Very slim differential fields of characteristic zero

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Introduction

Let T be the theory of a field of characteristic zero and T^* the model companion of $T \cup \{\text{“}\delta \text{ is a derivation”}\}$, if it exists. If T has some model-theoretic property P, does T^* also have P?

- if T has quantifier elimination in $\mathcal{L}_{\text{ring}}$, then $T = \mathrm{ACF}_0$ and $T^* = \mathrm{DCF}_0$ has quantifier elimination in $\mathcal{L}_{\text{ring}}(\delta)$.
- if T is stable, then, assuming the stable fields conjecture, $T = \mathrm{ACF}_0$, $T^* = \mathrm{DCF}_0$ and T^* is stable.
- if T is simple, then, assuming the simple fields conjecture, T is the theory of a bounded and PAC field, so T^* is bounded, PAC, and differentially large, and hence simple.

In the above examples, we see that model-theoretic properties are transferred from T to T^* by first showing that if T has this property, it must be the theory of a specific type of field. Can these properties be shown to transfer more generally, and without assuming the aforementioned conjectures?

Fields with operators

Instead of just considering derivations, we consider an axiomatic approach to fields with operators.

Let $\mathfrak{D} \supseteq \mathcal{L}_{\text{ring}}$ and $\tilde{T}$ a $\mathfrak{D}$-theory extending the theory of fields. We say that $\tilde{T}$ is derivation-like if the following conditions hold:

(1) If $\tilde{M} \models \tilde{T}$ and N is a separable field extension of $\tilde{M}$, then N can be expanded to a $\mathfrak{D}$-structure $\tilde{N} \models \tilde{T}$ with $\tilde{N} \geq \tilde{M}$.

(2) If N is separably algebraic over $\tilde{M}$, then there is a unique such $\mathfrak{D}$-structure on N.

(3) If $\tilde{M} \models \tilde{T}$ and $\tilde{A}$ and $\tilde{B}$ are submodels of $\tilde{M}$, then the field compositum $\tilde{A}\tilde{B}$ is also a submodel of $\tilde{M}$.

(4) Suppose $\tilde{M}, \tilde{N} \models \tilde{T}$ inside a field F. Suppose also that, inside F, $\tilde{M}$ and $\tilde{N}$ are linearly disjoint over a common submodel $\tilde{A}$. Then there is a unique $\mathfrak{D}$-structure on the field compositum $\tilde{M}\tilde{N}$ that makes $\tilde{M}\tilde{N}$ into a model of $\tilde{T}$ extending $\tilde{M}$ and $\tilde{N}$.

Examples

Several commuting derivations; (commuting) free operators coming from a local algebra in the sense of Moosa–Scanlon; derivations of Frobenius; operators coming from ring schemes in the sense of Gogolok–Kowalski under the assumption $\mathrm{Fr}_{\mathcal{B}}(\ker \pi)) = 0$.

Field automorphisms are a nonexample—they fail conditions (1) and (2).

Very $\mathcal{L}$-slimness

Consider first the question of whether simplicity is transferred from K to (K, δ). To prove this via the Kim–Pillay theorem, we need to understand nonforking independence in K. We introduce the following modification of a notion of Junker–Koenigsmann.

Let $\mathcal{L} \supseteq \mathcal{L}_{\text{ring}}$ and K an $\mathcal{L}$-structure. K is $\mathcal{L}$-slim if $\mathrm{acl}(A) = A^{\text{alg}}$ for every $\mathcal{L}$-substructure A. A^{alg} is the field-theoretic relative algebraic closure of A inside K.

K is very $\mathcal{L}$-slim if every $\mathcal{L}$-structure elementarily equivalent to K is $\mathcal{L}$-slim; equivalently if a sufficiently saturated model of $\mathrm{Th}(K)$ is $\mathcal{L}$-slim.

For subsets A, B, C of an $\mathcal{L}$-structure K, write

$$A \mathop{\smile\!\!\!\!\!|}^{\mathcal{L}}_{C} B \iff \langle A\rangle_{\mathcal{L}} \text{ is algebraically independent from } \langle B\rangle_{\mathcal{L}} \text{ over } \langle C\rangle_{\mathcal{L}}$$

Lemma

Suppose $\mathcal{L}$ is such that the $\mathcal{L}$-structure generated by AB is the compositum of the $\mathcal{L}$-structures generated by A and B.

Then K is very $\mathcal{L}$-slim if and only if $\mathop{\smile\!\!\!\!\!|}^{\mathcal{L}}$ satisfies existence:

$$\text{for any } A, B, C, \text{ there is } A' \equiv_C A \text{ such that } A' \mathop{\smile\!\!\!\!\!|}^{\mathcal{L}}_{C} B$$

The above assumption is needed for transitivity of $\mathop{\smile\!\!\!\!\!|}^{\mathcal{L}}$.

Independence relations

In this case, $\mathop{\smile\!\!\!\!\!|}^{\mathcal{L}}$ is an independence relation: this means it satisfies invariance under automorphisms, transitivity, finite character, local character, symmetry, and existence.

The proof of the Kim–Pillay theorem tells us that

$$\mathrm{tp}_{\mathcal{L}}(A/BC) \text{ does not divide over } C \implies A \mathop{\smile\!\!\!\!\!|}^{\mathcal{L}}_{C} B \qquad (\dagger)$$

Theorem

Let T be the $\mathcal{L}$-theory of a very $\mathcal{L}$-slim structure, and T^* the model companion of $T \cup \tilde{T}$, which we assume exists. Then

(i) $\mathrm{acl}(A) = \mathrm{acl}_T\langle A\rangle_{\mathcal{L}(\mathfrak{D})} = \langle A\rangle^{\text{alg}}_{\mathcal{L}(\mathfrak{D})}$

(ii) If T is simple, then T^* is simple. Nonforking independence in T^* is given by

$$A \mathop{\smile\!\!\!\!\!|}^{*}_{C} B \iff \mathrm{acl}(A) \mathop{\smile\!\!\!\!\!|}_{\mathrm{acl}(C)} \mathrm{acl}(B)$$

(iii) If T has quantifier elimination in $\mathcal{L}$, then T^* has quantifier elimination in $\mathcal{L}(\mathfrak{D})$.

(iv) If T is stable, then T^* is stable.

Proof

(i) Let $F = \langle A\rangle^{\text{alg}}_{\mathcal{L}(\mathfrak{D})}$ and $d \notin F$. We show $\mathrm{tp}(d/F)$ is not algebraic by finding infinitely many realisations. Let $H = \langle Fd\rangle_{\mathcal{L}(\mathfrak{D})}$ and use existence of $\mathop{\smile\!\!\!\!\!|}^{\mathcal{L}}$ to find $H' \equiv_F H$ such that $H' \mathop{\smile\!\!\!\!\!|}^{\mathcal{L}}_{F} H$. This induces an $\mathcal{L}(\mathfrak{D})$-structure on H' and we can use condition (4) of $\mathfrak{D}$ to amalgamate the $\mathfrak{D}$-structures on H' and H to their compositum $H'H$, and finally extend to K using (1). The copy of d in H' gives another realisation of $\mathrm{tp}(d/F)$ and we can then iterate.

(ii) Most of the properties of $\mathop{\smile\!\!\!\!\!|}^{*}$ being an independence relation follow immediately from the corresponding property for $\mathop{\smile\!\!\!\!\!|}$. Condition (3) is needed for transitivity. Existence and the independence theorem follow from similar amalgamation arguments as in (i) using fact ($\dagger$).

(iii) is also a similar amalgamation argument.

(iv) Stationarity of $\mathop{\smile\!\!\!\!\!|}^{*}$ follows from stationarity of $\mathop{\smile\!\!\!\!\!|}$ and the fact that quantifier elimination transfers.

Example

Let K be a bounded PAC field. For each $n > 1$, let $N(n)$ be the degree over K of the Galois extension composite of all Galois extensions of K of degree n.

Let $C = (c_{n,i})_{n>1, 0\leq i \leq N(n)}$ be a new set of constants and let T be the $\mathcal{L}_{\text{ring}}(C)$-theory which adds to $\mathrm{Th}_{\mathcal{L}_{\text{ring}}}(K)$ sentences saying that the polynomial

$$x^{N(n)} + c_{n,N(n)-1}x^{N(n)-1} + \ldots + c_{n,0}$$

is irreducible and the extension it defines is Galois and contains all Galois extensions of degree n.

Then T is a model complete, simple $\mathcal{L}_{\text{ring}}(C)$-theory and K is large. A similar proof to that of Junker–Koenigsmann shows that T is very $\mathcal{L}_{\text{ring}}(C)$-slim.

Then for the first two examples given earlier of $\mathfrak{D}$, $T \cup \tilde{T}$ has a model companion which is simple. One can show that results of bounded PAC fields established by Chatzidakis–Pillay transfer to this model companion; in particular, elimination of imaginaries transfers.

References

[1] Zoé Chatzidakis and Anand Pillay.
Generic structures and simple theories.
Annals of Pure and Applied Logic, 95(1):71–92, 1998.

[2] Markus Junker and Jochen Koenigsmann.
Schlanke körper (slim fields).
The Journal of Symbolic Logic, 75(2):481–500, 2010.

[3] Omar León Sánchez and Marcus Tressl.
Differentially large fields.
To appear in Algebra and Number Theory.

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