REAL ANALYSIS: HOMEWORK SET 4

DUE WED. NOV. 7

Exercise 1. Let $X = Y = \mathbb{N}$ with σ-algebras $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$ and $\mu = \nu$ be counting measure. Let $f(n,m) = \begin{cases} 1 & m = n \\ -1 & m = n+1 \\ 0 & \text{otherwise} \end{cases}$. Show that both iterated integrals $\int_X \int_Y f d\nu d\mu$ and $\int_X \int_Y f d\nu d\mu$ exist but are not equal. Why does this not contradict Fubini?

Exercise 2. In this exercise you will give the missing details in the proof of Fubini theorem for the completed product measure. Let $(X, \mathcal{M}, \nu)$ and $(Y, \mathcal{N}, \nu)$ be two complete measure spaces and $(X \times Y, \mathcal{L}, \lambda)$ the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$.

(1) Show that if $E \in \mathcal{M} \times \mathcal{N}$ with $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.
(2) Show that if f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e, then f_x and f^y are integrable for a.e x and y, and $\int f_x d\nu = \int f^y d\mu = 0$ for a.e. x and y.

Exercise 3. Let $E = [0,1] \times [0,1]$. For each of the following functions check whether $\int_E f dm$, $\int_0^1 \int_0^1 f(x,y) dx dy$ and $\int_0^1 \int_0^1 f(x,y) dy dx$ exist and are equal.

(1) $f(x,y) = \frac{(x^2 - y^2)}{(x^2+y^2)^2}$
(2) $f(x,y) = (1 - xy)^- a$ (with $a > 0$).
(3) $f(x,y) = \begin{cases} \frac{1}{(x-1/2)^3} & 0 < y < |x - \frac{1}{2}| \\ 0 & \text{otherwise} \end{cases}$.

Exercise 4. Show that if f is Lebesgue integrable on $(0,a)$ and $g(x) = \int_x^a t^{-1} f(t) dt$, then g is integrable on $(0,a)$ and $\int_0^a g(x) dx = \int_0^a f(x) dx$.

Exercise 5. Show that for any $s > 0$ we have $\int e^{-sx} x^{-1} \sin^2(x) dx = \frac{1}{4} \log(1 + 4s^{-2})$ (by integrating $e^{-sx} \sin(2xy)$ with respect to x and y).

Exercise 6. Let $f(x) = \frac{\sin(x)}{x}$.

(1) Show that $\lim_{b \to \infty} \int_0^b f(x) dx = \frac{\pi}{2}$ by integrating $e^{-xy} \sin(x)$ with respect to x and y.
(2) Show that $\int_0^\infty |f(x)| dx = \infty$ and with this in mind make sure your arguments in the previous part were done correctly.

Exercise 7. Prove the following formula for any $a > 0$ by expanding the integrand as a series and integrate term by term

$$\int_0^\infty \frac{x^{a-1}}{e^x - 1} dx = \gamma(a)\zeta(a), \quad \text{with } \zeta(a) = \sum_{n=1}^\infty \frac{1}{n^a}.$$

Exercise 8. Prove the formula $\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}$ (Hint: Write $\Gamma(x)\Gamma(y)$ as a double integral and make a change of variables in the exponent).

Exercise 9. Let f be continuous on $[0, \infty)$, for $\alpha > 0$ and $x \geq 0$ the αth fractional integral of f is defined by

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t).$$

(1) Show that $I_{\alpha+\beta} f = I_\alpha(I_\beta f)$ for $\alpha, \beta > 0$.
(2) Show that for $n \in \mathbb{N}$, $I_n f$ is an n'th order anti derivative of f.

Exercise 10. Let σ denote the measure on S^{n-1} corresponding to polar coordinates in $\mathbb{R}^n$. Let $f(x) = \prod_{j=1}^n x_j^{k_j}$ be a monomial (with degrees $k_j \in \mathbb{N} \cup \{0\}$) and think of it as a function on S^{n-1}. Show that if one of the degrees k_j is odd then $\int f d\sigma = 0$ and if all degrees are even then

$$\int f d\sigma = 2 \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_n)}{\Gamma(\beta_1 + \cdots + \beta_n)}, \quad \text{where } \beta_j = \frac{k_j + 1}{2}.$$