

**ANSWER KEY 3**  
**UTILITY FUNCTIONS, THE CONSUMER'S PROBLEM, DEMAND CURVES**

ECON 210  
GUSE  
REVISED OCT 3, 2017

- (1) *Perfect Substitutes*. Suppose that Jack's utility is entirely based on number of hours spent camping (c) and skiing (s). Sketch Jack's indifference curves.

$$u(c, s) = 3c + 2s$$

- (a) What is Jack's MRS of hours spent camping for hours spent skiing? *ANSWER*. To get the number of camping hours Jack is willing to forego in order to spend another hour skiing take the ratio of the partial derivatives of the utility function as follows.

$$\frac{\partial u / \partial s}{\partial u / \partial c} = \frac{2}{3}$$

Since Jack only get  $\frac{2}{3}$  as much happiness per hour skiing as he does per camping hour, he is only will to give up  $\frac{2}{3}$  hours of camping to get an additional hour of skiing.

- (b) Let  $p_c = 1$  be the price to Jack of spending an hour camping and  $p_s$  the price per hour of skiing. Solve for  $p_s^*$ , the price at which a utility-maximizing Jack might mix his time between the two activities. If  $p_s > p_s^*$  what will Jack do? *ANSWER*. To find  $p_s^*$  just set MRS equal to the price ratio as follows.

$$\begin{aligned}\frac{p_s^*}{p_c} &= \frac{\partial u / \partial s}{\partial u / \partial c} \\ p_s^* &= \frac{2}{3}\end{aligned}$$

Since Jack only gets  $\frac{2}{3}$  as much happiness per hour skiing as he does camping, the price of skiing cannot be any higher than  $\frac{2}{3}$  the price of camping in order to lure Jack away from his tent under the stars to the those crowded ski slopes. Once the price of skiing gets any higher than that, he is back to the woods full-time.

- (c) Write down Jack's demand functions for both activities.

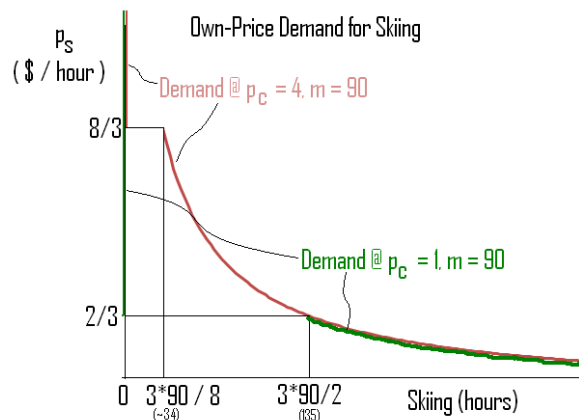


FIGURE 1. Jack Demand curves for skiing drawn for two difference prices of camping. For prices of skiing below  $\frac{2}{3}$ , these two curves are exactly the same. However if the price of camping is 4, Jack will tolerate higher prices of skiing up to  $\frac{8}{3}$ .

ANSWER.

$$x_s(p_s, p_c, m) = \begin{cases} \frac{m}{p_s} & \text{if } \frac{p_s}{p_c} < \frac{2}{3} \\ \left[0, \frac{m}{p_s}\right] & \text{if } \frac{p_s}{p_c} = \frac{2}{3} \\ 0 & \text{if } \frac{p_s}{p_c} > \frac{2}{3} \end{cases}$$

$$x_c(p_s, p_c, m) = \begin{cases} 0 & \text{if } \frac{p_s}{p_c} < \frac{2}{3} \\ \left[0, \frac{m}{p_c}\right] & \text{if } \frac{p_s}{p_c} = \frac{2}{3} \\ \frac{m}{p_c} & \text{if } \frac{p_s}{p_c} > \frac{2}{3} \end{cases}$$

Since Jack has constant MRS - a feature of perfect substitutes utility - he almost always either spends all his income on one good or the other depending whether the price ratio is greater or less than that constant MRS. A possible exception is when the price ratio is exactly equal to the MRS in which Jack may mix his time between the two activities.

- (d) Let  $p_c = 1$  and  $m = 90$ . Sketch Jack's demand curve for skiing (with the price of skiing on the vertical axis). Re-draw the same curve when  $p_c = 4$ . See Figure 1.

- (2) *Perfect Complements*. Katherine is only made happy by eating portions of ice cream containing *exactly* four parts chocolate to three parts vanilla. She simply will not eat anything else and will not eat ice-cream in any other ratio. Assume however, that she is willing modifying imperfectly balanced dishes of ice cream by disposal.

- (a) Write down a utility function for Katherine and sketch her indifference curves.

ANSWER. We know the utility function from our class discussion will take the form  $u(v, c) = \min\{av, bv\}$ . The trick is to figure out what  $a$  and  $b$  are. We also know from class that the indifference curve for perfect complements are "L"-shaped, which means,

for example, that,  $(3, 4)$ , the bundle consisting of 3 scoops of vanilla and 4 chocolate yields the same utility as  $(3 + k, 4)$  and  $(3, 4 + k)$  for any  $k > 0$ . Our choice of  $a$  and  $b$  must be consistent with this fact...

$$\min\{a3, b4\} = \min\{a(3 + k), b4\} \quad \forall k > 0$$

$$\min\{a3, b4\} = \min\{a3, b(4 + k)\} \quad \forall k > 0.$$

Note that the right-hand side of the top equality must be equal to  $b4$  if we make  $k$  large enough. Therefore, the only way for the first equality to be true for all  $k > 0$  is for  $\min\{a3, b4\} = b4$ . Likewise the only way for the second equality to hold for all  $k > 0$  is for  $\min\{a3, b4\} = a3$ . Putting these together we get  $b4 = a3$ . Any choice of  $a$  and  $b$  satisfying this relationship is a legitimate choice of parameters. Take for example,  $a = 4$  and  $b = 3$  giving us

$$u(v, c) = \min\{4v, 3c\}$$

Similarly any monotonic transformation of this will do ...

$$u(v, c) = \min\{4v, 3c\}$$

$$u(v, c) = \min\{v, \frac{3}{4}c\}$$

$$u(v, c) = \min\{\log(4v), \log(3c)\}$$

$$u(v, c) = \min\{4000v, 3000c\}$$

are all possible correct answers.

- (b) If the price of chocolate ice cream is \$3 per pint and price of vanilla is \$2 per pint, what is Katherine's utility-maximizing consumption of chocolate ice-cream when her income is \$9? *ANSWER* I think the best was to approach solving for the demands in the case of perfect complements is to keep in mind a particular optimally proportioned bundles which I will call the *unit bundle*. Let's take  $(3, 4)$ , the bundle with 3 vanilla and 4 chocolate. We know that if she is not choosing this particular bundle, she is choosing a scaled up or scaled down version of it with chocolate and vanilla in the same ratio. That is she will consume  $(\alpha 3, \alpha 4)$  for some  $\alpha > 0$ .

Now let's calculate the "price",  $P$ , of one of these unit bundles of both flavors. There are 3 pints of vanilla at \$2 per pint and 4 pints of chocolate at \$3 per pint for a total of \$18 per unit bundle  $(3, 4)$ . Now since she has only \$9, she can only afford  $\frac{1}{2}$  of one of these bundles which would contain **2 pints of chocolate**.

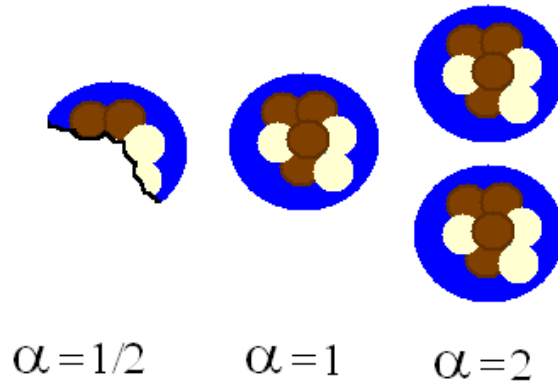


FIGURE 2. Examples of consumption bundles Katherine may demand depending on how many unit bundles she can afford.

- (c) Write down Katherine's demand functions for both flavors of ice-cream. *ANSWER* Using similar logic as we did in the previous question, first write down an expression for the "price",  $P$  of a unit bundle

$$P = 3p_v + 4p_c$$

The number of such bundles Katherine demands for arbitrary  $m$  is

$$\frac{m}{P} = \frac{m}{3p_v + 4p_c}$$

Demands for  $c$  and  $v$  are simply the demand for unit bundles multiplied by the amount of chocolate and vanilla per bundle.

$$v(p_v, p_c, m) = \frac{3m}{3p_v + 4p_c}$$

$$c(p_v, p_c, m) = \frac{4m}{3p_v + 4p_c}$$

- (d) Draw Katherine's Engel curve for chocolate ice-cream under the prices given above and show how it changes when the price of vanilla decreases to \$1 per pint. *ANSWER* The demands we need are

$$c(m, |p_c = 3, p_v = 2) = \frac{4m}{18}$$

$$c(m, |p_c = 3, p_v = 1) = \frac{4m}{15}$$

As we see from Figure 3, the Engel curve generate when the price of vanilla is held fixed at the higher price of  $p_v = 2$  appears steeper. This is because it takes more more to buy additional unit bundles and therefore additional pints of chocolate.

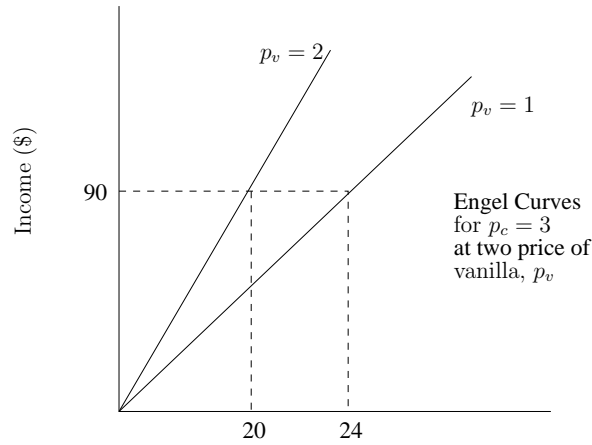


FIGURE 3. Here we see that when the price of vanilla drops from 2 to 1, the Engel curve for Chocolate shifts out (or more precisely rotates out) reflecting the fact that now for every income level the consumer would demand more chocolate ice cream. The price of chocolate is being held at  $p_c = 3$  for both curves.

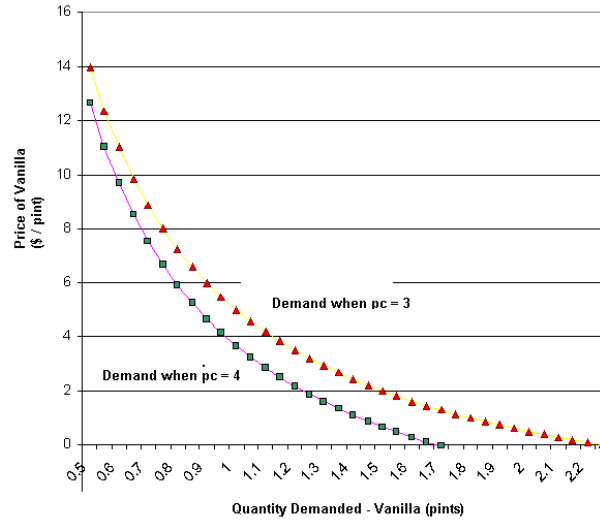


FIGURE 4. Katherine's own-price demand curve for ice cream.

- (e) Sketch Katherine's demand curve for vanilla ice-cream and show how it changes when the price of chocolate ice-cream increases to \$4 per pint. *ANSWER* The demand curves we need are

$$v(p_v, |p_c = 3, m = 9) = \frac{27}{12 + 3p_v}$$

$$v(p_v, |p_c = 4, m = 9) = \frac{27}{16 + 3p_v}$$

We see that raising the price of chocolate shifts the demand curve for vanilla inward; Katherine demands less vanilla at every price. Consider for example when the price of vanilla is \$5 / pint. If the price of chocolate is \$3 she demands  $\frac{27}{12+15} = 1$ . But if the price of chocolate is fixed at \$4, she would demand  $\frac{27}{16+15} = .87$

(3) *Cobb-Douglas*. Consider the utility function  $u(x_1, x_2) = x_1^a x_2^b$ ,  $a > 0$ ,  $b > 0$ .

(a) Find the MRS in terms of  $x_1$ ,  $x_2$ ,  $a$  and  $b$ . *ANSWER*.

$$\begin{aligned} MRS(x_1, x_2) &= \frac{\partial u / \partial x_1}{\partial u / \partial x_2} \\ &= \frac{a x_1^{a-1} x_2^b}{b x_1^a x_2^{b-1}} \\ &= \frac{a x_2}{b x_1} \end{aligned}$$

(b) Define a new utility function  $v(x_1, x_2) = \log(u(x_1, x_2))$ . Show that the MRS for  $v$  is everywhere the same as the MRS for  $u$ .

*ANSWER*. First  $\log(u(x_1, x_2)) = \log(x_1^a x_2^b) = a \log x_1 + b \log x_2$  Therefore we have

$$\begin{aligned} MRS(x_1, x_2) &= \frac{\partial v / \partial x_1}{\partial v / \partial x_2} \\ &= \frac{a/x_1}{b/x_2} \\ &= \frac{a x_2}{b x_1} \end{aligned}$$

QED.

(c) Let  $a = .4$  and  $b = .6$ . Write down the demand functions for goods 1 and 2. Sketch the price-expansion path and the demand curve with price on the vertical axis for good 1. Sketch the income expansion path and the Engel curve with income on the vertical axis for good 1.

*ANSWER*. The demand functions are gotten by solving the consumer's problem. Let  $(x_1^*, x_2^*)$  represent the optimal choice. The first order condition is

$$MRS = MRT$$

$$\frac{a x_2^*}{b x_1^*} = \frac{p_1}{p_2}$$

We also know, by monotonicity, that the optimal consumption bundle will lie on the budget line. so

$$p_1 x_1^* + p_2 x_2^* = m$$

Using these two equations to solve for  $x_1^*$  and  $x_2^*$  we get

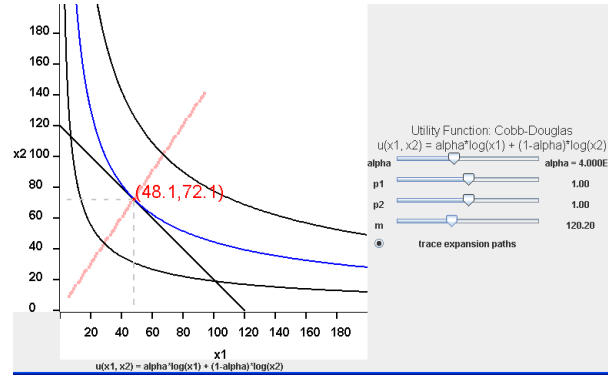


FIGURE 5. Income Expansion Path. This shows a snapshot of the utility function java tool with an IEP in pink.

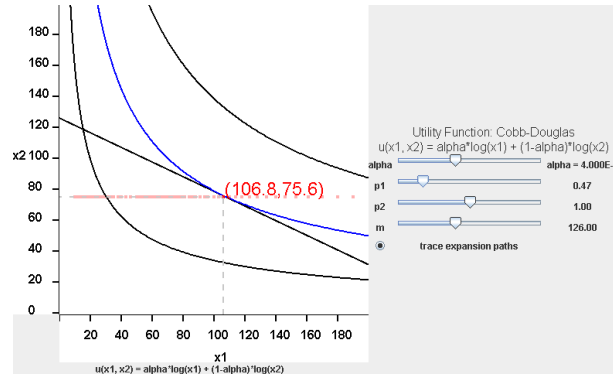


FIGURE 6. The price expansion path is shown in pink

$$x_1^*(p_1, p_2, m) = \frac{am}{(a+b)p_1}$$

$$x_2^*(p_1, p_2, m) = \frac{bm}{(a+b)p_2}$$

if  $a = .4$  and  $b = .6$  we get

$$x_1^*(p_1, p_2, m) = \frac{.4m}{p_1}$$

$$x_2^*(p_1, p_2, m) = \frac{.6m}{p_2}$$

Note that CD demands do not depend on cross prices. Using the nifty java - utility viewer provided by the Prof. Guse, we can sketch the price and income expansion paths.

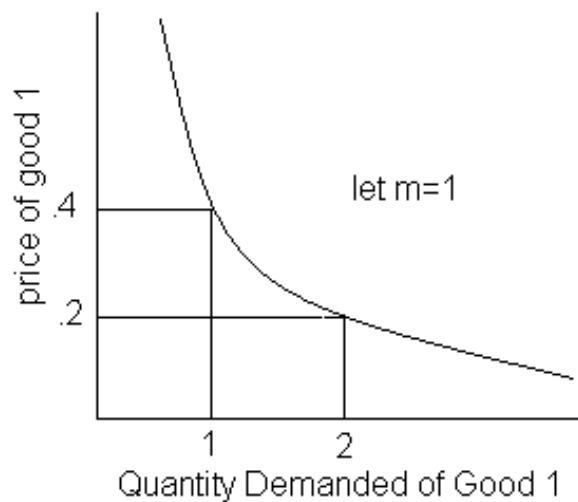


FIGURE 7. Demand For Good 1 (own-price).

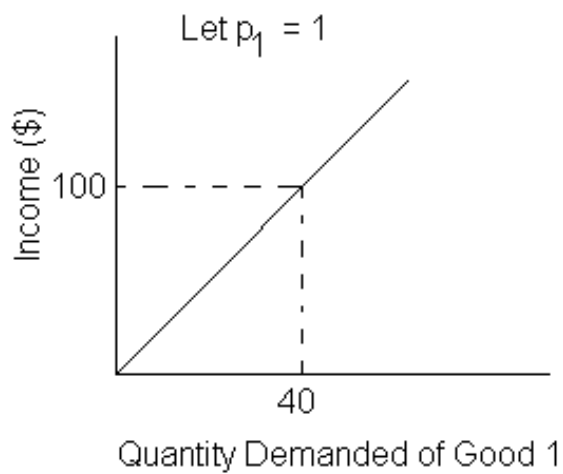


FIGURE 8. Engel Curve for Good 1.

Inspired by the Income Expansion and Price Expansion paths respectively, we have the Engel and own-price demand demand curves for good 1.

- (4)
- (5) *Quasi-linear Preferences* Consider the utility function  $u(x_1, x_2) = x_2 - (x_1 - 5)^2$ . Sketch the indifference curves for this utility function.



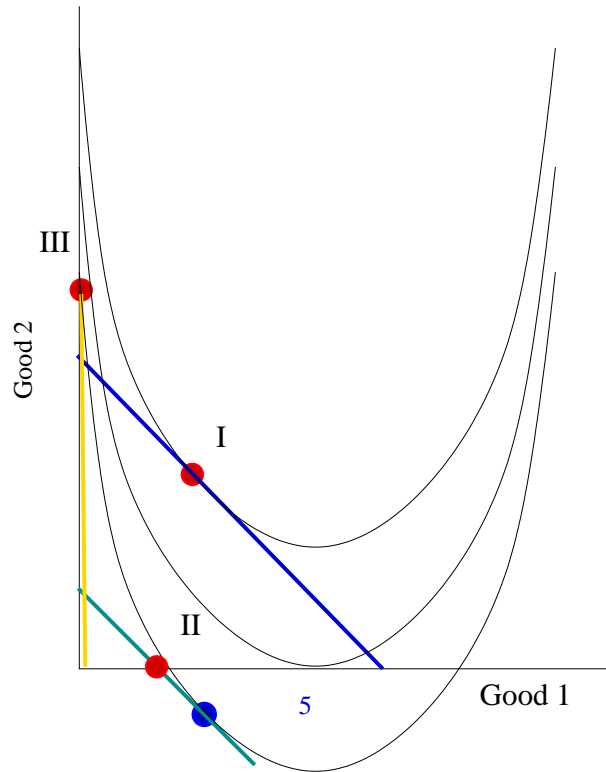


FIGURE 9. Indifference curves appear as parabolas with the given utility function. Figure shows the three main ways that different budget lines might interact with these preferences to produce (I) an interior solution, (II) a corner solution along the good 1 axis and (III) a corner solution along the Good 2 axis. Red dots highlight optimal choice for each case. Blue dot indicates the  $(x_1, x_2)$  combination that would satisfy the FOC in case II. Note that it involves a negative quantity of good 2 (and is therefore not feasible).

- (a) Describe in words how this consumer feels about the two goods.

*ANSWER.* More of good 2 is always better. More of good 1 is better up to 5, after that less is better. Note this means preferences do not conform to our usual “nice” assumption of monotonicity.

- (b) Write down an expression for the MRS. (i.e. find  $\frac{\partial u / \partial x_1}{\partial u / \partial x_2}$ ) At what rate is this consumer willing to give up good 2 to get another unit of good 1 when  $x_1 = 4$ . How about when  $x_1 = 6$ .

*ANSWER*  $MRS = \frac{-2(x_1-5)}{1} = 2(5-x_1)$ . When  $x_1 = 4$ , the consumer is willing to give up good 2 for good 1 at the rate of 2. However when  $x_1 = 6$  the sign on MRS flips; the consumer is now willing to give up 2 units of good 1 per unit of good 1 *less*.

- (c) Solve for the demands of both goods.

*ANSWER* Setting  $MRS = MRT$  we get

$$(1) \quad \begin{aligned} 2(5 - x_1) &= \frac{p_1}{p_2} \\ x_1 &= 5 - \frac{p_1}{2p_2} \end{aligned}$$

Plugging this value of  $x_1$  back in to the budget line equation, we get an expression for the demand of good 2.

$$(2) \quad \begin{aligned} x_2 &= \frac{m - p_1 x_1}{p_2} \\ &= \frac{m - p_1 \left(5 - \frac{p_1}{2p_2}\right)}{p_2} \\ &= \frac{m - 5p_1 + \frac{p_1^2}{2p_2}}{p_2} \end{aligned}$$

Equations (1) and (2) together represent the solution to the choice problem *when they are both non-negative*. This is represented in Figure 9 by case I. However, if, for some combination of budget parameters  $(p_1, p_2, m)$  either  $x_1$  or  $x_2$  are negative, then the solution is at a corner. For example, if  $m$  is too low (i.e. less than  $5p_1 - \frac{p_1^2}{2p_2}$ ) then equation (2) says that  $x_2$  is negative. But that cannot be. This is case II. Conceptually equations (1) and (2) would place the solution at the blue dot in Figure 9. However, the actual solution cannot have negative quantities. Similarly equation (1) would yield a negative quantity for good 1, if  $p_1 > 10 \times p_2$ . That is illustrated by case III.

Putting this all together, we can write down the complete demand equations

$$\begin{aligned} x_1(p_1, p_2, m) &= \begin{cases} 5 - \frac{p_1}{2p_2} & \text{if (I) } m > p_1(5 - \frac{p_1}{2p_2}) \text{ AND } p_1 \leq 10p_2 \\ \frac{m}{p_1} & \text{if (II) } m \leq p_1(5 - \frac{p_1}{2p_2}) \\ 0 & \text{if (III) } p_1 > 10p_2 \end{cases} \\ x_2(p_1, p_2, m) &= \begin{cases} \frac{m - p_1(5 - \frac{p_1}{2p_2})}{p_2} & \text{if (I) } m > p_1(5 - \frac{p_1}{2p_2}) \text{ AND } p_1 \leq 10p_2 \\ 0 & \text{if (II) } m \leq p_1(5 - \frac{p_1}{2p_2}) \\ \frac{m}{p_2} & \text{if (III) } p_1 > 10p_2 \end{cases} \end{aligned}$$

(6)

- (7) Show as rigorously as possible that in a 2-good world, both goods cannot be inferior. Use any of our usual “nice” assumptions on preferences (rational, monotonic, convex) that you need.

*ANSWER.* Suppose preferences are monotonic so that the optimal consumption bundle is always on the budget line. Now consider some starting optimal bundle  $(x_1, x_2)$  when the budget parameters are  $(p_1, p_2, m)$ . If this bundle is on the budget line then

$$p_1x_1 + p_2x_2 = m$$

Now consider a new budget with higher income level  $m' > m$ . Let  $(x'_1, x'_2)$  stand for the new optimal consumption bundle which also must be on the new higher budget line.

$$p_1x'_1 + p_2x'_2 = m'$$

To prove that both goods cannot be inferior, use proof by contradiction and suppose that both goods *are* inferior. This would mean that

$$x'_1 < x_1$$

$$x'_2 < x_2$$

multiplying each of through by  $p_1$  and  $p_2$  respectively we get

$$p_1x'_1 < p_1x_1$$

$$p_2x'_2 < p_2x_2$$

which gives us

$$p_1x'_1 + p_2x'_2 < p_1x_1 + p_2x_2$$

$$\Rightarrow m' < m$$

which contradicts our assumption that  $m < m'$ . QED.