Problems on *statistics and data analysis (MVComp2)*

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Problem sheet 4

To be handed in during the exercise group on 30.May.2016

1. *Hölder's inequality and the divergence of moments* *(10 points)*

Let us start with the following lemma:

Lemma Let $p, q > 1$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{I}$$

Then, for any $a, b \geq 0$ the inequality (Young's inequality)

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab \tag{II}$$

holds, with equality if and only if $a^p = b^q$.

(a) Suppose that X and Y are two random variables. Using the Lemma, prove Hölder's inequality, i.e. that

$$\langle |XY| \rangle \leq \langle |X|^p \rangle^{\frac{1}{p}} \langle |Y|^q \rangle^{\frac{1}{q}} . \tag{III}$$

(b) Prove that $|\langle XY \rangle| \leq \langle |XY| \rangle$. Then use this and (III) to explain the result that you get when you set $p = q = 2$: did you already know this result?

(c) Let us now work for simplicity with variables $X, Y \geq 0$. Write explicitly how (III) becomes for $X = x^r$ (x is a positive random variable and $r > 0$), and $Y = 1$. Then set $s = rp$ and rewrite the inequality in terms of s and r. What does this tell you about the divergence of the moment $\langle x^s \rangle$ with $s \geq r$, if the moment $\langle x^r \rangle$ diverges?

(d) Prove the following inequality

$$\langle |X + Y|^p \rangle^{\frac{1}{p}} \leq \langle |X|^p \rangle^{\frac{1}{p}} + \langle |Y|^p \rangle^{\frac{1}{p}} . \tag{IV}$$

Hint: rewrite $\langle |X + Y|^p \rangle = \langle |X + Y| \, |X + Y|^{p-1} \rangle$ and use the triangle inequality $|X + Y| \leq |X| + |Y|$, together with Hölder's inequality.

2. *Complex Gaussian distribution* *(10 points)*

Suppose we have a complex-valued random variable z with statistically independent real and imaginary parts x and y, each following a Gaussian distribution.

(a) Suppose z has expectation value 0. Define a covariance matrix $C = zz^+$, where the superscript $+$ denotes Hermitian conjugation. Show that C is diagonal if you express it in terms of x and y.

(b) Is the determinant of C always positive? Give an explicit expression for the inverse of C.

3. *Cauchy distribution* *(10 points)*

(a) Show that the characteristic function of a standard Cauchy distribution

$$p(x) = \frac{1}{\pi(1+x^2)}$$

is proportional to $\exp(-|x|t)$.

(b) Using what you found in (a), show that the sum of Cauchy distributed random numbers is again Cauchy distributed.

(c) Does the Chebyshev inequality apply to Cauchy-distributed numbers? If yes, give an example, if not a counterexample. What about the law of large numbers?

4. *Python exercise* *(10 points)*

Write a Python script:

(a) to generate a sample of Gaussian distributed random numbers using the inversion method.

(b) same as in (a), but for an exponential distribution.

In the file `python_tutorial3.py` on Moodle you can find the definitions of the functions needed to numerically solve equations of the type $g(x) = y$ in the unknown x.