Single System
Classical Information

- Quantum states representated by vectors, where operations are represented by unitary matrices
- In general, quantum states are represented by density matrices, allows more general class of measurements and operations.
- Assume X as finite no. of classical states
  Eg: X is a bit, then it's classical set is $\Sigma = \{0, 1\}$.
- Probablistic state of X, Eg:

let $Pr(X=0) = \frac{3}{4}$ and $Pr(X=1) = \frac{1}{4}$

succient way to represent is column vector:

$$\begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} \begin{matrix} \leftarrow \text{entry for } (0) \\ \leftarrow \text{entry for } (1) \end{matrix}$$

also called probablity vector but sum = 1 always.

I.) Dirac notation Eg: If $\Sigma = \{0, 1\}$, then

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ "cat of zero" and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ "cat of one"

Standard basis vector: $\begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} = \frac{3}{4}|0\rangle + \frac{1}{4}|1\rangle$

- Deterministic operations $f : \Sigma \to \Sigma$

transforms the classical state a into $f(a)$, for each $a \in \Sigma$

Eg: $M\,|a\rangle = |f(a)\rangle$

"multiplying matrix M by ket a gives vector ket f of a"

This matrix has exactly one 1 in each column, and 0 for all other entries:

$$\underset{\substack{\uparrow \\ \text{row}}}{M}(b, a) = \begin{cases} 1 & b = f(a) \\ 0 & b \neq f(a) \end{cases}$$

row column

this action of this operation is described by matrix-vector multiplication $\quad v \mapsto Mv$

"notation of mapping one thing to another"

Eg: "rows first columns second"

for $M(b, a) = \begin{cases} 1 & b = f(a) \\ 0 & b \neq f(a) \end{cases}$

for $\Sigma = \{0, 1\}$, there are four f's of the form $f : \Sigma \to \Sigma$:

a	$f_1(a)$
0	0
1	0

a	$f_2(a)$
0	0
1	1

a	$f_3(a)$
0	1
1	0

a	$f_4(a)$
0	1
1	1

Hence the matrices corresponding to these f's:

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad M_4 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$M\,|a\rangle = |f(a)\rangle$$

here $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

"bra a"

II.) Dirac Notation $\langle a|$ the row vector having a 1 in the entry to $a \in \Sigma$, 0 for all others

Eg: If $\Sigma = \{0, 1\}$ then

$$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \text{and} \quad \langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$\langle a|\,|b\rangle = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases} = \langle a|b\rangle$$

"bra ket"

row vect × column vect = Scalar value, column vect × row vect = matrix eg: c × r

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

r × c

$$\langle 0|0\rangle = \cancel{(} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

- Deterministic operations $f : \Sigma \to \Sigma$

$$M\,|a\rangle = |f(a)\rangle$$

can be represented as

$$M = \sum |f(b)\rangle\langle b|$$

In action on standard basis vectors work as:

$$M\,|a\rangle = \left(\sum_{b=\Sigma} |f(b)\rangle\langle b| \right) |a\rangle$$

$$= \left(\sum_{b=\Sigma} |f(b)\rangle\langle b|a\rangle \right)$$

$$= |f(a)\rangle$$

- Probabilistic operations : "may introduce randomness"

Eg: If the classical state = 0, do nothing
If the classical state = 1, flip the bit with 50% chance

Probability operations are described by stochastic matrices:

- All entries are non -ve real numbers
- The enteries in every column sum to 1

Eg: $\begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- Composing operations $M_2(M_1 V) = (M_2 M_1) V$
"associative property"

but not commutative i.e $M_2 M_1 \neq M_1 M_2$

Single System
Quantum Information

A quantum state of a system is represented by a column vector whose indices are placed in correspondence with the classical states of that system.

- The entries are complex numbers
- The sum of the absolute value squared of the entries must equal 1

- The Euclidean norm for vectors with complex numbers entries is defined :

$$V = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \Rightarrow \|V\| = \sqrt{\sum_{k=1}^{n} |\alpha_k|^2}$$

Quantum state vectors also called qubit states

Eg: Plus/minus states :

$$|+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \text{ and}$$

$$|-\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$$

A state without a special name:

$$\frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle$$

another Eg: A quantum state of a system with classical states a, b, c, d:

$$\frac{1}{2}|a\rangle - \frac{i}{2}|b\rangle + \frac{1}{\sqrt{2}}|d\rangle = \begin{bmatrix} 1/2 \\ -i/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

III.) Dirac Notation used for arbitrary Vectors

kets → columns Vector | bra → row Vector

Eg: The notation $|\psi\rangle$ is commonly used to refer to an arbitrary vector:

$$\underset{\text{"ket"}}{|\psi\rangle} = \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle = \begin{pmatrix} \frac{1+2i}{3} \\ -\frac{2}{3} \end{pmatrix}$$

For any column Vector $|\psi\rangle$, the row Vector $\langle\psi|$ is the conjugate transpose of $|\psi\rangle$:

$$\langle\psi| = |\psi\rangle^{\dagger}$$

$$\therefore \langle \psi | = \frac{1-2i}{3} \langle 0| - \frac{2}{3} \langle 1|$$

"bra"

$$= \begin{pmatrix} \frac{1-2i}{3} & -\frac{2}{3} \end{pmatrix}$$

- Measuring quantum states (std. basis measurements)

The possible outcomes are the classical states

The probability for each classical state to be the outcome is the absolute value squared of the corresponding quantum state vector entry

Eg: Measuring the quantum state

$$|+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$$

yields an outcome as:

$$\Pr(\text{outcome is } 0) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

$$\Pr(\text{outcome is } 1) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

Eg 2: $\frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle$

yields an outcome:

$$Pr(\text{outcome is } 0) = \left|\frac{1+2i}{3}\right|^2 = \frac{5}{9}$$

$$Pr(\text{outcome is } 1) = \left|-\frac{2}{3}\right| = \frac{4}{9}$$

Measuring a system changes its quantum state: if we obtain the classical state a, the new quantum state becomes $|a\rangle$.

$$\begin{array}{ccc} & \frac{1+2i}{3}|0\rangle - \frac{2}{3}|1\rangle & \\ \swarrow \text{probability } \frac{5}{9} & & \searrow \text{probability} \\ |0\rangle & & |1\rangle \end{array}$$

- Unitary Operations

The set of allowable operations that can be performed on a quantum state is different than it is for classical information

Operations on quantum state vectors are represented by unitary matrices.

Defination:

A sq^r matrix U having complex number entries in unitary if it satisfies the equalities

$$U^\dagger U = 1 = U U^\dagger$$

where $U^\dagger$ is the conjugate transpose of U and 1 is the identity matrix

$$U^{-1} = U^\dagger \text{ (only for sq}^r\text{ matrix)}$$

The condition that an $n \times n$ matrix U is unitary equivalent to

$$\|Uv\| = \|v\|$$ for every n-dimensional column vector v with complex number entries.

If v is a quantum state vector, then Uv is also a quantum state vector

- Qubit unitary operations

1. Pauli operations - by Pauli matrices.

bit flip (NOT) phase flip

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Common alternative notations : $X = \sigma_x$, $Y = \sigma_y$, $Z = \sigma_z$

2. Hadamard operation

$$H = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

it's unitary i.e $HH^{\dagger} = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. Phase operation

$$P_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$ also unitary

for any choice of real no. θ.

imp operation :

$$S = P_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$ and

$$T = P_{\pi/4} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

Eg 1:

$$H|0\rangle = |+\rangle \qquad H|1\rangle = |-\rangle$$
$$H|+\rangle = |0\rangle \qquad H|-\rangle = |1\rangle$$

Eg2:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

$$T|0\rangle = |0\rangle \qquad T|1\rangle = \frac{1+i}{\sqrt{2}}|1\rangle$$

$$T|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1+i}{2}|1\rangle$$

$$HT|+\rangle = \left(\frac{1}{2} + \frac{1+i}{2\sqrt{2}}\right)|0\rangle + \left(\frac{1}{2} - \frac{1+i}{2\sqrt{2}}\right)|1\rangle$$

composing unitary operations are represented by matrix multiplication

Eg: sq^r root of NOT

$$HSH = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}$$

applying this twice, gives NOT operation

$$(HSH)^2 = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

And now we have learnt how quantum information works in isolation.

And further we will learn how quantum information works on multiple systems