1 Introduction

Weil restriction is a result of classical algebra:

Theorem 1.1 (Weil, Grothendieck)**.** *If A is a ring and B an A-algebra, with B finite and free as an A-module, then base change $F\colon \mathsf{Alg}_A \to \mathsf{Alg}_B$, $R \mapsto R \otimes_A B$ has a left adjoint, W.*

Definition 1.2. $W\colon \mathsf{Alg}_B \to \mathsf{Alg}_A$ is a left adjoint to F if for any B-algebra C and any A-algebra R, we have the following natural bijection:

$$\mathrm{Hom}_A(W(C), R) \to \mathrm{Hom}_B(C, F(R))$$

Weil restriction has several uses in number theory and geometry, but we are interested in a specific use. It is used to show that algebraic extensions of large fields are again large.

Definition 1.3. A field K is large if whenever V is a smooth K-variety, then $V(K) = \emptyset$ or $V(K)$ is Zariski dense in V.

Proof. Let L/K be a finite extension of fields, and V a smooth L-variety. Then $W(V)$ is a smooth K-variety. By taking $R = K$ and $C = L[V]$, the adjunction gives a bijection

$$\mathrm{Hom}_K(K[W(V)], K) \to \mathrm{Hom}_L(L[V], L)$$

The left hand side is in correspondence with $W(V)(K)$, and the right with $V(L)$. If $V(L) \neq \emptyset$, then $W(V)(K) \neq \emptyset$, so $W(V)(K)$ is Zariski dense in $W(V)$ by largeness, so $V(L)$ is Zariski dense in V. ∎

We will show that a similar adjunction result holds when we add extra structure. This structure comes from what are known as $\mathcal{D}$-rings.

2 $\mathcal{D}$-rings

$\mathcal{D}$-rings are a framework for talking about classes of rings with a finite collection of free operators. The behaviour of the operators is determined by the choice of $\mathcal{D}$.

Fix a field k, a finite-dimensional k-algebra $\mathcal{D}$, and a k-basis $\{\varepsilon_1, \ldots, \varepsilon_l\}$ of $\mathcal{D}$. We define $a_{ijk} \in k$ by $\varepsilon_i \varepsilon_j = \sum_{k=1}^{l} a_{ijk} \varepsilon_k$.

Definition 2.1. $(R, e_1, \ldots, e_l)$ is a $\mathcal{D}$-ring if R is a k-algebra, $e_i\colon R \to R$ and the map

$$\begin{aligned} e\colon R &\to R \otimes_k \mathcal{D} \\ r &\mapsto e_1(r) \otimes \varepsilon_1 + \ldots + e_l(r) \otimes \varepsilon_l \end{aligned}$$

is a k-algebra homomorphism.

Equivalently, we can say that (R, e) is a $\mathcal{D}$-ring if R is a k-algebra and $e\colon R \to R \otimes_k \mathcal{D}$ is a k-algebra homomorphism.

This condition is equivalent to saying that the maps e_i are k-linear and satisfy a certain multiplicativity condition, ie that $e_k(rs) = \sum_{i,j} a_{ijk} e_i(r) e_j(s)$.

Example.

- n endomorphisms: $e\colon R \to R \otimes_k k^n \cong R^n$
- Derivations

Don't mention commuting case.

We will work with $\mathcal{D}$-rings using the homomorphism e, but we should think of these things intuitively as algebras with operators that satisfy certain laws.

Definition 2.2. Write $\mathcal{D} = \prod_{i=1}^{t} B_i$ where each B_i is a local k-algebra. We assume the residue field of B_i is actually k. Define π_i as the composition $\mathcal{D} \to B_i \to k$. This map lifts to $\pi_i^R\colon R \otimes_k \mathcal{D} \to R$. For a $\mathcal{D}$-ring (R, e), we define $\sigma_i = \pi_i^R \circ e\colon R \to R$. Then $\sigma_1, \ldots, \sigma_t$ are the associated endomorphisms of (R, e).

3 $\mathcal{D}$-base change

How do we extend the base change functor to $\mathcal{D}$-rings?

First we need the appropriate notion of a structure-preserving map:

Definition 3.1. If (R, u) and (S, v) are two $\mathcal{D}$-rings and $\theta\colon R \to S$ is a map, then we say θ is a $\mathcal{D}$-homomorphism if it is a k-algebra homomorphism and the following diagram commutes:

$$\begin{tikzcd} R \otimes_k \mathcal{D} \arrow[r, "\theta\otimes\mathrm{id}_{\mathcal{D}}"] & S \otimes_k \mathcal{D} \\ R \arrow[u, "u"] \arrow[r, "\theta"] & S \arrow[u, "v"'] \end{tikzcd} \iff \begin{tikzcd} R \arrow[r, "\theta"] & S \\ R \arrow[u, "u_i"] \arrow[r, "\theta"] & S \arrow[u, "v_i"'] \end{tikzcd} \text{ for each } i = 1, \ldots, l$$

Let (A, e) be a $\mathcal{D}$-ring and (B, f) an (A, e)-algebra — this means that the algebra structure map ϕ is a $\mathcal{D}$-homomorphism.

What is the appropriate category here? $\mathsf{Alg}_{(A,e)}$ is the category of (A, e)-algebras with (A, e)-algebra homomorphisms — A-algebra homomorphisms which are also $\mathcal{D}$-homomorphisms.

Given an (A, e)-algebra (R, u), there is a unique $\mathcal{D}$-ring structure on $R \otimes_A B$ that makes the natural maps $\mathcal{D}$-ring homomorphisms. We call this $u \otimes f$. We then define the $\mathcal{D}$-base change functor as:

$$\begin{aligned} F^{\mathcal{D}}\colon \mathsf{Alg}_{(A,e)} &\to \mathsf{Alg}_{(B,f)} \\ (R, u) &\mapsto (R \otimes_A B, u \otimes f) \end{aligned}$$

Example. If $\mathcal{D} = k$, so that $\mathcal{D}$-rings are rings with an endomorphism, then $u \otimes f$ is just the usual tensor product of endomorphisms.

4 A counterexample

Does $F^{\mathcal{D}}$ have a left adjoint?

$F^{\mathcal{D}}$ does not always have a left adjoint. Let $\mathcal{D} = k$ so that $\mathcal{D}$-rings are precisely rings with an endomorphism. Then the $\mathcal{D}$-base change functor is just the tensor product of endomorphisms.

Let A be a ring, $B = A[\varepsilon]/(\varepsilon^2)$, $e = \mathrm{id}_A$ and $f(a + b\varepsilon) = a$. Suppose $F^{\mathcal{D}}$ had a left adjoint $W^{\mathcal{D}}$. Let $C = B[x]$ and g the unique endomorphism of C extending f and sending $x \mapsto \varepsilon$. Then $W^{\mathcal{D}}(C, g)$ is an (A, e)-algebra so write it as (R, u) where R is an A-algebra and u extends e.

Let η be the component of the unit of the adjunction at (C, g). That is, in the adjunction bijection, let $R = W^{\mathcal{D}}(C, g)$ and let η be the image of $\mathrm{id}_{W^{\mathcal{D}}(C,g)}$.

$$\eta \colon (C, g) \to F^{\mathcal{D}}(W^{\mathcal{D}}(C, g)) = (R \otimes_A B, u \otimes f)$$

η is a B-algebra homomorphism and a difference ring homomorphism. Then $\eta(g(x)) = (u \otimes f)(\eta(x))$.

$$\begin{aligned}
\eta(g(x)) &= (u \otimes f)(\eta(x)) \\
\eta(\varepsilon) &= u(r_1) \otimes f(1) + u(r_2) \otimes f(\varepsilon) \\
1 \otimes \varepsilon &= u(r_1) \otimes 1
\end{aligned}$$

Since $1 \otimes 1$ and $1 \otimes \varepsilon$ is an R-basis of $R \otimes_A B$, this is a contradiction and hence no left adjoint exists.

5 The $\mathcal{D}$-Weil descent

The issue here is that the following matrix is not invertible:

$$\begin{bmatrix} \mathrm{pr}_1(f(1)) & \mathrm{pr}_1(f(\varepsilon)) \\ \mathrm{pr}_2(f(1)) & \mathrm{pr}_2(f(\varepsilon)) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

where the pr_i are the functions that select the coefficient of the ith basis element.

Definition 5.1. In general, for an endomorphism f of B and a basis $b_1, \ldots, b_r$, we define

$$M_f = \begin{bmatrix} \mathrm{pr}_1(f(b_1)) & \ldots & \mathrm{pr}_1(f(b_r)) \\ \vdots & & \\ \mathrm{pr}_r(f(b_1)) & \ldots & \mathrm{pr}_r(f(b_r)) \end{bmatrix}$$

If M_f is invertible, then the left adjoint exists.

In the case of a general $\mathcal{D}$, we construct similar matrices for each of the associated endomorphisms of (B, f). If each of these matrices is invertible, then the left adjoint exists.

We construct it by putting an appropriate $\mathcal{D}$-ring structure on the classical Weil descent $W(C)$.

Theorem 5.2. *If for each associated endomorphism σ_i of (B, f) has M_{σ_i} invertible, then $F^{\mathcal{D}}$ has a left adjoint.*

6 Final remarks

Is this condition on the matrices necessary? That is, if $F^{\mathcal{D}}$ has a left adjoint, must each associated endomorphism of (B, f) have invertible matrix?

Proposition 6.1. *If A is a field, and $F^{\mathcal{D}}$ has a left adjoint, then each M_{σ_i} is invertible.*

Proof. If A is a field, then yes. We do this in the case $\mathcal{D} = k$. Let $b_1, \ldots, b_r$ be an A-basis of B.

Let $C = B[x]$ and g extends f on B and maps $x \mapsto b_i$. Let $W^{\mathcal{D}}(C, g) = (R, u)$ and let η be the component of the unit at (C, g). Then η is a B-algebra homomorphism and a difference ring homomorphism. Write $\eta(x) = \sum_{i=1}^{r} s_i \otimes b_i$ for $s_i \in R$.

$$\begin{aligned}\eta(g(x)) &= \eta(b_i) \\ &= 1 \otimes b_i\end{aligned}$$

and

$$\begin{aligned}u \otimes f(\eta(x)) &= \sum_{i=1}^{r} u(s_i) \otimes f(b_i) \\ &= \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_j(f(b_i)) u(s_i) \otimes b_j\end{aligned}$$

Now, equating coefficients of the basis $1 \otimes b_i$ gives us a matrix equation

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = M \begin{bmatrix} u(s_1) \\ \vdots \\ u(s_r) \end{bmatrix}$$

This is a linear system defined over A with a solution in some A-algebra (namely R), and since A is a field, this system has a solution in A. So M is onto, and hence invertible. ■

This condition of the matrix being invertible is a bit strange. However, if each associated endomorphism of (A, e) is an automorphism, then this condition is equivalent to each associated endomorphism of (B, f) being an automorphism.

Theorem 6.2. *Suppose A is a field and each associated endomorphism of e is an automorphism. Then $F^{\mathcal{D}}$ has a left adjoint if and only if each associated endomorphism of f is an automorphism.*

Lemma 6.3. *1. Let (C, g) be a (B, f)-algebra with associated endomorphisms $\rho_1, \ldots, \rho_t$. Then $W^{\mathcal{D}}(C, g)$ is an (A, e)-algebra with associated endomorphisms $\rho_1^W, \ldots, \rho_t^W$, where these are calculated using the $\mathcal{D}$-Weil descent for $\mathcal{D} = k$. In addition, if $\rho_i = \mathrm{id}_C$, then $\rho_i^W = \mathrm{id}_{W(C)}$.*

2. If (C, g) has $g_i g_j = g_j g_i$ for all i, j, then $W^{\mathcal{D}}(C, g)$ has $g_i^W g_j^W = g_j^W g_i^W$ for all i, j.

Example.

1. Let $\mathcal{D} = k^n$ so $\mathcal{D}$-rings are rings with n endomorphisms. By the lemma, $W^{\mathcal{D}}$ restricts to the category of inversive difference rings — rings with n commuting automorphisms.

2. Let $\mathcal{D} = k[\varepsilon_1, \ldots, \varepsilon_l]/(\varepsilon_1, \ldots, \varepsilon_l)^2$. By the lemma, $W^{\mathcal{D}}$ restricts to the category of $\mathcal{D}$-rings whose associated endomorphisms are trivial and whose operators commute — this is precisely the category of rings with n commuting derivations.

Remark. We define the notion $\mathcal{D}$-largeness to be analogous to largeness but for $\mathcal{D}$-fields. The $\mathcal{D}$-Weil descent then allows us to prove that algebraic extensions of $\mathcal{D}$-large fields are again $\mathcal{D}$-large, in much the same way as for pure fields.