REAL ANALYSIS: HOMEWORK SET 3

DUE WED. OCT. 17

Exercise 1. Let $\{f_n\} \subseteq L^+$, $f_n \to f$ a.e. and $\int f = \lim \int f_n < \infty$.

1. Show that for any measurable set $E \in \mathcal{M}$, $\int_E f = \lim \int_E f_n$.
2. Show that this need not be true if $\lim \int f_n = \int f = \infty$.

Exercise 2. Let $f \in L^+$ satisfy $\int f < \infty$. Show that for every $\epsilon > 0$ there is $E \in \mathcal{M}$ with $\mu(E) < \infty$ such that $\int_E f > \int f - \epsilon$.

Exercise 3. Let $\{f_n\} \subseteq L^1(\mu)$ and $f_n \to f$ uniformly.

1. Assume that $\mu(X) < \infty$ and show that $f \in L^1(\mu)$ and $\int f_n \to \int f$.
2. Give an example showing that this can fail when $\mu(X) = \infty$.

Exercise 4. Let $f_n, g_n, f, g \in L^1$ with $f_n \to f$ and $g_n \to g$ a.e. Assume that $|f_n| \leq g_n$ and $\int g_n \to \int g$ and show that $\int f_n \to \int f$.[1]

Exercise 5. Let $f_n, f \in L^1$, $f_n \to f$ a.e. Show that $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.[2]

Exercise 6. Let $f \in L^2(m)$ and $F(x) = \int_{(-\infty,x)} f dm$. Show that F is continuous.

Exercise 7. Compute the following limits (use the appropriate convergence theorems to justify your calculations).

1. $\lim_{n\to\infty} \int_0^\infty (1 + \frac{x}{n})^{-n} \sin(\frac{x}{n}) dx$
2. $\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} dx$
3. $\lim_{n\to\infty} \int_0^\infty n \sin(\frac{x}{n})[x(1 + x^2)]^{-1} dx$
4. $\lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2x^2} dx$
   Note: the answer depends on whether $a > 0, a = 0$ or $a < 0$.

Exercise 8. Recall that $f_n \to f$ almost uniformly, if $\forall \epsilon > 0$ there is E with measure $\mu(E) < \epsilon$ such that $f_n \to f$ uniformly on E^c. Show that if $f_n \to f$ almost uniformly then $f_n \to f$ in measure, and $f_n \to f$ a.e

[1] Hint: Rework the proof of the dominant convergence theorem
[2] Hint:use previous exercise.

Exercise 9. Assume that $|f_n| < g \in L^1$ and $f_n \to f$ in measure, show that $f_n \to f$ in L^1.

Exercise 10. Let $f : [a, b] \to \mathbb{C}$ be Lebesgue measurable. Show that for any $\epsilon > 0$ there is a compact set $E \subset [a, b]$ with $m(E^c) < \epsilon$ and such that $f_{|_E}$ is continuous. (Note: this does not mean that f is continuous at E).