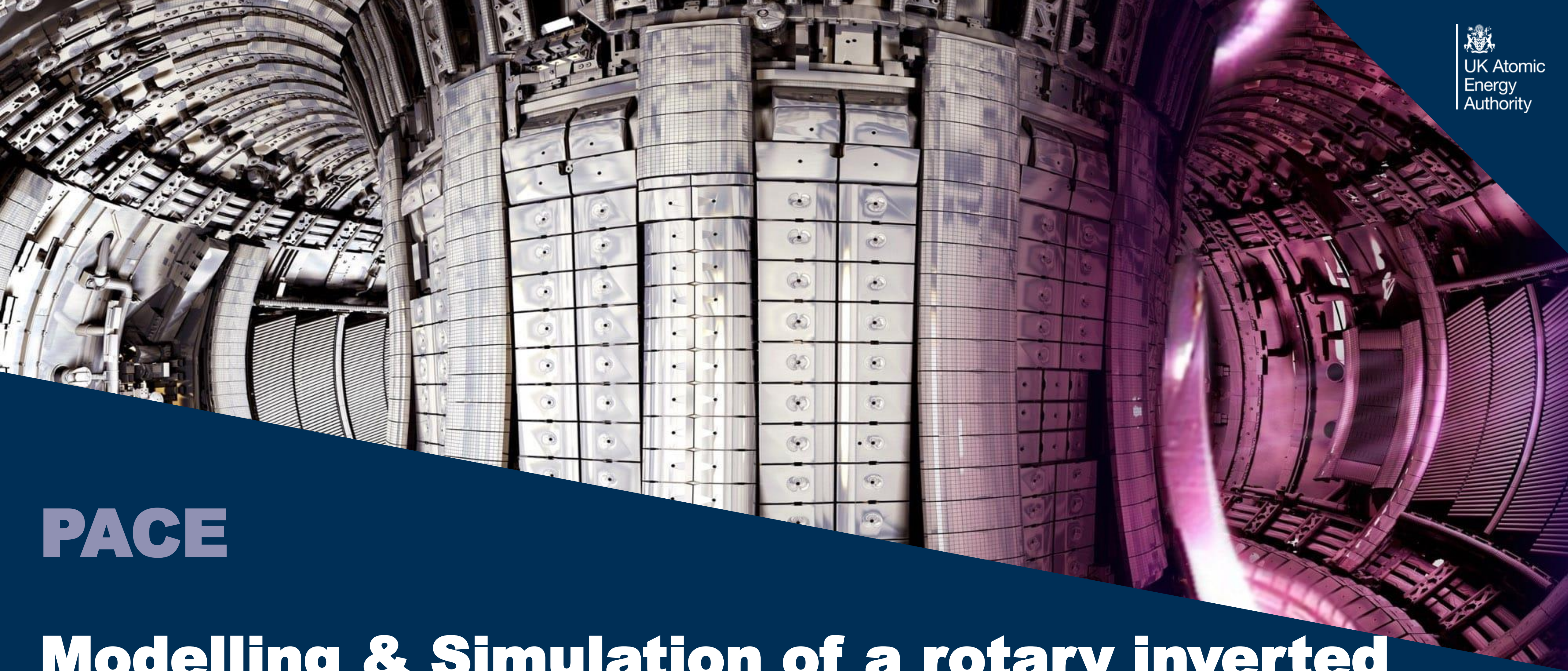

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PACE

Modelling & Simulation of a rotary inverted pendulum

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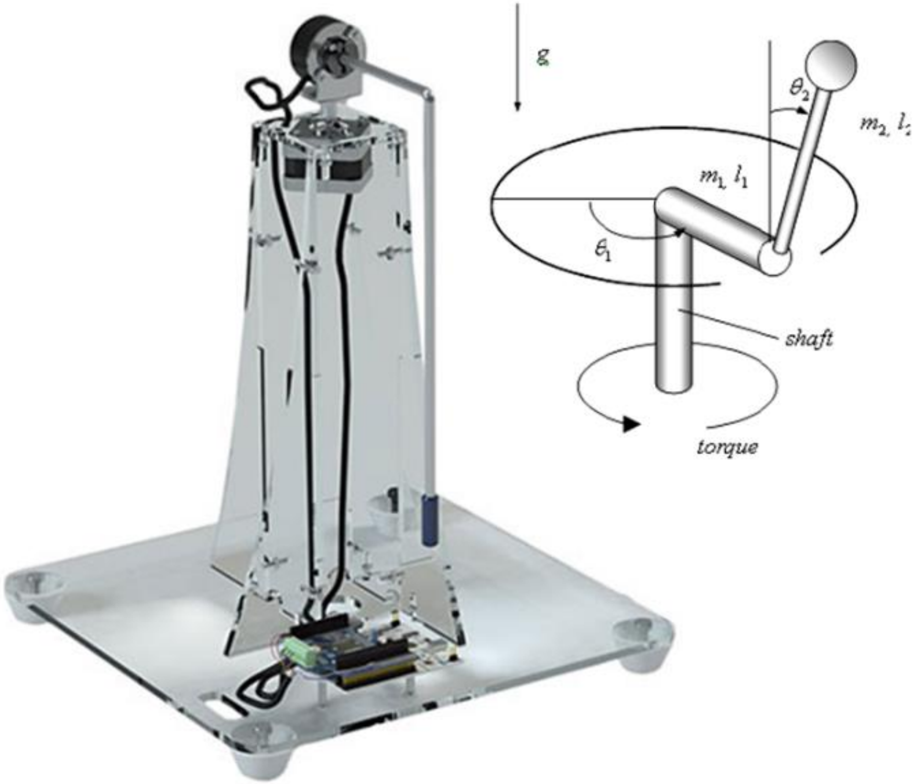
Modelling

Scope

These slides outline the steps required to derive the non-linear equations of motion (aka the dynamic model) of a rotary pendulum with only one actuator.

Challenging topics involved:

- Vector calculus
- Rigid Body Kinematics & Dynamics
    - Lagrangian Mechanics
- Simulation of Ordinary Differential Equations (ODEs)



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Important Kinematics

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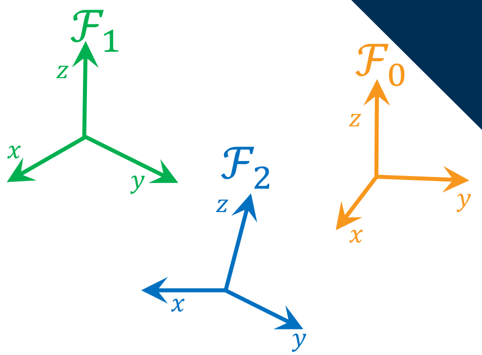
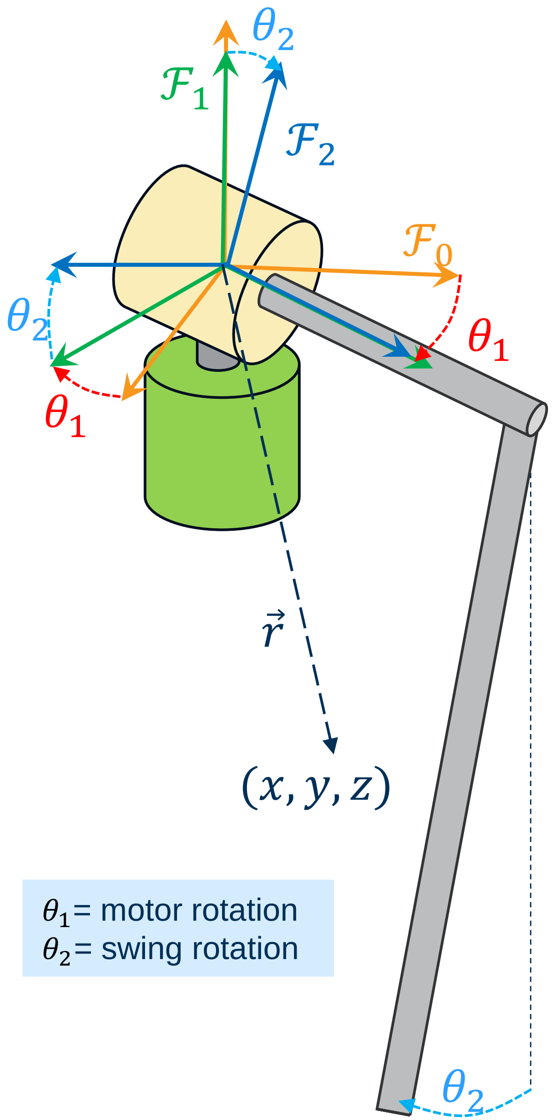
There are 3 frames of reference: $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$.

Inertial (globally fixed) frame — Encoder-fixed frame — Pendulum-fixed frame





θ_1= motor rotation
θ_2= swing rotation

- Let's define some position vector $\vec{r}$. We can express this as a numerical (x, y, z) vector, but these values depend on the frame of reference that we choose to express the vector in.

- Let's define: "$\vec{r}$ expressed in frame $\mathcal{F}_0$" as $\boldsymbol{r}^{\mathcal{F}_0} \in \mathbb{R}^3$.

- For our pendulum system, it can be shown that:

$$\boldsymbol{r}^{\mathcal{F}_1} = \mathbf{R}_{10}(\theta_1)\boldsymbol{r}^{\mathcal{F}_0}$$
$$\boldsymbol{r}^{\mathcal{F}_2} = \mathbf{R}_{21}(\theta_2)\boldsymbol{r}^{\mathcal{F}_1}$$

$$\boldsymbol{r}^{\mathcal{F}_2} = \underbrace{\mathbf{R}_{21}\mathbf{R}_{10}}\boldsymbol{r}^{\mathcal{F}_0}$$

$$\mathbf{R}_{20}(\theta_1, \theta_2) = \begin{bmatrix} \cos\theta_1 \cos\theta_2 & -\sin\theta_1 & \sin\theta_2 \cos\theta_1 \\ \sin\theta_1 \cos\theta_2 & \cos\theta_1 & \sin\theta_1 \sin\theta_2 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

This is known as a $z \rightarrow y'$ (yaw-pitch) transformation

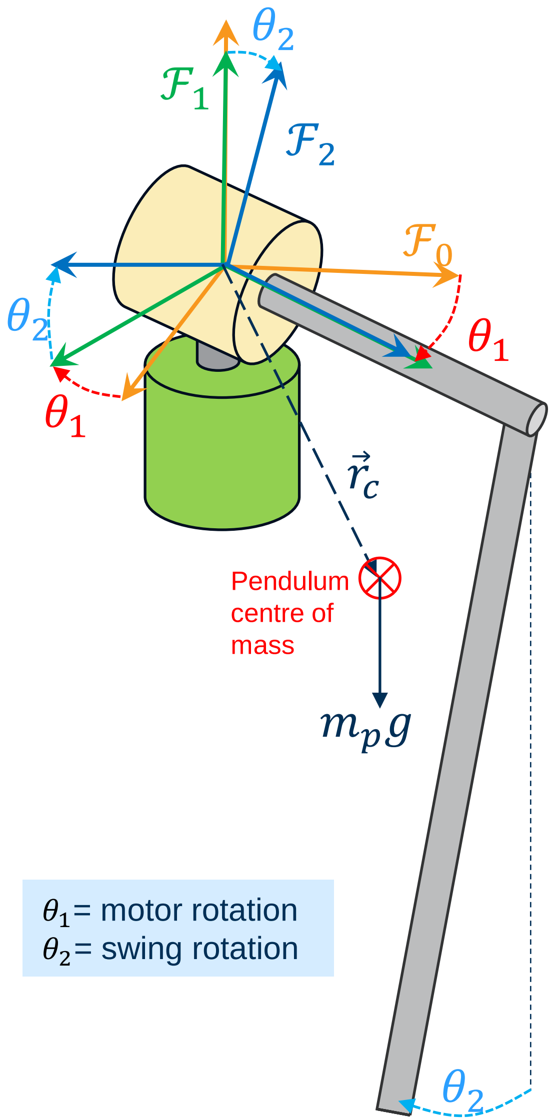
Rotation matrices R

$$\mathbf{R}_{10} = \mathbf{R}_z(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_{21} = \mathbf{R}_y(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

$\mathbf{R}_{20}$ describes the orientation of the pendulum body with respect to the inertial frame of reference.

Calculating the energy of the pendulum



θ_1= motor rotation
θ_2= swing rotation

Gravitational Potential Energy

$$V = m_p g \boxed{\boldsymbol{r}_c\big|_z}$$

Vertical position of the centre of mass, expressed in the inertial frame.

$\boldsymbol{r}_c = \boldsymbol{r}_c^{\mathcal{F}_0} = \mathbf{R}_{20}(\theta_1, \theta_2)^\top \boldsymbol{r}_c^{\mathcal{F}_2}$

Rotational Kinetic energy

$$T_{rot_p} = \frac{1}{2}\boldsymbol{\omega}_p^\top \boxed{\boldsymbol{I}_p} \boxed{\boldsymbol{\omega}_p}$$

Pendulum inertia tensor, <u>calculated in the next slide</u>

Angular velocity of pendulum, expressed in the inertial frame. This can be shown to be:

$$\boldsymbol{\omega}_p = \boldsymbol{\omega}_p^{\mathcal{F}_0} = \mathbf{R}_z(\theta_1)^\top \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

Linear Kinetic energy

$$T_{lin_p} = \frac{1}{2}\boxed{m_p}\dot{\boldsymbol{r}}_c^\top \boxed{\dot{\boldsymbol{r}}_c}$$

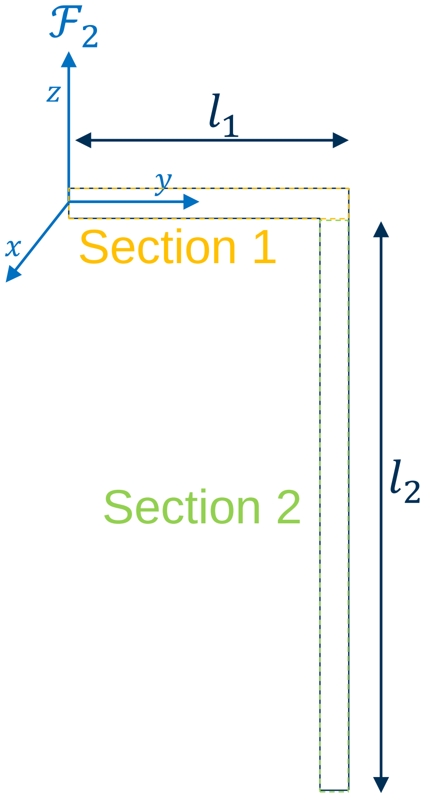
Pendulum Mass

Linear velocity of pendulum centre of mass, expressed in the inertial frame. This can be shown to be:

$\dot{\boldsymbol{r}}_c = \dot{\boldsymbol{r}}_c^{\mathcal{F}_0} = \dot{\mathbf{R}}_{20}(\theta_1, \theta_2)^\top \boldsymbol{r}_c^{\mathcal{F}_2}$

We can now define the systems 'Lagrangian': $L = T_{rot_p} + T_{lin_p} - V$

Aside: Approximating the pendulum Inertia Tensor I_p



$\boldsymbol{I}_p$ can be thought of as the 'rotational mass' of an object. This depends on the axis of rotation, which makes $\boldsymbol{I}_p$ a 3X3 matrix.

$$\boldsymbol{I}_p^{\mathcal{F}_2} = \boldsymbol{I}_1^{\mathcal{F}_2} + \boldsymbol{I}_2^{\mathcal{F}_2}$$

Pendulum inertia tensor resolved in the moving (or 'body') frame $\mathcal{F}_2$

Section 1

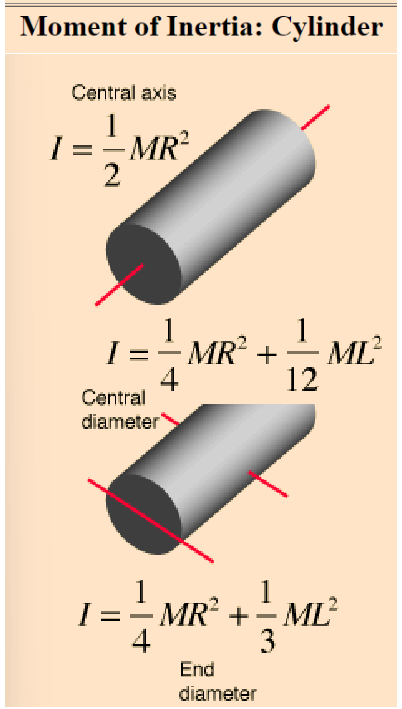
$$\boldsymbol{I}_1^{\mathcal{F}_2} = \begin{bmatrix} \frac{1}{4}m_1r^2 + \frac{1}{3}m_1l_1^2 & 0 & 0 \\ 0 & \frac{1}{2}m_1r^2 & 0 \\ 0 & 0 & \frac{1}{4}m_1r^2 + \frac{1}{3}ml_1^2 \end{bmatrix}$$

Section 2

$$\boldsymbol{I}_2^{\mathcal{F}_2} = \begin{bmatrix} \frac{1}{4}m_2r^2 + \frac{1}{12}m_2l_2^2 & 0 & 0 \\ 0 & \frac{1}{4}m_2r^2 + \frac{1}{12}m_2l_2^2 & 0 \\ 0 & 0 & \frac{1}{2}m_2r^2 \end{bmatrix} + \underbrace{m_2\left(\begin{bmatrix} 0 \\ l_1 \\ \frac{-l_2}{2} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 \\ l_1 \\ \frac{-l_2}{2} \end{bmatrix} \boldsymbol{E}_3 - \begin{bmatrix} 0 \\ l_1 \\ \frac{-l_2}{2} \end{bmatrix} \begin{bmatrix} 0 \\ l_1 \\ \frac{-l_2}{2} \end{bmatrix}^{\mathsf{T}}\right)}_{\text{From the parallel axis theorem}}$$

Assumption: solid cylinder

(can improve this using measurements)



Finding the state-space 'Equations of Motion'

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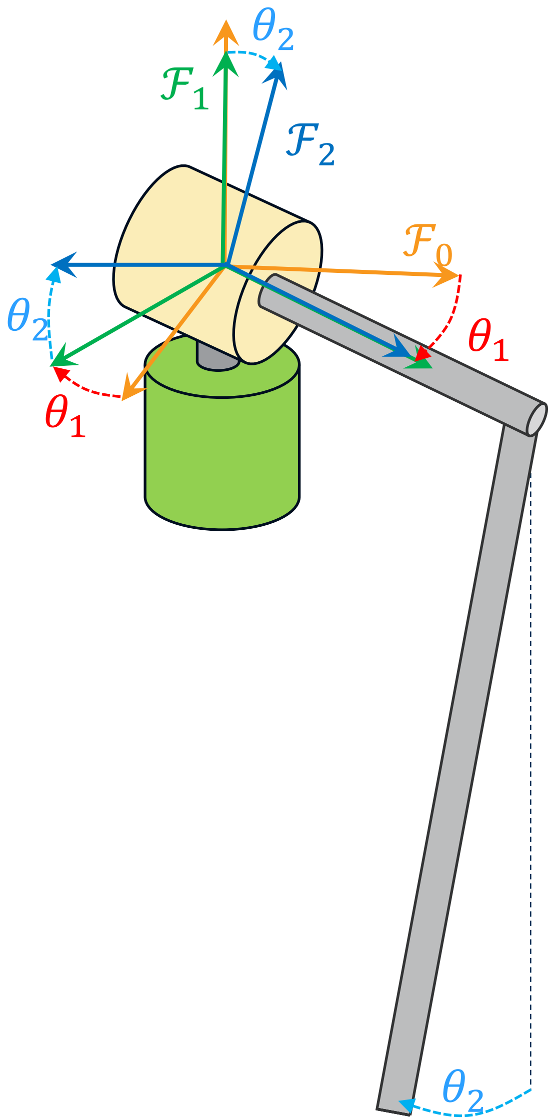


Let's define our state coordinates, $\boldsymbol{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$.

→ $\boldsymbol{q}$ completely describes the position of our pendulum

Euler-Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = \boldsymbol{F}$$

The 'external force' vector. If we ignore friction damping, then $\boldsymbol{F} = \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$, where u is the torque applied by the motor.

This equation basically creates the "F=ma" of the system: the equation that describes the behaviour of the pendulum. Also known as a "Dynamic Model".

By substituting L into this and expanding / rearranging, the result ends up looking something like this:

"Mass Matrix" $\boxed{\boldsymbol{M}(\boldsymbol{q})}\ddot{\boldsymbol{q}} + \boxed{\boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})} = \boldsymbol{F}$

Vector describing gravity & Coriolis effects

≡

Often it is nice to express this in the *equivalent* 'first order ODE form', by defining a new set of variables

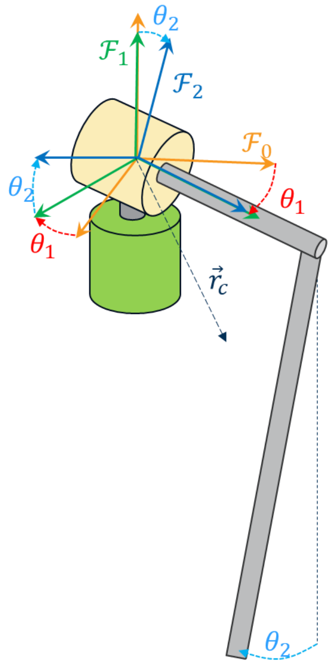
$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{M}^{-1}(\boldsymbol{F} - \boldsymbol{N}) \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{\boldsymbol{q}} \\ -\boldsymbol{M}^{-1}\boldsymbol{N}) \end{bmatrix}}_{\boldsymbol{f}(\boldsymbol{x})} + \underbrace{\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}}_{\boldsymbol{g}(\boldsymbol{x})} u$$

$$\boxed{\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u}$$

This model form is a "Control affine" system, meaning the input u appears linearly

Simulating the free-response system (u=0)

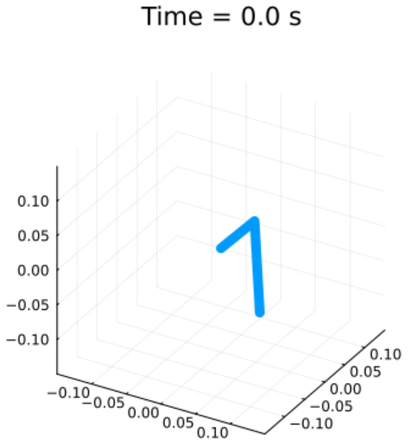


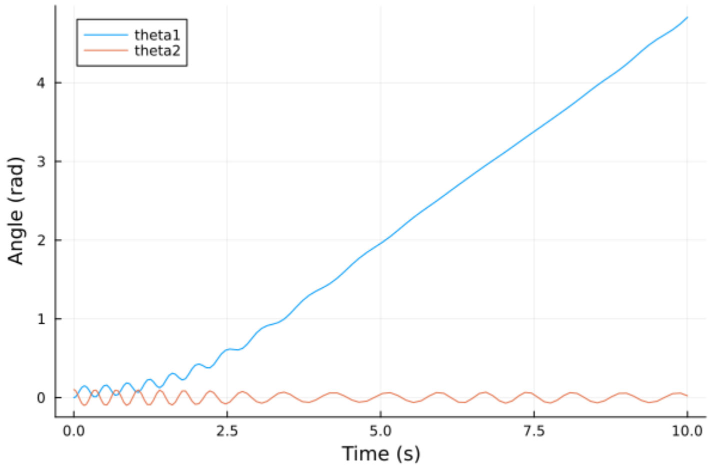
Given a starting state $\boldsymbol{x}(\boldsymbol{t}=\boldsymbol{0})=\boldsymbol{x}_0$, the model equations $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})$ can be 'solved' (aka 'simulated') using a variety of techniques (aka 'ODE solvers').

- This can be done in Python, Matlab, C++, Julia and more.
  - I use ODE solvers in Julia's DifferentialEquations.jl toolbox, this is just my personal preference.

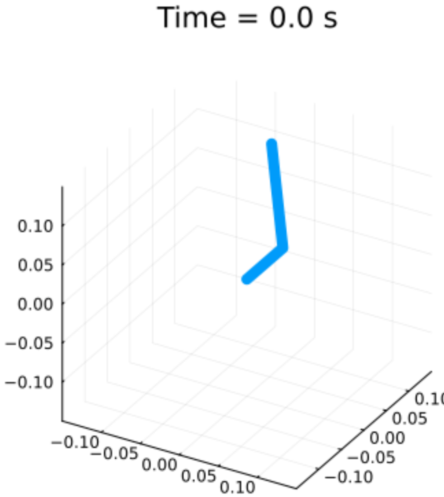
You can then create an animation of the model simulation.

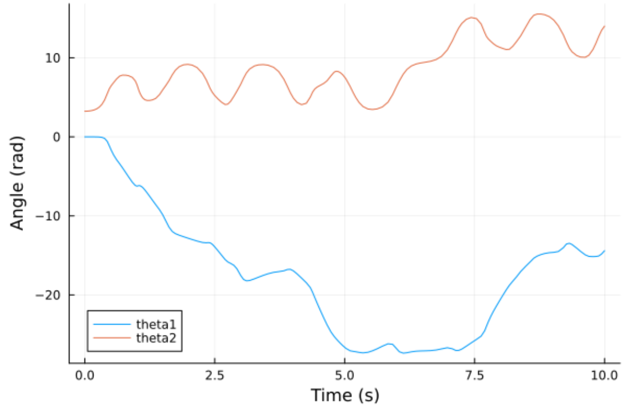
Example 1: $\boldsymbol{x}_0=\begin{bmatrix}\theta_{1_0}\\ \theta_{2_0}\\ \dot{\theta}_{1_0}\\ \dot{\theta}_{2_0}\end{bmatrix}=\begin{bmatrix}0\\ 0.1\\ 0\\ 0\end{bmatrix},\quad u=0$





Example 2: $\boldsymbol{x}_0=\begin{bmatrix}0\\ \pi+0.1\\ 0\\ 0\end{bmatrix},\quad u=0$



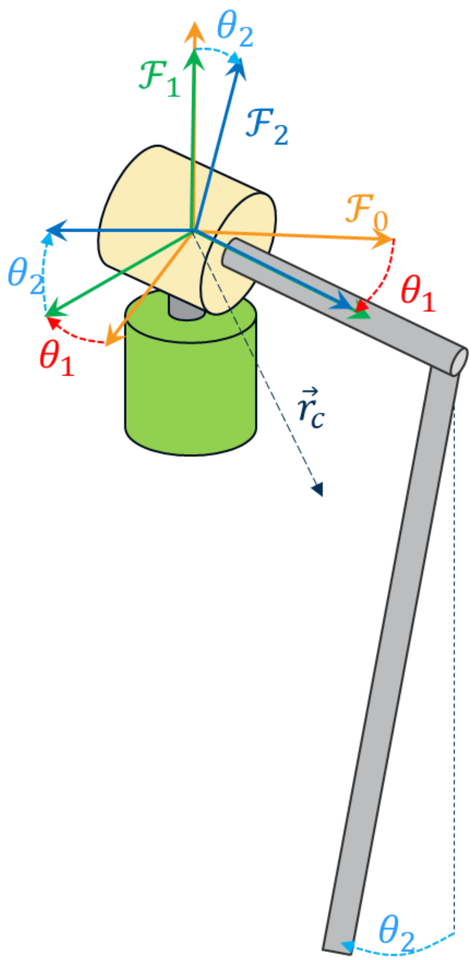


Very chaotic!
(no damping is modelled)

Adding damping to the model

In reality, there will be *viscous damping* at the joints.

➢ This is a friction torque proportional to the joint velocity:



$$\tau_{fr_i} \approx -\boxed{d_{fr_i}}\dot{\theta}_i$$

A damping constant of proportionality for joint i

In vector form:

$$\boldsymbol{\tau}_{fr} \approx -\boxed{\boldsymbol{D}}\dot{\boldsymbol{\theta}}$$

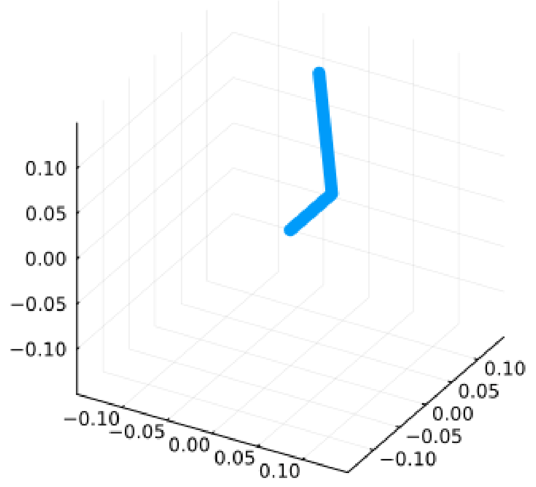
A diagonal Damping matrix

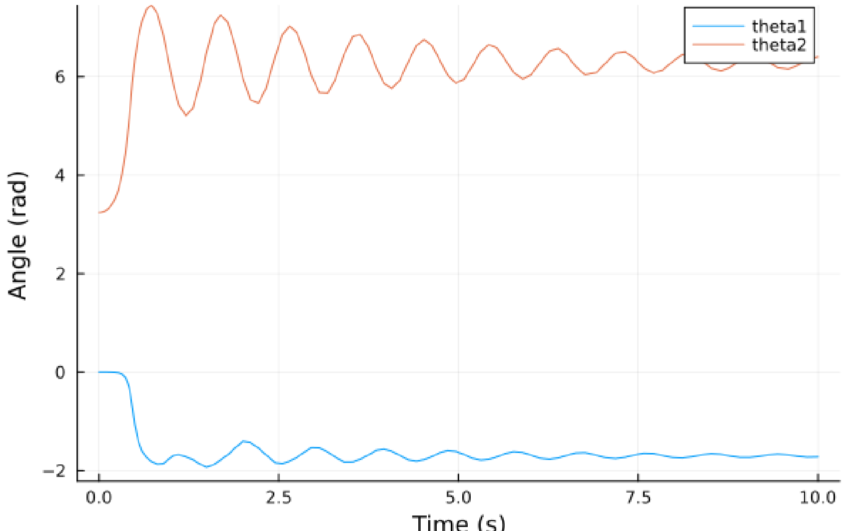
The model now becomes:

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{D}\dot{\boldsymbol{\theta}} = \boldsymbol{F}$$

$$\dot{\boldsymbol{x}} = \underbrace{\begin{bmatrix} \dot{\boldsymbol{q}} \\ \boldsymbol{M}^{-1}(\boldsymbol{F} - \boldsymbol{N} + \boldsymbol{D}\dot{\boldsymbol{\theta}}) \end{bmatrix}}_{f(x,u)}$$

Time = 0.0 s

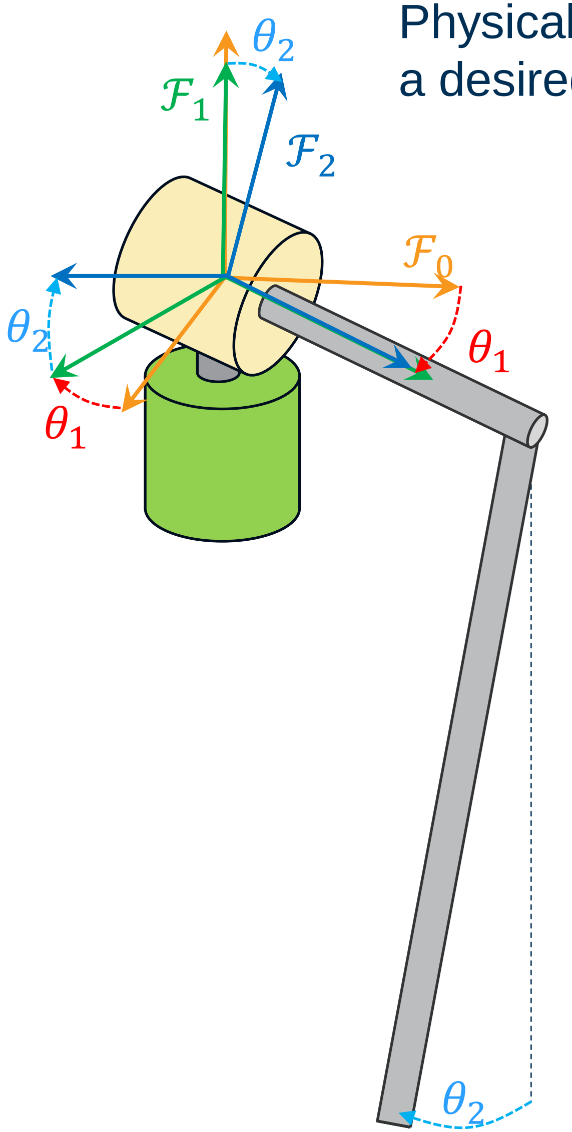




Stepper motor stabilization control

What if our control input is $u = \ddot{\theta}_1$, instead of the motor torque?

Let's assume we can precisely control the acceleration of the stepper motor $\ddot{\theta}_1$. Physically, this means the motor imparts a torque on the system that ensures $\ddot{\theta}_1$ follows a desired acceleration, $u(t)$. We can consider this as a *constraint!*



-> Lets express $\ddot{\theta}_1 = \underbrace{[1 \;\; 0]}_{\boldsymbol{A}} \ddot{\boldsymbol{q}}$

Dynamics equations

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{N} + \boldsymbol{A}^\top \tau_1 = 0 \qquad (1)$$

rearrange

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(-\boldsymbol{N} - \boldsymbol{A}^\top \tau_1) = 0$$

Sub into

Constraint equations

$$\boldsymbol{A}\ddot{\boldsymbol{q}} - u(t) = 0$$

$$\boldsymbol{A}\boldsymbol{M}^{-1}(-\boldsymbol{N} - \boldsymbol{A}^\top \tau_1) - u(t) = 0$$

rearrange

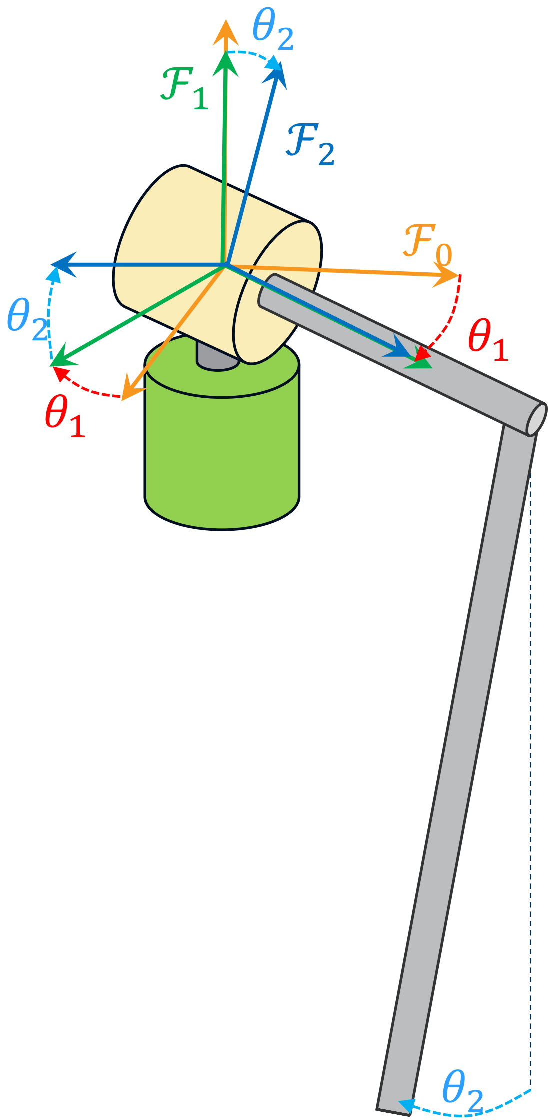
$$\tau_1 = -(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^\top)^{-1}(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{N} + u)$$

Sub into (1)

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{N} - \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^\top)^{-1}(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{N} + u) = 0$$

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{N}_a = \boldsymbol{B}_a u \quad \begin{cases} \boldsymbol{N}_a = \boldsymbol{N} - \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^\top)^{-1}\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{N} \\ \boldsymbol{B}_a = \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^\top)^{-1} \end{cases}$$

What if our control input is $u = \ddot{\theta}_1$, instead of the motor torque?



$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{N}_a = \boldsymbol{B}_a u$$

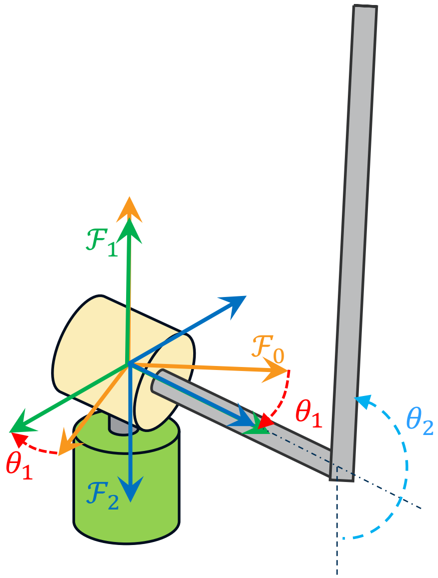
$$\begin{cases} \boldsymbol{N}_a = \boldsymbol{N} - \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^\top)^{-1}\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{N} \\ \boldsymbol{B}_a = \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{M}^{-1}\boldsymbol{A}^\top)^{-1} \end{cases}$$

First order form

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\boldsymbol{q}} \\ \ddot{\boldsymbol{q}} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{\boldsymbol{q}} \\ -\boldsymbol{M}^{-1}\boldsymbol{N}_a \end{bmatrix}}_{\boldsymbol{f}(\boldsymbol{x})} + \underbrace{\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{B}_a \end{bmatrix}}_{\boldsymbol{g}(\boldsymbol{x})} u$$

LQR

What should $u(t)$ be if we want to stabilise the pendulum?



$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{B}u$$

$$\Delta\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \approx \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_{equil}} \Delta\boldsymbol{x} + \boldsymbol{B}u$$

$$\boldsymbol{x}_{equil} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}_{equil} = \begin{bmatrix} \theta_1 \\ k\pi \\ 0 \\ 0 \end{bmatrix}, where\ k = any\ integer$$

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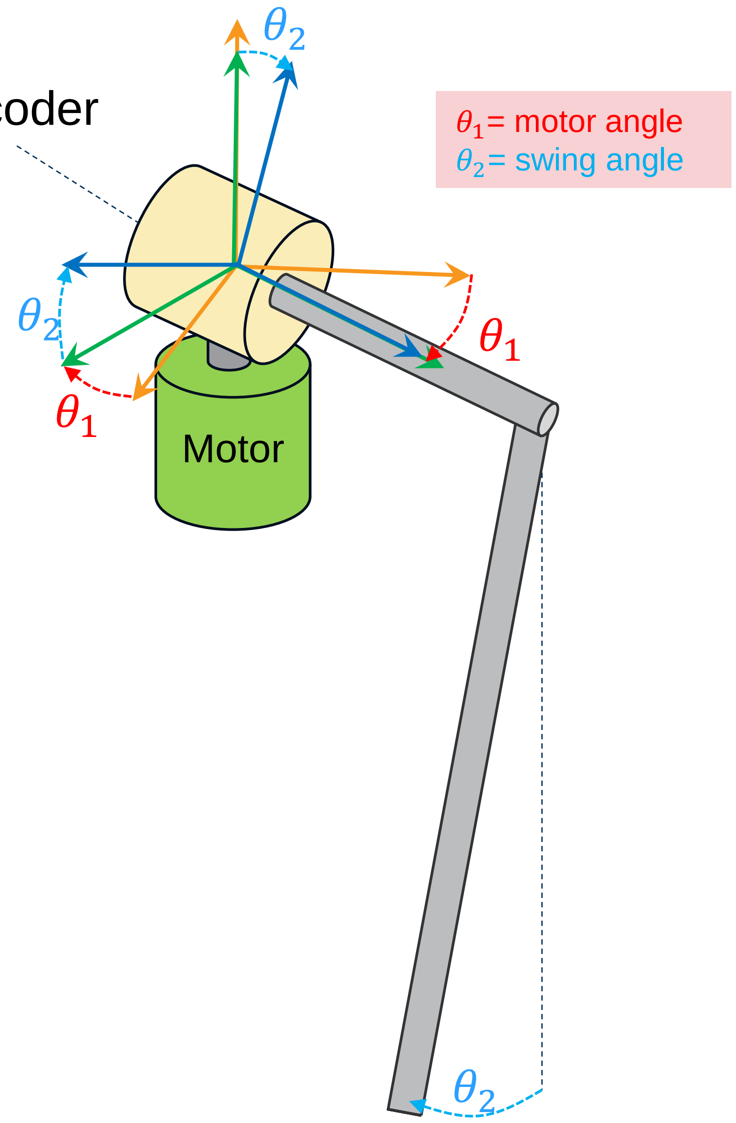

Swing-up

Trajectory Optimisation

We want to ‘swing up’ the pendulum.

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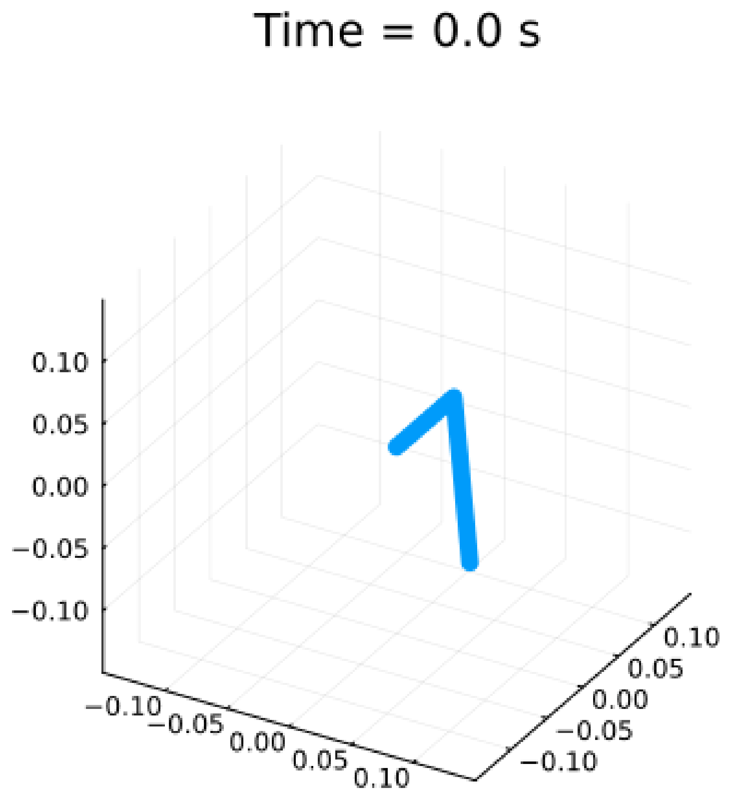
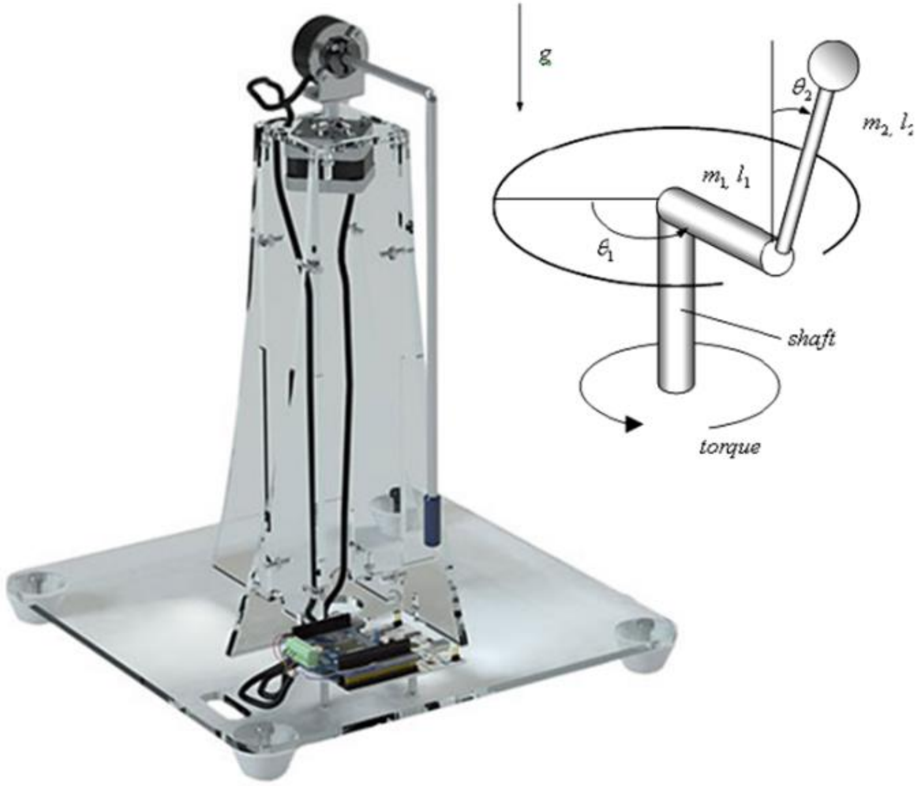
How can we do this intelligently? Can we consider:

- Motor Torque / Velocity / acceleration limits
- Spatial constraints

One approach is **trajectory optimisation**





How?

System Identification

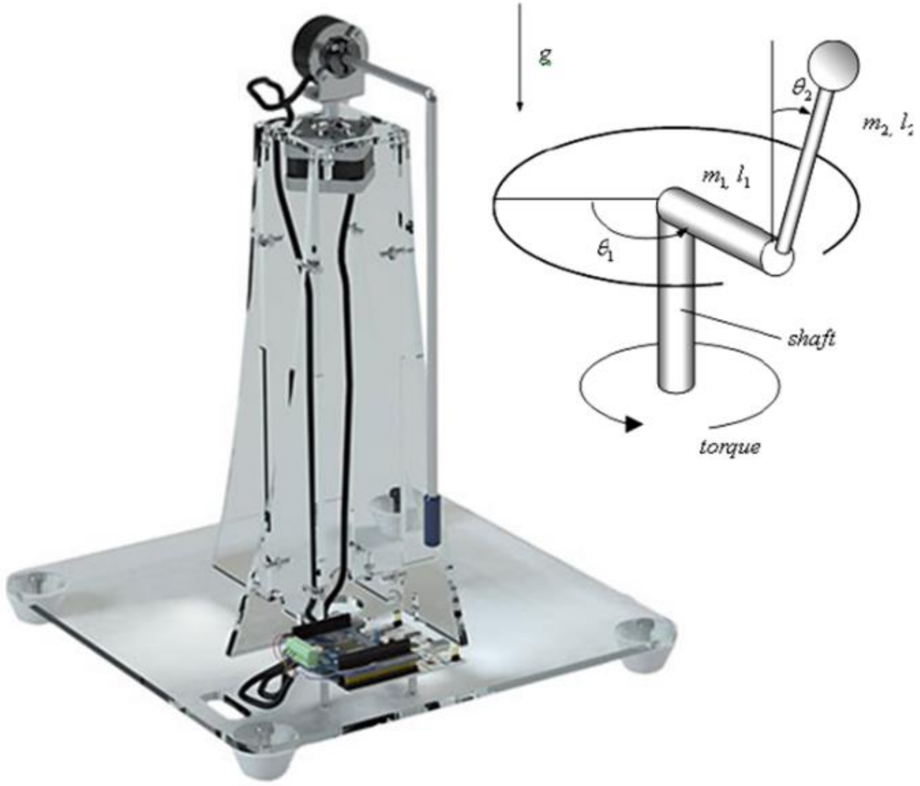
Why System Identification?

Our analytical model may be poor due to making bad estimates of:

$$m_1, m_2, d_{fr_2}, \boldsymbol{r}_c^{\mathcal{F}_2}$$

Can we improve our model parameters using observed data? YES

$$\boldsymbol{p} = m_1, m_2, d_{fr_2}, \boldsymbol{r}_c^{\mathcal{F}_2}$$



How?

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Next steps

- Improve model parameters & verify
- Implement on real set-up