THE UNIFORM COMPANION FOR LARGE FIELDS WITH FREE OPERATORS IN CHARACTERISTIC ZERO

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ABSTRACT. Generalising the uniform companion for large fields with a single derivation, we construct a theory $\mathrm{UC}_{\mathcal{D}}$ of fields of characteristic 0 with free operators—operators determined by a homomorphism from the field to its tensor product with $\mathcal{D}$, a finite-dimensional, local $\mathbb{Q}$-algebra—which is the model companion of any theory of a field with free operators whose underlying field is large and model complete. In addition, we show that simplicity is transferred from the theory of the underlying field to the theory of the field with operators, and use this to study the model theory of bounded, PAC fields with free operators.

1. INTRODUCTION

In [32], Tressl showed that there is an inductive theory UC in the language of differential rings with a single derivation, $\mathcal{L}_{\mathrm{ring}}(\delta)$, such that whenever T is a model complete $\mathcal{L}_{\mathrm{ring}}$-theory of large fields of characteristic 0, $T \cup \mathrm{UC}$ is the model companion of the $\mathcal{L}_{\mathrm{ring}}(\delta)$-theory of differential fields whose underlying field is a model of T. He called UC the uniform companion. This brought the theories DCF_0 and CODF into a common framework, and also showed that the theory of p-adically closed fields of fixed p-rank with a derivation and the theory of pseudofinite fields with a derivation both admit a model companion.

The results of this paper generalise the above result of Tressl from the case of a single derivation to that of so-called free operators. Fields with free operators, or $\mathcal{D}$-fields as we will now refer to them, were introduced by Moosa and Scanlon in [25]. In [27] they showed that $\mathcal{D}$-fields are an appropriate framework for unifying the model theory of various classes of algebraically closed fields with operators: derivations, endomorphisms, and others. We give a brief outline of the set-up; see Section 2 for details. Fix a field k of characteristic 0 and a finite-dimensional k-algebra $\mathcal{D}$ with a k-algebra homomorphism $\pi \colon \mathcal{D} \to k$. Then a $\mathcal{D}$-field is a field F which is also a k-algebra equipped with a k-algebra homomorphism $F \to F \otimes_k \mathcal{D}$ which is a section to $\mathrm{id}_F \otimes \pi$. Note that $\delta \colon F \to F$ is a k-linear derivation if and only if $x \mapsto x + \delta(x)\varepsilon$ is a k-algebra homomorphism $F \to F[\varepsilon]/(\varepsilon^2)$, and $\sigma \colon F \to F$ is a k-linear field endomorphism if and only if (somewhat trivially) $x \mapsto (x, \sigma(x))$ is a k-algebra homomorphism $F \to F \times F$. In the case $k = \mathbb{Q}$ we recover the usual notions of derivations and endomorphisms.

The authors of [27] then show that the theory of algebraically closed fields of characteristic 0 with a $\mathcal{D}$-field structure admits a model companion, called $\mathcal{D}$-CF$_0$. Our aim, therefore, is to find a theory UC$_\mathcal{D}$ in the language of $\mathcal{D}$-fields such that whenever T is a model complete theory of large fields, $T \cup$ UC$_\mathcal{D}$ is the model companion of $T \cup$ "$\mathcal{D}$-fields". It will then follow that $\mathcal{D}$-CF$_0 =$ ACF$_0 \cup$ UC$_\mathcal{D}$.

Thus, this paper aims to generalise previous results in two senses. In the first sense, we generalise Tressl's uniform companion from the differential case (in a single derivation) to the free operators case. In the second, the result of Moosa–Scanlon that ACF$_0 \cup$ "$\mathcal{D}$-fields" admits a model companion from the case of ACF$_0$ to that of an arbitrary model complete theory of large fields of characteristic 0.

There is a complication however. When $\mathcal{D} = k \times k$, $\mathcal{D}$-fields correspond to fields with an endomorphism. As Kikyo and Shelah proved in [17], if T is a model complete theory with the strict order property, T_σ has no model companion. In particular, RCF$_\sigma$ and Th$(\mathbb{Q}_p)_\sigma$ have no model companion, and thus one cannot construct a uniform companion for theories of large fields with an endomorphism. In general, a $\mathcal{D}$-field (F, ∂) has associated ring homomorphisms from F to certain finite-dimensional F-algebras (which particular F-algebras depends on $\mathcal{D}$—see Definition 2.3) which may or may not induce endomorphisms $F \to F$. In Section 7 we will see that, when $\mathcal{D}$ is not a local ring and there are induced endomorphisms (in particular if Assumption 2.4 holds), the Kikyo–Shelah theorem implies that, whenever T is a model complete theory with the strict order property, $T \cup$ "$\mathcal{D}$-fields" has no model companion. Hence no uniform companion exists in these instances.

On the other hand, in the case when the algebra $\mathcal{D}$ is a local ring, our main result states that we can find a uniform companion for theories of $\mathcal{D}$-fields, which we call UC$_\mathcal{D}$.

Theorem 1. *Suppose $\mathcal{D}$ is local. Let T be a model complete theory of large fields, and suppose it is the model companion of some T_0. Then*

(i) *$T \cup$ UC$_\mathcal{D}$ is the model companion of $T_0 \cup$ "$\mathcal{D}$-fields";*
(ii) *if T is the model completion of T_0, then $T \cup$ UC$_\mathcal{D}$ is the model completion of $T_0 \cup$ "$\mathcal{D}$-fields";*
(iii) *if T has quantifier elimination, $T \cup$ UC$_\mathcal{D}$ has quantifier elimination.*

This yields the uniform companion in the following cases:

- several (not necessarily commuting) derivations;
- truncated, non-iterative higher derivations;
- operators combining these two.

The existence of the uniform companion will follow from two facts:

(1) Let $M, N \models$ UC$_\mathcal{D}$, and A a common $\mathcal{D}$-subring of them. If M and N have the same existential theory over A as fields, then they do as $\mathcal{D}$-fields.
(2) Every $\mathcal{D}$-field which is large as a field can be extended to a model of UC$_\mathcal{D}$, and this extension is elementary as an extension of fields.

In [32], Tressl establishes his results for differential fields via these two facts. Our proof of these will be more geometric and takes ideas from [27]. Indeed, the axiom scheme UC$_\mathcal{D}$ is very similar to the geometric axiomatisation of $\mathcal{D}$-CF$_0$ in [27].

Remark. In [32], Tressl constructs his uniform companion in the setting of several *commuting* derivations $\delta_1, \ldots, \delta_m$. As pointed out above, here we obtain, as an instance of the general result of this paper, the uniform companion in the case of

several not necessarily commuting derivations. Thus, in the context of derivations, the results of this paper and those of Tressl differ, agreeing only in the case of a *single* derivation. However, the case of not necessarily commuting derivations does appear in a recent paper of Fornasiero and Terzo [10] where they consider generic derivations on algebraically bounded structures—a wider context than the large and model complete fields considered here; see Remark 5.12 for more details.

In Section 4 we prove alternative characterisations of $UC_{\mathcal{D}}$, one still geometric in flavour, and one similar to the notion of differential largeness from [21]—this suggests that the analogous notion of $\mathcal{D}$-largeness may be defined as $UC_{\mathcal{D}} +$ "large", but we do not pursue this here. From these alternative characterisations, we will show that algebraic extensions of large fields which are models of $UC_{\mathcal{D}}$ are again models of $UC_{\mathcal{D}}$ with the unique induced $\mathcal{D}$-structure on the algebraic extension. Hence the algebraic closure of a large field which is a model of $UC_{\mathcal{D}}$ is a model of $\mathcal{D}$-CF_0 from [27]. This gives a class of $\mathcal{D}$-fields with minimal $\mathcal{D}$-closures—though it is as yet unknown if $\mathcal{D}$-closures exist in general. Here by a $\mathcal{D}$-closure we mean a prime model extension in the theory $\mathcal{D}$-CF_0.

The fact that NIP transfers from T to $T \cup UC_{\mathcal{D}}$ is an immediate consequence of the transfer of quantifier elimination. In Section 5 we show the following.

Theorem 2. *Suppose T is a model complete theory of large fields. If T is simple, then $T \cup UC_{\mathcal{D}}$ is simple.*

This is less immediate and requires the notion of slimness from [16] to get a handle on what nonforking independence looks like in the underlying theory of fields. The authors of [16] show that model complete, large fields are very slim, and hence that, in such a field, algebraic independence is an independence relation. From the proof of the Kim–Pillay theorem, we then get that if two $\mathcal{D}$-fields are independent in the sense of nonforking, they are algebraically independent as fields. This fact will allow us to amalgamate independent (in the sense of nonforking) $\mathcal{D}$-fields, and thus prove that if the independence theorem holds in T, it must also hold in $T \cup UC_{\mathcal{D}}$.

In Section 6 we study the PAC substructures in $\mathcal{D}$-CF_0 using the definition from [12] of being existentially closed in every $\mathcal{L}_{\mathrm{ring}}(\partial)$-regular extension (that is, an extension of $\mathcal{D}$-fields $A \leq B$ where $\mathrm{acl}(A) \cap \mathrm{dcl}(B) = \mathrm{dcl}(A)$), and we show that they are characterised as those $\mathcal{D}$-fields which are models of $UC_{\mathcal{D}}$ and PAC as fields. We use this in conjunction with Section 5 to prove simplicity and elimination of imaginaries for the theory of a bounded $\mathcal{D}$-field which is a PAC substructure in $\mathcal{D}$-CF_0, extending the corresponding differential results from Section 5 of [13] to the case of $\mathcal{D}$-fields.

As mentioned above, in Section 7 we prove that when $\mathcal{D}$ is not local and the associated homomorphisms induce endomorphisms, $T \cup$ "$\mathcal{D}$-fields" has no model companion when T is model complete and has the strict order property. As an application we show that, when the base field k is finitely generated over $\mathbb{Q}$ and $\mathcal{D}$ is not local, there is some prime p such that $\mathrm{Th}(\mathbb{Q}_p) \cup$ "$\mathcal{D}$-fields" has no model companion. Thus, when k is finitely generated over $\mathbb{Q}$, the uniform companion exists if and only if $\mathcal{D}$ is local.

Assumptions. All rings are commutative and unital. Ring homomorphisms preserve the unit. All fields are of characteristic 0.

2. Preliminaries

In this section we review the basic definitions of $\mathcal{D}$-fields and large fields and the model-theoretic set-up we will be working in. See [27] and [4] for more details.

2.1. $\mathcal{D}$-**fields.** Fix a base field k of characteristic 0. Let $\mathcal{D}$ be a finite-dimensional k-algebra, and let $\varepsilon_0, \ldots, \varepsilon_l$ be a k-basis of $\mathcal{D}$. Let $\pi \colon \mathcal{D} \to k$ be a k-algebra homomorphism that sends $\varepsilon_0 \mapsto 1$ and $\varepsilon_i \mapsto 0$ for $i = 1, \ldots, l$. If R is a k-algebra, $1 \otimes \varepsilon_0, \ldots, 1 \otimes \varepsilon_l$ is an R-basis of $R \otimes_k \mathcal{D}$. Write $\pi^R \colon R \otimes_k \mathcal{D} \to R$ for the k-algebra homomorphism $\mathrm{id}_R \otimes \pi$.

Definition 2.1. Let R be a k-algebra and $\partial_i \colon R \to R$ a sequence of unary functions on R for $i = 1, \ldots, l$. We say that $(R, \partial_1, \ldots, \partial_l)$ is a $\mathcal{D}$-ring if the map $\partial \colon R \to R \otimes_k \mathcal{D}$ given by

$$r \mapsto r \otimes \varepsilon_0 + \partial_1(r) \otimes \varepsilon_1 + \ldots + \partial_l(r) \otimes \varepsilon_l$$

is a k-algebra homomorphism. Equivalently, we will say that (R, ∂) is a $\mathcal{D}$-ring if $\partial \colon R \to R \otimes_k \mathcal{D}$ is a k-algebra homomorphism such that $\pi^R \circ \partial = \mathrm{id}_R$.

If R is a k-algebra and S is an R-algebra given by $a \colon R \to S$, we say that $\partial \colon R \to S \otimes_k \mathcal{D}$ is a $\mathcal{D}$-operator along $a \colon R \to S$ if it is a k-algebra homomorphism and $\pi^S \circ \partial = a$. Then (R, ∂) is a $\mathcal{D}$-ring if and only if ∂ is a $\mathcal{D}$-operator along id_R.

The ring structure of $\mathcal{D}$ determines the additive and multiplicative rules of the functions ∂_i. Indeed, let $a_{ijk}, b_i \in k$ be the elements defined by $\varepsilon_i \varepsilon_j = \sum_{k=0}^{l} a_{ijk} \varepsilon_k$ and $1_{\mathcal{D}} = \sum_{i=0}^{l} b_i \varepsilon_i$. Then k-linearity of ∂ corresponds to k-linearity of each ∂_i. Multiplicativity of ∂ corresponds to the following "product rule" being satisfied for each k: $\partial_k(rs) = \sum_{i,j=0}^{l} a_{ijk} \partial_i(r) \partial_j(s)$. That ∂ preserves the unit corresponds to the equation $\partial_i(1_R) = b_i$.

Note that being a $\mathcal{D}$-ring imposes no additional relations between the functions ∂_i. For example, commutativity of the operators is not imposed by being a $\mathcal{D}$-ring (though a particular $\mathcal{D}$-ring may indeed have $\partial_i \partial_j = \partial_j \partial_i$).

We can axiomatise the theory of $\mathcal{D}$-rings in the language

$$\mathcal{L}_{\mathrm{ring}}(\partial) = \{+, -, \cdot, 0, 1, (c_a)_{a \in k}, \partial_1, \ldots, \partial_l\},$$

where c_a is a constant symbol for the element $a \in k$.

Example 2.2. (1) Take $\mathcal{D}$ to be the algebra of dual numbers, $k[\varepsilon]/(\varepsilon^2)$, with the standard k-algebra structure and basis $1, \varepsilon$. Then (R, ∂_1) is a $\mathcal{D}$-ring precisely when R is a k-algebra and ∂_1 is a k-linear derivation of R.
  (2) Let $\mathcal{D} = k[\varepsilon_1, \ldots, \varepsilon_l]/(\varepsilon_1, \ldots, \varepsilon_l)^2$ with basis $1, \varepsilon_1, \ldots, \varepsilon_l$. Then $(R, \partial_1, \ldots, \partial_l)$ is a $\mathcal{D}$-ring if R is a k-algebra and each ∂_i is a k-linear derivation of R. As explained before, these derivations will in general be noncommuting.
  (3) Take $\mathcal{D} = k^{l+1}$ with the product k-algebra structure and the standard basis. Then $(R, \partial_1, \ldots, \partial_l)$ is a $\mathcal{D}$-ring if and only if R is a k-algebra and each ∂_i is a k-linear endomorphism of R. These endomorphisms will in general be noncommuting.
  (4) We can combine the above examples. Let $\mathcal{D} = k[\varepsilon]/(\varepsilon^2) \times k$ with basis $(1, 0), (\varepsilon, 0), (0, 1)$. Then a $\mathcal{D}$-ring $(R, \partial_1, \partial_2)$ is a k-algebra with a derivation ∂_1 and an endomorphism ∂_2.

(5) Let $\mathcal{D} = k[\varepsilon]/(\varepsilon^{l+1})$ with basis $1, \varepsilon, \ldots, \varepsilon^l$. Then $\mathcal{D}$-rings are k-algebras with non-iterative, truncated higher derivations $(\partial_1, \ldots, \partial_l)$. That is, they satisfy the following: $\partial_i(xy) = \sum_{r+s=i} \partial_r(x)\partial_s(y)$.

We refer the reader to [27] for more examples.

Since $\mathcal{D}$ is a finite-dimensional k-algebra, it can be written as a finite product of local, finite-dimensional k-algebras $\mathcal{D} = \prod_{i=0}^{t} B_i$. Let $\mathfrak{m}_i$ be the unique maximal ideal of B_i. Then the residue field is a finite field extension of k: $B_i/\mathfrak{m}_i = k[x]/(P_i)$ for some k-irreducible polynomial P_i. We define the k-algebra homomorphisms $\pi_i \colon \mathcal{D} \to k[x]/(P_i)$ by the compositions $\mathcal{D} \to B_i \to k[x]/(P_i)$, and we let $\pi_i^R = \mathrm{id}_R \otimes \pi_i$ be the k-algebra homomorphism $R \otimes_k \mathcal{D} \to R[x]/(P_i)$ for any k-algebra R. Note that the k-algebra homomorphism $\pi \colon \mathcal{D} \to k$ gives a maximal ideal of $\mathcal{D}$ with residue field k. So π must correspond to one of the π_i, say π_0, and B_0 has residue field k.

Definition 2.3. Suppose $\partial \colon R \to S \otimes_k \mathcal{D}$ is a $\mathcal{D}$-operator along $a \colon R \to S$. Composing ∂ and the map π_i^S gives the following k-algebra homomorphism:

$$R \xrightarrow{\ \partial\ } S \otimes_k \mathcal{D} \xrightarrow{\ \pi_i^S\ } S[x]/(P_i).$$

This is called the ith associated homomorphism, σ_i, of ∂.

Now, $\sigma_0 = \pi_0^S \circ \partial = \pi^S \circ \partial = a$ and the associated homomorphism corresponding to B_0 is always a.

Suppose now that (R, ∂) is a $\mathcal{D}$-ring. If $\alpha \in R$ is a root of P_i, we have a map $R[x]/(P_i) \to R$. The composition of σ_i with this map gives an endomorphism of R, $\sigma_{i,\alpha} \colon R \to R$. This endomorphism is α-definable in $\mathcal{L}_{\mathrm{ring}}(\partial)$.

As explained in the introduction, the presence of associated homomorphisms which induce nontrivial endomorphisms (in the sense of the paragraph above) means that the uniform companion does not exist. If there are associated homomorphisms which do not induce nontrivial endomorphisms, then we do not know whether or not the uniform companion exists; we defer a more in-depth discussion of this to Section 7. Much of the following work will be done in the case $\mathcal{D}$ is local, so that the only associated homomorphism, σ_0, is the identity endomorphism.

In [27] the authors make the following assumption throughout.

Assumption 2.4. The residue field of each B_i is k.

As a result of this assumption, all the associated homomorphisms of a $\mathcal{D}$-ring are actually endomorphisms, and much is simplified. All of the main motivating examples fit into this situation. In this paper, we will make the stronger assumption that $\mathcal{D}$ is local (Assumption 3.3) from Section 3 onwards. As mentioned above, in Section 7 we will discuss what happens in the absence of this.

2.2. **Algebraic geometry.** Our axioms for the uniform companion will be geometric in nature. Since we are mainly interested in a model-theoretic analysis, we will mostly consider affine algebraic varieties.

Definition 2.5. A field K is large if every K-irreducible variety with a smooth K-rational point has a Zariski dense set of K-rational points. Equivalently, K is large if it is existentially closed in $K((t))$.

Large fields were first introduced by Pop [30]. Most fields considered model-theoretically "tame" are large. For example, algebraically closed fields, real closed fields, and fields with a nontrivial henselian valuation are large. On the other hand, number fields and function fields are not large.

Large fields play an important role in Tressl's uniform companion for differential fields in [32]. They will play a similar role here. Firstly, we recall some results that we state here for completeness.

Fact 2.6. *Let K be a field, and V a K-irreducible variety. Then V has a Zariski dense set of K-rational points if and only if K is existentially closed in the function field $K(V)$.*

This is well-known—see Fact 2.3 of [31]. The following is then an immediate corollary.

Fact 2.7. *Let K be a large field and V a K-irreducible variety. Then V has a smooth K-rational point if and only if some extension of the function field $K(V)$ is an elementary extension of K.*

2.3. **The prolongation of an affine variety.** For the geometric axioms of the uniform companion, we need the notion of a prolongation of a variety. We will give a similar presentation to [27]; see [25] for a more in-depth description of this object as a Weil restriction of a base change, and Remark 2.9 of [26] for a construction of the prolongation as an adjunction.

Let (K, ∂) be a $\mathcal{D}$-field and X a variety over K. The prolongation of X, τX, is a variety over K with the defining property that for any field extension $L \geq K$, there is an identification $\tau X(L) \leftrightarrow X(L \otimes_k \mathcal{D})$, where X is viewed as a variety over $K \otimes_k \mathcal{D}$ via the base change coming from the map $\partial \colon K \to K \otimes_k \mathcal{D}$. If (L, δ) is a $\mathcal{D}$-field extension of (K, ∂), then the above identification induces a map at the level of L-points $\nabla \colon X(L) \to \tau X(L)$.

If $X = \operatorname{Spec} K[x]/I$ is affine, where x is a tuple of variables, then $\tau X = \operatorname{Spec} K[x^0, \ldots, x^l]/I'$ where I' is constructed as follows (recall that $\dim_k \mathcal{D} = l+1$). For each $f \in I$, let $f^\partial \in (K \otimes_k \mathcal{D})[x]$ be the polynomial obtained by applying ∂ to the coefficients of f. Now compute

$$f^\partial \left(\sum_{j=0}^{l} x^j \varepsilon_j \right) = \sum_{j=0}^{l} f^{(j)}(x^0, \ldots, x^l) \varepsilon_j.$$

Then I' is the ideal generated by all the $f^{(j)}$. With respect to these coordinates, the map $\nabla \colon X(L) \to \tau X(L)$ is given by $\nabla(a) = (a, \delta_1(a), \ldots, \delta_l(a))$.

If Assumption 2.4 holds, the k-algebra homomorphisms $\pi_i \colon \mathcal{D} \to k$ induce morphisms $\hat{\pi}_i \colon \tau X \to X^{\sigma_i}$, where X^{σ_i} is just X base changed via the associated endomorphism $\sigma_i \colon K \to K$. See [25] section 4.1 for a discussion of these morphisms.

If Assumption 2.4 does not hold, but $\alpha \in K$ is a root of P_i, then the endomorphism $\sigma_{i,\alpha}$ induces a morphism $\hat{\pi}_{i,\alpha} \colon \tau X \to X^{\sigma_{i,\alpha}}$. This morphism does not follow from section 4.1 of [25]—see the discussion after Lemma 7.5 of [27]. We will not need these morphisms in this paper, but they do appear in the axiomatisation of $\mathcal{D}\text{-CF}_0$ when Assumption 2.4 does not hold; see Theorem 7.6 of [27].

Example 2.8.   (1) If $\mathcal{D} = k[\varepsilon]/(\varepsilon^2)$, then the prolongation of X is the twisted tangent bundle from [28]. The morphism $\hat{\pi}_0$ is the usual coordinate projection.

(2) If $\mathcal{D} = k \times k$, so that $\mathcal{D}$-rings are precisely rings with a single endomorphism σ, then the prolongation of X is $X \times X^\sigma$ which appears in the geometric axioms for ACFA in [7]. The morphism $\hat{\pi}_0$ is the projection to X, and $\hat{\pi}_1$ is the projection to X^σ.

Remark 2.9. The prolongation of X does not always exist. If the induced map $\operatorname{Spec} \mathcal{D} \to \operatorname{Spec} k$ is a universal homeomorphism, then τX exists for every variety X. If this is not the case, then τX still exists whenever X is quasi-projective. See the discussion after Definition 4.1 of [25]. From Section 3, we will work under the assumption that $\mathcal{D}$ is local. Hence $\operatorname{Spec} \mathcal{D} \to \operatorname{Spec} k$ is a universal homeomorphism, and the prolongation exists for every variety.

3. The uniform companion

Before we discuss the axioms for the uniform companion, we need some results about extending $\mathcal{D}$-ring structures. We carry over the notation from the previous section. In particular, k is a field of characteristic 0, and $\mathcal{D}$ is a finite-dimensional k-algebra. We write the local decomposition of $\mathcal{D}$ as $\prod_{i=0}^{t} B_i$ and the residue field of B_i as $k[x]/(P_i)$.

Lemma 3.1. *Suppose R is a k-algebra, S is an R-algebra given by $a \colon R \to S$, and $\partial \colon R \to S \otimes_k \mathcal{D}$ is a $\mathcal{D}$-operator along a (that is, $\pi^S \circ \partial = a$). Let $\sigma_i \colon R \to S[x]/(P_i)$ be the associated homomorphisms of ∂, $\pi_i^S \circ \partial$. Let $\tau_i \colon S \to S[x]/(P_i)$ be k-algebra homomorphisms extending σ_i. If S is 0-smooth over R, there is an extension of ∂ to a $\mathcal{D}$-ring structure on S with associated homomorphisms τ_i. If S is 0-étale over R, there is a unique such extension.*

Proof. Let $(S[x]/(P_i))^{\sigma_i}$ be the R-algebra which as a ring is just $S[x]/(P_i)$ but whose R-algebra structure is given by $\sigma_i \colon R \to S[x]/(P_i)$.

Let $(S \otimes_k B_i)^\partial$ be the R-algebra which as a ring is just $S \otimes_k B_i$ but whose R-algebra structure is given by

$$R \xrightarrow{\ \partial\ } S \otimes_k \mathcal{D} \longrightarrow S \otimes_k B_i.$$

Similarly, $(S \otimes_k \mathcal{D})^\partial$ has R-algebra structure $\partial \colon R \to S \otimes_k \mathcal{D}$.

Then τ_i is an R-algebra homomorphism $S \to (S[x]/(P_i))^{\sigma_i}$. The quotient map $(S \otimes_k B_i)^\partial \to (S[x]/P_i)^{\sigma_i}$ is a surjective R-algebra homomorphism with nilpotent kernel $S \otimes_k \mathfrak{m}_i$. Since $R \to S$ is 0-smooth (0-étale), there is a (unique) R-algebra homomorphism $S \to (S \otimes_k B_i)^\partial$ whose composition with the quotient map is τ_i. Combining these for each i gives a (unique) R-algebra homomorphism $S \to (S \otimes_k \mathcal{D})^\partial$, that is, a (unique) $\mathcal{D}$-ring structure on S extending ∂ with associated homomorphisms τ_i. ∎

Remark 3.2.   (1) Separable extensions are 0-smooth. Separable algebraic extensions are 0-étale. Localisations are 0-étale. See 25.3 and 26.9 of [22] for details.
(2) When $\mathcal{D}$ is local, this lemma appears as Lemma 2.7 in [4].

As a consequence of Lemma 3.1, the problem of extending $\mathcal{D}$-field structures to field extensions reduces to that of extending the associated homomorphisms. In general, extending endomorphisms in a given class of fields is not always possible; for instance, in the class of real fields. If $\mathcal{D}$ is local however, there are no associated homomorphisms, and so $\mathcal{D}$-field structures can always be extended to a field

extension, and extended uniquely to algebraic extensions (recall our assumption of characteristic 0). These facts will allow us to construct the uniform companion in the local case. We will see in Section 7 a partial converse.

Assumption 3.3. From here until the end of Section 6, we assume $\mathcal{D}$ is local.

Example 3.4. The algebras in (1), (2), and (5) from Example 2.2 are local. We can combine these algebras using fibred products and tensor products to form more local examples. See Examples 3.4 and 3.5 of [27].

Since $\mathcal{D}$ is local, any $\mathcal{D}$-ring R has only one associated homomorphism: the identity id_R. Hence for any affine K-variety X, there is only one induced morphism $\tau X \to X$, which we call $\hat{\pi}$. With respect to the coordinates described in Section 2.3, this is just the morphism induced by the inclusion $K[x]/I \to K[x^0, \ldots, x^l]/I'$ where $x \mapsto x^0$.

We are now able to define the axiom scheme $\mathrm{UC}_{\mathcal{D}}$.

Definition 3.5. Let (K, ∂) be a $\mathcal{D}$-field. We say that (K, ∂) is a model of $\mathrm{UC}_{\mathcal{D}}$ if the following holds:

if X is a K-irreducible affine variety and $Y \subseteq \tau X$ is a K-irreducible affine variety such that $\hat{\pi}(Y)$ is Zariski dense in X, and Y has a smooth K-rational point, then for every nonempty Zariski open set $U \subseteq Y$ over K there exists some $a \in X(K)$ such that $\nabla(a) \in U(K)$.

This axiom scheme can be expressed in a first-order fashion in the language $\mathcal{L}_{\mathrm{ring}}(\partial)$. This is nowadays a standard argument, but we provide some details; we follow the argument used in [32]. We will make use of Theorem 4.2 from there, which collects results about ideals of polynomials from [33].

Theorem (Theorem 4.2 of [32]). *Let $n, d \in \mathbb{N}$, $x = (x_1, \ldots, x_n)$. Then there are bounds $B = B(n,d)$, $C = C(n,d)$, and $E = E(n,d)$ in $\mathbb{N}$ such that for each field K, each ideal I of $K[x]$ generated by polynomials of degree at most d, and all $f_1, \ldots, f_p \in K[x]$ of degree at most d, the following are true.*

   (i) *If I is generated by $f_1, \ldots, f_p$, then for every $g \in I$ of degree at most d, there are $c_1, \ldots, c_p \in K[x]$ of degree at most E such that $g = c_1 f_1 + \ldots + c_p f_p$.*
   (ii) *I is prime if and only if $1 \notin I$ and for all $f, g \in K[x]$ of degree at most B, if $fg \in I$, then $f \in I$ or $g \in I$.*
   (iii) *For all $m \in \{1, \ldots, n\}$, the ideal $I \cap K[x_1, \ldots, x_m]$ is generated by at most C polynomials of degree at most C.*

Let $n, d, m \in \mathbb{N}$, $x = (x_1, \ldots, x_n)$, $f_1, \ldots, f_m \in K[x]$, $g_1, \ldots, g_m, h \in K[x^0, \ldots, x^l]$, all of degree at most d. The polynomials f_i generate the ideal that defines X, I_X, the polynomials g_i generate the ideal that defines Y, I_Y, and the nonvanishing of the polynomial h will define the open subset U. The fact that K-irreducibility of X and Y can be expressed as first-order $\mathcal{L}_{\mathrm{ring}}$-axioms comes from (i) and (ii). That $Y \subseteq \tau X$ can be checked by verifying that the polynomials that define τX are elements of I_Y; that this is a first-order condition follows from (i). Indeed the polynomials that define τX can be computed in terms of the coefficients of the f_i (see the discussion at the end of Section 3 of [27]). That $\hat{\pi} \colon Y \to X$ is dominant says that $I_Y \cap K[x] = I_X$. By (iii) there is a bound, depending on n and d, on the number and degree of the polynomials needed to generate $I_Y \cap K[x]$. That this equality is first-order comes from (i). The existence of a smooth K-point for Y can

be verified by the Jacobian condition on the g_i. Note also that the dimension of Y is the Krull dimension of I_Y, which is definable in terms of the coefficients of the g_i. Finally, that there is a point $\nabla(a)$ in the nonempty Zariski open set U is equivalent to the fact that that the polynomial h either is an element of I_Y, which is first-order by (i), or does not vanish at some $\nabla(a)$.

The fact that the theory $\mathrm{UC}_\mathcal{D}$ is the desired uniform companion will follow from the next two theorems (Theorem 3.6 and Theorem 3.7). These form the $\mathcal{D}$-field analogue of Theorem 6.2 from [32], where the differential counterpart is stated.

Theorem 3.6. *Let* $M, N \models \mathrm{UC}_\mathcal{D}$, *and* A *a common* $\mathcal{D}$-*subring. If* M *and* N *have the same existential theory over* A *as fields, then they do as* $\mathcal{D}$-*fields.*

Proof. Let F_1 and F_2 be the quotient fields of A inside M and N, respectively. By Remark 3.2(1), the $\mathcal{D}$-structure on A extends uniquely to F_1 and F_2, and so they are isomorphic as $\mathcal{D}$-fields. So we may assume $F = F_1 = F_2$ is contained in both M and N. Let L_1 be the relative algebraic closure of F in M, and similarly for L_2 in N. Since M and N have the same existential theory over A as fields, there is a field isomorphism $L_1 \to L_2$ over A. Again by Remark 3.2(1), the $\mathcal{D}$-structure on F extends uniquely so they are isomorphic as $\mathcal{D}$-fields. Then we may also assume that $L = L_1 = L_2$ is contained in both M and N.

Suppose that $M \models \exists \bar{x} \phi(\bar{x})$ where $\phi(\bar{x})$ is a quantifier-free $\mathcal{L}_{\mathrm{ring}}(\partial)$-formula with parameters in A and $\bar{x} = (x_1, \ldots, x_m)$. As usual, we can assume that ϕ is of the form $\bigwedge_{i=1}^n f_i(\bar{x}) = 0$, where the f_i are $\mathcal{L}_{\mathrm{ring}}(\partial)$-terms with coefficients in F. Let c_0 be such that $M \models \phi(c_0)$. Let Ξ be the set of all finite words on $\{\partial_1, \ldots, \partial_l\}$. For each r let Ξ_r be an enumeration of the words of length at most r such that Ξ_r is an initial segment of Ξ_{r+1}, and let $n_r = |\Xi_r|$. Let $\nabla_r \colon M \to M^{n_r}$ be given by $b \mapsto (\xi(b) \colon \xi \in \Xi_r)$. Let r be minimal such that $\phi(M) = \{b \in M^m \colon \nabla_r(b) \in Z\}$ where $Z \subseteq M^{mn_r}$ is a Zariski closed set over F. If $r = 0$, then ϕ is actually an $\mathcal{L}_{\mathrm{ring}}$-formula and since M and N have the same existential theory over A as fields, we have a solution in N. So assume $r > 0$ and let $c = \nabla_{r-1}(c_0)$.

Let $X = \mathrm{loc}(c/L)$, $Y = \mathrm{loc}(\nabla c/L)$. Note that $Y \subseteq \tau X$ and that $\hat{\pi}(Y)$ is Zariski dense in X. Let $g_1, \ldots, g_s$ be polynomials that generate the vanishing ideal of Y over L. By the primitive element theorem, let $\alpha \in L$ be such that $F(\alpha)$ is a field of definition for X and Y. After clearing denominators, we can rewrite the polynomials $g_i(\bar{u})$ instead as $G_i(v, \bar{u}) \in F[v, \bar{u}]$, where $G_i(\alpha, \bar{u}) = g_i(\bar{u})$. Let $\mu(v) \in F[v]$ be the minimal polynomial of α.

We claim that $(\alpha, \nabla c) \in M$ is a smooth solution to the system

(†) $$\mu(v) = 0, G_1(v, \bar{u}) = 0, \ldots, G_s(v, \bar{u}) = 0.$$

As ∇c is L-generic in Y, it must be a smooth solution to the system $g_1(\bar{u}) = 0, \ldots, g_s(\bar{u}) = 0$. Let $J(\nabla c)$ be the Jacobian for $g_1, \ldots, g_s$ at ∇c, and let d be its rank. Since $\mu(v)$ contains none of the $\bar{u}$ variables, the Jacobian of the system (†) at $(\alpha, \nabla c)$ is of the form

$$\begin{pmatrix} \frac{d\mu}{dv}(\alpha) & 0 \\ * & J(\nabla c) \end{pmatrix}.$$

Since μ is the minimal polynomial of α, $\frac{d\mu}{dv}(\alpha) \neq 0$, and hence this matrix has rank $d + 1$. Note also that the variety defined by (†) is a finite union of conjugates of Y, and hence has the same dimension. So $(\alpha, \nabla c)$ is a smooth point of (†).

The system ($\dagger$) is defined over F, and hence we get a smooth solution (β, b) in N (since M and N have the same existential theory as fields over F). Since β is a root of μ, we have an F-isomorphism $\sigma \colon F(\alpha) \to F(\beta)$ with $g_i^\sigma(\bar{u}) = G_i(\beta, \bar{u})$, and so b is a smooth point of Y^σ.

Since Y is L-irreducible, Y^σ is L-irreducible, and since L is relatively algebraically closed in N, Y^σ is N-irreducible. Likewise, X^σ is N-irreducible.

As α and β are algebraic over F, by Remark 3.2 the $\mathcal{D}$-field structure on F extends uniquely to $F(\alpha)$ and $F(\beta)$. Hence σ is a $\mathcal{D}$-field isomorphism.

Proposition 4.8 of [25] tells us that $\tau X^\sigma \simeq \tau X$, and that this isomorphism respects points of the form ∇z. We also have that $Y^\sigma \subseteq \tau X^\sigma$ and that $\hat{\pi}(Y^\sigma)$ is Zariski dense in X^σ. Since σ fixes F it also fixes Z.

Since $N \vDash \mathrm{UC}_\mathcal{D}$, there is a point $a \in X^\sigma(N)$ with $\nabla a \in Y^\sigma(N)$. Let a_0 be the first m coordinates of a. We claim that a_0 is a realisation of ϕ, which will conclude the proof. As in the proof of Theorem 4.5 of [27], we prove first that $\nabla_{r-1}(a_0) = a$. Write $a = (a_\xi : \xi \in \Xi_{r-1})$. We prove by induction on the length of ξ that $\xi(a_0) = a_\xi$. For $\xi = \mathrm{id}$ this is clear. Suppose now that $\xi = \partial_i \xi'$. Since $\nabla_{r-1}(c_0) = c$, we have that $\partial_i c_{\xi'} = c_\xi$. This is an algebraic fact about ∇c over F. Since ∇a satisfies all the algebraic relations ∇c does over F (since σ fixes F), we also have $\partial_i a_{\xi'} = a_\xi$. By the inductive hypothesis, $\partial_i a_{\xi'} = \partial_i \xi'(a_0) = \xi(a_0)$.

Since c_0 realises ϕ, $\nabla_r(c_0) \in Z$. This is an algebraic fact about $\nabla c = \nabla \nabla_{r-1}(c_0)$ over F. Since ∇a satisfies all the algebraic relations ∇c does over F, we have $\nabla_r(a_0) \in Z$. So $N \models \phi(a_0)$. $\blacksquare$

Theorem 3.7. *Every $\mathcal{D}$-field that is large as a field has a $\mathcal{D}$-field extension which is a model of $\mathrm{UC}_\mathcal{D}$ and an elementary extension at the level of fields.*

Proof. Let (F, ∂) be a $\mathcal{D}$-field that is large as a field, and let X, Y be F-irreducible varieties where $Y \subseteq \tau X$, $\hat{\pi}(Y)$ is Zariski dense in X, and Y has a smooth F-rational point. Let U be a nonempty Zariski-open subset of Y defined over F.

Let $b \in Y(L)$ be an F-generic point in some field extension L of F. Since $\hat{\pi}(Y)$ is Zariski dense in X, we get that $\hat{\pi}(b) = a$ is generic in X. As Y has a smooth F-rational point and F is large, by Fact 2.7 some extension K of $F(b)$ is an $\mathcal{L}_{\mathrm{ring}}$-elementary extension of F. We need to define a $\mathcal{D}$-field structure on K such that $\nabla(a) = b$.

As mentioned in Section 2, there is an identification $\tau X(K) \leftrightarrow X(K \otimes_k \mathcal{D})$. Let b' be the tuple from $K \otimes_k \mathcal{D}$ that corresponds to $b \in \tau X(K)$ under this identification.

As in the proof of Theorem 4.5 of [27], we can extend $\partial \colon F \to F \otimes_k \mathcal{D} \subseteq K \otimes_k \mathcal{D}$ to a k-algebra homomorphism $\partial \colon F[a] \to K \otimes_k \mathcal{D}$ with $\partial(a) = b'$. Indeed, since $b \in \tau X(L)$, we have $p^\partial(b') = 0$ for all $p \in I(X/F)$. Since a is F-generic in X, $I(X/F) = I(a/F)$. As $p^\partial(b') = 0$ for all $p \in I(a/F)$, ∂ extends to $F[a] = F[x]/I(a/F)$ by setting $\partial(a) = b'$. Since $\hat{\pi}(b) = a$, we have that $\pi^K(b') = a$ so $\pi^K \circ \partial \colon F[a] \to K$ is inclusion. So ∂ is now a $\mathcal{D}$-operator along inclusion $F[a] \subseteq K$. Now we can use Lemma 3.1 to extend ∂ to a $\mathcal{D}$-ring structure on K extending the one on F with $\nabla(a) = b$. Since b is F-generic in Y, we must have that $b = \nabla a \in U(K)$.

Then we can iterate this construction transfinitely to get an extension of F that is elementary as an extension of fields, which is also a model of $\mathrm{UC}_\mathcal{D}$. $\blacksquare$

Proposition 3.8. *The $\mathcal{L}_{\mathrm{ring}}(\partial)$-theory $\mathrm{UC}_\mathcal{D}$ is inductive. If U is an $\mathcal{L}_{\mathrm{ring}}(\partial)$-theory of large $\mathcal{D}$-fields satisfying the properties in the previous two theorems (Theorems 3.6 and 3.7), then U contains $\mathrm{UC}_\mathcal{D}$. If U is in addition inductive, $U =*

$\mathrm{UC}_\mathcal{D} \cup$ *"large fields", where containment and equality here mean in the sense of deductive closure.*

Proof. It is clear that the union of an increasing chain of models of $\mathrm{UC}_\mathcal{D}$ is also a model of $\mathrm{UC}_\mathcal{D}$, hence $\mathrm{UC}_\mathcal{D}$ is inductive. Using Theorems 3.6 and 3.7, the rest of the argument is the same as in Proposition 6.3 of [32]. ∎

Having now established Theorems 3.6 and 3.7, the following theorem is proved in precisely the same way as its differential counterpart: Theorem 7.1 of [32].

Theorem 3.9. *Let C be a set of new constant symbols and T a model complete theory of large fields in the language $\mathcal{L}_{\mathrm{ring}}(C)$. Let T^* be a theory in a language $\mathcal{L}^* \supseteq \mathcal{L}_{\mathrm{ring}}(C)$ such that $T^* \supseteq T$ and is an extension by definitions of T, that is, the only sentences added to T^* are those defining the new symbols in $\mathcal{L}^*$ as $\mathcal{L}_{\mathrm{ring}}(C)$-formulas. In addition, let A be an $\mathcal{L}^*(\partial)$-structure such that when A is viewed as an $\mathcal{L}_{\mathrm{ring}}(\partial)$-structure it is a $\mathcal{D}$-field.*

If $T^* \cup \mathrm{diag}(A \restriction_{\mathcal{L}^*})$ is complete, then $T^* \cup \mathrm{UC}_\mathcal{D} \cup \mathrm{diag}(A)$ is complete.

In the next theorem we argue why Theorem 3.9 justifies calling $\mathrm{UC}_\mathcal{D}$ the uniform companion for theories of large $\mathcal{D}$-fields. This is the precise formulation of Theorem 1 from the introduction.

Theorem 3.10. *Let C a set of new constant symbols and T a model complete theory of large fields in the language $\mathcal{L}_{\mathrm{ring}}(C)$. Let T^* be an $\mathcal{L}^*$-theory which is an extension by definitions of T.*

Assume T^* is a model companion of an $\mathcal{L}^*$-theory T_0^* which extends the theory of fields. Then:

- (i) *$T^* \cup \mathrm{UC}_\mathcal{D}$ is a model companion of the $\mathcal{L}^*(\partial)$-theory $T_0^* \cup$ "$\mathcal{D}$-fields";*
- (ii) *if T^* is a model completion of T_0^*, then $T^* \cup \mathrm{UC}_\mathcal{D}$ is a model completion of $T_0^* \cup$ "$\mathcal{D}$-fields";*
- (iii) *if T^* has quantifier elimination, then $T^* \cup \mathrm{UC}_\mathcal{D}$ has quantifier elimination;*
- (iv) *if T is complete and M is a $\mathcal{D}$-field which is a model of T, then $T^* \cup \mathrm{UC}_\mathcal{D} \cup \mathrm{diag}(\mathcal{C})$ is complete, where $\mathcal{C}$ is the $\mathcal{L}_{\mathrm{ring}}(C)(\partial)$-substructure generated by $\emptyset$ in M, that is, the $\mathcal{D}$-subring of M generated by the elements $(c^M)_{c \in C}$.*

Proof. First note that $T^* \cup \mathrm{UC}_\mathcal{D}$ and $T_0^* \cup$ "$\mathcal{D}$-fields" have the same universal theory (equivalently, that a model of one can be embedded in a model of the other). Let $M \models T^* \cup \mathrm{UC}_\mathcal{D}$. Since T^* and T_0^* have the same universal theory, there is an $\mathcal{L}^*$-structure N such that $M \restriction_{\mathcal{L}^*} \leq N$. By Remark 3.2, we can extend the $\mathcal{D}$-structure on M to one on N, so that $M \leq N$ as $\mathcal{L}^*(\partial)$-structures. For the other direction, let $M \models T_0^* \cup$ "$\mathcal{D}$-fields". Then there is some $\mathcal{L}^*$-structure $N \models T^*$ such that $M \restriction_{\mathcal{L}^*} \leq N$. Use Remark 3.2 to extend the $\mathcal{D}$-structure on M to one on N, so that $N \models T^* \cup$ "$\mathcal{D}$-fields" and then use Theorem 3.7 to embed this in a model of $T^* \cup \mathrm{UC}_\mathcal{D}$.

To show (i), it is enough to show that $T^* \cup \mathrm{UC}_\mathcal{D}$ is model complete, or equivalently, that if $M \models T^* \cup \mathrm{UC}_\mathcal{D}$, then $T^* \cup \mathrm{UC}_\mathcal{D} \cup \mathrm{diag}(M)$ is complete. Since T^* is model complete, $T^* \cup \mathrm{diag}(M \restriction_{\mathcal{L}^*})$ is complete. Then $T^* \cup \mathrm{UC}_\mathcal{D} \cup \mathrm{diag}(M)$ is complete by Theorem 3.9.

For (ii), let M be a model of $T_0^* \cup$ "$\mathcal{D}$-fields". We need to show that $T^* \cup \mathrm{UC}_\mathcal{D} \cup \mathrm{diag}(M)$ is complete. But $M \vDash T_0^*$, and so $T^* \cup \mathrm{diag}(M \restriction_{\mathcal{L}^*})$ is complete since T^* is a model completion of T_0^*. Then apply Theorem 3.9.

For (iii), let $M \models T^* \cup \mathrm{UC}_\mathcal{D}$, and let $A \leq M$ be an $\mathcal{L}^*(\partial)$-substructure. We need to show that $T^* \cup \mathrm{UC}_\mathcal{D} \cup \mathrm{diag}(A)$ is complete. By quantifier elimination for T^*, we have that $T^* \cup \mathrm{diag}(A \restriction_{\mathcal{L}^*})$ is complete; the result follows by Theorem 3.9.

For (iv), since T is complete, $T^* \cup \mathrm{diag}(\mathcal{C} \restriction_{\mathcal{L}^*}) \subseteq T^*$ is complete. By Theorem 3.9, $T^* \cup \mathrm{UC}_\mathcal{D} \cup \mathrm{diag}(\mathcal{C})$ is complete. ∎

We can now collect the consequences of these theorems, similarly to the differential set-up in Section 8 of [32].

Corollary 3.11. (1) $\mathrm{ACF}_0 \cup \mathrm{UC}_\mathcal{D}$ is $\mathcal{D}$-CF_0 from [27] in the case $\mathcal{D}$ is local. It eliminates quantifiers in $\mathcal{L}_{\mathrm{ring}}(\partial)$.
  (2) $\mathrm{RCF} \cup \mathrm{UC}_\mathcal{D}$ is complete and the model companion of the theory of real closed $\mathcal{D}$-fields. It admits quantifier elimination in $\mathcal{L}_{\mathrm{ring}}(\leq)(\partial)$.
  (3) $p\mathrm{CF}_d \cup \mathrm{UC}_\mathcal{D}$ is complete and the model companion of p-adically closed $\mathcal{D}$-fields of fixed rank d. It has quantifier elimination in $\mathcal{L}_{\mathrm{ring}}(P_n : n \in \mathbb{N})(\partial)$ where P_n is a predicate for the nth powers.
  (4) Let $\mathrm{Psf}_0(C)$ be the $\mathcal{L}_{\mathrm{ring}}(C)$-theory of pseudofinite fields of characteristic 0 with sentences saying that the polynomial $x^n + c_{n,1}x^{n-1} + \ldots + c_{n,n}$ is irreducible for each $n > 1$. Recall that $\mathrm{Psf}_0(C)$ is model complete. We then get that $\mathrm{Psf}_0(C) \cup \mathrm{UC}_\mathcal{D}$ is the model completion of $\mathrm{Psf}_0(C) \cup$ "$\mathcal{D}$-fields".

4. Alternative characterisations of the uniform companion

In this section we will describe some additional characterisations of $\mathrm{UC}_\mathcal{D}$. One in particular will use the notion of a D-variety, and will allow us to show that an algebraic extension of a large field which is a model of $\mathrm{UC}_\mathcal{D}$ is also a model of $\mathrm{UC}_\mathcal{D}$. In particular, the algebraic closure of such a $\mathcal{D}$-field will be a model of $\mathcal{D}$-CF_0 from [27]. Assumption 3.3 remains in place: the k-algebra $\mathcal{D}$ is a local ring.

Definition 4.1. Let (K, ∂) be a $\mathcal{D}$-ring. A D-variety over (K, ∂) is a pair (V, s) where V is a variety over K and $s : V \to \tau V$ is an algebraic morphism over K which is a section to the canonical projection $\hat{\pi} : \tau V \to V$. We say that (V, s) is K-irreducible if V is K-irreducible, affine if V is affine, etc.

Given a $\mathcal{D}$-field extension (L, δ) of (K, ∂), the (L, δ)-rational sharp points of (V, s) are defined as $(V, s)^\sharp(L, \delta) = \{a \in V(L) : \nabla a = s(a)\}$.

As before, we will mainly be interested in affine D-varieties. If V is an affine variety, a D-variety structure on V is equivalent to a $\mathcal{D}$-ring structure on its coordinate ring, $K[V]$. A K-rational sharp point is equivalent to a $\mathcal{D}$-ring homomorphism $K[V] \to K$. This is the natural $\mathcal{D}$-field analogue of D-varieties as defined for differential rings; see for example [19].

We now establish some basic results about D-varieties. For the following, if (R, ∂) is a $\mathcal{D}$-ring and $\mathfrak{a}$ is an ideal of R, $\mathfrak{a}$ is called a $\mathcal{D}$-ideal if $\partial(\mathfrak{a}) \subseteq \mathfrak{a} \otimes_k \mathcal{D}$, or equivalently, if $\partial_i(\mathfrak{a}) \subseteq \mathfrak{a}$ for each $i = 1, \ldots, l$.

Lemma 4.2. Let (R, ∂) be a $\mathcal{D}$-ring, and $\mathfrak{a} \subseteq R$ a radical $\mathcal{D}$-ideal. Then the minimal prime ideals above $\mathfrak{a}$ are $\mathcal{D}$-ideals.

Proof. Let $\mathfrak{p}$ be a minimal prime ideal above $\mathfrak{a}$, and consider the localisation $R_\mathfrak{p}$. Since $\mathfrak{a}$ is radical, so is $\mathfrak{a}R_\mathfrak{p}$ (see Proposition 3.11 of [3]). Suppose $\mathfrak{q} \subseteq \mathfrak{p}R_\mathfrak{p}$ is a prime ideal of $R_\mathfrak{p}$ that also lies above $\mathfrak{a}R_\mathfrak{p}$. Then by part iv) of the same proposition, we must have $\mathfrak{q} = \mathfrak{p}R_\mathfrak{p}$, and hence $\mathfrak{p}R_\mathfrak{p}$ is a minimal prime above $\mathfrak{a}R_\mathfrak{p}$. It is also the

unique maximal ideal of $R_{\mathfrak{p}}$, and hence is the only prime ideal lying above $\mathfrak{a}R_{\mathfrak{p}}$. Then $\mathfrak{a}R_{\mathfrak{p}} = \sqrt{\mathfrak{a}R_{\mathfrak{p}}} = \mathfrak{p}R_{\mathfrak{p}}$ (the radical of an ideal is the intersection of the prime ideals lying above it).

By Remark 3.2 we know that ∂ extends uniquely to a $\mathcal{D}$-structure on $R_{\mathfrak{p}}$ with $\partial(\frac{a}{b}) = \partial(a)\partial(b)^{-1}$. Since $\mathfrak{a}$ is a $\mathcal{D}$-ideal it is clear that $\mathfrak{a}R_{\mathfrak{p}}$ is also a $\mathcal{D}$-ideal.

Then $\mathfrak{p}R_{\mathfrak{p}}$ is a $\mathcal{D}$-ideal, and hence its contraction to R, $\mathfrak{p}$, is also a $\mathcal{D}$-ideal. ∎

Lemma 4.3. *Let (V, s) be an affine D-variety over (K, ∂). Then*

    a) *any nonempty Zariski-open $U \subseteq V$ defined over K is a D-subvariety of (V, s);*

    b) *any K-irreducible component of V is a D-subvariety of (V, s).*

Proof. a) Let $K[V]$ be the coordinate ring of V. Then s corresponds to $\partial_s \colon K[V] \to K[V] \otimes_k \mathcal{D}$. Let U be a basic open subset of V given by the nonvanishing of some f. By Remark 3.2 we then get that ∂_s extends uniquely to $K[V]_f \to K[V]_f \otimes_k \mathcal{D}$. That is, s restricts to $U \to \tau U$. Now if $U = \bigcup_{i \in I} U_i$ is a union of basic open subsets, s restricts to $U_i \to \tau U_i \subseteq \tau U$, and these restrictions agree on $U_i \cap U_j$ since this is also a basic open. Glueing the morphisms $U_i \to \tau U$ gives a morphism $U \to \tau U$ which is a restriction of s.

b) by Lemma 4.2. ∎

Theorem 4.4. *Suppose (K, ∂) is a $\mathcal{D}$-field and K is large. Then the following are equivalent:*

    (1) *$K \vDash \mathrm{UC}_{\mathcal{D}}$;*

    (2) *whenever (V, s) is an affine, K-irreducible D-variety, if V has a smooth K-rational point, then the set of K-rational sharp points of (V, s) is Zariski dense in V;*

    (3) *whenever (V, s) is an affine, K-irreducible D-variety, if V has a smooth K-rational point, then (V, s) has a K-rational sharp point;*

    (4) *whenever (V, s) is a smooth, affine, K-irreducible D-variety, if V has a K-rational point, then (V, s) has a K-rational sharp point;*

    (5) *whenever (L, δ) is a $\mathcal{D}$-field extension of (K, ∂) such that K is existentially closed in L as a field, then (K, ∂) is existentially closed in (L, δ) as a $\mathcal{D}$-field.*

Proof. (1) $\implies$ (2): Suppose $(K, \partial) \vDash \mathrm{UC}_{\mathcal{D}}$ and let (V, s) be a K-irreducible D-variety with a smooth K-rational point. Let $X = V$ and $Y = s(V)$. Note that X and Y are isomorphic. Then Y has a smooth K-rational point, $Y \subseteq \tau X$, and $\hat{\pi} \colon Y \to X$ is an isomorphism. So, since $K \models \mathrm{UC}_{\mathcal{D}}$, Y has a Zariski dense set of K-rational points of the form $\nabla(a)$ for $a \in X(K)$, and hence for each such $a \in X(K)$, $\nabla(a) = s(a)$.

(2) $\implies$ (3) is clear.

(3) $\implies$ (1): Let X, Y, U be as in the statement of $\mathrm{UC}_{\mathcal{D}}$. Let $b \in L \geq K$ be a K-generic point of Y, so that $a = \hat{\pi}(b) \in L$ is K-generic in X by dominance. Since $b \in \tau X(K(b))$, let $b' \in K(b) \otimes_k \mathcal{D}$ be the point corresponding to b under the correspondence $\tau X(K(b)) \leftrightarrow X(K(b) \otimes_k \mathcal{D})$. Then $P^{\partial}(b') = 0$ for all $P \in I(X/K)$, and so ∂ extends to a homomorphism $\partial \colon K[a] \to K(b) \otimes_k \mathcal{D}$ with $\partial(a) = b'$. Extend this to a $\mathcal{D}$-ring structure $\partial \colon K(b) \to K(b) \otimes_k \mathcal{D}$ using Lemma 3.1. In this $\mathcal{D}$-ring structure, $\nabla(a) = b$. Now each $\partial_i(b_j) \in K(b)$ so $\partial_i(b_j) = \frac{P_{ij}(b)}{Q_{ij}(b)}$ for some polynomials $P_{ij}, Q_{ij} \in K[x]$. Let $Q \in K[x]$ be the product of all Q_{ij}. Note that

∂ restricts to $K[b] \to K[b]_{Q(b)} \otimes_k \mathcal{D}$. Again by Lemma 3.1, we must have that ∂ extends to $K[b]_{Q(b)} \to K[b]_{Q(b)} \otimes_k \mathcal{D}$. Let U' be the open subset of Y corresponding to $Q(x)$. This extension of ∂ gives a D-variety structure $s \colon U' \to \tau U'$.

Since K is large and V has a smooth K-point, $U \cap U'$ has a smooth K-point. By Lemma 4.3, $(U \cap U', s|_{U \cap U'})$ is a K-irreducible D-variety with a smooth K-rational point. By (3) there is $(c, d_1, \ldots, d_l) \in (U \cap U')(K)$ with $\nabla(c, d_1, \ldots, d_l) = s(c, d_1, \ldots, d_l)$. Then $c \in X(K)$ with $\nabla(c) = (c, d_1, \ldots, d_l) \in U(K)$.

(3) $\implies$ (4) is clear.

(4) $\implies$ (3): Let (V, s) be a D-variety over K with V K-irreducible and $a \in V(K)$ a smooth K-rational point. Let $W \subseteq V$ be the smooth locus of V. Then W is a smooth, K-irreducible D-subvariety of V. The point a is a K-rational point of W and so by (4), W has a K-rational sharp point. Then V has a K-rational sharp point.

(1) $\implies$ (5): Let (L, δ) be a $\mathcal{D}$-field extension of $(K, \partial) \models \mathrm{UC}_{\mathcal{D}}$ such that K is existentially closed in L as a field. Then there is a field extension $L \leq M$ such that M is an elementary extension of K as a field; note that M is then also a large field. Extend the $\mathcal{D}$-field structure on L to one on M, and use Theorem 3.7 to find a $\mathcal{D}$-field extension $(N, d) \models \mathrm{UC}_{\mathcal{D}}$ such that $K \prec M \prec N$ as fields. This last fact implies that K and N have the same existential theory as fields over K. So by Theorem 3.6, they have the same existential theory as $\mathcal{D}$-fields over (K, ∂). Then (K, ∂) is existentially closed in (N, d), and hence in (L, δ).

(5) $\implies$ (1): Assume (K, ∂) has property (5). By Theorem 3.7, there is $(L, \delta) \models \mathrm{UC}_{\mathcal{D}}$ extending it such that $K \prec L$ as fields. Then K is existentially closed in L as fields, and so (K, ∂) is existentially closed in (L, δ) as $\mathcal{D}$-fields by (5). Since $\mathrm{UC}_{\mathcal{D}}$ is inductive, we must also have $(K, \partial) \models \mathrm{UC}_{\mathcal{D}}$. $\blacksquare$

Remark 4.5. As mentioned in the introduction, condition (5) of the previous theorem suggests that, in analogy with the notion of differential largeness from [21], one may define $\mathcal{D}$-largeness as $\mathrm{UC}_{\mathcal{D}}$ + "large". This definition has not been made though. This is mainly due to the fact that the $\mathcal{D}$-field analogues of the properties of differential largeness appearing in Sections 4 and 5 of [21] (for instance, that K is differentially large exactly when it is existentially closed in $K((t))$ with the natural derivations) have not yet been established. This line of work has been left for possible future research.

We will now show that algebraic extensions of large models of $\mathrm{UC}_{\mathcal{D}}$ are again large and models of $\mathrm{UC}_{\mathcal{D}}$. Similar to the differential case (Theorem 5.11 of [21]), this will rely on the $\mathcal{D}$-Weil descent, established in [24].

We recall some of the properties of the $\mathcal{D}$-Weil descent. Let $(L, \delta)/(K, \partial)$ be an extension of $\mathcal{D}$-fields where L/K is a finite field extension. Let (V, s) be an affine D-variety over (L, δ); as mentioned above, this is equivalent to a $\mathcal{D}$-ring structure, δ^s, on the coordinate ring, $L[V]$, extending δ on L. The classical Weil descent of V, V^W, is a K-variety such that there is a natural bijection

$$V(L) \leftrightarrow V^W(K).$$

Stated algebraically, this is equivalent to the natural bijection

$$\mathrm{Hom}_L(L[V], L) \leftrightarrow \mathrm{Hom}_K(K[V^W], K).$$

In [24] it is shown that there is a unique $\mathcal{D}$-ring structure, ∂^s, on $K[V^W]$ extending ∂ on K such that the above natural bijection restricts to a natural bijection

$$\mathrm{Hom}_{(L,\delta)}((L[V],\delta^s),(L,\delta)) \leftrightarrow \mathrm{Hom}_{(K,\partial)}((K[V^W],\partial^s),(K,\partial)).$$

The $\mathcal{D}$-ring structure ∂^s corresponds to $s^W: V^W \to \tau(V^W)$ and makes (V^W, s^W) into a D-variety over (K,∂). As mentioned above, a $\mathcal{D}$-ring homomorphism $L[V] \to L$ corresponds to an L-rational sharp point of (V,s). Geometrically then, we have that the first natural bijection restricts to the natural bijection

$$(V,s)^\sharp(L,\delta) \leftrightarrow (V^W, s^W)^\sharp(K,\partial).$$

Theorem 4.6. *Let $(L,\delta)/(K,\partial)$ be an algebraic extension of $\mathcal{D}$-fields where $(K,\partial) \models \mathrm{UC}_\mathcal{D}$ and K is a large field. Then $(L,\delta) \models \mathrm{UC}_\mathcal{D}$ and L is large.*

Proof. Consider first the case when L/K is a finite extension. We verify condition (4) of Theorem 4.4. Let (V,s) be a smooth, L-irreducible D-variety defined over (L,δ) with an L-rational point. Now apply the $\mathcal{D}$-Weil descent to get a D-variety (V^W, s^W) over (K,∂). Since V is affine and smooth, V^W is affine and smooth (see Proposition 5 of Section 7.6 of [5]). By the bijection $V(L) \leftrightarrow V^W(K)$, V^W has a K-rational point. Let (U,t) be the irreducible component of (V^W, s^W) containing the K-rational point. Since (K,∂) satisfies condition (4), (U,t) has a K-rational sharp point, and hence (V^W, s^W) has a K-rational sharp point. By the bijection $(V,s)^\sharp(L,\delta) \leftrightarrow (V^W, s^W)^\sharp(K,\partial)$, (V,s) has an L-rational sharp point.

If L/K is algebraic, let F be an intermediate extension such that V, s, and the L-rational point are all defined over F and F/K is finite. Then by the above, $(V,s)^\sharp(F,\delta) \neq \emptyset$, and hence $(V,s)^\sharp(L,\delta) \neq \emptyset$. ∎

5. Transfer of neo-stability properties

We continue to work under Assumption 3.3. In Section 3 we saw that if T is a model complete $\mathcal{L}_{\mathrm{ring}}(C)$-theory of large fields and T^* is an expansion by definitions of T with quantifier elimination in a language $\mathcal{L}^*$, then the model companion of $T^* \cup$ "$\mathcal{D}$-fields", namely $T^* \cup \mathrm{UC}_\mathcal{D}$, also has quantifier elimination in the language $\mathcal{L}^*(\partial)$. An immediate consequence of this fact is the following.

Corollary 5.1. *Suppose T is the complete, model complete $\mathcal{L}_{\mathrm{ring}}(C)$-theory of a large field. If T is NIP, so is (any completion of) $T \cup \mathrm{UC}_\mathcal{D}$.*

Proof. Let T^* be an expansion by definitions of T with quantifier elimination where the $\mathcal{L}^*$-terms are the same as the $\mathcal{L}_{\mathrm{ring}}(C)$-terms (for instance, if T^* is the Morleyisation of T). Let $\mathfrak{C}$ be a monster model of $T^* \cup \mathrm{UC}_\mathcal{D}$.

Suppose $\phi(x,y)$ is an $\mathcal{L}^*(\partial)$-formula with IP: so there are $(a_i)_{i\in\omega}$, $(b_I)_{I\subseteq\omega}$ in $\mathfrak{C}$ with $\mathfrak{C} \models \phi(a_i, b_I) \iff i \in I$. By Theorem 3.10, $T^* \cup \mathrm{UC}_\mathcal{D}$ has quantifier elimination, and we may assume $\phi(x,y)$ is quantifier-free. Now, since the $\mathcal{L}^*$-terms are the same as the $\mathcal{L}_{\mathrm{ring}}(C)$-terms, the $\mathcal{L}^*(\partial)$-terms in the variables x,y are then just polynomials in the variables $\{\nabla_r(x), \nabla_r(y), \nabla_r(c): r \in \mathbb{N}, c \in C\}$. So there are $r \in \mathbb{N}$ and a quantifier-free $\mathcal{L}^*$-formula ϕ^* such that $\mathfrak{C} \models \phi(x,y)$ if and only if $\mathfrak{C} \models \phi^*(\nabla_r(x), \nabla_r(y))$. Then

$$\mathfrak{C} \models \phi^*(\nabla_r(a_i), \nabla_r(b_I)) \iff \mathfrak{C} \models \phi(a_i, b_I) \iff i \in I.$$

Therefore the tuples $(\nabla_r(a_i))_{i\in\omega}$ and $(\nabla_r(b_I))_{I\subseteq\omega}$ witness that ϕ^* has IP. ∎

Remark 5.2.   (1) This result generalises the fact of Michaux and Rivière that
CODF is NIP from Theorem 2.2 of [23].
    (2) In Corollary 4.3 of [11], Guzy and Point show that NIP is transferred from
    a topological field (possibly with extra structure) to the model companion
    of the field with a derivation. The imposition of a topological structure
    allows them to consider fields with genuine extra structure, as opposed to
    the definitional expansions considered here.

A similar argument to Corollary 5.1 shows that stability transfers via its char-
acterisation of no formula having the order property. But in fact stability yields
something stronger.

Lemma 5.3. *Suppose T is the complete, model complete $\mathcal{L}_{\mathrm{ring}}(C)$-theory of a large
field. If T is stable, then $T \cup \mathrm{UC}_{\mathcal{D}} = \mathcal{D}\text{-}\mathrm{CF}_0$.*

Proof. A stable, large field of characteristic 0 is algebraically closed by Theorem D
of [14]. The result then follows as $\mathrm{ACF}_0 \cup \mathrm{UC}_{\mathcal{D}} = \mathcal{D}\text{-}\mathrm{CF}_0$.   ∎

We will now prove a similar result to Corollary 5.1 for the transfer of simplicity.
The proof will be via the Kim–Pillay theorem [18] and hence we need to understand
nonforking independence in T. We introduce the notion of very $\mathcal{L}$-slim which is a
slight modification of the definition of very slim of Junker–Koenigsmann from [16].
First, a restriction on the language we work with and some notation we will need.

Assumption 5.4. Let C be a set of constant symbols. For the remainder of this
section, we suppose that $\mathcal{L} = \mathcal{L}_{\mathrm{ring}}(C)$.

Let K be a $\mathcal{D}$-field in the language $\mathcal{L}(\partial)$. For $A \subseteq K$, the field generated by A
and C inside K is denoted by $\langle A \rangle_{\mathcal{L}}$ (that is, the quotient field of the $\mathcal{L}$-structure
generated by A). Similarly, the $\mathcal{D}$-field generated by A and C inside K is denoted
by $\langle A \rangle_{\mathcal{L}(\partial)}$. Note then that $\langle XY \rangle_{\mathcal{L}} = \langle X \rangle_{\mathcal{L}} \cdot \langle Y \rangle_{\mathcal{L}}$, where $\cdot$ denotes the compositum
in the sense of fields. By the multiplicative rules for $\mathcal{D}$-fields this implies that
$\langle XY \rangle_{\mathcal{L}(\partial)} = \langle X \rangle_{\mathcal{L}(\partial)} \cdot \langle Y \rangle_{\mathcal{L}(\partial)}$.

Definition 5.5. Let K be a field in the language $\mathcal{L}$. We say that K is $\mathcal{L}$-slim if for
every $\mathcal{L}$-substructure F, we have $\mathrm{acl}_{\mathcal{L}}^{K}(F) = F^{\mathrm{alg}}$. Equivalently, if for every subset
A, we have $\mathrm{acl}_{\mathcal{L}}^{K}(A) = \langle A \rangle_{\mathcal{L}}^{\mathrm{alg}}$. By F^{alg} we always mean the relative, field-theoretic
algebraic closure of F in K.

We say that K is very $\mathcal{L}$-slim if every $\mathcal{L}$-structure elementarily equivalent to K
is $\mathcal{L}$-slim.

Remark 5.6.   (1) We recover the definition of (very) slim from [16] by setting
C to be empty, and only considering fields in the language of rings. In
other words, a field with no extra structure is (very) slim exactly when it
is (very) $\mathcal{L}_{\mathrm{ring}}$-slim.
    (2) Let $\mathcal{C}$ be the field generated by the constants C inside K. Then by Lemma 2.12
    of [15], K is very $\mathcal{L}$-slim if and only if it is algebraically bounded over $\mathcal{C}$.

As mentioned in [16] for slim fields, to check whether K is very $\mathcal{L}$-slim it is
enough to check whether a sufficiently saturated model of its theory is $\mathcal{L}$-slim.

We take the following definition from Kim–Pillay [18], though phrased in the
terminology of Adler [2].

Definition 5.7. Let $\mathfrak{C}$ be a saturated and strongly homogeneous structure. A relation $\underset{}{\overset{*}{\smile}}$ on triples of small subsets of $\mathfrak{C}$ is called an independence relation if it is invariant under automorphisms and satisfies:

(1) *normality:* $X \underset{A}{\overset{*}{\smile}} B \implies X \underset{A}{\overset{*}{\smile}} AB$;
(2) *monotonicity:* $X \underset{A}{\overset{*}{\smile}} B \implies X \underset{A}{\overset{*}{\smile}} B'$ for $B' \subseteq B$;
(3) *base monotonicity:* $X \underset{A}{\overset{*}{\smile}} D \implies X \underset{B}{\overset{*}{\smile}} D$ for $A \subseteq B \subseteq D$;
(4) *transitivity:* $X \underset{A}{\overset{*}{\smile}} B$ and $X \underset{B}{\overset{*}{\smile}} D \implies X \underset{A}{\overset{*}{\smile}} D$ for $A \subseteq B \subseteq D$;
(5) *symmetry:* $X \underset{A}{\overset{*}{\smile}} B \iff B \underset{A}{\overset{*}{\smile}} X$;
(6) *full existence:* for any X, A, B there is $X' \equiv_A X$ such that $X' \underset{A}{\overset{*}{\smile}} B$ (here and throughout $X' \equiv_A X$ means that X' and X have the same type over A);
(7) *finite character:* if $X_0 \underset{A}{\overset{*}{\smile}} B$ for all finite $X_0 \subseteq X$, then $X \underset{A}{\overset{*}{\smile}} B$;
(8) *local character:* there is a cardinal κ such that for all X and A, there is $A_0 \subseteq A$ with $|A_0| < \kappa$ such that $X \underset{A_0}{\overset{*}{\smile}} A$.

An independence relation $\underset{}{\overset{*}{\smile}}$ satisfies the independence theorem over M if the following holds:

(9) *independence theorem over M:* if $A_1 \underset{M}{\overset{*}{\smile}} A_2$, $a_1 \underset{M}{\overset{*}{\smile}} A_1$, $a_2 \underset{M}{\overset{*}{\smile}} A_2$, and $a_1 \equiv_M a_2$, then there is $a \models \mathrm{tp}(a_1/MA_1) \cup \mathrm{tp}(a_2/MA_2)$ with $a \underset{M}{\overset{*}{\smile}} A_1 A_2$.

In simple theories, nonforking independence is an independence relation that satisfies the independence theorem over models. As shown in [16], algebraic independence in very slim fields is an independence relation—in general, algebraic independence does not satisfy full existence (called existence in [16]).

We recall the definition of algebraic independence in fields. If $\mathbb{U}$ is a big field and A, B, D are subfields with $D \leq A, B$, we say A and B are algebraically independent over D if every finite tuple from A which is algebraically independent over D is also algebraically independent over B. We write $A \underset{D}{\overset{\mathrm{alg}}{\smile}} B$ if this holds.

If A and B are linearly disjoint over D, then they are algebraically independent. The converse holds if at least one of the extensions A/D or B/D is regular, or equivalently, since we are in characteristic 0, relatively algebraically closed.

Definition 5.8. For an $\mathcal{L}$-structure K, define the following relation on triples of subsets of K:

$$A \underset{D}{\overset{\mathcal{L}}{\smile}} B \iff \langle A \rangle_{\mathcal{L}} \underset{\langle D \rangle_{\mathcal{L}}}{\overset{\mathrm{alg}}{\smile}} \langle B \rangle_{\mathcal{L}}.$$

We say that A and B are $\mathcal{L}$-algebraically independent over D.

Theorem 2.1 of [16] says that a field is very slim if and only if algebraic independence is an independence relation. The following result is the very $\mathcal{L}$-slim analogue.

Lemma 5.9. *K is very $\mathcal{L}$-slim if and only if $\underset{}{\overset{\mathcal{L}}{\smile}}$ is an independence relation.*

Proof. Invariance, monotonicity, base monotonicity, symmetry, and transitivity are clear. The fact that $\langle XY \rangle_{\mathcal{L}} = \langle X \rangle_{\mathcal{L}} \cdot \langle Y \rangle_{\mathcal{L}}$ implies normality. Indeed, $A \underset{D}{\overset{\mathcal{L}}{\smile}} B \implies \langle A \rangle_{\mathcal{L}} \underset{\langle D \rangle_{\mathcal{L}}}{\overset{\mathrm{alg}}{\smile}} \langle B \rangle_{\mathcal{L}}$. Then by normality for $\underset{}{\overset{\mathrm{alg}}{\smile}}$ we have $\langle A \rangle_{\mathcal{L}} \underset{\langle D \rangle_{\mathcal{L}}}{\overset{\mathrm{alg}}{\smile}} \langle B \rangle_{\mathcal{L}} \langle D \rangle_{\mathcal{L}}$, so $\langle A \rangle_{\mathcal{L}} \underset{\langle D \rangle_{\mathcal{L}}}{\overset{\mathrm{alg}}{\smile}} \langle BD \rangle_{\mathcal{L}}$ and $A \underset{D}{\overset{\mathcal{L}}{\smile}} BD$. Finite and local character also follow from the corresponding property for algebraic independence.

Now the same proof as in Theorem 2.1 of [16] shows that K is very $\mathcal{L}$-slim if and only if algebraic independence satisfies full existence when the base is an $\mathcal{L}$-structure. ∎

Remark 5.10. Remark 1.20 of [1] says that nondividing independence (which is not in general an independence relation) implies any independence relation. Indeed this fact is implicit in the proof of the Kim–Pillay theorem, see Theorem 4.2 Claim I of [18]. Hence if T is the theory of a simple, very $\mathcal{L}$-slim field, nonforking independence implies $\mathcal{L}$-algebraic independence.

The following result will allow us to use the notion of very $\mathcal{L}$-slimness in the context of Section 3.

Lemma 5.11. *Let K be an $\mathcal{L}$-structure which is a field of characteristic 0. Suppose K is large and model complete (that is, its $\mathcal{L}$-theory is model complete). Then K is very $\mathcal{L}$-slim.*

Proof. The same proof as in Theorem 5.4 of [16] works here. In part 2 of that proof, when they take a subfield k, we instead take an $\mathcal{L}$-substructure k, that is, a subfield containing the constants C. Model completeness then implies that ϕ is an existential $\mathcal{L}_{\mathrm{ring}}(C)$-formula with parameters from k. But this is the same as an existential $\mathcal{L}_{\mathrm{ring}}$-formula with parameters from k since k contains C. In part 5 of the proof, we do not need perfectness since we are in characteristic 0. The rest of the proof is the same. ∎

Remark 5.12. As noted in Remark 5.6(2), K is very $\mathcal{L}$-slim if and only if it is algebraically bounded over the field generated by the constants C. Thus the results of this paper, specialised to the case of not necessarily commuting derivations, appear as an instance of the work of Fornasiero–Terzo [10]; there arbitrary expansions of fields are considered, here only expansions by constants.

We now fix some notation that will be in place for the rest of this section. Let T be the model complete $\mathcal{L}$-theory of a large field of characteristic 0, $T_{\mathcal{D}}$ the $\mathcal{L}(\partial)$-theory $T \cup \text{``}\mathcal{D}\text{-fields''}$, and $T^+ = T \cup \mathrm{UC}_{\mathcal{D}}$ the model companion of $T_{\mathcal{D}}$, which we know exists by Section 3. By acl we mean the model theoretic algebraic closure in the sense of T^+, and by acl_T we mean in the sense of T.

We will use the following result about amalgamating $\mathcal{D}$-field structures.

Fact 5.13. *Suppose (K, ∂) and (L, γ) are two $\mathcal{D}$-fields containing a common $\mathcal{D}$-subfield (F, ∂) with K and L linearly disjoint over F inside some common field extension (which is not necessarily a $\mathcal{D}$-field). Then there is a unique $\mathcal{D}$-structure on the compositum KL extending ∂ and γ.*

Proof. This is just Lemma 5.1 of [27]. We drop the inversiveness condition since $\mathcal{D}$ is local so there are no (nontrivial) associated endomorphisms. ∎

Lemma 5.14. *Let $(K, \partial) \models T^+$. For $A \subseteq K$, $\mathrm{acl}(A) = \mathrm{acl}_T\left(\langle A \rangle_{\mathcal{L}(\partial)}\right)$.*

Proof. Let $F = \mathrm{acl}_T\left(\langle A \rangle_{\mathcal{L}(\partial)}\right)$. Clearly $F \subseteq \mathrm{acl}(A)$. For the other containment, suppose $d \notin F$. We will show that $\mathrm{tp}(d/F)$ is not algebraic by finding infinitely many realisations.

The $\mathcal{L}$-structure K is very $\mathcal{L}$-slim so $\mathcal{L}$-algebraic independence in K is an independence relation. By full existence there is an $\mathcal{L}$-structure K' which is an $\mathcal{L}$-elementary extension of K containing L such that $L \equiv^{\mathcal{L}}_F K$ (recall that this means that L and K have the same $\mathcal{L}$-type over F) with L and K $\mathcal{L}$-algebraically independent over F inside K'. Now F is relatively algebraically closed in K, so L

and K are linearly disjoint over F inside K'. Note also that F is a $\mathcal{L}(\partial)$-structure and L has a $\mathcal{D}$-field structure coming from the partial $\mathcal{L}$-elementary map $K \to L$. By Fact 5.13 we can amalgamate their $\mathcal{D}$-field structures to the compositum LK and finally we extend this $\mathcal{D}$-field structure to K' using Remark 3.2. So $K' \models T_{\mathcal{D}}$. Since T^+ is the model companion of $T_{\mathcal{D}}$, embed K' inside some $K'' \models T^+$. Let $\alpha \colon K \to L$ be the $\mathcal{L}(\partial)$-isomorphism fixing F. By model completeness of T^+, this is an $\mathcal{L}(\partial)$-elementary map. Then $\alpha(d) \models \mathrm{tp}(d/F)$ and $\alpha(d) \neq d$ as otherwise, $d \mathop{\smash{\underset{F}{\overset{\mathrm{alg}}{\;}}}}\!\!\!\downarrow d$ which would imply $d \in F^{\mathrm{alg}} = F$.

Now we can iterate to find the infinitely many realisations of $\mathrm{tp}(d/F)$. ∎

We are now in a position to prove the transfer of simplicity. The proof idea is standard; see the proof of Wagner that T_A is simple if T is stable in Theorem 6.3.22 of [34], and that of Kruckman and Ramsey that generic expansions and generic Skolemisations of T are NSOP$_1$ if T is NSOP$_1$ in Corollary 4.6 of [20].

Theorem 5.15. *If T is simple, then (any completion of) T^+ is simple.*

Proof. Let $\mathop{\smash{\displaystyle\mathop{\downarrow}}}$ be nonforking independence for T. For a monster model $\mathfrak{C}$ of T^+ and small subsets A, B, D of $\mathfrak{C}$, we define $\mathop{\smash{\displaystyle\mathop{\downarrow}}}^+$ as follows:

$$A \mathop{\smash{\underset{D}{\downarrow}}}^{\!+} B \iff \mathrm{acl}(A) \mathop{\smash{\underset{\mathrm{acl}(D)}{\downarrow}}} \mathrm{acl}(B).$$

We will show that $\mathop{\smash{\displaystyle\mathop{\downarrow}}}^+$ satisfies the conditions of the Kim-Pillay theorem, and hence that T^* is simple.

Invariance, monotonicity, transitivity, symmetry, finite character, local character. These follow from the corresponding property of $\mathop{\smash{\displaystyle\mathop{\downarrow}}}$. For *normality*, the comment after Assumption 5.4 gives $\langle BD \rangle_{\mathcal{L}(\partial)} = \langle B \rangle_{\mathcal{L}(\partial)} \cdot \langle D \rangle_{\mathcal{L}(\partial)}$. Then Lemma 5.14 gives $\mathrm{acl}(BD) \subseteq \mathrm{acl}_T(\mathrm{acl}(B), \mathrm{acl}(D))$.

Full existence. Let a, A, B be given inside some $M \models T^+$. We will find a' in some elementary extension of M with $a' \equiv_A a$ and $a' \mathop{\smash{\underset{A}{\downarrow}}}^{\!+} M$, which implies $a' \mathop{\smash{\underset{A}{\downarrow}}}^{\!+} B$ by monotonicity. We may also assume that $A = \mathrm{acl}(A)$, and hence that A is a $\mathcal{D}$-field and is algebraically closed in the sense of T. Note that A is field-theoretically, relatively algebraically closed in M. Since T is simple, use full existence for $\mathop{\smash{\displaystyle\mathop{\downarrow}}}$ to find a saturated and homogeneous $\mathcal{L}$-elementary extension N of M such that $M' \mathop{\smash{\underset{A}{\downarrow}}} M$ with $M' \subseteq N$ and $M' \equiv_A^{\mathcal{L}} M$. Use the $\mathcal{L}$-isomorphism over A to put a $\mathcal{D}$-field structure on M'. Now M and M' are algebraically independent over A, and hence linearly disjoint over A since $A \subseteq M$ is a regular extension. They then amalgamate to a $\mathcal{D}$-field structure on the compositum MM', which we may then extend to a $\mathcal{D}$-field on N. So $N \models T_{\mathcal{D}}$ and extends to $N' \models T^+$. Let $a' \in M'$ correspond to $a \in M$. By model completeness of T^+, the $\mathcal{L}(\partial)$-isomorphism $M \to M'$ is an $\mathcal{L}(\partial)$-elementary map. Then $a' \equiv_A a$ in the sense of $\mathcal{L}(\partial)$. Since $M' \models T^+$ and T^+ is model complete, it is acl-closed, and so $\mathrm{acl}(a'A) \subseteq M'$. By monotonicity, $\mathrm{acl}(a'A) \mathop{\smash{\underset{\mathrm{acl}(A)}{\downarrow}}} \mathrm{acl}(M)$ and so $a' \mathop{\smash{\underset{A}{\downarrow}}}^{\!+} M$.

Independence theorem. Let $M \prec N \models T^+$, $A_1 \mathop{\smash{\underset{M}{\downarrow}}}^{\!+} A_2$, $a_1 \mathop{\smash{\underset{M}{\downarrow}}}^{\!+} A_1$, $a_2 \mathop{\smash{\underset{M}{\downarrow}}}^{\!+} A_2$, and $\mathrm{tp}(a_1/M) = \mathrm{tp}(a_2/M)$. We will show that in some elementary extension of N, there is $a \mathop{\smash{\underset{M}{\downarrow}}}^{\!+} A_1 A_2$ realising $\mathrm{tp}(a_1/A_1) \cup \mathrm{tp}(a_2/A_2)$. By Löwenheim–Skolem and full existence for $\mathop{\smash{\displaystyle\mathop{\downarrow}}}^+$, we may assume that A_1, A_2, a_1, a_2 are all models of T^+ containing M and contained inside N.

By the independence theorem for T, there are an $\mathcal{L}$-elementary, saturated, homogeneous extension N' of N and an element $a \in N'$ with $a \mathop{\smash{\underset{M}{\downarrow}}} A_1 A_2$ and

$a \models \mathrm{tp}_{\mathcal{L}}(a_1/A_1) \cup \mathrm{tp}_{\mathcal{L}}(a_2/A_2)$. In fact, by full existence for T, we can ensure that $a \underset{M}{\downarrow} N$.

Now for $i = 1, 2$, let N_i' be the copy of N coming from the $\mathcal{L}$-elementary map $A_i a_i \mapsto A_i a$. By full existence for T, let $N_i \equiv^{\mathcal{L}}_{A_i a} N_i'$ with $N_1 \underset{A_1 a}{\downarrow} N$ and $N_2 \underset{A_2 a}{\downarrow} NN_1$. From $a \underset{M}{\downarrow} N$ we get $a \underset{A_1}{\downarrow} N$ and $a \underset{A_2}{\downarrow} N$. Then $N \underset{A_1}{\downarrow} N_1$ and $N \underset{A_2}{\downarrow} N_2$ by transitivity. From $a \underset{M}{\downarrow} N$ we get $a \underset{A_1}{\downarrow} A_2$, and so $A_1 a \underset{A_1}{\downarrow} A_2$. Along with $A_1 \underset{M}{\downarrow} A_2$, transitivity gives $A_1 a \underset{M}{\downarrow} A_2$, so that $A_1 a \underset{a}{\downarrow} A_2$ by base monotonicity. This implies $A_1 \underset{a}{\downarrow} A_2$ and $N_1 \underset{a}{\downarrow} A_2$. This last part implies $N_1 \underset{a}{\downarrow} A_2 a$ and along with $N_2 \underset{A_2 a}{\downarrow} NN_1$ implies $N_1 \underset{a}{\downarrow} N_2$. Also, $N \underset{A_1 A_2}{\downarrow} N_1$ by base monotonicity since $A_1 A_2 \subseteq N$. From $NN_1 \underset{A_2 a}{\downarrow} N_2$, we get $N \underset{A_2 a N_1}{\downarrow} N_2$, and hence $N \underset{A_2 N_1}{\downarrow} N_2$ since $a \in N_1$. Combining this with $N \underset{A_1 A_2}{\downarrow} N_1$ gives $N \underset{A_1 A_2}{\downarrow} N_1 N_2$.

Now define $\mathcal{D}$-field structures ∂_1 on N_1 and ∂_2 on N_2 such that $(N_i, A_i, a, \partial_i)$ is $\mathcal{L}(\partial)$-isomorphic to (N, A_i, a_i, ∂). So $N_i \models T^+$. Now since $N_1 \underset{a}{\downarrow} N_2$, N_1 and N_2 are algebraically independent over a, and hence linearly disjoint over a (since both are regular extensions of a) and so their $\mathcal{D}$-field structures can be amalgamated to the compositum $N_1 N_2$. Extend the $\mathcal{D}$-field structure on $N_1 N_2$ to $\mathrm{acl}_T(N_1 N_2) = (N_1 N_2)^{\mathrm{alg}}$. From $N \underset{A_1 A_2}{\downarrow} N_1 N_2$ we get $N \underset{\mathrm{acl}_T(A_1 A_2)}{\downarrow} \mathrm{acl}_T(N_1 N_2)$ and hence we may amalgamate the $\mathcal{D}$-field structures on $\mathrm{acl}_T(N_1 N_2)$ and N to the compositum $\mathrm{acl}_T(N_1 N_2)N$ and extend this to N'. So $N' \models T_{\mathcal{D}}$.

Extend N' to $N'' \models T^+$. By model completeness of T^+, the $\mathcal{L}(\partial)$-isomorphisms $(N_i, A_i, a, \partial_i) \simeq (N, A_i, a_i, \partial)$ are $\mathcal{L}(\partial)$-elementary maps, and hence $a \models \mathrm{tp}(a_1/A_1) \cup \mathrm{tp}(a_2/A_2)$. From $a \underset{M}{\downarrow} N$ and monotonicity, we get $a \underset{M}{\downarrow} \mathrm{acl}(A_1 A_2)$, and so $a \underset{M}{\overset{+}{\downarrow}} A_1 A_2$.   ∎

Remark 5.16. In [9], Chatzidakis proves that for any field F^*, if $F \prec F^*$, A and B are acl-closed, and $A \underset{F}{\downarrow} B$, then A and B are linearly disjoint over F. This fact allows us to amalgamate $\mathcal{D}$-fields which are independent over models. However, in the above proofs (and in Theorem 6.5 below), we must amalgamate $\mathcal{D}$-fields which are independent over arbitrary acl-closed sets. Hence the notion of $\mathcal{L}$-slimness to facilitate this.

6. Pseudo $\mathcal{D}$-closed fields

We now apply some of the results obtained in Sections 4 and 5 to the study of PAC substructures in the theory $\mathcal{D}\text{-CF}_0$. We continue to assume that $\mathcal{D}$ is local (Assumption 3.3) and that we are working in a language $\mathcal{L}$ of the form $\mathcal{L}_{\mathrm{ring}}(C)$ (Assumption 5.4). Recall that a field K is called pseudo algebraically closed (PAC) if every absolutely irreducible variety over K has a K-rational point. PAC fields are large—a K-irreducible variety with a smooth K-rational point is absolutely irreducible—and a field is PAC if and only if it is existentially closed in every regular extension.

In [8], Chatzidakis and Pillay show that the $\mathcal{L}$-theory of a bounded, PAC field is simple, and that if it, in addition, has finite degree of imperfection, then it eliminates imaginaries (after naming constants). Recall that a field is bounded if it has only finitely many separable algebraic extensions of each degree. Hoffman and León Sánchez in [13] then prove the analogous results for bounded, pseudo differentially closed fields of characteristic 0. Their result gives an example of a differential field whose theory is simple and unstable. In this section we will prove analogous results in the case of $\mathcal{D}$-fields.

PAC substructures of a given theory have been defined as generalisations of PAC fields in various ways. We use the definition presented in [12].

Definition 6.1. Let T be an arbitrary complete L-theory, and $\mathfrak{C}$ a monster model. An extension of L-substructures $A \leq B$ of $\mathfrak{C}$ is called L-regular if $\mathrm{dcl}^{\mathrm{eq}}(B) \cap \mathrm{acl}^{\mathrm{eq}}(A) = \mathrm{dcl}^{\mathrm{eq}}(A)$. An L-substructure A of $\mathfrak{C}$ is called a PAC substructure if A is existentially closed in every L-regular extension.

Consider now the $\mathcal{L}_{\mathrm{ring}}(\partial)$-theory $\mathcal{D}\text{-}\mathrm{CF}_0$. This theory eliminates imaginaries (see Theorem 5.12 of [27]), $\mathrm{dcl}(A)$ is the $\mathcal{D}$-field generated by A, and $\mathrm{acl}(A)$ is the (full) field-theoretic algebraic closure of the $\mathcal{D}$-field generated by A (Proposition 5.5 of [27]). Then an extension of $\mathcal{D}$-fields is $\mathcal{L}_{\mathrm{ring}}(\partial)$-regular exactly when the field extension is field-theoretically, relatively algebraically closed (and so regular in the field sense since we are in characteristic 0).

We now prove three conditions equivalent to being a PAC substructure in $\mathcal{D}\text{-}\mathrm{CF}_0$.

Theorem 6.2. Let (K, ∂) be a $\mathcal{D}$-field. The following are equivalent:

  (1) (K, ∂) is a PAC substructure in the theory $\mathcal{D}\text{-}\mathrm{CF}_0$;
  (2) K is a PAC field and $(K, \partial) \models \mathrm{UC}_{\mathcal{D}}$;
  (3) if (V, s) is a D-variety over K and V is absolutely irreducible, then (V, s) has a K-rational sharp point;
  (4) (K, ∂) is existentially closed in every $\mathcal{D}$-field extension (L, δ) which is R-regular, that is, where $\mathrm{tp}^{\mathcal{D}\text{-}\mathrm{CF}_0}(a/K)$ is stationary for every finite tuple $a \in L$.

Proof. (1) $\Longrightarrow$ (2). Let L be any regular field extension of K, and let δ be any $\mathcal{D}$-structure on L extending ∂. Then (K, ∂) is existentially closed in (L, δ) as $\mathcal{D}$-fields, and hence K is existentially closed in L as fields. So K is PAC. Now since K is large, there is a $\mathcal{D}$-field extension $(F, \gamma) \models \mathrm{UC}_{\mathcal{D}}$ of (K, ∂) such that K is elementary in F as fields. In particular, $K \subseteq F$ is regular. By (1), (K, ∂) is existentially closed in (F, γ). Since $\mathrm{UC}_{\mathcal{D}}$ is inductive, $(K, \partial) \models \mathrm{UC}_{\mathcal{D}}$.

(2) $\Longrightarrow$ (1). Let (L, δ) be an $\mathcal{L}_{\mathrm{ring}}(\partial)$-regular $\mathcal{D}$-field extension of (K, ∂), so that L/K is a regular field extension. Since K is PAC, K is existentially closed in L as fields. By characterisation (5) of Theorem 4.4, (K, ∂) is existentially closed in (L, δ) as $\mathcal{D}$-fields.

(2) $\Longrightarrow$ (3). If V is absolutely irreducible, then K is regular in its function field $K(V)$. V has a smooth $K(V)$-rational point, and since K is existentially closed in $K(V)$, V has a smooth K-rational point. By characterisation (3) of $\mathrm{UC}_{\mathcal{D}}$ in Theorem 4.4, (V, s) has a K-rational sharp point.

(3) $\Longrightarrow$ (2). Let V be an absolutely irreducible variety defined over K. Extend the $\mathcal{D}$-field structure on K to one on $K(V)$. As in the proof of Theorem 4.4 (3) $\Longrightarrow$ (1), there is an affine, open subset $U \subseteq V$ defined over K such that this $\mathcal{D}$-ring structure restricts to one on $K[U]$. This gives a D-variety structure s on U, making (U, s) an absolutely irreducible D-variety defined over (K, ∂). By (3), (U, s) has a K-rational sharp point, and hence V has a K-rational point. So K is a PAC field. We again use characterisation (3) of Theorem 4.4 and the fact that a K-irreducible variety with a smooth K-rational point is absolutely irreducible to get that $(K, \partial) \models \mathrm{UC}_{\mathcal{D}}$.

(1) $\Longleftrightarrow$ (4) is the content of Lemma 3.36 in [12]; R-regular extensions are the same as $\mathcal{L}_{\mathrm{ring}}(\partial)$-regular extensions since $\mathcal{D}\text{-}\mathrm{CF}_0$ is stable and eliminates imaginaries. ∎

We say that a $\mathcal{D}$-field is pseudo $\mathcal{D}$-closed if any of the equivalent conditions of Theorem 6.2 hold.

Remark 6.3. Apart from condition (3), this is just the $\mathcal{D}$-field analogue of Theorem 5.16 from [21]. There the authors need to consider differential varieties as they work with several commuting derivations. In a single derivation, it is enough to consider D-varieties; see Proposition 5.6 of [29] for instance.

Theorem 5.2 of [13] states that the theory of a bounded, pseudo differentially closed field (that is, a PAC substructure of $\mathrm{DCF}_{0,m}$) is simple and eliminates imaginaries. We will now prove the $\mathcal{D}$-field analogue. Let (K, ∂) be a bounded, pseudo $\mathcal{D}$-closed field. For each $n > 1$, let $N(n)$ be the degree over K of the Galois extension composite of all Galois extensions of K of degree n. Let $C = (c_{n,i})_{n>1, 0 \leq i < N(n)}$ be the set of constant symbols in our language $\mathcal{L} = \mathcal{L}_{\mathrm{ring}}(C)$, and consider the set of $\mathcal{L}$-sentences $\Sigma_C = \{\sigma_n \colon n > 1\}$ where σ_n says that the polynomial $x^{N(n)} + c_{n,N(n)-1} x^{N(n)-1} + \ldots + c_{n,0}$ is irreducible and the extension this polynomial defines is Galois and contains all Galois extensions of K of degree n. This is the same set-up used by Chatzidakis and Pillay in Section 4 of [8] in their treatment of bounded, PAC fields. Let $T^+ = \mathrm{Th}(K, \partial) \cup \Sigma_C$. Note then that $T^+ \supseteq \mathrm{Th}_{\mathcal{L}_{\mathrm{ring}}}(K) \cup \Sigma_C \cup \mathrm{UC}_{\mathcal{D}}$.

For the next two proofs, we will at times need to refer to notions in both the sense of T^+ and the sense of $\mathcal{D}$-CF_0. In the second case, we will always include this as a superscript; if no superscript is given, the notion should be understood in the sense of whatever model of T^+ we are working in. The full, field-theoretic algebraic closure of A is denoted by $\tilde{A}$, and A^{alg} denotes the relative, field-theoretic algebraic closure of A. If $A \supseteq C$, then A^{alg} is equal to $\mathrm{acl}_{\mathcal{L}}(A)$ since models of T^+ are very $\mathcal{L}$-slim.

Lemma 6.4. *Let $(F, \partial, C) \models T^+$, and $A \leq B \leq F$ with A acl-closed in the sense of T^+. Then*
$$\mathrm{acl}^{\mathcal{D}\text{-}\mathrm{CF}_0}(B) = \mathrm{acl}(B) \cdot \mathrm{acl}^{\mathcal{D}\text{-}\mathrm{CF}_0}(A).$$

Proof. Since B is a $\mathcal{D}$-field containing C and F is very $\mathcal{L}$-slim, $\mathrm{acl}(B) = B^{\mathrm{alg}}$, and $\mathrm{acl}^{\mathcal{D}\text{-}\mathrm{CF}_0}(B) = \tilde{B}$. So we need to show $\tilde{B} = B^{\mathrm{alg}} \cdot \tilde{A}$. The proof of Proposition 4.6(2) of [8] shows that the restriction maps $\mathrm{Gal}(F) \to \mathrm{Gal}(A)$ and $\mathrm{Gal}(F) \to \mathrm{Gal}(\mathrm{acl}(B))$ are isomorphisms, and hence the restriction map $\mathrm{Gal}(\mathrm{acl}(B)) \to \mathrm{Gal}(A)$ is an isomorphism. Therefore, any automorphism of $\tilde{B}$ that fixes $B^{\mathrm{alg}} \cdot \tilde{A}$ must also fix $\tilde{B}$. Since we are in characteristic 0, $\tilde{B}/B^{\mathrm{alg}} \cdot \tilde{A}$ is a Galois extension, and so $\tilde{B} = B^{\mathrm{alg}} \cdot \tilde{A}$. ∎

Theorem 6.5. *Let $(F, \partial, C) \models T^+$ and $(E, \partial, C) \subseteq (F, \partial, C)$. Then*

(1) $\mathrm{acl}(E) = E^{\mathrm{alg}}$;
(2) *if $E = \mathrm{acl}(E)$, then $T^+ \cup \mathrm{diag}(E)$ is complete;*
(3) *T^+ is model complete;*
(4) *the independence theorem holds for T^+ over algebraically closed sets;*
(5) *T^+ is simple and forking is given by forking independence in $\mathcal{D}$-CF_0;*
(6) *T^+ has elimination of imaginaries.*

Proof. (1). By Lemma 5.14 since F is very $\mathcal{L}$-slim (it is model complete, large, and characteristic 0).

(2). By Proposition 4.6(2) of [8], $\mathrm{Th}_{\mathcal{L}_{\mathrm{ring}}}(K) \cup \Sigma_C \cup \mathrm{diag}_{\mathcal{L}}(E)$ is complete. Then Theorem 3.9 tells us that $\mathrm{Th}_{\mathcal{L}_{\mathrm{ring}}}(K) \cup \Sigma_C \cup \mathrm{UC}_{\mathcal{D}} \cup \mathrm{diag}(E)$ is complete. So $T^+ \cup \mathrm{diag}(E)$ is complete.

(3). By Theorem 3.10(i) since $\mathrm{Th}_{\mathcal{L}_{\mathrm{ring}}}(K) \cup \Sigma_C$ is model complete (Proposition 4.6(1) of [8]).

(4). This follows from the proof of Theorem 5.15 and the fact that the independence theorem over algebraically closed sets holds for bounded PAC fields (Theorem 4.7 of [8]). Note that the proof of Theorem 5.15 only uses the fact that M is a model if the independence theorem for T holds only for models.

(5). By Theorem 5.15 and the corresponding result for bounded PAC fields (Corollary 4.8 of [8]) we know that T^+ is simple and forking independence is given by linear disjointness after closing under acl—the relative algebraic closure of the $\mathcal{D}$-field it generates. We can then use general properties of linear disjointness of regular extensions, along with Lemma 6.4, to show that, if A, B, and D are all acl-closed, then A and B are linearly disjoint over D if and only if $\tilde{A}$ is linearly disjoint from $\tilde{B}$ over $\tilde{D}$. This is precisely forking independence in $\mathcal{D}$-CF_0 (see Theorem 5.9 of [27]).

(6). This proof is essentially a combination of Theorem 5.12 of [27] and Theorem 5.6 of [13]. Nonetheless, some details will be provided. We will assume that (F, ∂, C) is a monster model of (some completion of) T^+, and that $(\mathfrak{D}, \partial)$ is a monster model of $\mathcal{D}$-CF_0 extending it. We write $\underset{}{\bigcup}$ for nonforking independence in (F, ∂, C). If we omit a superscript from an operator, we mean in the sense of (F, ∂, C).

We need the notion of dimension from Definition 5.10 of [27]. If K is a $\mathcal{D}$-field, then $\dim_{\mathcal{D}}(a/K) = (\mathrm{trdeg}(\nabla_r(a)/K) \colon r < \omega) \in \omega^{\omega}$, where $\nabla_r(a)$ is the tuple applying words of length at most r on $\{\partial_1, \ldots, \partial_l\}$ to a. Note that $\dim_{\mathcal{D}}(a/K) = \dim_{\mathcal{D}}(a/\tilde{K})$. Using Lemma 5.11 of [27], we then get that if L/k is a regular extension, $\dim_{\mathcal{D}}(a/k) = \dim_{\mathcal{D}}(a/L)$ if and only if $\mathrm{acl}^{\mathcal{D}\text{-}\mathrm{CF}_0}(ka)$ is linearly disjoint from $\tilde{L}$ over $\tilde{k}$ if and only if $a \underset{k}{\bigcup} L$.

Let $e \in (F, \partial, C)^{\mathrm{eq}}$ given by a 0-definable function f and a finite real tuple $a \in F$, that is, $f(a) = e$. Let $E = \mathrm{acl}^{\mathrm{eq}}(e) \cap F$ and let Q be the set of realisations of $\mathrm{tp}(a/E)$. Having established that $\dim_{\mathcal{D}}$ measures nonforking independence in T^+, the same proof as in Theorem 5.12 of [27] allows us to find some $u \in Q$ such that $f(u) = e$ and $u \underset{E}{\bigcup} a$.

We now follow the rest of the argument in Theorem 5.6 of [13]. Let $D = \{d \in Q \colon f(d) = e\}$. If $D = Q$, then $e \in \mathrm{dcl}^{\mathrm{eq}}(E)$ and we get weak elimination of imaginaries.

If $D \subsetneq Q$, let $d_0 \in Q \setminus D$ and $d \equiv_E d_0$ with $d \underset{E}{\bigcup} D$. If $f(d) = e$, then $d \in D$ and hence $d \in \mathrm{acl}(E) = E$. So $d \in \mathrm{acl}^{\mathrm{eq}}(e)$. Since $f(d) = e$, $e \in \mathrm{dcl}^{\mathrm{eq}}(d)$, and we get weak elimination of imaginaries.

So assume $f(d) \neq e$. Now $u \equiv_E d$, $u \underset{E}{\bigcup} a$, and $u \underset{E}{\bigcup} d$. By the independence theorem over algebraically closed sets, we get $m \models \mathrm{tp}(u/Ea) \cup \mathrm{tp}(d/Eu)$ with $m \underset{E}{\bigcup} au$. But this contradicts $f(d) \neq e$. Finally, since we are in a theory of fields and we have weak elimination of imaginaries, we have elimination of imaginaries.   ∎

7. THE NON-LOCAL CASE

Recall that, throughout Sections 3–6, we assumed the k-algebra $\mathcal{D}$ was a local ring. In this section we make some observations in the case when $\mathcal{D}$ is *not* local.

In particular, when the base field k is finitely generated over $\mathbb{Q}$, we get a full characterisation of when the uniform companion exists: it exists if and only if $\mathcal{D}$ is local.

We start with the general case: k is a field of characteristic 0, $\mathcal{D}$ is a finite-dimensional k-algebra, and $\mathcal{D} = \prod_{i=0}^{t} B_i$ where each B_i is a local, finite-dimensional k-algebra. We no longer impose Assumption 3.3 (that $\mathcal{D}$ is local) or even Assumption 2.4 (that the residue field of each B_i is k). For $i > 0$, the residue field of B_i is $k[x]/(P_i)$ for some k-irreducible polynomial P_i, and that of B_0 is k. For an $\mathcal{L}$-theory T, the $\mathcal{L}(\partial)$-theory $T \cup$ "$\mathcal{D}$-fields" is denoted by $T_\mathcal{D}$, and the $\mathcal{L}(\sigma)$-theory $T \cup$ "σ is an endomorphism" is denoted by T_σ.

A result of Kikyo and Shelah [17] states that if a model complete theory has the strict order property, then the theory obtained by adding an automorphism has no model companion. In particular, if $\mathcal{D} = k \times k$, $\mathcal{D}$-fields correspond to fields with an endomorphism, and so $\mathrm{RCF}_\mathcal{D} = \mathrm{RCF}_\sigma$ and $\mathrm{Th}(\mathbb{Q}_p)_\mathcal{D} = \mathrm{Th}(\mathbb{Q}_p)_\sigma$ have no model companion. In fact, the Kikyo–Shelah theorem implies that $T_\mathcal{D}$ has no model companion when $\mathcal{D}$ is not local and T has a model in which one of the polynomials P_i has a root. We first prove this for the case when $\mathcal{D}$ has at least one local component with residue field k, and then reduce the more general statement to this case.

Proposition 7.1. *Assume $\mathcal{D}$ is such that one of the local components B_i has residue field k for $i > 0$. If $T_\mathcal{D}$ has a model companion, then T_σ has a model companion.*

Proof. Note that by a particular choice of the basis $\varepsilon_0, \ldots, \varepsilon_l$, we may assume that the associated endomorphism σ_i corresponding to B_i appears as one of the operators ∂_j. So $\mathcal{L}(\sigma) \subseteq \mathcal{L}(\partial)$.

Write T^+ for the model companion of $T_\mathcal{D}$ and T^- for its reduct to $\mathcal{L}(\sigma)$. We will show that T^- is the model companion of T_σ; clearly their universal parts coincide, so it suffices to prove T^- is model complete.

Let $(K, \sigma) \models T^-$. We will show that $T^- \cup \mathrm{diag}_{\mathcal{L}(\sigma)}(K)$ is complete. Use Lemma 3.1 to equip K with a $\mathcal{D}$-ring structure whose ith associated homomorphism is σ and whose jth associated homomorphism is inclusion $K \to K[x]/(P_j)$ for $j \neq i$. Then $K \models T_\mathcal{D}$, and it embeds in some $L \models T^+$. Since T^+ is model complete, $T^+ \cup \mathrm{diag}_{\mathcal{L}(\partial)}(L)$ is complete, and hence its reduct to $\mathcal{L}(\sigma)(K)$, $T^- \cup \mathrm{diag}_{\mathcal{L}(\sigma)}(K)$, is complete. ∎

We now weaken the assumption that the residue field of some B_i is k to the assumption that T has a model K in which one of the polynomials P_i has a root. If $T_\mathcal{D}$ has a model companion, then $T_\mathcal{D} \cup \mathrm{diag}(K)$ has a model companion. Let $\mathcal{E}$ be the K-algebra $\mathcal{D} \otimes_k K$. As mentioned in the proof of Theorem 3.2 of [4], if L is an $\mathcal{E}$-field, then $\mathcal{E}$-field extensions of L coincide with $\mathcal{D}$-field extensions of L. Hence if $T_\mathcal{D} \cup \mathrm{diag}(K)$ has a model companion, so does $T_\mathcal{E}$. But $\mathcal{E}$ now satisfies the assumption in Proposition 7.1. So we have proved the following.

Corollary 7.2. *If T is model complete and has a model with the strict order property in which one of the P_i has a root, then $T_\mathcal{D}$ has no model companion.*

Real closed fields and $\mathbb{Q}_p$ have the strict order property, and so this result means if any P_i has a root in some real closed field or some p-adically closed field, there is no uniform companion. In particular, if the base field k is a finitely generated field extension of $\mathbb{Q}$, we get a full converse to the main theorem.

Corollary 7.3. *Suppose k is a finitely generated field extension of $\mathbb{Q}$. Then there is a uniform companion for theories of large $\mathcal{D}$-fields if and only if $\mathcal{D}$ is a local ring.*

Proof. Suppose $\mathcal{D}$ is not local. Then the splitting field of the polynomial $P_1 \in k[x]$ is a finitely generated extension of $\mathbb{Q}$, and so by Theorem 1 of [6], embeds in some $\mathbb{Q}_p$. Hence P_1 has a root in $\mathbb{Q}_p$. Then by Theorem 7.2, $\mathrm{Th}(\mathbb{Q}_p)_\mathcal{D}$ has no model companion. ∎

Remark 7.4. The base field k does have an impact on when the uniform companion exists. If k is algebraically closed, then the only model complete theory of fields containing k is ACF_0, and hence the existence of a uniform companion for $\mathcal{D}$-fields is equivalent to the existence of the model companion of $\mathrm{ACF}_0 \cup$ "$\mathcal{D}$-fields"; this exists for all $\mathcal{D}$ by Theorem 7.6 of [27].

However, for other fields k the question is still open. For instance, suppose $k = \mathbb{R}$. No model of $\mathrm{Th}(\mathbb{Q}_p)$ can be an $\mathbb{R}$-algebra, and so $\mathrm{Th}(\mathbb{Q}_p) \cup$ "$\mathcal{D}$-fields" is inconsistent. Hence the above method does not show that there is no uniform companion in the case $\mathcal{D} = \mathbb{R} \times \mathbb{C}$, for instance.

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