# Analysis and Design of Algorithms

## Chapter 5: Divide and Conquer

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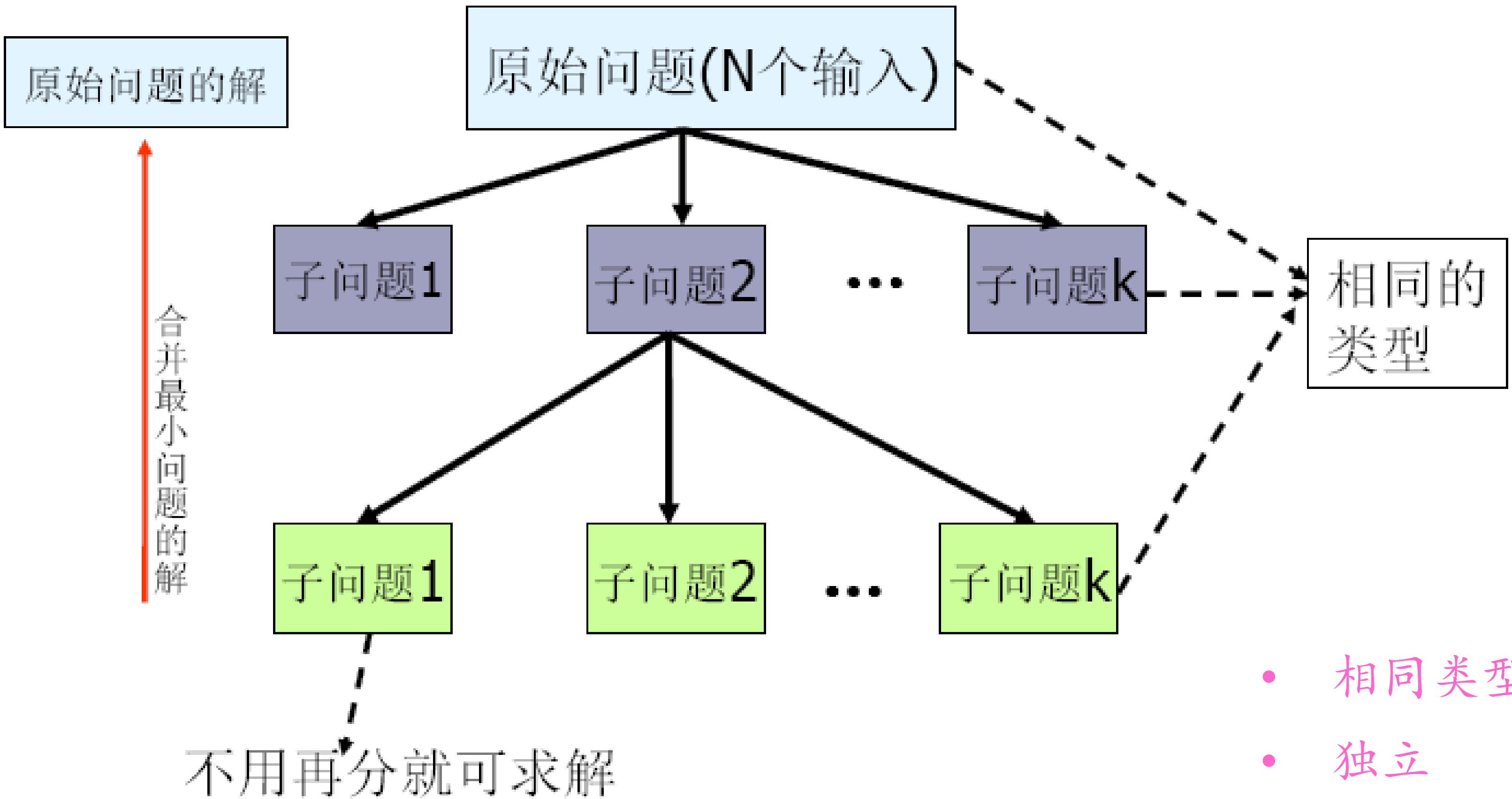
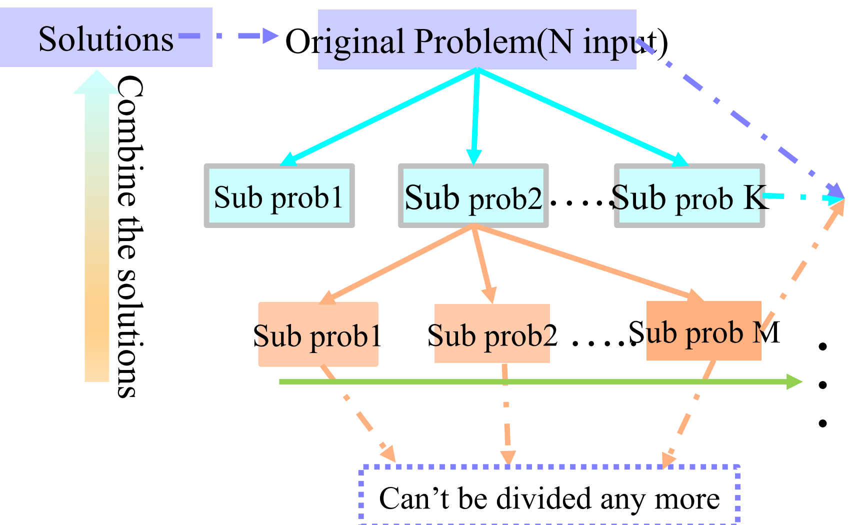
# *Divide and Conquer Approach*




- 相同类型
- 独立
- 规模均衡

# *Divide and Conquer Approach*




Share the same type

- Share the same type
- Dependent
- Similar size of input



分而治之，逐个击破。

# Divide and Conquer Approach

- ***Three Steps of The Divide and Conquer Approach***

  - *Divide the problem into two or more smaller subproblems*
  - *Conquer the subproblems by solving them recursively*
  - *Combine the solutions to the subproblems into the solutions to the original problem*

# Divide and Conquer Approach

## Algorithm analysis—general divide-and-conquer recurrence

- *A problem's instance of size **n** is divided into **a** instances of size **n/b** (**assuming n is a power of b**)*
- ***a** is the number of the problems needs of be solved*
- ***f(n)** is a function that counts for the time spent on dividing the problem into smaller ones and on combing their solutions*

$$T(n) = \begin{cases} O(1) & n = 1 \\ aT(n/b) + f(n) & n > 1 \end{cases}$$

# Multiplication of Large Integers

## *Idea of Multiplication of Large Integers :*karatsuba

Consider the problem of multiplying two (large) $n$-digit integers represented by arrays of their digits such as:

A = 12345678901357986429   B = 87654321284820912836

- *brute-force algorithm*

$$
\begin{array}{r}
a_1\ a_2 \dots\ a_n \\
b_1\ b_2 \dots\ b_n \quad \\
(d_{10})\ d_{11} d_{12} \dots d_{1n} \\
(d_{20})\ d_{21} d_{22} \dots d_{2n} \qquad\quad \\
\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\
(d_{n0})\ d_{n1} d_{n2} \dots d_{nn} \qquad\qquad\qquad
\end{array}
$$

- Efficiency: $\boldsymbol{n^2}$ one-digit multiplications

# Multiplication of Large Integers

## *First Divide-and-Conquer Algorithm*

| | | |
|---|---|---|
| ***X=*** | A | B |
| ***Y=*** | C | D |

- if $X = A\ 10^{n/2} + B$, and $Y = C\ 10^{n/2} + D$ *--divide X, Y into two parts*

  where $X$ and $Y$ are $n$-digit, A, B, C, D are $n/2$-digit numbers

  $$X * Y = AC \cdot 10^{n} + (AD + BC) \cdot 10^{n/2} + BD$$

**Idea of the algorithm**:

- If $n = 2^k$, recurrence
- Otherwise, stop, when $n$=1 or $n$ is small enough to multiply the numbers of that size directly

# *Multiplication of Large Integers*

## *First Divide-and-Conquer Algorithm*

- *Analysis:*
- Basic operation: one-digit multiplication
- $$T(n) = \begin{cases} O(1) & n = 1 \\ 4T(n/2) + O(n) & n > 1 \end{cases}$$

Solution: ***$T(n) = O(n^2)$*** ✘**NO Promotion**

① If $n=2^k$, then
$C(2^k)=4C(2^{k-1})=4[4C(2^{k-2})]=4^2C(2^{k-2})$
$=\ldots =4^kC(2^{k-k})=4^k$
Given $\mathbf{n=2^k}$, we have $\mathbf{k=\log_2 n}$**;**
Then, $C(2^k)=4^{\log n}=2^{2\log n}=n^2 =C(n)$

② If n <> $2^k$ , by Smoothness Rule, $C(n) \in O(n^2)$

# *Multiplication of Large Integers*

## *Second Divide-and-Conquer Algorithm*

$$X * Y = A * C \cdot 10^{n} + (A * D + B * C) \cdot 10^{n/2} + B * D$$

- The idea is to decrease the number of multiplications from 4 to 3:

  Since we have: $(A + B) * (C + D) = AC + (AD + BC) + BD$

  ., $(AD + BC) = (A + B) * (C + D) - A^{*}C - B^{*}D$

$$X * Y = A * C \cdot 10^{n} + [(A + B) * (C + D) - A^{*}C - B^{*}D] \cdot 10^{n/2} + B * D$$

which requires only 3 multiplications at the expense of (4-1) extra addition/subtraction.

# *Multiplication of Large Integers*

## *Second Divide-and-Conquer Algorithm*

- The idea is to decrease the number of multiplications from 4 to 3:

  $(A + B) * (C + D) = AC + (AD + BC) + BD$

  i.e., $(AD + BC) = (A + B) * (C + D) - A^* C - B^*D$

  which requires only 3 multiplications at the expense of (4-1) extra add/sub.

- *Analysis:*

  Recurrence for the number of multiplications *T*(*n*):

$$T(n) = \begin{cases} O(1) & n = 1 \\ 3T(n/2) + O(n) & n > 1 \end{cases}$$

If $n=2^k$, then
$C(2^k)=3C(2^{k-1})=3[3C(2^{k-2})]=3^2C(2^{k-2})$
$=\dots =3^kC(2^{k-k})=3^k$
$k=\log_2 n$
$C(2^k)=C(n)=3^{\log n}=n^{\log 3}\approx n^{1.585}$

Solution:

$T(n) = O(3^{\log_2 n}) = O(n^{\log_2 3}) \approx O(n^{1.585})$ ✓ Promotion

# *Multiplication of Large Integers*

☆ $X = a* 2^{n/2}+b; Y = c* 2^{n/2}+d;$

☆$X*Y=(a* 2^{n/2}+b)*(c* 2^{n/2}+d) =ac * 2^n +(a*d+b*c) * 2^{n/2}+bd$

☆因为 $a*d+b*c = (a+c)(b+d)-ac-bd$

☆所以 $XY = ac\ 2^n + ((a+c)(b+d)-ac-bd)\ 2^{n/2} + bd$

☆ 两个XY的复杂度都是$O(n^{\log 3})$，但考虑到a+c,b+d可能得到m+1位的结果，使问题的规模变大，故不选择第2种方案。

☆ 如果将大整数分成更多段，用更复杂的方式把它们组合起来，将有可能得到更优的算法。

☆ 最终的，这个思想导致了快速傅利叶变换(Fast Fourier Transform)的产生。该方法也可以看作是一个复杂的分治算法。

# Definition of Matrix Multiplication

- ***Idea***
  - *brute-force alg.*: ***$O(n^3)$***

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

*For $C_{ij}$, **n** multiplications and **n-1** summations*

*So for n elements in C, $T(n)= O(n^3)$*

# Strassen's Matrix Multiplication

- ## Idea
  - *Divide and Conquer— idea1*

divide A, B, and C into 4 equal-size sub-matrix,

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$T(n) = \begin{cases} O(1) & n = 1 \\ 8T(n/2) + O(n^2) & n > 1 \end{cases}$$

**$T(n)=O(n^3)$**

# Strassen's Matrix Multiplication

## Idea

- *Divide and Conquer — idea2* *to reduce the times for multiply*

Strassen observed [1969] that the product of two matrices can be computed as follows

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} M_5 + M_4 - M_2 + M_6 & M_1 + M_2 \\ M_3 + M_4 & M_5 + M_1 - M_3 - M_7 \end{bmatrix}$$

$$M_1 = A_{11}(B_{12} - B_{22})$$
$$M_2 = (A_{11} + A_{12})B_{22}$$
$$M_3 = (A_{21} + A_{22})B_{11}$$
$$M_4 = A_{22}(B_{21} - B_{11})$$

$$M_5 = (A_{11} + A_{22})(B_{11} + B_{22})$$
$$M_6 = (A_{12} - A_{22})(B_{21} + B_{22})$$
$$M_7 = (A_{11} - A_{21})(B_{11} + B_{12})$$

# *Strassen's Matrix Multiplication*

## *Analysis of Strassen's Matrix Multiplication*

- *Number of multiplications:*

  $M(n) = 7M(n/2)$,

  $M(1) = 1$

If $n=2^k$, then
$M(n)=7M(n/2)=7^2M(n/2^2)\ldots\ldots$
$=7^kM(1)=7^K$

- Solution:

**$M(n) = 7^{\log_2 n} = n^{\log_2 7} \approx n^{2.807}$ vs. $n^3$ of brute-force alg.**

- If $n$ is not a power of 2, matrices can be padded with zeros.
- ☆ Practical implementation of Strassen's alg. usually switch to brute-force method after matrix sizes become smaller than some "crossover point"

# *Strassen's Matrix Multiplication*

- *Standard vs Strassen: Practical:*

| | N | Multiplications | Additions |
|---|---|---|---|
| Standard alg. | 100 | 1,000,000 | 990,000 |
| Strassen's alg. | 100 | 411,822 | 2,470,334 |
| Standard alg. | 1000 | 1,000,000,000 | 999,000,000 |
| Strassen's alg. | 1000 | 264,280,285 | 1,579,681,709 |
| Standard alg. | 10,000 | $10^{12}$ | $9.99*10^{11}$ |
| Strassen's alg. | 10,000 | $0.169*10^{12}$ | $10^{12}$ |

# Strassen's Matrix Multiplication

- *More algorithms for matrix Multiplication:*
  - Algorithms with better asymptotic efficiency are known but they are even more complex.

| 时间 | 复杂度 | 作者 |
|---|---|---|
| <1969 | O(n^3) | |
| 1969 | O(n^2.81) | Strassen |
| 1978 | O(n^2.79) | Pan |
| 1979 | O(n^2.7799) | Bini,Lotti etc. |
| 1981 | O(n^2.55) | Schonhage |
| 1984 | O(n^2.52) | Victor Pan |
| 1987 | O(n^2.48) | Strassen |
| 1987 | O(n^2.376) | Coppersmith and Winograd |

# Binary Search

问题描述：

已知一个按非降次序排列的元素表$a_0, a_1, \ldots, a_{n-1}$，判定某个给定元素*K*是否在该表中出现，

若是，则找出该元素在表中的位置并返回其所在位置的下标*j*；否则，返回值-1。

## Idea of Binary Search

Very efficient algorithm for searching in sorted array with non-decreasing order

$$K$$

vs

$$A[0] \;.\;.\;.\; A[m] \;.\;.\;.\; A[n\text{-}1]$$

选取一个下标m，可得到三个子问题：

$I_1=(m, a_0, \ldots, a_{m-1}, K)$

$I_2=(1, a_m, K)$

$I_3=(n-m-1, a_{m+1}, \ldots, a_{n-1}, K)$

- If $K = A[m]$, stop (successful search);
- otherwise, continue searching by the same method
  in $A[0..m\text{-}1]$ if $K < A[m]$
  and in $A[m\text{+}1..n\text{-}1]$ if $K > A[m]$

如果对所求解的问题（或子问题）所选的下标 *m* 都是中间元素的下标，m= $\lfloor (l+r)/2 \rfloor$，则由此产生的算法就是二分检索算法。

# *Binary Search*

- ***Example***

  - K=70

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 14 | 27 | 31 | 39 | 42 | 55 | 70 | 74 | 81 | 85 | 93 | 98 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| l | | | | | | m | | | | | | r |
| | | | | | | | l | | m | | | r |
| | | | | | | | l,m | r | | | | |

# Binary Search

- ***Example***

  If A(1:9)=(-15，-6，0，7，9，23，54，82，101)

  search in A: *k* =101，-14，82。

  Searching process:

| k=101 | | | k=-14 | | | k=82 | | |
|---|---|---|---|---|---|---|---|---|
| low | high | mid | low | high | mid | low | high | mid |
| 1 | 9 | 5 | 1 | 9 | 5 | 1 | 9 | 5 |
| 6 | 9 | 7 | 1 | 4 | 2 | 6 | 9 | 7 |
| 8 | 9 | 8 | 1 | 1 | 1 | 8 | 9 | 8 |
| 9 | 9 | 9 | 2 | 1 | | | | |
| | | 找到 | | | 找不到 | | | 找到 |
| successful search | | | unsuccessful search | | | successful search | | |

# *Binary Search*

## *Binary Search – an Iterative Algorithm*

ALGORITHM BinarySearch(A[0..n-1], K)

$l \leftarrow 0$; $r \leftarrow n$-1

while $l \leq r$ do // $l$ and $r$ crosses over→ can't find K

 $m \leftarrow \lfloor (l+r)/2 \rfloor$

 if $K$ = A[$m$] return $m$ //the key is found

 else if $K$ < A[$m$] $r \leftarrow m$-1 //the key is on the left half of the array

 else $l \leftarrow m$+1 // the key is on the right half of the array

return -1

# *Binary Search*

## *Binary Search – a Recursive Algorithm*

Basic operation:

while循环中 *k* 与 *A* 中元素的比较运算

*three-way comparison*

```
ALGORITHM BinarySearchRecur(A[0..n-1], l, r, K)
if l > r
    return –1
else
   m ← ⌊(l + r) / 2⌋
    if K = A[m]
            return m
    else if K < A[m]
            return BinarySearchRecur(A[0..n-1], l, m-1, K)
    else
            return BinarySearchRecur(A[0..n-1], m+1, r, K)
```

# *Binary Search*

## *Analysis of Binary Search*

- *Basic operation:* *key comparison* (*three-way comparison* )
- *Worst-case* (*successful or fail*) *:*

  $$C_w(n) = C_w(\lfloor n/2 \rfloor) + 1 \,,$$
  $$C_w(1) = 1$$

  - Solution:

  $$C_w(n) = \Theta(\log n)$$

- *Best-case:*
  - successful $C_b(n) = 1$

    一次找到，即K＝A[n/2]

for $n = 2^k$,

$C(2^k) = C(2^{k-1}) + 1$ for $k>0$

$C(2^0) = 1$

backward substitutions:

$C(2^k) = C(2^{k-1}) + 1$

$= [C(2^{k-2}) + 1] + 1$

$= C(2^{k-2}) + 2 = \ldots = C(2^{k-i}) + i$ …

$= C(2^{k-k}) + k = 1 + k$

then, $C(n) = \log_2 n + 1$

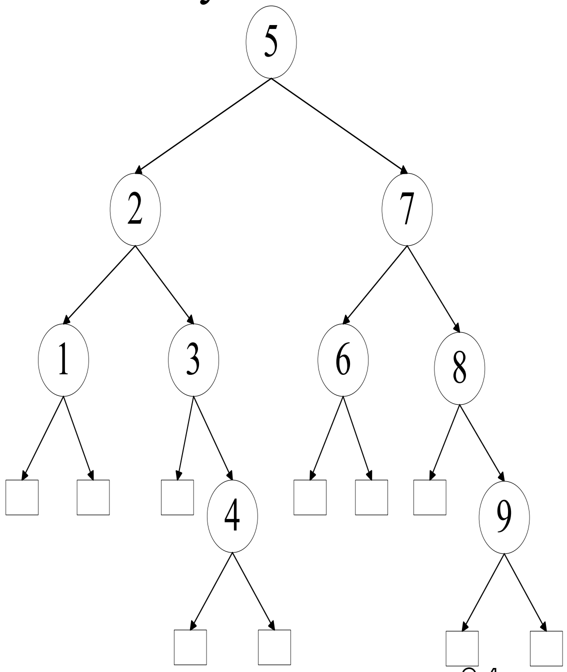
# Binary Search

## Analysis of Binary Search

### *Average-case:*

- Consider the searching process as a binary tree. In looking at the binary tree, we see that there are $i$ comparisons needed to search $2^{i-1}$ elements on level $i$ of the tree.
- For a list with $n = 2^k-1$ elements, there are $k$ levels in the binary tree.
- The average case for all successful search:

$$A(n) = \frac{1}{n}\sum_{i=1}^{k} i2^{i-1} \approx \log(n+1)-1$$



# Binary Search

## Analysis of Binary Search

- 区分以下情况进行分析

1. 成功检索：指所检索的$K$恰好在A中出现
   由于A中共有n个元素，故成功检索恰好有n种可能的情况
2. 不成功检索：指 $K$ 不出现在A中
   根据取值，不成功检索共有n+1种可能的情况 (取值区间)

   $K < A(1)$ 或 $A(i) < K < A(i+1), 1 \le i < n-1$ 或 $K > A(n)$

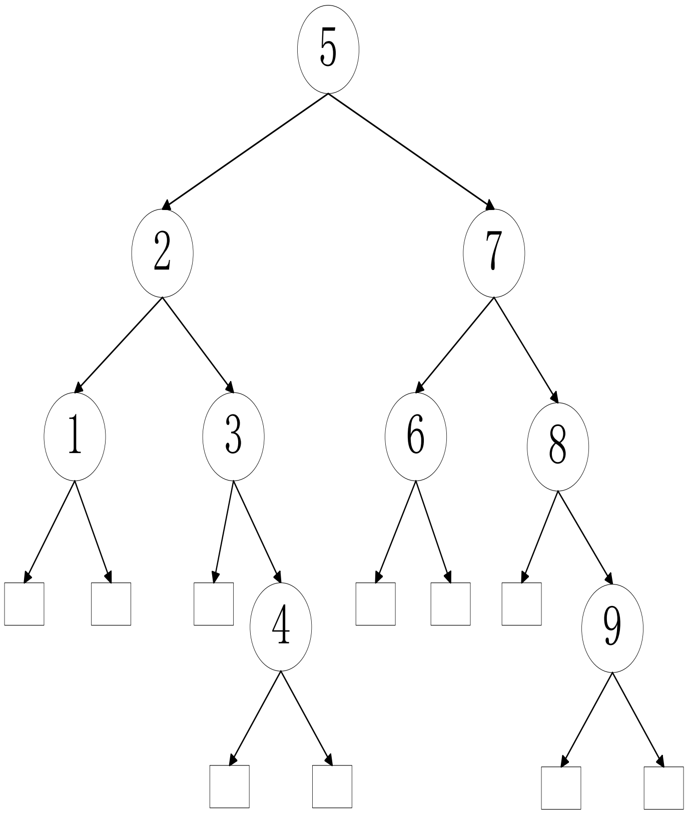
# *Binary Search*

- 二元比较树

 算法执行过程的主体是k与一系列中间元素A(mid)比较。

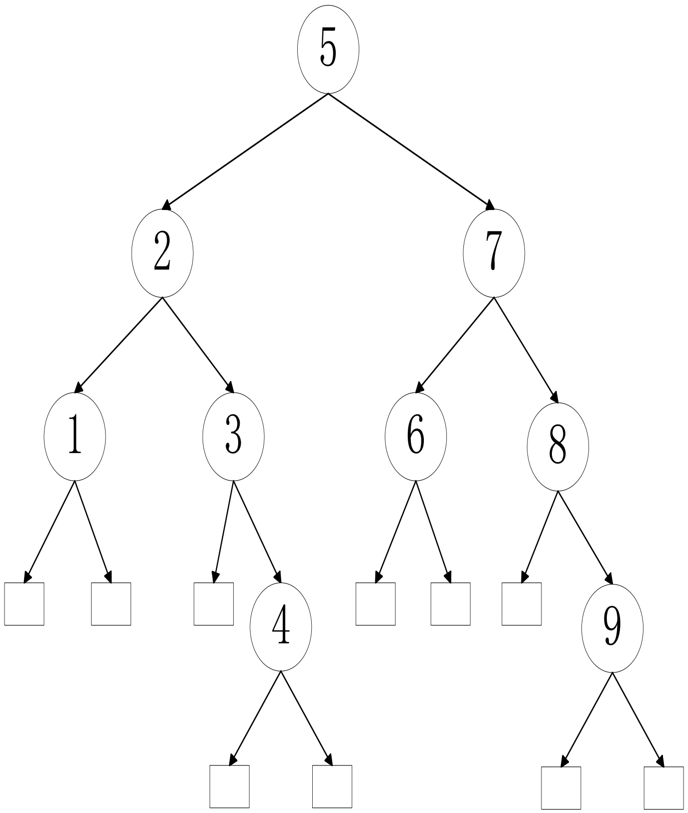
用一棵二元树描述该过程，称为二元比较树

- 结点：内结点和外结点
  - 内结点：
    - 代表一次元素比较
    - 用 圆形结点表示
    - 存放一个 mid值(下标)
    - 代表成功检索情况
  - 外结点：
    - 用方形结点表示，
    - 表示不成功检索情况
- 路径：代表检索中元素的比较序列



# Binary Search

- 二元比较树的查找过程
  - 若*K*在A中出现，则算法的执行过程在一个圆形的**内结点**处结束
  - 若*K*不在A中出现，则算法的执行过程在一个方形的**外结点**处结束




□ 注：外结点不代表元素的比较，因为比较过程在该外结点的上一级的内结点处结束。

# Binary Search

## Analysis of Binary Search

逐个查找法的复杂度AverageCase

$$T_{avg}(n) = \sum_{size(I)=n} p(I)T(I)$$

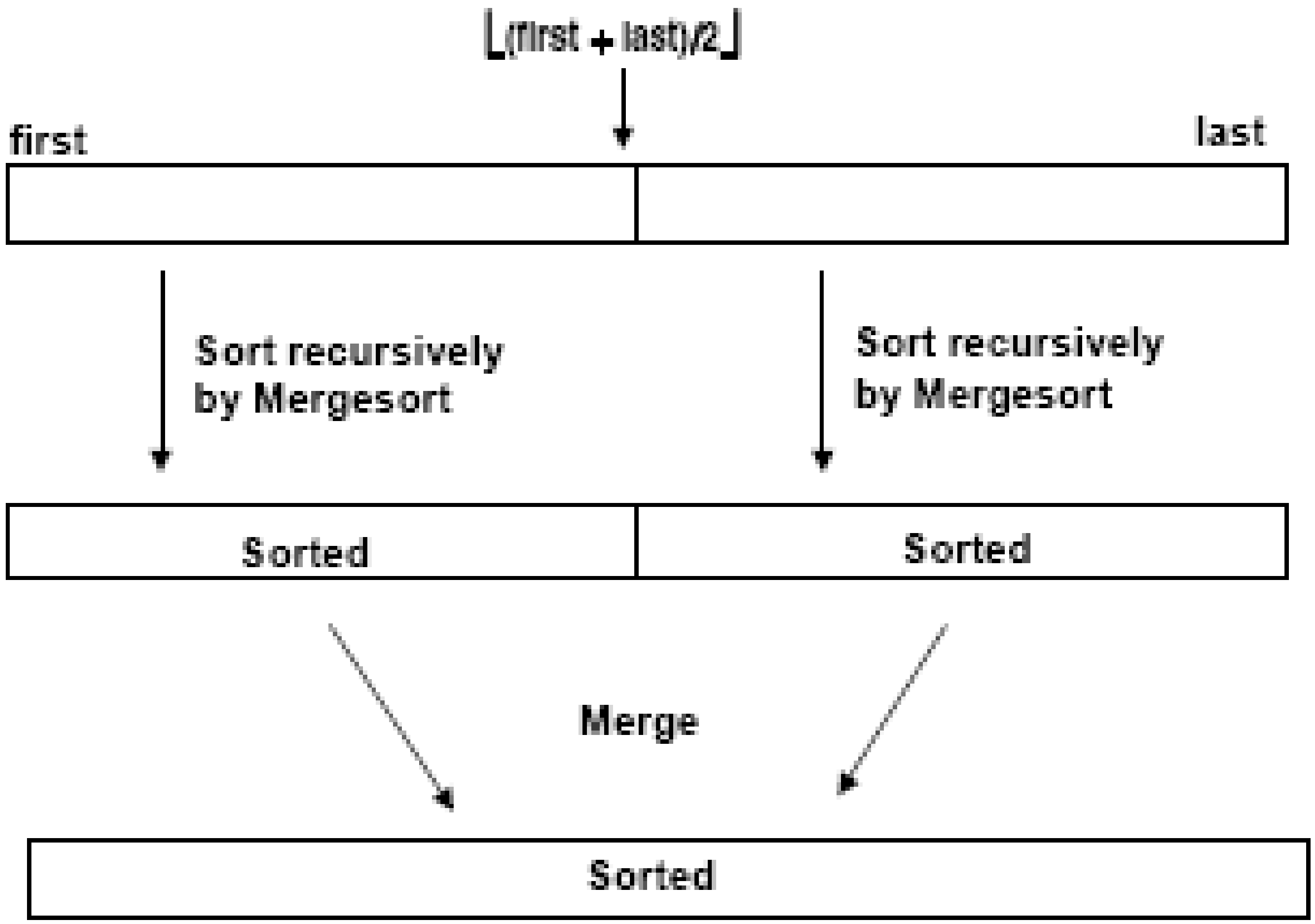
$$= \left(1 \cdot \frac{p}{n} + 2 \cdot \frac{p}{n} + 3 \cdot \frac{p}{n} + \cdots + n \cdot \frac{p}{n}\right) + n \cdot (1 - p)$$

$$= \frac{p}{n} \sum_{i=1}^{n} i + n(1 - p) = \frac{p(n+1)}{2} + n(1 - p)$$

# Mergesort

- ***Idea of Mergesort***
  - *Step1 Divide: divide array A[0..n-1] in two about equal halves and make copies of each half in arrays B and C*
  - *Step 2 Conquer:*
    - *If number of elements in B and C is 1, directly solve it (go to step 3)*
    - *Sort arrays B and C recursively (go to step 1)*
  - *Step 3 Combine: Merge sorted arrays B and C into a single sorted A*
  - *Repeat the following until no elements remain in one of the arrays:*
    - *compare the first elements in the remaining unprocessed portions of the arrays B and C*
    - *copy the smaller of the two into A, while incrementing the index indicating the unprocessed portion of that array*
  - *Once all elements in one of the arrays are processed, the remaining unprocessed elements from the other array are copied into the end of A.*

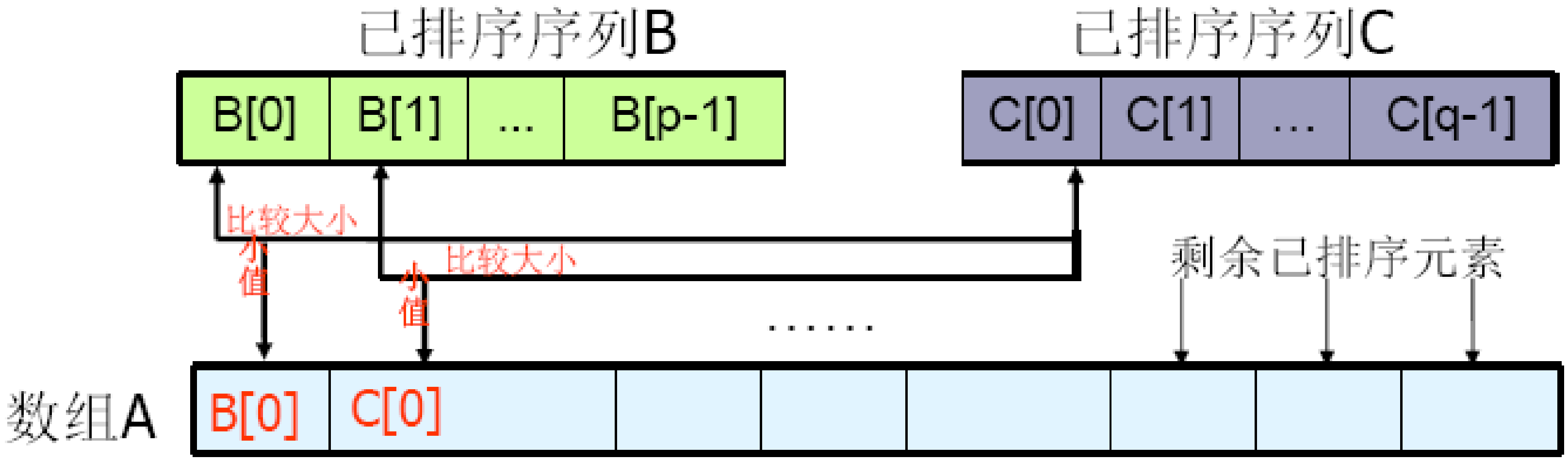
# *Mergesort*



⌊(first + last)/2⌋
first
last
Sort recursively by Mergesort
Sort recursively by Mergesort
Sorted
Sorted
Merge
Sorted

# Mergesort

*MERGE*



# Mergesort

## The Mergesort Algorithm

**ALGORITHM** *Mergesort*$(A[0..n-1])$

//Sorts array $A[0..n-1]$ by recursive mergesort
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $A[0..n-1]$ sorted in nondecreasing order
**if** $n > 1$
  copy $A[0..\lfloor n/2 \rfloor - 1]$ to $B[0..\lfloor n/2 \rfloor - 1]$
  copy $A[\lfloor n/2 \rfloor..n-1]$ to $C[0..\lceil n/2 \rceil - 1]$
  *Mergesort*$(B[0..\lfloor n/2 \rfloor - 1])$
  *Mergesort*$(C[0..\lceil n/2 \rceil - 1])$
  *Merge*$(B, C, A)$

中分点 mid← $\lfloor$(low+high)/2$\rfloor$

# Mergesort

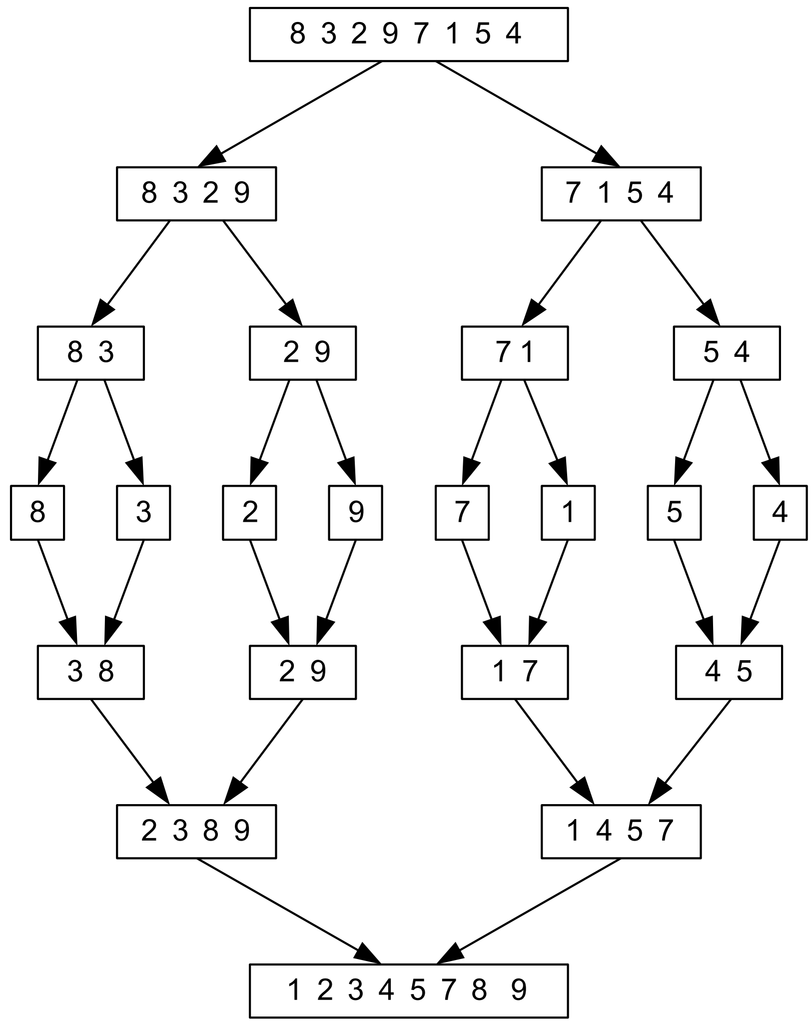
## The Mergesort Algorithm ('cont)

**ALGORITHM** $Merge(B[0..p-1], C[0..q-1], A[0..p+q-1])$

//Merges two sorted arrays into one sorted array
//Input: Arrays $B[0..p-1]$ and $C[0..q-1]$ both sorted
//Output: Sorted array $A[0..p+q-1]$ of the elements of $B$ and $C$
$i \leftarrow 0$; $j \leftarrow 0$; $k \leftarrow 0$
**while** $i < p$ **and** $j < q$ **do**
  **if** $B[i] \le C[j]$
    $A[k] \leftarrow B[i]$; $i \leftarrow i + 1$
  **else** $A[k] \leftarrow C[j]$; $j \leftarrow j + 1$
  $k \leftarrow k + 1$
**if** $i = p$
  copy $C[j..q-1]$ to $A[k..p+q-1]$
**else** copy $B[i..p-1]$ to $A[k..p+q-1]$

# Mergesort

- ***Example:***
  - 8 3 2 9 7 1 5 4



# Mergesort

- ***Analysis of Mergesort***
  - *Number of basic operations (key comparisons):*

$$C(n) = 2C(n/2) + C_{merge}(n) \text{ for } n>1$$

$$C(1) = 0$$

Where $C_{merge}(n) = n-1$

  - **$C(n) = \Theta(n \log n)$**

# Quicksort

## Idea of Quicksort ------ Partition

- 快速分类是一种基于划分的分类方法；

- *Divide:* *Partition array A[l..r] into 2 subarrays, A[l..s-1] and A[s+1..r] such that each element of the first array is ≤A[s] and each element of the second array is ≥ A[s]. (<u>computing the index of s is part of partition</u>.)*
  - *Implication: A[s] will be in its final position in the sorted array.*

- 快速分类：通过反复地对待排序集合进行划分达到分类目的的分类算法。

- *Conquer:*
  - *Sort the two subarrays A[l..s-1] and A[s+1..r] by recursive calls to quicksort*

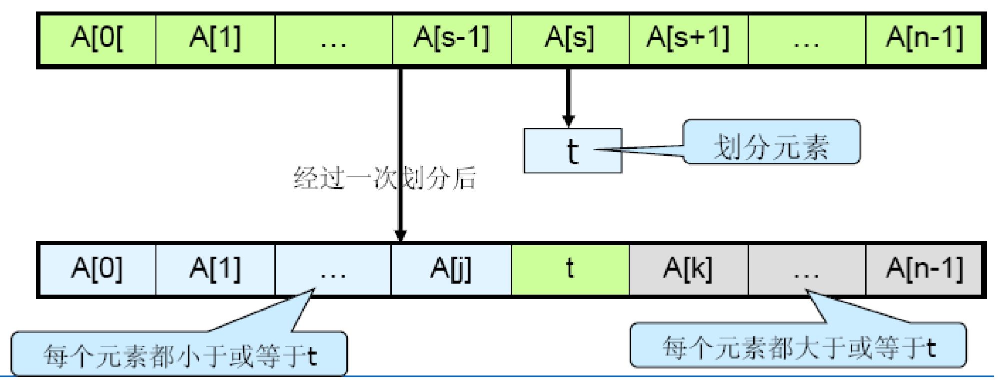
- A[l..s-1] 中所有元素小于等于 A[s+1..r] 中任何元素，所以这两个集合可独立进行划分

- *Combine:* *No work is needed, because A[s] is already in its correct place after the partition is done, and the two subarrays have been sorted.*

# Quicksort

## Idea of Quicksort ('cont) Pseudo Code of the Algorithm

1. *Select a pivot w.r.t. whose value we are going to divide the list. (typically, p = A[l])*
2. *Rearrange the list so that*
   - *all elements in the first s positions are smaller than or equal to **p***
   - *all elements in the remaining n-s positions are larger than or equal to **p***




3. *After exchanging **p** with the last element in the first sublist(i.e., ≤ sublist) , the pivot **p** is now in its final position*
4. *Sort the two sublists recursively using quicksort (i.e. **go to step 1**).*

# Quicksort

## Idea of Quicksort (‘cont)

### Strategy for pivot selection

1. *Randomly selected*

2. *Simplest Strategy: selecting the array’s first element* $A[l]$

# Quicksort

## Idea of Quicksort (‘cont)

- ***Procedure for rearranging elements in a partition***

  *------ based on two-scans of the subarray*

1. *Left-to-right scan: index i, starts with the second element,*
   - *Wants elements smaller than the pivot to be in the first part*
   - *Skip over elements that are smaller than the pivot*
   - *Stop on encountering the first element greater than or equal to pivot*

2. *Right-to-left scan: index j, starts with the last element,*
   - *Wants elements larger than the pivot to be in the second part of the subarray*
   - *Skip over elements that are larger than the pivot*
   - *Stop on encountering the first element smaller than or equal to pivot*

# *Quicksort*

- *three cases for scan stopping*

*i, j not crossed,* $i < j$

→ $i$ $j$ ←

| | | | | | |
|---|---|---|---|---|---|
| p | all ≤ p | ≥ p | ... | ≤ p | all ≥ p |

→ $i{+}{+};\ j\,{-}{-}$

*i, j crossed,* $i > j$

→ $i$ $j$ ←

| | | | | |
|---|---|---|---|---|
| p | all ≤ p | ≤ p | ≥ p | all ≥ p |

*Partition achieved*

*pointing to the same,* $i = j$

→ $i = j$ ←

| | | | |
|---|---|---|---|
| p | all ≤ p | = p | all ≥ p |

*Partition achieved,* $s = i = j$

# Quicksort

## The Quicksort Algorithm

ALGORITHM Quicksort(A[*l*..*r*])
//Sorts a subarray by quicksort
//Input: A subarray A[*l*..*r*] of A[0..*n*-1],defined by its left and right indices *l* and *r*
//Output: The subarray A[*l*..*r*] sorted in nondecreasing order
if $l < r$
    s ← Partition (A[*l*..*r*]) // s is a split position
    Quicksort(A[*l*..*s*-1])
    Quicksort(A[*s*+1..*r*]

# Quicksort

## The Quicksort Algorithm - Partitioning

```
template<class Type>
int Partition (Type a[], int l, int r)
{
        int i = l, j = r + 1;
        Type x=a[l];
        // 将< x的元素交换到左边区域
       // 将> x的元素交换到右边区域
        while (true) {
          while (a[++i] <x);
          while (a[- -j] >x);
          if (i >= j) break;
          Swap(a[i], a[j]);
          }
 //将x交换到它在排序序列中应在的位置上
       a[l] = a[j];
       a[j] = x;
return j;
}
```

Number of comparisons:

$n + 1$ (if indices $i$, j, cross over)

$n$ (if indices $i$, j, coincide )

# Quicksort

- *Example* A : **65** 70 75 80 85 60 55 50 45

| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | i | j |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A: | 65 | 70 | 75 | 80 | 85 | 60 | 55 | 50 | 45 | +∞ | 2 | 9 |
| A: | 65 | 45 | 75 | 80 | 85 | 60 | 55 | 50 | 70 | +∞ | 3 | 8 |
| A: | 65 | 45 | 50 | 80 | 85 | 60 | 55 | 75 | 70 | +∞ | 4 | 7 |
| A: | 65 | 45 | 50 | 55 | 85 | 60 | 80 | 75 | 70 | +∞ | 5 | 6 |
| A: | 65 | 45 | 50 | 55 | 60 | 85 | 80 | 75 | 70 | +∞ | 6 | 5 |
| A: | 60 | 45 | 50 | 55 | 65 | 85 | 80 | 75 | 70 | +∞ | | |

*j......i*

交换划分元素

划分元素定位于此

# Quicksort

- ***Analysis of Quicksort***
  - *basic operation:* *key comparison*
  - *Based on whether the partitioning is balanced.*

# *Quicksort*

## *Analysis of Quicksort*

*Number of comparisons for a partition:*

$n + 1$ (if indices *i, j*, cross over)

$n$ (if indices *i, j*, coincide )

- **Best case**: *split in the middle — **Θ ( n log n)***

$$C_b(n) = 2C_b(n/2) + \Theta(n)$$ //2 subproblems of size n/2 each

$$C_b(1) = 0$$

for $n = 2^k$, backward substitutions, could get it

# Quicksort

## Analysis of Quicksort

- **Worst case:** *sorted array!* — $\Theta\ (n^2)$

$C_w(n) = C_w(n\text{-}1) + \Theta\ (n)$ //2 subproblems of size 0 and n-1

A [0…$n$-1] is a strictly increasing array, and A [0] is used as pivot, the left-to-right scan stops on A [1], right-to-left scan goes all the way to A [0],

| | $\rightarrow i$ | | $j \leftarrow$ |
|---|---|---|---|
| A [0] | A [1] | ... | A [n-1] |

$n$+1 Comparisons

$C_w = (n+1) + n + \ldots + 3 = (n+1)(n+2)/2 - 3 = \Theta\ (n^2)$

# Quicksort

## Analysis of Quicksort

- *Average case*: *random arrays* — ***O( n log n)***

# Quicksort

## Improvements

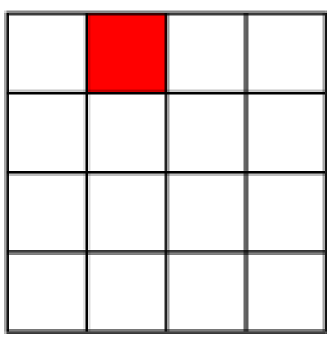
- 快速排序算法的性能取决于划分的对称性。通过修改算法partition，可以设计出采用随机选择策略的快速排序算法。
  - 随机选取划分元素

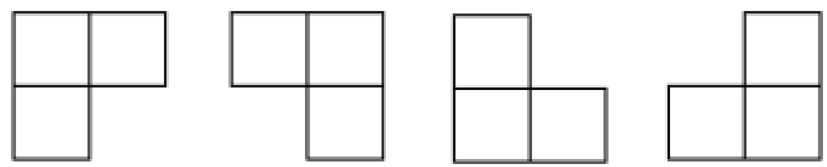
    在快速排序算法的每一步中，当数组还没有被划分时，可以在a[p:r]中随机选出一个元素作为划分基准，这样可以使划分基准的选择是随机的，从而可以期望划分是较对称的。

```
template<class Type>
int RandomizedPartition (Type a[], int p, int r)
{
        int i = Random(p,r);
        Swap(a[i], a[p]);
        return Partition (a, p, r);
}
```

# 棋盘覆盖（Chessboard Cover）问题

在一个$2^k \times 2^k$ 个方格组成的棋盘中，恰有一个方格与其它方格不同，称该方格为一特殊方格，且称该棋盘为一特殊棋盘。在棋盘覆盖问题中，要用图示的4种不同形态的L型骨牌覆盖给定的特殊棋盘上除特殊方格以外的所有方格，且任何2个L型骨牌不得重叠覆盖。

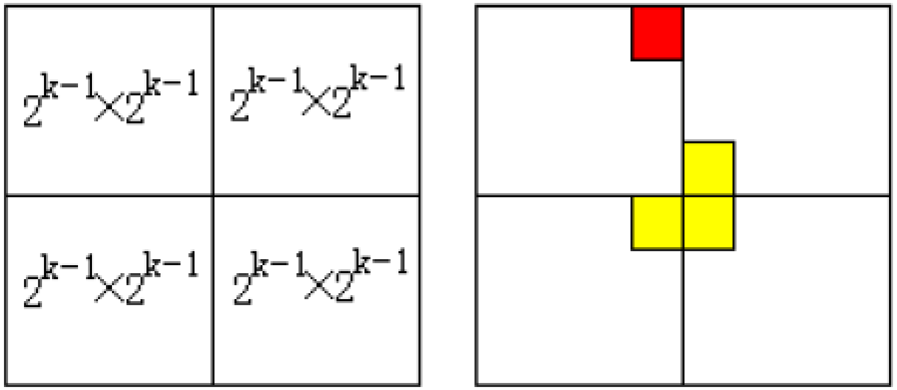




# 棋盘覆盖问题

**分治策略：**

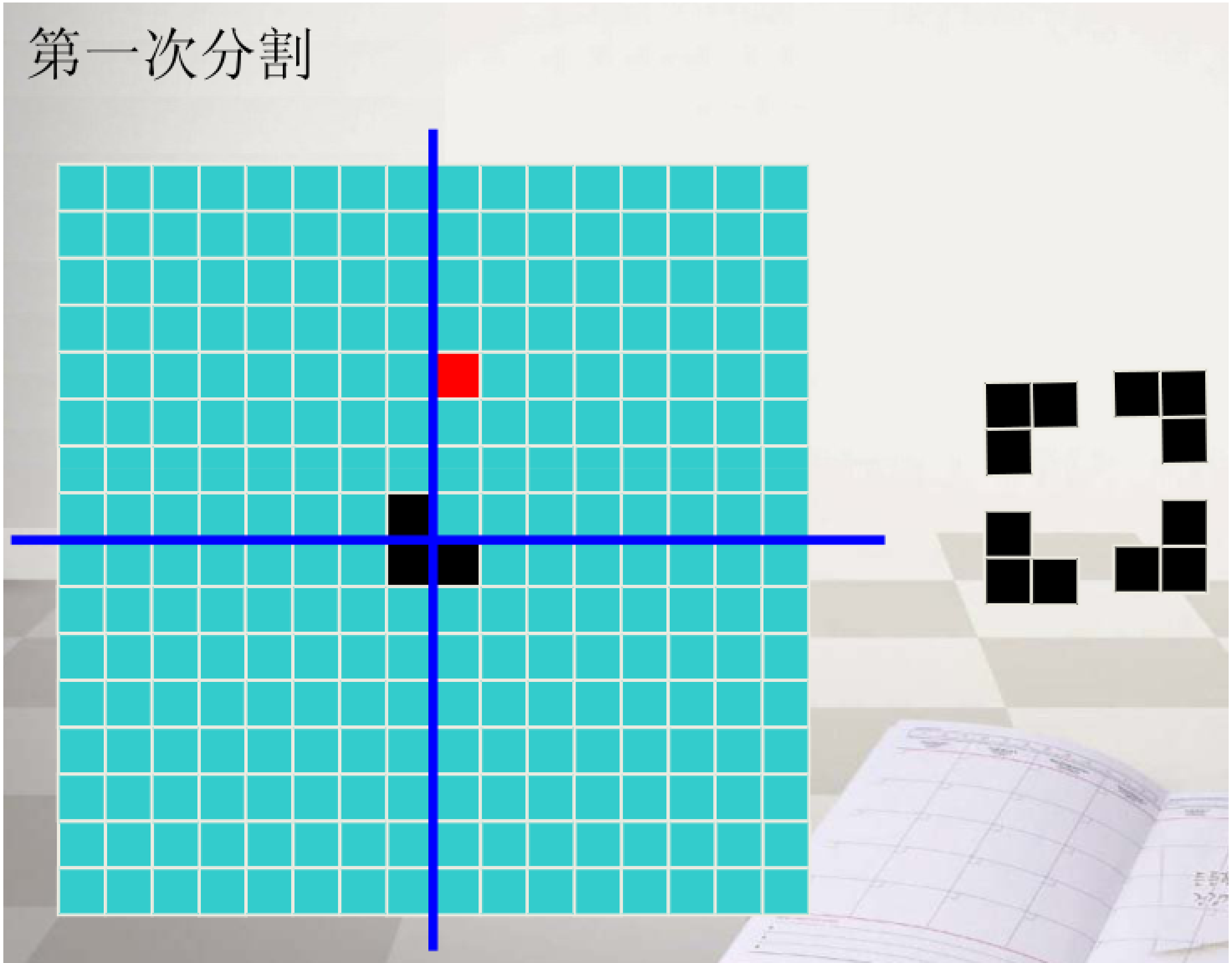
特殊方格必位于4个较小子棋盘之一中，其余3个子棋盘中无特殊方格。为了将这3个无特殊方格的子棋盘转化为特殊棋盘，可以用一个L型骨牌覆盖这3个较小棋盘的会合处，从而将原问题转化为4个较小规模的棋盘覆盖问题。递归地使用这种分割，直至棋盘简化为棋盘1×1。



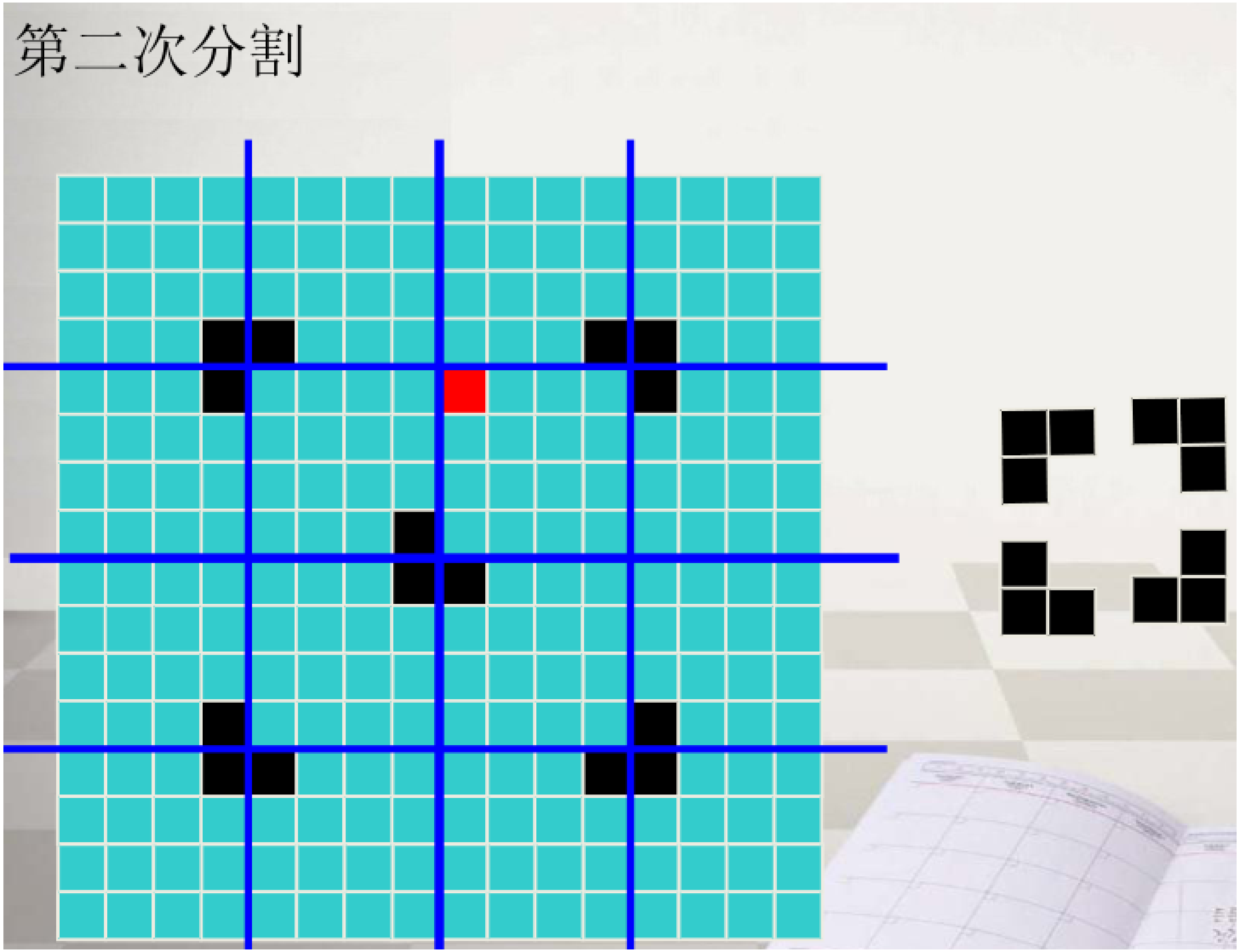

(a) (b)

52

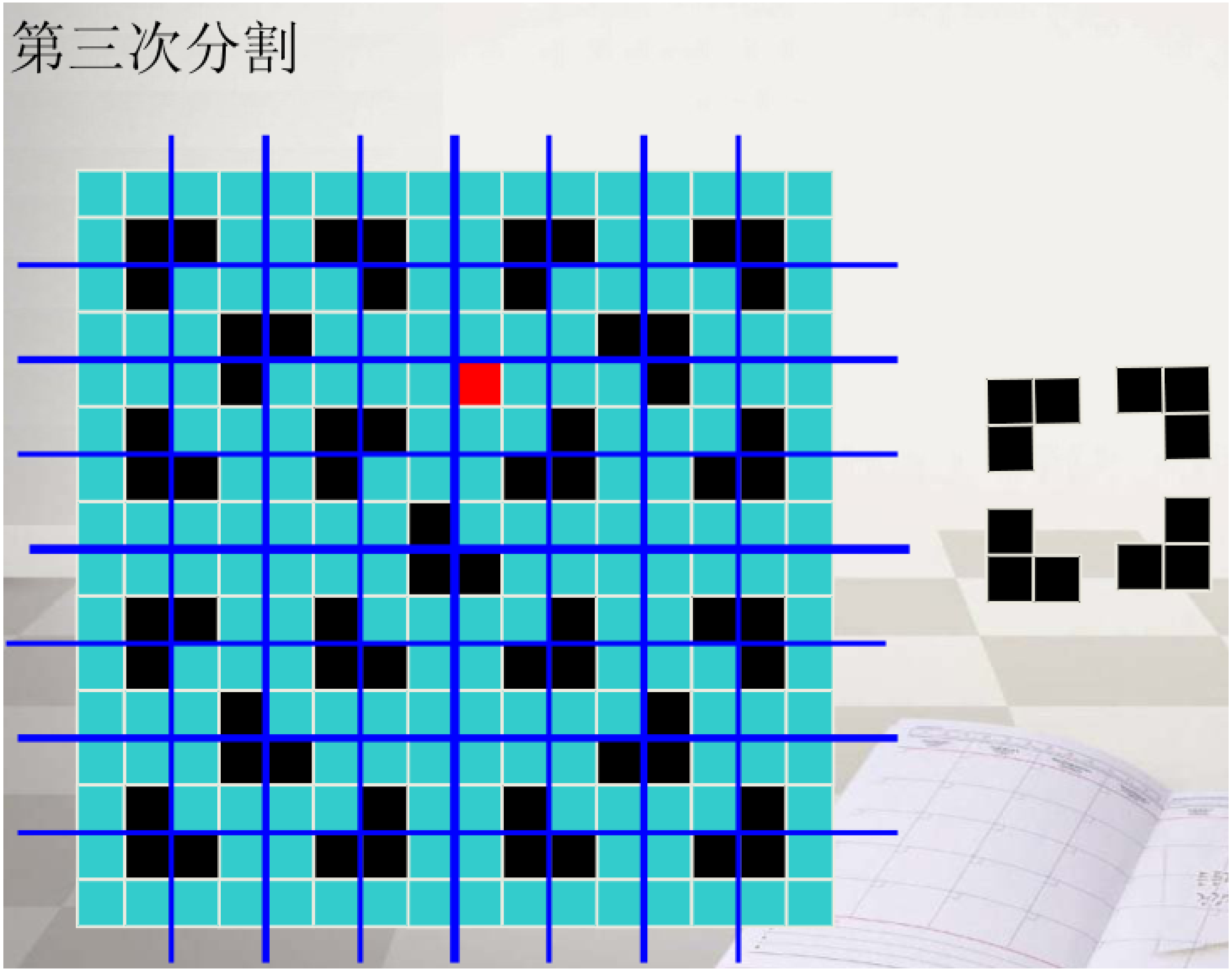
# 棋盘覆盖问题



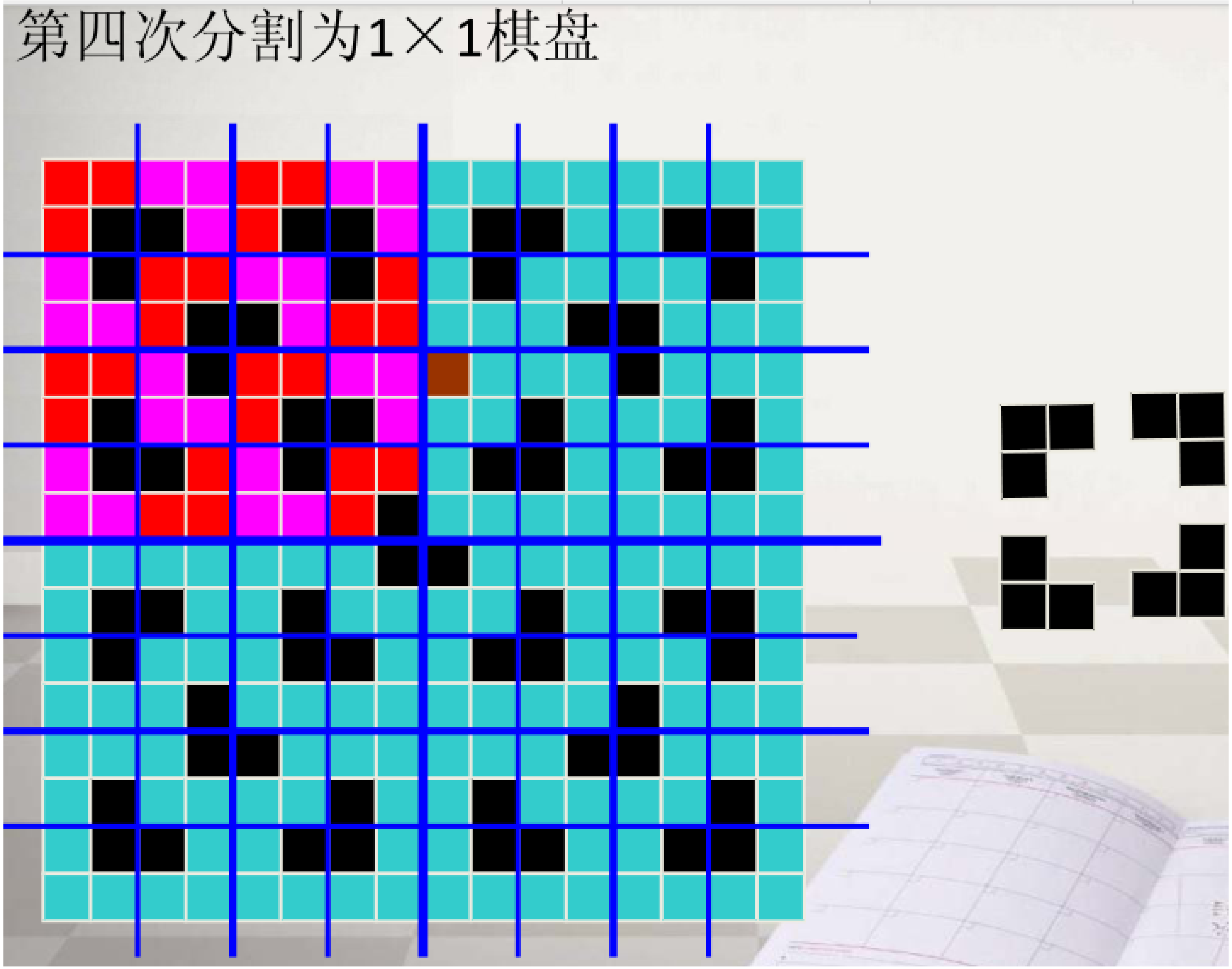
# 棋盘覆盖问题



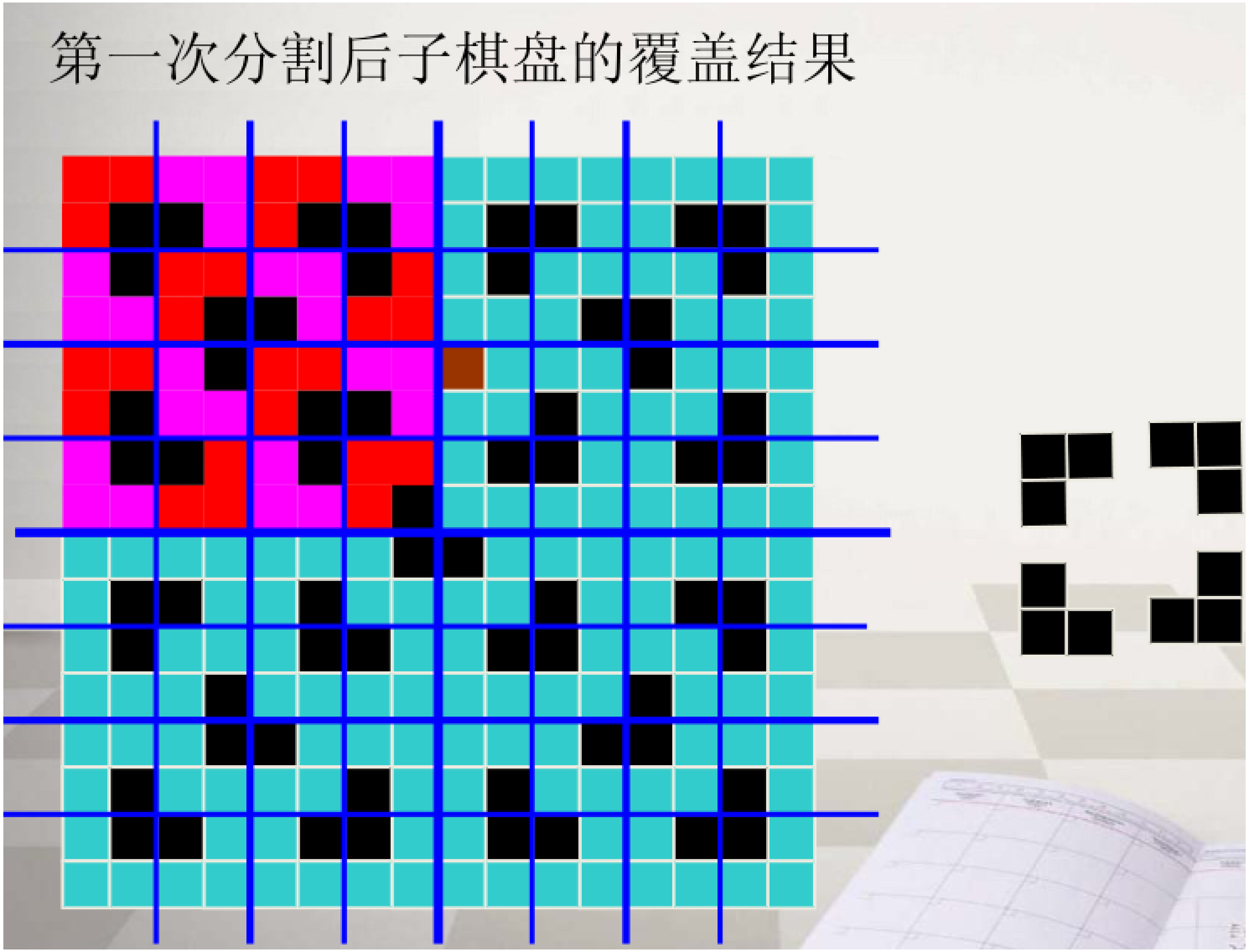
# 棋盘覆盖问题



# 棋盘覆盖问题



# 棋盘覆盖问题



# 棋盘覆盖问题

**分析算法所用数据及数据结构：**

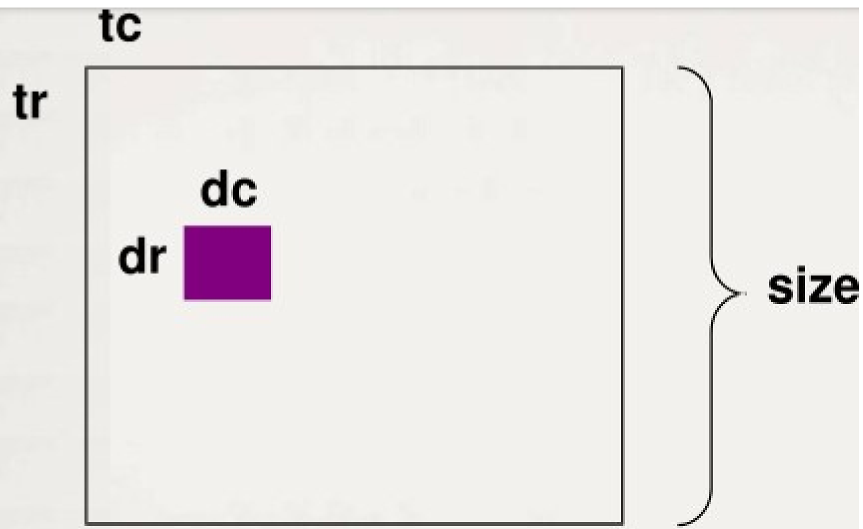
（1）棋盘：可以用一个二维数组board[size][size]表示一个棋盘，其中，size=2^k。为了在递归处理的过程中使用同一个棋盘，将数组board设为全局变量；

（2）子棋盘：当整个棋盘用二维数组board[size][size]表示，其中的子棋盘由棋盘左上角的下标tr、tc和棋盘大小s表示；

（3）特殊方格：用board[dr][dc]表示特殊方格，**dr和dc**是该特殊方格在二维数组board中的下标;

（4） L型骨牌：将所有L型骨牌从1开始连续编号，用一个全局变量t表示。

# 棋盘覆盖问题




棋盘覆盖问题中的数据结构

算法的输入参数是:

tr:棋盘左上角方格的行号

tc:棋盘左上角方格的列号

dr:特殊方格所在的行号

dc:特殊方格所在的列号

size:size= $2^k$, 棋盘规格为 $2^k \times 2^k$

# 棋盘覆盖问题

```
void chessBoard(int tr, int tc, int dr, int dc, int size)
  {
    if (size == 1) return;
    int t = tile++,  // L型骨牌号
      s = size/2;  // 分割棋盘
    // 覆盖左上角子棋盘
    if (dr < tr + s && dc < tc + s)        // 特殊方格在此棋盘中
      chessBoard(tr, tc, dr, dc, s);
    else {  // 此棋盘中无特殊方格,则用 t 号L型骨牌覆盖该子棋盘的右下角
      board[tr + s - 1][tc + s - 1] = t;
      chessBoard(tr, tc, tr+s-1, tc+s-1, s);}

    // 覆盖右上角子棋盘
    if (dr < tr + s && dc >= tc + s)       // 特殊方格在此棋盘中
      chessBoard(tr, tc+s, dr, dc, s);
    else {    // 此棋盘中无特殊方格,则用 t 号L型骨牌覆盖左下角
```

# 棋盘覆盖问题

```
    board[tr + s - 1][tc + s] = t;
    chessBoard(tr, tc+s, tr+s-1, tc+s, s);}

 // 覆盖左下角子棋盘
 if (dr >= tr + s && dc < tc + s)        // 特殊方格在此棋盘中
    chessBoard(tr+s, tc, dr, dc, s);
 else {      //此棋盘中无特殊方格,则用 t 号L型骨牌覆盖右上角
    board[tr + s][tc + s - 1] = t;
    chessBoard(tr+s, tc, tr+s, tc+s-1, s);}

 // 覆盖右下角子棋盘
 if (dr >= tr + s && dc >= tc + s)       // 特殊方格在此棋盘中
    chessBoard(tr+s, tc+s, dr, dc, s);
 else {    //此棋盘中无特殊方格,则用 t 号L型骨牌覆盖左上角
    board[tr + s][tc + s] = t;
    chessBoard(tr+s, tc+s, tr+s, tc+s, s);}
}
```

# 棋盘覆盖问题

**复杂度分析：**

$$T(k)=\begin{cases} O(1) & k=0 \\ 4T(k-1)+O(1) & k>0 \end{cases}$$

$T(n)=O(4^k)$ 渐进意义下的最优算法

如何推导出所需要的骨牌个数呢？

(4^k -1)/3

# Summary

1. 分治法是一种一般性的算法设计技术，他将问题的实例划分为若干个较小的实例(最好用有相同的规模)，对这些小的问题求解，然后合并这些解，得到原始问题的解。
2. 分治法的时间效率满足：$T(n)=aT(n/b)+f(n)$
3. 合并排序是一种分治排序算法，任何情况下，该算法的时间效率都是$\Theta(n\log n)$，它的键值比较次数非常接近理论的最小值，缺点是需要大量的额外存储空间。
4. 快速排序也是一种分治排序算法，具有出众的时间效率nlogn，最差效率是平方级的。
5. 折半查找是一种对有序数组进行查找的算法，效率为logn
6. n位大整数乘法的分治算法，大约需要做$n^{1.585}$次乘法。
7. Stressan 算法也是分治算法

# 思考题

1. 设a[0:n-1]是一个已排好序的数组。设计搜索算法，使得当搜索元素在数组中时，i和j相同，均为x在数组中的位置；搜索元素x不在数组中时，返回小于x的最大元素的位置i和大于x的最小元素位置j。

   并对自己的程序进行复杂性分析。

2. 给定2个大整数u和v，分别有m位和n位数字，且m<=n。用通常的乘法求uv的值需要O(mn)时间。当m比n小得多时，试设计一个算法，在上述情况下用$O(nm^{\log(3/2)})$时间求出uv值。