Problems on *statistics and data analysis (MVComp2)*

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Problem sheet 1

To be handed in during the exercise group on 02.May.2016

1. *Transformation of variables* *(10 points)*

(a) Given the expectation value $E[x]$ of a random variable x, prove that

$$E[ax] = a\,E[x]$$
$$E[x + a] = E[x] + a$$

i.e., that the expectation value is a linear operation. Is the same property valid for every estimator of the expectation value?

(b) The absolute and apparent magnitudes of a galaxy, M and m, are related to its distance r via the relation $M = m - 25 - 5\log_{10} r$. Assuming the apparent magnitude m to be known and the PDF of M to be a Gaussian, what is the PDF of $r(M)$? Is it true that $E[r(M)] = r(E[M])$?

(c) Consider a standard Gaussian variable Y and the transformation $Z = Y^2 - 2Y$: find the PDF of the random variable Z. Pay attention to the transformation, which is not bijective: first restrict yourselves to the regions where it is bijective and then sum up the results that you find there.

2. *Moments* *(10 points)*

(a) Show the "memorylessness" of the exponential distribution $p(x) = \lambda \mathrm{e}^{-\lambda x}$, $(\lambda > 0)$, i.e. show that

$$P\{X > m + k | X > k\} = P\{X > m\}, \quad \forall m, k > 0.$$

(b) Calculate the moment-generating function of the following distributions:

- Poisson, $\mathrm{Pois}(\lambda, n) = \mathrm{e}^{-\lambda}\frac{\lambda^n}{n!}$ $(\lambda, n > 0)$
- Gamma, $\Gamma(\alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\mathrm{e}^{-\lambda x}$ $(\alpha, \lambda > 0)$
- Exponential, $\mathrm{Exp}(\lambda) = \lambda \mathrm{e}^{-\lambda x}$ $(\lambda > 0)$
  How is the result for $\mathrm{Exp}(\lambda)$ related to the one for $\Gamma(\alpha, \lambda)$?

(c) Using the moment-generating function and always considering the variables X and Y as independent, show that

- if X follows $Pois(\lambda_1)$ and Y follows $Pois(\lambda_2)$, then $X + Y$ follows $Pois(\lambda_1 + \lambda_2)$
- if X follows $\Gamma(\alpha_1, \lambda)$ and Y follows $\Gamma(\alpha_2, \lambda)$, then $X + Y$ follows $\Gamma(\alpha_1 + \alpha_2, \lambda)$.

3. *Convolution of distributions* *(10 points)*

Determine the distribution of the product of two random numbers generated independently from a uniform distribution in $[0, 1]$. (In other words, if X, Y follow a uniform distribution $U(0, 1)$ and are independent, determine the distribution of XY). Then:

(a) Considering the random variables $Z = -\log X$ and $W = -\log Y$, find their distributions.

(b) Determine the distribution of $Z + W$.

(c) Find the distribution of $XY = \mathrm{e}^{-(Z+W)}$.

(d) If $X_1, \cdots X_n \sim U(0, 1)$ are independent, what is the distribution of $X_1 X_2 \cdots X_n$?

4. *Python exercise* *(10 points)*

Write a Python script that executes the following two tasks:

- Extract two independent random numbers x, y from a Gaussian distribution with the same variance σ^2 and combine them to a sum $s = x + y$. By plotting the outcome, verify that the variable s follows a probability distribution given by a Gaussian with doubled mean and variance.
- Extract two independent random numbers x, y from a Gaussian distribution with the same variance σ^2 and combine them to a product $q = xy$. By plotting the outcome, verify that the variable q follows a probability distribution given by a Bessel function of the second kind.