Problem sheet 7

1) Combining measurements

$\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2$

$$F_{\mu\nu} = -\left\langle \frac{\partial \ln \mathcal{L}}{\partial \alpha_\mu \partial \alpha_\nu} \right\rangle$$

$$= -\left\langle \frac{\partial \ln(\mathcal{L}_1 \mathcal{L}_2)}{\partial \alpha_\mu \partial \alpha_\nu} \right\rangle$$

$$= -\left\langle \frac{\partial (\ln \mathcal{L}_1 + \ln \mathcal{L}_2)}{\partial \alpha_\mu \partial \alpha_\nu} \right\rangle$$

$$= -\left\langle \frac{\partial \ln \mathcal{L}_1}{\partial \alpha_\mu \partial \alpha_\nu} \right\rangle - \left\langle \frac{\partial \ln \mathcal{L}_2}{\partial \alpha_\mu \partial \alpha_\nu} \right\rangle$$

$$= F_{\mu\nu}^{(1)} + F_{\mu\nu}^{(2)} \quad \#$$

Many independent data points (n):

$\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2 \ldots \mathcal{L}_n$

$\rightarrow F_{\mu\nu} = \sum_{i=1}^{n} F_{\mu\nu}^{(i)}$, so F increases

Both for marginalised and conditional errors, the idea is that $\sigma^2 \sim \frac{1}{F}$

$\rightarrow$ the bigger F, the smaller the error

$\rightarrow$ many data points decrease the error.

2) $C \to C' = A C A^{-1}$

$$F'_{\alpha\beta} = \frac{1}{2} \operatorname{tr}\left(C'^{-1} \frac{\partial C'}{\partial \mu_\alpha} C'^{-1} \frac{\partial C'}{\partial \mu_\beta} \right)$$

$$= \frac{1}{2} \operatorname{tr}\left((ACA^{-1})^{-1} \frac{\partial (ACA^{-1})}{\partial \mu_\alpha} (ACA^{-1})^{-1} \frac{\partial (ACA^{-1})}{\partial \mu_\beta} \right)$$

$$= \frac{1}{2} \operatorname{tr}\left(A C^{-1} \cancel{A^{-1} A} \frac{\partial C}{\partial \mu_\alpha} \cancel{A^{-1} A} C^{-1} \cancel{A^{-1} A} \frac{\partial C}{\partial \mu_\beta} A^{-1} \right)$$

$$= \frac{1}{2} \operatorname{tr}\left(A C^{-1} \frac{\partial C}{\partial \mu_\alpha} C^{-1} \frac{\partial C}{\partial \mu_\beta} A^{-1} \right)$$

$$\underset{\substack{\downarrow \\ \text{cyclicity of the trace}}}{=} \frac{1}{2} \operatorname{tr}\left(\cancel{A^{-1} A} C^{-1} \frac{\partial C}{\partial \mu_\alpha} C^{-1} \frac{\partial C}{\partial \mu_\beta} \right) = F_{\alpha\beta}$$

3) Generalised fit of a polynomial

a)
$$\chi^2 = \sum_{i=0}^{n} \left(\frac{y_i - y(x_i)}{\sigma_i} \right)^2$$
$$= \sum_{i=0}^{n} \left(\frac{y_i}{\sigma_i} - \frac{1}{\sigma_i} \sum_{\alpha=0}^{m} a_\alpha x_i^\alpha \right)^2$$

b)
$$\partial_\alpha \chi^2 = \sum_{i=0}^{n} 2 \left(\frac{y_i}{\sigma_i} - \frac{1}{\sigma_i} \sum_{\beta=0}^{m} a_\beta x_i^\beta \right) \frac{x_i^\alpha}{\sigma_i}$$
$$= 2 \left(\sum_{i=0}^{n} \frac{y_i x_i^\alpha}{\sigma_i^2} - \sum_{i=0}^{n} \sum_{\beta=0}^{m} \frac{a_\beta x_i^{\alpha+\beta}}{\sigma_i^2} \right)$$
$$\overset{!}{=} 0$$

$$\rightarrow \sum_{i=0}^{n} \sum_{\beta=0}^{m} \frac{a_\beta x_i^{\alpha+\beta}}{\sigma_i^2} = \sum_{i=0}^{n} \frac{y_i x_i^\alpha}{\sigma_i^2}$$

c) In matrix-vector notation:

$$A p = q$$

with:
$$A_{\mu\nu} = \sum_{i=0}^{n} \frac{x_i^{\mu+\nu}}{\sigma_i^2}$$
$$p_\mu = a_\mu$$
$$q_\mu = \sum_{i=0}^{n} \frac{y_i x_i^\mu}{\sigma_i^2}$$

d) $$A_{\mu\nu} = n \left\langle \frac{x_i^{\mu+\nu}}{\sigma_i^2} \right\rangle$$

$$q_\mu = n \left\langle \frac{y_i x_i^\mu}{\sigma_i^2} \right\rangle$$

$$\rightarrow \begin{pmatrix} \left\langle \frac{1}{\sigma_i^2} \right\rangle & \cdots & \left\langle \frac{x_i^m}{\sigma_i^2} \right\rangle \\ \vdots & \ddots & \\ \left\langle \frac{x_i^m}{\sigma_i^2} \right\rangle & \cdots & \left\langle \frac{x_i^{2m}}{\sigma_i^2} \right\rangle \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} \left\langle \frac{y_i}{\sigma_i^2} \right\rangle \\ \vdots \\ \left\langle \frac{y_i x_i^m}{\sigma_i^2} \right\rangle \end{pmatrix}$$

e) $$\begin{pmatrix} a_0 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} \left\langle \frac{1}{\sigma_i^2} \right\rangle & \cdots & \left\langle \frac{x_i^m}{\sigma_i^2} \right\rangle \\ \vdots & \ddots & \\ \left\langle \frac{x_i^m}{\sigma_i^2} \right\rangle & \cdots & \left\langle \frac{x_i^{2m}}{\sigma_i^2} \right\rangle \end{pmatrix}^{-1} \begin{pmatrix} \left\langle \frac{y_i}{\sigma_i^2} \right\rangle \\ \vdots \\ \left\langle \frac{y_i x_i^m}{\sigma_i^2} \right\rangle \end{pmatrix}$$

f) $m = 0$:

$$a_0 = \left\langle \frac{1}{\sigma_i^2} \right\rangle^{-1} \left\langle \frac{y_i}{\sigma_i^2} \right\rangle = \frac{\sum_{i=0}^{n} \frac{y_i}{\sigma_i^2}}{\sum_{i=0}^{n} \frac{1}{\sigma_i^2}}$$

weighted average!

4) Posterior distribution

a) Likelihood $\mathcal{L}$, posterior dist. $p(\mu|x)$, prior $p(\mu)$

$$p(\mu|x) = \mathcal{L}(x|\mu)\, p(\mu) \quad \text{with } p(\mu) = \text{const.}$$

$$\mathcal{L} = \frac{1}{(2\pi)^{N/2}\sqrt{\det C}} \exp\left(-\frac{1}{2}\sum_{i,j} x_i (C^{-1})_{ij} x_j\right)$$

$$P(x_\mu, x_\nu)\, dx_\mu\, dx_\nu = \frac{1}{2\pi\sqrt{\det C}} \exp\left(-\frac{1}{2} x_\mu (C^{-1})_{\mu\nu} x_\nu\right) dx_\mu\, dx_\nu$$

$$= \frac{\sqrt{\det F}}{2\pi} \exp\left(-\frac{1}{2} x_\mu F_{\mu\nu} x_\nu\right) dx_\mu\, dx_\nu$$

using $C^{-1} = F$, $\det(C^{-1}) = \frac{1}{\det C} = \det F$

b) Correlation coefficient $r_{\mu\nu}$:

$$r_{\mu\nu} = \frac{(F^{-1})_{\mu\nu}}{\sqrt{(F^{-1})_{\mu\mu}(F^{-1})_{\nu\nu}}}$$

where $F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$, $F^{-1} = \frac{1}{\det F}\begin{pmatrix} F_{22} & -F_{12} \\ -F_{21} & F_{11} \end{pmatrix}$

c) $$p(x_\mu)\, dx_\mu = \int p(x_\mu, x_\nu)\, dx_\nu$$

$$\rightarrow p(x_1)\, dx_1 = \frac{\sqrt{\det F}}{2\pi} \int \exp\left\{-\frac{1}{2}\left(x_1^2 F_{11} + 2 x_1 x_2 F_{12} + x_2^2 F_{22}\right)\right\} dx_2$$

$$\left(\text{using } \int e^{-ax^2 + bx}\, dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}\right)$$

$$= \frac{\sqrt{\det F}}{2\pi}\, e^{-\frac{1}{2} x_1^2 F_{11}} \sqrt{\frac{2\pi}{F_{22}}}\, e^{\frac{x_1^2 F_{12}^2}{2 F_{22}}}$$

$$= \sqrt{\frac{2\pi \det F}{F_{22}}}\, e^{-\frac{1}{2}\left(F_{11} - \frac{F_{12}^2}{F_{22}}\right) x_1^2}$$

$$\rightarrow \boxed{\sigma_m^2 = \left(F_{11} - \frac{F_{12}^2}{F_{22}}\right)^{-1}}$$

d) $x_2 = 0$

Calculate exponent: $$\begin{pmatrix} x_1 \\ 0 \end{pmatrix}^T F \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = (x_1 \;\; 0) \begin{pmatrix} F_{11} x_1 \\ F_{22} x_1 \end{pmatrix} = F_{11} x_1^2$$

$$\rightarrow p(x_1)\, dx_1 = \frac{\sqrt{\det F}}{2\pi}\, e^{-\frac{1}{2} F_{11} x_1^2}$$

$$\rightarrow \boxed{\sigma_c^2 = F_{11}^{-1}}$$

5) Estimating

$$y = \prod_i^N x_i$$

x_i are factors that account for different aspects that may affect the number of civilizations in our galaxy, with which communication might be possible

$\rightarrow$ $\ln y = \sum_i \ln x_i$

$\rightarrow$ error on $\ln y$ $\propto \sqrt{N}$

$\rightarrow$ $\boxed{\text{relative error} \quad \frac{\Delta \ln y}{y} \sim \frac{1}{\sqrt{N}}}$