Chapter 1. Preference and Choice

1.A. Introduction

Two approaches to modeling individual choice behavior:

1. Preference-based Approach: preference as primative (rationality axioms) $\Longrightarrow$ consequences on choices

2. Choice-based Approach: choice behavior as primative (axioms on behavior)

Traditional: Preference-based Approach is preferred.

Some attractive features of Choice-based Approach: allows more room for general forms of behavior, assumptions on observable behavior, doesn't require introspection

1.B. Preference Relations

X: Set of Alternatives. For example, if Alice just graduated from Wuhan University majoring in economics, then her set of alternatives is: $X = \{$go to graduate school and study economics, go to a Big-4 firm, go to work for the government, ..., run a small business$\}$.

We use capital letters (like X and B) for a set of alternatives, small letters (like x and y) for a specific choice alternative.

Defining Preference Relations Denote by $\succsim$ *the preference relation* defined on the set X, allowing the comparison of any x and y in X.

$x \succsim y$: pronounced as "x is preferred to y" or "x is at least as good as y." The first usage is more common.

Strict preference $\succ$: $x \succ y \iff x \succsim y$ but not $y \succsim x$ (i.e., $y \not\succsim x$) ("x is strictly preferred to y.")

Indifference $\sim$: $x \sim y \iff x \succsim y$ and $y \succsim x$ ("x is indifferent to y.")

Rational Preference Not all preference relations make sense. For example, consider that Alice strictly prefers "Hot and Dry Noodles" to "Doupi" (dòu pí), strictly prefers

"Doupi" to "Xiaolongbao" (xiǎo lóng bāo), and prefers "Xiaolongbao" to "Hot and Dry Noodles." Alice must have a hard time choosing her breakfast from $X = \{$Hot and Dry Noodles, Doupi, Xiaolongbao$\}$.

Definition 1.B.1 (Rational preference)**.** The preference relation $\succsim$ is **rational** if it possesses these two properties:

(i) Completeness: $\forall x, y \in X$, $x \succsim y$ or $y \succsim x$. (rules out $x \not\succsim y$ and $y \not\succsim x$)

(ii) Transitivity: $\forall x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Question 1. In the example above, which property does Alice's preference relation violate?

Answer: Transitivity. Since Alice strictly prefers "Doupi" to "Xiaolongbao", and prefers "Xiaolongbao" to "Hot and Dry Noodles," she must prefer "Doupi" to "Hot and Dry Noodles." This contradicts that she strictly prefers "Hot and Dry Noodles" to "Doupi."

Implications on $\succ$ and $\sim$ The following propositions follow from the definition of *rational preference.*

Proposition 1.B.1. *If $\succsim$ is rational, then:*

(i) $\succ$ is both irreflexive ($x \succ x$ never holds) and transitive.

(ii) $\sim$ is reflexive ($x \sim x$), transitive and symmetric (if $x \sim y$, then $y \sim x$).

(iii) if $x \succ y \succsim z$, then $x \succ z$. (slightly stronger than transitivity in (i))

Proof.

(i) *Irreflexive.* Suppose $x \succ x$, then

$$x \succsim x \text{ and } x \not\succsim x \quad (\text{definition of } \succ),$$

which is never true.

Transitive. Suppose $x \succ y, y \succ z$ and $z \succsim x$, then

$$y \succ z \implies y \succsim z \quad \text{(definition of } \succ\text{)},$$

and

$$y \succsim z \ \&\ z \succsim x \implies y \succsim x \quad \text{(transitivity of } \succsim\text{)}.$$

This contradicts that $x \succ y$.

(ii) *Reflexive.* By completeness of $\succsim$, $x \succsim x$. Then, $x \sim x$ by definition of $\sim$.

Transitive. Suppose $x \sim y, y \sim z$, then

$$\begin{aligned} & x \succsim y, y \succsim z \ \&\ y \succsim x, z \succsim y \quad \text{(definition of } \sim\text{)} \\ \implies & x \succsim z, z \succsim x \quad \text{(transitivity of } \succsim\text{)} \\ \implies & x \sim z \quad \text{(definition of } \sim\text{)} \end{aligned}$$

Symmetric. Suppose $x \sim y$, then $x \succsim y$ and $y \succsim x$ (definition of $\sim$). Using the definition of $\sim$ again, $y \sim x$.

(iii) Suppose $x \succ y, y \succ z$ and $z \succsim x$, then

$$y \succsim z \ \&\ z \succsim x \implies y \succsim x \quad \text{(transitivity of } \succsim\text{)}.$$

This contradicts that $x \succ y$.

□

Utility Functions It seems unnecessarily abstract to use always the preference relation $\succsim$. Since human beings are better at comparing the order of numbers, we assign each choice with a number. In doing that, we are using some *utility function* to represent the preference relation.

Definition 1.B.2. A function $u : X \to \mathbb{R}$ is a utility function representing preference relation $\succsim$ if

$$x \succsim y \iff u(x) \geq u(y) \text{ for all } x, y \in X. \tag{1}$$

The utility function is nothing but assigning each choice x with a number $u(x)$. Obviously, the function u satisfying Condition (1) is not unique.

Example. $u(x) \geq u(y) \iff \alpha u(x) \geq \alpha u(y)$ for all $\alpha > 0$.

> *Question* 2. When can a preference relation be represented by a utility function?
>
> *Answer: Only if* the preference relation is rational. See the next proposition.

Proposition 1.B.2. *If the preference relation $\succsim$ can be represented by a utility function (i.e. $\exists u(\cdot)$ s.t. $u(x) \geq u(y)$ iff $x \succsim y$), then $\succsim$ is rational (i.e. complete & transitive).*

Proof. Suppose there exists some $u(\cdot)$ such that $u(x) \geq u(y)$ iff $x \succsim y$.

Completeness: $u(x), u(y) \in \mathbb{R} \implies u(x) \geq u(y)$ or $u(y) \geq u(x) \iff x \succsim y$ or $y \succsim x$

Transitivity: $x \succsim y$ & $y \succsim z \iff u(x) \geq u(y)$ & $u(y) \geq u(z) \implies u(x) \geq u(z) \iff x \succsim z$. $\square$

> *Question* 3. If $\succsim$ is rational, does there exist a utility function u representing $\succsim$?
>
> *Answer:* Not always. Rationality is just a necessary condition for the existence of a utility representation, but not sufficient. See the counterexample below.

Definition (Lexicographic Preference)**.** Let $X = \mathbb{R}^2$. The preference relation $\succsim$ is a *lexicographic preference* if for all $x, y \in X$, $x \succsim y$ whenever (i) $x_1 > y_1$ or (ii) $x_1 = y_1$ and $x_2 \geq y_2$.

Claim. The lexicographic preference on $\mathbb{R}^2$ do *not* have a utility representation.

Let's look at an example of the lexicographic preference before moving into the proof. As the lexicographic preference is defined on $\mathbb{R}^2$, it is used to describe a decision making situation with *two-dimensional comparisons.* For example, Alice is considering buying a new phone. The relevant attributes include brand name, price, CPU, and so on. For simplicity, suppose Alice only cares about the brand (Apple or Huawei) and price. Alice is a Apple fan and strictly prefers an iPhone to a Huawei Phone regardless of the price. For Alice,

$$(\text{Apple}, 5000) \succ (\text{Apple}, 8000) \succ (\text{Huawei}, 5000).$$

In this case, Alice's desicion making criteria satisfy the requirements of the lexicographic preference. Although her preference is rational, it can not be modelled by a utility function.

Proof.

1. Lexicographic Preference is rational.

   Completeness. For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2_+$:

   a) If $x_1 > y_1$, then $x \succsim y$

   b) If $y_1 > x_1$, then $y \succsim x$

   c) If $x_1 = y_1$, then either $x_2 \geq y_2 \implies x \succsim y$ or $y_2 \geq x_2 \implies y \succsim x$

   Transitivity. Let $x, y, z \in \mathbb{R}^2_+$. Suppose $x \succsim y$ and $y \succsim z$. Then, one of the following cases must prevail:

   a) $x_1 > y_1$ and $y_1 > z_1$

   b) $x_1 > y_1$, $y_1 = z_1$ and $y_2 \geq z_2$

   c) $x_1 = y_1$, $x_2 \geq y_2$ and $y_1 > z_1$

   d) $x_1 = y_1$, $x_2 \geq y_2$, $y_1 = z_1$ and $y_2 \geq z_2$

   In each case, $x \geq z$ since

   a) $x_1 > y_1 > z_1 \implies x_1 > z_1$

   b) $x_1 > y_1 = z_1 \implies x_1 > z_1$

   c) $x_1 = y_1 > z_1 \implies x_1 > z_1$

   d) $x_1 = y_1 = z_1$ and $x_2 \geq y_2 \geq z_2 \implies x_1 = z_1$ and $x_2 \geq z_2$

2. There does not exist $u(\cdot)$ that represents Lexicographic Preference.

   We prove by contradiction. Suppose $\exists u(\cdot)$ that represents $\succsim$.

   For any $x_1 \in \mathbb{R}_+$, $u(x_1, 2) > u(x_1, 1)$ (definition of Lexicographical Preference).

   Therefore, $\exists r(x_1) \in \mathbb{Q}$ s.t. $u(x_1, 2) > r(x_1) > u(x_1, 1)$.

   Consider $x_1 > x_1'$. $r(x_1) > u(x_1, 1) > u(x_1', 2) > r(x_1')$. That is, $r(x_1) > r(x_1')$.

   Hence, we have a function $r : \mathbb{R}^2_+ \to \mathbb{Q}$ that is strictly increasing.

Thus, $r(\cdot)$ provides a one-to-one mapping from an uncountable set ($\mathbb{R}^2_+$) to a countable set $\mathbb{Q}$. This is impossible.

$\square$

1.C. Choice Rules

In reality, the preferences are in Decision Maker (DM)'s mind and we cannot observe them. What we can observe are DM's *choices.* To put the preference theory to work, we need deduce DM's preferences from her decisions.

A *choice structure* $(\mathscr{B}, C(\cdot))$ consists of two ingredients:

(i) $\mathscr{B}$ is a family (a set) of nonempty subsets of X: that is, every $B \in \mathscr{B}$ is a set $B \in X$.

- In consumer theory (Chapter 2 & 3), B are budget sets.
- $\mathscr{B}$ does not need to include all possible subsets of X. The convention is to use a fancy capital letter (like $\mathscr{B}$) for a set of sets.

(ii) $C(\cdot)$ is a choice rule that assigns a nonempty subset of chosen elements $C(B) \subset B$ for every $B \in \mathscr{B}$.

- $C(B)$ is a set of *acceptable alternatives.*

Example 1.C.1. $X = \{x, y, z\}, \mathscr{B} = \{\{x, y\}, \{x, y, z\}\}$

Choice Structure 1 $(\mathscr{B}, C_1(\cdot))$: $C_1(\{x, y\}) = \{x\}$, $C_1(\{x, y, z\}) = \{x\}$

Choice Structure 2 $(\mathscr{B}, C_2(\cdot))$: $C_2(\{x, y\}) = \{x\}$, $C_2(\{x, y, z\}) = \{x, y\}$

- Under $(\mathscr{B}, C_2(\cdot))$, y is acceptable only if z is available.

You might find the choice structure 2 unreasonable. How can the decision maker *not* choose y when the choice set is $\{x, y\}$, but choose y simply when a new item z is added. Consider the following conversation.

Waiter:	Coffee or Tea?
Customer:	Coffee, please.
Waiter:	Sure. Oh sorry, actually we also serve coke. Do you want some coke?
Customer:	*Since coke is available, I'd prefer tea rather than coffee.*

We introduce the following restrictions to eliminate the case that "Since coke is available, I'd prefer tea rather than coffee."

Weak Axiom of Revealed Preference (Reasonable restrictions)

Definition 1.C.1. The choice structure ($\mathscr{B}$, $C(\cdot)$) satisfies the weak axiom of revealed preference (W.A.R.P) if the following property holds:
If for some $B \in \mathscr{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathscr{B}$ with $x, y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$.

In the last example, $(\mathscr{B}, C_2(\cdot))$ violates weak axiom of revealed preference since $x \in C_2(\{x, y, z\})$, $x, y \in \{x, y\}$, $y \in C_2(\{x, y\})$ but $x \notin C_2(\{x, y\})$ [Think of $\{x, y, z\}$ as B and $\{x, y\}$ as B' in definition]
IDEA: Agent's choice between x and y should not be affected by irrelevant options/alternatives.

Revealed Preference: Preference inferred from/ revealed through Choice

Definition 1.C.2. Given a choice structure ($\mathscr{B}$, $C(\cdot)$), the revealed preference relation $\succsim^*$ is defined by

$$x \succsim^* y \iff \exists B \in \mathscr{B} \text{ s.t. } x, y \in B \text{ and } x \in C(B).$$

Remark.

1. $x \succsim^* y$ reads "x is revealed at least as good as y".
2. $x \succ^* y$: $\exists B \in \mathscr{B}$ s.t. $x, y \in B$ and $x \in C(B)$, and $y \notin C(B)$. ("x is revealed preferred to y")
3. $\succsim^*$ needs not to be complete or transitive.

4. Unlike "revealed preference", "preference" is defined without reference to B.

5. **Restatement of W.A.R.P**: If $x \succsim^* y$, then $y \not\succ^* x$. (only imposed on $B \in \mathscr{B}$)

Example 1.C.2. Recall Example 1.C.1.

$(\mathscr{B}, C_1(\cdot))$: $x \succ^* y$ and $x \succ^* y$, $x \succ^* z$

$(\mathscr{B}, C_2(\cdot))$: $x \succ^* y$ and $y \succsim^* x \implies$ contradicts W.A.R.P

Useful alternative statement of W.A.R.P $x, y \in B$, $x \in C(B)$, $y \in C(B')$ & $x \notin C(B')$, then $x \notin B'$.

Proof. Proof by contradiction. If $x \in B'$ & $y \in C(B')$, W.A.R.P $\implies x \in C(B')$. □

1.D. Relationship between Preference Relations & Choice Rules

More precisely, we want to know the relationship between *rational preference* and *weak axiom of revealed preference*, the two restrictions we impose on preference and choice rules (revealed preference).

(i) Does rational preference imply Weak Axiom of Revealed Preference? (YES)

(ii) Does WARP implies Rational Preference? (MAYBE)

Preference Generated Choice Structure Consider rational preference $\succsim$ on X.

Define: $C^*(B, \succsim) = \{x \in B\colon x \succsim y \text{ for every } y \in B\}$

- Elements of $C^*(B, \succsim)$ are DM's most preferred alternatives in B.

- Assumption: $C^*(B, \succsim)$ is nonempty for all $B \in \mathscr{B}$.

We say that the preference $\succsim$ *generates* the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$.

Remark. $\succsim$ is defined independently of B. This already hints W.A.R.P is implied.

Proposition 1.D.1. *Suppose $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot, \succsim))$ satisfies the weak axiom.*

Proof. Suppose $x, y \in B$ and $x \in C^*(B, \succsim)$.

Also suppose $x, y \in B'$ and $y \in C^*(B', \succsim) \implies y \succsim z, \forall z \in B'$.

Since $x \succsim y, y \succsim z, \forall z \in B' \implies x \in C^*(B', \succsim) \implies$ W.A.R.P is satisfied. $\square$

Definition 1.D.1. Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ rationalizes $C(\cdot)$ relative to $\mathscr{B}$ if $C(B) = (C^*(B, \succsim))$ for all $B \in \mathscr{B}$, that is, if $\succsim$ generates the choice structure $(\mathscr{B}, C(\cdot))$.

Remark.

1. If a rational preference relation rationalizes the choice rule, we can interpret the DM's choices as if she were a preference maximizer.

2. In general, there may be more than one rationalizing preference relation $\succsim$ for a given choice structure $(\mathscr{B}, C(\cdot))$.

   Example. $X = \{x, y\}, \mathscr{B} = \{\{x\}, \{y\}\}, C(\{x\}) = x, C(\{y\}) = y$. Any rational preference relation rationalizes $C(\cdot)$.

Example 1.D.1. $X = \{x, y, z\}$, $\mathscr{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$[1], $C(\{x, y\}) = \{x\}, C(\{y, z\}) = \{y\}, C(\{x, z\}) = \{z\}$.

This choice structure satisfies the W.A.R.P.

- The choice is not challenged by having to choose from $\{x, y, z\}$.

- W.A.R.P is defined by $\mathscr{B}$.

Proof. Use Restatement of W.A.R.P : If $x \succsim^* y$, then $y \not\succ^* x$.

From the choice rules, we know $x \succsim^* y$, $y \succsim^* z$, $z \succsim^* x$. There is no choice rule indicating $y \succ^* x$, $z \succ^* y$, or $x \succ^* z$. Thus, W.A.R.P is not violated. $\square$

However, it cannot be rationalized by a rational preference. Part of rationalizing $(\cdot)$ is that $C^*(B, \succsim) = C(B), \forall B \in \mathscr{B}$. Since $C(\{x, y\}) = \{x\}$, it means $x \succsim y$ & $y \not\succsim x$, i.e. $x \succ y$. Similary, $y \succ x$ & $z \succ x$. Therefore, $\succsim$ is not a transitive preference.

Proposition 1.D.2. *If $(\mathscr{B}, C(\cdot))$ is a choice structure such that*

[1] $\{x, y, z\}$ is not empirically relevant.

(i) the W.A.R.P is satisfied, [$x \succsim^* y$, then $y \not\succ^* x$]

(ii) $\mathscr{B}$ includes all subsets of X of up to three elements,

then $\exists$ rational $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$, i.e.,

$$C(B) = (C^*(B, \succsim)), \forall B \in \mathscr{B}.$$

Furthermore, this rational preference relation is unique.

Proof. Natural candidate $\succsim^*$ (revealed via $(\mathscr{B}, C(\cdot))$)

Step (i) $\succsim^*$ is a rational preference.

Step (ii) $\succsim^*$ rationalizes $(\mathscr{B}, C(\cdot))$, i.e., $C(B) = C^*(B, \succsim^*), \forall B \in \mathscr{B}$.

Step (iii) $\succsim^*$ is unique in rationalizing $(\mathscr{B}, C(\cdot))$.

(i) Rational $\succsim^*$

Transitivity. Suppose $x \succsim^* y$ and $y \succsim^* z$.

Consider $\{x, y, z\} \in \mathscr{B}$. If suffices to prove that $\{x\} \in C(\{x, y, z\})$ for $x \succsim^* z$.

$C(\{x, y, z\}) \neq \varnothing$.

Suppose $\{x\} \in C(\{x, y, z\})$. Then $x \succsim^* z$.

Suppose $\{y\} \in C(\{x, y, z\})$. Since $x \succsim^* y$, W.A.R.P implies $x \in C(\{x, y, z\})$, then $x \succsim^* z$.

Suppose $\{z\} \in C(\{x, y, z\})$. Since $y \succsim^* z$, W.A.R.P implies $y \in C(\{x, y, z\})$. By previous case, $x \in C(\{x, y, z\})$, then $x \succsim^* z$.

Completeness. All 2-element subsets belong to $\mathscr{B}$ and $C\{x, y\} \neq \varnothing, \forall (x, y) \in X \implies x \succsim^* y$ or $y \succsim^* x$.

(ii) $\succsim^*$ rationalize $C(B), \forall B \in \mathscr{B}$.

Step (a). $C(B) \subseteq C^*(B, \succsim^*)$

Step (b). $C(B) \supseteq C^*(B, \succsim^*)$

(a) Suppose $x \in C(B)$. Then $x \succsim^* y, \forall y \in B \iff x \in C^*(B, \succsim^*)$

(b) Suppose $x \in C^*(B, \succsim^*)$. Then $x \succsim^* y, \forall y \in B$.

By definition of $\succsim^*$, $\forall y \in B, \exists B_y \in \mathscr{B}$ (e.g. $B_y = \{x, y\}$) such that $x, y \in B_y$ and $x \in C(B_y)$.

Since $C(B) \neq \varnothing$, either $x \in C(B)$ or $\exists y \in B \setminus \{x\}$ such that $y \in C(B)$. Then by W.A.R.P and $x \succsim^* y$, $x \in C(B)$.

(iii) Uniqueness of $\succsim^*$.

$\mathscr{B}$ includes all 2-element subsets of X. The choice behavoir in $C(\cdot)$ completely pins down whether $x \succsim y$ or $y \succsim x$ for the $\succsim$ which rationalizes $C(\cdot)$. So, $\succsim^*$ is unique.

□

Summary of Chapter 1

- A preference relation $\succsim$ is a binary relation on the choice set X.
- $\succsim$ is rational if Completeness & Transitivity.
- Choice function $C(\cdot)$ is defined on $\mathscr{B}$, NOT on X.
- Assumptions: W.A.R.P & $C(\cdot) \neq \varnothing$
- Rational Preference implies W.A.R.P.

  But for W.A.R.P to imply Rational Preference, it requires $C(\cdot) \neq \varnothing$ and that $\mathscr{B}$ includes all 2 & 3-element subsets of X.