Advanced Microeconomics

Assignment 3

(Tentative) Due date: November 11, 2019 (before class)

Submission method: Please submit your assignment to me in class, or via E-mail: sherryecon@qq.com.

- 纸质版：要求字迹工整，可辨认。
- 电子版：附件要求 .pdf格式。邮件标题格式为“作业编号-学号-姓名”，如：作业1-201901010101-张三。

Grading: Your assignment will be graded based on your effort, not the accuracy of your answers.

The exercises are embedded in the Chapter 3 lecture notes (red boxes). You are advised to read the relevant sections when you work on the exercises.

The same set of exercises are provided below:

3.B.2 The preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ is said to be *weakly monotone* if and only if $x \geq y$ implies that $x \succsim y$. Show that if $\succsim$ is transitive, locally nonsatiated, and weakly monotone, then it is monotone.

Solutions. We want to show that if $x \gg y$, then $x \succ y$. Define $\varepsilon = \min\{x_1 - y_1, x_2 - y_2, ..., x_L - y_L\}$, then $\forall z \in X$, if $\|z - y\| < \varepsilon$, then $x \gg z$. By local nonsatiation, $\exists z^* \in X$, such that $\|z - y\| < \varepsilon$ and $z^* \succ y$. On the other hand, $x \succsim z^*$ due to weak monotonicity. Hence, by transitivity, we have $x \succ y$.

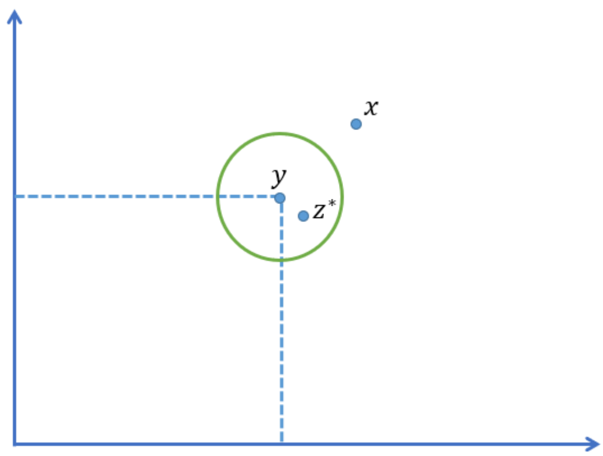

Figure 1: 3.B.2

3.C.6 Suppose that in a two-commodity world, the consumer's utility function takes the form $u(x) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho}$. This utility function is known as the *constant elasticity of substitution* (or *CES*) utility function.

(a) Show that when $\rho = 1$, indifference curves become linear.

(b) Show that as $\rho \to 0$, this utility function comes to represent the same preference as the (generalized) Cobb-Douglas utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$.

(c) Show that as $\rho \to -\infty$, indifference curves become "right angles"; that is, this utility function has in the limit the indifference map of the Leontief utility function $u(x_1, x_2) = \min\{x_1, x_2\}$.

Solution.

(a) When $\rho = 1, u(x) = \alpha_1 x_1 + \alpha_2 x_2$. Take the total differentiation of the utility function, we have $du(x) = \alpha_1 dx_1 + \alpha_2 dx_2$. On an indifference curve, $du(x) = 0$, and hence $\alpha_1 dx_1 + \alpha_2 dx_2 = 0$. Rearranging terms gives

$$\frac{dx_2}{dx_1} = -\frac{\alpha_1}{\alpha_2},$$

which implies that the indifference curve is a straight line.

(b) Consider $\hat{u}(x) := \ln(u(x)) = (1/\rho)\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)$, which represents the same preference as $u(x)$ does. By L'Hôpital's rule, we have

$$\begin{aligned}\lim_{\rho \to 0} \hat{u}(x) &= \lim_{\rho \to 0} \frac{\alpha_1 x_1^\rho \ln(x_1) + \alpha_2 x_2^\rho \ln(x_2)}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho} \\ &= \frac{\alpha_1 \ln(x_1) + \alpha_2 \ln(x_2)}{\alpha_1 + \alpha_2} \equiv \tilde{u}(x).\end{aligned}$$

Since $\exp[(\alpha_1 + \alpha_2)\tilde{u}(x)] = x_1^{\alpha_1} x_2^{\alpha_2}$, we obtain the Cobb-Douglas utility function.

(c) Without loss of generality, we assume $x_1 \leq x_2$, then it suffices to show that $\lim_{\rho \to -\infty} [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho} = x_1$.

Consider $\rho < 0$. Since $x_1 \leq x_2$, we have

$$[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho} \geq [\alpha_1 x_1^\rho + \alpha_2 x_1^\rho]^{1/\rho},$$

i.e., $u(x) \geq (\alpha_1 + \alpha_2)^{1/\rho} x_1$.

Hence,

$$\lim_{\rho \to -\infty} u(x) \geq \lim_{\rho \to -\infty} (\alpha_1 + \alpha_2)^{1/\rho} x_1 = x_1.$$

On the other hand, since $x_1, x_2 \geq 0$, we have

$$[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho} \leq [\alpha_1 x_1^\rho]^{1/\rho},$$

i.e., $u(x) \leq \alpha_1^{1/\rho} x_1$.

Hence,

$$\lim_{\rho \to -\infty} u(x) \leq \alpha_1^{1/\rho} x_1 = x_1.$$

Therefore, we have $\lim\limits_{\rho \to -\infty} u(x) = x_1$.

3.D.5 Consider again the CES utility function of Exercise 3.C.6, and assume that $\alpha_1 = \alpha_2 = 1$.

(a) Compute the Walrasian demand and indirect utility functions for this utility function.

(b) Verify that these functions satisfy all the properties of Propositions 3.D.2 and 3.D.3.

(c) Derive the Walrasian demand correspondence and indirect utility function for the case of linear utility and the case of Leontief utility (see Exercise 3.C.6). Show that the CES Walrasian demand and indirect utility functions approach these as ρ approaches 1 and $-\infty$, respectively.

(d) The *elasticity of substitution between goods* 1 and 2 is defined as

$$\xi_{12}(p, w) = -\frac{\partial[x_1(p,w)/x_2(p,w)]}{\partial[p_1/p_2]} \frac{p_1/p_2}{x_1(p,w)/x_2(p,w)}.$$

Show that for the CES utility function, $\xi_{12}(p,w) = 1/(1-\rho)$, thus justifying the name. What is $\xi_{12}(p,w)$ for the linear, Leontief, and Cobb-Douglas utility functions?

Solution.

(a) Formulate the utility maximization problem as follows:

$$\max_{x_1, x_2 \geq 0} u(x) = (x_1^\rho + x_2^\rho)^{1/\rho},$$
$$\text{s.t. } p_1 x_1 + p_2 x_2 \leq w.$$

The constraint should hold in equality at the optimum, since any wealth left could have been spent to increase utility.

Set up the Lagrangian

$$\mathcal{L}(x_1, x_2, \lambda) = (x_1^\rho + x_2^\rho)^{1/\rho} - \lambda(p_1 x_1 + p_2 x_2 - w).$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1}(\rho x_1^{\rho-1}) - \lambda p_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1}(\rho x_2^{\rho-1}) - \lambda p_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = p_1 x_1 + p_2 x_2 - w = 0$$

The first two FOCs imply that

$$\frac{x_1}{x_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}.$$

Together with the third FOC, one can solve for the Walrasian demand function, which is given by

$$x(p,w) = \left(\frac{wp_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}, \frac{wp_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\right).$$

Substitute $x(p,w)$ back into the utility function, we obtain the indirect utility function

$$\begin{aligned} v(p,w) &= \left[\left(\frac{wp_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\right)^\rho + \left(\frac{wp_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\right)^\rho\right]^{1/\rho} \\ &= \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w. \end{aligned}$$

(b) **Proposition 3.D.2**

(i) **Homogeneity of degree zero** of the demand function. For any p, w and $\alpha > 0$, we have

$$x_1(\alpha p, \alpha w) = \frac{(\alpha p_1)^{\frac{1}{\rho-1}}}{(\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}}\alpha w = \frac{p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w = x_1(p,w),$$

$$x_2(\alpha p, \alpha w) = \frac{(\alpha p_2)^{\frac{1}{\rho-1}}}{(\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}}\alpha w = \frac{p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w = x_2(p,w).$$

(ii) **Walras' law**. Direct calculation gives

$$\begin{aligned}
p_1x_1+p_2x_2 &= p_1\frac{p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}w+p_2\frac{p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}w \\
&= \frac{p_1^{\frac{\rho}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}w+\frac{p_2^{\frac{\rho}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}}w \\
&= w,
\end{aligned}$$

(iii) The **uniqueness** is trivial: $x(p,w)$ is unique given the explicit expression.

Proposition 3.D.3

(i) **Homogeneity of degree zero** in price of the indirect utility function. For any $\alpha > 0$, we have

$$\begin{aligned}
v(\alpha p, w) &= \left((\alpha p_1)^{\frac{\rho}{\rho-1}}+(\alpha p_2)^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}}\alpha w \\
&= \left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w \\
&= v(p,w).
\end{aligned}$$

(ii) **Monotonicity** of the indirect utility function. Since for any $p \gg 0, w > 0$,

$$\begin{aligned}
\frac{\partial v(p,w)}{\partial w} &= \left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} > 0, \\
\frac{\partial v(p,w)}{\partial p_l} &= \frac{1-\rho}{\rho}\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-2\rho}{\rho}} w\left(\frac{\rho}{\rho-1}p_l^{\frac{1}{\rho-1}}\right) \\
&= -\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-2\rho}{\rho}} w p_l^{\frac{1}{\rho-1}} < 0 \text{ for } l = 1,2.
\end{aligned}$$

Hence, the indirect function is strictly increasing in w and strictly decreasing in p_l for all l.

(iii) **Quasiconvexity**. To prove quasiconvexity, we claim that, by homogeneity of degree zero, it suffices to prove that for any $\bar{v} \in \mathbb{R}$ and $w > 0$, the set $\{p \in \mathbb{R}^2_{++} : v(p,w) \le \bar{v}\}$ is convex.

For $\rho \to 0$, the utility function is Cobb-Douglas, and the indirect utility function is given by $v(p,w) = \frac{w^2}{4p_1p_2}$ which is convex in p [1], and hence quasiconvex.

[1] The Hessian matrix of the function $v(p,w) = \frac{w^2}{4p_1p_2}$ is $w^2\begin{bmatrix}\frac{1}{2p_1^3p_2} & \frac{1}{4p_1^2p_2^2} \\ \frac{1}{4p_1^2p_2^2} & \frac{1}{2p_1p_2^3}\end{bmatrix}$, which is positive definite when $p \gg 0$.

For $\rho < 0$, since $p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is concave in p [2], the set $\{p : v(p,w) \le \bar{v}\} = \{p : p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} \ge \bar{v}^{\frac{\rho}{1-\rho}}\}$ is convex.

For $\rho \in (0,1)$, since $p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is convex in p [3], the set $\{p : v(p,w) \le \bar{v}\} = \{p : p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} \le \bar{v}^{\frac{\rho}{1-\rho}}\}$ is convex.

To justify the claim, note that quasiconvexity states that

$$v(p,w) \le \bar{v}, v(p',w') \le \bar{v} \implies v(\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w') \le \bar{v}.$$

And homogeneity of degree zero implies $v(p,w) = v\left(\frac{p}{w}, 1\right)$. Hence, quasiconvexity is equivalent to

$$v\left(\frac{p}{w}, 1\right) \le \bar{v}, v\left(\frac{p'}{w'}, 1\right) \le \bar{v} \implies v\left(\frac{\alpha p + (1-\alpha)p'}{\alpha w + (1-\alpha)w'}, 1\right) \le \bar{v}. \quad (1)$$

Let $p_1 = \frac{p}{w}$, $p_2 = \frac{p'}{w'}$, then

$$\frac{\alpha p + (1-\alpha)p'}{\alpha w + (1-\alpha)w'} = \beta p_1 + (1-\beta)p_2,$$

where

$$\beta = \frac{\alpha w}{\alpha w + (1-\alpha)w'} \in (0,1).$$

Now if the set $\{p \in \mathbb{R}^L_{++} : v(p,w) \le \bar{v}\}$ is convex, we have

$$v(p_1, 1) \le \bar{v}, v(p_2, 1) \le \bar{v} \implies v(\beta p_1 + (1-\beta)p_2, 1) \le \bar{v}.$$

Therefore, we have established (1).

(iv) **Continuity** follows from the functional form of $v(p,w)$.

(c) For linear utility function, $u(x) = x_1 + x_2$, one can solve for the Walrasian demand function by substituting $x_2 = \frac{w - p_1 x_1}{p_2}$ into the objective function, which gives

$$x(p,w) = \begin{cases} (w/p_1, 0), & \text{if } p_1 < p_2; \\ (0, w/p_2), & \text{if } p_2 < p_1; \\ (w/p_1)(\lambda, 1-\lambda), \lambda \in [0,1], & \text{if } p_1 = p_2. \end{cases}$$

[2]The Hessian matrix of the function $g(p) = p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is $\begin{bmatrix} \frac{\rho}{(\rho-1)^2} p_1^{\frac{2-\rho}{\rho-1}} & 0 \\ 0 & \frac{\rho}{(\rho-1)^2} p_2^{\frac{2-\rho}{\rho-1}} \end{bmatrix}$, which is negative definite when $\rho < 0$ and $p \gg 0$.

[3]The Hessian matrix of the function $g(p) = p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is $\begin{bmatrix} \frac{\rho}{(\rho-1)^2} p_1^{\frac{2-\rho}{\rho-1}} & 0 \\ 0 & \frac{\rho}{(\rho-1)^2} p_2^{\frac{2-\rho}{\rho-1}} \end{bmatrix}$, which is positive definite when $\rho \in (0,1)$ and $p \gg 0$.

And the indirect utility function is given by

$$v(p,w) = \max(w/p_1, w/p_2).$$

Similarly, for Leontief utility function $u(x) = \min\{x_1, x_2\}$, the Walrasian demand function is given by

$$x(p,w) = (\frac{w}{p_1+p_2}, \frac{w}{p_1+p_2}).$$

And the indirect utility function is given by

$$v(p,w) = \frac{w}{p_1+p_2}.$$

Consider $\rho < 1$ and $\rho \to 1$.

If $p_1 < p_2$, $(p_2/p_1)^{\frac{\rho}{\rho-1}} \to 0$, $(p_1/p_2)^{\frac{\rho}{\rho-1}} \to \infty$ as $\rho \to 1^-$. Thus,

$$\begin{aligned}\lim_{\rho\to 1^-} x_1(p,w) &= \lim_{\rho\to 1^-} \frac{p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w \\ &= \lim_{\rho\to 1^-} \frac{p_1^{-1}}{1 + (p_2/p_1)^{\frac{\rho}{\rho-1}}} w \\ &= w/p_1,\end{aligned}$$

and

$$\begin{aligned}\lim_{\rho\to 1^-} x_2(p,w) &= \lim_{\rho\to 1^-} \frac{p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w \\ &= \lim_{\rho\to 1^-} \frac{p_2^{-1}}{(p_1/p_2)^{\frac{\rho}{\rho-1}} + 1} w \\ &= 0.\end{aligned}$$

Similarly, if $p_1 > p_2$,

$$\begin{aligned}\lim_{\rho\to 1^-} x_1(p,w) &= 0, \\ \lim_{\rho\to 1^-} x_2(p,w) &= w/p_2.\end{aligned}$$

Finally, if $p_1 = p_2$, then

$$\lim_{\rho\to 1^-} x_l(p,w) = \frac{w}{2p_1}, \text{ for } l = 1, 2.$$

Therefore, the CES Walrasian demands converge to the Walrasian demand of the linear preference as $\rho \to 1$.

As for the indirect utility function, if $p_1 < p_2$, then

$$\begin{aligned}\lim_{\rho\to 1^-} v(p,w) &= \lim_{\rho\to 1^-} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w \\ &= \lim_{\rho\to 1^-} \left(1 + (p_2/p_1)^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} \frac{w}{p_1} \\ &= \frac{w}{p_1}.\end{aligned}$$

If $p_1 > p_2$, then

$$\begin{aligned}\lim_{\rho\to 1^-} v(p,w) &= \lim_{\rho\to 1^-} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w \\ &= \lim_{\rho\to 1^-} \left((p_1/p_2)^{\frac{\rho}{\rho-1}} + 1\right)^{\frac{1-\rho}{\rho}} \frac{w}{p_2} \\ &= \frac{w}{p_2}.\end{aligned}$$

If $p_1 = p_2$, then

$$\begin{aligned}\lim_{\rho\to 1^-} v(p,w) &= \lim_{\rho\to 1^-} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w \\ &= \lim_{\rho\to 1^-} 2^{\frac{1-\rho}{\rho}} \frac{w}{p_1} \\ &= \frac{w}{p_1},\end{aligned}$$

which belongs to the set of the Walrasian demands of the linear preference when $p_1 = p_2$.

Therefore, $\lim_{\rho\to 1^-} v(p,w) = \max(w/p_1, w/p_2)$ which agrees with the indirect utility function of the linear preference.

Now consider the case when $\rho \to -\infty$. Since $\frac{\rho}{\rho-1} \to 1$, we have

$$\begin{aligned}\lim_{\rho\to -\infty} x_1(p,w) &= \lim_{\rho\to -\infty} \frac{p_1^{-1}}{1 + (p_2/p_1)^{\frac{\rho}{\rho-1}}} w \\ &= \frac{w}{p_1+p_2},\end{aligned}$$

and

$$\begin{aligned}\lim_{\rho\to -\infty} x_2(p,w) &= \lim_{\rho\to -\infty} \frac{p_2^{-1}}{(p_1/p_2)^{\frac{\rho}{\rho-1}} + 1} w \\ &= \frac{w}{p_1+p_2}.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{\rho\to-\infty} v(p,w) &= \lim_{\rho\to-\infty}\left(p_1^{\frac{\rho}{\rho-1}}+p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w\\ &= \frac{w}{p_1+p_2}.\end{aligned}$$

Therefore, the CES Walrasian demand function and the indirect utility function converge to those of the Leontief preference as $\rho \to -\infty$.

(d) Recall that the FOC of the utility maximization problem with CES utility is

$$\frac{x_1}{x_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}.$$

Hence, the elasticity of substitution between goods 1 and 2 is

$$\begin{aligned}\xi_{12}(p,w) &= -\frac{\partial[x_1(p,w)/x_2(p,w)]}{\partial[p_1/p_2]}\frac{p_1/p_2}{x_1(p,w)/x_2(p,w)}\\ &= -\frac{1}{\rho-1}\left(\frac{p_1}{p_2}\right)^{\frac{2-\rho}{\rho-1}}\frac{p_1}{p_2}\left(\frac{p_1}{p_2}\right)^{\frac{-1}{\rho-1}}\\ &= \frac{1}{1-\rho},\end{aligned}$$

which is a constant.

Similarly, we have $\xi_{12}(p,w) = \lim\limits_{\rho\to1}\frac{1}{1-\rho} = \infty$ for linear utility, $\xi_{12}(p,w) = \lim\limits_{\rho\to-\infty}\frac{1}{1-\rho} = 0$ for Leontief utility, and $\xi_{12}(p,w) = \lim\limits_{\rho\to0}\frac{1}{1-\rho} = 1$ for Cobb-Douglas utility.

3.E.6 Consider the constant elasticity of substitution utility function studied in Exercises 3.C.6 and 3.D.5 with $\alpha_1 = \alpha_2 = 1$. Derive its Hicksian demand function and expenditure function. Verify the properties of Propositions 3.E.2 and 3.E.3.

Solution. Consider the following expenditure minimization problem:

$$\begin{aligned}\min_{x_1,x_2\geq0}\ & p_1x_1+p_2x_2,\\ \text{s.t. } & u(x) = (x_1^\rho+x_2^\rho)^{1/\rho}\geq u.\end{aligned}$$

The constraint should be binding at optimum since a sufficiently small reduction in consumption can reduce expenditure without violating the constraint. We then set up the Lagrangian as follows:

$$\mathcal{L}(x_1,x_2,\lambda) = -p_1x_1-p_2x_2-\lambda[u-(x_1^\rho+x_2^\rho)^{1/\rho}].$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= -p_1 + \lambda \rho^{-1}(x_1^\rho + x_2^\rho)^{1/\rho - 1}\rho x_1^{\rho-1} = 0,\\
\frac{\partial \mathcal{L}}{\partial x_2} &= -p_2 + \lambda \rho^{-1}(x_1^\rho + x_2^\rho)^{1/\rho - 1}\rho x_2^{\rho-1} = 0,\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= u - (x_1^\rho + x_2^\rho)^{1/\rho} = 0.
\end{aligned}$$

From the first two FOCs, we obtain $\frac{x_1}{x_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}$. Together with the third FOC, we derive the Hicksian demand function as follows:

$$\begin{aligned}
h_1(p,u) &= u p_1^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}},\\
h_2(p,u) &= u p_2^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}}.
\end{aligned}$$

The expenditure function is thus given by

$$e(p,u) = p \cdot h(p,u) = u \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}.$$

We check the properties of the expenditure function as follows.

Proposition 3.E.2

(i) For any $\alpha > 0$, we have

$$\begin{aligned}
e(\alpha p, u) &= u \left((\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}\\
&= \alpha u \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}\\
&= \alpha e(p,u).
\end{aligned}$$

Hence, the expenditure function is **homogeneous of degree one in p.**

(ii) Since for any $u > 0$ and $p \gg 0$

$$\frac{\partial e(p,u)}{\partial u} = \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} > 0,$$

and

$$\frac{\partial e(p,u)}{\partial p_l} = u \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} p_l^{\frac{1}{\rho-1}} > 0, \text{ for } l = 1, 2,$$

the expenditure function is **strictly increasing in u and p_l for all l.**

(iii) Since the Hessian matrix

$$D_p^2 e(p,u) = \begin{bmatrix} \frac{u}{\rho-1} p_1^{\frac{2-\rho}{\rho-1}} p_2^{\frac{\rho}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} & -\frac{u}{\rho-1} (p_1 p_2)^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} \\ -\frac{u}{\rho-1} (p_1 p_2)^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} & \frac{u}{\rho-1} p_2^{\frac{2-\rho}{\rho-1}} p_1^{\frac{\rho}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} \end{bmatrix}$$

is negative definite, the expenditure function is **strictly concave in p.**

(iv) The **continuity** of $e(p,u)$ is trivial.

We check the properties of the Hicksian demand function as follows.

Proposition 3.E.3

(i) Since for any $\alpha > 0$ and $l = 1, 2$,

$$\begin{aligned} h_l(\alpha p, u) &= u(\alpha p_l)^{\frac{1}{\rho-1}} \left((\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} \\ &= u p_1^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} \\ &= h_1(p,u), \end{aligned}$$

the Hicksian demand function is **homogeneous of degree zero in p.**

(ii) Since

$$\begin{aligned} u(h_1(p,u), h_2(p,u)) &= \left\{ \left[u p_1^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}}\right]^\rho + \left[u p_2^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}}\right]^\rho \right\}^{1/\rho} \\ &= u, \end{aligned}$$

there is **no excess utility**.

(iii) The **uniqueness** of the value of Hicksian demand is trivial.

3.E.9 Use the relations in (3.E.1) to show that the properties of the indirect utility function identified in Proposition 3.D.3 imply Proposition 3.E.2. Likewise, use the relations in (3.E.1) to prove that Proposition 3.E.2 implies Proposition 3.D.3.

Solution. We first show that by relations in (3.E.1), the properties of the indirect utility function in Proposition 3.D.3 imply the properties of the expenditure function in Proposition 3.E.2.
Let $p \gg 0, p' \gg 0, u \in \mathbb{R}, u' \in \mathbb{R}$, and $\alpha \geq 0$.

(i) **Homogeneity:** Let $\alpha > 0$. Define $w = e(p, u)$, then $u = v(p, w)$ by the second relation in (3.E.1). Hence,

$$e(\alpha p, u) = e(\alpha p, v(p, w)) = e(\alpha p, v(\alpha p, \alpha w)) = \alpha w = \alpha e(p, u),$$

where the second equality follows from the homogeneity of $v(p, w)$ and the third from the first relation in (3.E.1). Therefore, the expenditure function is homogeneous of degree one in p.

(ii) **Monotonicity:** Let $u' > u$. Define $w = e(p, u)$ and $w' = e(p, u')$, then $u = v(p, w)$ and $u' = v(p, w')$. Since $v(p, w)$ is strictly increasing in w, we must have $w' > w$, that is, $e(p, u') > e(p, u)$. Next let $p' \geq p$. Define $w = e(p, u)$ and $w' = e(p', u)$, then by the second relation in (3.E.1), $u = v(p, w) = v(p', w')$. Since $v(p, w)$ is strictly increasing in w and nonincreasing in p_l for any l, we must have $w' \geq w$, that is, $e(p', u) \geq e(p, u)$. Therefore, the expenditure function $e(p, u)$ is strictly increasing in u and nondecreasing in p_l for any l.

(iii) **Concavity:** Let $\alpha \in [0, 1]$. Define $w = e(p, u)$ and $w' = e(p', u)$, then $u = v(p, w) = v(p', w')$. Define $p'' = \alpha p + (1 - \alpha)p'$ and $w'' = \alpha w + (1 - \alpha)w'$. Then, by the quasiconvexity of $v(p, w)$, $v(p'', w'') \leq u$. Hence, since $v(p, w)$ is strictly increasing in w and $v(p'', e(p'', u)) = u$, we must have $e(p'', u) \geq w''$, that is,

$$e(\alpha p + (1 - \alpha)p', u) \geq \alpha e(p, u) + (1 - \alpha)e(p', u).$$

Therefore, the expenditure function $e(p, u)$ is concave in p.

(iv) **Continuity:** Suppose the sequence $\{(p^n, u^n)\}_{n=1}^{\infty}$ converges to (p, u), we show that $\lim_{n\to\infty} e(p^n, u^n) = e(p, u)$. Suppose to the contrary that $\lim_{n\to\infty} e(p^n, u^n) = w \neq e(p, u)$ for some $w \in \mathbb{R}$. On the one hand, by the second relation in (3.E.1), $v(p^n, e(p^n, u^n)) = u^n$, which converges to u by assumption. On the other hand, since $v(p, w)$ is continuous in (p, w) and is strictly increasing in w, $\lim_{n\to\infty} e(p^n, u^n) = w \neq e(p, u)$ implies that $\lim\limits_{n\to\infty} v(p^n, e(p^n, u^n)) = v(p, w) \neq v(p, e(p, u)) = u$, which is a contradiction. Hence, we must have $\lim_{n\to\infty} e(p^n, u^n) = e(p, u)$, i.e., $e(p, u)$ is continuous in (p, u).

Now we show that by relations in (3.E.1), the properties of the expenditure function in Proposition 3.E.2 imply the properties of the indirect utility function in Proposition 3.D.3.

Let $p \gg 0, p' \gg 0, w \in \mathbb{R}, w' \in \mathbb{R}$, and $\alpha \geq 0$.

(i) **Homogeneity:** Let $\alpha > 0$. Define $u = v(p, w)$, then by the first relation in (3.E.1), $e(p, u) = w$. Hence,

$$v(\alpha p, \alpha w) = v(\alpha p, \alpha e(p, u)) = v(\alpha p, e(\alpha p, u)) = u = v(p, w),$$

where the second equality follows from the homogeneity of $e(p, u)$ and the third from the second relation in (3.E.1). Therefore, the indirect utility function $v(p, w)$ is homogeneous of degree zero.

(ii) **Monotonicity:** Let $w' > w$. Define $u = v(p, w)$ and $u' = v(p, w')$, then $e(p, u) = w$ and $e(p, u') = w'$. Since $e(p, u)$ is strictly increasing in u, we must have $u' > u$, that is, $v(p, w') > v(p, w)$. Therefore, the indirect utility function is strictly increasing in w. Next let $p' \geq p$. Define $u = v(p, w)$ and $u' = v(p', w)$, then $e(p, u) = e(p', u') = w$. Since $e(p, u)$ is strictly increasing in u and nondecreasing in p_l for any l, we must have $u' \leq u$, that is $v(p', w) \leq v(p, w)$. Therefore, the indirect utility function is nonincreasing in p.

(iii) **Quasiconvexity:** Let $\alpha \in [0, 1]$. Define $u = v(p, w)$ and $u' = v(p', w')$, then $e(p, u) = w$ and $e(p, u') = w'$. Without loss of generality, assume that $u' \geq u$. Define $p'' = \alpha p + (1-\alpha)p'$ and $w'' = \alpha w + (1-\alpha)w'$, and we show that $v(p'', w'') \leq u'$. Since $u' = v(p'', e(p'', u'))$ and $v(p, w)$ is strictly increasing in w, it suffices to show that $e(p'', u') \geq w''$. This is proved as follows:

$$\begin{aligned} e(p'', u') &\geq \alpha e(p, u') + (1-\alpha)e(p', u') \\ &\geq \alpha e(p, u) + (1-\alpha)e(p', u) \\ &= \alpha w + (1-\alpha)w' = w'', \end{aligned}$$

where the first inequality follows from the concavity of $e(p, u)$ in p, and the second from the monotonicity of $e(p, u)$ in u. Therefore, the indirect utility function $v(p, w)$ is quasiconvex.

(iv) **Continuity:** Suppose the sequence $\{(p^n, w^n)\}_{n=1}^{\infty}$ converges to (p, w), we show that $\lim_{n\to\infty} v(p^n, w^n) = v(p, w)$. Suppose to the contrary that $\lim_{n\to\infty} v(p^n, w^n) = u \neq v(p, w)$ for some $u \in \mathbb{R}$. On the one hand, $e(p^n, v(p^n, w^n)) = w^n$, which converges to w by assumption. On the other hand, as $e(p, u)$ is continuous in (p, u) and is strictly increasing in u, $e(p^n, v(p^n, w^n))$ converges to $e(p, u) \neq e(p, v(p, w)) = w$, which is a contradiction. Hence, we must have $\lim_{n\to\infty} v(p^n, w^n) = v(p, w)$, i.e., $v(p, w)$ is continuous in (p, w).

3.G.1 Prove that Proposition 3.G.1 is implied by Roy's identity (Proposition 3.G.4).

Solution. Define $w = e(p, u)$, then $v(p, w) = u$. Differentiate both sides with respect to p, and we have

$$\nabla_p v(p, w) + \nabla_w v(p, w)\nabla_p e(p, u) = 0.$$

By Roy's identity, $\nabla_p v(p, w) = -x(p, w)\nabla_w v(p, w)$, we have

$$\begin{aligned} & - x(p, w)\nabla_w v(p, w) + \nabla_w v(p, w)\nabla_p e(p, u) = 0, \\ \Longrightarrow\ & \nabla_w v(p, w)[\nabla_p e(p, u) - x(p, e(p, u))] = 0. \end{aligned}$$

Since $\nabla_w v(p, w) > 0$ and $x(p, e(p, u)) = h(p, u)$, we obtain $h(p, u) = \nabla_p e(p, u)$.

3.G.8 The indirect utility function $v(p, w)$ is logarithmically homogeneous if $v(p, \alpha w) = v(p, w) + \ln \alpha$ for $\alpha > 0$ [in other words, $v(p, w) = \ln(v^*(p, w))$, where $v^*(p, w)$ is homogeneous of degree one in w]. Show that if $v(\cdot, \cdot)$ is logarithmically homogeneous, then $x(p, 1) = -\nabla_p v(p, 1)$.

Solution. For any $w > 0$, by Roy's identity, we have

$$x(p, w)\nabla_w v(p, w) = -\nabla_p v(p, w).$$

Since $v(\cdot, \cdot)$ is logarithmically homogeneous, we rewrite the above equation as

$$x(p, w)\nabla_w (v(p, 1) + \ln(w)) = -\nabla_p v(p, w).$$

Hence,

$$x(p, w)/w = -\nabla_p v(p, w).$$

Evaluate at $w = 1$, we obtain

$$x(p, 1) = -\nabla_p v(p, 1).$$

3.G.15 Consider the utility function

$$u = 2x_1^{1/2} + 4x_2^{1/2}.$$

(a) Find the demand functions for goods 1 and 2 as they depend on prices and wealth.

(b) Find the compensated demand function $h(\cdot)$.

(c) Find the expenditure function, and verify that $h(p, u) = \nabla_p e(p, u)$.

(d) Find the indirect utility function, and verify Roy's identity.

Solution.

(a) Consider the following utility maximization problem

$$\max_{x_1,x_2\geq 0} u(x) = 2x_1^{1/2} + 4x_2^{1/2},$$
$$\text{s.t. } p_1x_1 + p_2x_2 \leq w.$$

The budget constraint should hold in equality at the optimum since the marginal utility of each good is always positive for any positive consumption.

We set up the Lagrangian

$$\mathcal{L}(x,\lambda) = 2x_1^{1/2} + 4x_2^{1/2} - \lambda[p_1x_1 + p_2x_2 - w].$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = x_1^{-1/2} - \lambda p_1 = 0,$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 2x_2^{-1/2} - \lambda p_2 = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = p_1x_1 + p_2x_2 - w = 0.$$

The first two FOCs imply

$$\frac{x_2}{x_1} = \left(\frac{2p_1}{p_2}\right)^2.$$

Together with the third FOC, we can derive the Walrasian demand function

$$x(p,w) = \left(\frac{p_2w}{p_1p_2 + 4p_1^2}, \frac{4p_1w}{4p_1p_2 + p_2^2}\right).$$

(b) Consider the expenditure minimization problem

$$\min_{x_1,x_2\geq 0} p_1x_1 + p_2x_2,$$
$$\text{s.t. } u(x) = 2x_1^{1/2} + 4x_2^{1/2} \geq u.$$

The constraint must hold in equality at the optimum since any sufficiently small reduction in consumption can reduce expenditure without violating the utility requirement.

Hence, we set up the Lagrangian as follows:

$$\mathcal{L}(x,\lambda) = -p_1x_1 - p_2x_2 - \lambda[u - 2x_1^{1/2} - 4x_2^{1/2}].$$

The first-order conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= -p_1 + \lambda x_1^{-1/2} = 0,\\ \frac{\partial \mathcal{L}}{\partial x_2} &= -p_2 + \lambda 2 x_2^{-1/2} = 0,\\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 2x_1^{1/2} + 4x_2^{1/2} - u = 0.\end{aligned}$$

The first two FOCs imply that

$$\frac{x_2}{x_1} = \left(\frac{2p_1}{p_2}\right)^2.$$

Together with the third FOC, we obtain the Hicksian demand function

$$h(p,u) = \left(\left(\frac{up_2}{8p_1+2p_2}\right)^2, \left(\frac{up_1}{4p_1+p_2}\right)^2\right).$$

(c) The expenditure function is given by

$$e(p,u) = p_1\left(\frac{up_2}{8p_1+2p_2}\right)^2 + p_2\left(\frac{up_1}{4p_1+p_2}\right)^2 = \frac{p_1p_2u^2}{4(4p_1+p_2)}.$$

Direct calculation gives

$$\begin{aligned}\frac{\partial e(p,u)}{\partial p_1} &= \frac{4p_2u^2(4p_1+p_2) - 16p_1p_2u^2}{16(4p_1+p_2)^2}\\ &= \left(\frac{up_2}{8p_1+2p_2}\right)^2\\ &= h_1(p,u),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial e(p,u)}{\partial p_2} &= \frac{4p_1u^2(4p_1+p_2) - 4p_1p_2u^2}{16(4p_1+p_2)^2}\\ &= \left(\frac{up_1}{4p_1+p_2}\right)^2\\ &= h_2(p,u).\end{aligned}$$

Hence, $h(p,u) = \nabla_p e(p,u)$.

(d) The indirect utility function is given by

$$v(p,w) = 2\left(\frac{p_2w}{p_1p_2+4p_1^2}\right)^{1/2} + 4\left(\frac{4p_1w}{p_2^2+4p_1p_2}\right)^{1/2}.$$

Direct calculation gives

$$\begin{aligned}\frac{\partial v(p,w)}{\partial w} &= \left(\frac{p_2}{p_1p_2+4p_1^2}\right)^{1/2} w^{-1/2} + \left(\frac{16p_1}{p_2^2+4p_1p_2}\right)^{1/2} w^{-1/2}\\ &= \frac{(p_2+4p)^{1/2}}{(p_1p_2)^{1/2}} w^{-1/2},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial v(p,w)}{\partial p_1} &= \left(\frac{p_2w}{p_1(p_2+4p_1)}\right)^{-1/2} \frac{-(p_2+8p_1)p_2w}{[p_1(p_2+4p_1)]^2}\\ &\quad + 2\left(\frac{4p_1w}{p_2(p_2+4p_1)}\right)^{-1/2} \frac{4wp_2(p_2+4p_1)-4p_2\cdot 4w}{[p_2(p_2+4p_1)]^2}\\ &= \frac{-(p_2+8p_1)(p_2w)^{1/2}}{[p_1(p_2+4p_1)]^{3/2}} + \frac{(p_1w)^{-1/2}4wp_2^2}{[p_2(p_2+4p_1)]^{3/2}}\\ &= \frac{-(p_2+8p_1)(p_2w)^{1/2}+(p_1w)^{-1/2}p_1^{3/2}4wp_2^{1/2}}{p_1^{3/2}(p_2+4p_1)^{3/2}}\\ &= \frac{-p_2^{3/2}w^{1/2}-4p_1p_2^{1/2}w^{1/2}}{p_1^{3/2}(p_2+4p_1)^{3/2}}\\ &= \frac{-(p_2w)^{1/2}}{p_1^{3/2}(p_2+4p_1)^{1/2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}-\frac{\partial v(p,w)/\partial p_1}{\partial v(p,w)/\partial w} &= -\frac{-(p_2w)^{1/2}}{p_1^{3/2}(p_2+4p_1)^{1/2}} \cdot \frac{(p_1p_2)^{1/2}}{(p_2+4p)^{1/2}} w^{1/2}\\ &= \frac{p_2w}{p_1(p_2+4p_1)}\\ &= x_1(p,w).\end{aligned}$$

Similarly, one can check that

$$-\frac{\partial v(p,w)/\partial p_2}{\partial v(p,w)/\partial w} = x_2(p,w).$$

Therefore, Roy's identity holds.

Additional.

3.D.6 Consider the three-good setting in which the consumer has utility function $u(x) = (x_1-b_1)^\alpha(x_2-b_2)^\beta(x_3-b_3)^\gamma$.

(a) Why can you assume that $\alpha+\beta+\gamma=1$ without loss of generality? Do so for the rest of the problem.

(b) Write down the first-order conditions for the UMP, and derive the consumer's Walrasian demand and indirect utility functions. This system of demands is known as the *linear expenditure system* and is due to Stone (1954).

(c) Verify that these demand functions satisfy the properties listed in Propositions 3.D.2 and 3.D.3.

Solution.

(a) Suppose $\alpha+\beta+\gamma \neq 1$ and $\alpha+\beta+\gamma > 0$, then we can define $\tilde{u}(x) = u(x)^{\frac{1}{\alpha+\beta+\gamma}} = (x_1-b_1)^{\frac{\alpha}{\alpha+\beta+\gamma}}(x_2-b_2)^{\frac{\beta}{\alpha+\beta+\gamma}}(x_3-b_3)^{\frac{\gamma}{\alpha+\beta+\gamma}}$ such that all exponents add up to 1. Since $\tilde{u}(x) = u(x)^{\frac{1}{\alpha+\beta+\gamma}}$ is an increasing transformation, $\tilde{u}(x)$ represents the same preference realtion as $u(x)$. Therefore, we may assume $\alpha+\beta+\gamma = 1$ without loss of generality.

(b) Consider the following utility maximization problem

$$\begin{aligned} &\max u(x) = (x_1-b_1)^{\alpha}(x_2-b_2)^{\beta}(x_3-b_3)^{\gamma}, \\ &\text{s.t. } p_1x_1 + p_2x_2 + p_3x_3 \leq w. \end{aligned}$$

The budget constraint should hold in equality at optimum since any amount of wealth left could have been spent to increase utility. Hence, we set up the Lagrangian as follows:

$$\mathcal{L}(x,\lambda) = (x_1-b_1)^{\alpha}(x_2-b_2)^{\beta}(x_3-b_3)^{\gamma} + \lambda[w - p_1x_1 - p_2x_2 - p_3x_3].$$

The first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \alpha(x_1-b_1)^{\alpha-1}(x_2-b_2)^{\beta}(x_3-b_3)^{\gamma} - \lambda p_1 = 0, \\ \frac{\partial \mathcal{L}}{\partial x_2} &= (x_1-b_1)^{\alpha}\beta(x_2-b_2)^{\beta-1}(x_3-b_3)^{\gamma} - \lambda p_2 = 0, \\ \frac{\partial \mathcal{L}}{\partial x_3} &= (x_1-b_1)^{\alpha}(x_2-b_2)^{\beta}\gamma(x_3-b_3)^{\gamma-1} - \lambda p_3 = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= w - p_1x_1 - p_2x_2 - p_3x_3 = 0. \end{aligned}$$

Solving the system of equaitons gives

$$\begin{aligned} x_1 &= \frac{\alpha}{p_1}(w - p_1b_1 - p_2b_2 - p_3b_3) + b_1, \\ x_2 &= \frac{\beta}{p_2}(w - p_1b_1 - p_2b_2 - p_3b_3) + b_2, \\ x_3 &= \frac{\gamma}{p_3}(w - p_1b_1 - p_2b_2 - p_3b_3) + b_3. \end{aligned}$$

The indirect utility function is given by

$$v(p,w) = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{\gamma}{p_3}\right)^{\gamma}(w - p_1b_1 - p_2b_2 - p_3b_3).$$

(c) **Proposition 3.D.2**

(i) It is easy to verify that $x_l(\theta p, \theta w) = x_l(p,w)$ for $l = 1,2,3$ and any $\theta > 0$. Hence, the Walrasian demand function satisfies **homogeneity of degree zero**.

(ii) To check **Walras' law**, calculate

$$p \cdot x = p_1[\frac{\alpha}{p_1}(w - p_1b_1 - p_2b_2 - p_3b_3) + b_1] + p_2[\frac{\beta}{p_2}(w - p_1b_1 - p_2b_2 - p_3b_3) + b_2] \\ + p_3[\frac{\gamma}{p_3}(w - p_1b_1 - p_2b_2 - p_3b_3) + b_3] = w.$$

(iii) The **uniqueness** is trivial.

Proposition 3.D.3

(i) By direct calculation, one can show that $v(\theta p, \theta w) = v(p,w), \forall \theta > 0$. Hence, the indirect utility function satisfies **homogeneity of degree zero**.

(ii) Since

$$\frac{\partial v(p,w)}{\partial w} = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{\gamma}{p_3}\right)^{\gamma} > 0,$$

$$\frac{\partial v(p,w)}{\partial p_1} = \alpha\left(\frac{\alpha}{p_1}\right)^{\alpha-1}\left(-\frac{\alpha}{p_1^2}\right)\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{\gamma}{p_3}\right)^{\gamma}(w - p_1b_1 - p_2b_2 - p_3b_3) \\ +(-b_1)\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{\gamma}{p_3}\right)^{\gamma} < 0,$$

and similarly, $\frac{\partial v(p,w)}{\partial p_2} < 0$ and $\frac{\partial v(p,w)}{\partial p_3} < 0$, the indirect utility function is **strictly increasing in w and strictly decreasing in p_l for $l = 1,2,3$.**

(iii) To prove **quasiconvexity**, it suffices to show that the set $\{p \in \mathbb{R}^3_{++} : v(p,w) \leq \bar{v}\}$ is convex for any $\bar{v} \in \mathbb{R}$ and $w > 0$ [4]. Consider $\ln(v(p,w)) = \alpha \ln \alpha + \beta \ln \beta + \gamma \ln \gamma + \ln(w - p_1b_1 - p_2b_2 - p_3b_3) - \ln(p_1) - \ln(p_2) - \ln(p_3)$, since $\ln(\cdot)$ is concave, $\ln(v(p,w))$ is convex in p, and hence the set $\{p : \ln(v(p,w)) \leq \ln(\bar{v})\}$ is convex. Therefore, the set $\{p : v(p,w) \leq \bar{v}\}$ is convex.

(iv) $v(p,w)$ is **continuous** in (p,w) by its given functional form.

[4] See a proof in Exercise 3.D.5.