Problem sheet 6

1) Shannon entropy and Gaussian distribution

a) $$L = -\int_a^b p(x) \ln[p(x)]\, dx + \lambda \left(\int_a^b p(x)\, dx - 1 \right)$$

$$\delta L = -\int_a^b \delta p(x) \ln[p(x)] - \int_a^b p(x) \frac{1}{p(x)} \delta p(x)\, dx + \lambda \int_a^b \delta p(x)\, dx \overset{!}{=} 0$$

$$\Rightarrow \quad -\ln[p(x)] - 1 + \lambda \overset{!}{=} 0 \quad \text{(being } \delta p(x) \text{ arbitrary)}$$

$$\Rightarrow \quad \boxed{p(x) = e^{\lambda - 1}}$$

$$\int_a^b e^{\lambda-1}\, dx \overset{!}{=} 1 \longrightarrow \boxed{\lambda = 1 - \ln(b-a)}$$

$$\Rightarrow \boxed{p(x) = e^{1 - \ln(b-a) - 1} = \frac{1}{b-a}}$$ uniform distribution

b) $$L = -\int_{-\infty}^{\infty} p(x) \ln[p(x)]\, dx + \lambda_1 \left(1 - \int_{-\infty}^{\infty} p(x)\, dx \right) + \lambda_2 \left(\sigma^2 - \int_{-\infty}^{\infty} x^2 p(x)\, dx \right)$$

$$\delta L = -\int_{-\infty}^{\infty} \delta p(x) \ln[p(x)] dx - \int_{-\infty}^{\infty} p(x) \frac{1}{p(x)} \delta p(x)\, dx - \lambda_1 \int_{-\infty}^{\infty} \delta p(x)\, dx - \lambda_2 \int_{-\infty}^{\infty} x^2 \delta p(x)\, dx \overset{!}{=} 0$$

$$\Rightarrow \quad -\ln[p(x)] - 1 - \lambda_1 - \lambda_2 x^2 = 0 \quad (\delta p(x) \text{ arbitrary})$$

$$\Rightarrow \quad \ln[p(x)] = -\lambda_2 x^2 - \lambda_1 - 1 \longrightarrow \boxed{p(x) = e^{-\lambda_2 x^2 - \lambda_1 - 1}}$$

$$\int_{-\infty}^{\infty} p(x)\, dx \overset{!}{=} 1$$

$$e^{-(\lambda_1+1)} \sqrt{\frac{\pi}{\lambda_2}} = 1$$

$$\lambda_2 = \pi \, e^{-2(\lambda_1+1)}$$

$$\int_{-\infty}^{\infty} x^2 p(x)\, dx \overset{!}{=} \sigma^2$$

$$\frac{e^{-(\lambda_1+1)} \sqrt{\pi}}{2 \lambda_2^{3/2}} = \sigma^2$$

$$\frac{e^{-(\lambda_1+1)} \sqrt{\pi}}{2 \pi^{3/2} e^{-3(\lambda_1+1)}} = \sigma^2$$

$$\frac{1}{2\pi} e^{2(\lambda_1+1)} = \sigma^2 \rightarrow \boxed{\lambda_1 = \frac{1}{2} \ln 2\pi\sigma^2 - 1}$$

$$\lambda_2 = \pi \, e^{-2\left(\frac{1}{2}\ln 2\pi\sigma^2 - 1 + 1\right)}$$

$$= \pi \, e^{-\ln 2\pi\sigma^2} = \boxed{\frac{1}{2\sigma^2}}$$

$$\Rightarrow \boxed{p(x) = \frac{1}{\sqrt{2\pi\sigma}} \, e^{-\frac{x^2}{2\sigma^2}}}$$

Gaussian distribution

2) Skewness of the exponential distribution

a) For exponential distribution $\exp(-x)$, $x = 0 \sim \infty$
it is true that $\langle x^n \rangle = n!$

$$\rightarrow \quad S \equiv \langle (x - \langle x \rangle)^3 \rangle = \langle x^3 - 3x^2 \langle x \rangle + 3x \langle x \rangle^2 - \langle x \rangle^3 \rangle =$$
$$= \langle x^3 \rangle - 3 \langle x^2 \rangle \langle x \rangle + 3 \langle x \rangle^3 - \langle x \rangle^3$$
$$= 3! - 3 \cdot 2! \cdot 1! + 3 \cdot 1! - 1! = 2.$$

c) You should get that the probability of having an estimate of s in $[3-\varepsilon, 3+\varepsilon]$ is $\sim 1\%$, as an order of magnitude.
You should also get that $\sim 10-15\%$ of the samples have value larger than 3.

d) These outliers become more likely if there are more samples.

e) S_{max} should be around 2,5

f) In this case you should get $S_{max} \sim 3{,}5$

g) Since $3 < 3.5$, according to what you found in f) you are still in the acceptance region, therefore you should not reject the hypothesis.

3) Drug testing

$P(+|D) = 0.99$, $P(-|N) = 0.98$, $P(D) = 0.005$

1) $P(-|D) = 1 - P(+|D) = 0.01$ $\quad P(+|N) = 1 - P(-|N) = 0.02$

2) $P(+) = P(+|D) \cdot P(D) + P(+|N) \cdot P(N) = 0.025$
with $P(N) = 1 - P(D)$

3) $P(D|+)\, P(+) = P(+|D)\, P(D)$ Bayes' law
$$\rightarrow P(D|+) = \frac{P(+|D)\ P(D)}{P(+)} \simeq 0.2$$

4) $P(D|+) \simeq 0.2$, small value. This is because $P(D)$ is a very small value.

4| <u>Monkeys</u>

a) M^N

b) $\binom{N}{n_1}$

c) $\binom{N}{n_1} \cdot \binom{N-n_1}{n_2} \cdot \binom{N-n_1-n_2}{n_3} \cdots \binom{n_M}{n_M} =$

$= \frac{N!}{n_1!\,\cancel{(N-n_1)!}} \cdot \frac{\cancel{(N-n_1)!}}{n_2!\,(N-n_1-n_2!)} \cdots$

$= \frac{N!}{n_1!\, n_2! \cdots n_M!}$

d) $p(\{n_i\}) = \frac{N!}{n_1!\, n_2! \cdots n_M!} \quad \frac{1}{M^N}$

e) $\ln p(\{n_i\}) = \ln \left(\frac{N!}{n_1! n_2! \cdots n_M!} \quad \frac{1}{M^N} \right)$

$= \ln(N!) - \sum_i^M \ln(n_i!) - \ln(M^N)$

$\underset{\text{(Stirling)}}{\simeq} + N \ln N - \cancel{N} - \sum_i^M (n_i \ln n_i - \cancel{n_i}) - N \ln M$

$= -N \ln M + N \ln N - \sum_i^M n_i \ln n_i$

f) $p_i = \frac{n_i}{N} \longrightarrow n_i = N p_i$

$\ln p(\{p_i\}) = -N \ln M + N \ln N - N \sum_i^M p_i (\ln p_i + \ln N)$

$= -N \ln M + \cancel{N \ln N} - N \sum_i^M p_i \ln p_i - \cancel{N \sum_i p_i \ln N}$

$\sum_i p_i = 1$

$= -N \ln M - N \sum_{i=1}^M p_i \ln p_i$