Problem sheet 2

1) <u>Bernoulli distribution</u>

$$p(n,p,x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$g_x(p,q) := \binom{n}{x} p^x q^{n-x}$$

$$x\, g_x(p,q) = p \frac{\partial}{\partial p} g_x(p,q)$$

$$\Rightarrow \langle x \rangle = \sum_{x=0}^{n} x\, g_x(p,q)\Big|_{q=1-p} = \sum_{x=0}^{n} p \frac{\partial}{\partial p} g_x(p,q)\Big|_{q=1-p} = p\frac{\partial}{\partial p}(p+q)^n\Big|_{q=1-p}$$

$$= p\, n\, (p+q)^{n-1}\Big|_{q=1-p} = pn.$$

$$x(x-1)\, g_x(p,q) = p^2 \frac{\partial^2}{\partial p^2} g_x(p,q)$$

$$\Rightarrow \langle x(x-1)\rangle = \langle x^2 - x\rangle = p^2 \frac{\partial^2}{\partial p^2}(p+q)^n\Big|_{q=1-p} = p^2 n(n-1)$$

$$\Rightarrow \langle x^2\rangle = p^2 n(n-1) + \langle x\rangle = p^2 n^2 - p^2 n + pn$$

$$x(x-1)(x-2)\, g_x(p,q) = p^3 \frac{\partial^3}{\partial p^3} g_x(p,q)$$

$$\Rightarrow \langle x(x-1)(x-2)\rangle = \langle x^3 - 3x^2 + 2x\rangle = p^3 \frac{\partial^3}{\partial p^3}(p+q)^n\Big|_{q=1-p} = p^3 n(n-1)(n-2)$$

$$\Rightarrow \langle x^3\rangle = p^3 n(n-1)(n-2) + 3\langle x^2\rangle - 2\langle x\rangle$$
$$= p^3 n^3 - 3p^3 n^2 + 2p^3 n + 3p^2 n^2 - 3p^2 n + pn$$

$$\langle (x-\langle x\rangle)^3\rangle = \langle x^3 - 3x^2\langle x\rangle + 3\cancel{x}\langle x\rangle^2 - \langle x\rangle^3\rangle = \langle x^3\rangle - 3\langle x^2\rangle\langle x\rangle + 2\langle x\rangle^3 =$$
$$= 2p^3 n - 3p^2 n + pn = pn(1-p)(1-2p)$$

$$\langle (x-\langle x\rangle)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = pn(1-p)$$

$$\Rightarrow \frac{\langle (x-\langle x\rangle)^3\rangle}{\langle (x-\langle x\rangle)^2\rangle^{3/2}} = \frac{pn(1-p)(1-2p)}{(pn)^{3/2}(1-p)^{3/2}} = \frac{1-2p}{(pn)^{1/2}(1-p)^{1/2}} \xrightarrow[n\to\infty]{} 0$$

2) Moments

$$\varphi(t) = \sum_n \langle x^n \rangle \frac{(it)^n}{n!}$$

a) $c \sum_n \frac{(it)^n}{n!} = c \exp(it)$ always converge

b) $\sum_n n! \frac{(it)^n}{n!} = \sum_n (it)^n$

$$\left| \frac{(it)^{n+1}}{(it)^n} \right| = |it| = |t|$$ $\rightarrow$ the series converges for $|t| < 1$

c) $\sum_n c^n n! \frac{(it)^n}{n!} = \sum_n c^n (it)^n$

$$\left| \frac{(ict)^{n+1}}{(ict)^n} \right| = |ict| = |ct|$$ $\longrightarrow$ the series converges for $|ct| < 1$

3) Moments of the exponential distribution

$$p(x) = e^{-x}, \qquad 0 \le x < \infty$$

$$\langle x \rangle = \int_0^\infty dx \; x \, e^{-x} = -e^{-x} x \Big|_0^\infty + \int e^{-x} dx$$

$$= -e^{-x} \Big|_0^\infty = 1 = 1! \quad \rightarrow \quad \text{so } \langle x^n \rangle = n! \text{ is true for } n=1.$$

Assuming that $\langle x^n \rangle = n!$ is true for unspecified n:

$$\langle x^{n+1} \rangle = \int_0^\infty dx \; x^{n+1} e^{-x} = -e^{-x} x^{n+1} + (n+1) \int e^{-x} x^n =$$

$$= (n+1)! \quad \rightarrow$$ so if $\langle x^n \rangle = n!$ is true for unspecified n, it is also true for $n+1$.

$\Rightarrow$ Induction is complete, $\langle x^n \rangle = n!$.

$$m(t) = \frac{1}{1-t}$$

$$\rightarrow \langle x^n \rangle = \frac{\partial^n}{\partial t^n} m \Big|_{t=0} = \frac{\partial^{n-1}}{\partial t^{n-1}} \left(\frac{1}{(1-t)^2} \right) \Bigg|_{t=0} = \left(\frac{\partial}{\partial t} \right)^{n-2} \frac{2}{(1-t)^3} \Bigg|_{t=0} =$$

$$= \ldots = \left(\frac{\partial}{\partial t} \right)^{n-i} \frac{i!}{(1-t)^{i+1}} \Bigg|_{t=0} = \frac{\partial}{\partial t} \frac{(n-1)!}{(1-t)^n} \Bigg|_{t=0} = \frac{n!}{(1-t)^{n+1}} \Bigg|_{t=0} = n!$$

4) Cumulants and moments

$$\phi(t) = \sum_n \langle x^n \rangle \frac{(it)^n}{n!} \quad , \quad K(t) = \sum_n k_n \frac{(it)^n}{n!}$$

$$\frac{\partial}{\partial t}\phi = \frac{\partial}{\partial t} \exp(K(t)) = \exp K \left(\frac{\partial}{\partial t} K\right) = \phi \frac{\partial}{\partial t}\left(\sum_n k_n \frac{(it)^n}{n!}\right) =$$

$$= \phi(t) \sum_n' k_n \, n \, i^n \frac{t^{n-1}}{n!} = \phi(t) \sum_n' k_n \frac{n}{t} \frac{(it)^n}{n!} =$$

$$= \sum_m' \langle x^m \rangle \frac{(it)^m}{m!} \sum_n k_n \frac{n}{t} \frac{(it)^n}{n!} \qquad (2)$$

$$\frac{\partial}{\partial t}\phi = \sum_n' \langle x^n \rangle \frac{n}{t} \frac{(it)^n}{n!} = \sum_n' \langle x^n \rangle \frac{i^n t^{n-1}}{(n-1)!} \qquad (1)$$

So, equating (1) and (2)

$$\rightarrow \sum_n' \langle x^n \rangle \frac{i^n t^{n-1}}{(n-1)!} = \sum_{m,l}' \frac{\langle x^m \rangle (it)^{m+l-1}}{m! \, (l-1)!} k_l \, i$$

Redefine $m + l - 1 = n - 1 \rightarrow m + l = n$

$\rightarrow$ For fixed n:

$$\frac{\langle x^n \rangle}{(n-1)!} i^n = \sum_{m \le n}' \langle x^m \rangle i^n k_{n-m} \frac{1}{m! \, (n-m-1)!}$$

$$\rightarrow \boxed{\langle x^n \rangle = \sum_{m \le n}' \langle x^m \rangle k_{n-m} \binom{n}{m} \frac{n-m}{n}}$$