Lines and line segments
$x_1, x_2 \in R$
$y = \theta x_1 + (1-\theta)x_2$
$y = x_2 + \theta(x_1 - x_2)$

Affine Sets (C)
$x_1, x_2 \in C, \theta \in R$
$\theta x_1 + (1-\theta)x_2 \in C$

Subspace
$V = C - x_0 = \{x - x_0 | x \in C\}$

Affine Hull of C
Set of all affine combinations of points in C
$\textbf{aff}C = \{\theta_1 x_1 + ... + \theta_k x_k |$
$x_1, ..., x_k \in C, \theta_1 + ... + \theta_k = 1\}$

Relative interior
$\textbf{relint}C = \{x \in C | B(x,r) \cap \textbf{aff}C \subseteq C$
for some r $> 0\}$

Relative boundary = **cl** C**relint** C

Convex Hull
Set of all convex combinations of points in C
$\textbf{conv}C = \{\theta_1 x_1 + ... + \theta_k x_k | x_i \in C, \theta_i \ge 0$
$i = 1, ..., k, \theta_1 x_1 + ... + \theta_k x_k = 1\}$

Cones $x \in C, \theta \ge 0 \Rightarrow \theta x \in C$
Convex Cone
$x_1, x_2 \in C, \theta_1, \theta_2 \ge 0 \Rightarrow \theta_1 x_1 + \theta_2 x_2 \in C$

Conic Hull
Set of all conic combinations of points in C
$\{\theta_1 x_1 + ... + \theta_k x_k | x_i \in C, \theta_i \ge 0, i = 1, ..., k\}$

empty set $\emptyset$, single pt $\{x_0\}$ whole space $\mathbf{R}^n$

Hyperplane
$\{x | a^T x = b\}$ OR $\{x | a^T(x - x_0) = 0\}$
where $a \in \mathbf{R}^n, a \ne 0, b \epsilon \mathbf{R}$

$\{x | a^T(x - x_0) = 0\} = x_0 + a^\perp$
where $a^\perp = \{v | a^T v = 0\}$

Halfspaces
Closed halfspace
$\{x | a^T x \le b\}$ OR $\{x | a^T(x - x_0) \le 0\}$, $a \ne 0$
boundary of halfspace is hyperplane

Open halfspace (interior of halfspace)
$\{x | a^T x < b\}$

Euclidean Ball
x_c is center, r is scalar radius
$B(x_c, r) = \{x | \ ||x - x_c||_2 \le r\}$
$B(x_c, r) = \{x | (x - x_c)^T(x - x_c) \le r^2\}$
$B(x_c, r) = \{x_c + ru | \ ||u||_2 \le 1\}$
where $r > 0$ and $||.||_2 = (u^T u)^{1/2}$

Ellipsoids
$\epsilon = \{x | (x - x_0)^T P^{-1}(x - x_c) \le 1\}$
where $P = P^T \succ 0$
P is symmetric and positive definite
$\epsilon = \{x_c + Au | \ ||u||_2 \le 1\}$
A is square and nonsingular
Ball is and ellipsoid P = r^2I
Degenerate ellipsoid: A is SPSD but singular
Affine dimension(degen ellipsoid) = **rank**(A)

Norm balls and norm cones
x_c is center, r is radius, $||.||$ any norm in $\mathbf{R}^n$
norm ball B = $\{x | \ ||x|| \le r\}$
norm cone C = $\{(x,t) | \ ||x|| \le t\} \subseteq \mathbf{R}^{n+1}$

Polyhedra = {halfspaces $\cap$ hyperplanes}
All affine sets are polyhedra.
polytope: bounded polyhedron
$P = \{x | a_j^T x \le b_j, j = 1, ..., m, c_j^T x = d_j, j =$
$1, ..., p\}$
$P = \{x | Ax \preceq b, Cx = d\}$ where

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \ C = \begin{bmatrix} c_1^T \\ \vdots \\ c_m^T \end{bmatrix}$$

Simplexes
$C = \textbf{conv}\{v_0, ..., v_k\}$
$= \{\theta_0 v_0 + ... + \theta_k v_k | \theta \succeq 0, 1_T \theta = 1\}$

then $x \in C$ iff $x = v_0 + By$

Convex hull description of polyhedra
$\textbf{conv}\{v_1, ..., v_k\} = \textbf{conv}$(finite set)
$= \{\theta_1 v_1 + ... + \theta_k v_k | \theta \succeq 0, 1^T \theta = 1\}$

Generalization of convex hull description
where $m \le k$ $\{\theta_1 v_1 + ... + \theta_k v_k | \ \theta_1 + ... + \theta_m =$
$1, \theta_i \ge 0, i = 1, ..., k\}$

Positive semidefinte cone
Symmetric Ms: $\mathbf{S}^n = \{X \in \mathbf{R}^{n \times n} | X = X^T\}$
SPSDMs: $\mathbf{S}_+^n = \{X \in \mathbf{S}^n | X \succeq 0\}$
SPDMs: $\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n | X \succ 0\}$

if $\theta_1, \theta_2 \ge 0$ and $A, B \in \mathbf{S}_+^n$
then $\theta_1 A + \theta_2 B \in \mathbf{S}_+^n$

Operations that preserve convexity
if S_α is convex for every $\alpha \in A$
then $\cap_{\alpha \in A} S_\alpha$ is convex (Intersection)

Closed convex set S: $\bigcap$of all halfspaces
$S = \bigcap\{\mathcal{H} | \mathcal{H}$ halfspace$, S \subseteq \mathcal{H}\}$

Affine functions
$f : \mathbf{R}^n \to \mathbf{R}^m$ is affine if $f(x) = Ax + b$
where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$

$S \subseteq \mathbf{R}^n$ is convex. Below both also convex
image of S under f $f(S) = \{f(x) | x \in S\}$
inverse image: $f^{-1}(S) = \{x | f(x) \in S\}$

Linear-fractional and perspective functions
The Perspective function P: $\mathbf{R}^{n+1} \to \mathbf{R}^n$
dom $P = \mathbf{R}^n \times \mathbf{R}_{++} \Rightarrow P(z,t) = z/t$

If C $\subseteq$ **dom**P is convex: $P(C) = \{P(x) | x \in C\}$
If C $\subseteq \mathbf{R}^n$ is convex:inverse under P is convex
$P^{-1} = \{(x,t) \in \mathbf{R}^{n+1} | x/t \in C, t > 0\}$

Convex Functions

Basic properties and examples

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if **dom** f is a convex set if
$\forall x, y \in$ **dom** f and $0 \le \theta \le 1$
$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$

Extended-value extensions

if f is convex $\tilde{f} : \mathbf{R}^n \to \mathbf{R} \bigcup \{\infty\}$

$$\tilde{f} = \begin{cases} f(x) x \in \textbf{dom } f \\ \infty \ x \notin \textbf{dom } f \end{cases}$$

dom $f = \{x | \tilde{f}(x) < \infty\} for 0 < \theta < 1$

$\tilde{f}(\theta x + (1-\theta)y) \le \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y)$

First-order conditions

f is differentiable: f is convex iff **dom** f is convex
$f(y) \ge f(x) + \nabla f(x)^T(y - x) \forall x, y \in$ **dom** f
global underestimator:1st-order Taylor approx.
local info: derive global info

Second-order conditions

f is twice differentiable: $\nabla^2 f$ exists
$\nabla^2 f(x) \succeq 0 \ \forall x \in$ **dom** f

f is concave iff **dom**f is convex $\& \nabla^2 f(x) \preceq 0$

Sublevel sets

α-sublevel set of $f : \mathbf{R}^n \to \mathbf{R}$
$C_\alpha = \{x \in$ **dom** $f | f(x) \le \alpha\}$
Sublevel sets of a convex function are convex
Superlevel set of a concave f are convex
$C_\alpha = \{x \in$ **dom** $f | f(x) \ge \alpha\}$
Converse not true

Epigraph

Graph of a function $f : \mathbf{R}^n \to \mathbf{R} \subseteq \mathbf{R}^{n+1}$
Epigraph of a function $f : \mathbf{R}^n \to \mathbf{R} \subseteq \mathbf{R}^{n+1}$
epi $f = \{(x,t) | x \in$ **dom** $f, f(x) \le t\}$

A function is convex iff its epigraph is a convex set. A function is concave iff its hypograph is a convex set.
hypo $f = \{(x,t) | t \le f(x)\}$

$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$

If f is convex, $x_1, ..., x_k \in$ **dom** f
$\theta_1, ..., \theta_k \ge 0 with \theta_1 + ... + \theta_k = 1$
$f(\theta_1 x_1 + ... + \theta_k x_k) \le \theta_1 f(x_1) + ... + \theta_k f(x_k)$

x is random variable $x \in$ **dom** $f$with p=1
f is convex $\Rightarrow f(\mathbf{E}x) \le \mathbf{E} f(x)$
prob$(x = x_1) = \theta$, **prob**$(x = x_2) = 1 - \theta$
f is not convex $\Rightarrow f(\mathbf{E}x) > \mathbf{E} f(x)$

Inequalities

Arithmetic-geometric mean inequality:
$\sqrt{ab} \le (a+b)/2 \ \forall a, b \ge 0$

Operations that preserve convexity

Nonnegative weighted sums
f, f_1, f_2 are convex functions and
Scaling: $\alpha \ge 0 \to \alpha f$ is convex
Addition: $f_1 + f_2$ is convex

Convex cone: set of convex functions (a nonnegative weighted sum of convex functions)
$f = w_1 f_1 + ... + w_m f_m$is convex

Nonnegative weighted sum of:
concave function $\rightarrow$ concave
strictly cvx/ccv functions $\rightarrow$strictly cvx/ccv

$f(x,y)$ is convex in x for each $y \in \mathcal{A}$
$w(y) \ge 0$ for each $y \in \mathcal{A}$
$g(x) = \int_{\mathcal{A}} w(y) f(x,y) dy \to$ is convex in x
image of a convex set under a linear mapping is convex **epi**(wf) = **epi** f

Composition with an affine mapping
$f : \mathbf{R}^n \to \mathbf{R}, A \in \mathbf{R}^{n \times m}$ and $b \in \mathbf{R}^n$
$g(x) = f(Ax + b)$, **dom** $g = \{x | Ax + b \in$ **dom** $f\}$
f: convex/concave $\Rightarrow$ g is convex/concave

Pointwise maximum and supremum
f pointwise maximum of convex functions f_1, f_2
$f(x) = max\{f_1(x), f_2(x)\} \to$ convex
dom f = dom $f_1 \bigcap$dom f_2
f pointwise supremum over infinite set of cvx fns
$g(x) = sup_{y \in \mathcal{A}} f(x,y) \to$ convex
dom $g = \{x | (x,y) \in$ **dom** $f \forall y \in$
$\mathcal{A}, sup_{y \in \mathcal{A}} f(x,y) < \inf\}$
epi $g = \bigcap_{y \in \mathcal{A}}$ **epi** $f(\cdot, y)$

Composition

$h : \mathbf{R}^k \to \mathbf{R}, g : \mathbf{R}^n \to \mathbf{R}^k, f = h \circ g : \mathbf{R}^n \to \mathbf{R}$
$f(x) = h(g(x))$,
dom $f = \{x \in$ **dom** $g | g(x) \in$ **dom** $h\}$

Optimization Problems

minimize $f_0(x)$ subject to $f_i(x) \le 0, i = 1, ..., m$
$h_i(x) = 0, i = 1, ..., p$
$x \in \mathbf{R}^n$ optimization variable
$f_0 : \mathbf{R}_n \to \mathbf{R}$ objective or cost function
$f_i(x) \le 0$ inequality constraints
$f_i : \mathbf{R}^n \to \mathbf{R}$ inequality constraint functions
$h_i(x) = 0$ equality constraints
$h_i : \mathbf{R}^n \to \mathbf{R}$ equality constraint functions
if no constraints $\rightarrow$ unconstrained problem

domain $\mathcal{D} = \bigcap_{i=0}^m$ **dom** $f_i \cap \bigcap_{i=1}^p$ **dom** h_i
$x \in \mathcal{D}$ is feasible if satisfies constraints
problem feasible if $\exists$ one feasible point
feasible set: set of all feasible points

optimal value $p^* = inf\{f_0(x) | f_i(x) \le 0, i =$
$1, ..., m, h_i(x) = 0, i = 1, ..., p\}$

Optimal and locally optimal points
optimal point x*: solves optimization problem
if x* is feasible and $f_0(x^*) = p^*$
optimal set: set of all optimal points
$X_{opt} = \{x | f_i(x) \le 0, i = 1, ..., m, h_i(x) = 0, i =$
$1, ..., p, f_0(x) = p^*\}$

If $\exists$ optimal point then problem is solvable
optimal value is attained or achieved
ϵ-suboptimal point x:feasible point x,
with $f_0(x) \le p^* + \epsilon$

[1] All the formulas listed here are taken from the book → Boyd, S. and Vandenberghe, L., 2004. Convex optimization. Cambridge university press.