Задачи 2

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1:

$$\begin{gathered}
\int \frac{dx}{\sin^2 x \cos^2 x} \\
\int \frac{dx}{\frac{1}{4} \cdot \sin^2 2x} \\
t = 2x,\ dx = \frac{dt}{2} \\
2 \int \frac{dt}{\sin^2 t} = -2 \operatorname{ctg} t + C = -2 \operatorname{ctg} 2x + C
\end{gathered} \tag{1}$$

2:

$$\begin{gathered}
x(t) = \int_1^{t^2} t \ \ln t \ dt \\
y(t) = \int_{t^2}^1 t^2 \ \ln t \ dt \\
\frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dx} \\
\frac{dx}{dt} = \frac{d}{dt} \int_1^{t^2} t \ln t \ dt = 4t^3 \ln t \\
\frac{dy}{dt} = \frac{d}{dt} \int_{t^2}^1 t^2 \ln t \ dt = -4t^5 \ln t \\
\implies \frac{dy}{dx} = -t^2
\end{gathered} \tag{2}$$

3:

$$\int_{-\ln 2}^{0} \sqrt{1-e^{2x}}dx$$

$$t = 1-e^{2x},\ dt = -2e^{2x}dx,\ x = \frac{1}{2}\ln(1-t)$$

$$x = -\ln 2,\ t = \frac{3}{4},\quad x = 0,\ t = 0$$

$$-\int_{0}^{3/4} \frac{\sqrt{t}}{-2e^{\ln(1-t)}}dt = \frac{1}{2}\int_{0}^{3/4} \frac{\sqrt{t}}{1-t}dt$$

$$g = \sqrt{t},\ dg = \frac{dt}{2\sqrt{t}},\ t = g^2$$

$$\int \frac{2g^2}{-g^2+1}dg = 2\int -1 + \frac{1}{-g^2+1}dg$$

$$a+b = 1,\ a-b = 0 \implies a = b = 1/2$$

$$2\int (\frac{1}{2(1-g)} + \frac{1}{2(1+g)} - 1)dg = 2(-g - \frac{1}{2}\ln|1-g| + \frac{1}{2}\ln|1+g|) + C$$

$$[-\sqrt{t} - \frac{1}{2}\ln|1-\sqrt{t}| + \frac{1}{2}\ln(1+\sqrt{t})]_{0}^{3/4} = \frac{-\sqrt{3}}{2} - \frac{1}{2}\ln(1-\frac{\sqrt{3}}{2}) + \frac{1}{2}\ln(1+\frac{\sqrt{3}}{2})$$

$$= \frac{-\sqrt{3}}{2} + \frac{1}{2}\ln(\frac{2+\sqrt{3}}{2-\sqrt{3}}) = \frac{-\sqrt{3}}{2} + \ln(2+\sqrt{3}) \tag{3}$$

4:

$$\int_{-2}^{-1} \frac{x^3+1}{x^2(1-x)}dx$$
$$\int \frac{x^3}{x^2(1-x)}dx + \int \frac{1}{x^2(1-x)}dx$$
$$\int \frac{x}{1-x}dx + \int \frac{1}{x^2(1-x)}dx$$
$$\text{Рассмотрел } \int \frac{x}{1-x}dx,\ t = 1-x,\ dx = -dt$$
$$-\int \frac{1-t}{t}dt = -\ln|t| + t - C =$$
$$1 - x - \ln|1-x| - C = -x - \ln|1-x| - C$$
$$\text{Теперь второй } \int \frac{1}{x^2(1-x)}dx$$
$$\frac{1}{x^2(1-x)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{1-x}$$
$$\frac{x}{x^3(1-x)} = \frac{ax^2 - ax^3}{x^3(1-x)} + \frac{bx - bx^2}{x^3(1-x)} + \frac{cx^3}{x^3(1-x)}$$
$$0 = -a + c;\ 0 = a - b;\ 1 = b \implies a = c = 1$$
$$\int \frac{dx}{1-x} + \int \frac{dx}{x} + \int \frac{dx}{x^2} = -\ln|1-x| + \ln x - \frac{1}{x} + C$$
$$\text{Подставил } [-x - 2\ln|1-x| + \ln x - \frac{1}{x}]_{-2}^{-1} =$$
$$(2 - 2\ln 2 + \ln -1) - (2.5 - 2\ln 3 + \ln -2) = -0.5 + 2\ln\frac{3}{2} + \ln 0.5 =$$
$$-0.5 + \ln\frac{9}{4} + \ln 0.5 = -0.5 + \ln\frac{9}{8} \tag{4}$$

5:

$$\int_{\frac{1-\sqrt{5}}{2}}^{1/2}(1-3x)\sqrt{1+x-x^2}\ dx\ (1)$$

$$\sqrt{1+x-x^2}=\sqrt{-(x-\frac{1}{2})^2+\frac{5}{4}}$$

$$\int(1-3x)\sqrt{-(x-\frac{1}{2})^2+\frac{5}{4}}\ dx$$

$$x-\frac{1}{2}=\frac{\sqrt{5}}{2}\sin t,\ dx=\frac{\sqrt{5}}{2}\cos t\ dt$$

$$\frac{5}{4}\int(1-3(\frac{1}{2}+\frac{\sqrt{5}}{2}\sin t))\cos^2 t\ dt$$

$$\frac{5}{4}\int(-\frac{1}{2}-3\frac{\sqrt{5}}{2}\sin t)\cos^2 t\ dt$$

$$-\frac{5}{8}\int\cos^2 t\ dt-15\frac{\sqrt{5}}{8}\int\sin t\cos^2 t\ dt$$

$$-\frac{5}{8}(\frac{1}{2}t+\frac{\sin 2t}{4})-\frac{15\sqrt{5}}{8}\int\cos^2 t\ d(\cos t)$$

$$-\frac{5}{8}(\frac{1}{2}t+\frac{\sin 2t}{4})+\frac{5\sqrt{5}}{8}\cos^3 t+C$$

$$t=\arcsin\frac{2x-1}{\sqrt{5}}$$

$$x=\frac{1}{2},\ t=0,\ x=\frac{1-\sqrt{5}}{2},\ t=-\frac{\pi}{2}$$

$$(1)=\frac{5\sqrt{5}}{8}-\frac{5\pi}{32} \tag{5}$$

6:

$$\int \frac{\cos x \ dx}{\sin x - 5\cos x}$$

$$t = \operatorname{tg} x,\ dt = \sec^2 x dx,\ x = \operatorname{arctg} t$$

$$\int \frac{\cos^3(\operatorname{arctg} t)\ dt}{(\sin(\operatorname{arctg} t) - 5\cos(\operatorname{arctg} t))}$$

$$\int \frac{\frac{1}{(1+t^2)\sqrt{1+t^2}}}{\frac{t-5}{\sqrt{1+t^2}}}\ dt$$

$$\int \frac{dt}{(1+t^2)(t-5)} = \int (\frac{a}{t-5} + \frac{bt+c}{t^2+1})\ dt$$

$$1 = a + at^2 + bt^2 - 5bt + ct - 5c,\ a = \frac{1}{26},\ b = -\frac{1}{26},\ c = -\frac{5}{26}$$

$$\frac{1}{26}\int (\frac{1}{t-5} - \frac{t+5}{t^2+1})\ dt$$

$$\frac{1}{26}(\ln|t-5| - \frac{1}{2}\ln(t^2+1) - 5\operatorname{arctg} t) + C$$

$$\frac{1}{26}(\ln|\operatorname{tg} x - 5| - \frac{1}{2}\ln\sec^2 x - 5x) + C \tag{6}$$

7:

$$\int_2^{+\infty} (\cos\frac{2}{x} - 1)\ dx$$

$$t = x - 2,\ dt = dx$$

$$\int_0^{+\infty} (\cos\frac{2}{t+2} - 1)\ dt$$

$$t \to +\infty,\ \frac{2}{t+2} \to 0$$

$$1 - \cos\alpha \sim \frac{\alpha^2}{2},\ \alpha \to 0$$

$$\cos(\frac{2}{t+2}) - 1 \sim -\frac{2}{(t+2)^2} \sim -\frac{2}{t^2}$$

$$\Longrightarrow \ \text{интеграл сходится} \tag{7}$$

8:

$$
\begin{gathered}
\int_0^1 \frac{2-\sqrt[3]{x}-x^3}{\sqrt[5]{x^3}}\ dx \\
t=\sqrt[15]{x},\ dx=15t^{14}\ dt \\
\int \frac{2-t^5-t^{45}}{t^9}\ 15t^{14}dt \\
30\int t^5\ dt-15\int t^{10}\ dt-15\int t^{50}\ dt \\
5t^6-\frac{15}{11}t^{11}-\frac{15}{51}t^{51}+C \\
\sqrt[15]{1}=1,\ \ \sqrt[15]{0}=0 \\
[5t^6-\frac{15}{11}t^{11}-\frac{15}{51}t^{51}]_0^1= \\
=5-\frac{15}{11}-\frac{15}{51}=\frac{625}{187}
\end{gathered}
\tag{8}
$$

9:

$$S = \frac{1}{2}\int_{t_0}^{t_1} (xy' - x'y)\ dt$$

$$x = \frac{1}{1+t^2},\ y = \frac{t(1-t^2)}{1+t^2}$$

$$S = \frac{1}{2}\int_{-1}^{1} \frac{-t^4-4t^2+1}{(t^2+1)^3} + \frac{2t^2(1-t^2)}{(t^2+1)^3}dt$$

$$\int \frac{-3t^4-2t^2+1}{(t^2+1)^3}dt = \frac{at^3+bt^2+ct+d}{(t^2+1)^2} + \int \frac{et+f}{t^2+1}dt$$

$$-3t^4-2t^2+1 = -at^4+3at^2-2bt^3+2bt-3ct^2+c-4dt+et^5+2et^3+et+ft^4+2ft^2+f$$

$$t^5:\ e = 0$$

$$t^4:\ -a+f = -3$$

$$t^3:\ -2b+2e = 0$$

$$t^2:\ 3a-3c+2f = -2$$

$$t:\ 2b-4d+e = 0$$

$$t^0:\ c+f = 1$$

$$a = c = 2,\ b = d = e = 0,\ f = -1$$

$$\frac{2t^3+2t}{(t^2+1)^2} - \int \frac{dt}{t^2+1}$$

$$S = [\frac{t}{t^2+1} - \frac{1}{2}\arctan t]_{-1}^{1} = 1 - \frac{\pi}{4}$$

9':

$$\text{Исследование кривой: } x(t) = \frac{1}{1+t^2},\ y(t) = \frac{t(1-t^2)}{1+t^2}$$

$$t^2 = \frac{1}{x} - 1,\ \to y(x) = x\sqrt{\frac{1}{x}-1}(2-\frac{1}{x})$$

$$y = 0,\ x = 0 \vee x = 1 \vee x = \frac{1}{2}$$

$$y' = \frac{-4x^2+2x+1}{2x\sqrt{x-x^2}},\ y' = 0 \text{ if } x = \frac{1+\sqrt{5}}{2} > \frac{1}{2}$$

$$\text{значит петля у кривой между } x = \frac{1}{2} \text{ и } x = 1$$

$$\begin{aligned} S = [\frac{t}{t^2+1} - \frac{1}{2}\arctan t]_{\frac{1}{2}}^{1} = \frac{1}{2} - \frac{\pi}{8} - \frac{2}{5} + \frac{1}{2}\arctan\frac{1}{2} = \\ = \frac{1}{10} - \frac{\pi}{8} + \frac{1}{2}\arctan\frac{1}{2} \end{aligned} \tag{9}$$

10:

$$0 \leq x \leq \frac{9}{16},\ y = \sqrt{1-x^2} + \arcsin x$$
$$l = \int_{x_0}^{x_1} \sqrt{1+y'^2}\ dx$$
$$y' = \frac{\sqrt{1-x^2}}{1+x}$$
$$l = \int_0^{\frac{9}{16}} \sqrt{1 + \frac{1-x^2}{(1+x)^2}}\ dx =$$
$$\int_0^{\frac{9}{16}} \frac{\sqrt{2}}{\sqrt{1+x}}\ dx = [\sqrt{2} \cdot 2\sqrt{1+x}]_0^{\frac{9}{16}} =$$
$$\frac{5\sqrt{2}}{2} - 2\sqrt{2} = \frac{\sqrt{2}}{2} \tag{10}$$

11:

$$S = 2\pi \int_a^b |y|\ dl,\ dl = \sqrt{1+y'^2}\ dx,\ y' = -\frac{1}{e^x}$$
$$S = 2\pi \int_0^{+\infty} |e^{-x}| \sqrt{1 + \frac{1}{e^{2x}}}\ dx =$$

e^{-x} неотрицательна на промежутке интегрирования; $t = e^{-x}, dt = -e^{-x}\ dx$

$$2\pi \int_1^0 -\sqrt{t^2+1}\ dt = \pi[t\sqrt{(t^2+1)} + \ln(|\sqrt{t^2+1}+t|)]_0^1 =$$
$$\pi(\sqrt{2} + \ln(1+\sqrt{2})) \tag{11}$$

12:

$$
\begin{gathered}
\sum_{i=1}^{n} \frac{1}{n(n+1)(n+2)} = \\
a_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)} \\
= \frac{1}{4}(\cancel{2} - \cancel{2} + \frac{2}{3}) + \frac{1}{4}(1 - \frac{4}{3} + \frac{1}{2}) \ldots = \frac{1}{4}(1 - \frac{2}{3} + \frac{1}{2}) \ldots \\
S_n = \frac{1}{4}(1 - \frac{1}{2(n+2)} + \frac{1}{2(n+3)}) = \frac{1}{4}(1 + \frac{1}{2(n+2)(n+3)}) = \frac{1}{4} \\
\text{Ряд сходится}
\end{gathered}
\tag{12}
$$

13:

$$
\begin{gathered}
\sum_{i=1}^{n} \frac{(-1)^n}{(2n + (-1)^n)^\alpha} \\
\text{По лейбницу: } \lim_{n\to\infty} \frac{1}{(2n + (-1)^n)^\alpha} \sim \lim_{n\to\infty} \frac{1}{2n^\alpha} \text{ ряд сходится при } \alpha > 1 \\
\text{Ряд из модулей: } |a_n| = \frac{1}{(2n + (-1)^n)^\alpha} \sim \frac{1}{2n^\alpha} \text{ сходится при } \alpha > 1,\ \alpha \leq 1 - \text{ расходится} \\
\text{Тогда исх. ряд при } \alpha > 1 - \text{ сходится абсолютно} \\
\text{Рассмотрю } a_{2n} + a_{2n-1} = \frac{1}{(4n+1)^\alpha} - \frac{1}{(4n-3)^\alpha} = \\
= \frac{(4n-3)^\alpha - (4n+1)^\alpha}{(4n+1)^\alpha (4n-3)^\alpha} = \frac{4n^\alpha (1 - \frac{3}{4n})^\alpha - 4n^\alpha (1 + \frac{1}{4n})^\alpha}{4^{2\alpha} (1 + \frac{1}{4n})^\alpha (1 - \frac{3}{4n})^\alpha} = \\
= \frac{(1 - \frac{3\alpha}{4n} + \mathcal{O}(\frac{1}{n^2})) - (1 + \frac{\alpha}{4n})}{4n^\alpha (1 - \frac{3\alpha}{4n} + \mathcal{O}(\frac{1}{n^2})) \cdot (1 + \frac{\alpha}{4n})} \sim \frac{-\alpha}{4n^{\alpha+1}}, \text{ ряд расходится при } \alpha \leq 0
\end{gathered}
\tag{13}
$$