Multiple System

Classical Information

- Classical states

Let

X be a system having classical state set Σ
Y be a system having classical state set Γ.

New compound state XY.

Classical state of XY, cartesian product

$$\Sigma \times \Gamma = \{(a,b) : a \in \Sigma \text{ and } b \in \Gamma\}$$

Suppose $X_1 \dots X_{10}$ are bits so their classical state sets are all the same:

$$\Sigma_1 = \Sigma_2 = \cdots\cdots = \Sigma_{10} = \{0,1\}$$

The classical state set of $(X_1 \dots X_{10})$ is the Cartesian product

$$\Sigma_1 \times \Sigma_2 \times \cdots\cdots \times \Sigma_{10} = \{0,1\}^{10}$$

- Probablistic state

Let a probablistic state of a pair of bits (X,Y):

$$\begin{aligned} Pr((X,Y) = (0,0)) &= 1/2 \\ Pr((X,Y) = (0,1)) &= 0 \\ Pr((X,Y) = (1,0)) &= 0 \\ Pr((X,Y) = (1,1)) &= 1/2 \end{aligned} \;\underset{\text{"can be represented as"}}{\mapsto}\; \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}$$

For a given probablistic state of (X,Y), we say that X and Y are independent if

$$Pr((X,Y) = (a,b)) = Pr(X=a)\, Pr(Y=b)$$

for all $a \in \Sigma$ and $b \in \Gamma$.

to represent as vector:

$$|\pi\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} p_{ab} |ab\rangle$$

The systems X and Y are independent if there exist probablity

$$|\phi\rangle = \sum_{a \in \Sigma} q_a |a\rangle \text{ and } |\psi\rangle = \sum_{b \in \gamma} r_b |b\rangle$$

such that $p_{ab} = q_a r_b$ for all $a \in \Sigma$ and $b \in \Gamma$.

Eg: The probabilistic state of a pair of bits (X, Y) represented by the vector

$$|\pi\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$$

is one in which X and Y are independent. The required condition is true for these probabilities vectors:

$$|\phi\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \text{ and } |\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$$

For the probabilistic state

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle \text{ of two bits } (X, Y),$$

we have that X and Y are not independent.

- Tensor products of Vectors $\otimes$

let $|\phi\rangle = \sum_{a \in \Sigma} \alpha_a |a\rangle$ and $|\psi\rangle = \sum_{b \in \Gamma} \beta_b |b\rangle$

is the vector

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} \alpha_a \beta_b |ab\rangle$$

Equivalent, the vector $|\pi\rangle = |\phi\rangle \otimes |\psi\rangle$ is defined by this condition

$\langle ab|\pi\rangle = \langle a|\phi\rangle\langle b|\psi\rangle$ (for all $a \in \Sigma$ and $b \in \Gamma$)

Eg: $|\phi\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle$ and $|\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$

$\therefore |\phi\rangle \otimes |\psi\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$

Alternative notations for tensor products:

$$|\phi\rangle|\psi\rangle = |\phi\rangle \otimes |\psi\rangle$$
$$|\phi \otimes \psi\rangle = |\phi\rangle \otimes |\psi\rangle$$

$$|a\rangle \otimes |b\rangle = |a\rangle|b\rangle = |ab\rangle$$

if (a, b) is an ordered pair:

$$|a\rangle \otimes |b\rangle = |(a, b)\rangle = |a, b\rangle$$

The tensor product of two vectors is bilinear

1. Linearity in the first argument

$$(|\phi_1\rangle + |\phi_2\rangle) \otimes |\psi\rangle = |\phi_1\rangle \otimes |\psi\rangle + |\phi_2\rangle \otimes |\psi\rangle$$

$$(\alpha|\phi\rangle) \otimes |\psi\rangle = \alpha(|\phi\rangle \otimes |\psi\rangle)$$

2. Linearity in the second arguement:

$$|\phi\rangle \otimes (|\psi_1\rangle + |\psi_2\rangle) = |\phi\rangle \otimes |\psi_1\rangle + |\phi\rangle \otimes |\psi_2\rangle$$

$$|\phi\rangle \otimes (\alpha|\psi\rangle) = \alpha(|\phi\rangle \otimes |\psi\rangle)$$

Notice that scalar "float freely"

- Tensor products generalize to three or mor systems

If $|\phi_1\rangle \dots |\phi_n\rangle$ are Vectors, then the tensor product

$$|\psi\rangle = |\phi_1\rangle \otimes \dots |\phi_n\rangle$$

is defined by

$$\langle a_1 \dots a_n|\psi\rangle = \langle a_1|\psi\rangle \dots \langle a_n|\psi_n\rangle$$

Equivalently, the tensor product of three or more vectors can be defined recussively:

$$|\phi_1\rangle \otimes \dots \otimes |\phi_n\rangle = (|\phi_1\rangle \otimes \dots \otimes |\phi_{n-1}\rangle) \otimes |\phi_n\rangle$$

The tensor product of 3 or more is multi linear.

- Measurements of probablistic states

- Same way as single systems, but collectively

Eg: Suppose that two bits (X, Y) are in the probablistic state

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

Measuring both bits yields the outcome 00 with probablity 1/2 and the vice versa...

Q Suppose two systems (X, Y) are together in some probablistic state. What happens when we measure X and do nothing to Y.

→ 1. The probablity to observe a particular classical state $a \in \Sigma$ when just X is measured is

$$Pr(X = a) = \sum_{b \in \Gamma} Pr((X, Y) = (a, b))$$

2. There may still exist uncertainity about the classical state of Y, depending on the outcome of the measurement:

$$Pr(Y = b \mid X = a) = \frac{Pr((X, Y) = (a, b))}{Pr(X = a)}$$

These formulae can be expressed in Dirac

Suppose (X, Y) is in some arbitrary probabilistic state:

$$\sum_{(a,b) \in \Sigma \times \Gamma} p_{ab} |ab\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} p_{ab} |a\rangle \otimes |b\rangle$$

$$= \sum_{a \in \Sigma} |a\rangle \otimes \left(\sum_{b \in \Gamma} p_{ab} |b\rangle \right)$$

1. The probability that a measurement of X yields an outcome $a \in \Sigma$ is

$$\Pr(X = a) = \sum_{b \in \Gamma} p_{ab}$$

2. Conditioned on the outcome $a \in \Sigma$, the probabilities state of Y becomes

$$\frac{\sum_{b \in \Gamma} p_{ab} |b\rangle}{\sum_{c \in \Gamma} p_{ac}}$$

Eg: Suppose (X, Y) is in the probabilistic state

$$\frac{1}{12}|00\rangle + \frac{1}{4}|01\rangle + \frac{1}{3}|10\rangle + \frac{1}{3}|11\rangle$$

we write vector as:

$$|0\rangle \otimes \left(\frac{1}{12}|0\rangle + \frac{1}{4}|1\rangle\right) + |1\rangle \otimes \left(\frac{1}{3}|0\rangle + \frac{1}{3}|1\rangle\right)$$

Case 1: the measurement outcome is 0

$$\Pr(\text{outcome is } 0) = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$$

conditioned on this outcome, the probabilistic state of Y:

$$\frac{1/12\,|0\rangle + 1/4\,|1\rangle}{1/3} = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle$$

Case 2: the measurement outcome is 1

$$\Pr(\text{outcome is } 1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

conditioned on this outcome, the probabilistic state of Y

$$\frac{1/3\,|0\rangle + 1/3\,|1\rangle}{2/3} = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$$

- Operations on probabilistic states

Probabilistic operations on compound systems are represented by stochastic matrices having rows and columns that correspond to the cartesian product of the individual systems' classical state sets.

Eg:

A controlled NOT operation on two bits X an Y:

If X = 1, then perform NOT operation on Y, else do nothing

X is the control bit that determines whether or not a NOT operation is applied to the target bit Y.

Action on standard basis | Matrix rep.

$$|00\rangle \rightarrow |00\rangle$$
$$|01\rangle \rightarrow |01\rangle$$
$$|10\rangle \rightarrow |11\rangle$$
$$|11\rangle \rightarrow |10\rangle$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Eg: Here is different operation on two bits (X, Y):

with probability $1/2$, set Y to be equal to X, else set X equal to Y

matrix representation:

$$\begin{pmatrix} 1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Q. Suppose we have two probablistic operations, each on its own system, described by stochastic matrices

1. M is an operation on X.
2. N is an operation on Y.

If we simultaneously perform the two operations, how do we describe the effect on the compound system (X, Y)?

- Tensor products of matrices

$$M = \sum_{a,b \in \Sigma} \alpha_{ab} |a\rangle\langle b| \quad \text{and} \quad N = \sum_{c,d \in \Gamma} \beta_{cd} |c\rangle\langle d|$$

is the matrix

$$M \otimes N = \sum_{a,b \in \Sigma} \sum_{c,d \in \Sigma} \alpha_{ab} \beta_{ab} |ac\rangle\langle bd|$$

Equivalently, $M \otimes N$ is defined by this condition

$$\langle ac | M \otimes N | bd \rangle = \langle a|M|b\rangle \langle c|N|d\rangle$$

(for all $a, b \in \Sigma$ and $b, d \in \Gamma$)

An alternative but equivalent way to define $M \otimes N$ is that it is the unique matrix that satisfies the eq^n

$$(M \otimes N) |\phi \otimes \psi\rangle = M|\phi\rangle \otimes N|\psi\rangle$$

for every choice of vectors $|\phi\rangle$ and $|\psi\rangle$

Tensor pdt of three or more matrices are defined in an analogous way.

If $M_1, \ldots M_n$ are matrices, then the tensor product $M_1 \otimes \ldots M_n$ is defined by:

$$\langle a_1 \cdots a_n | M_1 \otimes \cdots \otimes M_n | b_1 \cdots b_n \rangle = \langle a_1 | M_1 | b_1 \rangle \cdots \langle a_n | M_n | b_n \rangle$$

Alternatively, the tensor product of three or more matrices can be defined recursively similar to what we observed for vectors

The tensor pdt of matrices is multiplicative

$$(M_1 \otimes \cdots \otimes M_n)(N_1 \otimes \cdots \otimes N_n) = (M_1 N_1) \otimes \cdots \otimes (M_n N_n)$$

Now for the answer

→ The action is described by the tensor pdt $M \otimes N$.

Tensor pdt represent independence - this time b/w operations.

A common situation that we encounter is one in which one operation is performed on one system and nothing on the other, same prescription is followed, noting that doing nothing is represented as identity matrix.

Multiple System

Quantum Information

$$|\phi\rangle = \sum_{a\in\Sigma} \alpha_a |a\rangle \qquad |\psi\rangle = \sum_{b\in\Gamma} \beta_b |b\rangle$$

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \cancel{\alpha_a \beta_b}\ \alpha_a \beta_b |ab\rangle$$

- Quantum states - cartesian products of the individual system's classical state sets.

Eg: If X and Y are qubits, the classical set for the pair (X, Y) is

$$\{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$$

These are e.g of quantum states vectors of the pair (X, Y):

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

$$\frac{3}{5}|00\rangle - \frac{4}{5}|11\rangle$$

$$|01\rangle$$

Tensor products of quantum state vectors are also quantum state vectors.

Let $|\phi\rangle$ be a quantum state vector of a system X and let $|\psi\rangle$ be a quantum state vector of a system Y. The tensor product

$$|\phi\rangle \otimes |\psi\rangle$$

is then a quantum state vector of the system (X, Y).

States of this form are called product states. They represent independence between the systems X and Y.

More generally, if $|\psi_1\rangle, \dots, |\psi_n\rangle$ are quantum state vectors of systems $X_1, \dots, X_n$, then

$$|\psi_1\rangle \otimes \dots\dots \otimes |\psi_n\rangle$$

is a quantum state vector representing a product state of the compound system $(X_1 \dots X_n)$.

Eg: The quantum state vector

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

is an example of a product state:

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

$$= \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right)$$

Eg:

The quantum state Vetor

$|\phi^-\rangle$ $\quad \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ of two qubit

is not a product state

This is one of four Bell states, which collectively from Bell basis.

The Bell basis:

$$|\phi^+\rangle = 1/\sqrt{2}\,|00\rangle + 1/\sqrt{2}\,|11\rangle$$

$$|\phi^-\rangle = 1/\sqrt{2}\,|00\rangle - 1/\sqrt{2}\,|11\rangle$$

$$|\psi^+\rangle = 1/\sqrt{2}\,|01\rangle + 1/\sqrt{2}\,|10\rangle$$

$$|\psi^-\rangle = 1/\sqrt{2}\,|01\rangle - 1/\sqrt{2}\,|10\rangle$$

Here are a couple of well known examples of quantum state vectors for three qubits

GHZ state

$$1/\sqrt{2}\,|000\rangle + 1/\sqrt{2}\,|111\rangle$$

W state

$$1/\sqrt{3}\,|001\rangle + 1/\sqrt{3}\,|010\rangle + 1/\sqrt{3}\,|100\rangle$$

- Measurements - same way as single systems provided all are measured

If $|\psi\rangle$ a quantum state of a system $(X_1 \cdots X_n)$ and every one of the systems is measured, then each n tuple

$$(a_1 \cdots a_n) \in \Sigma_1 \times \cdots \Sigma_n$$

(or string $a_1 \cdots a_n$) is obtained with probability

$$|\langle a_1 \cdots a_n | \psi \rangle|^2$$

Q. Suppose two systems (X, Y) are together in some quantum state. What happens when we measure X and do nothing to Y?

→ A quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} \alpha_{ab} |ab\rangle$$

If both X and Y are measured, then each outcome $(a, b) \in \Sigma \times \Gamma$ appears with probability

$$|\langle ab|\psi\rangle|^2 = |\alpha_{ab}|^2$$

If just X is measured, the probability for each outcome $a \in \Sigma$ to appear must therefore be equal to

$$Pr(\text{outcome is } a) = \sum_{b \in \Gamma} |\langle ab|\psi\rangle|^2$$

$$= \sum_{b \in \Gamma} |\alpha_{ab}|^2$$

Similar to the probabilistic setting, the quantum state of Y changes as a result...

A quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_{ab}|ab\rangle$$

We can express the vector $|\psi\rangle$ as

$$|\psi\rangle = \sum_{a\in\Sigma} |a\rangle \otimes |\phi_a\rangle$$

where

$$|\phi_a\rangle = \sum_{b\in\Gamma} \alpha_{ab}|b\rangle$$

for each $a \in \Sigma$.

1. The probability to obtain each outcome $a \in \Sigma$ is

$$\Pr(\text{outcome is } a) = \sum_{b\in\Gamma} |\alpha_{ab}|^2 = \| \, |\phi_a\rangle \|^2$$

2. As a result of the standard basis measurement of X giving the outcome a, the quantum state of (X, Y) becomes

$$|a\rangle \otimes \frac{|\phi_a\rangle}{\| |\phi_a\rangle \|}$$

Eg:
Suppose than (X, Y) is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{i}{2\sqrt{2}}|10\rangle - \frac{1}{2\sqrt{2}}|11\rangle$$

and X is measured.

We begin by writing.

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle\right) +$$

$$|1\rangle \otimes \left(\frac{1}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right)$$

The probability for the measurement to result in the outcome 0 is

$$\left\| \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle \right\|^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

in which case the state of (X, Y) becomes

$$|0\rangle \otimes \frac{1/\sqrt{2}\,|0\rangle + 1/2\,|1\rangle}{\sqrt{3/4}}$$

$$= |0\rangle \otimes \left(\sqrt{\frac{2}{3}}\,|0\rangle + \frac{1}{\sqrt{3}}\,|1\rangle\right)$$

The probability for the measurement to result in the outcome is

$$\left\| \frac{i}{2\sqrt{2}}\,|0\rangle - \frac{1}{2\sqrt{2}}\,|1\rangle \right\|^2 = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

in which case the state of (X, Y) becomes

$$|1\rangle \otimes \frac{i/2\sqrt{2}\,|0\rangle - 1/2\sqrt{2}\,|1\rangle}{\sqrt{1/4}}$$

$$= |1\rangle \otimes \left(\frac{i}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle\right)$$

Suppose that (X, Y) is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{2}\,|01\rangle + \frac{i}{2\sqrt{2}}\,|10\rangle - \frac{i}{2\sqrt{2}}\,|11\rangle$$

and Y is measured.

We begin by writing

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle\right) \otimes |0\rangle +$$

$$\left(\frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right) \otimes |1\rangle$$

The probability of the measurement to result in the outcome 0 is

$$\left\| \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle \right\|^2 = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

in which case the state of (X, Y) becomes

$$\frac{1/\sqrt{2}\,|0\rangle + i/2\sqrt{2}\,|1\rangle}{\sqrt{5/8}} = \left(\sqrt{\frac{4}{5}}|0\rangle + \frac{i}{\sqrt{5}}|1\rangle\right)$$

$$\otimes |0\rangle$$

The probability of the measurement to result in the outcome 1 is

$$\left\| \frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle \right\|^2 = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

in which case the state of (X, Y) becomes

$$\frac{\frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle}{\sqrt{3/8}} = \left(\sqrt{\frac{2}{3}}|0\rangle - \frac{1}{\sqrt{3}}|1\rangle\right) \otimes |1\rangle$$

- Unitary operations - on compound systems are represented by unitary matrices whose rows and columns correspond to the Cartesian product of the classical state sets of the individual systems.

The combined action of a collection of unitary operations applied independently to a collection of systems is represented by the tensor product of the unitary matrices.

Eg: Suppose X and Y are qubits

Performing a Hadamard operation on X and doing nothing to Y is equivalent to performing this unitary operation on (X, Y).

$$H \otimes 1 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

$$1 \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

Not every unitary operation on a compound system can be expressed as a tensor pdt of unitary operations.

Eg: Suppose that X and Y are systems that share the Same Classical State set Σ. The swap operation on the pair (X, Y) exchange the contents of the systems.

$$\text{SWAP} \,|\phi \otimes \psi\rangle = |\psi \otimes \phi\rangle$$

It can be expressed as Dirac notation

$$\text{SWAP} = \sum_{a,b \in \Sigma} |a\rangle\langle b| \otimes |b\rangle\langle a|$$

$SWAP|\phi^+\rangle = |\phi^+\rangle \quad = 1/\sqrt{2}\,|00\rangle + 1/\sqrt{2}\,|11\rangle$

$SWAP|\phi^-\rangle = |\phi^-\rangle \quad = 1/\sqrt{2}\,|00\rangle - 1/\sqrt{2}\,|11\rangle$

$SWAP|\psi^+\rangle = |\psi^+\rangle \quad = 1/\sqrt{2}\,|01\rangle + 1/\sqrt{2}\,|10\rangle$

$SWAP|\psi^-\rangle = -|\psi^-\rangle \quad = 1/\sqrt{2}\,|01\rangle - 1/\sqrt{2}\,|10\rangle$

- Controlled Operations:

Suppose that X is a qubit and Y is an arbitrary system.

For every unitary operation U on Y, a controlled U operation is a unitary operation on the pair (X, Y) defined as

$$|0\rangle\langle 0| \otimes 1_Y + |1\rangle\langle 1| \otimes U = \begin{pmatrix} 1_Y & 0 \\ 0 & U \end{pmatrix}$$

Eg: A controlled - NOT operation (where the first qubit is the control):

$$|0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes \sigma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

A controlled-NOT operation (where the second qubit is the control):

$$1 \otimes |0\rangle\langle 0| + \sigma_x \otimes |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

A controlled σ_z (or controlled Z) operator,

$$|0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$