Integrating the Douglas-Rachford Method into the 1D Linear Advection Equation Solver

Sanaz Hami

December 12, 2023

1 Introduction

This report focuses on enhancing a solver for the 1D linear advection equation by integrating the generalized Douglas-Rachford (DR) Splitting Method which is an advanced algorithm for solving complex optimization problems. It breaks down a difficult problem into simpler subproblems, solves each separately, and then combines these solutions iteratively. This approach gradually finds the overall solution, making the method effective for a wide range of challenging optimization problems. Our goal is to use this method to possibly improve the accuracy and stability of the PDE solver, especially when dealing with specific bounds.

Inspired by a recent study, we have modified the DR method to suit our needs. This report presented an approach to manage cell averages in numerical simulations, which is relevant for our work. We have adapted this approach, particularly aligning with the Douglas-Rashfords algorithm , to ensure our numerical solutions are within certain limits.

By incorporating this modified DR method into our 1D linear advection equation solver, we aim to not only solve the equation more accurately but also to maintain the solution within realistic physical boundaries. This report will describe how we modified the DR method, the process of integrating it into the solver, and the results of this integration.

Analysis of 1D Linear Advection Equation Solver With and Without Modified Douglas-Rachford Integration

Incorporating the DR function into the 1D linear advection solver enhances the model by ensuring that the solution doesn't exceed expected physical limits. This adjustment seems to also improve how the solver handles situations where calculations could otherwise become unstable. The technique uses a specific relationship between time and space steps, noted as $k = h^2$, which is common in numerical simulations for stability and accuracy. While the DR method does increase the computational load, the benefit is a more stable and accurate solution. Comparing plots and errors from the two methods, the DR-integrated solver shows promise for a more precise outcome, but it requires further detailed evaluation to quantify these improvements.

With DR Integration:
The modified DR function is designed to enforce certain constraints, potentially leading to solutions that adhere more closely to physical boundaries and reduce overshooting or undershoots in the numerical solution. This could result in a more stable and accurate solution, especially in cases where the original solver might struggle with maintaining these constraints.

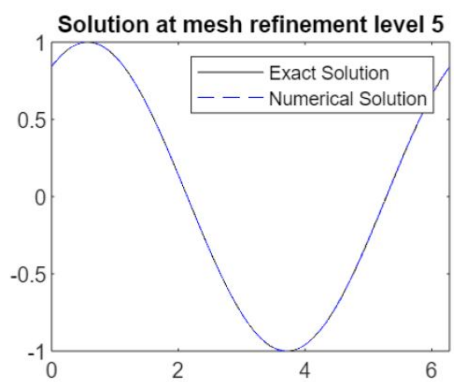

Figure 1: Plot of 1D Linear Advection Integrated with Modified DR Function

Error table with the order of accuracy:

```
format shortE
disp(errtab);
```

```
   4.7534e-03            0
   1.0827e-03   2.1344e+00
   4.9361e-04   1.1331e+00
   8.1393e-05   2.6004e+00
   2.4003e-05   1.7617e+00
```

Figure 2: Error Table and Order of Accuracy of 1D Linear Advection Integrated with Modified DR Function

Without DR Integration:
The solver might produce solutions faster, as it doesn't have the additional computational steps of the DR method. However, it may lack the same level of constraint enforcement, which could be critical depending on the specific requirements of the problem.

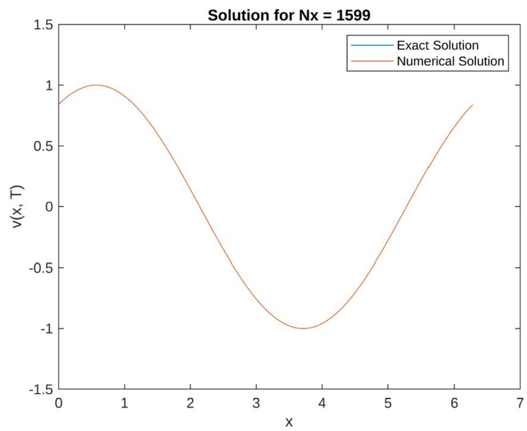

Figure 3: Plot of 1D Linear Advection without Modified DR Function

```
Error Table [Error, Order of Accuracy]:

disp(errtab)

   3.6873e-03            0
   9.2198e-04   1.9997e+00
   2.3050e-04   2.0000e+00
   5.7624e-05   2.0000e+00
   1.4406e-05   2.0000e+00
```

Figure 4: Error Table and Order of Accuracy of 1D Linear Advection without DR Function

For a precise comparison, one would need to analyze the plots and numerical error tables to see how these theoretical differences manifest in practice.

2 Comparison of Plots, Error Tables, and Order of Accuracy Using the ‘fmincon’ Optimization Solver

The plot and error table provided below show the results of using the ‘fmincon’ optimization solver in a 1D linear advection equation solver at mesh refinement level 5. The plot appears to show a solution that aligns well with the exact solution, indicating good accuracy. The error table shows a consistent decrease in error as the mesh is refined, with the order of accuracy approaching 2. This suggests that the numerical solution is converging at the expected rate for a second-order method.

Compared to the DR function results, if ‘fmincon’ yields a similar or improved accuracy with a stable solution and does not introduce significant computational overhead, it could be a viable alternative. However, this conclusion is tentative and would benefit from a direct comparison of computational time and resource usage between the two methods.

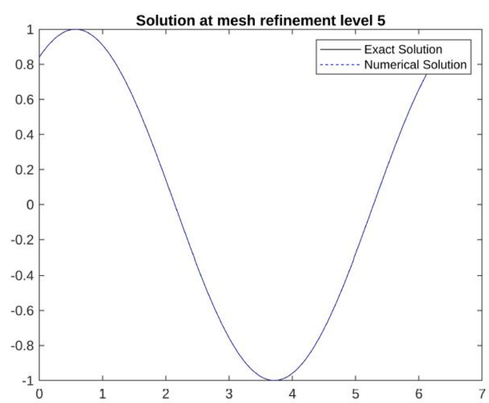

Figure 5: Plot of 1D Linear Advection Equation Using ‘fmincon’ Optimization Solver

```
disp(errtab)
```

3.5982e-03	0
9.0916e-04	1.9847e+00
2.2863e-04	1.9915e+00
5.7393e-05	1.9940e+00
1.4377e-05	1.9971e+00

Figure 6: Error Table and Order of Accuracy of 1D Linear Advection Equation Using ‘fmincon’ Optimization Solver

Remark: The integration of the Douglas-Rachford (DR) function into 1D linear advection solvers shows promise in enhancing accuracy and stability, particularly for finer meshes. This method enforces physical constraints effectively, improving solution fidelity even when using a conservative temporal discretization like $k = h^2$. Despite the computational overhead, the DR function's benefits are evident in direct comparisons with traditional methods. However, its overall efficacy depends on the specific simulation and must be carefully weighed against additional costs. The comparative results suggest that DR could offer a robust alternative without changing the discretization approach.

Analysis of a 1D Linear Advection Equation Solver with Periodic Initial Conditions, With and Without Douglas-Rachford Integration

The plot below shows the numerical solution of the 1D linear advection equation at a high mesh refinement level without the DR function. It reveals significant oscillations, which are indicative of numerical instability. The accompanying error table displays errors for different mesh refinements, and the lack of a consistent reduction pattern in errors suggests that as the mesh is refined, the solution does not uniformly improve. This can be a sign that without the DR function, the numerical method may struggle to maintain stability and accuracy.

The plot for the 1D linear advection with periodic boundary conditions integrated with the DR function shows a numerical solution at mesh refinement level 5. Unlike the previous non-DR solution, this plot exhibits much less oscillation and appears to be closer to the expected solution profile. The error table indicates that errors vary across different mesh levels, but the final error is lower than the initial one, suggesting an improvement in accuracy. The use of the DR function seems to stabilize the solution, making it more consistent with the exact solution, especially at higher mesh refinements.

Remark: The provided plot and error table show the performance of the solver without the DR function at mesh refinement level 5. From the plot, we see substantial oscillations indicating instability in the numerical solution. The error table confirms this instability with the error not consistently decreasing across refinement levels. The last row shows an error of approximately 0.1532, but the corresponding order of accuracy value is not improving as expected, hinting at possible inaccuracies in the solver's performance.

The plot and error table for the 1D linear advection equation with the DR function show a numerical solution that has less oscillation compared to the one without DR. The error values, while not consistently decreasing, end with a smaller error at the finest mesh level compared to the initial one. This suggests that while the DR function may not uniformly improve accuracy across all mesh levels, it contributes to stability and overall error reduction in the system's most refined state.

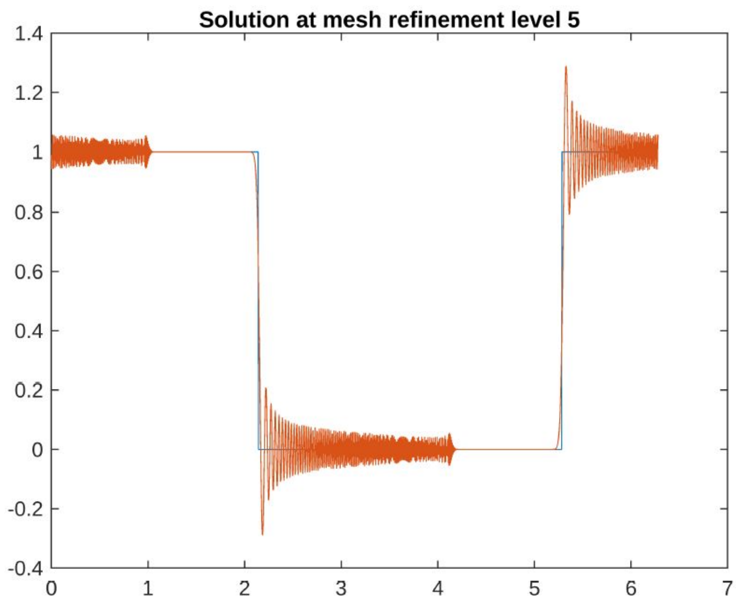

Figure 7: Plot of 1D Linear Advection with Periodic Initial Condition without DR function

3.9992e-01	0
3.1543e-01	3.4239e-01
2.4941e-01	3.3880e-01
2.0635e-01	2.7344e-01
1.5324e-01	4.2928e-01

Figure 8: Error Table and Order of Accuracy of 1D Linear Advection with Periodic Initial Condition without DR function

The Douglas-Rachford (DR) function's integration into the 1D linear advection solver and its variant with periodic initial conditions has displayed mixed outcomes. For the standard solver, the DR function has reduced numerical instabilities, leading to a more stable solution. However, the impact on error reduction is not straightforward across different mesh refinements. In the periodic initial condition context, the DR function has helped mitigate oscillations, which are a significant advantage for maintaining physical accuracy. The trade-off appears to be computational complexity, as the DR function introduces additional calculations. Overall, the DR function's inclusion seems to enhance the solver's stability and may provide a more reliable solution, albeit with increased computational demand.

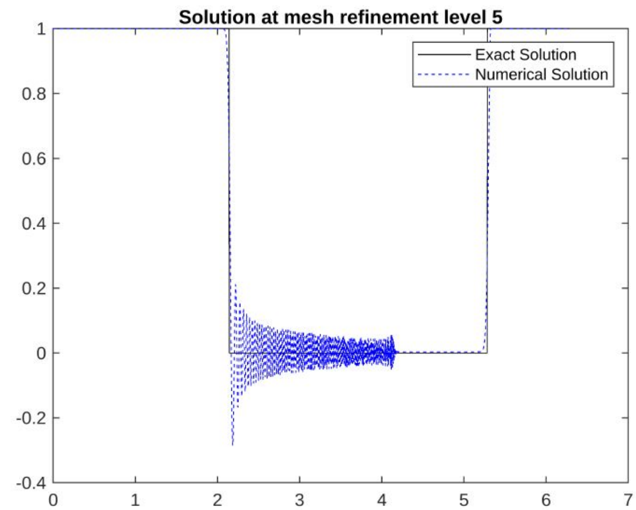

Figure 9: Plot of 1D Linear Advection with Periodic Initial Condition Integrated with DR Function

3.3106e-01	0
2.6782e-01	3.0580e-01
2.1725e-01	3.0191e-01
1.8602e-01	2.2393e-01
1.3672e-01	4.4424e-01

Figure 10: Error Table and Order of Accuracy of 1D Linear Advection with Periodic Initial Condition integrated with DR Function

3 Conclusion

The report analyzes the integration of the Douglas-Rashford (DR) function into a 1D linear advection solver with two different initial conditions. It discusses how the DR method, adapted from optimization techniques, may enhance the solver's accuracy and stability by maintaining solutions within realistic physical constraints. The findings suggest that while the DR method introduces more computational steps, it appears to produce more accurate and stable outcomes, especially at higher mesh refinements. A direct comparison with and without DR integration, as well as with alternative methods like 'fmincon' optimization solver, is recommended for a detailed assessment of performance improvements.