Problems on *statistics and data analysis (MVComp2)*

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Problem sheet 8

To be handed in during the exercise group on 27.June.2016

1. *polytropic atmosphere* *(10 points)*

In the lecture we've shown that the density ρ of an ideal gas with the quation of state $p \propto \rho$ in a gravitational potential Φ is given by $\rho \propto \exp(-\Phi)$, which yields the barometric formula $\rho \propto \exp(-h)$ in a homogeneous gravitational field. Please derive a corresponding relation between ρ and Φ for a polytropic equation of state with $p \propto \rho^\alpha$ with the polytropic index $\alpha \neq 1$:

$$\rho \propto -\Phi^{\frac{1}{\alpha-1}}. \tag{I}$$

As an intermediate result, please show that

$$\frac{\nabla\rho^\alpha}{\rho} = \frac{\alpha}{\alpha-1}\nabla\rho^{\alpha-1}. \tag{II}$$

2. *matrix derivatives* *(10 points)*

A is a nonsingular, matrix-valued function of the variable x.

(a) Please start by showing

$$\partial_x(\mathrm{tr}A) = \mathrm{tr}(\partial_x A), \tag{III}$$

(b) and continue with

$$\partial_x A^{-1} = -A^{-1}(\partial_x A)A^{-1}, \tag{IV}$$

(c) and with

$$\partial_x A^n = nA^{n-1}\partial_x A. \tag{V}$$

with positive integer n.

(d) Then, please show that

$$\partial_x \exp(A) = \exp(A)\partial_x A \tag{VI}$$

by using a series expansion of the exponential,

(e) and then show the relation

$$\partial_x \ln(A) = A^{-1}\partial_x A, \tag{VII}$$

by using the inverse of the logarithm.

(f) Finally, please show that

$$\partial_x \ln \det A = \mathrm{tr}\left(A^{-1}\partial_x A\right), \tag{VIII}$$

(g) and the relation with fixed y_i,

$$\partial_x \left(y_i(A^{-1})_{ij}y_j\right) = -\mathrm{tr}\left(A^{-1}\partial_x A A^{-1} Y\right) \tag{IX}$$

with the definition $Y_{ij} = y_i y_j$.

3. *lognormal process* *(10 points)*

Please use the central limit theorem for designing a random process where the logarithm of the random variable is Gaussian-distributed.

4. *MCMC-sampling from a Gaussian* *(10 points)*

Implement the Metropolis-Hastings-algorithm to sample from a Gaussian: Make the Markov-chain move on the surface of a parabola and show that the distribution of sampling points follows a Gaussian distribution. For bonus points, measure the correlation along the chain.