1

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a)

$$\int \omega(r)\, d^2 r = 1$$

$$k\Bigg[\underbrace{\int_0^{1/2}\left(1-6\left(\frac{r}{h}\right)^2+6\left(\frac{r}{h}\right)^3\right)dr}_{=I_1} + \underbrace{\int_{1/2}^{1} 2\left(1-\frac{r}{h}\right)^3 dr}_{=I_2} = 1$$

$$I_1 = \int_0^{1/2}\left(1-6\left(\frac{r}{h}\right)^2+6\left(\frac{r}{h}\right)^3\right)dr$$

$$\frac{r}{h} = x \Rightarrow dr = h\,dx$$

$$r=0 \longrightarrow x=0$$

$$r=1/2 \longrightarrow x = 1/2h$$

$$I_1 = h\int_0^{1/2h}\left(1-6x^2+6x^3\right)dx$$

$$= h\left[x-\frac{6x^3}{3}+\frac{6x^4}{4}\right]_0^{1/2h} = h\left[1/2h - 2\left(\frac{1}{2h}\right)^3+\frac{3}{2}\left(\frac{1}{2h}\right)^4\right]$$

$$= \frac{1}{2}-\frac{1}{4h^2}+\frac{3}{32h^3}$$

$$\text{—— (i)}$$

Now

$$I_2 = 2\int_{1/2}^{1}\left(1-\frac{r}{h}\right)^3 dr \qquad \left[\begin{array}{ll}\frac{r}{h}=x & dr = dx\cdot h\\ r=1/2 \longrightarrow & x=\frac{1}{2h}\\ r=1 \longrightarrow & x = 1/h\end{array}\right.$$

$$I_2 = 2h\int_{\frac{1}{2h}}^{1/h}\left(1-3x+3x^2-x^3\right)dx$$

$$= 2h\left[x-\frac{3}{2}x^2+x^3-\frac{x^4}{4}\right]_{1/2h}^{1/h}$$

$$= 2\hbar \left[\left(\frac{1}{\hbar} - \frac{3}{2}\frac{1}{\hbar^2} + \left(\frac{1}{\hbar}\right)^3 - \frac{1}{4}\left(\frac{1}{\hbar}\right)^4 \right) - \left[\left(\frac{1}{2\hbar}\right) - \frac{3}{8}\left(\frac{1}{\hbar}\right)^2 + \frac{1}{8}\left(\frac{1}{\hbar^3}\right) - \frac{1}{64}\left(\frac{1}{\hbar}\right)^4 \right] \right]$$

$$= 2\hbar \left[\frac{1}{2\hbar} - \frac{9}{8}\left(\frac{1}{\hbar^2}\right) + \frac{7}{8}\left(\frac{1}{\hbar^3}\right) - \frac{15}{64}\left(\frac{1}{\hbar^4}\right) \right]$$

$$= \left[1 - \frac{9}{4}\left(\frac{1}{\hbar}\right) + \frac{7}{4}\left(\frac{1}{\hbar^2}\right) - \frac{15}{16}\left(\frac{1}{\hbar^3}\right) \right] \quad \text{—— (ii)}$$

So, from (i) & (ii)

$$K \left\{ \left[1 - \frac{9}{4}\left(\frac{1}{\hbar}\right) + \frac{7}{4}\left(\frac{1}{\hbar^2}\right) - \frac{15}{16}\left(\frac{1}{\hbar^3}\right) \right] + \left[\frac{1}{2} - \frac{1}{4}\left(\frac{1}{\hbar^2}\right) + \frac{3}{32}\left(\frac{1}{\hbar^3}\right) \right] \right\} = 1$$

$$K \left[\frac{3}{2} - \frac{9}{4}\left(\frac{1}{\hbar}\right) + \frac{3}{2}\left(\frac{1}{\hbar^2}\right) - \frac{27}{32}\left(\frac{1}{\hbar^3}\right) \right] = 1$$

$$K \left[\frac{48\hbar^3 - 72\hbar^2 + 48\hbar - 27}{32\hbar^3} \right] = 1$$

$$\boxed{K = 32 \left[\frac{\hbar^3}{48\hbar^3 - 72\hbar^2 + 48\hbar - 27} \right]}$$