REAL ANALYSIS: HOMEWORK SET 5

DUE TUE. NOV. 20

Exercise 1. Let ν be a signed measure on $(X, \mathcal{M})$

(1) Show that $L^1(\nu) = L^1(|\nu|)$ and that for any $f \in L^1(\nu)$ we have $|\int f d\nu| \leq \int f d|\nu|$
(2) Show that for any $E \in \mathcal{M}$,
$$|\nu|(E) = \sup\{|\int_E f d\nu| : |f| \leq 1\}.$$

Exercise 2. Let ν be a signed measure on $(X, \mathcal{M})$ and let $E \in \mathcal{M}$

(1) Show that $\nu^+(E) = \sup\{\nu(F) | F \in \mathcal{M}, F \subseteq E\}$ and $\nu^-(E) = -\inf\{\nu(F) | F \in \mathcal{M}, F \subseteq E\}$.
(2) Show that
$$|\nu|(E) = \sup\left\{\sum_{j=1}^{n} |\nu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n \text{ disjoint, with } \bigcup_j E_j = E\right\}.$$

Exercise 3. Let $\{\nu_j\}$ be a sequence of positive measures and μ a positive measure. Show that if $\nu_j \perp \mu$ for all j then $(\sum_j \nu_j) \perp \mu$ and that if $\nu_j \ll \mu$ for all j then $(\sum_j \nu_j) \ll \mu$.

Exercise 4. Let μ be a positive measure. A subset $\mathcal{F} \subseteq L^1(\nu)$ is called uniformly integrable if for any $\epsilon > 0$ there is $\delta > 0$ such that if $\mu(E) < \delta$ and $f \in \mathcal{F}$ then $|\int_E f d\mu| < \epsilon$.

(1) Show that any finite subset of $L^1(\mu)$ is uniformly integrable.
(2) Show that any sequence $\{f_n\}$ in $L^1(\mu)$ that converges in the L^1 metric to some $f \in L^1(\mu)$ is uniformly integrable.

Exercise 5. Let $(X, \mathcal{M}) = ([0, 1], \mathbb{B}_{[0,1]})$, let m denote Lebesgue measure and μ the counting measure.

(1) Show that $m \ll \mu$ but that $dm \neq f d\mu$ for any f
(2) Show that μ has no Lebesgue decomposition with respect to m
(3) Why does this not contradict the Lebesgue-Radon-Nikodym Theorem.

Exercise 6. Let $f \in L^1(\mathbb{R}^n)$ non zero and Hf the Hardy-Littlewood maximal function.

(1) Show that there exists $C, R > 0$ such that when ever $|x| > R$ we have $Hf(x) \geq C|x|^{-n}$.
(2) Conclude that there is $C' > 0$ such that for α sufficiently small
$$m(\{x|Hf(x) > \alpha\}) \geq \frac{C'}{\alpha},$$
(This shows that the estimate in the maximal theorem is essentially sharp.)

Exercise 7. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and recall that the Lebesgue set of f is
$$L_f = \{x \in \mathbb{R}^n | \lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dy = 0.\}.$$
Show that if f is continuous at x then $x \in L_f$