2$  is not continuous.



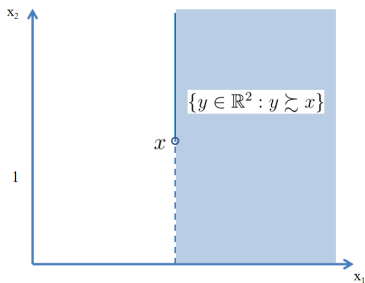
## Continuous Preference

**Claim.** Lexicographic Preference Relation on  $\mathbb{R}^2$  cannot be represented by  $u(\cdot)$ .

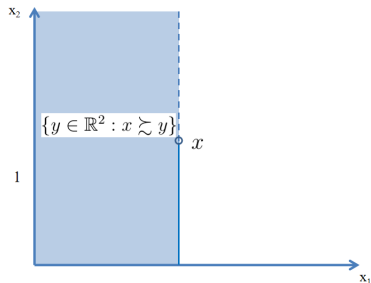


Lexicographic Preference

**Continuous Preference** Alternatively, we could use the fact that upper and lower contour sets of a continuous preference must be closed.



Upper Contour Set



Lower Contour Set

## Continuous Preference

**Proposition 3.C.1.** *Suppose that the preference relation  $\succsim$  on  $X$  is continuous. Then there exists continuous utility function  $u(x)$  that represents  $\succsim$ , i.e.,  $u(x) \geq u(y)$  if and only if  $x \succsim y$ .*

For this course, we will only prove a simplified version of Proposition 3.C.1, which assumes that the preference relation  $\succsim$  is also monotone.

## Continuous Preference

*Remark.*  $u(x)$  is not unique, any increasing transformation  $v(x) = f(u(x))$  will represent  $\succsim$ . We can also introduce countably many jumps in  $f(\cdot)$ .

**Claim.** If  $u(\cdot)$  is a continuous utility function representing  $\succsim$ , then  $\succsim$  is continuous.

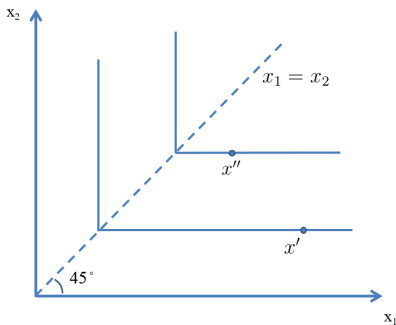
## **Assumptions of differentiability of $u(x)$**

The assumption of differentiability is commonly adopted for technical convenience, but is not applicable to all useful models.

## Assumptions of differentiability of $u(x)$

Here is an example of preference that is not differentiable.

**Example** (Leontief Preference).  $x \succsim y$  if and only if  $\min\{x_1, x_2\} \geq \min\{y_1, y_2\}$ .



Leontief Preference

## Implications of $\succsim$ and $u$

- (i)  $\succsim$  is convex  $\iff u : X \rightarrow \mathbb{R}$  is quasi-concave.
- (ii) continuous  $\succsim$  on  $\mathbb{R}_+^L$  is homothetic  $\iff u(x)$  is H.D.1.
- (iii) continuous  $\succsim$  on  $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$  is quasilinear with respect to Good 1  $\iff u(x) = x_1 + \phi(x_2, \dots, x_L)$

## Quasiconcave Utility

**Definition.** The utility function  $u(\cdot)$  is *quasiconcave* if the set  $\{y \in \mathbb{R}_+^L : u(y) \geq u(x)\}$  is convex for all  $x$  or, equivalently, if  $u(\alpha x + (1 - \alpha)y) \geq \min\{u(x), u(y)\}$  for all  $x, y$  and all  $\alpha \in [0, 1]$ . If  $u(\alpha x + (1 - \alpha)y) > \min\{u(x), u(y)\}$  for  $x \neq y$  and  $\alpha \in (0, 1)$ , then  $u(\cdot)$  is *strictly quasiconcave*.



### 3.D. Utility Maximization Problem (UMP)

Assume throughout that preference is *rational*, *continuous*, *locally nonsatiated*, and  $u(x)$  continuous.

Consumer's *Utility Maximization Problem (UMP)*:

$$\max_{x \in \mathbb{R}_+^L} u(x)$$

$$\text{s.t. } p \cdot x \leq w$$

## Existence of Solution

**Proposition 3.D.1.** *If  $p \gg 0$  and  $u(\cdot)$  is continuous, then the utility maximization problem has a solution.*

## Existence of Solution

Here, we provide two counter examples where the solution of UMP does not exist.

### Counter Examples.

(i)  $B_{p,w}$  is not closed:  $p \cdot x < w$

(ii)  $u(x)$  is not continuous:

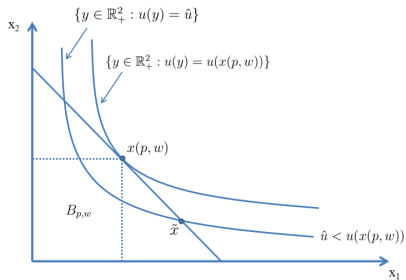
$$u(x) = \begin{cases} p \cdot x & \text{for } p \cdot x < w \\ 0 & \text{for } p \cdot x = w \end{cases}$$

## **Walrasian demand correspondence/functions**

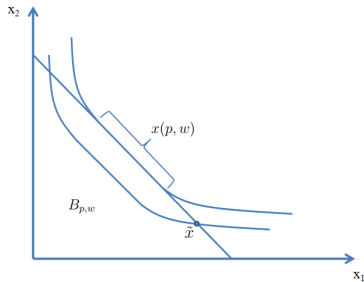
The solution of UMP, denoted by  $x(p, w)$ , is called *Walrasian* (or *ordinary* or *market*) *demand correspondence*.

When  $x(p, w)$  is single valued for all  $(p, w)$ , we refer to it as *Walrasian* (or *ordinary* or *market*) *demand function*.

# Walrasian demand correspondence/functions



Single solution



Multiple solutions

## Properties of Walrasian demand correspondence

**Proposition 3.D.2.** *Suppose that  $u(x)$  is a continuous utility function representing a locally nonsatiated preference relation  $\succsim$  defined on the consumption set  $X = \mathbb{R}_+^L$ . Then the Walrasian demand correspondence  $x(p, w)$  possesses the following properties:*

- (i) *Homogeneity of degree zero in  $(p, w)$  :  $x(\alpha p, \alpha w) = x(p, w)$  for any  $p, w$  and scalar  $\alpha > 0$ .*
- (ii) *Walras' Law:  $p \cdot x = w$  for all  $x \in x(p, w)$ .*

## Properties of Walrasian demand correspondence

### Proposition 3.D.2 (continued).

(iii) *Convexity/uniqueness: If  $\succsim$  is convex, so that  $u(\cdot)$  is quasiconcave, then  $x(p, w)$  is a convex set. Moreover, if  $\succsim$  is strictly convex, so that  $u(\cdot)$  is strictly quasiconcave, then  $x(p, w)$  consists of a single element.*