Problem Set 1

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1. Profit Maximization

a. Set up the Lagrangian for this problem, letting λ denote the multiplier on the constraint.

$$\mathcal{L}(y,k,l,\lambda) = py - rk - wl + \lambda * (k^a l^b - y)$$

, where $0 < a < 1$, $0 < b < 1$, and $0 < a + b < 1$.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values y^*, k^*, and l^* that solve the firm's problem, together with the associated value λ^* for the multiplier.

$$\mathcal{L}_1(k^*,l^*,y^*,\lambda^*) = r - a\lambda k^{a-1}l^b = 0. \tag{1.1}$$

$$\mathcal{L}_2(k^*,l^*,y^*,\lambda^*) = w - b\lambda k^a l^{b-1} = 0. \tag{1.2}$$

$$\mathcal{L}_3(k^*,l^*,y^*,\lambda^*) = p - \lambda = 0. \tag{1.3}$$

$$\mathcal{L}(k^*,l^*,y^*,\lambda^*) = y - k^a l^b \geq 0 \tag{2}$$

$$\lambda^* \geq 0 \tag{3}$$

$$\lambda^*(y - k^a l^b) = 0. \tag{4}$$

c. Assume that the constraint binds at the optimum (can you tell under what conditions this will be true?), and use your results from above to solve for y^*, k^*, l^*, and λ^* in terms of the model's parameters: a, b, p, r, and w.
Answer: When the complementary slackness condition (4):

$$\lambda^*(y - k^a l^b) = 0$$

must hold and be satisfied; where if $\lambda^* \geq 0$, then the constraint must bind and if the constraint does not bind, then $\lambda^* = 0$.

Since here, (1.3) indicates $\lambda^* = p \geq 0$, the constraint $k^a l^b = y$ binds.

$$y^*(k^*, l^*, a, b) = k^{*a} l^{*b}.$$
$$(1.1) \Rightarrow r = \lambda a k^{a-1} l^b$$
$$(1.2) \Rightarrow w = \lambda b k^a l^{b-1}$$
$$(1.3) \Rightarrow \lambda^*(p) = p$$
$$(1.1)/(1.2) \Rightarrow \frac{r}{w} = \frac{al}{bk}$$
$$(1.2) \Rightarrow w = bpk^a l^{b-1} = bp(\frac{awl}{br})^a l^{b-1} = pa^a w^a r^{-a} b^{1-a} * l^{a+b-1}$$
$$\Rightarrow l^*(a, b, p, r, w) = \sqrt[a+b-1]{\frac{a^{-a} w^{1-a} r^a b^{a-1}}{p}}$$
$$k^*(a, b, p, r, w) = \sqrt[a+b-1]{\frac{r^{1-b} b^{-b} w^b a^{b-1}}{p}}$$
$$y^*(a, b, p, r, w) = (\sqrt[a+b-1]{\frac{r^{1-b} b^{-b} w^b a^{b-1}}{p}})^a (\sqrt[a+b-1]{\frac{a^{-a} w^{1-a} r^a b^{a-1}}{p}})^b$$

d. Finally, use your solutions from above to answer the following questions:

i. What happens to the optimal y^*, k^*, and l^* when the output price p rises, holding all other parameters fixed? In each case, does the optimal choice rise, fall, or stay the same?
Answer: l^*, k^*, y^* all fall if the output price p rises.

ii. What happens to the optimal y^*, k^*, and l^* when the rental rate for capital r rises, holding all other parameters fixed?
Answer: l^* rises, k^* falls if the rental rate for capital r rises, but y^* depends on the ratio of a/b because it is a product of $l^* * k^*$.

iii. What happens to the optimal y^*, k^*, and l^* when the wage rate w rises, holding all other parameters fixed?
Answer: l^* falls, k^* rises if the wage rate w rises, but y^* depends on the ratio of a/b because it is a product of $l^* * k^*$.

iv. What happens to the optimal y^*, k^*, and l^* when p, r, and w all double at the same time?
Answer: l^* and k^* double but y^* times 4 when p, r, and w all double at the same time.

2. Utility Maximization

a. Set up the Lagrangian for the consumer's problem: choose c_1 and c_2 to maximize utility subject to the budget constraint, letting λ denote the multiplier on the constraint.

$$\mathcal{L}(c_1, c_2, p_1, p_2, \lambda) = c_1^a c_2^{1-a} + \lambda * [I - (p_1 c_1 + p_2 c_2)]$$

, where $0 < a < 1$.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^* and c_2^* that solve the consumer's problem, together with the associated value λ^* for the multiplier.

$$\begin{aligned} \mathcal{L}_1(c_1^*, c_2^*, \lambda^*) &= a c_1^{a-1} c_2^{1-a} - \lambda p_1 = 0 \\ \mathcal{L}_2(c_1^*, c_2^*, \lambda^*) &= (1-a) c_1^a c_2^{-a} - \lambda p_2 = 0. \end{aligned} \tag{1}$$

$$\mathcal{L}(c_1^*, c_2^*, \lambda^*) = I - (p_1 c_1 + p_2 c_2) \geq 0. \tag{2}$$

$$\lambda^* \geq 0. \tag{3}$$

$$\lambda^*[I - (p_1 c_1 + p_2 c_2)] = 0 \tag{4}$$

c. Assume that the budget constraint binds at the optimum (again, can you tell under what conditions this will be true?), and use your results from above to solve for c_1^*, c_2^*, and λ^* in terms of the model's parameters: I, p_1, p_2, and a.
<u>Answer</u>: When the complementary slackness condition (4):

$$\lambda^*[I - p_1 c_1 + p_2 c_2] = 0$$

must hold and be satisfied; where if $\lambda^* \geq 0$, then the constraint must bind and if the constraint does not bind, then $\lambda^* = 0$. Since the constraint binds,

$$\lambda \geq 0, and$$
$$I = p_1 c_1 + p_2 c_2$$
$$(1) \Rightarrow \frac{p_1}{p_2} = \frac{a c_1^{a-1} c_2^{1-a}}{1 - a c_1^a c_2^{-a}} = \frac{a}{1-a} \frac{c_2}{c_1} \Rightarrow p_1 c_1 = \frac{a}{1-a} p_2 c_2$$
$$(4) \Rightarrow I = p_1 c_1 + p_2 c_2 = (\frac{a}{1-a} + 1) p_2 c_2 = \frac{1}{1-a} p_2 c_2$$
$$c_2^*(p_1, p_2, a, I) = (1-a) \frac{I}{p_2}$$
$$c_1^*(p_1, p_2, a, I) = a \frac{I}{p_1}$$
$$\lambda^*(p_1, p2, a, I) = -\frac{a^{a+1}(1-a)^{1-a} I^{a-1}}{p_1 p_2}$$

d. Finally, using the answer we get,
$a = p_1 c_1^*/I$ and $1 - a = p_2 c_2^*/I$, indicating the preference parameter a is directly related to goods consumption.

3. Utility Maximization (Again)

a. Set up the Lagrangian for the consumer's problem: choose c_1 and c_2 to maximize utility subject to the budget constraint, letting λ denote the multiplier on the constraint.

$$\mathcal{L}(c_1, c_2, p_1, p_2, \lambda) = a\ln(c_1) + (1-a)\ln(c_2) + \lambda * [I - (p_1c_1 + p_2c_2)]$$

,where ln denotes the natural logarithm and where $0 < a < 1$ as before.

b. Next, write down the conditions that, according to the Kuhn-Tucker theorem, must be satisfied by the values c_1^* and c_2^* that solve the consumer's problem, together with the associated value λ^* for the multiplier.

$$\mathcal{L}_1(c_1^*, c_2^*, \lambda^*) = \frac{a}{c_1} - \lambda p_1 = 0 \tag{1.1}$$

$$\mathcal{L}_2(c_1^*, c_2^*, \lambda^*) = a\ln(c_1) - \lambda p_2 = 0. \tag{1.2}$$

$$\mathcal{L}(c_1^*, c_2^*, \lambda^*) = I - (p_1c_1 + p_2c_2) \geq 0. \tag{2}$$

$$\lambda^* \geq 0. \tag{3}$$

$$\lambda^*[I - (p_1c_1 + p_2c_2] = 0 \tag{4}$$

c. <u>Answer</u>: When the complementary slackness condition (4):

$$\lambda^*[I - p_1c_1 + p_2c_2] = 0$$

must hold and be satisfied; so if $\lambda \geq 0$, the constrain binds at $I = p_1c_1 + p_2c_2$.

$$(1.1) \Rightarrow \frac{a}{c_1} - \lambda p_1 = 0$$

$$(1.2) \Rightarrow \frac{1-a}{c_2} - \lambda p_2 = 0$$

$$so,\ \frac{p_1}{p_2} = \frac{ac_2}{(1-a)c_1} \Rightarrow p_1c_1 = \frac{a}{1-a}p_2c_2$$

$$(4) \Rightarrow I = p_1c_1 + p_2c_2 = \frac{1}{1-a}p_2c_2$$

$$c_1^*(p_1, p_2, a, I) = a\frac{I}{p_1}$$

$$c_2^*(p_1, p_2, a, I) = (1-a)\frac{I}{p_2}$$

$$\lambda^*(p_1, p2, a, I) = \frac{1}{I}$$

d. Using the answer above, we get $a = p_1c_1^*/I$ and $1 - a = p_2c_2^*/I$.