Chapter 2. Consumer Choice

2.A. Introduction

In this chapter, we perform analysis of choice structure in the context of consumption. In other words, we analyze consumer demand for commodities.

2.B. Commodities

The decision problem faced by the consumer is to choose the consumption levels of various goods or services. We call the goods and services *commodities*. A *commodity vector* (or commodity *bundle*) is a point

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix} \in \mathbb{R}^L$$

- Number of commodities L, indexed by $l = 1, 2, ..., L$.
- $\mathbb{R}^L$ is the commodity *space*.
- x_l is the amount of commodity l consumed.

Remark. Time (see the example below) and location (see Figure 3), could be built into the definition of a commodity.

For example, x_1 could be bread today, and x_2 could be bread tomorrow. (In this example, we ignore other commodities.) Alice who plans to consume 5 slices of bread today and 6 slices of bread tomorrow would have a commodity vector

$$x = \begin{bmatrix} x_1 = 5 \\ x_2 = 6 \end{bmatrix} \in \mathbb{R}^2.$$

2.C. Consumption Set

The *consumption set* is a subset of the commodity space $\mathbb{R}^L$, denoted by $X \subset \mathbb{R}^L$, whose elements are the consumption bundles that the individual can conceivably consume given the physical and institutional constraints imposed by his environment.

Below are some examples of 2 commodities, i.e., $L = 2$, with *Physical* Constraints:

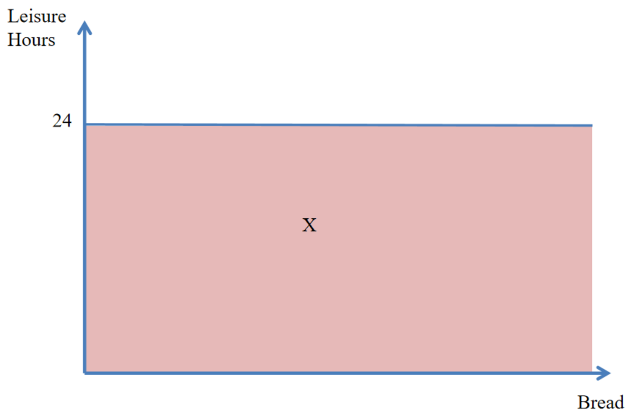

Figure 1: Possible consumption levels of bread and leisure in a day

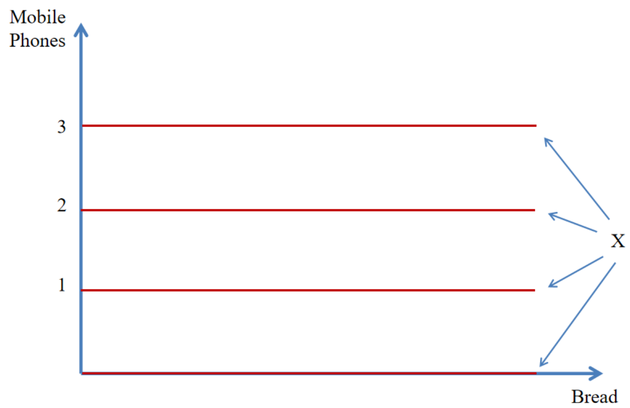

Figure 2: Possible consumption levels of bread and mobile phones

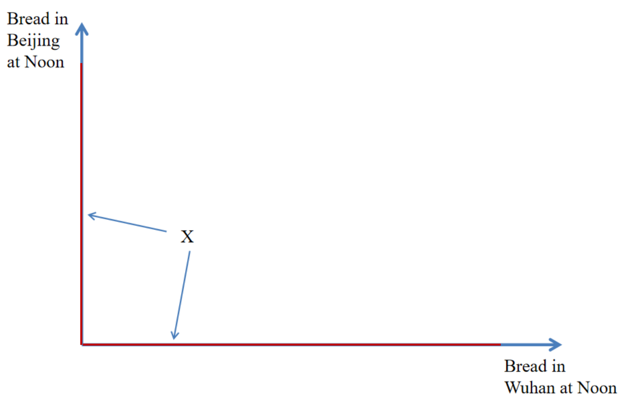

Figure 3: Possible consumption levels of bread in Beijing and Wuhan at noon

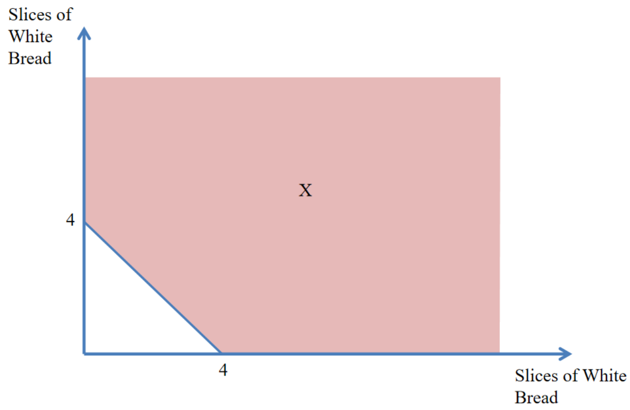

Figure 4: Possible consumption levels of bread where the minimum survival amount is 4 slices and only 2 types of bread are available

There could also be *Institutional* Constraints.

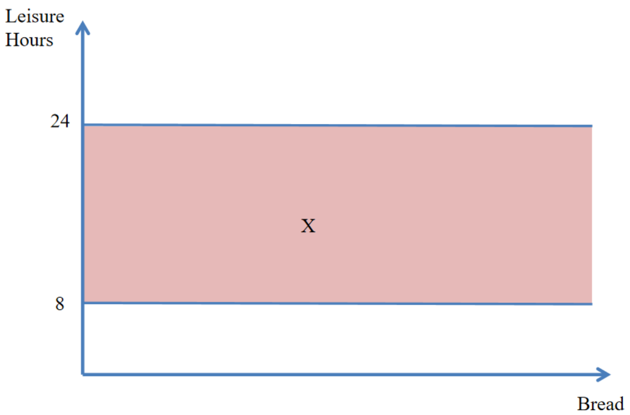

Figure 5: Possible consumption levels of bread and leisure in a day with a law requiring that no one work more than 16 hours a day

Practically, to keep our discussion in this section as straightforward as possible, we adopt the simplest consumption set:

$$X = \mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x_l \geq 0 \text{ for } l = 1, 2, ..., L\}.$$

Below is an illustration of the consumption set $\mathbb{R}^L_+$ in 2 dimensions, i.e., $\mathbb{R}^2_+$.

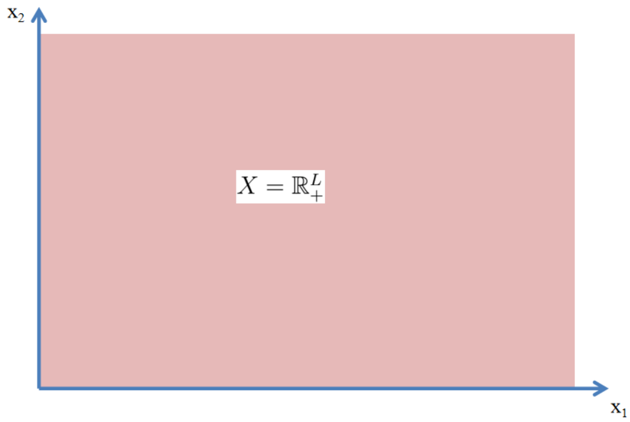

Figure 6: The consumption set $\mathbb{R}^L_+$

Remark. X is convex: $x \in X$, $x' \in X \implies \alpha x + (1-\alpha)x' \in X$.

Proof. Given any two commodities $l, k = 1, ..., L$.

$x_l \geq 0, x_k \geq 0 \implies \alpha x_l + (1-\alpha)x_k \geq 0$ □

Much of the theory to be developed applies also for more general convex consumption sets (for example, the consumption sets illustrated in Figures 1, 4, 5).[1]

2.D. Competitive Budgets (Affordability)

In addition to the physical and institutional constraints, the consumer also faces *economic* constraint: affordability.

To formalize the economics constraint, we assume that L commodities are all traded at public dollar prices and that consumers are *price takers*. Formally, prices are represented by the *price vector*:

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L$$

Assumption. $p \gg 0$, *i.e.*, $p_l > 0, \forall l$.

Throughout the course, we make the above assumption, even though the assumption may not be reasonable. There exist scenarios in real life that $p_l = 0$, or even $p_l < 0$. We provide two counter examples below.

[1] You should check by yourselves that the consumption sets in Figures 1, 4, 5 are convex.

Counter Examples.

1. Someone invites you: for you, $p_l = 0$.

2. Sometimes parents pay kid to read books: for the kid, $p_l < 0$.

Economic-Affordability Constraint The affordability of a consumption bundle depends on

1. market prices: $p = (p_1, \cdots, p_L)$

2. consumer's wealth level (in dollars): w

The consumption bundle $x \in \mathbb{R}^L_+$ is affordable if

$$p \cdot x = p_1 x_1 + ... + p_L x_L \leq w.$$

Walrasian budget set

Definition 2.D.1. The Walrasian, or competitive budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ is the set of all feasible consumption bundles for the consumer who faces market prices p and has wealth w.

The consumer's problem is to choose *consumption bundle* x from $B_{p,w}$.

The set $\{x \in \mathbb{R}^L_+ : p \cdot x = w\}$ is called the *budget hyperplane*.

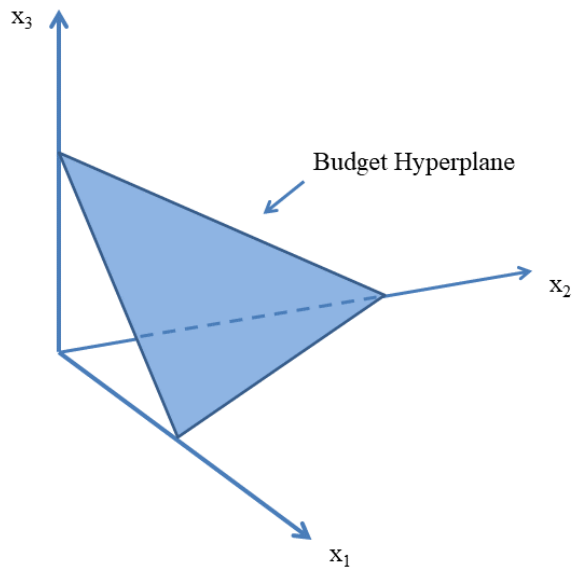

Figure 7: Budget Hyperplane (3 commodities)

When $L = 2$, Budget Hyperplane is Budget Line. The slope $-\frac{p_1}{p_2}$ captures the rate of exchange between the two commodities.

- $\frac{p_1}{p_2}$ describes the units of x_2 the consumer can obtain by giving up one unit of x_1: one unit of $x_1 \implies p_1$ of money $\implies \frac{p_1}{p_2}$ units of x_2

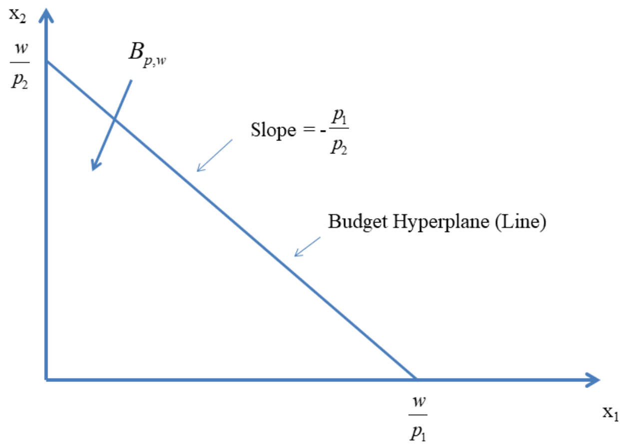

Figure 8: Budget hyperplane (line) for two commodities

The price vector p, drawn from any point $\overline{x}$ on the budget hyperplane, must be orthogonal to any vector starting at $\overline{x}$ and lying on the budget hyperplane.

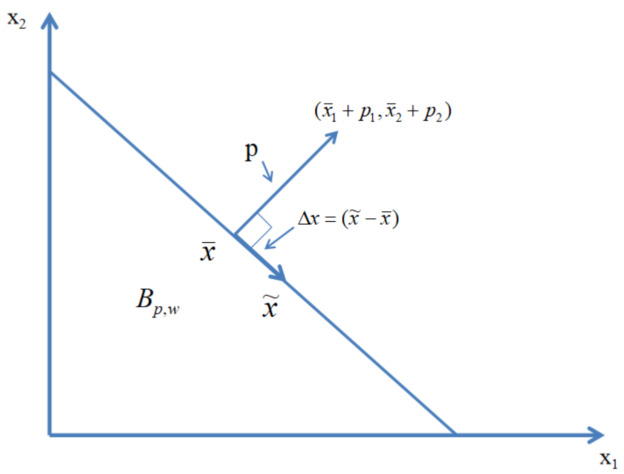

Figure 9: The geometric relationship between p and the budget hyperplane

To check the orthogonality, we need to check whether $p \cdot \triangle x = 0$, where $\triangle x = \widetilde{x} - \overline{x}$ and $\widetilde{x}$, $\overline{x}$ are on the budget hyperplane. This is true because $p \cdot \widetilde{x} = p \cdot \overline{x} = w$.

Walrasian budget set $B_{p,w}$ is convex.

Proof. We need to show that for all $x, x' \in B_{p,w}$, $x'' = \alpha x + (1-\alpha)x' \in B_{p,w}$.

First, $x, x' \in \mathbb{R}^L_+ \implies x'' \in \mathbb{R}^L_+$. Second, since $p \cdot x \leq w$ and $p \cdot x' \leq w$, we have $p \cdot x'' = p \cdot [\alpha x + (1-\alpha)x'] = \alpha(p \cdot x) + (1-\alpha)(p \cdot x') \leq w$.

Thus, $x'' \in B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$. □

Exercise 2.D.1

A consumer lives for two periods, denoted 1 and 2, and consumes a single consumption good in each period. His wealth when born is $w > 0$. What is his (lifetime) Walrasian budget set?

Exercise 2.D.2

A consumer consumes one consumption good x and hours of leisure h. The price of the consumption good is p, and the consumer can work at a wage rate of $s = 1$. What is the consumer's Walrasian budget set?

Remark. The convexity of $B_{p,w}$ depends on the convexity of the consumption set. $B_{p,w}$ will be convex as long as X is.

Proof. We need to show that for all $x, x' \in B_{p,w}$, $x'' = \alpha x + (1-\alpha)x' \in B_{p,w}$.

First, $x, x' \in X \implies x'' \in X$ since X is convex. Second, since $p \cdot x \leq w$ and $p \cdot x' \leq w$, we have $p \cdot x'' = p \cdot \alpha x + (1-\alpha)x' = \alpha(p \cdot x) + (1-\alpha)(p \cdot x') \leq w$.

Thus, $x'' \in B_{p,w} = \{x \in X : p \cdot x \leq w\}$. □

2.E. Demand Functions and Comparative Statics

The consumer's *Walrasian* (or *market*, or *ordinary*) *demand correspondence* $x(p, w)$ assigns a set of chosen consumption bundles for each (p, w).

When $x(p, w)$ is single-valued, we refer to it as a *demand function*.

Assumption.

1. *$x(p, w)$ is homogeneous of degree zero.*

2. *$x(p, w)$ satisfies Walras' law.*

General definition of Homogeneous Functions:

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is Homogeneous of Degree k if for any $\alpha > 0$,

$$f(\alpha x_1, \alpha x_2, ..., \alpha x_n) = \alpha^k f(x_1, x_2, ..., x_n).$$

Example.

1. $f(x, y) = xy$ is Homogeneous of Degree 2.
2. $f(x, y, z) = \frac{x}{y} + \frac{2z}{x}$ is Homogeneous of Degree 0.
3. $f(x_1, x_2) = Ax_1^a x_2^b$ is Homogeneous of Degree $a + b$.
4. $f(x_1, x_2) = x_1 + x_2^2$ is not a Homogeneous Function.

Homogeneity in Example 1 to 3 could be easily checked. For Example 4, we provide a proof.

Proof. We prove by contradiction.

Suppose $f(x_1, x_2) = x_1 + x_2^2$ is Homogeneous of Degree k. We must have

$$\begin{aligned} & f(\alpha x_1, \alpha x_2) = \alpha^k f(x_1, x_2) \\ \Longrightarrow\ & \alpha x_1 + (\alpha x_2)^2 = \alpha^k (x_1 + x_2^2) \quad \forall \alpha > 0,\ x_1, x_2 \in \mathbb{R} \end{aligned}$$

In particular, taking $\alpha = 2$ gives

$$2x_1 + 4x_2^2 = 2^k x_1 + 2^k x_2.$$

Then, for $(x_1, x_2) = (1, 0)$ and $(x_1, x_2) = (0, 1)$, we have $k = 1$ and $k = 2$ respectively, which constitutes a contradiction.

Definition 2.E.1. The Walrasian demand correspondence $x(p, w)$ is homogeneous of degree zero (H.D.$\emptyset$) if $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and $\alpha > 0$.

Remark. A change from (p, w) to $(\alpha p, \alpha w)$ does not change the consumer's set of feasible consumption bundles, i.e., $B_{p,w} = B_{\alpha p, \alpha w}$. H.D.$\emptyset$ means that individual's choice depends only on the set of feasible points.

Remark. Implication of H.D.∅: it is without loss to *normalize* the level of one of the $L+1$ independent variables at an arbitrary level. One common normalization is $p_l = 1$ for some L. Another is $w = 1$.

Definition 2.E.2. The Walrasian demand correspondence $x(p,w)$ satisfies Walras' law if for every $p \gg 0$ and $w > 0$, we have $p \cdot x = w$ for all $x \in x(p,w)$.

Remark. Walras' law says that the consumer fully expends his wealth.

Question: Is Walras' law reasonable?

Answer: It's more reasonable if w refers the life-time income and x refers to life-time demands. Even then, it's still controversial.

Exercise 2.E.1

Suppose $L = 3$, and consider the demand function $x(p,w)$ defined by

$$
\begin{aligned}
x_1(p,w) &= \frac{p_2}{p_1+p_2+p_3}\frac{w}{p_1} \\
x_2(p,w) &= \frac{p_3}{p_1+p_2+p_3}\frac{w}{p_2} \\
x_3(p,w) &= \frac{\beta p_1}{p_1+p_2+p_3}\frac{w}{p_3}
\end{aligned}
$$

Does this demand function satisfy homogeneity of degree zero and Walras' law when $\beta = 1$? What about when $\beta \in (0,1)$?

For the remainder of the section, we assume that $x(p,w)$ is single-valued, continuous, and differentiable.

$\mathbf{x(p,w)}$ and Choice-base Approach (in Chapter 1) Recall that a choice structure $(\mathscr{B}, C(\cdot))$ consists of two ingredients:

(i) $\mathscr{B}$ is a family of nonempty subsets of X. Every $B \in \mathscr{B}$ is a budget set.

(ii) $C(\cdot)$ is a choice rule. It maps every set $B \in \mathscr{B}$ to a nonempty set $C(B) \subset B$.

The family of Walrasian budget sets is

$$\mathscr{B}^{\mathscr{W}} = \{B_{p,w} : p \gg 0, w > 0\}.$$

Remark. $\mathscr{B}^{\mathscr{W}}$ does not include all possible subsets of X.

Since the price-wealth pair (p, w) determines the Walrasian budget set $B_{p,w}$ faced by consumer, we have

$$C(B_{p,w}) = x(p, w).$$

Hence, $(\mathscr{B}^{\mathscr{W}}, x(p, w))$ is a choice structure.

2.E.1. Comparative statics (with respect to p and w)

The examination of a change in outcome in response to a change in underlying economic parameters is known as *comparative statics* analysis.

This section examines how the consumer's choice would vary with changes in his wealth and in prices.

Wealth Effects For fixed prices $\overline{p}$, $x(\overline{p}, w)$ is called the consumer's *Eagel function*. Its image in $\mathbb{R}^L_+$, $E_{\overline{p}} = \{x(\overline{p}, w) : w > 0\}$ is the *wealth expansion path*.

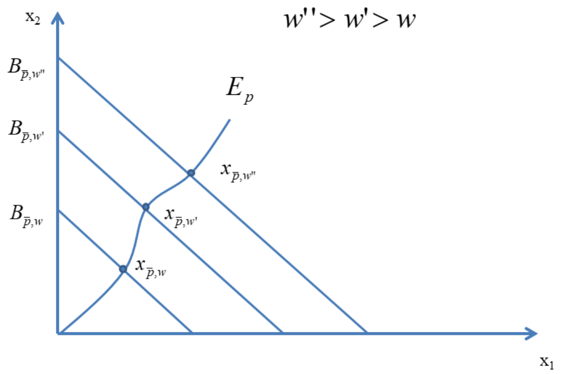

Figure 10: Wealth expansion path at $\overline{p}$

The derivative $\frac{\partial x_l(p,w)}{\partial w}$ is the *wealth effect* for the l^{th} good.

- A commodity l is *normal* at (p, w) if $\frac{\partial x_l(p,w)}{\partial w} \geq 0$.
- A commodity l is *inferior* at (p, w) if $\frac{\partial x_l(p,w)}{\partial w} < 0$.

In matrix notation, the wealth effects are $D_w x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L$.

Price Effects The demand function for good l could be represented as a function of p_l, keeping other things equal, i.e., $x(p_l, \overline{p}_{-l}, \overline{w})$.

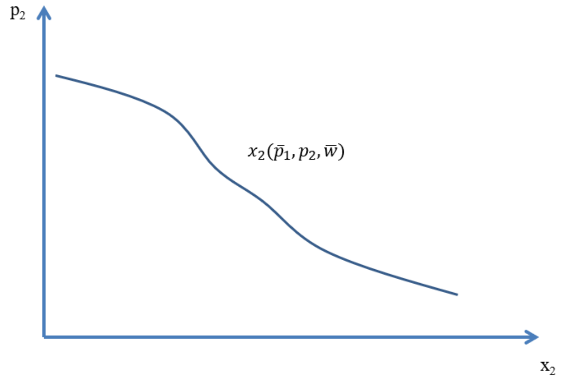

Figure 11: Demand for good 2 as a function of its price

Another useful representation of the consumers' demand at different prices p_l is the locus of points demanded in $\mathbb{R}^L_+$, for fixed p_{-l} and w. This is known as an *offer curve.*

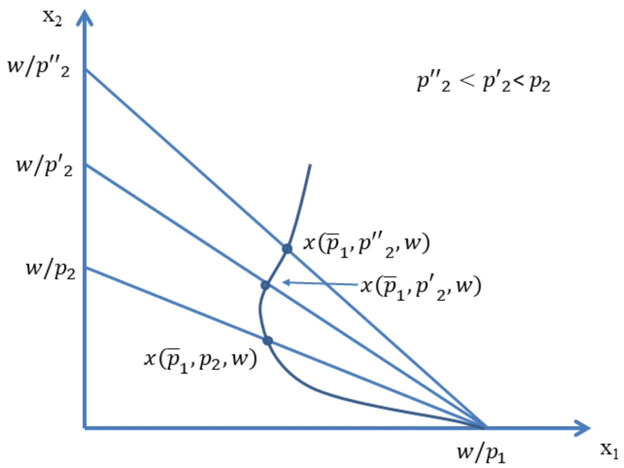

Figure 12: Offer Curve

The derivative $\frac{\partial x_l(p,w)}{\partial p_k}$ is the *price effect* of p_k on the demand for good l.

- Good l is a *Giffen good* if $\frac{\partial x_l(p,w)}{\partial p_l} > 0$. (Example: potatoes at low wealth level)

In matrix notation, the price effects are $D_p x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} \\ & \ddots & \\ \frac{\partial x_L(p,w)}{\partial p_1} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}$.

2.E.2. Implications of homogeneity and Walras' law for price and wealth effects

Implication of H.D.∅

Proposition 2.E.1. *If the Walrasian demand function $x(p, w)$ is homogeneous of degree zero, then for all p and w:*

$$\sum_{k=1}^{L} \frac{\partial x_l(p, w)}{\partial p_k} p_k + \frac{\partial x_l(p, w)}{\partial w} w = 0, \text{ for } l = 1, ..., L. \tag{2.E.1}$$

In matrix notation, it is expressed as

$$D_p x(p, w)p + D_w x(p, w)w = 0. \tag{2.E.2}$$

Proof.

$$x(p, w) \text{ is } H.D.\emptyset \implies x_l(\alpha p, \alpha w) = x_l(p, w), \text{ for } l = 1, ..., L$$

Differentiating both sides of the equation with respect to α gives

$$\frac{\partial x_l(\alpha p, \alpha w)}{\partial \alpha} = 0 \implies \sum_{k=1}^{L} \frac{\partial x_l(\alpha p, \alpha w)}{\partial (\alpha p_k)} p_k + \frac{\partial x_l(\alpha p, \alpha w)}{\partial (\alpha w)} w = 0.$$

Setting $\alpha = 1$ implies the result. □

Dividing the expression by x_l:

$$\sum_{k=1}^{L} \frac{\partial x_l(p, w)}{\partial p_k} \frac{p_k}{x_l(p, w)} + \frac{\partial x_l(p, w)}{\partial w} \frac{w}{x_l(p, w)} = 0, \text{ for } l = 1, ..., L.$$

i.e.,

$$\sum_{k=1}^{L} \varepsilon_{lk}(p, w) + \varepsilon_{lw}(p, w) = 0, \text{ for } l = 1, ..., L. \tag{2.E.3}$$

Note: $\varepsilon_{lk}(p, w) = \frac{\partial x_l(p,w)/x_l(p,w)}{\partial p_k/p_k}$ indicates % change in x_l given % change in p_k. Similarly, $\varepsilon_{lw}(p, w) = \frac{\partial x_l(p,w)/x_l(p,w)}{\partial w/w}$ indicates % change in x_l given % change in w.

Intuition: The above equation describes the precentage change in x_l if all prices and wealth changes 1%. Basically, the equation captures the definition of H.D.∅.

TWO implications of Walras' Law $(p \cdot x(p,w) = w)$

Proposition 2.E.2 (Cournot Aggregation). *If the Walrasian demand function $x(p,w)$ satisfies the Walras' Law, then for all p and w :*

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0, \ \textit{for } k = 1, 2, ..., L, \tag{2.E.4}$$

or written in matrix notation,

$$p \cdot D_p x(p,w) + x(p,w)^T = 0^T. \tag{2.E.5}$$

Proof.

$$p \cdot x(p,w) = w \implies \frac{\partial}{\partial p_k}(p \cdot x(p,w)) = 0 \implies p \cdot \frac{\partial x(p,w)}{\partial p_k} + x_k(p,w) = 0$$
$$\implies \sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0. \quad \square$$

Intuition: Total expenditure cannot change in response to a change in prices.

Proposition 2.E.3 (Eagel Aggregation). *If the Walrasian demand function $x(p,w)$ satisfies Walras' Law, then for ALL p and w:*

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1, \tag{2.E.6}$$

or, written in matrix natation,

$$p \cdot D_w x(p,w) = 1. \tag{2.E.7}$$

Proof.

$$p \cdot x(p,w) = w \implies \frac{\partial}{\partial w}(p \cdot x(p,w)) = 1 \implies p \cdot \frac{\partial x(p,w)}{\partial w} = 1 \implies \sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1. \quad \square$$

Intuition: Total expenditure must change by an amount equal to any wealth change.

Exercise 2.E.3

Use Proposition 2.E.1 to 2.E.3 to show that $p \cdot D_p x\,(p,w)\,p = -w$.

Exercise 2.E.5

Suppose that $x(p, w)$ is a demand function which is homogeneous of degree one with respect to w and satisfies Walras' law and homogeneity of degree zero. Suppose also that all the cross-price effects are zero, that is $\partial x_l(p, w)/\partial p_k = 0$ whenever $k \neq l$. Show that this implies that for every l, $x_l(p, w) = \alpha_l w/p_l$, where $\alpha_l > 0$ is a constant independent of (p, w).

Exercise 2.E.7

A consumer in a two-good economy has a demand function $x(p, w)$ that satisfies Walras' law. His demand function for the first good is $x_1(p, w) = \alpha w/p_1$. Derive his demand function for the second good. Is his demand function homogeneous of degree zero?

Exercise 2.E.8

Show that the elasticity of demand for good l with respect to price p_k, $\varepsilon_{lk}(p, w)$, can be written as $\varepsilon_{lk}(p, w) = d\ln(x_l(p, w))/d\ln(p_k)$, where $\ln(\cdot)$ is the natural logarithm function. Derive a similar expression for $\varepsilon_{lw}(p, w)$. Conclude that if we estimate the parameters $(\alpha_0, \alpha_1, \alpha_2, \gamma)$ of the equation $\ln(x_l(p, w)) = \alpha_0 + \alpha_1 \ln p_1 + \alpha_2 \ln p_2 + \gamma \ln w$, these parameter estimates provide us with estimates of the elasticities $\varepsilon_{l1}(p, w)$, $\varepsilon_{l2}(p, w)$, and $\varepsilon_{lw}(p, w)$.

2.F. Weak Axiom of Revealed Preference and Law of Demand

Implicit assumptions: $x(p, w)$ is single-valued, H.D.$\emptyset$, and satisfies Walras' Law.

Definition 2.F.1. The Walrasian demand function $x(p, w)$ satisfies the weak axiom of revealed preference (W.A.R.P) if the following holds for any two price-wealth situations (p, w) and (p', w'):

If $p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w)$,[2] then $p' \cdot x(p, w) > w'$.

Definition stated using language in Chapter 1 Let $B_{p,w}$ denote the budget set given p and w; and $B_{p',w'}$ denote the budget set given p' and w'. $p \cdot x(p', w') \leq w$ means that

[2]Note that $x(p, w)$ is the demand given (p, w) and $x(p', w')$ is the demand given (p', w').

$x(p', w')$ is also affordable under $B_{p,w}$. Through the choice given $B_{p,w}$, $x(p, w)$ is revealed preferred to $x(p', w')$. Therefore, by W.A.R.P, it must not be revealed that $x(p', w')$ is preferred to $x(p, w)$. In other words, if $x(p, w)$ is not chosen given the budget $B_{p',w'}$, it must be that it is not affordable, i.e., $p' \cdot x(p, w) > w'$, or $x(p, w) \notin B_{p',w'}$.

The below figure illustrates an example of demand function $x(p, w)$ that satisfies W.A.R.P.

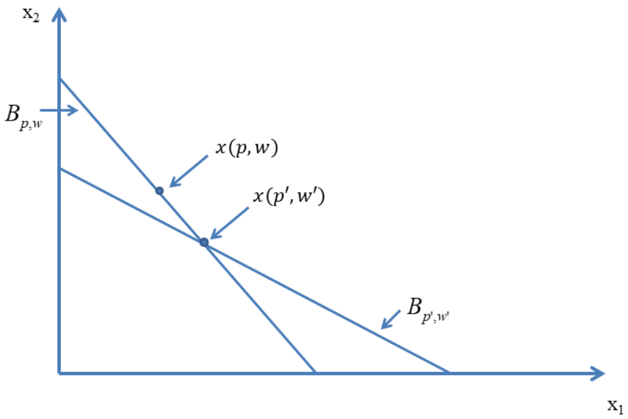

Figure 13: Demand satisfying W.A.R.P

Violation of W.A.R.P W.A.R.P may be violated only if both $x(p, w)$ and $x(p', w')$ belong to both $B_{p,w}$ and $B_{p',w'}$.

The below figure illustrates an example of demand function $x(p, w)$ that violates W.A.R.P.

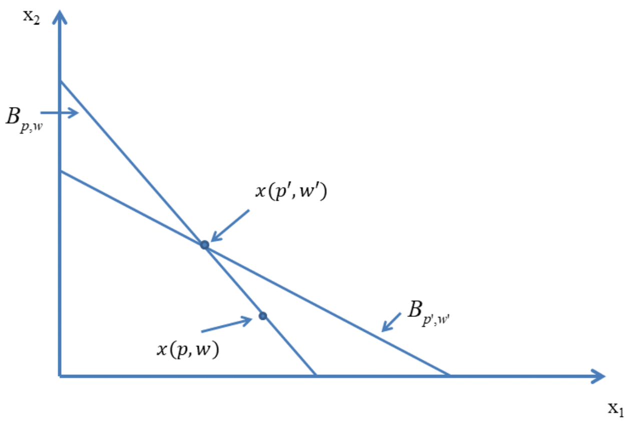

Figure 14: Demand violating W.A.R.P

Implications of W.A.R.P

Uncompensated price change: p_1 to p_1' An uncompensated price change is a change in price without a corresponding change in wealth. Such a price change would affect the consumer in two ways:

- change the relative cost of commodities;
- change the consumer's real wealth.

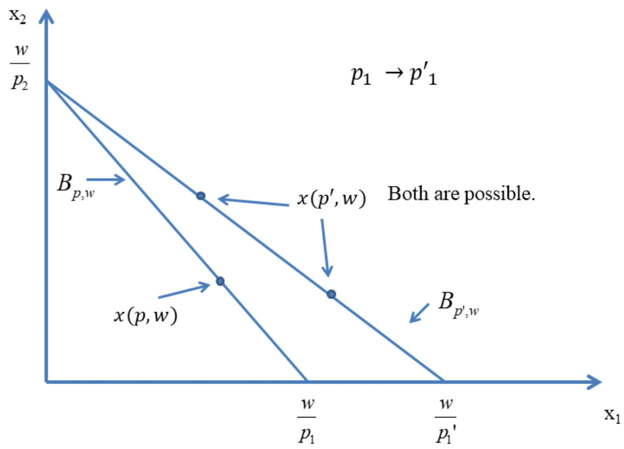

Figure 15: Uncompensated price change

No prediction on change in demand can be drawn.

Compensated price change Imagine a situation in which a change in prices is accompanied by a change in the consumer's wealth that makes her initial consumption bundle just affordable at the new prices. That is, $w' = p' \cdot x(p, w)$. The wealth adjustment is $\Delta w = \Delta p \cdot x(p, w)$. This kind of wealth adjustment is called *Slutsky wealth compensation*. The price changes that are accompanied by compensating wealth changes are called *(Slutsky) compensated price changes.*

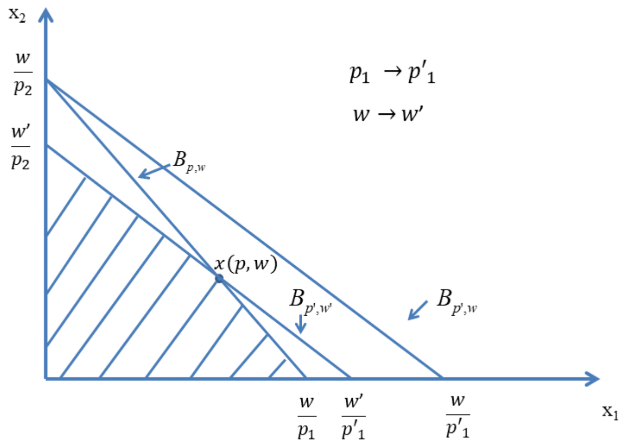

Figure 16: Compensated price change

The shaded area is revealed not as good as $x(p, w)$. So, the bundles in the area won't be picked after price change. *This implies x_1 must increase after compensated price change.* Another perspective is that $x(p', w') \succsim x(p, w)$, so it cannot be $x(p, w) \succ x(p', w')$.

How to check whether W.A.R.P. is Satisfied. It would be easier to check whether W.A.R.P is satisfied for compensated price changes.

Question. What does it mean that W.A.R.P is satisfied for all compensated price changes?

Recall,

1. W.A.R.P: If $p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w)$, then $p' \cdot x(p, w) > w'$.

2. Compensated price change means $p' \cdot x(p, w) = w'$.

W.A.R.P is satisfied for all compensated price change means: For any price change from (p, w) to (p', w') such that $p' \cdot x(p, w) = w'$, if $x(p', w') \neq x(p, w)$, then $p \cdot x(p', w') > w$. [New bundle chosen after compensated price change is unaffordable under original price and wealth.]

W.A.R.P is violated for some compensated price change means: There exists a price change from (p, w) to (p', w') such that $p' \cdot x(p, w) = w', x(p', w') \neq x(p, w)$ and

$p \cdot x(p', w') \leq w$.

Figure 17 below depicts an example compensated price change. Changes in demand to $x(p', w')$ in the red area constitutes violation of W.A.R.P.

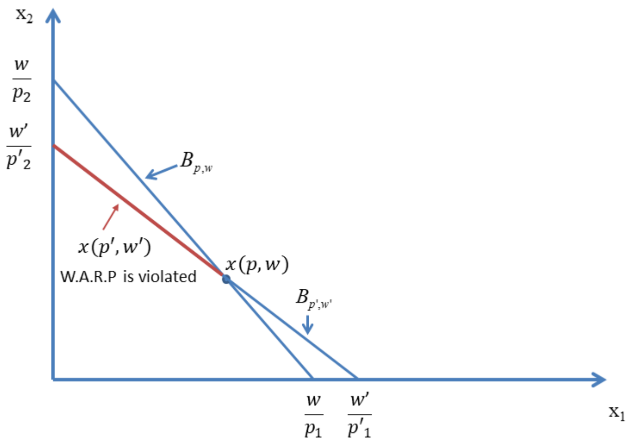

Figure 17: Compensated price change

Next, we present a useful lemma which makes it easier to check whether a demand function satisfies W.A.R.P.

Lemma 1. *W.A.R.P holds for all price changes if and only if it holds for all compensated price changes.*

Proof. *Necessary* (only if) part is obvious.

Sufficiency (if): Suppose that W.A.R.P is violated for some price change. We'll show that it must also be violated for some compensated price change.

Suppose that W.A.R.P is violated for the two price-wealth pairs (p', w') and (p'', w''). Then, we must have $x(p', w') \neq x(p'', w'')$, $p' \cdot x(p'', w'') \leq w'$ and $p'' \cdot x(p', w') \leq w''$.

If one of the weak inequalities holds in equality, then either the change from (p', w') to (p'', w'') or the change from (p'', w'') to (p', w') is a compensated price change.

Therefore, we restrict attention to the case of $p' \cdot x(p'', w'') < w' = p' \cdot x(p', w')$ and $p'' \cdot x(p', w') < w'' = p'' \cdot x(p'', w'')$, as shown in the following figure:

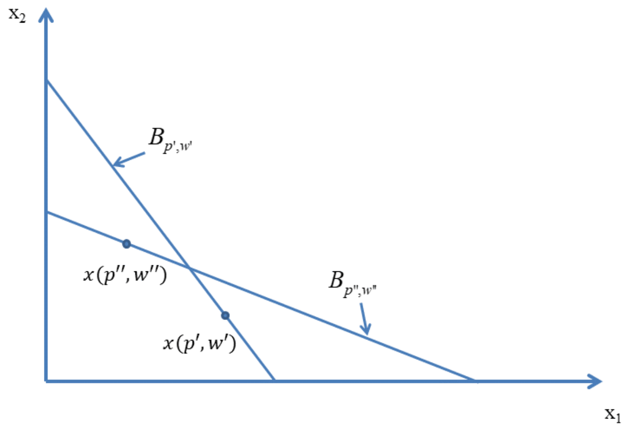

Figure 18: Uncompensated price change

Note that there exists $\alpha \in (0,1)$ such that

$$(\alpha p' + (1-\alpha)p'') \cdot x(p', w') = (\alpha p' + (1-\alpha)p'') \cdot x(p'', w'').$$ [3]

Consider a new budget $B_{p,w}$ with $p = \alpha p' + (1-\alpha)p''$ and $w = (\alpha p' + (1-\alpha)p'') \cdot x(p', w')$. This construction is illustrated in the following figure:

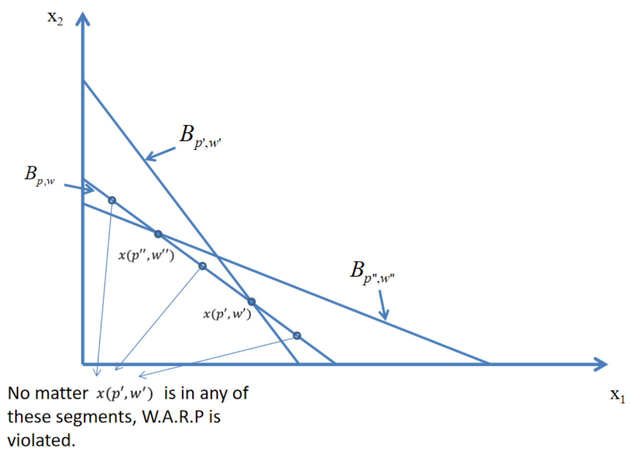

Figure 19: Construction of a compensated price change

The respective price changes from (p', w') to (p, w) and from (p'', w'') to (p, w) are compensated. From Figure 19, we see that wherever $x(p, w)$ is located on the budget curve of $B_{p,w}$, it is affordable under budget $B_{p',w'}$ or $B_{p'',w''}$, so W.A.R.P is violated for the compensated price change.

[3] You can verify this by showing that when $\alpha = 0$, LHS < RHS and when $\alpha = 1$, LHS > RHS.

Now, we formally prove that $w' > p' \cdot x(p,w)$ or $w'' > p'' \cdot x(p,w)$ must hold.

Suppose $w' \leq p' \cdot x(p,w)$ and $w'' \leq p'' \cdot x(p,w)$. Then

$$\begin{aligned}
\alpha w' + (1-\alpha)w'' &\leq \alpha p' \cdot x(p,w) + (1-\alpha)p'' \cdot x(p,w) \\
&= [\alpha p' + (1-\alpha)p''] \cdot x(p,w) \\
&= w \\
&= [\alpha p' + (1-\alpha)p''] \cdot x(p',w') \\
&= \alpha p' \cdot x(p',w') + (1-\alpha)p'' \cdot x(p',w') \\
\implies w'' &\leq p'' \cdot x(p',w'),
\end{aligned}$$

which constitutes a contradiction with the initial supposition $p'' \cdot x(p',w') < w''$.[4]

Therefore, W.A.R.P is violated for some compensated price change. □

In Proposition 2.F.1 below, we show that the weak axiom can be equivalently stated in terms of the demand response to compensated price changes.

Proposition 2.F.1. *Suppose that the Walrasian demand function $x(p,w)$ is homogeneous of degree zero and satisfies Walras' Law, Then $x(p,w)$ satisfies W.A.R.P if and only if the following property holds:*

For ANY compensated price change from an initial situation (p,w) to a new price-wealth pair $(p',w') = (p', p \cdot x(p,w))$, we have

$$(p'-p) \cdot [x(p',w') - x(p,w)] \leq 0, \tag{2.F.1}$$

with strict inequality whenever $x(p,w) \neq x(p',w')$.

Before proving the result, let's rewrite (2.F.1).

$$(p'-p) \cdot [x(p',w') - x(p,w)] = p' \cdot x(p',w') - p' \cdot x(p,w) - p \cdot [x(p',w') - x(p,w)].$$

Note that $p' \cdot x(p',w') - p' \cdot x(p,w) = 0$ because we consider compensated price changes.

[4] Alternative proof: Since $w' = p' \cdot x(p',w')$ and $w'' > p'' \cdot x(p',w')$, we have $\alpha w' + (1-\alpha)w'' > \alpha p' \cdot x(p',w') + (1-\alpha)p'' \cdot x(p',w') = p \cdot x(p',w') = w = p \cdot x(p,w) = \alpha p' \cdot x(p,w) + (1-\alpha)p'' \cdot x(p,w)$. Therefore, one of the following must hold: $p' \cdot x(p,w) < w'$ or $p'' \cdot x(p,w) < w''$.

Therefore, (2.F.1) is equivalent to

$$p \cdot [x(p', w') - x(p, w)] \geq 0 \quad (> 0 \text{ if } x(p', w') \neq x(p, w)). \tag{*}$$

Below, we provide a formal proof of Proposition 2.F.1.

Proof.

(i) *W.A.R.P implies Equation (*).*

If $x(p, w) = x(p', w')$, LHS of (*) is 0 and the inequality holds obviously.

Suppose $x(p, w) \neq x(p', w')$. Since $p' \cdot x(p, w) = w'$, $x(p, w)$ is affordable under (p', w'), yet it is not chosen. W.A.R.P implies that $x(p', w')$ is not affordable under (p, w), i.e., $p \cdot x(p', w') > p \cdot x(p, w)$. This is (*).

(ii) *If Equation (*) holds for all compensated price changes, then W.A.R.P holds for all compensated price changes.* (And by means of Lemma 1, W.A.R.P also holds for all uncompensated price changes.)

Equivalently, we prove that if W.A.R.P is violated for some compensated price changes, then (*) is also violated.

Suppose W.A.R.P is violated for some compensated price changes, then there exists a compensated price change from (p, w) to (p', w'), $p' \cdot x(p, w) = w'$, such that $x(p', w') \neq x(p, w)$ and $p \cdot x(p', w') \leq w = p \cdot x(p, w)$. Then, $p \cdot [x(p', w') - x(p, w)] \leq 0$, and (*) is violated. □

Remark. The inequality (2.F.1) can be interpreted as a form of the Law of Demand: Demand and price move in opposite directions. Since it only holds for compensated price changes, it is called the *Compensated Law of Demand.*

- As illustrated in Figure 15, W.A.R.P does not generate definitive prediction on the demand changes in response to *uncompensated* price changes.

Weak Axiom and Differentiable Demand Consider a differentiable change in price dp, compensated by the change in wealth

$$dw = x(p, w) \cdot dp.$$

Proposition 2.F.1 implies

$$dp \cdot dx \leq 0. \tag{2.F.5}$$

By chain rule, the differential change in demand induced by this compensated price change is

$$\begin{aligned}
& dx = D_p x(p,w)dp + D_w x(p,w)dw \\
\implies & dx = D_p x(p,w)dp + D_w x(p,w)dw \\
\implies & dx = D_p x(p,w)dp + D_w x(p,w)[x(p,w) \cdot dp] \\
\implies & dx = \left[D_p x(p,w) + D_w x(p,w) x(p,w)^T\right] dp \qquad (2.F.8)
\end{aligned}$$

Define

$$S(p,w) = D_p x(p,w) + D_w x(p,w) x(p,w)^T$$

as the *substitution matrix* or *Slutsky matrix*. In matrix notation, it is

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ & \ddots & \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix},$$

where the $(l,k)^{th}$ entry is

$$s_{l,k}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w).$$[5]

$s_{l,k}(p,w)$ are known as *substitution effects*.

Implications of the substitution effects $s_{l,k}(p,w)$ is the change in demand for good l given a change in p_k and a compensating change in w.

Wealth Change:

$$\begin{aligned}
& w = \sum_{l=1}^{L} x_l(p,w)p_l \ \& \ w' = \sum_{l \neq k} x_l(p,w)p_l + x_k p_k' \\
\implies & w' - w = (p_k' - p_k) x_k(p,w) \\
\implies & \Delta w = \Delta p_k x_k(p,w)
\end{aligned}$$

[5] $S_{lk}(p,w)$ is not directly observable, but can be inferred if we can estimate $x(p,w)$.

Suppose $\Delta p_k = 1$, then $\Delta w = x_k(p,w)$. The effect of the change of $\Delta p_k = 1$ and the compensating change in wealth gives $s_{lk}(p,w)$:

$$\begin{aligned}
&\underbrace{\frac{\partial x_l(p,w)}{\partial p_k}\Delta p_k}_{\text{price effect due to } \Delta p_k} + \underbrace{\frac{\partial x_l(p,w)}{\partial w}\Delta w}_{\substack{\text{wealth effect due to} \\ \text{compensating change in wealth}}} \\
=&\frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w}x_k(p,w) \\
=&s_{lk}(p,w).
\end{aligned}$$

Negative semidefiniteness of Slutsky matrix (2.F.8) and (2.F.5) gives

$$dp^T S(p,w)dp \le 0, \forall p.$$

The result is summarized in Proposition 2.F.2 below.

Proposition 2.F.2. *If a differentiable Walrasian demand function $x(p,w)$ satisfies Walras' Law, homogeneous of degree zero, and W.A.R.P, then at any (p,w), the Slutsky matrix $S(p,w)$ satisfies $v \cdot S(p,w)v \le 0$ for any $v \in \mathbb{R}^L$. i.e. $S(p,w)$ is negative semidefinite.*

Remark. Proposition 2.F.2 does not imply, in general, that the matrix $S(p,w)$ is symmetric.

- For $L = 2$, $S(p,w)$ is necessarily symmetric. (Exercise 2.F.11)

Exercise 2.F.11

Show that for $L = 2$, $S(p,w)$ is always symmetric. [Hint: Use Proposition 2.F.3.]

- When $L > 2$, $S(p,w)$ is not necessarily symmetric, under the assumptions so far (H.D.∅, Walras' Law, and W.A.R.P).
- Symmetry of $S(p,w)$ is connected with maximization of rational preferences. (It will be introduced in Chapter 3.)

Corollary. *The substitution effect of good l with respect to its own price is always nonpositive, i.e., $s_{ll}(p,w) \le 0$.*

Proof. Since $S(p,w)$ is negative semidefinite, i.e., $v \cdot S(p,w)v \leq 0$ for any $v \in R^L$.
Pick $v^T = \begin{bmatrix} v_1 & \cdots & v_{l-1} & v_l & v_{l+1} & \cdots & x_L \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$.
Then, $v \cdot S(p,w)v = s_{ll}(p,w) \leq 0$. □

Remark. An implication of $s_{ll}(p,w) \leq 0$ is that a good can be a Giffen good at (p,w) only if it is inferior.

Proof. $s_{ll}(p,w) = \frac{\partial x_l(p,w)}{\partial p_l} + \frac{\partial x_l(p,w)}{\partial w} x_l(p,w) \leq 0$.
Then, if $\frac{\partial x_l(p,w)}{\partial p_l} > 0$ (Giffen good), we must have $\frac{\partial x_l(p,w)}{\partial w} < 0$ (inferior). □

Remark. Suppose a differentiable Walrasian demand function $x(p,w)$ satisfies Walras' law, homogeneous of degree zero, and the Slutsky matrix $S(p,w)$ is negative semidefinite. It is not necessarily true that $x(p,w)$ satisfies W.A.R.P. That is, *Negative semidefiniteness of $S(p,w)$ is not sufficient for W.A.R.P.*

Below we provide a counter example. (Exercise 2.F.16 in the book)

Example. Consider a setting where $L = 3$ and a consumer whose consumption set is $\mathbb{R}$. Suppose that his demand function $x(p,w)$ is

$$\begin{aligned} x_1(p,w) &= \frac{p_2}{p_3} \\ x_2(p,w) &= -\frac{p_1}{p_3} \\ x_3(p,w) &= \frac{w}{p_3}. \end{aligned}$$

The demand satisfies

(a) $x(p,w)$ is H.D.$\emptyset$ and satisfies Walras' law.

(b) $x(p,w)$ violates W.A.R.P.

(c) $v \cdot S(p,w)v = 0$ for all $v \in \mathbb{R}^3$.

Solution.

(a) H.D.∅ can be checked as follows:

$$x_1(\alpha, p, \alpha w) = \alpha p_2/\alpha p_3 = p_2/p_3 = x_1(p, w),$$
$$x_2(\alpha, \alpha, \alpha w) = -\alpha p_1/\alpha p_3 = -p_1/p_3 = x_2(p, w),$$
$$x_3(\alpha_2, \alpha_W) = \alpha w/\alpha p_3 = w/p_3 = x_3(p, w).$$

As for Walras' law,

$$p_1 x_1(p_1, x_1) + p_2 x_2(p, w) + p_3 x_3(p, w) = p_1 p_2 - p_2 p_1 + p_3 w/p_3 = w.$$

(b) Let $p = (1, 2, 1), w = 1, p' = (1, 1, 1)$, and $w' = 2$, then $x(p, w) = (2, -1, 1)$ and $x(p', w') = (1, -1, 2)$. Thus, $p' \cdot x(p, w) = 2 \leq w'$ and $p \cdot x(p', w') = 1 \leq w$. Hence, W.A.R.P is violated.

(c) First, we compute $S(p, w)$.

$$S(p, w) = \begin{bmatrix} 0 & 1/p_3 & -p_2/p_3^2 \\ -1/p_3 & 0 & p_1/p_3^2 \\ p_2/p_3^2 & -p_1/p_3^2 & 0 \end{bmatrix}$$

Then,

$$v \cdot S(p, w) = \begin{bmatrix} -\frac{v_2}{p_3} + \frac{p_2 v_3}{p_3^2} & \frac{v_1}{p_3} - \frac{p_1 v_3}{p_3^2} & -\frac{p_2 v_1}{p_3^2} + \frac{p_1 v_2}{p_3^2} \end{bmatrix}$$
$$v \cdot S(p, w)v = -\frac{v_2 v_1}{p_3} + \frac{p_2 v_3 v_1}{p_3^2} + \frac{v_1 v_2}{p_3} - \frac{p_1 v_3 v_2}{p_3^2} - \frac{p_2 v_1 v_3}{p_3^2} + \frac{p_1 v_2 v_3}{p_3^2} = 0.$$

Remark. The sufficient condition is $v \cdot S(p, w)v < 0$ whenever $v \neq \alpha p$ for any scalar α. That is, $S(p, w)$ must be negative definite for all vectors other than those that are proportional to p.

The proof is out of the scope of this course. See Samuelson (1947) or Kihlstrom, Mas-Colell, and Sonnenschein (1976) for an advanced treatment.

More properties on Slutsky matrix

Proposition 2.F.3. *Suppose that the Walrasian demand function $x(p,w)$ is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then, $p \cdot S(p,w) = 0$ and $S(p,w)p = 0$ for any (p,w).*

Proof.

$$\begin{aligned} p \cdot S(p,w) &= p \cdot D_p x(p,w) + p \cdot D_w x(p,w)\, x(p,w)^T \\ &= p \cdot D_p x(p,w) + x(p,w)^T \ \text{(by Proposition 2.E.3)} \\ &= 0^T \ \text{(by Proposition 2.E.2)} \end{aligned}$$

$$\begin{aligned} S(p,w)\, p &= D_p x(p,w)\, p + D_w x(p,w)\, x(p,w)^T p \\ &= -D_w x(p,w)\, w + D_w x(p,w)\, w \ \text{(by Proposition 2.E.1 and Walras' law)} \\ &= 0 \quad \square \end{aligned}$$

It follows from Proposition 2.F.3 that the negative semidefiniteness of $S(p,w)$ established in Proposition 2.F.2 cannot be extented to negative definiteness. As an example, see Exercise 2.F.17.

Exercise 2.F.17

In an L-commodity world, a consumer's Walrasian demand function is

$$x_k(p,w) = \frac{w}{\sum_{l=1}^{L} p_l} \text{ for } k = 1, ..., L.$$

(a) In this demand function homogeneous of degree zero in (p,w)?

(b) Does it satisfy Walras' law?

(c) Does it satisfy the weak axiom?

(d) Compute the Slutsky substitution matrix for this demand function. Is it negative semidefinite? Negative definite? Symmetric?

Choice-based Approach and Preference-based Approach

Remark. $\mathscr{B}^{\mathscr{W}} = \{B_{p,w} : p \gg 0, w > 0\}$ does NOT contain all two- and three-element subsets of X. Therefore, choice-based approach $\neq$ preference-based approach.

Example 2.F.1. In a three-commodity world, consider the three budget sets determined by the price vectors $p^1 = (2, 1, 2)$, $p^2 = (2, 2, 1)$, $p^3 = (1, 2, 2)$ and wealth $= 8$ (the same for the three budgets). Suppose that the respective (unique) choices are $x^1 = (1, 2, 2)$, $x^2 = (2, 1, 2)$, $x^3 = (2, 1, 1)$. For these three budgets, any two pairs of choices satisfy W.A.R.P but x^3 is revealed preferred to x^2, x^2 is revealed preferred to x^1, and x^1 is revealed preferred to x^3.

- We check W.A.R.P for budget 1 and 2, the satisfaction of W.A.R.P for the rest of the pairs could be shown similarly. W.A.R.P is satisfied for budget 1 and 2 since we have $p^2 \cdot x^1 = 8$, $x_1 \neq x_2$ and $p^1 \cdot x^2 = 9 > 8$.
- For revealed preference, x^2 is revealed preferred to x^1 since $p^2 \cdot x^1 = 8$, implying that x^1 is affordable under budget 2 but not chosen. Other pairs could be similarly checked.

Summary of Chapter 2 Taking choice as the primative, we look at the implications of these assumptions:

(i) $x(p, w)$ is homogeneous of degree zero

(ii) $x(p, w)$ satisfies Walras' Law

(iii) $x(p, w)$ satisfies the W.A.R.P $\implies$ Compensated Law of Demand

(iv) $x(p, w)$ is differentiable $\implies$ negative semidefinite Slutsky matrix

References

Kihlstrom, Richard, Andreu Mas-Colell, and Hugo Sonnenschein. 1976. "The Demand Theory of the Weak Axiom of Revealed Preference." *Econometrica* :971–978.

Samuelson, Paul Anthony. 1947. *Foundations of Economic Analysis.*