1) Extreme value statistics

Maximum distribution: $\quad p_+(x) = n\, P(x)^{n-1}\, p(x)$

Minimum distribution: $\quad p_-(x) = n\,\left(1 - P(x)\right)^{n-1} p(x)$

where $\quad P(x) = \int_{-\infty}^{x} p(x')\, dx'$.

For the Gram-Charlier distribution:

$$p_+(x) = n\, P(x)^{n-1} p(x)$$

$$= n\left[\phi\left(\frac{x}{\sigma}\right) - \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left[\frac{K_3}{3!\,\sigma^2} H_2\left(\frac{x}{\sigma}\right) + \frac{K_4}{4!\,\sigma^3} H_3\left(\frac{x}{\sigma}\right)\right]\right]^{n-1} \cdot$$

$$\cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left[1 + \frac{K_3}{3!\,\sigma^3} H_3\left(\frac{x}{\sigma}\right) + \frac{K_4}{4!\,\sigma^4} H_4\left(\frac{x}{\sigma}\right)\right]$$

$$p_-(x) = n\left(1 - P(x)\right)^{n-1} p(x)$$

$$= n\left[1 - \phi\left(\frac{x}{\sigma}\right) + \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left[\frac{K_3}{3!\,\sigma^2} H_2\left(\frac{x}{\sigma}\right) + \frac{K_4}{4!\,\sigma^3} H_3\left(\frac{x}{\sigma}\right)\right]\right]^{n-1} \cdot$$

$$\cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\left[1 + \frac{K_3}{3!\,\sigma^3} H_3\left(\frac{x}{\sigma}\right) + \frac{K_4}{4!\,\sigma^4} H_4\left(\frac{x}{\sigma}\right)\right]$$

a) $\quad K_3 = 10^{-2}, \quad K_4 = 0, \quad n = 10^3, \quad \sigma = 1$

$$p_+(x) = 10^3\left[\phi(x) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\left[\frac{1}{600} H_2(x)\right]\right]^{10^3-1} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\left[1 + \frac{1}{600} H_3(x)\right]$$

$$p_-(x) = 10^3\left[1 - \phi(x) + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\left[\frac{1}{600} H_2(x)\right]\right]^{10^3-1} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\left[1 + \frac{1}{600} H_3(x)\right]$$

b) $\quad K_3 = 0, \quad K_4 = 10^{-3}, \quad n = 10^3, \quad \sigma = 1$

$$p_+(x) = 10^3\left[\phi(x) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\left[\frac{1}{24000} H_3(x)\right]\right]^{10^3-1} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\left[1 + \frac{1}{24000} H_4(x)\right]$$

$$p_-(x) = 10^3\left[1 - \phi(x) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\left[\frac{1}{24000} H_3(x)\right]\right]^{10^3-1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\left[1 + \frac{1}{24000} H_4(x)\right]$$

3) Extreme values from the uniform distribution

a) $$p_+(x) = n\left(P(x)\right)^{n-1} p(x)$$
$$p_-(x) = n\left(1-P(x)\right)^{n-1} p(x)$$

$$P(x) = \int_{-1}^{x} p(x')\,dx' = \int_{-1}^{x} \frac{dx'}{2} = \frac{1}{2}x'\Big|_{-1}^{x} = \frac{1}{2}(x+1)$$

$$\Rightarrow \quad p_+(x) = \frac{n}{2^n}(x+1)^{n-1}$$
$$p_-(x) = \frac{n}{2^n}(1-x)^{n-1}$$

b) $$p_-(-x) = \frac{n}{2^n}\left(1-(-x)\right)^{n-1} = \frac{n}{2^n}(1+x)^{n-1} = \frac{n}{2^n}(x+1)^{n-1} = p_+(x)$$

c) Position of maximum for $p_\pm(x)$:

$$\frac{d}{dx}p_\pm(x) = 0 \quad \Rightarrow \quad \frac{d}{dx}p_+(x) = \frac{n}{2^n}(n-1)(x+1)^{n-2} \overset{!}{=} 0 \quad \Leftrightarrow \quad x=-1 \quad \text{for } n>2$$
$$\frac{d}{dx}p_-(x) = \frac{n}{2^n}(n-1)(1-x)^{n-2}(-1) \overset{!}{=} 0 \Leftrightarrow x=+1 \quad \text{for } n>2$$

These solutions are minima! In fact, $p_+(x=-1)=0$
$p_-(x=+1)=0$

Since $p_+(x)$ monotonically increasing, $p_-(x)$ monotonically decreasing, maximum positions at the other boundary:

$$x_{max,+} = +1 \quad \text{for } p_+$$
$$x_{max,-} = -1 \quad \text{for } p_-$$

$$\Rightarrow \quad p_{+,max}(x) = p_+(x_{max,+}) = \frac{n}{2^n}(1+1)^{n-1} = \frac{n}{2}$$
$$p_{-,max}(x) = p_-(x_{max,-}) = \frac{n}{2^n}\left(1-(-1)\right)^{n-1} = \frac{n}{2}$$

d)

$$\langle x\rangle_+ = \int_{-1}^{1} dx\ x\ p_+(x) = \int_{-1}^{1} dx\ x\ \frac{n}{2^n}(x+1)^{n-1}$$

$$\underset{\substack{z = x+1 \\ dz = dx}}{\overset{=}{\downarrow}} \quad \frac{n}{2^n}\int_0^2 dz\ (z-1)\ z^{n-1} = \frac{2n}{n+1} - 1$$

$$\langle x\rangle_- = \int_{-1}^{1} dx\ x\ p_-(x) = \int_{-1}^{1} dx\ x\ \frac{n}{2^n}(1-x)^{n-1}$$

$$\underset{\substack{z = 1-x \\ dz = -dx}}{\overset{=}{\downarrow}} \quad -\int_2^0 dz\ (1-z)\ \frac{n}{2^n}\ z^{n-1} = 1 - \frac{2n}{n+1}$$

$$\Rightarrow \quad \langle x\rangle_- = -\langle x\rangle_+$$

For growing n, the expectation values' dependence on n becomes weaker. As $n\to\infty$,

$$\langle x\rangle_+ \to +1$$

$$\langle x\rangle_- \to -1 .$$

4) <u>Gumbel distribution</u>

$$p(x) = e^{-x - e^{-x}}$$

$$\langle x^n \rangle = \int dx\, x^n p(x) = \int dx\, x^n e^{-x - e^{-x}} = \int dx\, x^n e^{-x} e^{-e^{-x}} =$$

$e^{-x} = z$ $\qquad = \int -\frac{dz}{z} (-\ln z)^n z\, e^{-z}$

$x = -\ln z$

$dx = -\frac{dz}{z}$ $\qquad = \int (-1)^{n+1} (\ln z)^n e^{-z} dz.$

$$(-1)^n \frac{d^n}{dz^n} \Gamma(z) \Big|_{z=1} = (-1)^n \frac{d^n}{dz^n} \int_0^\infty dt\, t^{z-1} e^{-t} \Big|_{z=1} =$$

$$= (-1)^n \frac{d^{n-1}}{dz^{n-1}} \int_0^\infty dt\, t^{z-1} \ln t\, e^{-t} \Big|_{z=1} =$$

$$= (-1)^n \frac{d^{n-2}}{dz^{n-2}} \int_0^\infty dt\, t^{z-1} (\ln t)^2 e^{-t} \Big|_{z=1} = \quad \dots$$

$$= (-1)^n \int_0^\infty dt\, t^{z-1} (\ln t)^n e^{-t} \Big|_{z=1} =$$

$$= \int_0^\infty (-1)^n (\ln t)^n e^{-t} dt = \langle x^n \rangle.$$