Advanced Microeconomics

Assignment 5

Due date: December 9, 2019 (before class)

Submission method: Please submit your assignment to me in class, or via E-mail: sherryecon@qq.com.

- 纸质版：要求字迹工整，可辨认。
- 电子版：附件要求 .pdf格式。邮件标题格式为“作业编号-学号-姓名”，如：作业1-201901010101-张三。

Grading: Your assignment will be graded based on your effort, not the accuracy of your answers.

The exercises are embedded in the Chapter 5 lecture notes (red boxes). You are advised to read the relevant sections when you work on the exercises.

The same set of exercises are provided below:

5.B.2 Suppose that $f(\cdot)$ is the production function associated with a single-output technology, and let Y be the production set of this technology. Show that Y satisfies constant returns to scale if and only if $f(\cdot)$ is homogeneous of degree one.

Solution Recall that the single-output production function $f(\cdot)$ gives rise to the production set

$$Y = \{(-z_1, -z_2, \cdots, -z_{L-1}, q) : q - f(z_1, \cdots, z_{L-1}) \leq 0 \text{ and } (z_1, \cdots, z_{L-1}) \geq 0\}.$$

Suppose first that Y satisfies constant returns to scale. Let $z \in \mathbb{R}^{L-1}_+$, then $(-z, f(z)) \in Y$ by definition. The property of constant returns to scale implies $(-\alpha z, \alpha f(z)) \in Y$ for any $\alpha > 0$, which in turn implies $\alpha f(z) \leq f(\alpha z)$ by the definition of Y. On the other hand, since $(-\alpha z, f(\alpha z)) \in Y$, by the constant returns to scale, we have $(\alpha^{-1}\alpha z, \alpha^{-1} f(\alpha z)) \in Y$, which implies $\alpha^{-1} f(\alpha z) \leq f(z)$, or $f(\alpha z) \leq \alpha f(z)$. In conclusion, $f(\alpha z) = \alpha f(z)$, i.e., the production function $f(\cdot)$ is homogeneous of degree one.

Conversely, suppose that $f(\cdot)$ is homogeneous of degree one. Let $(-z, q) \in Y$ and $\alpha > 0$, then $q \leq f(z)$ by definition, and $f(\alpha z) = \alpha f(z)$ by the homogeneity of degree one. This implies $\alpha q \leq \alpha f(z) = f(\alpha z)$, and hence $(-\alpha z, \alpha q) \in Y$, i.e., Y satisfies constant returns to scale. □

5.B.3 Show that for a single-output technology, Y is convex if and only if the production function $f(\cdot)$ is concave.

Solution Recall that the single-output production function $f(\cdot)$ gives rise to the production set

$$Y = \{(-z_1, -z_2, \cdots, -z_{L-1}, q) : q - f(z_1, \cdots, z_{L-1}) \leq 0 \text{ and } (z_1, \cdots, z_{L-1}) \geq 0\}.$$

Suppose first that Y is convex. Let $z_1, z_2 \in \mathbb{R}_+^{L-1}$ and $\alpha \in [0,1]$, then $(-z_1, f(z_1)) \in Y$ and $(-z_2, f(z_2)) \in Y$. By convexity,

$$\Big(-(\alpha z_1 + (1-\alpha)z_2), \alpha f(z_1) + (1-\alpha) f(z_2)\Big) \in Y.$$

By definition, $\alpha f(z_1) + (1-\alpha) f(z_2) \leq f(\alpha z_1 + (1-\alpha)z_2)$, i.e. $f(z)$ is concave.
Conversely, suppose that $f(z)$ is concave. Let $(-z_1, q_1) \in Y$, $(-z_2, q_2) \in Y$, and $\alpha \in [0,1]$, then $q_1 \leq f(z_1)$ and $q_2 \leq f(z_2)$. Hence,

$$\alpha q_1 + (1-\alpha) q_2 \leq \alpha f(z_1) + (1-\alpha) f(z_2).$$

As $f(z)$ is concave,

$$\alpha f(z_1) + (1-\alpha) f(z_2) \leq f(\alpha z_1 + (1-\alpha) z_2).$$

Therefore,

$$\alpha q_1 + (1-\alpha) q_2 \leq f(\alpha z_1 + (1-\alpha) z_2).$$

Hence,

$$\Big(-(\alpha z_1 + (1-\alpha)z_2), \alpha q_1 + (1-\alpha) q_2\Big) = \alpha(-z_1, q_1) + (1-\alpha)(-z_2, q_2) \in Y.$$

$\square$

5.C.9 Derive the profit function $\pi(p)$ and supply function (or correspondence) $y(p)$ for the single-output technologies whose production functions $f(z)$ are given by

(a) $f(z) = \sqrt{z_1 + z_2}$.

(b) $f(z) = \sqrt{\min\{z_1, z_2\}}$.

(c) $f(z) = (z_1^\rho + z_2^\rho)^{1/\rho}$, for $\rho \leq 1$.

Solution

(a) Let the price vector be $p = (p_1, p_2, p_3) \gg 0$. The profit maximization problem is

$$\max_{z_1, z_2 \geq 0} \ (p_1, p_2, p_3) \cdot (-z_1, -z_2, q),$$
$$\text{s.t. } q \leq f(z) = \sqrt{z_1 + z_2}.$$

Clearly, the constraint must hold in equality, since otherwise one can increase the production scale q without violating the constraint and earn a higher profit. Hence, we can substitute $q = \sqrt{z_1 + z_2}$ into the objective function and set up the Lagrangian:

$$\mathcal{L}(z_1, z_2) = -p_1 z_1 - p_2 z_2 + p_3 \sqrt{z_1 + z_2}$$

If $z_1 + z_2 = 0$, then the non-negativity constraint implies $z_1 = z_2 = 0$, and thus the profit is zero. Suppose $z_1 + z_2 \neq 0$, the Kuhn-Tucker first-order conditions are

$$\frac{p_3}{2\sqrt{z_1 + z_2}} - p_1 \leq 0 \text{ with equality if } z_1 > 0,$$
$$\frac{p_3}{2\sqrt{z_1 + z_2}} - p_2 \leq 0 \text{ with equality if } z_2 > 0.$$

(i) If $p_1 > p_2$, then $\frac{p_3}{2\sqrt{z_1+z_2}} \leq p_2 < p_1$, implying $z_1 = 0$. Since we are discussing the case where $z_1 + z_2 \neq 0$, we have $z_2 > 0$. And thus, $\frac{p_3}{2\sqrt{z_2}} = p_2 \implies z_2 = \frac{p_3^2}{4p_2^2}$. Therefore, the supply function and the profit function are given by

$$y(p) = \left(0, -\frac{p_3^2}{4p_2^2}, \frac{p_3}{2p_2}\right),$$
$$\pi(p) = 0 - \frac{p_3^2}{4p_2} + \frac{p_3^2}{2p_2} = \frac{p_3^2}{4p_2} > 0.$$

(ii) If $p_2 > p_1$, by symmetry, we have

$$y(p) = \left(-\frac{p_3^2}{4p_1^2}, 0, \frac{p_3}{2p_1}\right),$$
$$\pi(p) = -\frac{p_3^2}{4p_1} - 0 + \frac{p_3^2}{2p_1} = \frac{p_3^2}{4p_1} > 0.$$

(iii) Finally, if $p_1 = p_2$, then $p_1 = p_2 = \frac{p_3}{2\sqrt{z_1+z_2}}$. z_1, z_2 satisfy $z_1 + z_2 = \frac{p_3^2}{4p_1^2}$. Thus we have

$$y(p) = (-z_1, -z_2, \sqrt{z_1 + z_2}) \text{ s.t. } z_1 \geq 0, z_2 \geq 0, z_1 + z_2 = \frac{p_3^2}{4p_1^2},$$
$$\pi(p) = -\frac{p_3^2}{4p_1} + \frac{p_3^2}{2p_1} = \frac{p_3^2}{4p_1}.$$

In summary, the supply function is given by

$$y(p) = \begin{cases} \left(0, -\frac{p_3^2}{4p_2^2}, \frac{p_3}{2p_2}\right), & \text{if } p_1 > p_2; \\ \left(-\frac{p_3^2}{4p_1^2}, 0, \frac{p_3}{2p_1}\right), & \text{if } p_1 < p_2; \\ (-z_1, -z_2, \frac{p_3}{2p_1}), z_1 + z_2 = \frac{p_3^2}{4p_1^2}, & \text{if } p_1 = p_2. \end{cases}$$

And the profit function is given by

$$\pi(p) = \begin{cases} \frac{p_3^2}{4p_2}, & \text{if } p_1 \geq p_2; \\ \frac{p_3^2}{4p_1}, & \text{if } p_1 < p_2. \end{cases}$$

(b) The profit maximization problem is

$$\max_{z_1, z_2 \geq 0} (p_1, p_2, p_3) \cdot (-z_1, -z_2, q),$$
$$\text{s.t. } q \leq f(z) = \sqrt{\min\{z_1, z_2\}}.$$

Clearly, the constraint must hold in equality, since otherwise one can increase the production scale q without violating the constraint and earn a higher profit. Hence, we can substitute $q = \sqrt{\min\{z_1, z_2\}}$ into the objective function and obtain

$$\max_{z_1, z_2 \geq 0} -p_1 z_1 - p_2 z_2 + p_3 \sqrt{\min\{z_1, z_2\}}.$$

If $z_1 \neq z_2$, we assume that $z_1 < z_2$ (the case in which $z_1 > z_2$ is similar). Then the objective function can be written as

$$-p_1 z_1 - p_2 z_2 + p_3 \sqrt{z_1},$$

to maximize which we must have $z_2 = 0$, but this will force $z_1 < 0$ which is infeasible. Therefore, we must have $z_1 = z_2$ in optimum. Then we can write the problem as

$$\max_{z_1 \geq 0} -(p_1 + p_2) z_1 + p_3 \sqrt{z_1}.$$

Set up the Lagrangian:

$$\mathcal{L}(z_1) = -(p_1 + p_2) z_1 + p_3 \sqrt{z_1}$$

If $z_1 = 0$, then the profit is zero.
If $z_1 > 0$, then the Kuhn-Tucker first-order conditions are given by

$$-(p_1 + p_2) + \frac{p_3}{2\sqrt{z_1}} = 0$$

which imply that $z_1 = \frac{p_3^2}{4(p_1+p_2)^2}$.
Hence, the supply function and the profit function are given by

$$y(p) = \left(-\frac{p_3^2}{4(p_1+p_2)^2}, -\frac{p_3^2}{4(p_1+p_2)^2}, \frac{p_3}{2(p_1+p_2)}\right),$$
$$\pi(p) = \frac{p_3^2}{4(p_1+p_2)}.$$

(c) The profit maximization problem is

$$\max_{z_1, z_2 \geq 0} (p_1, p_2, p_3) \cdot (-z_1, -z_2, q)$$
$$\text{s.t. } q \leq f(z) = (z_1^\rho + z_2^\rho)^{1/\rho}.$$

Clearly, the constraint must hold in equality, since otherwise one can increase the production scale q without violating the constraint and earn a higher profit. Hence, we can substitute $q = (z_1^\rho + z_2^\rho)^{1/\rho}$ into the objective function.

(i) If $\rho = 1$, the objective function becomes

$$\max_{z_1, z_2 \geq 0} -p_1 z_1 - p_2 z_2 + p_3(z_1 + z_2).$$

And the Lagrangian function is

$$\mathcal{L}(z_1, z_2) = -p_1 z_1 - p_2 z_2 + p_3(z_1 + z_2).$$

The Kuhn-Tucker conditions are

$$-p_1 + p_3 \leq 0 \text{ with equality if } z_1 > 0,$$
$$-p_2 + p_3 \leq 0 \text{ with equality if } z_2 > 0,$$
$$z_1 \geq 0; z_2 \geq 0.$$

If $p_1 > p_2$, then $z_1 = 0$. Then the objective function can be written as

$$(p_3 - p_2)z_2.$$

If $p_2 < p_3$, then there is no optimal solution, i.e., $y(p) = \emptyset$, because by setting z_2 to be infinity, one can achieve infinite profit, i.e., $\pi(p) = \infty$. If $p_2 > p_3$, then $z_2 = 0$ is optimal, and hence $y(p) = (0, 0, 0), \pi(p) = 0$. If $p_2 = p_3$, then any $z_2 \geq 0$ is optimal, and hence $y(p) = (0, -z_2, z_2), z_2 \geq 0$ and $\pi(p) = 0$.

By symmetry, if $p_1 < p_2$, then $z_2 = 0$; if $p_1 < p_3$, then $y(p) = \emptyset$ and $\pi(p) = +\infty$. If $p_1 > p_3$, then $y(p) = (0, 0, 0), \pi(p) = 0$; if $p_1 = p_3$, then $y(p) = (-z_1, 0, z_1), z_1 \geq 0$ and $\pi(p) = 0$.

If $p_1 = p_2$, the objective function becomes $(p_3 - p_1)(z_1 + z_2)$. If $p_1 = p_2 > p_3$, then $z_1 = z_2 = 0$ is optimal, and thus $y(p) = (0, 0, 0)$ and $\pi(p) = 0$; if $p_1 = p_2 = p_3$, then $y(p) = (-z_1, -z_2, z_1 + z_2), z_1 \geq 0, z_2 \geq 0$ and $\pi(p) = 0$; if $p_1 = p_2 < p_3$, there is no solution and $y(p) = \emptyset$, $\pi(p) = \infty$.

In summary, when $\rho = 1$, the supply function and profit function are given by

$$y(p) = \begin{cases} \emptyset, & \text{if } \min\{p_1, p_2\} < p_3; \\ (0, 0, 0), & \text{if } \min\{p_1, p_2\} > p_3; \\ (0, -z_2, z_2), z_2 \geq 0, & \text{if } p_1 > p_2 = p_3; \\ (-z_1, 0, z_1), z_1 \geq 0, & \text{if } p_2 > p_1 = p_3; \\ (-z_1, -z_2, z_1 + z_2), z_1, z_2 \geq 0. & \text{if } p_1 = p_2 = p_3. \end{cases}$$

$$\pi(p) = \begin{cases} \infty, & \text{if } \min\{p_1, p_2\} < p_3; \\ 0. & \text{otherwise.} \end{cases}$$

(ii) If $\rho < 1$, set up the Lagrangian:

$$\mathcal{L}(z_1, z_2) = -p_1 z_1 - p_2 z_2 + p_3 (z_1^\rho + z_2^\rho)^{1/\rho}.$$

The Kuhn-Tucker conditions are

$$\begin{aligned} & -p_1 + p_3 z_1^{\rho-1} (z_1^\rho + z_2^\rho)^{1/\rho - 1} \leq 0, \text{ with equality if } z_1 > 0, \\ & -p_2 + p_3 z_2^{\rho-1} (z_1^\rho + z_2^\rho)^{1/\rho - 1} \leq 0, \text{ with equality if } z_1 > 0, \\ & z_1 \geq 0; z_2 \geq 0. \end{aligned}$$

We first consider the interior solution $z_1 > 0, z_2 > 0$. The two FOCs imply that

$$z_2^{\rho-1} = \frac{p_2}{p_1} z_1^{\rho-1}.$$

Substitute this into the second FOC and simplify the equation, we obtain

$$p_3 = \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}}.$$

Hence, if the above equation indeed holds for the given prices, then the supply function is

$$y(p) = \left(-z_1, -\left(\frac{p_2}{p_1}\right)^{\frac{1}{\rho-1}} z_1, \left[z_1^\rho + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho-1}} z_1^\rho \right]^{1/\rho} \right),$$

where $z_1 > 0$, and the profit function is

$$\pi(p) = -p_1 z_1 - p_2 \left(\frac{p_2}{p_1}\right)^{\frac{1}{\rho-1}} z_1 + \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} \left[z_1^\rho + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho-1}} z_1^\rho\right]^{1/\rho} = 0.$$

We consider the boundary solutions when $p_3 = \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$. If $\rho < 0$, then $p_3 > \max\{p_1, p_2\}$. Hence, we can choose $z_1 = 0$ and let $z_2 \to \infty$ (or $z_2 = 0, z_1 = \infty$), which gives $\pi = \infty$. If $0 < \rho < 1$, then $p_3 < \min\{p_1, p_2\}$ and the optimal boundary solution is $y(p) = (0, 0, 0)$ which gives $\pi(p) = 0$, while any other boundary solutions will give $\pi < 0$. Hence, when $p_3 = \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$, if $\rho < 0$, we have $y(p) = \emptyset$ and $\pi(p) = \infty$; if $\rho \in (0, 1)$, we have $y(p) = \left(-z_1, -\left(\frac{p_2}{p_1}\right)^{\frac{1}{\rho-1}} z_1, \left[z_1^\rho + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho-1}} z_1^\rho\right]^{1/\rho}\right)$ for $z_1 \geq 0$ and $\pi(p) = 0$.

If $p_3 \neq \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$, there is no interior solution. We then consider only the boundary solutions. If $p_3 > \min\{p_1, p_2\}$, there is no optimal solution (i.e., $y(p) = \emptyset$) and $\pi(p) = \infty$, which can be achieved by letting the quantity of the cheaper input go to infinity and the other be of zero amount. If $p_3 = p_1 < p_2$, then the optimal boundary solution is $y(p) = (-z_1, 0, z_1), z_1 \geq 0$ and $\pi(p) = 0$; similarly, if $p_3 = p_2 \leq p_1$, then the optimal boundary solution is $y(p) = (0, -z_2, z_2), z_2 \geq 0$ and $\pi(p) = 0$. If $p_3 < \min\{p_1, p_2\}$, then the optimal boundary solution is $y(p) = (0, 0, 0)$ and $\pi(p) = 0$.

In summary, if $\rho < 1$, then the supply function and profit function are given by

$$y(p) = \begin{cases} \emptyset, & \text{if } \min\{p_1, p_2\} < p_3; \\ (0,0,0), & \text{if } \min\{p_1, p_2\} > p_3; \\ (0, -z_2, z_2), z_2 \geq 0, & \text{if } p_1 \geq p_2 = p_3; \\ (-z_1, 0, z_1), z_1 \geq 0, & \text{if } p_2 > p_1 = p_3; \\ \left(-z_1, -\left(\frac{p_2}{p_1}\right)^{\frac{1}{\rho-1}} z_1, \left[z_1^\rho + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho-1}} z_1^\rho\right]^{1/\rho}\right), z_1 \geq 0, & \\ \qquad \text{if } p_3 = \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}, 0 < \rho < 1. & \end{cases}$$

$$\pi(p) = \begin{cases} \infty, & \text{if } \min\{p_1, p_2\} < p_3; \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

5.C.10 Derive the cost function $c(w,q)$ and conditional factor demand functions (or correspondences) $z(w.q)$ for each of the following single-output constant return technologies with production functions given by

(a) $f(z) = z_1 + z_2$ (perfect substitutable inputs)

(b) $f(z) = \min\{z_1, z_2\}$ (leontief technology)

(c) $f(z) = (z_1^\rho + z_2^\rho)^{1/\rho}, \rho \leq 1$ (constant elasticity of substitution technology)

Solution Let the factor price vector be $w = (w_1, w_2) \gg 0$.

(a) The cost minimization problem is

$$\begin{aligned} \min_{z_1, z_2 \geq 0} \ & w_1 z_1 + w_2 z_2 \\ \text{s.t. } & f(z) = z_1 + z_2 \geq q. \end{aligned}$$

Clearly, the constraint must hold in equality in optimum, since otherwise one can reduce the amount of inputs without violating the constraint and reduce the cost. Therefore, we set up the Lagrangian as follows:

$$\mathcal{L}(z_1, z_2, \lambda, \mu_1, \mu_2) = -w_1 z_1 - w_2 z_2 + \lambda(z_1 + z_2 - q) + \mu_1 z_1 + \mu_2 z_2.$$

The Kuhn-Tucker first-order conditions are

$$\begin{gathered} w_1 = \lambda + \mu_1, \\ w_2 = \lambda + \mu_2, \\ z_1 + z_2 = q, \\ \mu_1 z_1 = 0,\ \ \mu_2 z_2 = 0, \\ \mu_1 \geq 0,\ \ \mu_2 \geq 0, \\ z_1 \geq 0, z_2 \geq 0. \end{gathered}$$

The first two FOCs imply

$$w_1 - \mu_1 = w_2 - \mu_2.$$

If $w_1 < w_2$, then $0 \leq \mu_1 < \mu_2$, and hence $z_2 = 0$. By the third FOC, $z_1 = q$. Therefore, the cost function is given by

$$c(w,q) = w_1 q,$$

and the (conditional) factor demand function is given by

$$z(w,q) = (q, 0).$$

By symmetry, if $w_1 > w_2$, we have

$$c(w, q) = w_2 q,$$

and

$$z(w, q) = (0, q).$$

If $w_1 = w_2$, then any solution $(z_1, z_2) \geq 0$ satisfying $z_1 + z_2 = q$ is optimal, and $c(w, q) = w_1 q = w_2 q$.

In summary, we have

$$z(w, q) = \begin{cases} (q, 0), & \text{if } w_1 < w_2; \\ (0, q), & \text{if } w_1 > w_2; \\ \{(z_1, z_2) \in \mathbb{R}_+^2 : z_1 + z_2 = q\}, & \text{if } w_1 = w_2. \end{cases}$$

$$c(w, q) = \begin{cases} w_1 q, & \text{if } w_1 \leq w_2; \\ w_2 q, & \text{if } w_1 > w_2. \end{cases}$$

(b) The cost minimization problem is

$$\begin{aligned} \min_{z_1, z_2 \geq 0} \; & w_1 z_1 + w_2 z_2 \\ \text{s.t. } & f(z) = \min\{z_1, z_2\} \geq q. \end{aligned}$$

Clearly, the constraint must hold in equality in optimum, since otherwise one can reduce the amount of inputs without violating the constraint and reduce the cost.

If $z_1 \neq z_2$, assume $z_1 < z_2$ (the other case where $z_1 > z_2$ is symmetric). Then the constraint implies $z_1 = q$. To minimize cost, one should set z_2 as close to 0 as possible but $z_2 \neq 0$. If $z_2 = 0$, then $z_1 < z_2 = 0$ is infeasible. Such z_2 does not exist. Hence, there is no optimal solution if $z_1 \neq z_2$.

If $z_1 = z_2$, then the constraint implies that $z_1 = z_2 = q$, i.e., $z(w, q) = (q, q)$. And the minimized cost is $c(w, q) = (w_1 + w_2)q$.

(c) We consider $\rho < 1$ only. The case when $\rho = 1$ is identical to part (a).

The cost minimization problem is

$$\begin{aligned} \min_{z_1, z_2 \geq 0} \; & w_1 z_1 + w_2 z_2 \\ \text{s.t. } & f(z) = (z_1^\rho + z_2^\rho)^{1/\rho} \geq q, \rho \leq 1 \end{aligned}$$

Clearly, the constraint must hold in equality in optimum, since otherwise one can reduce the amount of inputs without violating the constraint and reduce the cost. We set up the Lagrangian function as follows:

$$\mathcal{L}(z_1, z_2, \lambda) = -w_1 z_1 - w_2 z_2 + \lambda[(z_1^\rho + z_2^\rho)^{1/\rho} - q].$$

Kuhn-Tucker conditions are

$$\begin{aligned} -w_1 + \lambda(z_1^\rho + z_2^\rho)^{\frac{1}{\rho}-1} z_1^{\rho-1} &\leq 0, \text{ with equality if } z_1 > 0. \\ -w_2 + \lambda(z_1^\rho + z_2^\rho)^{\frac{1}{\rho}-1} z_2^{\rho-1} &\leq 0, \text{ with equality if } z_2 > 0. \\ (z_1^\rho + z_2^\rho)^{1/\rho} &= q. \end{aligned}$$

For $z_1, z_2 > 0$, the two FOCs imply

$$\frac{w_1}{w_2} = \left(\frac{z_1}{z_2}\right)^{\rho-1},$$

which further implies

$$z_1 = \left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}} z_2.$$

Substitute this into the third FOC, we have

$$\left[\left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\rho-1}} z_2^\rho + z_2^\rho\right]^{\frac{1}{\rho}} = q,$$

which can be solved for

$$z_2 = q w_2^{\frac{1}{\rho-1}} (w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{-\frac{1}{\rho}}.$$

Then, we can obtain

$$z_1 = q w_1^{\frac{1}{\rho-1}} (w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}})^{-\frac{1}{\rho}}.$$

Hence, the factor demand function is given by

$$z(w, q) = q\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} \left(w_1^{\frac{1}{\rho-1}}, w_2^{\frac{1}{\rho-1}}\right),$$

and the cost function is given by

$$c(w, q) = q\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{1-\frac{1}{\rho}}.$$

We show below that there exists no boundary solution.

When $z_1 = 0$, we have $z_2 = q$. And the cost function is $c(w, q) = qw_2$ which is larger than $q\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{1-\frac{1}{\rho}}$ for $\rho < 1$. Similarly, when $z_2 = 0$, we have $z_1 = q$. And the cost function is $c(w, q) = qw_1$ which is larger than $q\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{1-\frac{1}{\rho}}$. $\square$

5.C.11 Show that $\partial z_l(w,q)/\partial q > 0$ if and only if marginal cost at q is increasing in w_l.

Solution Assume that $c(w,q)$ is twice differentiable and $z(w,q)$ is differentiable, then Proposition 5.C.2 implies

$$\frac{\partial z_l(w,q)}{\partial q} = \frac{\partial}{\partial q}\frac{\partial c(w,q)}{\partial w_l} = \frac{\partial}{\partial w_l}\frac{\partial c(w,q)}{\partial q}.$$

Therefore, $\frac{z_l(w,q)}{\partial q} > 0$ if and only if $\frac{\partial}{\partial w_l}\frac{\partial c(w,q)}{\partial q} > 0$, i.e., the marginal cost at q is increasing in w_l. □

5.D.1 Show that $AC(\bar{q}) = C'(\bar{q})$ at any $\bar{q}$ satisfying $AC(\bar{q}) \leq AC(q)$ for all q. Does this result depend on the differentiability of $C(\cdot)$ everywhere?

Solution We only need $C(\cdot)$ to be differentiable at $\bar{q}$. Everywhere differentiability is unnecessary. Differentiate the average cost function and evaluate at $\bar{q}$, we have

$$AC'(\bar{q}) = \frac{\partial}{\partial q}\frac{C(q)}{q}\bigg|_{q=\bar{q}} = \frac{C'(\bar{q})\bar{q} - C(\bar{q})}{\bar{q}^2}.$$

Then, if $AC(q)$ is minimized at $\bar{q}$, we have $AC'(\bar{q}) = 0$, or $C'(\bar{q})\bar{q} - C(\bar{q}) = 0$, which implies

$$C'(\bar{q}) = \frac{C(\bar{q})}{\bar{q}} = AC(\bar{q}).$$

□

5.D.2 Depict the supply locus for a case with partially sunk costs, that is, where $C(q) = K + C_v(q)$ if $q > 0$ and $0 < C(0) < K$.

Solution The total cost function is given by

$$C(q) = \begin{cases} C(0), & \text{if } q = 0; \\ K + C_v(q), & \text{if } q > 0, \end{cases}$$

which can be depicted by the following diagram:

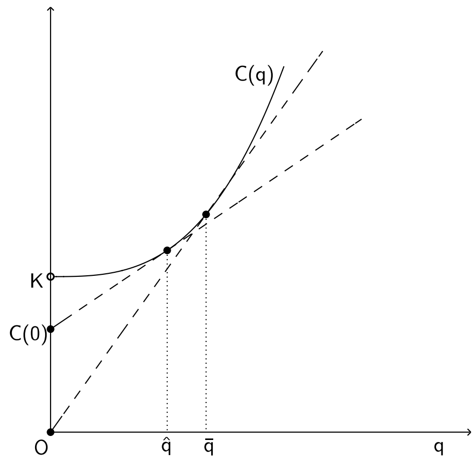

Figure 1: Total cost curve with partially sunk costs

Marginal cost $C'(q)$ can be represented by the slope of each point on the total cost curve. We define two different notions of average cost. Average total cost is defined as $ATC(q) = \frac{K+C_v(q)}{q}, \forall q > 0$, which can be represented by the slope of the line segment connecting the origin with any point on the total cost curve where $q > 0$. Average production cost is defined as $APC(q) = \frac{K+C_v(q)-C(0)}{q}, \forall q > 0$, which can be represented by the slope of the line segment connecting the point $(0, C(0))$ with any point on the total cost curve where $q > 0$. Unlike $ATC(q)$, $APC(q)$ excludes the sunk cost $C(0)$ from the calculation. The two dashed lines in the above diagram are tangent to the total cost curve, one passing through the origin and the other point $(0, C(0))$. The tangent points have the property that $ATC(\bar{q}) = C'(\bar{q})$ and $APC(\hat{q}) = C'(\hat{q})$. We depict the marginal cost curve and average cost curves in the following diagram:

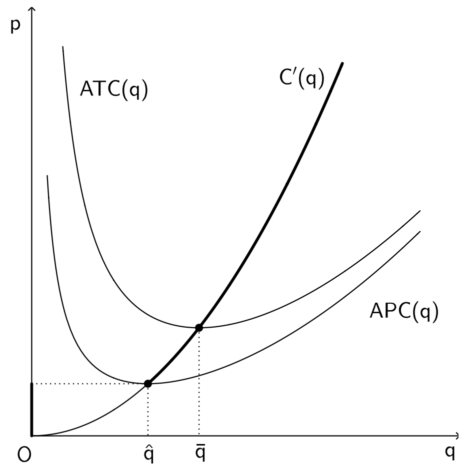

Figure 2: Marginal cost curve, average cost curves, and supply curve

Since the sunk cost does not affect the production decision, the producer will produce positive quantity if and only if the price is higher than the average production cost, and the quantity is determined by equalizing marginal cost with the price level. Hence, the quantity supplied is zero when $p < C'(\hat{q})$, and is determined by $p = C'(q)$ when $p \geq C'(\hat{q})$, which is depicted by the bold curve in the above diagram. Note that the supply curve consists of two pieces. □

5.D.3 Suppose that a firm can produce good L from $L-1$ factor inputs ($L > 2$). Factor prices are $w \in \mathbb{R}^{L-1}$ and the price of output is p. The firm's differentiable cost function is $c(w, q)$. Assume that this function is strictly convex in q. However, although $c(w, q)$ is the cost function when all factors can be freely adjusted, factor 1 cannot be adjusted in the short run.

Suppose that the firm is initially at a point where it is producing its long-run profit-maximizing output level of good L given prices w and p, $q(w, p)$ [i.e., the level that is optimal under the long-run cost conditions described by $c(w, q)$], and that all inputs are optimally adjusted [i.e., $z_l = z_l(w, q(w, p))$ for all $l = 1, ..., L-1$, where $z_l(\cdot, \cdot)$ is the long-run input demand function]. Show that the firm's profit-maximizing output response to a marginal increase in the price of good L is larger in the long run than in the short run. [*Hint*: Define a short-run cost function $c_s(w, q|z_1)$ that gives the minimized costs of producing output level q given that input 1 is fixed at level z_1.]

Solution Let $\bar{w}$ be the initial input price vector, $\bar{p}$ the initial output price, and $\bar{z}_1$ the initial level of input 1. The long-run profit maximization problem

$$\max_{q \geq 0} \; pq - c(w, q)$$

implies

$$p = \frac{\partial c(w, q)}{\partial q}, \text{for } q > 0,$$

which implicitly defines $q(w, p)$.

Differentiate both sides of the above equation with respect to p, and evaluate at $(\bar{w}, \bar{p})$, we have

$$\frac{\partial^2 c(\bar{w}, q(\bar{w}, \bar{p}))}{\partial q^2} \frac{\partial q(\bar{w}, \bar{p})}{\partial p} = 1,$$

which implies

$$\frac{\partial q(\bar{w}, \bar{p})}{\partial p} = \left(\frac{\partial^2 c(\bar{w}, q(\bar{w}, \bar{p}))}{\partial q^2} \right)^{-1}. \tag{1}$$

Similarly, the short-run profit maximization problem

$$\max_{q_s \geq 0} \; pq_s - c_s(w, q_s | z_1)$$

implies

$$p = \frac{\partial c_s(w, q_s | z_1)}{\partial q_s},$$

which implicitly defines $q_s(w, p | z_1)$.

Differentiate both sides of the above equation with respect to p and evaluate at $(\bar{w}, \bar{p}, \bar{z}_1)$, we have

$$\frac{\partial^2 c_s(\bar{w}, q(\bar{w}, \bar{p}) | \bar{z}_1)}{\partial q^2} \frac{\partial q_s(\bar{w}, \bar{p} | \bar{z}_1)}{\partial p} = 1,$$

which implies

$$\frac{\partial q_s(\bar{w}, \bar{p} | \bar{z}_1)}{\partial p} = \left(\frac{\partial^2 c_s(\bar{w}, q(\bar{w}, \bar{p}) | \bar{z}_1)}{\partial q^2} \right)^{-1}. \tag{2}$$

For any q, the short-run cost minimization problem has more constraints than the long-run, and hence

$$c(\bar{w}, q) \leq c_s(\bar{w}, q), \forall q > 0,$$

and

$$c(\bar{w}, \bar{q}) = c_s(\bar{w}, \bar{q}),$$

where $\bar{q} = q(\bar{w}, \bar{p})$.

Therefore, the function $f(q) = c(\bar{w}, q) - c_s(\bar{w}, q)$ obtains its maximum at $\bar{q}$. The second-order necessary condition implies

$$f''(q) = \frac{\partial^2 c(\bar{w}, \bar{q})}{\partial q^2} - \frac{\partial^2 c_s(\bar{w}, \bar{q})}{\partial q^2} \leq 0.$$

Therefore, by Eq. (1) and (2), we have

$$\frac{\partial q(\bar{w}, \bar{p})}{\partial p} \geq \frac{\partial q_s(\bar{w}, \bar{p}|\bar{z}_1)}{\partial p}.$$

That is, if the output price increases marginally, the long-run profit maximizing output is larger than the short-run. This completes the proof. □