# REAL ANALYSIS: HOMEWORK SET 6

DUE FRI. NOV. 30

**Exercise 1.** Let $X$ be a normed vector space. Show that the closure of any subspace is also a subspace.

**Exercise 2.** Let $X$ be a Banach space.

(1) Let $T \in L(X, X)$ satisfy $\|T - I\| < 1$. Show that $T$ is invertible and that $\sum_n (I - T)^n$ converges in $L(X, X)$ to $T^{-1}$.
(2) Let $T \in L(X, X)$ be invertible. Show that if $S \in L(X, X)$ satisfies $\|T - S\| < \frac{1}{\|T^{-1}\|}$ then $S$ is invertible and conclude that the set of invertible operators in $L(X, X)$ is open.

**Exercise 3.** Let $X$ be a normed infinite dimensional vector space and let $\mathcal{M} \subseteq X$ be a proper closed subspace.

(1) Show that for every $\epsilon > 0$ there is $x \in X$ with $\|x\| = 1$ and $\inf\{\|x + y\| \,|y \in M\} \geq 1 - \epsilon$.
(2) Show that if $\mathcal{M}$ is finite dimensional then it is closed.
(3) Show that there is a sequence $(x_j) \subseteq X$ with $\|x_j\| = 1$ and $\|x_j - x_k\| \geq \frac{1}{2}$ for all $j \neq k$.
(4) Conclude that the unit sphere $\{x \in X | \, \|x\| = 1\}$ is closed and bounded but not compact.

**Exercise 4.** Show that a linear functional $f : X \to \mathbb{C}$ on a normed space $X$ is bounded if and only if $f^{-1}(0)$ is closed.

**Exercise 5.** Let $f, g \in L^p$ for $1 \leq p < \infty$. Under what condition do we have equality in Minkowski's inequality $\|f + g\|_p \leq \|f\|_p + \|g\|_p$? (The answer is different for $p = 1$ and $p > 1$).

**Exercise 6.** Let $\{f_n\} \subseteq L^\infty(X)$. Show that $\|f_n - f\|_\infty \to 0$ if and only if there a measurable $E$ with $\mu(E^c) = 0$ such that $f_n \to f$ uniformly on $E$.

**Exercise 7.** Suppose $1 \leq p < q < \infty$. Show that $L^p(X) \not\subset L^q(X)$ if and only if $X$ contains sets of arbitrarily small measure and that $L^q(X) \not\subset L^p(X)$ if and only if $X$ contains sets of arbitrarily large finite measure.

**Exercise 8.** Let $1 \leq q < \infty$ and let $f \in L^p \cap L^\infty$. Show that $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$

**Exercise 9.** Let $1 \leq p < \infty$ and let $f_n, f \in L^p$.

(1) Show that if $f_n \to f$ in $L^p$ then $f_n \to f$ in measure and hence there is a subsequence $f_{n_k} \to f$ a.e.
(2) Show that if $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ then $f_n \to f$ in $L^p$.
(3) Assume that $f_n \to f$ a.e and show that $f_n \to f$ in $L^p$ iff $\|f_n\|_p \to \|f\|$ (use the generalized version of the dominated convergence theorem)