Problem sheet 8

1) Polytropic atmosphere

Navier - Stokes equ.: $\partial_t \vec{v} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p - \nabla\phi + \mu\Delta\vec{v}$

Assume $\vec{v}=0$ and $\partial_t\vec{v}=0$ (stationarity):

$$0 = -\frac{1}{\rho}\nabla P - \nabla\phi$$

For a polytropic equ. of state $P\sim\rho^\alpha$ we get

$$0 = -\frac{\nabla(\rho^\alpha)}{\rho} - \nabla\phi \qquad (*)$$

Now we show that $\frac{1}{\rho}\nabla(\rho^\alpha) = \frac{\alpha}{\alpha-1}\nabla(\rho^{\alpha-1})$.

Proof:
$$\left.\begin{aligned} \frac{1}{\rho}\nabla(\rho^\alpha) &= \frac{1}{\rho}\alpha\rho^{\alpha-1}\nabla\rho = \alpha\rho^{\alpha-2}\nabla\rho \\ \nabla(\rho^{\alpha-1}) &= (\alpha-1)\rho^{\alpha-2}\nabla\rho \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \frac{\nabla(\rho^\alpha)}{\rho}\frac{1}{\nabla(\rho^{\alpha-1})} = \frac{\alpha}{\alpha-1} \Longrightarrow \frac{\nabla(\rho^\alpha)}{\rho} = \frac{\alpha}{\alpha-1}\nabla(\rho^{\alpha-1}) \quad (\square)$$

Therefore from $(*)$ and $(\square)$ we get

$$\nabla\phi = -\frac{\alpha}{\alpha-1}\nabla(\rho^{\alpha-1})$$

$$\Rightarrow \phi \sim -\rho^{\alpha-1} \Longrightarrow \rho \sim -\phi^{\frac{1}{\alpha-1}}$$

2) Matrix derivatives

a) $\frac{\partial}{\partial x}(\mathrm{tr}\, A) = \frac{\partial}{\partial x}\sum_{i=1}^{N} a_{ii}(x) = \sum_{i=1}^{N}\frac{\partial}{\partial x} a_{ii}(x) = \mathrm{tr}\,(\partial_x A)$

b) $0 = \partial_x(A^{-1}A) = (\partial_x A^{-1})A + A^{-1}(\partial_x A)$

$\rightarrow \quad \partial_x A^{-1} = -A^{-1}(\partial_x A)A^{-1}$

c) $\partial_x A^n = \partial_x(\underbrace{A \cdots A}_{n\text{-times}}) = (\partial_x A)A^{n-1} + \cdots + A^{n-1}\partial_x A$

$\underset{\downarrow}{=} n A^{n-1}\partial_x A$

assuming A is symmetric, so that $[A, \partial_x A] = 0$

d) $\partial_x \exp(A) = \partial_x\left(\sum_{k=0}^{\infty}\frac{A^k}{k!}\right) = \sum_{k=0}^{\infty}\left(\partial_x \frac{A^k}{k!}\right) = \sum_{k=1}^{\infty}\left(\partial_x\frac{A^k}{k!}\right)$

$= \sum_{k=1}^{\infty}\left(\frac{A^{k-1}}{(k-1)!}\partial_x A\right) \underset{\downarrow}{=} \left(\sum_{l=0}^{\infty}\frac{A^l}{l!}\right)\partial_x A = \exp(A)\,\partial_x A$

$l = k-1$

e) $\partial_x A = \partial_x\left(\exp(\ln(A))\right) \overset{d)}{=} \exp(\ln(A))\,\partial_x \ln(A) = A\,\partial_x \ln(A)$

$\rightarrow \quad \partial_x \ln A = A^{-1}\partial_x A$

f) $\mathrm{tr}\,(A^{-1}\partial_x A) \overset{e)}{=} \mathrm{tr}\,(\partial_x \ln A) \overset{a)}{=} \partial_x(\mathrm{tr}(\ln A))$

$\underset{\downarrow}{=} \partial_x(\ln \det A)$

from lecture, known that $\mathrm{tr}(\ln A) = \ln(\det A)$

g) $\partial_x(y_i A^{-1}_{ij} y_j) = y_i(\partial_x A^{-1})_{ij} y_j \overset{b)}{=} y_i(-A^{-1}(\partial_x A)A^{-1})_{ij} y_j$

$= (-A^{-1}(\partial_x A)A^{-1})_{ij}\underbrace{y_j y_i}_{Y_{ji}} = -\mathrm{tr}\,(A^{-1}(\partial_x A)A^{-1}Y)$