Problem sheet 3

1) <u>Gram - Charlier series</u>

a) $$H_n\left(\frac{x}{\sigma}\right) = (-\sigma)^n e^{\frac{x^2}{2\sigma^2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2\sigma^2}}$$

$$H_1\left(\frac{x}{\sigma}\right) = -\sigma e^{\frac{x^2}{2\sigma^2}}\left(-\frac{2x}{2\sigma^2}\right) e^{-\frac{x^2}{2\sigma^2}} = \frac{x}{\sigma}$$

$$H_2\left(\frac{x}{\sigma}\right) = \sigma^2 e^{\frac{x^2}{2\sigma^2}}\left(-\frac{1}{\sigma^2} + \left(\frac{x}{\sigma^2}\right)^2\right) e^{-\frac{x^2}{2\sigma^2}} = \left(\frac{x}{\sigma}\right)^2 - 1$$

$$H_3\left(\frac{x}{\sigma}\right) = -\sigma^3 e^{\frac{x^2}{2\sigma^2}}\left(\frac{2x}{\sigma^4} - \frac{x}{\sigma^2}\left(-\frac{1}{\sigma^2} + \left(\frac{x}{\sigma^2}\right)^2\right)\right) e^{-\frac{x^2}{2\sigma^2}} = \left(\frac{x}{\sigma}\right)^3 - 3\frac{x}{\sigma}$$

$$H_4\left(\frac{x}{\sigma}\right) = \sigma^4 e^{\frac{x^2}{2\sigma^2}}\left(\frac{3}{\sigma^4} - \frac{3x^2}{\sigma^6} - \frac{x}{\sigma^2}\left(\frac{3x}{\sigma^4} + \left(\frac{x}{\sigma^2}\right)^3\right)\right) e^{-\frac{x^2}{2\sigma^2}} =$$

$$= \left(\frac{x}{\sigma}\right)^4 - 6\left(\frac{x}{\sigma}\right)^2 + 3$$

b) $$P(x) = \int_{-\infty}^{x} p(x')\,dx' = \int_{-\infty}^{x'} dx' \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\left(1 + \frac{K_3}{3!\sigma^6}x'^3 - \frac{3K_3}{3!\sigma^4}x' +\right.\right.$$

$$\left.\left. + \frac{K_4}{4!\sigma^8}x'^4 - \frac{6K_4}{4!\sigma^6}x'^2 + \frac{3K_4}{4!\sigma^4}\right)\right)$$

$= \dots$

$$= \Phi\left(\frac{x}{\sigma}\right) - \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right)\left[\frac{K_3}{3!\sigma^2} H_2\left(\frac{x}{\sigma}\right) + \frac{K_4}{4!\sigma^3} H_3\left(\frac{x}{\sigma}\right)\right]$$

c) $$\frac{d}{dx}p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\frac{d}{dx}\left(e^{-\frac{x^2}{2\sigma^2}}\left(1+\frac{K_3}{3!\sigma^3}H_3\left(\frac{x}{\sigma}\right)+\frac{K_4}{4!\sigma^4}H_4\left(\frac{x}{\sigma}\right)\right)\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\left(-\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\left(1+\frac{K_3}{3!\sigma^3}H_3\left(\frac{x}{\sigma}\right)+\frac{K_4}{4!\sigma^4}H_4\left(\frac{x}{\sigma}\right)\right)+\right.$$

$$\left.+\;e^{-\frac{x^2}{2\sigma^2}}\left(\frac{K_3}{3!\sigma^3}\frac{3}{\sigma}H_2\left(\frac{x}{\sigma}\right)+\frac{K_4}{4!\sigma^4}\frac{4}{\sigma}H_3\left(\frac{x}{\sigma}\right)\right)\right) =$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}\left(-\frac{x}{\sigma^2}-\frac{K_3}{3!\sigma^5}H_3\left(\frac{x}{\sigma}\right)-\frac{K_4}{4!\sigma^6}H_4\left(\frac{x}{\sigma}\right)+\frac{K_3}{2!\sigma^4}H_2\left(\frac{x}{\sigma}\right)+\frac{K_4}{3!\sigma^5}H_3\left(\frac{x}{\sigma}\right)\right) =$$

$$= -\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}\left(\frac{x}{\sigma^2}+\frac{K_3}{3!\sigma^4}\left(\frac{x}{\sigma}H_3\left(\frac{x}{\sigma}\right)-3H_2\left(\frac{x}{\sigma}\right)\right)+\frac{K_4}{4!\sigma^5}\left(\frac{x}{\sigma}H_4\left(\frac{x}{\sigma}\right)-4H_3\left(\frac{x}{\sigma}\right)\right)\right)$$

Consider: $H_5\left(\frac{x}{\sigma}\right)=\left(\frac{x}{\sigma}\right)^5-10\left(\frac{x}{\sigma}\right)^3+15\left(\frac{x}{\sigma}\right)$

so we have that $\left(\frac{x}{\sigma}\right)H_4\left(\frac{x}{\sigma}\right)-4H_3\left(\frac{x}{\sigma}\right)=\left(\frac{x}{\sigma}\right)^5-6\left(\frac{x}{\sigma}\right)^3+3\left(\frac{x}{\sigma}\right)-4\left(\frac{x}{\sigma}\right)^3+12\left(\frac{x}{\sigma}\right)$

$$=\left(\frac{x}{\sigma}\right)^5-10\left(\frac{x}{\sigma}\right)^3+15\left(\frac{x}{\sigma}\right)=H_5\left(\frac{x}{\sigma}\right)$$

and $\left(\frac{x}{\sigma}\right)H_3\left(\frac{x}{\sigma}\right)-3H_2\left(\frac{x}{\sigma}\right)=\left(\frac{x}{\sigma}\right)^4-3\left(\frac{x}{\sigma}\right)^2-3\left(\frac{x}{\sigma}\right)^2+3=H_4\left(\frac{x}{\sigma}\right)$

$$\Rightarrow \boxed{\frac{d}{dx}p(x) = -\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}\left(\frac{x}{\sigma^2}+\frac{K_3}{3!\sigma^4}H_4\left(\frac{x}{\sigma}\right)+\frac{K_4}{4!\sigma^5}H_5\left(\frac{x}{\sigma}\right)\right)}.$$

d) $$\langle x \rangle = \int_{-\infty}^{\infty} dx\, p(x)\, x = 0 + \int_{-\infty}^{\infty} dx\, \frac{K_3}{3!\,\sigma^3}\left(\left(\frac{x}{\sigma}\right)^3 - 3\frac{x}{\sigma}\right) e^{-\frac{x^2}{2\sigma^2}} x \; +$$

$$+ \int_{-\infty}^{\infty} dx\, \frac{K_4}{4!\,\sigma^4}\left(\left(\frac{x}{\sigma}\right)^4 - 6\left(\frac{x}{\sigma}\right)^2 + 3\right) e^{-\frac{x^2}{2\sigma^2}} x =$$

$$= \ldots =$$

$$= 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, p(x)\, x^2 = \sigma^2 + \int_{-\infty}^{\infty} \frac{K_3}{3!\,\sigma^3}\left(\left(\frac{x}{\sigma}\right)^3 - 3\frac{x}{\sigma}\right) e^{-\frac{x^2}{2\sigma^2}} x^2\, dx \; +$$

$$+ \frac{K_4}{4!\,\sigma^4}\int_{-\infty}^{\infty} dx \left(\left(\frac{x}{\sigma}\right)^4 - 6\left(\frac{x}{\sigma}\right)^2 + 3\right) e^{-\frac{x^2}{2\sigma^2}} x^2 =$$

$$= \ldots =$$

$$= \sigma^2$$

$\rightarrow$ NO dependence on K_3 or K_4 for mean and variance

2) Fit of a third order polynomial

$$y(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$$

$$\chi^2 = \sum_{i=1}^{N} \left| y(x_i) - y_i \right|^2 = \sum_{i=1}^{N} \left(p_0 + p_1 x_i + p_2 x_i^2 + p_3 x_i^3 - y_i \right)^2$$

$$\frac{\partial \chi^2}{\partial p_0} = 2 \sum_i \left(p_0 + p_1 x_i + p_2 x_i^2 + p_3 x_i^3 - y_i \right) \cdot 1 \overset{!}{=} 0$$

$\rightarrow$ (division by N) $$\boxed{p_0 + p_1 \langle x_i \rangle + p_2 \langle x_i^2 \rangle + p_3 \langle x_i^3 \rangle = \langle y_i \rangle}$$

$$\frac{\partial \chi^2}{\partial p_1} = 2 \sum_i \left(p_0 + p_1 x_i + p_2 x_i^2 + p_3 x_i^3 - y_i \right) x_i \overset{!}{=} 0$$

$$\rightarrow \boxed{p_0 \langle x_i \rangle + p_1 \langle x_i^2 \rangle + p_2 \langle x_i^3 \rangle + p_3 \langle x_i^4 \rangle = \langle x_i y_i \rangle}$$

and so on,

$$\frac{\partial \chi^2}{\partial p_2} \overset{!}{=} 0 \rightarrow \boxed{p_0 \langle x_i^2 \rangle + p_1 \langle x_i^3 \rangle + p_2 \langle x_i^4 \rangle + p_3 \langle x_i^5 \rangle = \langle x_i^2 y_i \rangle}$$

$$\frac{\partial \chi^2}{\partial p_3} \overset{!}{=} 0 \rightarrow \boxed{p_0 \langle x_i^3 \rangle + p_1 \langle x_i^4 \rangle + p_2 \langle x_i^5 \rangle + p_3 \langle x_i^6 \rangle = \langle x_i^3 y_i \rangle}$$

a) Combine the conditions :

$$\begin{pmatrix} \langle x_i^6 \rangle & \langle x_i^5 \rangle & \langle x_i^4 \rangle & \langle x_i^3 \rangle \\ \langle x_i^5 \rangle & \langle x_i^4 \rangle & \langle x_i^3 \rangle & \langle x_i^2 \rangle \\ \langle x_i^4 \rangle & \langle x_i^3 \rangle & \langle x_i^2 \rangle & \langle x_i \rangle \\ \langle x_i^3 \rangle & \langle x_i^2 \rangle & \langle x_i \rangle & 1 \end{pmatrix} \begin{pmatrix} p_3 \\ p_2 \\ p_1 \\ p_0 \end{pmatrix} = \begin{pmatrix} \langle x_i^3 y_i \rangle \\ \langle x_i^2 y_i \rangle \\ \langle x_i y_i \rangle \\ \langle y_i \rangle \end{pmatrix}$$

b) The condition for the matrix of moments to be invertible is that its determinant is different from 0.
Both this condition and the actual calculation of the inverse can be carried out explicitly (but the calculations are very lengthy) so as to be expressed in terms of $\langle x_i \rangle$ manifestly.