REAL ANALYSIS: HOMEWORK SET 2

DUE WED. OCT. 03

Exercise 1. Let μ denote a complete Lebesgue-Stieltjes measure on $(\mathbb{R}, \mathcal{M})$. Let $E \in \mathcal{M}$ with $\mu(E) < \infty$. Show that for every $\epsilon > 0$ there is a set A that is a finite union of open intervals, such that $\mu(E \setminus A) + \mu(A \setminus E) < \epsilon$.

Exercise 2. Let $E \in \mathcal{L}$ be a Lebesgue measurable set. Let $N \subset [0,1]$ be the (non measurable) set obtained by taking a set of representatives for the equivalence classes of the relation $x \sim y$ if $x - y \in \mathbb{Q}$.

(1) Show that $E \subseteq N$ implies that $m(E) = 0$
(2) Show that if $m(E) > 0$ then E contains a non measurable set [1]

Exercise 3. Let $E \in \mathcal{L}$ with $m(E) > 0$.

(1) Show that for any $0 \leq \alpha < 1$ there is an open interval I with $m(E \cap I) > \alpha I$.
(2) Show that the set $E - E = \{x - y | x, y \in E\}$ contains an interval centered at zero. [2]

Exercise 4. Let $(X, \mathcal{M})$ be a measurable space and $f, g : X \to \bar{R}$ measurable functions.

(1) Show that $f \cdot g$ is measurable when we define $0 \cdot (\pm\infty) = 0$.
(2) Fix $a \in \bar{\mathbb{R}}$ and define a the function $g : X \to \bar{R}$ by

$$h(x) = \begin{cases} a & f(x) = -g(x) = \pm\infty \\ f(x) + g(x) & \text{otherwise} \end{cases}.$$

Show that h is measurable.

Exercise 5. Let $(X, \mathcal{M})$ be a measurable space and $\{f_n\}_{n \in \mathbb{N}}$ a sequence of measurable functions on X. Show that the set $\{x | \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable.

Exercise 6. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. Show that f is Borel measurable.

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[1] Hint: for $E \subset [0,1]$ write $E = \cup_{r \in \mathbb{Q}} E \cap N_r$
[2] Hint: If I is an interval with $m(E \cap I) > \frac{3}{4}$ show that $(-\frac{m(I)}{2}, \frac{m(I)}{2}) \subseteq E - E$.

Exercise 7. Let $f : [0,1] \to [0,1]$ be the Cantor function (defined on the Cantor set $\mathcal{C}$ via $f(\sum_{j=1}^{\infty} \frac{a_j}{3^j}) = \sum_{j=1}^{\infty} \frac{a_j/2}{2^j}$ and constant on any interval on $[0,1] \setminus \mathcal{C}$). Let $g(x) = f(x) + x$.

(1) Show that $g : [0,1] \to [0,2]$ is a bijection and that $h = g^{-1} : [0,2] \to [0,1]$ is continuous.
(2) Show that $m(g(\mathcal{C})) = 1$.
(3) Let $A \subseteq g(\mathcal{C})$ be a Lebesgue nonmeasurable set (explain why does such a set exists) and let $B = g^{-1}(A)$. Show that B is Lebesgue measurable but not Borel measurable.
(4) Give an example of $F, G : \mathbb{R} \to \mathbb{R}$ with F Lebesgue measurable and G continuous such that $f \circ g$ is not Lebesgue measurable.