(A) normalisation of a Gaussian probability density

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\left(\int p(x)dx\right)^2 = \int p(x)dx \cdot \int p(y)dy = \int dx\,dy\; p(x)\cdot p(y)$$

$$= \frac{1}{(2\pi)} \int dx\,dy\; \exp\left(-\frac{x^2+y^2}{2}\right)$$

$r^2 = x^2+y^2$, $\quad x = r\cos\varphi$, $y = r\sin\varphi$

$$dx\,dy = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{pmatrix} \cdot dr\,d\varphi = \det\begin{pmatrix} \cos\varphi & \sin\varphi \\ -r\cdot\sin\varphi & r\cdot\cos\varphi \end{pmatrix} dr\,d\varphi$$

$$= r \cdot \det\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} dr\,d\varphi = r\,dr\,d\varphi.$$

$$\rightarrow \quad = \frac{1}{2\pi} \int r\,dr\,d\varphi\; \exp\left(-\frac{r^2}{2}\right) = \int r\,dr\; \exp\left(-\frac{r^2}{2}\right)$$

substitution $t = \frac{r^2}{2}$ $\quad \frac{dt}{dr} = r \rightarrow r\,dr = dt$

$$= \int_0^\infty dt\; \exp(-t) = -\exp(-t)\Big|_0^\infty = \exp(-t)\Big|_\infty^0 = 1.$$

(B) normalisation if the variance is not equal to unity

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\int p(x)\,dx = 1 = \int dx \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = \int \frac{dy}{\sigma} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

$\rightarrow p(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right)$ by substitution $x = \frac{y}{\sigma}$

$\rightarrow$ normalisation $\int dy\, p(y) \equiv 1.$

(c) mean and variance of a Gaussian probability density

$\int dx\, p(x) \cdot x = 0$ because $p(x)$ is even and x is odd,

$\int dx\, p(x) \cdot x^2 = \sigma^2$, because:

$$\int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) x^2 = \sqrt{\frac{\lambda}{\pi}} \int dx \exp(-\lambda x^2) \cdot x^2$$

by substitution $\lambda = \frac{1}{2\sigma^2}$:

$$= -\sqrt{\frac{\lambda}{\pi}} \int dx \frac{d}{d\lambda} \exp(-\lambda x^2) = -\sqrt{\frac{\lambda}{\pi}} \frac{d}{d\lambda} \int dx \exp(-\lambda x^2) = -\sqrt{\frac{\lambda}{\pi}} \frac{d}{d\lambda} \sqrt{\frac{\pi}{\lambda}}$$

$$= -\sqrt{\lambda} \frac{d}{d\lambda} \frac{1}{\sqrt{\lambda}} = \sqrt{\lambda} \cdot \frac{\sqrt{\lambda}^{-3}}{2} = \frac{1}{2\lambda} = \sigma^2 = \langle x^2 \rangle.$$

higher order moments: try $\frac{d^n}{d\lambda^n}$ + correct.

(d) higher order moments of a Gaussian

work your way up by partial integration

$$\int dx\, p(x) = 1 = \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \underbrace{\exp\left(-\frac{x^2}{2\sigma^2}\right)}_{u} \cdot \underbrace{1}_{v'}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \cdot x \Big|_{-\infty}^{+\infty}$$

$$- \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{x^2}{2\sigma^2}\right) \cdot \left(-\frac{x}{\sigma^2}\right) \cdot x$$

$$= +\frac{1}{\sigma^2} \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \cdot x^2$$

$\longrightarrow \langle x^2 \rangle = \sigma^2$

let's try to go higher:

$$\int dx\, p(x)\, x^2 = \sigma^2 = \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \underbrace{\exp\left(-\frac{x^2}{2\sigma^2}\right)}_{u} \cdot \underbrace{x^2}_{v'}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \frac{x^3}{3} \Big|_{-\infty}^{+\infty} - \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \left(-\frac{x}{\sigma^2}\right) \frac{x^3}{3}$$

① higher order moments of a Gaussian, cont'ed

$$\langle x^2\rangle = \sigma^2 = \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\frac{x^4}{3\sigma^2}$$

$$\rightarrow \quad \langle x^4\rangle = 3\sigma^4 = 3(\sigma^2)^2$$

in general:

$\langle x^{2n}\rangle = \alpha(\sigma^2)^n$ with some number α. then,

$$\langle x^{2n}\rangle = \cancel{\alpha(\sigma^2)^n} = \int dx \frac{1}{\sqrt{2\pi\sigma^2}}\cdot \overset{u}{\exp\left(-\frac{x^2}{2\sigma^2}\right)}\cdot \overset{v'}{x^{2n}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\cdot \frac{x^{2n+1}}{2n+1}\Bigg|_{-\infty}^{+\infty} - \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left(-\frac{x}{\sigma^2}\right)\frac{x^{2n+1}}{2n+1}$$

$$= +\int \frac{dx}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\cdot \frac{x^{2n+2}}{\sigma^2\cdot(2n+1)}$$

$$\rightarrow \quad \alpha\cdot(\sigma^2)^n\cdot\sigma^2\cdot(2n+1) = \alpha\cdot(\sigma^2)^{n+1}\cdot(2n+1) = \langle x^{2n+2}\rangle$$

there's an additional factor $2n+1$ from each moment:

$$\langle x^{2n+2}\rangle = (2n+1)\sigma^2\cdot\langle x^{2n}\rangle$$

$$= \underbrace{(2n+1)(2n-1)\cdot\ldots\cdot 3\cdot 1}_{\text{odd-numbered factorials.}}\cdot(\sigma^2)^{n+1}$$

what about the odd moments?

$$\int dx\, p(x)\cdot x = \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \overset{u}{\exp\left(-\frac{x^2}{2\sigma^2}\right)}\cdot \overset{v'}{x}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\cdot\frac{x^2}{2}\Bigg|_{-\infty}^{+\infty} - \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\frac{-x}{\sigma^2}\frac{x^2}{2}$$

$$= \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-x^2)\cdot\frac{x^3}{2\sigma^2}$$

$$\rightarrow \quad \langle x^3\rangle = 2\sigma^2\,\langle x\rangle, \qquad \langle x^3\rangle = 0 \text{ if } \langle x\rangle = 0.$$

$$\int dx\, p(x)\, x^{2n-1} = \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \overset{u}{\exp\left(-\frac{x^2}{2\sigma^2}\right)}\, \overset{v'}{x^{2n-1}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\cdot\frac{x^{2n}}{2n}\Bigg|_{-\infty}^{+\infty} - \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left(\frac{-x}{\sigma^2}\right)\frac{x^{2n}}{2n}$$

$$= \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\cdot\frac{x^{2n+1}}{2n\sigma^2}$$

Ⓓ higher order moments of a Gaussian, cont'ed

$$\longrightarrow \langle x^{2n-1}\rangle \cdot 2n\cdot\sigma^2 = \langle x^{2n+1}\rangle \qquad = 0, \text{ if } \langle x\rangle = 0$$

$$\langle x^{2n+1}\rangle = 2n\cdot(2n-2)\cdots 4\cdot 2\cdot \langle x\rangle \cdot (\sigma^2)^n$$

but what about $\langle x\rangle$ now?

$$\int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(\frac{-x^2}{2\sigma^2}\right) x$$

split up: $$\int_{-\infty}^{0} dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) x + \int_0^{\infty} dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) x$$

$x \to -x$

$$= \int_{\infty}^{0} dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) x + \int_0^{\infty} dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) x$$

$$= -\int_0^{\infty} dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) x + \int_0^{\infty} dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) x$$

$$= 0 \quad \checkmark$$

or, alternatively:

$$\int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \cdot x = \int dx \frac{1}{\sqrt{2\pi\sigma^2}} (-2\sigma^2) \frac{d}{dx} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

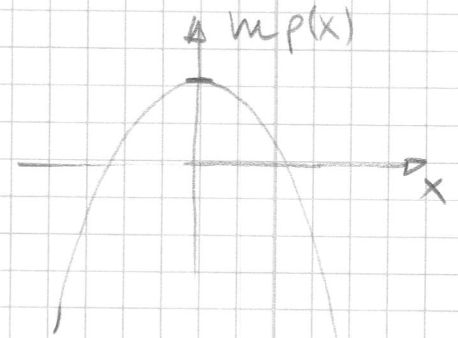
$$= -\frac{2\sigma^2}{\sqrt{2\pi\sigma^2}} \frac{d}{dx} \int dx \exp\left(\frac{-x^2}{2\sigma^2}\right) = -2\sigma^2 \frac{d}{dx} 1 = 0. \quad :)$$

Ⓔ curvature of the Gaussian distribution

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \longrightarrow \ln p(x) = \frac{1}{2}\ln(2\pi\sigma^2) - \frac{x^2}{2\sigma^2}$$

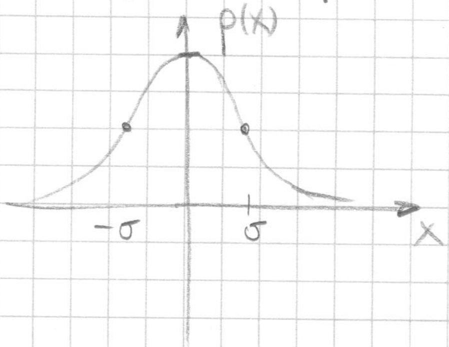
what about $\frac{d^2}{dx^2} \ln p(x)$?

$$\frac{d^2}{dx^2} \ln p(x) = -\frac{1}{\sigma^2} = \text{const.}$$



parabola! curvature $\sim \frac{1}{\sigma^2}$

(F) inflection points of the Gaussian distribution



$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

$$\frac{d^2}{dx^2} p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{d}{dx}\left(\exp\left(\frac{-x^2}{2\sigma^2}\right) \cdot \frac{-x}{\sigma^2}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \cdot \underbrace{\left(\frac{x^2}{\sigma^4} - \frac{1}{\sigma^2}\right)}_{=0} = 0 \rightarrow x = \pm\sigma.$$

(G) characteristic function of the Gaussian distribution

$$\varphi(t) = \int dx\, p(x) \exp(-itx) = \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\left(\frac{x^2}{2\sigma^2} + itx\right)\right)$$

complete the square in the exponent

$$\frac{x^2}{2\sigma^2} + itx = \left(\frac{x}{\sqrt{2}\sigma} + \frac{i\sigma t}{\sqrt{2}}\right)^2 + \frac{\sigma^2 t^2}{2}$$

$$\rightarrow \varphi(t) = \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\left(\frac{x}{\sqrt{2}\sigma} + \frac{i\sigma t}{\sqrt{2}}\right)^2\right) \cdot \underbrace{\exp\left(-\frac{\sigma^2 t^2}{2}\right)}_{\text{does not depend on } x}$$

$$= \exp\left(-\frac{\sigma^2 t^2}{2}\right) \cdot \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\left(\frac{x}{\sqrt{2}\cdot\sigma} + \frac{i\sigma t}{\sqrt{2}}\right)^2\right)$$

substitute $y = \sqrt{2}\left(\frac{x}{\sigma} + i\sigma t\right)$, $dy = \frac{\sqrt{2}}{\sigma} dx$

$$= \exp\left(-\frac{\sigma^2 t^2}{2}\right) \cdot \underbrace{\int \frac{dy}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)}_{\equiv 1.} = \exp\left(-\frac{\sigma^2 t^2}{2}\right)$$

$$\rightarrow \varphi(t) = \langle \exp(-itx) \rangle = \exp\left(-\frac{\sigma^2 t^2}{2}\right)$$

(H) moment-generating function of the Gaussian distribution

$$m(t) = \int dx\, p(x) \cdot \exp(-tx) = \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2} - tx\right)$$

complete the square

minus sign here!

$$\frac{x^2}{2\sigma^2} + tx = \left(\frac{x}{\sqrt{2}\sigma} + \frac{\sigma t}{\sqrt{2}}\right)^2 - \frac{\sigma^2 t^2}{2}$$

$$\rightarrow m(t) = \int dx \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(+\frac{\sigma^2 t^2}{2}\right) \cdot \exp\left(-\left(\frac{x}{\sqrt{2}\sigma} + \frac{\sigma t}{\sqrt{2}}\right)^2\right)$$

substitute $y = \sqrt{2}\left(\frac{x}{\sigma} + \sigma t\right)$, $dy = \frac{\sqrt{2}}{\sigma} \cdot dx$

$$= \exp\left(+\frac{\sigma^2 t^2}{2}\right) \cdot \int \frac{dy}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) = \exp\left(+\frac{\sigma^2 t^2}{2}\right)$$

try out to get the variance and other moments from $m(t)$:

$m(t=0) = 1 = \langle x^0 \rangle$ normalisation ✓

$$\left.\frac{d}{dt} m\right|_{t=0} = \left.\sigma^2 t \exp\left(+\frac{\sigma^2 t^2}{2}\right)\right|_{t=0} = 0 = \langle x \rangle \quad ✓$$

$$\left.\frac{d^2}{dt^2} m\right|_{t=0} = \left.\sigma^2 \exp\left(+\frac{\sigma^2 t^2}{2}\right) + (\sigma^2 t)^2 \exp\left(+\frac{\sigma^2 t^2}{2}\right)\right|_{t=0} = \sigma^2 ✓$$

(I) cumulants

introduce mean μ to the Gaussian distribution

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$\rightarrow$ characteristic function

$$\varphi(t) = \int dx\, p(x) \exp(itx) = \ldots = \exp\left(-\frac{\sigma^2 t^2}{2} + i\mu t\right)$$

$\ln \varphi(t)$ is a second-order polynomial:

$$\varkappa_1 = \left.\frac{1}{i} \frac{d}{dt} \varphi(t)\right|_{t=0} = \mu$$

$$\varkappa_2 = \left.\frac{1}{i^2} \frac{d^2}{dt^2} \varphi(t)\right|_{t=0} = \sigma^2 \qquad \varkappa_3 = \ldots = 0!$$

(J) multivariate Gaussian - properties

$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n \det C}} \exp\left(-\frac{1}{2} x^t C^{-1} \vec{x}\right)$$

$C_{ij} = \langle x_i x_j \rangle$ or $C = \langle \vec{x}^t \vec{x} \rangle$ covariance matrix

① positive definite

② symmetric

look at quadratic form:

$$\vec{x}^t C^{-1} \vec{x} = \sum_{ij} x_i (C^{-1})_{ij} x_j = \sum_{ij} \underbrace{x_i x_j}_{= D_{ij}} (C^{-1})_{ij} = \sum_{ij} D_{ij} (C^{-1})_{ij} = tr(D C^{-1})$$

reformulate $\det C$

$\ln \det C = tr \ln C$, because in the eigenframe we can write

$$\ln \det C = \ln \prod_i \lambda_i = \sum_i \ln \lambda_i = tr \ln C$$

$$\rightarrow \ln p(\vec{x}) = \frac{n}{2} \ln(2\pi) + \frac{1}{2} tr \ln C - \frac{1}{2} tr(C^{-1} D)$$
$$= \frac{n}{2} \ln(2\pi) + \frac{1}{2} tr \left[\ln C - C^{-1} D \right]$$

(K) invariance and principal axis frame

imagine a transform $\vec{x} \rightarrow \vec{y} = R\vec{x}$ which is linear

$$p(\vec{y}) = \frac{1}{\sqrt{(2\pi)^n \det C_y}} \exp\left(\vec{y}^t C_y^{-1} \vec{y}\right)$$

$$C_y = \langle \vec{y} \vec{y}^t \rangle = \langle R\vec{x} R^t \vec{x}^t \rangle = R \langle \vec{x} \vec{x}^t \rangle R^t = R C_x R^t$$

$$\det C_y = \det(R C_x R^t) = \det(R) \det(C_x) \det(R^t)$$
$$= \det(C_x) \cdot \det(R R^t) = \det(C_x) \text{ if } R \cdot R^t = id$$

$\rightarrow$ R must be orthogonal $R^{-1} \hat{=} R^t$ for det to be constant

(remember $\det(C)$ is the "volume" inside the 1σ contour)

Ⓚ invariance and principal axis frame

$$p(y) = \frac{1}{\sqrt{(2\pi)^n \det C_y}} \exp\left(-\frac{1}{2}\vec{y}^t C_y^{-1} \vec{y}\right)$$

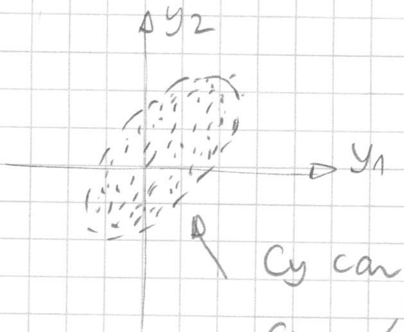
$$= \frac{1}{\sqrt{(2\pi)^n \det C_x}} \exp\left(-\frac{1}{2}\vec{x}^t \underbrace{R^t C_y^{-1} R}_{= C_x} \vec{x}\right) = p(x)$$

if R is orthogonal:

$$R^t C_y^{-1} R = R^t (R C_x R^t)^{-1} R = \underbrace{R^t R^{t-1}}\cdot C_x \cdot \underbrace{R^{-1} \cdot R} = C_x$$

Ⓛ principal axis system and principal components.

$C_y = \langle \vec{y}\vec{y}^t \rangle$ is the covariance matrix of the random variable $\vec{y}$



C_y can be estimated from data

$C_y = \langle \vec{y}\vec{y}^t \rangle \longrightarrow \tilde{C}_y = \frac{1}{N}\sum_i \vec{y}_i \cdot \vec{y}_i^t$ from samples $\vec{y}_i$

C_y is symmetric $\rightarrow$ there's an orthogonal transformation which diagonalises C_y:

$$R^t C_y R = \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{pmatrix}$$

with eigenvalues σ_i^2, which are necessarily > 0 due to positive definiteness

$\vec{y} = R \cdot \vec{x}$ transforms the random vector. We already showed that $p(\vec{y}) = p(\vec{x})$, meaning that $p(\vec{x})$ factorises completely:

$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n \det C_x}} \exp\left(-\frac{1}{2}\vec{x}^t C_x \vec{x}\right) = \prod_i \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{x_i^2}{2\sigma_i^2}\right)$$

the largest σ_i^2 is called 1st principal component, ...

(M) Cholesky-decompositions: 2 applications

(1) generate multivariate samples from uncorrelated random numbers by linear combination.

$$C_y = \langle \vec{y}\vec{y}^t \rangle = R \langle \vec{x} \cdot \vec{x}^t \rangle R^t = R C_x R^t$$

if $\vec{y} = R\vec{x}$. now chose R such that $C_x = \mathrm{id}$:

$$C_y = R \cdot R^t \sim R \text{ is the cholesky decomposition of } C_y$$

now: sample $\vec{x}$-values with covariance $C_x = \mathrm{id}$ (unit variance and mutually independent)

+ transform to $\vec{y} = R\vec{x}$ with the desired covariance C_y

(2) suppose there's a relation $C_y = R C_x R^t$ between any two covariances C_y and C_x $\rightarrow$ can one solve for R?

introduce Cholesky-transforms $C_y = Y \cdot Y^t$ and $C_x = X \cdot X^t$

$\rightarrow \quad YY^t = RXX^t \cdot R^t = RX \cdot (RX)^t$

$\rightarrow \quad Y = RX$ and therefore $R = Y \cdot X^{-1}$ transforms C_x to C_y.

conditionalisation

$$p_c(x) = p(x, y=0) - \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp\left(-\frac{1}{2} \underbrace{\begin{pmatrix}x\\0\end{pmatrix}^t C^{-1} \begin{pmatrix}x\\0\end{pmatrix}}_{=\chi^2}\right)$$

$$C = \begin{pmatrix} \langle x^2\rangle & \langle xy\rangle \\ \langle xy\rangle & \langle y^2\rangle \end{pmatrix} \longrightarrow C^{-1} = \frac{1}{\det C} \begin{pmatrix} \langle y^2\rangle & -\langle xy\rangle \\ -\langle xy\rangle & \langle x^2\rangle \end{pmatrix}$$

$$\det C = \langle x^2\rangle \langle y^2\rangle - \langle xy\rangle^2$$

$$\chi^2 = \begin{pmatrix}x\\0\end{pmatrix}^t C^{-1} \begin{pmatrix}x\\0\end{pmatrix} = \frac{1}{\det C} \langle y^2\rangle x^2 = \frac{\langle y^2\rangle}{\langle x^2\rangle\langle y^2\rangle - \langle xy\rangle^2} \cdot x^2$$

$$= \frac{1}{1-r^2} \frac{x^2}{\langle x^2\rangle}$$

$$p_c(x) = \frac{1}{\sqrt{(2\pi)^2 \det C}} \cdot \exp\left(-\frac{1}{2} \frac{1}{1-r^2} \frac{x^2}{\langle x^2\rangle}\right)$$

new variance: $\langle x^2\rangle \longrightarrow (1-r^2)\cdot\langle x^2\rangle \leq \langle x^2\rangle$

(equality for $r=0$, uncorrelated case).

normalisation is not 1 $\longrightarrow$ cuts through a Gaussian

(this would be different if $\int dy\, p(x,y)\cdot \delta_D(y) = p_c(x)$.)

marginalisation

$$p_m(x) = \int dy\, p(x,y) = \frac{1}{\sqrt{(2\pi) \det C}} \int dy \exp\left(-\frac{1}{2} \underbrace{\begin{pmatrix}x\\y\end{pmatrix}^t C^{-1} \begin{pmatrix}x\\y\end{pmatrix}}_{=\chi^2}\right)$$

$$\chi^2 = \begin{pmatrix}x\\y\end{pmatrix}^t C^{-1} \begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{\det C} \begin{pmatrix}x\\y\end{pmatrix}^t \begin{pmatrix} \langle y^2\rangle & -\langle xy\rangle \\ -\langle xy\rangle & \langle x^2\rangle \end{pmatrix} \begin{pmatrix}x\\y\end{pmatrix}$$

$$= \frac{1}{\det C} \left(\langle y^2\rangle \cdot x^2 - 2xy\langle xy\rangle + \langle x^2\rangle \cdot y^2\right)$$

complete the square

$$= \left(y\sqrt{\langle x^2\rangle} - x \cdot \frac{\langle xy\rangle}{\sqrt{\langle x^2\rangle}}\right)^2 - x^2 \frac{\langle xy\rangle^2}{\langle x^2\rangle} + x^2\langle y^2\rangle$$

$$= \left(y\sqrt{\langle x^2\rangle} - x \frac{\langle xy\rangle}{\sqrt{\langle x^2\rangle}}\right)^2 + x^2\left(\langle y^2\rangle - \frac{\langle xy\rangle^2}{\langle x^2\rangle}\right)$$

$$p_m(x) = \int dy \frac{1}{\sqrt{(2\pi)^2 \det C}} \exp\left(-\frac{1}{2} \frac{1}{\det C} \left(y\sqrt{\langle x^2\rangle} - x \frac{\langle xy\rangle}{\sqrt{\langle x^2\rangle}}\right)^2\right.$$

$$\cdot \exp(\ldots)$$

the second integral does not depend on y.

for the first term: $t \equiv \frac{\sqrt{\langle x^2\rangle}}{\det C} \cdot y$

$$dt = \frac{\sqrt{\langle x^2\rangle}}{\det C} \cdot dy \rightarrow \frac{dy}{\sqrt{\det C}} = \sqrt{\frac{\det C}{\langle x^2\rangle}}\, dt$$

- $\int \frac{dy}{\sqrt{\det C}} \exp\left(-\frac{1}{2}\frac{1}{\det C}\left(y\cdot\sqrt{\langle x^2\rangle} - x\frac{\langle xy\rangle}{\langle x^2\rangle}\right)\right) \cdot \cancel{\exp(\ldots)}$

  shifts the mean $\quad = \sqrt{\frac{2\pi}{\langle x^2\rangle}}$

- $$pm(x) = \frac{\sqrt{\det C}}{\sqrt{2\pi\langle x^2\rangle}} \exp\left(-\frac{1}{2}\frac{1}{\det C}\underbrace{\left(\langle y^2\rangle - \frac{\langle xy\rangle}{\langle x^2\rangle}\cdot x^2\right)}\right)$$

$$= \left(\langle y^2\rangle\langle x^2\rangle - \langle xy\rangle^2\right)\cdot\frac{x^2}{\langle x^2\rangle}$$

$$= +\langle y^2\rangle\cdot(1-r^2)\cdot\frac{x^2}{\langle x^2\rangle}$$

$$= \frac{\langle y^2\rangle}{\langle x^2\rangle}\cdot(1-r^2)\cdot x^2$$

$$pm(x) = \sqrt{\frac{\det C}{2\pi\langle x^2\rangle}} \cdot \exp\left(-\frac{1}{2}\frac{1}{\det C}\left(\frac{\langle y^2\rangle}{\langle x^2\rangle}\cdot(1-r^2)\right)x^2\right)$$