Problem sheet 1

1) Transformation of variables

a) For a continuous distribution:

$$E[ax] = \int a\, x\, p(x)\, dx = a \int x\, p(x)\, dx = a\, E[x]$$

$$E[x+a] = \int (x+a)\, p(x)\, dx = \int x\, p(x)\, dx + a \underbrace{\int p(x)\, dx}_{=1} = E[x] + a$$

Analogously for discrete distribution.
An estimate of the expectation value does not need to be linear, depending on how it is constructed.

b) Given probability distribution $f(x)\,dx$ and change of variable $y(x)$, it is true that $f(x)\,dx = g(y)\,dy$, where $g(y)$ is the prob. distribution in y.

$$\rightarrow \quad g(y) = f(x) \left| \frac{dx}{dy} \right|$$

→ the absolute value ENSURES the POSITIVITY of the new pdf.

$$A\, e^{-\frac{(M-M_0)^2}{2\sigma^2}}$$

So if the PDF of M is gaussian, the PDF of $r(M)$ is

$$f(r) = \left| \frac{dM}{dr} \right| A \exp\left(- \frac{(M(r) - M_0)^2}{2\sigma^2} \right)$$

$$= \left(\frac{5}{\ln 10 \;\, r} \right) A \exp\left(- \frac{(m - 25 - 5 \log r - M_0)^2}{2\sigma^2} \right)$$

$$= \frac{A'}{r} \exp\left(- \frac{(m - 25 - 5 \log r - M_0)^2}{2\sigma^2} \right)$$

Defining r_0 such that $M_0 = m - 25 - 5 \log r_0$

$$\rightarrow \quad \boxed{f(r) = \frac{A'}{r} \exp\left(- \frac{\left(5 \log \frac{r}{r_0}\right)^2}{2\sigma^2} \right)}$$

LOG-NORMAL distribution

In general, $\boxed{E[g(x)] \neq g(E[x])}$

In this particular case, with some calculations one can explicitly see that

$$\left.\begin{aligned} r\left(E[M]\right) &= 10^{\frac{m - E[m] - 25}{5}} \\ E\left(r[M]\right) &= \int r\, p(r)\, dr = \int A' e^{\frac{-\left(5 \log \frac{r}{r_0}\right)^2}{2\sigma^2}} \end{aligned}\right] \neq$$

c) For a bijective transformation $z = g(y)$

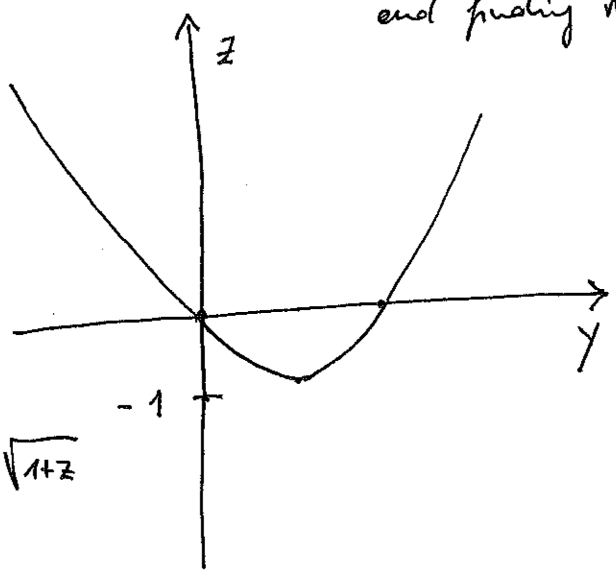
$$f_z(z) = \frac{f_Y(Y)}{\left| \frac{dz}{dY}\Big|_z \right|} = \frac{f_Y\left[g^{-1}(z)\right]}{\left| g'(Y)\Big|_z \right|}$$

For a non-bijective transformation $z = g(Y)$

$$f_z(z) = \sum_{i=1}^{NR} \frac{f_Y\left(Y_i(z)\right)}{\left| \frac{dz}{dY}\left(Y_i(z)\right) \right|}$$

where NR = number of ROOTS, that you get ~~...~~ fixing z and finding roots in Y.

In our case $z = Y^2 - 2Y$



→ 2 roots:

$$Y_1 = 1 - \sqrt{1+z}$$
$$Y_2 = 1 + \sqrt{1+z}$$

→ evaluating $\frac{dz}{dY} = 2Y - 2 \begin{cases} -2\sqrt{1+z} \\ 2\sqrt{1+z} \end{cases}$

→ Result: $f_z = 0$ for $z < -1$

$$f_z = \frac{1}{\left|2\sqrt{1+z}\right|}\left(f_Y\left(1 - \sqrt{1+z}\right) + f_Y\left(1 + \sqrt{1+z}\right)\right) \quad \text{for } z \geq -1$$

(f_Y is a Gaussian, so one could be even more explicit and write down Gaussians with these arguments)

2) Moments

a) Cumulative distribution function of the exponential distr. $\lambda e^{-\lambda x}$

$$F(x) = 1 - e^{-\lambda x}$$

$$P\{X > m+k \mid X > k\} = \frac{P\{X > k \mid X > m+k\}\, P\{X > m+k\}}{P\{X > k\}} =$$

Bayes' law

$$= \frac{P\{X > m+k\}}{P\{X > k\}} = \frac{1 - P\{X \le m+k\}}{1 - P\{X \le k\}} =$$

$$= \frac{e^{-\lambda(m+k)}}{e^{-\lambda k}} = e^{-\lambda m} = P\{X > m\} \quad \#$$

b) Poisson: $m_X(z) = E[e^{zX}]$

$$= \sum_{i=0}^{\infty} e^{zi} \frac{\lambda^i}{i!} e^{-\lambda} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{(\lambda e^z)^i}{i!} = e^{-\lambda} e^{\lambda e^z} = e^{\lambda(e^z - 1)}$$

Gamma: $m_X(z) = E[e^{zX}]$

$$= \int_0^\infty e^{zx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda - z)x}\, dx$$

$$= \frac{\lambda^\alpha}{(\lambda - z)^\alpha} \int_0^\infty \frac{(\lambda - z)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda - z)x}\, dx = \left(\frac{\lambda}{\lambda - z}\right)^\alpha$$

$\downarrow$

This is an integrated pdf of a Γ function, $\Gamma(\alpha, \lambda - z)$, which after integration gives 1.
This is true only if $\lambda - z > 0$
$\rightarrow$ the domain of the moment-generating function is limited to $z < \lambda$.

Exponential: the exponential distribution is a particular case of Gamma distribution, with parameters $\alpha = 1$ and λ

$$\rightarrow \quad m_X(z) = E[e^{zX}] = \left(\frac{\lambda}{\lambda - z}\right)$$

c) Given two independent random variables X, Y, it is true that

$$\boxed{m_{X+Y}(z) = m_X(z)\, m_Y(z)}$$

Proof: $S = X+Y$

$$M_S(z) = E\left[e^{zS}\right] = E\left[e^{z(X+Y)}\right] = E\left[e^{zX}\right] E\left[e^{zY}\right] = M_X(z)\, M_Y(z) \quad \#$$

$$\int e^{z(x+y)} f(x) f(y)\, dx\, dy = \int e^{zx} f(x)\, dx \int e^{zy} f(y)\, dy$$

Now, $X \sim Pois(\lambda_1)$
$Y \sim Pois(\lambda_2)$

$\rightarrow m_{X+Y}(z) = m_X(z)\, m_Y(z) = e^{\lambda_1(e^z-1)} e^{\lambda_2(e^z-1)} = e^{(\lambda_1+\lambda_2)(e^z-1)}$

which is the moment-generating function of a Poisson distribution with parameter $\lambda_1+\lambda_2$ $\rightarrow$ $X+Y$ follows $Pois(\lambda_1+\lambda_2)$

If $X \sim \Gamma(\alpha_1, \lambda)$
$Y \sim \Gamma(\alpha_2, \lambda)$

$\rightarrow m_{X+Y}(z) = m_X(z)\, m_Y(z) = \left(\frac{\lambda}{\lambda-z}\right)^{\alpha_1} \left(\frac{\lambda}{\lambda-z}\right)^{\alpha_2} = \left(\frac{\lambda}{\lambda-z}\right)^{\alpha_1+\alpha_2}$

which is the moment-generating function of a Gamma distribution $\Gamma(\alpha_1+\alpha_2, \lambda)$ $\rightarrow$ $X+Y$ follows $\Gamma(\alpha_1+\alpha_2, \lambda)$

3) Convolution of distributions

$q = XY$

$$p(q) = \int dx \int dy \, \delta_D(xy - q) = \log q$$

a) To find the distribution of Z, let us calculate the cumulative distribution function. Z has positive values, therefore $\forall\, t > 0$ we have

$$F_Z(t) = P\{-\log X \leq t\} = P\{X \geqslant e^{-t}\} = 1 - e^{-t}$$

and $F_Z(t) = 0$ for $t < 0$.

$\rightarrow$ Z (and analogously W) have exponential distribution, with parameter $\lambda = 1$.

b) $$f_{Z+W}(u) = \int_{\mathbb{R}} f_Z(z) \, f_W(u - z) \, dz$$

$$= \int_0^u e^{-z} e^{-(u-z)} dz = e^{-u} \int_0^u dz = u e^{-u}$$ for $u \geqslant 0$, otherwise 0.

c) We can find the probability density of $XY = e^{-(Z+W)}$ by calculating the cumulative function.
Being XY included within 0 and 1, for all $t \in (0,1)$ we have

$$F_{XY}(t) = P\{e^{-(Z+W)} \leq t\} = P\{Z + W \geqslant -\log t\} =$$

$$= \int_{-\log t}^{\infty} u e^{-u} du \underset{\text{change of variable } V := e^{-u}}{=} \int_0^t (-\log V) \, dV$$

Being F_{XY} the integral of a continuous function, it has a first derivative and therefore the density of XY is given by

$$f_{XY}(t) = F'_{XY}(t) = -\log t$$

d) Tough question!

Let $Y_i := -\log X_i$. It can be shown that $Y_1 + \dots + Y_n \sim \Gamma(n,1)$

This is difficult to show, but given that, we have

$$F_{\prod_{i=1}^n X_i}(t) = P\left\{ e^{-\sum_{i=1}^n X_i} \leq t \right\} = P\left\{ \sum_{i=1}^n X_i \geq -\log t \right\} =$$

$$= \int_{-\log t}^{\infty} \frac{1}{(n-1)!} u^{n-1} e^{-u} \, du = \int_0^t \frac{1}{(n-1)!} (-\log v)^{n-1} \, dv$$

$\rightarrow$ the probability density is then

$$f_{\prod_{i=1}^n X_i}(t) = F'_{\prod_{i=1}^n X_i}(t) = \frac{1}{(n-1)!} (-\log t)^{n-1}$$