(1) Computational statistics and data analysis, sheet 4

1. Hölder's inequality

- $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ Young's inequality with $\frac{1}{p}+\frac{1}{q}=1$

  set $a = \frac{|x|}{\langle |x|^p\rangle^{\frac{1}{p}}}$ and $b = \frac{|y|}{\langle |y|^q\rangle^{\frac{1}{q}}}$ $\rightarrow$

$$\frac{|x\,y|}{\langle |x|^p\rangle^{\frac{1}{p}} \cdot \langle |y|^q\rangle^{\frac{1}{q}}} \leq \frac{1}{p}\frac{|x|^p}{\langle |x|^p\rangle} + \frac{1}{q}\frac{|y|^q}{\langle |y|^q\rangle} \quad \Big| \langle \dots \rangle$$

$$\frac{\langle |xy|\rangle}{\langle |x|^p\rangle^{\frac{1}{p}} \langle |y|^q\rangle^{\frac{1}{q}}} \leq \frac{1}{p} + \frac{1}{q} = 1 \quad \text{by construction}$$

$\rightarrow \langle |xy|\rangle \leq \langle |x|^p\rangle^{\frac{1}{p}} \cdot \langle |y|^q\rangle^{\frac{1}{q}}$ Hölder-inequality

- moment $\langle x\rangle = \sum_i p_i x_i$ +
  triangle inequality $|x+y| \leq |x|+|y|$.

  $\langle |x|\rangle = \sum_i p_i |x_i| \leq |\sum_i p_i x_i| = |\langle x\rangle|$.

  set $x = x \cdot y$

- $p=q=2$: $\langle |xy|\rangle = \sqrt{\langle x^2\rangle \cdot \langle y^2\rangle}$ Cauchy-Schwarz inequality

- $x = x^r$, $y=1$, $s = r \cdot p \rightarrow p = \frac{r}{s}$

  $\langle x^r \cdot 1\rangle \leq \langle x^{r\cdot p}\rangle^{\frac{1}{p}} \langle 1^q\rangle^{\frac{1}{q}} = \langle x^s\rangle^{\frac{1}{p}} = \langle x^s\rangle^{\frac{r}{s}}$

$\rightarrow \langle x^r\rangle \leq \langle x^s\rangle^{\frac{r}{s}}$

  if $\langle x^r\rangle$ diverges, it will force $\langle x^s\rangle$, $s \geq r$, to diverge too

- $\langle |x+y|^p\rangle = \langle |x+y||x+y|^{p-1}\rangle \leq \langle |x||x+y|^{p-1}\rangle + \langle |y||x+y|^{p-1}\rangle$

  with the triangle-inequality $|x+y| \leq |x|+|y|$.

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with $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{q}=1-\frac{1}{p}=\frac{p-1}{p} \rightarrow q=\frac{p}{p-1}$

$$\langle |x+y|^p\rangle \leq \langle |x|\cdot|x+y|^{p-1}\rangle + \langle |y|\cdot|x+y|^{p-1}\rangle$$
$$\leq \langle |x|^p\rangle^{\frac{1}{p}}\cdot\langle |x+y|^{(p-1)\cdot q}\rangle^{\frac{1}{q}} + \langle |y|^p\rangle\cdot\langle |x+y|^{(p-1)\cdot q}\rangle^{\frac{1}{q}}$$

with Hölder's inequality $\langle |xy|\rangle \leq \langle |x|^p\rangle^{\frac{1}{p}}\cdot\langle |y|^q\rangle^{\frac{1}{q}}$

$$\leq \langle |x|^p\rangle^{\frac{1}{p}}\cdot\langle |x+y|^p\rangle^{\frac{1}{q}} + \langle |y|^p\rangle\cdot\langle |x+y|^p\rangle^{\frac{1}{q}}$$

with $(p-1)q=p$. from $\frac{1}{p}+\frac{1}{q}=1$.

$$=\left[\langle |x|^p\rangle^{\frac{1}{p}}+\langle |y|^p\rangle^{\frac{1}{p}}\right]\cdot\langle |x+y|^p\rangle^{\frac{p-1}{p}}$$

$$=\left[\langle |x|^p\rangle^{\frac{1}{p}}+\langle |y|^p\rangle^{\frac{1}{p}}\right]\cdot\frac{\langle |x+y|^p\rangle}{\langle |x+y|^p\rangle^{\frac{1}{p}}}$$

$$\rightarrow \langle |x+y|^p\rangle^{\frac{1}{p}} \leq \langle |x|^p\rangle^{\frac{1}{p}}+\langle |y|^p\rangle^{\frac{1}{p}}$$

2. complex Gaussian distribution.

z ~ complex valued Gaussian random variable, $z=x+iy$

$C=\langle z\cdot z^+\rangle$

- $C=\langle z\cdot z^+\rangle=\langle (x+iy)(x-iy)\rangle=\langle x^2+y^2\rangle$

~ covariance matrix of two independent Gaussian distributions, whose results are added. → variances add.

$\det(C)=\langle x^2+y^2\rangle>0$

$$C_{xy}^{-1}=\begin{pmatrix}\frac{1}{\langle x^2\rangle} & 0\\ 0 & \frac{1}{\langle y^2\rangle}\end{pmatrix}.$$

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3. Cauchy - distribution

$$p(x) = \frac{1}{\pi} \frac{1}{1+x^2}$$

- $\int dx\, p(x) \exp(itx) = \langle \exp(itx) \rangle = \int \frac{dx}{\pi} \frac{1}{1+x^2} \exp(itx)$

$\sim$ difficult to solve, complex integration, residuals at ± 1.

opposite direction (think of the natural line profile)

$$\int_{-\infty}^{0} dt \exp(t) \exp(itx) + \int_{0}^{\infty} dt \exp(-t) \exp(itx)$$

$$= \int_{-\infty}^{0} dt \exp((ix+1)t) + \int_{0}^{\infty} dt \exp((ix-1)t)$$

$$= \frac{\exp(ix+1)t}{ix+1} \Big|_{-\infty}^{0} + \frac{\exp(ix-1)t}{ix-1} \Big|_{0}^{\infty}$$

$$= \frac{1}{ix+1} - \frac{1}{ix-1} = \frac{ix-1 - (ix+1)}{(ix+1)(ix-1)} = \frac{2}{x^2+1}$$

- if independent random numbers are added, the characteristic functions are multiplied:

$$\varphi_{x_1+x_2}(t) = \langle \exp i(x_1+x_2)t \rangle = \langle \exp ix_1t \rangle \langle \exp ix_2t \rangle = \varphi_{x_1}(t) \cdot \varphi_{x_2}(t)$$

for $\varphi(t) = \exp(-t) \to \varphi_{x_1+x_2} = \exp(-t) \cdot \exp(-t) = \exp(-2t)$.

is still a Cauchy distribution $\to$

sums of Cauchy - distributed random numbers are again Cauchy - distributed

- no: for the Chebyshev inequality one needs a finite variance, but $\langle x^2 \rangle \to \infty$ for Cauchy - distributions.

the law of large numbers as a generalisation of the Chebyshev - inequality for sums does not apply either.