1 Introduction

I will present some recent work on a uniform companion for theories of $\mathcal{D}$-fields. This aims to generalise the work of Tressl in 2005 and then of León Sánchez and Tressl in 2020. Throughout, $\mathcal{L}$ is the language of rings. Everything is characteristic 0.

Theorem 1.1 (Tressl). *There exists an inductive theory UC such that whenever T is a model complete theory of large fields in a language $\mathcal{L}(C)$, then $T \cup UC$ is the model companion of $T \cup$ "differential fields".*

They then define the notion of differential largeness to be any differential field that is large as a field and is a model of UC, and then show various properties about differentially large fields. For instance, the algebraic closure of such a differential field is differentially closed. This gives a class of differential fields whose differential closures are minimal.

I will present similar results but for a different class of operators. These operators should be thought of as a generalisation of the single derivation case.

2 $\mathcal{D}$-fields

$\mathcal{D}$-fields are a framework for talking about classes of rings with a finite collection of free operators. The behaviour of the operators is determined by the choice of $\mathcal{D}$.

We fix a finite-dimensional $\mathbb{Q}$-algebra, $\mathcal{D}$, a basis of $\mathcal{D}$, $\{\varepsilon_0, \ldots, \varepsilon_l\}$, and a homomorphism $\pi\colon \mathcal{D} \to \mathbb{Q}$ such that $\pi(\varepsilon_0) = 1$ and $\pi(\varepsilon_i) = 0$. We define $a_{ijk} \in k$ by $\varepsilon_i \varepsilon_j = \sum_{k=1}^{l} a_{ijk}\varepsilon_k$.

Definition 2.1. $(K, \partial_1, \ldots, \partial_l)$ is a $\mathcal{D}$-field if K is a field, $\partial_i\colon K \to K$, and the map

$$\begin{aligned}\partial\colon K &\to K \otimes_{\mathbb{Q}} \mathcal{D}\\ a &\mapsto a \otimes \varepsilon_0 + \partial_1(a) \otimes \varepsilon_1 + \ldots + \partial_l(a) \otimes \varepsilon_l\end{aligned}$$

is a homomorphism.

This condition is equivalent to saying that the maps ∂_i are linear and satisfy a certain multiplicativity condition, ie that $\partial_k(rs) = \sum_{i,j} a_{ijk}\partial_i(r)\partial_j(s)$.

We axiomatise the theory of $\mathcal{D}$-fields in the language $\mathcal{L} \cup \{\partial_1, \ldots, \partial_l\}$ by the theory of fields along with sentences saying that each ∂_i is linear and $\partial_k(rs) = \sum_{i,j} a_{ijk}\partial_i(r)\partial_j(s)$. Moosa and Scanlon showed that the theory of $\mathcal{D}$-fields has a model companion, $\mathcal{D}$-CF_0.

Note that no other conditions are imposed on these operators. For instance, they do not have to commute.

Example.

1. Let $\mathcal{D} = \mathbb{Q}[\varepsilon]/(\varepsilon^2)$ with basis $\{1, \varepsilon\}$ and $\pi(a + b\varepsilon) = a$. Then (K, ∂) is a $\mathcal{D}$-field precisely when ∂ is a derivation on K.

2. Let $\mathcal{D} = \mathbb{Q}[\varepsilon_1, \ldots, \varepsilon_n]/(\varepsilon_1 \ldots, \varepsilon_n)^2$. Then $(K, \partial_1, \ldots, \partial_n)$ is a $\mathcal{D}$-field precisely when each ∂_i is a derivation.

3. Let $\mathcal{D} = \mathbb{Q}[\varepsilon]/(\varepsilon^3)$ with basis $\{1, \varepsilon, \varepsilon^2\}$. Then $(K, \partial_1, \partial_2)$ is a $\mathcal{D}$-field precisely when (∂_1, ∂_2) is a higher derivation of length 2, that is, when ∂_1 is a derivation and $\partial_2(xy) = x\partial_2(y) + \partial_1(x)\partial_1(y) + \partial_2(x)y$.

4. Let $\mathcal{D} = \mathbb{Q}^{n+1}$ with standard basis and π projection onto the first coordinate. Then $(K, \partial_1, \dots, \partial_n)$ is a $\mathcal{D}$-field precisely when each ∂_i is an endomorphism of K. The endomorphisms do not necessarily commute.

As we see from the previous example, we can fit endomorphisms into this framework of $\mathcal{D}$-fields. However, we already know that some theories of fields cannot be companionised after adding an automorphism, eg RCF. Hence we ensure there are no endomorphisms by imposing the following assumption:

Assumption 2.2. *For the remainder of this talk, we assume that $\mathcal{D}$ is a local $\mathbb{Q}$-algebra.*

3 Large fields

Definition 3.1. A field K is large if whenever V is a smooth K-variety, then $V(K) = \emptyset$ or $V(K)$ is Zariski dense in V.

Example. ACF, RCF, $\mathbb{Q}_p$ are all large. $\mathbb{Q}$ and function fields are not large.

4 $\mathrm{UC}_{\mathcal{D}}$

Definition 4.1. Let (K, ∂) be a $\mathcal{D}$-field. Let V be an affine variety over K. The prolongation of V, τV, is the variety with the characteristic property that for any field extension L, $\tau V(L) \leftrightarrow V(L \otimes_{\mathbb{Q}} \mathcal{D})$, where V can be thought of as a variety over $K \otimes_{\mathbb{Q}} \mathcal{D}$ via ∂.

If L is any $\mathcal{D}$-field extension, there is a map $\nabla \colon V(L) \to \tau V(L)$ given by $\nabla(a) = (a, \partial_1(a), \dots, \partial_l(a))$.

There is also a canonical projection map $\hat{\pi} \colon \tau V \to V$.

Example. Suppose $\mathcal{D} = \mathbb{Q}[\varepsilon]/(\varepsilon^2)$. Write the homomorphism $\partial \colon K \to K \otimes_{\mathbb{Q}} \mathcal{D} = K[\varepsilon]/(\varepsilon^2)$ as $\partial(r) = r + \delta(r)\varepsilon$. If $\delta = 0$, then τV is just the tangent bundle of V. If $\delta \neq 0$, then τV is the twisted tangent bundle referred to in the geometric axioms for DCF.

Definition 4.2. A D-variety is a variety V with a morphism $s \colon V \to \tau V$ such that $\hat{\pi} \circ s = \mathrm{id}$. We say that $a \in V$ is a sharp point, $a \in (V, s)^\sharp$, if $s(a) = \nabla(a)$.

Definition 4.3. We say that a $\mathcal{D}$-field (K, ∂) is a model of $\mathrm{UC}_{\mathcal{D}}$ if the following holds:

> every affine K-irreducible D-variety (V, s) defined over K with a smooth K-rational point has a K-rational sharp point.

Theorem 4.4. *1. Suppose N and M are two models of $\mathrm{UC}_{\mathcal{D}}$ with a common $\mathcal{D}$-subring A. If M and N have the same universal theory as pure fields, then they have the same universal theory as $\mathcal{D}$-fields.*

2. *Every $\mathcal{D}$-field that is large as a pure field can be extended to a model of* $\mathrm{UC}_{\mathcal{D}}$*, and this extension is elementary as an extension of fields.*

Corollary 4.5. *If T is a model complete theory of large fields, then $T \cup \mathrm{UC}_{\mathcal{D}}$ is the model companion of $T \cup$ "$\mathcal{D}$-fields". In addition, if T has quantifier elimination (possibly in some larger language), so does $T \cup \mathrm{UC}_{\mathcal{D}}$.*

Example. This corollary lets us companionise various theories of $\mathcal{D}$-fields:

1. $\mathrm{ACF} \cup \mathrm{UC}_{\mathcal{D}}$ is the theory $\mathcal{D}$-CF_0 from Moosa-Scanlon;

2. $\mathrm{RCF} \cup \mathrm{UC}_{\mathcal{D}}$ is the model companion of $\mathrm{RCF} \cup$ "$\mathcal{D}$-fields". It has qe once we add a symbol for the order;

3. $p\mathrm{CF} \cup \mathrm{UC}_{\mathcal{D}}$ is the model companion of $p\mathrm{CF} \cup$ "$\mathcal{D}$-fields". It has qe once we add unary predicates P_n for the nth powers of the field;

5 $\mathcal{D}$-largeness

Definition 5.1. A $\mathcal{D}$-field is called $\mathcal{D}$-large if it is large as a pure field and a model of $\mathrm{UC}_{\mathcal{D}}$.

Corollary 5.2. *Algebraic extensions of $\mathcal{D}$-large $\mathcal{D}$-fields are again $\mathcal{D}$-large.*

Proof. Suppose K is $\mathcal{D}$-large and L an algebraic extension. Let (V, s) be an affine D-variety defined over L which has a smooth L-rational point. We use the $\mathcal{D}$-Weil descent: there is a D-variety (W, t) defined over K such that there is a correspondence $V(L) \leftrightarrow W(K)$ that restricts to a correspondence $(V, s)^\sharp(L) \leftrightarrow (W, t)^\sharp(K)$. The first correspondence preserves smoothness. ■

This shows that the algebraic closure of a $\mathcal{D}$-large field is a model of $\mathcal{D}$-CF_0.

6 Preserving NIP and simplicity

Definition 6.1. Let T be a complete theory and $\mathbb{U}$ a monster model. A formula $\phi(x, y)$ has the independence property if there are $(a_i)_{i\in\omega}$ and $(b_I)_{I\subseteq\omega}$ such that $\models \phi(a_i, b_I) \iff i \in I$. T is NIP if no formula has the independence property.

Suppose as before T is a model complete theory of large fields. If T is NIP, is $T \cup \mathrm{UC}_{\mathcal{D}}$ NIP?

We can assume T has qe, and hence that $T \cup \mathrm{UC}_{\mathcal{D}}$ has qe. Suppose T was not NIP. Then some $\mathcal{L}(\partial)$-formula $\phi(x, y)$ has the independence property. We may assume ϕ is quantifier-free. For an element a, let $\nabla_r(a)$ be the tuple that applies to a every word of length at most r in the letters $\partial_1, \dots, \partial_l$.

Then $\phi(x, y)$ is equivalent modulo $T \cup \mathrm{UC}_{\mathcal{D}}$ to an $\mathcal{L}$-formula $\psi(\nabla_r(x), \nabla_r(y))$ for large enough r. ψ has the independence property in T with parameters $(\nabla_r(a_i))_{i\in\omega}$ and $(\nabla_r(b_I)_{I\subseteq\omega}$.

How about if T is simple?

Here it is not so obvious. We are currently attempting to prove this via the Kim-Pillay theorem.