(1) computational statistics and data analysis – exercise sheet 5

1. rings of Saturn $\rightarrow$ Jeffrey's prior

if one is completely uncertain about the value of a random variable x, one must be uncertain in the scaled value λx as well: $p(x)dx \rightarrow p(\lambda x)\, d(\lambda x) = \lambda p(\lambda x) dx = p(x)dx$

which is fulfilled by $p(x)\,dx = \frac{dx}{x} = d\ln x$

2. confidence intervals in n-dimensional Gaussian distributions

(1d) $\int_{-1}^{+1} dx\, \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = \mathrm{erf}\left(\frac{1}{\sqrt{2}}\right)$ for $\sigma = 1$.

(2d) $\cancel{2\pi} \int_0^r r\,dr\, \frac{1}{\cancel{(\sqrt{2\pi})^2}} \exp\left(-\frac{r^2}{2}\right) = \mathrm{erf}\left(\frac{1}{\sqrt{2}}\right)$, no correlation, $\sigma = 1$

substitution $t = \frac{r^2}{2} \rightarrow \frac{dt}{dr} = r$, $dt = r\,dr$

$= \int dt\, \exp(-t)\Big|_0^t = -\exp(-t)\Big|_0^t = 1 - \exp(-t) = 1 - \exp\left(-\frac{r^2}{2}\right)$

solve: $1 - \exp\left(-\frac{r^2}{2}\right) = \mathrm{erf}\left(\frac{1}{\sqrt{2}}\right)$

(3d) climb the ladder by integration by parts.

$\int_0^r \underset{u'}{r\,dr}\, \underset{v}{\exp\left(-\frac{r^2}{2}\right)} = \frac{r^2}{2}\exp\left(-\frac{r^2}{2}\right)\Big|_0^r + \int_0^r dr\, \frac{r^3}{2} \exp\left(-\frac{r^2}{2}\right)$

$\longrightarrow$ for 4d

$\frac{1}{2}\int_0^r \underset{u'}{r^3 dr}\, \underset{v}{\exp\left(-\frac{r^2}{2}\right)} = \frac{r^4}{8}\exp\left(-\frac{r^2}{2}\right)\Big|_0^r + \int_0^r dr\, \frac{r^5}{2}\exp\left(-\frac{r^2}{2}\right)$

$\int_0^r \underset{u'}{r^2 dr}\, \underset{v}{\exp\left(-\frac{r^2}{2}\right)} = \frac{r^3}{3}\exp\left(-\frac{r^2}{2}\right)\Big|_0^r + \int_0^r dr\, \frac{r^4}{3}\exp\left(-\frac{r^2}{2}\right)$

$\longrightarrow$ for 5d.

and $\int_0^r \underset{u'}{1\cdot dr}\, \underset{v}{\exp\left(-\frac{r^2}{2}\right)} = r\exp\left(-\frac{r^2}{2}\right)\Big|_0^r + \int_0^r dr\, r^2 \exp\left(-\frac{r^2}{2}\right)$

$= \sqrt{2\pi}$

$\longrightarrow$ for 3d,

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$$\int r^2 dr\, \exp\left(-\frac{r^2}{2}\right) = \sqrt{2\pi} - r\exp\left(-\frac{r^2}{2}\right)$$

$$4\pi \int_0^r \frac{r^2 dr}{\sqrt{2\pi}} \exp\left(-\frac{r^2}{2}\right) = \mathrm{erf}\left(\frac{1}{\sqrt{2}}\right) \quad \text{with spherical coords.: } d^3x = 4\pi r^2 dr$$

$$\rightarrow \sqrt{8\pi} \cdot \int_0^r r^2 dr\, \exp\left(-\frac{r^2}{2}\right) = \sqrt{8\pi} \cdot \left(\sqrt{2\pi} - r\exp\left(-\frac{r^2}{2}\right)\right) = \mathrm{erf}\left(\frac{1}{\sqrt{2}}\right)$$

3. Hölder's inequality

$$\langle |xy| \rangle \leq \langle |x|^p \rangle^{\frac{1}{p}} \cdot \langle |y|^q \rangle^{\frac{1}{q}} \quad \text{with } \frac{1}{p} + \frac{1}{q} = 1$$

set $x = y$, $p = q = \frac{1}{2}$

$$\langle x^2 \rangle \leq \langle |x|^p \rangle^{\frac{1}{p}} \cdot \langle |x|^p \rangle^{\frac{1}{p}} = \langle |x|^p \rangle^{\frac{2}{p}}$$

$$\rightarrow \sqrt{\langle x^2 \rangle} \leq \langle |x|^p \rangle^{\frac{1}{p}}$$

for a Gaussian even moments $\langle x^{2p} \rangle = (2p-1)!!\, \langle x^2 \rangle^p$

$$\sqrt{\langle x^2 \rangle} = \sigma \leq \left((2p-1)!! \cdot \langle x^2 \rangle^p\right)^{\frac{1}{p}} = \sigma \cdot \underbrace{\left((2p-1)!!\right)^{\frac{1}{p}}}_{\geq 1,\ \forall p.}$$

4. Cauchy-distribution

$$p(x,y) = \frac{1}{(1+x^2+y^2)^{3/2}}$$

always has correlations between x and y because

$$p(x,y) \neq p(x) \cdot p(y) \quad \text{or} \quad \ln p(x,y) \neq \ln p(x) + \ln p(y)$$

$$\int dx\, dy \frac{1}{(1+x^2+y^2)^{3/2}} = 2\pi \int_0^\infty r\, dr \frac{1}{(1+r^2)^{3/2}} =$$

$$= 2\pi \int_0^\infty dr \frac{d}{dr} \frac{1}{\sqrt{1+r^2}} = -2\pi \left.\frac{1}{\sqrt{1+r^2}}\right|_0^\infty = 2\pi \left.\frac{1}{\sqrt{1+r^2}}\right|_\infty^0$$

$$= 2\pi.$$

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marginalisation

$$p_m(x) = \int_{-\infty}^{+\infty} dy \, \frac{1}{(1+x^2+y^2)^{3/2}} =$$

use
$$\frac{d}{dy} \frac{y}{\sqrt{1+x^2+y^2}} = \frac{1}{\sqrt{1+x^2+y^2}} - \frac{y^2}{\sqrt{1+x^2+y^2}^{\,3}} = \frac{1+x^2+y^2-y^2}{\sqrt{1+x^2+y^2}^{\,3}}$$
$$= \frac{1+x^2}{\sqrt{1+x^2+y^2}^{\,3}}.$$

$$\longrightarrow \frac{1}{\sqrt{1+x^2+y^2}^{\,3}} = \frac{1}{1+x^2} \cdot \frac{d}{dy} \frac{y}{\sqrt{1+x^2+y^2}}$$

$$\int_{-\infty}^{+\infty} dy \, \frac{1}{(1+x^2+y^2)^{3/2}} = \frac{1}{1+x^2} \cdot \int_{-\infty}^{+\infty} dy \, \frac{d}{dy} \frac{y}{\sqrt{1+x^2+y^2}}$$
$$= \frac{1}{1+x^2} \left. \frac{y}{\sqrt{1+x^2+y^2}} \right|_{-\infty}^{+\infty} = \frac{2}{1+x^2}.$$

5. Minkowski-inequality

$$\langle (x+y)^p \rangle^{\frac{1}{p}} \leq \langle x^p \rangle^{\frac{1}{p}} + \langle y^p \rangle^{\frac{1}{p}}$$

$y = \alpha x$:

$$\langle (x+y)^p \rangle^{\frac{1}{p}} = \langle (1+\alpha)^p x^p \rangle^{\frac{1}{p}} = (1+\alpha) \langle x^p \rangle^{\frac{1}{p}}$$
$$\leq \langle x^p \rangle^{\frac{1}{p}} + \langle \alpha^p \cdot x^p \rangle^{\frac{1}{p}} = \langle x^p \rangle^{\frac{1}{p}} + \alpha \cdot \langle x^p \rangle^{\frac{1}{p}} = (1+\alpha) \langle x^p \rangle^{\frac{1}{p}}$$

$\rightarrow$ equality

$p = 2$, and square:

$$\langle (x+y)^2 \rangle = \langle x^2 \rangle + 2 \langle xy \rangle + \langle y^2 \rangle \leq$$
$$\langle x^2 \rangle + 2\sqrt{\langle x^2 \rangle \cdot \langle y^2 \rangle} + \langle y^2 \rangle \longrightarrow$$

$$\langle xy \rangle \leq \sqrt{\langle x^2 \rangle \cdot \langle y^2 \rangle}$$