REAL ANALYSIS: HOMEWORK SET 7

DUE FRI. DEC. 7

Exercise 1. Let $\phi_n \in (\ell^\infty)^*$ be given by $\phi_n(f) = \frac{1}{n}\sum_{j=1}^n f(j)$ (for $f \in \ell^\infty$). Using Alaoglu's theorem, stating that for any normed vector space X the set $\{\varphi \in X^* | \, \|\varphi\| \leq 1\}$ is compact in the weak* topology, show that there is a subsequence $\{\phi_{n_k}\}$ converging weak* to some $\phi \in (\ell^\infty)^*$ and that the limit ϕ does not arise from an element of ℓ^1.

Exercise 2. Let $\{f_n\} \subseteq L^p$ with $\|f_n\|_p$ uniformly bounded and $f_n \to f$ a.e.

(1) For $1 < p < \infty$ show that $f_n \to f$ weakly in L^p
(2) For $p = 1$, find counter examples in $L^1(\mathbb{R}, m)$ and in ℓ^1.
(3) For $p = \infty$, μ σ-finite, show that $f_n \to f$ in weak* topology of $(L^1)^* = L^\infty$.

Exercise 3. Let $1 < p < \infty$, and let $\{f_n\} \subseteq L^p$.

(1) Assume that $\|f_n\|_p$ is uniformly bounded and show that $f_n \to 0$ weakly iff $\int f_n g \to 0$ for all simple $g \in L^q$.
(2) Give an example in $L^p(\mathbb{R})$ showing that this is not true without the uniform bound on the norms.
(3) Show that in ℓ^p we have that $f_n \to f$ weakly if and only if $\|f_n\|_p$ is uniformly bounded and $f_n \to f$ pointwise.

Exercise 4. Let K be a nonnegative measurable function on $(0, \infty)$ with $\phi(s) = \int_0^\infty K(x)x^{s-1}dx < \infty$ for all $0 < s < 1$.

(1) Let $p, q \in (0, \infty)$ with $p^{-1} + q^{-1} = 1$ and $f, g \in L^+(0, \infty)$ and show that

$$\int_0^\infty \int_0^\infty K(xy)f(x)g(y)dxdy \leq \phi(p^{-1}) \left[\int_0^\infty x^{p-2}f(x)^p dx\right]^{1/p} \left[\int g(x)^q dx\right]^{1/q}.$$

(2) Show that the operator $Tf(x) = \int K(xy)f(y)dy$ is bounded on $L^2(0, \infty)$ with $\|T\| \leq \phi(\frac{1}{2})$.
(3) Compute T and ϕ for the special case where $K(x) = e^{-x}$.

Exercise 5. Let μ be a Radon measure on X (a locally compact Hausdorff space). We define the support of μ the complement N^c where N is the union of all open sets U in X with $\mu(U) = 0$. Show that $\mu(N) = 0$ and that $x \in \text{supp}\mu$ iff $\int f d\mu > 0$ for every $f \in C_c(X)$ with values in $[0, 1]$ and $f(x) > 0$.