The Weil descent functor in the category of algebras with free operators

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Introduction

Given a ring A and an A-algebra B, we can form the base change functor $F\colon \mathrm{Alg}_A \to \mathrm{Alg}_B$ defined by $F(R) = R \otimes_A B$. If B is free and finitely generated as an A-module, this functor has a left adjoint, W, which we call Weil restriction or Weil descent.

The Weil restriction has been applied to problems in number theory and geometry. We are interested in Pop's application of this result to the study of large fields [4]. Pop used the Weil restriction to show that algebraic extensions of large fields are again large. We would like to show a similar result in the case of $\mathcal{D}$-fields.

$\mathcal{D}$-rings

Fix a field k and a finite-dimensional k-algebra $\mathcal{D}$. Then a $\mathcal{D}$-ring is a k-algebra R equipped with a k-algebra homomorphism $e\colon R \to R \otimes_k \mathcal{D}$. $\mathcal{D}$-rings are a general framework for rings with a finite collection of "free" operators.

- If $\mathcal{D} = k$, then $\mathcal{D}$-rings are precisely k-algebras with a k-algebra endomorphism.
- If $\mathcal{D} = k[\varepsilon]/(\varepsilon^2)$, then $\mathcal{D}$-rings are k-algebras with a k-algebra endomorphism σ and a k-linear derivation δ which is twisted by σ.

$\mathcal{D}$-rings were introduced and studied in [3].

Largeness and $\mathcal{D}$-largeness

A field K is large if whenever C is an irreducible affine curve over K with a smooth K-rational point, then C has infinitely many K-rational points. If L/K is a finite field extension, then the adjunction gives rise to the bijection

$$\mathrm{Hom}_{\mathrm{Alg}_K}(W(R), K) \to \mathrm{Hom}_{\mathrm{Alg}_L}(R, L)$$

Thus if R is the coordinate ring of an affine irreducible curve C over L, we get a bijection between the L-rational points of C and the K-rational points of $\mathrm{Spec}(W(R))$.

Our long-term aim now is to formulate an analogous notion of $\mathcal{D}$-largeness for $\mathcal{D}$-fields. A short-term aim is to prove that the $\mathcal{D}$-base change functor, $F^{\mathcal{D}}$, has a left adjoint $W^{\mathcal{D}}$ and thus find the following bijection

$$\mathrm{Hom}_{\mathrm{Alg}_{(K,e)}}(W^{\mathcal{D}}(R,u),(K,e)) \to \mathrm{Hom}_{\mathrm{Alg}_{(L,f)}}((R,u),(L,f))$$

This follows the argument used by the authors in [2] to prove the differential case.

The $\mathcal{D}$-base change functor

In [1], the authors show that given a $\mathcal{D}$-ring (A,e) and an (A,e)-algebra (B,f), for any $\mathcal{D}$-ring (R,u), there exists a unique $\mathcal{D}$-ring structure on $R \otimes_A B$ such that the natural maps respect the appropriate $\mathcal{D}$-ring structures.

In fact, this result extends to show that there is a $\mathcal{D}$-base change functor

$$F^{\mathcal{D}}\colon \mathrm{Alg}_{(A,e)} \to \mathrm{Alg}_{(B,f)}$$

that acts like the classical base change on the underlying algebras. Thus we wish to prove this functor has a left adjoint if B is free and finitely generated as an A-module.

A counterexample

$F^{\mathcal{D}}$ does not always have a left adjoint. Let $\mathcal{D} = k$ so that $\mathcal{D}$-rings are precisely rings with an endomorphism. Then the $\mathcal{D}$-base change functor is just the tensor product of endomorphisms.

Let A be a ring, $B = A[\varepsilon]/(\varepsilon^2)$, $e = \mathrm{id}_A$ and $f(a + b\varepsilon) = a$. Suppose $F^{\mathcal{D}}$ had a left adjoint $W^{\mathcal{D}}$. Let $C = B[x]$ and g the unique endomorphism of C extending f and sending $x \mapsto \varepsilon$. Then $W^{\mathcal{D}}(C,g)$ is an (A,e)-algebra so write it as (R,u) where R is an A-algebra and u extends e.

Let η be the component of the unit of the adjunction at (C,g).

$$\eta\colon (C,g) \to F^{\mathcal{D}}(W^{\mathcal{D}}(C,g)) = (R \otimes_A B, u \otimes f)$$

η is a B-algebra homomorphism and a difference ring homomorphism. Then $\eta(g(x)) = (u \otimes f)(\eta(x))$. Since B is finitely generated and free as an A-module, write $\eta(x) = r_1 \otimes 1 + r_2 \otimes \varepsilon$.

$$\begin{aligned} \eta(g(x)) &= (u \otimes f)(\eta(x)) \\ \eta(\varepsilon) &= u(r_1) \otimes f(1) + u(r_2) \otimes f(\varepsilon) \\ 1 \otimes \varepsilon &= u(r_1) \otimes 1 \end{aligned}$$

Since $1 \otimes 1$ and $1 \otimes \varepsilon$ is an R-basis of $R \otimes_A B$, this is a contradiction and hence no left adjoint exists.

The matrix associated to an endomorphism

The reason this example fails is that while $\{1, \varepsilon\}$ is an A-basis of B, $\{f(1), f(\varepsilon)\}$ is no longer an A-basis of B. This is equivalent to saying that the matrix

$$\begin{bmatrix} \lambda_1(f(1)) & \lambda_1(f(\varepsilon)) \\ \lambda_2(f(1)) & \lambda_2(f(\varepsilon)) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is not invertible, where $\lambda_i(b)$ is the coefficient of the ith basis element in the linear combination for b.
We call this matrix the matrix associated to f with respect to the basis $b = \{1, \varepsilon\}$ and denote it M_b^f.

Theorem 1.

1. The invertibility of M_b^f is independent of the basis b.
2. If f is an automorphism, then M_b^f is invertible.
3. If $e = f|_A$ is an automorphism, then M_b^f is invertible if and only if f is an automorphism.

Note that invertibility of M_b^f is not equivalent to f being an automorphism.

The associated endomorphisms

Just as we can associated a matrix to an endomorphism, we can associate a collection of matrices to a $\mathcal{D}$-ring structure $e\colon R \to R \otimes_k \mathcal{D}$. Since $\mathcal{D}$ is a finite-dimensional k-algebra, write $\mathcal{D} = \prod_{i=1}^t B_i$ where B_i is a local finite-dimensional k-algebra. We assume that the residue field of B_i is k. Let $\pi_i\colon \mathcal{D} \to B_i \to k$, and $\pi_i^R = \mathrm{id}_R \otimes \pi_i\colon R \otimes_k \mathcal{D} \to R$. Then $\sigma_i = \pi_i^R \circ e$ is a k-algebra endomorphism of R for each i. These are the associated endomorphisms of the $\mathcal{D}$-ring (R,e).

Theorem 2. The $\mathcal{D}$-Weil descent

If $M_b^{\sigma_i}$ is invertible for each associated endomorphism σ_i of (B,f), then $F^{\mathcal{D}}$ has a left adjoint.

Proof. Given a (B,f)-algebra (C,g), we construct a $\mathcal{D}$-structure on the classical Weil descent of C, $W(C)$. We then show that the unit of the classical adjunction respects the appropriate $\mathcal{D}$-structures.
This is first done in the case when (C,g) is a $\mathcal{D}$-polynomial algebra with the standard $\mathcal{D}$-ring structure. Then the $\mathcal{D}$-ring structure we put on $W(C)$—which is also a polynomial algebra—is a twisting of the standard one by the matrices $(M_b^{\sigma_i})^{-1}$. ∎

A partial converse

If $F^{\mathcal{D}}$ has a left adjoint, must each $M_b^{\sigma_i}$ be invertible? We do not know yet in general, but we do know under some conditions on the structure of such a left adjoint. For each $z \in B \otimes_k \mathcal{D}$, we let $g_z\colon B[x] \to B[x] \otimes_k \mathcal{D}$ extend f on B and map $x \mapsto z$.

Theorem 3.

Suppose $F^{\mathcal{D}}$ has a left adjoint $W^{\mathcal{D}}$. For each $z \in B \otimes_k \mathcal{D}$, if the underlying A-algebra of $W^{\mathcal{D}}(B[x], g_z)$ is faithfully flat as an A-module, then each matrix $M_b^{\sigma_i}$ is invertible. Note that this condition is always satisfied if A is a field.

Theorem 4. The difference Weil descent

Suppose $(K,\sigma) \leq (L,\tau)$ is a difference field extension, L/K is a finite extension of fields, and σ is an automorphism. Then $F^{\mathcal{D}}$ has a left adjoint if and only if τ is an automorphism.

References

[1] Özlem Beyarslan, Daniel Hoffmann, Moshe Kamensky, and Piotr Kowalski.
Model theory of fields with free operators in positive characteristic.
Transactions of the American Mathematical Society, 372(8):5991–6016, 2019.

[2] Omar León Sánchez and Marcus Tressl.
Differentially Large Fields.
arXiv:2005.00888, Preprint, 2020.

[3] Rahim Moosa and Thomas Scanlon.
Model theory of fields with free operators in characteristic zero.
Journal of Mathematical Logic, 14(2):1450009, 2014.

[4] Florian Pop.
Embedding Problems Over Large Fields.
Annals of Mathematics, 144(1):1–34, 1996.

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