REAL ANALYSIS: HOMEWORK SET 1
DUE WED. SEP. 19

Exercise 1. Show that if $\mathcal{M}_\alpha, \alpha \in I$ is a collection of σ-algebras then the intersection $\mathcal{M} = \cap_{\alpha \in I} \mathcal{M}_\alpha$ is also a σ-algebra.

Exercise 2. Let $\mathcal{A}$ be an algebra. Show that $\mathcal{A}$ is a σ-algebra if and only if for any $\{E_j\}_{j=1}^\infty \subseteq \mathcal{A}$ with $E_j \subseteq E_{j+1}$ also $\bigcup_j E_j \in \mathcal{A}$ (that is, $\mathcal{A}$ is closed under countable increasing unions).

Exercise 3. Let μ denote a finitely additive measure on a σ-algebra $\mathcal{M}$. Show that μ is a measure if and only if

$$\mu(\bigcup_j E_j) = \lim_{j\to\infty} \mu(E_j),$$

for any countable nested collections $E_1 \subseteq E_2 \subseteq \ldots$ in $\mathcal{M}$.

Exercise 4. Let μ^* denote an outer measure on X and $\{A_j\}_{j=1}^\infty$ a sequence of disjoint μ^*-measurable sets. Show that for any $E \subseteq X$,

$$\mu^*(E \cap \bigcup_j A_j) = \sum_j \mu^*(E \cap A_j).$$

Exercise 5. In this exercise we will show that we can't replace the countable covers in the construction of an outer measure by a finite cover. Define the outer Jordan content of a set $A \subseteq \mathbb{R}$ as

$$J^*(A) = \inf\{\sum_{j=1}^N |I_j| : N \in \mathbb{N},\ A \subseteq \bigcup_j I_j\}.$$

where each $I_j = (a_j, b_j)$ is an open interval, and $|I_j| = b_j - a_j$ is the length of the interval.

- Show that $J^*(A) = J^*(\overline{A})$ where $\overline{A}$ is the closure of A
- Give an example of a countable set $A \subset [0, 1]$ with $J^*(A) = 1$.
- Conclude that J^* is not countably sub-additive

Exercise 6. Let $\mathbb{B}_\mathbb{R}$ denote the Borel σ-algebra on $\mathbb{R}$ (that is, the algebra generated by open sets).

- Show that any open set $\mathcal{O} \subseteq \mathbb{R}$ can be written as a disjoint union of countably many open intervals.
- Show that $\mathbb{B}_\mathbb{R}$ is generated by the set $\mathcal{E}_1$ of open intervals.
- Show that $\mathbb{B}_\mathbb{R}$ is generated by the set $\mathcal{E}_2$ of closed intervals.

- Show that $\mathbb{B}_{\mathbb{R}}$ is generated by the set $\mathcal{E}_3$ of open rays
$$\mathcal{E}_3 = \{(a, \infty) | a \in \mathbb{R}\}.$$

Exercise 7. Let μ^* denote the outer measure on $\mathbb{R}$ generated by the length function on open intervals, that is,

$$\mu^*(A) = \inf\{\sum_{j=1}^{\infty} |I_j| : A \subseteq \bigcup_j I_j\}.$$

- Show that any open ray (a, ∞) is μ^*-measurable.
- Conclude that any Borel set is μ^*-measurable.
- Show that a set $A \subseteq \mathbb{R}$ is μ^*-measurable if and only if for any $\epsilon > 0$ there is an open set $\mathcal{O} \supseteq A$ with $\mu^*(\mathcal{O} \setminus A) < \epsilon$.