2) Errors in a non-linear fit

$$F_{\alpha\alpha} = -\left\langle \frac{\partial^2 \ln L}{\partial \alpha^2} \right\rangle = -\left\langle \frac{\partial^2}{\partial \alpha^2} \ln \exp\left(-\frac{\chi^2}{2}\right) \right\rangle$$

$$= \frac{1}{2} \left\langle \frac{\partial^2 \chi^2}{\partial \alpha^2} \right\rangle = \frac{1}{2} \left\langle \frac{\partial^2}{\partial \alpha^2} \sum_i \left[\frac{y_i - \exp(-\alpha x_i)}{\sigma_i} \right]^2 \right\rangle$$

$$\underset{\sigma_i = \sigma}{=} \frac{1}{2\sigma^2} \left\langle \frac{\partial^2}{\partial \alpha^2} \sum_i \left[y_i^2 - 2 y_i \exp(-\alpha x_i) + \exp(-2\alpha x_i) \right] \right\rangle$$

$$= \frac{1}{2\sigma^2} \left\langle \sum_i \left[-2 y_i x_i^2 \exp(-\alpha x_i) + 4 x_i^2 \exp(-2\alpha x_i) \right] \right\rangle$$

$$\underset{x_i \text{ not random}}{=} \frac{1}{2\sigma^2} \sum_i \left[x_i \left(\exp(-\alpha x_i) \langle -2 y_i \rangle + 4 \exp(-2\alpha x_i) \right) \right]$$

$$\underset{\langle y_i \rangle = \exp(-\alpha x_i)}{=} \frac{1}{2\sigma^2} \sum_i \left[x_i^2 \left(-2 \exp(-2\alpha x_i) + 4 \exp(-2\alpha x_i) \right) \right]$$

$$= \frac{1}{2\sigma^2} \sum_i x_i^2 \exp(-2\alpha x_i) \, (-2)$$

$$= -\frac{1}{\sigma^2} \sum_i x_i^2 \exp(-2\alpha x_i)$$

exercise sheet 09 - question 3, bivariate Cauchy-distribution

$$p(x,y) = (1+x^2+y^2)^{-\frac{3}{2}}$$

(a) 1d is sufficient. $p(x) = \left(1+\frac{x^2}{a^2}\right)^{-1}$

curvature: $\frac{d^2 \ln p}{dx^2}\Big|_{x=0} = \frac{d}{dx}\left(\frac{1}{p}\cdot\frac{dp}{dx}\right)\Big|_{x=0} = \frac{1}{p}\frac{d^2p}{dx^2} - \frac{1}{p^2}\cdot\left(\frac{dp}{dx}\right)^2\Big|_{x=0}$

- $\frac{dp}{dx} = -\left(1+\frac{x^2}{a^2}\right)^{-2}\cdot\frac{2x}{a^2}$

- $\frac{d^2p}{dx^2} = 2\left(1+\frac{x^2}{a^2}\right)^{-3}\frac{4x^2}{a^4} - \left(1+\frac{x^2}{a^2}\right)^{-2}\cdot\frac{2}{a^2}$

$$\rightarrow \frac{d^2\ln p}{dx^2} = \left(1+\frac{x^2}{a^2}\right)\cdot\left(1+\frac{x^2}{a^2}\right)^{-2}\cdot\left[\left(1+\frac{x^2}{a^2}\right)^{-1}\cdot\frac{8x^2}{a^4} - \frac{2}{a^2}\right]$$
$$+ \left(1+\frac{x^2}{a^2}\right)^{+2}\cdot\left[\left(1+\frac{x^2}{a^2}\right)^{-2}\frac{2x}{a^2}\right]^2$$
$$= \left(1+\frac{x^2}{a^2}\right)^{-2}\cdot\frac{8x^2}{a^4} - \left(1+\frac{x^2}{a^2}\right)^{-1}\cdot\frac{2}{a^2} + \left(1+\frac{x^2}{a^2}\right)^{-2}\cdot\frac{4x^2}{a^4}$$
$$= \left(1+\frac{x^2}{a^2}\right)^{-2}\frac{12x^2}{a^4} - \left(1+\frac{x^2}{a^2}\right)\frac{2}{a^2}$$

at $x=0$: $\frac{d^2\ln p}{dx^2}\Big|_{x=0} = \frac{2}{a^2}$ → scaling with a

confidence intervals

$\int_{-q}^{+q} dx\, p(x) = \mathrm{erf}\left(\frac{q}{\sqrt{2}}\right)$ ~ defines Gaussian nσ-intervals

$$\int_{-q}^{+q} dx\, p(x) = \int_{-q}^{+q} dx \left(1+\frac{x^2}{a^2}\right)^{-1} = a\left[\arctan\left(\frac{q}{a}\right) - \arctan\left(-\frac{q}{a}\right)\right]$$
$$= \frac{2}{a}\left(\arctan\left(\frac{q}{a}\right) - 1\right) = \mathrm{erf}\left(\frac{n}{\sqrt{2}}\right).$$

substitute $a = 2\cdot\left(\frac{d^2\ln p}{dx^2}\Big|_{x=0}\right)^{-\frac{1}{2}}$ and solve

→ relationship between confidence intervals and curvature exercise sheet 09 - question 3, bivariate cauchy-distribution

(b) be careful! $\partial^2_{xy} \ln p = 0$ does not imply statistical independence between x and y, unlike in the case of the gaussian.

Gaussian $\rightarrow$ covariance $\langle xy \rangle = 0$ equivalent to $\partial^2_{xy} \ln p = 0$ because both conditions imply $p(x,y) = p(x) \cdot p(y)$

measurement of covariance not possible for the Cauchy-distribution, divergent $\langle xy \rangle = \int dx \int dy \; p(x,y)$.

copula ~ not easy: the cumulative distribution mixes x and y in a way that makes the edge-distributions complicated.

4) Volume and surface of a sphere in n dimensions

We can construct the volume $V_n(R)$ by adding infinitely thin spherical shells of radius $0 \le r \le R$. In equation form, this reads:

$$\boxed{V_n(R) = \int_0^R S_{n-1}(r)\, dr.}$$

Let us equate the two expressions we have for $V_n(R)$,

1) $$V_n(R) = \int \cdots \int_{x_1^2 + x_2^2 + \ldots + x_n^2 \le R^2} dx_1\, dx_2 \ldots dx_n = C_n R^n$$

2) $$V_n(R) = \int_0^R S_{n-1}(r)\, dr = \int_0^R n C_n r^{n-1} dr$$

$$\Rightarrow \quad 1) = 2) \quad \leftrightarrow \quad \boxed{dx_1\, dx_2 \ldots dx_n = r^{n-1} dr\, d\Omega_{n-1}}.$$

Using this, we get that

$$\int \cdots \int_{x_1^2 + x_2^2 + \ldots + x_n^2 \le R^2} dx_1\, dx_2 \ldots dx_n = \int_0^R r^{n-1} dr \int d\Omega_{n-1} = n C_n \int_0^R r^{n-1} dr$$

$$\Rightarrow \boxed{\int d\Omega_{n-1} = n C_n .} \qquad (*)$$

Let us now integrate the n-dimensional Gaussian over the full n-dimensional space in both rectangular and hyper-spherical coordinates, we get:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1^2 + x_2^2 + \ldots + x_n^2)} = \int_0^{\infty} r^{n-1} dr \int d\Omega_{n-1}\, e^{-r^2}$$

The integrand on the RHS depends only on r. Therefore we can perform the integral over $d\Omega_{n-1}$. Using $(*)$:

$$\int_{-\infty}^{\infty} e^{-x_1^2} dx_1 \int_{-\infty}^{\infty} e^{-x_2^2} dx_2 \cdots \int_{-\infty}^{\infty} e^{-x_n^2} dx_n = n C_n \int_0^{\infty} r^{n-1} e^{-r^2} dr$$

All the integrals above can be evaluated explicitly:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \quad , \quad \int_0^{\infty} r^{n-1} e^{-r^2}\,dr = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)$$

$$\Rightarrow \quad \pi^{n/2} = C_n \frac{n}{2}\,\Gamma\left(\frac{n}{2}\right) = C_n\,\Gamma\left(1+\frac{n}{2}\right)$$

where we used the property of the Gamma function $x\Gamma(x) = \Gamma(x+1)$.
Solving for C_n, we obtain

$$\boxed{C_n = \frac{\pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)}}$$

$$\Rightarrow \quad \boxed{V_n(R) = C_n R^n = \frac{\pi^{n/2} R^n}{\Gamma\left(1+\frac{n}{2}\right)}}$$

$$\Rightarrow \quad \boxed{S_{n-1}(R) = n C_n R^{n-1} = \frac{n\,\pi^{n/2} R^{n-1}}{\Gamma\left(1+\frac{n}{2}\right)}} = \frac{2\pi^{n/2} R^{n-1}}{\Gamma\left(\frac{n}{2}\right)}$$

using $\Gamma\left(1+\frac{n}{2}\right) = \frac{n}{2}\Gamma\left(\frac{n}{2}\right)$

$$\left.\begin{aligned} V_3(R) &= \frac{4}{3}\pi R^3 \\ S_2(R) &= 4\pi R^2 \end{aligned}\right\} \text{familiar results}$$

$$\left.\begin{aligned} V_2(R) &= \pi R^2 \\ S_1(R) &= 2\pi R \end{aligned}\right\} \text{familiar results}$$

In the following we present an argument to convince ourselves that indeed $\lim_{n\to\infty} V_n(R) = 0$ and $\lim_{n\to\infty} S_n(R) = 0$.

Consider a hypercube in n-dimensional space measuring one unit on each side. The total volume of this hypercube is 1. We can fit a hypersphere of diameter 1 (or radius $\frac{1}{2}$) inside the hypercube such that the surface of the hypersphere just touches each of the walls of the hypercube. Then $1 - V_n(\frac{1}{2})$ is the volume inside the cube but outside the hypersphere.

In particular, as n becomes large, $1 - V_n(\frac{1}{2})$ rapidly approaches 1, which is consistent with the assertion that $\lim_{n\to\infty} V_n(R) = 0$. This simply means that as the number of dimensions become larger and larger, the amount of space outside the hypersphere (but inside the cube) becomes relatively more and more important.

This is already happening as you go from 2 to 3 dimensions. If you inscribe a circle in a unit square and a sphere in a unit cube, and compute the total volume in three dimensions (area in two dimensions) outside the sphere (circle) but inside the cube (square), then you take the ratio of volumes (areas) of the sphere (circle) to that of the cube (square), this ratio actually decreases as you go from 2D to 3D.