Continuity of Link Scheduling Rate Region for Wireless Networks with Propagation Delays

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Abstract—**We study the link scheduling problem of wireless networks with signal propagation delays into consideration. Recently, when the propagation delays are integers, the rate region using slotted scheduling with a proper timeslot size has been characterized explicitly. We study the general case that the propagation delays can be real values and the scheduling can be unslotted. As a practical communication device cannot transmit signals in arbitrarily short time intervals, we focus on scheduling where an active interval's length is bounded below by a given value. We first reveal some properties of continuity of the scheduling rate region concerning the propagation delays. We then show that for a network with rational propagation delays, the continuous (unslotted) scheduling rate region is the same as that of slotted scheduling with a proper timeslot size when the bound on the active interval length is sufficiently small. Moreover, for a network with possibly irrational propagation delays, we provide an approximation of the network by Dirichlet's theorem so that the continuous scheduling rate region of the original network can be approximated by the slotted scheduling rate region for a network with integer delays.**

I. INTRODUCTION

For territory wireless radio communications, the signal propagation delay is usually ignored in the network scheduling problem as the time frame for transmitting signals is much longer than the signal propagation delay between devices [1]–[3]. However, for scenarios like underwater acoustic communications and outer space radio communications, the signal propagation delay between communication devices can be longer than seconds. It has been shown that by taking the signal propagation delay into consideration in network scheduling, significant performance advantages in terms of energy consumption and throughput can be obtained [4]–[9].

A network with propagation delays can be formulated as a weighted, directed graph [10]. One class of scheduling problem associated with a network with propagation delays is called *slotted scheduling*, where time is slotted with a unit timeslot size and a network link is either active or inactive for a whole timeslot [6], [8], [11]. Recently, the rate region of slotted scheduling is characterized explicitly when the propagation delays are integers [10].

A real-world network may not have all the propagation delays integer. A network with general propagation delays can be rounded to one with integer propagation delays, and a slotted schedule can be found for the integer network using algorithms in [6], [10]. When applying the slotted schedule to the original network, however, only a middle part of each timeslot is usable as the two ends of a timeslot may have collisions due to the different propagation delays. In the existing research, the usable part in a timeslot is lower bounded by 0 in general [6]. In other words, in general it is not clear whether the approximation by rounding real numbers is effective or not.

Another class of scheduling problem of interest is called *unslotted (or continuous) scheduling*, where the link activation does not have the timeslot limitation. It has been demonstrated for some simple network models (e.g., unicast and one-hop communications) with propagation delays, unslotted scheduling may have advantages over slotted scheduling [12], [13]. The graphical approach for characterizing slotted scheduling rate region in [10] cannot be directly extended to continuous scheduling, and the continuous scheduling rate region is still open in general.

In this paper, we study slotted and continuous scheduling for networks with propagation delays, and try to answer the above questions. We formulate the scheduling problem in Section II-A for a network with general propagation delays. As a practical communication device cannot transmit signals in arbitrarily short time intervals, we define ω-scheduling to capture this property, where the length of any active interval is bounded below by a positive value ω. We focus on ω-scheduling in this paper.

In the real world, the propagation delays can only be measured or estimated subject to a certain accuracy bound [14], [15]. It is crucial to know whether a scheduling scheme designed for a measured/estimated model can be adopted in real-world networks. We study this problem regarding the continuity properties of the ω-scheduling rate region when the delays vary (see Section II-C). When the delay difference of two networks is small (in terms of a particular norm to be specified in the paper), the rate regions of these two networks are also close in a certain sense. Our results imply that if the measurement or estimation error is sufficiently small, using the measured/estimated model to design schedules is valid.

In Section III, we study the relation of ω-scheduling and slotted scheduling. For a network with rational propagation delays, the rate region of ω-scheduling is the same as that of slotted scheduling with a proper timeslot size when ω is sufficiently small. Therefore, the approach in [10] also determines the rate region of ω-scheduling for a network with rational propagation delays when ω is small enough.

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Section IV discusses how to approximate a network with general propagation delays to one with only rational propagation delays using Dirichlet's theorem. We show that the ω-scheduling rate region of a general network can always be approximated by the rate region of slotted scheduling for a network with integer delays. Our approximation implies a general approach of using slotted scheduling for networks with general propagation delays, achieving nearly optimal performance. Moreover, we provide a class of new networks where considering propagation delays can achieve unbounded sum rate gain compared with the traditional scheduling without considering delay.

In addition to the above results, we obtain various properties of continuous and slotted scheduling, and discuss their relations. Our work provides a general theoretical framework for understanding both continuous and slotted scheduling for networks with general propagation delays.

Omitted proofs can be found in Appendix.

II. Network Model and Continuous Scheduling

We introduce the network models with propagation delays and the continuous scheduling problem. We also study some basic properties of the link scheduling rate region, including the continuity with respect to propagation delays.

Throughout this paper, we denote $\mathbb{R}$ the set of real numbers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{Z}$ the set of integer numbers, and $\mathbb{N}$ the set of nonnegative integer numbers. Besides, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ represent the sets of positive and nonnegative real numbers, respectively. The *length* of an interval $\Theta \subset \mathbb{R}$ is its Lebesgue measure $\lambda(\Theta)$, so that an open interval and its closure have the same length. Time is continuous, and a time interval is an interval of $\mathbb{R}$.

A. Network Models

Consider a network formed by N nodes indexed by $1, 2, \ldots, N$, each of which can transmit and receive communication signals. The *signal propagation delay* from node i to node j is a positive real number $d(i,j)$: If node i sends a signal to node j during a time interval Θ, node j will receive this signal during $\Theta + d(i,j)$.

The network has a *link set* $\mathcal{L} \subset \{1, \ldots, N\} \times \{1, \ldots, N\}$. Links are directional, i.e., (i,j) and (j,i) are two different links of opposite directions. A link can have one of the two states *active* and *inactive* in a time interval. If link (i,j) is active during a time interval Θ, node i transmits a signal during Θ, which is expected to be received by node j.

Each link $l \in \mathcal{L}$ is associated with a *collision set* $\mathcal{I}(l) \subset \mathcal{L}$. Every link in $\mathcal{I}(l)$ has potential to affect the reception of the signal transmitted on l: Suppose a link $l = (i,j)$ is active during a time interval Θ, if a link $(i',j') \in \mathcal{I}(l)$ is also active during $\Theta + d(i,j) - d(i',j)$, the signals transmitted by both node i and node i' will propagate to node j during $\Theta + d(i,j)$, which means a *collision occurs*.

Example 1. We use a three-node network to demonstrate the definitions above. The link set is $\mathcal{L} = \{l_1 := (1,2), l_2 := (3,2)\}$, and the signal propagation delays are $d(1,2) = \sqrt{2}$ and $d(3,2) = \sqrt{5}$. The two links can interfere each other, i.e., the collision sets are $\mathcal{I}(l_1) = \{l_2\}$ and $\mathcal{I}(l_2) = \{l_1\}$.

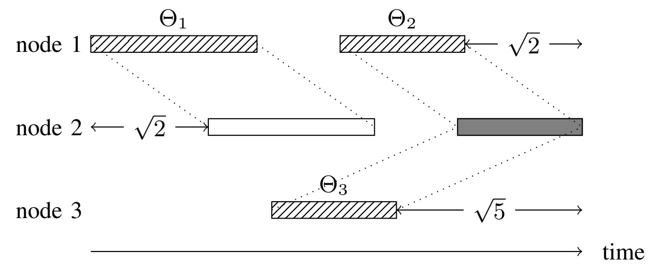

Fig. 1: Link $l_1 = (1,2)$ is active during Θ_1 and Θ_2, and link $l_2 = (3,2)$ is active during Θ_3. Link l_1 has no collision during Θ_1 but has a collision during Θ_2.

Define three time intervals $\Theta_1 = [0,2]$, $\Theta_2 = [3,4.5]$ and $\Theta_3 = [3+\sqrt{2}-\sqrt{5}, 4.5+\sqrt{2}-\sqrt{5}]$. Suppose that link l_1 is active during Θ_1 and Θ_2 and link l_2 is active during Θ_3. See Fig. 1 for an illustration of the signal propagation, where link l_1 has no collision during Θ_1, but has a collision during Θ_2.

The link-wise network $\mathcal{N}$ is essentially a weighted, directed graph. We are interested in the link scheduling problem determining when each link is active with the collision constraints. As in [10], to study this problem, it is sufficient to consider the link-wise network $\mathcal{N}$ defined by a triple $(\mathcal{L}, \mathcal{I}, D)$ where:

- $\mathcal{I} = (\mathcal{I}(l), l \in \mathcal{L})$ is called the *collision profile*; and
- $D = (D(l,l'), l, l' \in \mathcal{L})$ is called the *(link-wise) delay matrix* with the value of $D(l,l')$ specified as follows: For $l = (i,j) \in \mathcal{L}$ and $l' = (i',j') \in \mathcal{I}(l)$, $D(l,l') = d(i,j) - d(i',j)$. When $l' \notin \mathcal{I}(l)$, the value of $D(l,l')$ is not necessarily to be specified and is assigned 0 for convention.

A collision can be verified using the link-wise network model. When a link l is active during a time interval Θ, a collision occurs if a certain link $l' \in \mathcal{I}(l)$ is also active during $\Theta + D(l,l')$. In our model, $D(l,l')$ can be any real number. Due to precision limitations, however, $D(l,l')$ is usually rational in practical measurements. If $D(l,l')$ is rational (integer) for all $l \in \mathcal{L}$ and $l' \in \mathcal{I}(l)$, we call $\mathcal{N}$ a *rational (integer) network*.

Example 2. For the network defined in Example 1, we calculate that $D(l_1,l_2) = d(1,2) - d(3,2) = \sqrt{2} - \sqrt{5}$ and $D(l_2,l_1) = d(3,2) - d(1,2) = \sqrt{5} - \sqrt{2}$. So the corresponding link-wise network $\mathcal{N}$ is defined by $\mathcal{L} = \{l_1, l_2\}$, $\mathcal{I} = (\{l_2\}, \{l_1\})$, and the link-wise delay matrix

$$D = \begin{bmatrix} 0 & \sqrt{2}-\sqrt{5} \\ \sqrt{5}-\sqrt{2} & 0 \end{bmatrix}.$$

B. Link Scheduling

For network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$, a (continuous) schedule of a link $l \in \mathcal{L}$, denoted as $\mathcal{S}_l$, is a sequence of disjoint, closed intervals in $\mathbb{R}$, which are called *active intervals* of link l. We also abuse the notation by using $\mathcal{S}_l$ to denote the union of all the active intervals of link l. A schedule for $\mathcal{N}$ is defined by $\mathcal{S} = (\mathcal{S}_l, l \in \mathcal{L})$. A schedule $\mathcal{S}$ has a *collision* if there exists a link $l \in \mathcal{L}$ such that $\lambda(\mathcal{S}_{l'} \cap (\mathcal{S}_l + D(l, l'))) > 0$ for certain $l' \in \mathcal{I}(l)$. Otherwise, we say the schedule $\mathcal{S}$ is *collision free*.

A practical communication device cannot transmit signals in arbitrarily short time intervals. We give a lower bound on the length of an active interval to capture this property. For a fixed value $\omega \in \mathbb{R}_{>0}$, we call a schedule $\mathcal{S}$ an *ω-schedule* if the length of any active interval in $\mathcal{S}$ is bounded below by ω. We focus on ω-schedules in this paper.

Given a collision-free schedule $\mathcal{S}$ and a link $l \in \mathcal{L}$, define

$$R_{\mathcal{S}}^{\mathcal{N}}(l) = \lim_{T \to \infty} \frac{\lambda(\mathcal{S}_l \cap [0, T])}{T}. \quad (1)$$

If $R_{\mathcal{S}}^{\mathcal{N}}(l)$ exists for all $l \in \mathcal{L}$, we call $R_{\mathcal{S}}^{\mathcal{N}} = (R_{\mathcal{S}}^{\mathcal{N}}(l), l \in \mathcal{L})$ the *rate vector* of $\mathcal{S}$ for $\mathcal{N}$. We may omit the super script in $R_{\mathcal{S}}^{\mathcal{N}}$ and $R_{\mathcal{S}}^{\mathcal{N}}(l)$ when $\mathcal{N}$ is implied.

Definition 1. For a network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$, a rate vector R is *ω-achievable* if for any $\epsilon > 0$, there exists a collision-free ω-schedule $\mathcal{S}$, such that for any $l \in \mathcal{L}$, $R_{\mathcal{S}}(l) \geq R(l) - \epsilon$. The ω-scheduling rate region of $\mathcal{N}$, denoted by $\widetilde{\mathcal{R}}(\omega, \mathcal{N})$, is the collection of all the ω-achievable rate vectors for $\mathcal{N}$.

When studying the influence of delay matrix D on the rate region for the same $\mathcal{L}$ and $\mathcal{I}$, we write $\widetilde{\mathcal{R}}(\omega, \mathcal{N})$ as $\widetilde{\mathcal{R}}(\omega, D)$.

C. Properties of Scheduling Rate Region

We study some properties of ω-scheduling rate regions. A schedule $\mathcal{S}$ is *periodic* if there exists a positive real number T such that for any active interval $\Theta \subset \mathcal{S}_l$, $\Theta + T \subset \mathcal{S}_l$ for all $l \in \mathcal{L}$, where T is called a *period*.

Proposition 1. *For a network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$,*

1) *$\widetilde{\mathcal{R}}(\omega, D)$ can be achieved by periodic schedules only.*
2) *$\widetilde{\mathcal{R}}(\omega, D) = \widetilde{\mathcal{R}}(\alpha\omega, \alpha D)$ for $\alpha \in \mathbb{R}_{>0}$.*

In practice, the delay matrix can only be measured or estimated subject to a certain accuracy bound. For a schedule designed for the measured delay matrix, its real-world adoption performance is of interest. We study this problem in terms of the following continuity properties of the scheduling rate region.

Theorem 2. *Consider networks $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$ and $\mathcal{N}' = (\mathcal{L}, \mathcal{I}, D')$. Let $\delta = \|D - D'\|_\infty$. Then for any $\omega > 2\delta$ and $R \in \widetilde{\mathcal{R}}(\omega, D)$, there exist $\omega' \in [\omega - 2\delta, \omega]$ and $R' \in \widetilde{\mathcal{R}}(\omega', D')$ such that $\|R - R'\|_\infty < 2\delta/\omega$.*

Proof outline. We generalize a technique in [6] for rounding a delay matrix to an integer matrix. Fix a collision-free ω-schedule $\mathcal{S}$ of $\mathcal{N}$. If $\mathcal{S}$ is applied to $\mathcal{N}'$, collisions may only occur at the two ends of each active interval of $\mathcal{S}$. If we reduce each active interval of $\mathcal{S}$ for length δ from both ends, the new schedule is collision free for $\mathcal{N}'$. □

Since for every $1 \leq p \leq \infty$, we have $\|x\|_\infty \leq \|x\|_p \leq n^{1/p}\|x\|_\infty$ for all $x \in \mathbb{R}^n$, a similar result of Theorem 2 also holds for p-norms.

Corollary 3. *Consider networks $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$ and $\mathcal{N}' = (\mathcal{L}, \mathcal{I}, D')$. Let $\delta = \|D - D'\|_p$, $1 \leq p \leq \infty$. For any $\omega > 2\delta$ and $R \in \widetilde{\mathcal{R}}(\omega, D)$, there exist $\omega' \in [\omega - 2\delta, \omega]$ and $R' \in \widetilde{\mathcal{R}}(\omega', D')$, such that $\|R - R'\|_p < 2|\mathcal{L}|^{1/p}\delta/\omega$.*

Theorem 2 has the following practical implication. Suppose we want to design ω-scheduling for a delay matrix D. However, we only know the delay matrix D' and $\|D - D'\|_\infty \leq \delta < \omega/2$. Then we can design $(\omega - 2\delta)$-scheduling for D', which can achieve a similar region as $\widetilde{\mathcal{R}}(\omega, D)$.

The ω-scheduling discussed in this section is also called *unslotted scheduling* in literature (see, e.g., [12], [13]). A special class of scheduling assumes that time is slotted into intervals of the same length, and is usually called *slotted scheduling* (see, e.g., [6], [8], [10]). In the next two sections, we will derive a theory to connect these two classes of scheduling.

III. Discrete Scheduling

For a schedule $\mathcal{S}$, if the lengths of all the intervals in $\mathcal{S}_l$ and $\mathbb{R} \setminus \mathcal{S}_l$, $l \in \mathcal{L}$, are multiples of a certain positive real number Δ, we call $\mathcal{S}$ a *discrete schedule* with *timeslot size* Δ. Since Δ also bounds the length of all the active intervals, $\mathcal{S}$ is indeed an ω-schedule for any $0 < \omega \leq \Delta$.

A. Discrete Delay Matrix

A discrete schedule $\mathcal{S}$ with timeslot size Δ is said to be *aligned* if for any two boundaries t and t' of the active intervals of different links in $\mathcal{S}$, $t - t' \equiv 0 \pmod{\Delta}$. An aligned discrete schedule is also called a *slotted schedule*. For a discrete schedule $\mathcal{S} = (\mathcal{S}_l, l \in \mathcal{L})$ with timeslot size Δ, the *alignment* of $\mathcal{S}$ is defined to be a schedule $\mathcal{S}' = (\mathcal{S}_l - t_l, l \in \mathcal{L})$, where $t_l \in [0, \Delta)$ is the value of any boundary of $\mathcal{S}_l$ modulo Δ.

We first verify that $\mathcal{S}'$ is aligned. For any link $l \in \mathcal{L}$, suppose t' is a boundary in $\mathcal{S}_l'$. By definition of $\mathcal{S}'$, there exists a boundary point t of active intervals in $\mathcal{S}_l$, such that $t' = t - t_l \equiv 0 \pmod{\Delta}$. Therefore, $\mathcal{S}'$ is aligned as all the boundaries of active intervals are multiples of Δ.

Though a discrete schedule can be applied to a general delay matrix, it is more convenient to use a discrete schedule for a *discrete delay matrix* D whose entries are multiples of the timeslot size Δ.

Lemma 4. *Consider a network $\mathcal{N} = (\mathcal{I}, \mathcal{L}, D)$ where D/Δ is an integer matrix. For any collision-free discrete schedule $\mathcal{S}$ for $\mathcal{N}$ with timeslot size Δ, the alignment $\mathcal{S}'$ of $\mathcal{S}$ is also collision free for $\mathcal{N}$, and achieves the same rate vector as $\mathcal{S}$.*

An aligned discrete schedule $\mathcal{S} = (\mathcal{S}_l, l \in \mathcal{L})$ with the timeslot size Δ can be more concisely represented by a binary matrix [6], [10]. We define such a binary matrix $S = (S(l, u), l \in \mathcal{L}, u \in \mathbb{Z})$ by

$$\begin{aligned} S(l, u) &= 1, \quad \text{if } [u\Delta, (u+1)\Delta] \subseteq \mathcal{S}_l, \\ S(l, u) &= 0, \quad \text{otherwise}. \end{aligned}$$

We call $[u\Delta, (u+1)\Delta]$ the *u-th timeslot*, $u \in \mathbb{Z}$. We also refer to such a matrix S as a discrete schedule, which has another criterion for being collision free.

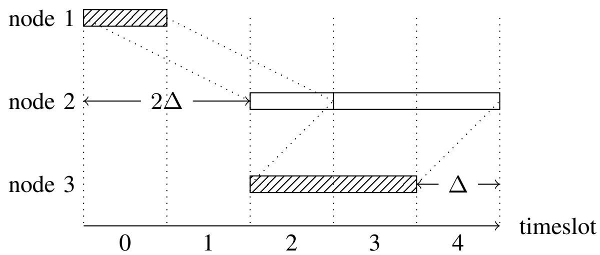

Fig. 2: Link l_1 is active at timeslot 0, link l_2 is active at timeslots 2 and 3. Correspondingly, $S(1,0) = S(2,2) = S(2,3) = 1$. Both links l_1 and l_2 are collision free.

Lemma 5. *Discrete schedule S is collision free if for all $l \in \mathcal{L}$ and $l' \in \mathcal{L}(l)$, $S(l,u) + S(l', u + \frac{D(l,l')}{\Delta}) < 2$.*

Example 3. Suppose we have a 3-node, 2-link network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$ with:

- $\mathcal{L} = \{l_1, l_2\}$, where $l_1 = (1,2)$ and $l_2 = (3,2)$;
- $\mathcal{I}(l_1) = \{l_2\}, \mathcal{I}(l_2) = \{l_1\}$;
- $d(1,2) = 2\Delta$, $d(3,2) = \Delta$ where $\Delta \in \mathbb{R}_{>0}$ is a constant, i.e.,

$$D = \begin{bmatrix} 0 & \Delta \\ -\Delta & 0 \end{bmatrix}.$$

Fig. 2 gives an example of a discrete schedule of $\mathcal{N}$ for timeslots $0, 1, \ldots, 4$, where the corresponding sub-matrix of the schedule is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

When S is collision free, the rate vector $R_S^{\mathcal{N}} = (R_S^{\mathcal{N}}(l), l \in \mathcal{L})$ has

$$R_S^{\mathcal{N}}(l) = \lim_{T \to \infty} \frac{1}{T} \sum_{u=0}^{T-1} S(l,u). \tag{2}$$

We may omit the super script in $R_S^{\mathcal{N}}$ and $R_S^{\mathcal{N}}(l)$ when $\mathcal{N}$ is implied. Similar to Definition 1, we can define the achievability by discrete scheduling with timeslot size Δ. Consider a network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$ where D/Δ is an integer matrix. The collection $\mathcal{R}(\Delta, \mathcal{N})$ of all such achievable rate vectors by discrete scheduling with timeslot size Δ is called the *discrete rate region* of $\mathcal{N}$ under timeslot size Δ. We also write $\mathcal{R}(\Delta, \mathcal{N})$ as $\mathcal{R}(\Delta, D)$ when $\mathcal{L}$ and $\mathcal{I}$ are implied in context.

Similar to Proposition 1, we can argue that $\mathcal{R}(\Delta, D) = \mathcal{R}(\alpha\Delta, \alpha D)$ for $\alpha \in \mathbb{R}_{>0}$. When $\alpha = 1/\Delta$, we obtain

$$\mathcal{R}(\Delta, D) = \mathcal{R}(1, D/\Delta), \tag{3}$$

where D/Δ is an integer matrix. The discrete scheduling rate region $\mathcal{R}(1, D/\Delta)$ has been explicitly characterized in [10], where a graph-based algorithm is provided to calculate the rate region. Because of (3), the same approach can be used to study $\mathcal{R}(\Delta, D)$. Moreover, in the following theorem, we show that for an integer delay matrix, using fractional timeslot sizes cannot increase the scheduling rate region.

Theorem 6. *For a network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$ with an integer delay matrix D and any $k > 0$ in $\mathbb{N}$,*

$$\mathcal{R}(1, D/k) = \mathcal{R}(k, D) \subseteq \mathcal{R}(1, D) = \mathcal{R}(1, kD) = \mathcal{R}(1/k, D).$$

The above theorem further implies that for a delay matrix D where D/Δ is an integer matrix and $k > 0$ in $\mathbb{N}$,

$$\mathcal{R}(k\Delta, D) \subseteq \mathcal{R}(\Delta, D) = \mathcal{R}(\Delta/k, D).$$

B. Continuous Scheduling vs Discrete Scheduling

For a delay matrix D where D/Δ is an integer matrix, we compare $\mathcal{R}(\Delta, D) = \mathcal{R}(1, D/\Delta)$ and $\widetilde{\mathcal{R}}(\omega, D) = \widetilde{\mathcal{R}}(\omega/\Delta, D/\Delta)$.

Theorem 7. *For a network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$ with an integer delay matrix, $\mathcal{R}(1, D) = \widetilde{\mathcal{R}}(\omega, D)$ for any $0 < \omega \le 1$.*

Proof outline. To show $\widetilde{\mathcal{R}}(\omega, D) \subset \mathcal{R}(1, D)$, for a collision-free ω-schedule $\mathcal{S}$, we find a new collision-free schedule $\mathcal{S}'$ such that $\mathcal{S}'_l \subset \mathcal{S}_l$ for all $l \in \mathcal{L}$, all the active and inactive intervals of $\mathcal{S}'$ are of rational lengths, and the rate vectors of $\mathcal{S}'$ and $\mathcal{S}$ are close. Then, we show that $\mathcal{S}'$ can be modified to a periodic, discrete schedule without changing much on the rate vector, where the latter has a rate vector in $\mathcal{R}(1, D)$. $\square$

Corollary 8. *For a network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$ such that D/Δ is an integer matrix and $0 < \omega \le \Delta$,*

$$\mathcal{R}(\Delta, D) = \widetilde{\mathcal{R}}(\omega, D).$$

Specially, if D is a rational matrix, then there exists $q \in \mathbb{N}$ such that qD is an integer delay matrix. For $0 < \omega \le 1/q$,

$$\mathcal{R}(1/q, D) = \widetilde{\mathcal{R}}(\omega, D).$$

Theorem 7 and its corollary demonstrate that for a discrete network model (including the rational model as a special case), its continuous rate region is the same as discrete ones with appropriate timeslot sizes. In other words, slotted scheduling can achieve the same rate region as the rate region by unslotted scheduling for discrete delay matrices.

IV. Approximation by Rational Models

For a delay matrix D where some of the entries are irrational, we present an approach to construct a rational delay matrix D' that has i) a similar rate region as D following Theorem 2 and ii) the same rate region by either continuous or discrete scheduling following Theorem 7.

A. Approximation of General Models

We first state Dirichlet's theorem on simultaneous approximation of a sequence of real numbers by rational numbers [17], [18]: Suppose $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n real numbers and $Q > 1$ is an integer, then there exist integers $q, p_1, p_2, \ldots, p_n$ with greatest common divisor (GCD) 1, such that

$$1 \le q < Q^n \text{ and } |\alpha_i - p_i/q| \le 1/(qQ),\ 1 \le i \le n.$$

There exist algorithms for finding such $q, p_1, p_2, \ldots, p_n$, e.g., the continued fraction algorithm [19], a lattice basis reduction

TABLE I: Dirichlet's Approximation of the matrix in (4).

Q	5	10	15	20
Q^3	125	1000	3375	8000
q	123	881	2728	4109
p_2	174	1246	3858	5811
p_3	213	1526	4725	7117
p_5	275	1970	6100	9188
$10^{-5}\max_i \lvert\sqrt{i}-p_i/q\rvert$	40	8.8	0.93	0.086
$10^{-5}/(qQ)$	160	11	2.4	1.2

algorithm [20], and its extension [21]. We use an example to illustrate the approximation.

Example 4. Consider the matrix

$$D = \begin{bmatrix} 0 & \sqrt{2} & \sqrt{3} \\ \sqrt{2} & 0 & \sqrt{5} \\ \sqrt{3} & \sqrt{5} & 0 \end{bmatrix}. \tag{4}$$

By Dirichlet's theorem, for any integer $Q > 1$, there exist integers q, p_i, $i = 2, 3, 5$ such that $1 \leq q < Q^3$ and $|\sqrt{i} - p_i/q| \leq 1/(qQ)$, $i = 2, 3, 5$. Table I gives the approximations of D with $Q = 5, 10, 15, 20$.

We can use Dirichlet's theorem to approximate a general delay matrix by a rational one and then apply Theorem 2 and Theorem 7 to show the following result.

Theorem 9. *For any network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$ and integer $Q > 1$, there exists a rational matrix D' such that $\|D - D'\|_\infty \leq \frac{1}{qQ}$ where $1 \leq q < Q^{|\mathcal{L}|^2}$ is the smallest integer making qD' integer. Moreover, for any $\omega \in [\frac{2}{qQ}, \frac{2+Q}{qQ}]$ and any $R \in \widetilde{\mathcal{R}}(\omega, D)$, there exists $R' \in \mathcal{R}(1, qD')$ such that $\|R - R'\|_\infty < \frac{2}{qQ\omega}$.*

Proof. By Dirichlet's theorem, there exist integers $q, p_{l,l'}$ for $l, l' \in \mathcal{L}$ such that $1 \leq q < Q^{|\mathcal{L}|^2}$ and $|D(l,l') - \frac{p_{l,l'}}{q}| \leq \frac{1}{qQ}$. Then we construct the rational delay matrix D' by $D'(l,l') = \frac{p_{l,l'}}{q}$. Therefore, $\|D - D'\|_\infty \leq \frac{1}{qQ}$. As the GCD of $q, p_{l,l'}$ is 1, q is the smallest integer such that qD' is integer.

By Theorem 2, as $\omega > \frac{2}{qQ}$ for any $R \in \widetilde{\mathcal{R}}(\omega, D)$, there exists $R' \in \widetilde{\mathcal{R}}(\omega - \frac{2}{qQ}, D')$ such that $\|R - R'\|_\infty < \frac{2}{qQ\omega}$. Moreover, as $\omega \leq \frac{2+Q}{qQ}$, i.e., $\omega - \frac{2}{qQ} \leq \frac{1}{q}$, by Corollary 8, $\widetilde{\mathcal{R}}(\omega - \frac{2}{qQ}, D') = \mathcal{R}(\frac{1}{q}, D') = \mathcal{R}(1, qD')$. ☐

B. Circular K-pair Network

We use a network motivated by an example in [6] to further demonstrate the approximation approach and show the advantages of considering propagation delays in scheduling.

Consider a network of $2K$ nodes located on a 2-dimensional plane. For $k = 1, \ldots, K$, node $2k-1$ and $2k$ locate at $(1, \theta_k)$ and $(1 + a, \theta_k)$ in the polar coordinate system, respectively. We call nodes $2k-1$ and $2k$ the kth pair. Communication links exist only within a pair, i.e., the link set is

$$\mathcal{L} = \{l_k := (2k-1, 2k), l'_k := (2k, 2k-1), k = 1, \ldots, K\}.$$

We assume that all the links can have collisions with each other, i.e., for $l \in \mathcal{L}$, the collision set $\mathcal{I}(l) = \mathcal{L} \setminus \{l\}$. Suppose that the signal propagation delay between two nodes is in proportion to the distance between the two nodes. We can then calculate the link-wise propagation delay matrix D, where

$$D(l_k, l'_k) = D(l'_k, l_k) = a, \ k = 1, \ldots, K,$$

for $1 \leq i \neq j \leq K$,

$$\begin{aligned} D(l_i, l_j) &= D(l'_i, l'_j) = a - 2\alpha_{i,j}, \\ D(l'_i, l_j) &= a - 2\alpha'_{i,j}, \\ D(l_i, l'_j) &= a - 2\alpha''_{i,j}, \end{aligned}$$

where

$$\begin{aligned} \alpha_{i,j} &= \frac{1}{2}\sqrt{(1+a)^2 + 1 + 2(1+a)\cos(\theta_i - \theta_j)}, \\ \alpha'_{i,j} &= \frac{1}{2}\sqrt{2 - 2\cos(\theta_i - \theta_j)}, \\ \alpha''_{i,j} &= \frac{1+a}{2}\sqrt{2 - 2\cos(\theta_i - \theta_j)}. \end{aligned}$$

Fix an integer $Q > 1$. By Dirichlet's theorem, there exist nonnegative integers $p_{i,j}$, $p'_{i,j}$, $p''_{i,j}$ and q, such that $|\alpha_{i,j} - p_{i,j}/q| \leq 1/(qQ)$, $|\alpha'_{i,j} - p'_{i,j}/q| \leq 1/(qQ)$, $|\alpha''_{i,j} - p''_{i,j}/q| \leq 1/(qQ)$ and $1 \leq q < Q^{3K(K-1)/2}$, for all $1 \leq i < j \leq K$. Then we approximate D by D_r, where

$$\begin{aligned} D_r(l_i, l'_i) &= D_r(l'_i, l_i) = a, \\ D_r(l_i, l_j) &= D_r(l'_i, l'_j) = a - 2p_{i,j}/q, \\ D_r(l'_i, l_j) &= a - 2p'_{i,j}/q, \\ D_r(l_i, l'_j) &= a - 2p''_{i,j}/q. \end{aligned}$$

Suppose that a is chosen such that qa is an odd integer: For any $q \in \mathbb{N}$, there exists an odd $n \in \mathbb{N}$, such that $q = n2^k$ for some $k \in \mathbb{N}$. Let $a = \frac{1}{2^k}$, then $qa = n$ is odd. Hence, qD_r is an integer matrix, and $\|D - D_r\|_\infty \leq 2/(qQ)$.

Define a discrete schedule S with period 2 as follows: $S(l, 0) = 1$ and $S(l, 1) = 0$ for all $l \in \mathcal{L}$. Such schedule is called an *even-odd* schedule in [6]. Following their discussion, as qa is odd and $2p_{i,j}, 2p'_{i,j}$ and $2p''_{i,j}$ are even, it can be shown that S is collision free for qD_r.

Note that $R_S = \left[1/2, 1/2, \cdots, 1/2\right]^\intercal \in \mathcal{R}(1, qD_r)$. By Theorem 7 and Proposition 1, $\mathcal{R}(1, qD_r) = \widetilde{\mathcal{R}}(1, qD_r) = \widetilde{\mathcal{R}}(\frac{1}{q}, D_r)$. Since $\|D - D_r\|_\infty \leq \frac{2}{qQ}$, by Theorem 2, there exist $\omega \in [(Q-2)/(qQ), 1/q]$ and rate vector $R \in \widetilde{\mathcal{R}}(\omega, D)$, such that $\|R - R_S\|_\infty < \frac{4}{qQ\omega} < \frac{4}{Q-2}$. Therefore, there exists an ω-schedule for the K-pair network with the sum rate at least $K(1 - \frac{8}{Q-2})$. In contrast, without considering the delays, the sum rate is at most 1 as all the links have collisions with each other.

V. Concluding Remarks

In a real-world system, the signal propagation delays can be time varying and cannot be accurately measured/estimated. The continuity properties of the scheduling rate region studied in this paper ensures that it is possible to take signal propagation delays into consideration in real-world where the delays are not accurately known. Our work also provides insights about how to design slotted and continuous scheduling for networks with large propagation delays.

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APPENDIX A
PROOFS OF SECTION II

Proof of Proposition 1. Proof of 1). Fix $R \in \widetilde{\mathcal{R}}(\omega, D)$ and $\epsilon > 0$. There exists a collision-free ω-schedule $\mathcal{S}$ such that $R_{\mathcal{S}}(l) \geq R(l) - \epsilon/2$ for every $l \in \mathcal{L}$. Moreover, there exists a sufficiently large T_0 such that for all $T \geq T_0$,

$$\left| R_{\mathcal{S}}(l) - \frac{\lambda(\mathcal{S}_l \cap [0,T])}{T} \right| \leq \epsilon/2.$$

Define

$$D_{\max} = \max_{l \in \mathcal{L}} \max_{l' \in \mathcal{I}(l)} D(i', j),$$

where $l = (i,j)$ and $l' = (i', j')$.

Fix any $T \geq \max\{T_0, D_{\max}, 2D_{\max}/\epsilon\}$. Define a schedule $\mathcal{S}'$ with period T such that for all l, $\mathcal{S}'_l \cap [0, T - D_{\max}) = \mathcal{S}_l \cap [0, T - D_{\max})$ and $\lambda(\mathcal{S}'_l \cap [T - D_{\max}, T)) = 0$. Note that S' is collision-free and hence

$$\begin{aligned}
R_{\mathcal{S}'}(l) &= \frac{\lambda(\mathcal{S}_l \cap [0,T])}{T} \\
&= \frac{\lambda(\mathcal{S}_l \cap [0, T - D_{\max}])}{T - D_{\max}} \left(1 - \frac{D_{\max}}{T}\right) \\
&\geq R_{\mathcal{S}}(l) - \epsilon/2 - \frac{D_{\max}}{T} \\
&\geq R_{\mathcal{S}}(l) - \epsilon.
\end{aligned}$$

Proof of 2). For any collision-free schedule $\mathcal{S}$, let $\alpha\mathcal{S} = (\alpha\mathcal{S}_l)$. We see an interval $\alpha\mathcal{T} \subset \alpha\mathcal{S}_l$ has a collision in $\alpha\mathcal{N}$ if and only if $\mathcal{T} \subset \mathcal{S}_l$ has a collision in $\mathcal{N}$. Moreover,

$$\begin{aligned}
R_{\alpha\mathcal{S}}^{\alpha\mathcal{N}}(l) &= \lim_{t \to \infty} \frac{\lambda(\alpha\mathcal{S}_l \cap [0,t])}{t} \\
&= \lim_{t \to \infty} \frac{\lambda(\mathcal{S}_l \cap [0, t/\alpha])}{t/\alpha} \\
&= R_{\mathcal{S}}^{\mathcal{N}}(l).
\end{aligned} \tag{5}$$

□

Proof of Theorem 2. By Definition 1 and Proposition 1, for a rate vector $R \in \widetilde{\mathcal{R}}(\omega, D)$ and fixed $\epsilon > 0$, there exists a collision-free ω-schedule $\mathcal{S}$ with period T, such that for all $l \in \mathcal{L}$, $R_{\mathcal{S}}(l) > R(l) - \epsilon$. Suppose $\mathcal{S}_l$ is formed by active intervals Θ_i, $i \in \mathbb{Z}$, where $\Theta_i = [a_i, b_i]$ with $a_i \leq b_i < a_{i+1} \leq b_{i+1}$ and $b_i - a_i \geq \omega > 2\delta$.

Define a schedule $\mathcal{S}'$ for $\mathcal{N}'$ by

$$\mathcal{S}'_l = \mathcal{S}_l \cap_{l' \in \mathcal{I}(l)} (\mathcal{S}_l + D(l,l') - D'(l,l')), \tag{6}$$

for all $l \in \mathcal{L}$. Define $\Theta_i^{'} = [a_i, b_i] \cap_{l' \in \mathcal{I}(l)} [a_i + D(l,l') - D'(l,l'), b_i + D(l,l') - D'(l,l')]$, which is non-empty as

$$\begin{aligned}
&\max_{l' \in \mathcal{I}(l) \cup \{l\}} (a_i + D(l,l') - D'(l,l')) \\
&\geq a_i + \delta \\
&> b_i - \delta \\
&\geq \min_{l' \in \mathcal{I}(l) \cup \{l\}} (b_i + D(l,l') - D'(l,l')).
\end{aligned}$$

Hence $\mathcal{S}'_l$ is formed by the active intervals $\Theta_i^{'}$, $i \in \mathbb{Z}$. As

$$\lambda({\Theta_i}') \geq b_i - a_i - 2\delta \geq \omega - 2\delta,$$

$\mathcal{S}'_l$ is an ω'-schedule with $\omega - 2\delta \leq \omega' \leq \omega$.

Now we show that $\mathcal{S}'_l$ has period T. For an active interval $\Theta \in \mathcal{S}_l$, denote

$$\begin{aligned}
\Theta_0 &= \Theta + T \in \mathcal{S}_l, \\
\Theta' &= \Theta \cap_{l' \in \mathcal{I}(l)} (\Theta + D(l,l') - D'(l,l')), \text{and} \\
\Theta'_0 &= \Theta' + T.
\end{aligned}$$

We have

$$\begin{aligned}
\Theta'_0 &= (\Theta + T) \cap_{l' \in \mathcal{I}(l)} (\Theta + T + D(l,l') - D'(l,l')) \\
&= \Theta_0 \cap_{l' \in \mathcal{I}(l)} (\Theta_0 + D(l,l') - D'(l,l')),
\end{aligned}$$

which is also an active interval in $\mathcal{S}'_l$ since $\Theta_0 \in \mathcal{S}_l$, indicating that $\mathcal{S}'_l$ also has period T.

For any $l' \in \mathcal{I}(l)$, as $\mathcal{S}'_{l'} \subseteq \mathcal{S}_{l'}$ and $\mathcal{S}'_l + D'(l,l') \subseteq \mathcal{S}_l + D(l,l')$, we have

$$(\mathcal{S}'_l + D'(l,l')) \cap \mathcal{S}'_{l'} \subseteq (\mathcal{S}_l + D(l,l')) \cap \mathcal{S}_{l'}.$$

As $\mathcal{S}_l$ is collision-free for D,

$$\lambda((\mathcal{S}'_l + D'(l,l')) \cap \mathcal{S}'_{l'}) \leq \lambda((\mathcal{S}_l + D(l,l')) \cap \mathcal{S}_{l'}) = 0.$$

We conclude that $\mathcal{S}'_l$ is a collision-free ω'-schedule with period T under D' with $\omega - 2\delta \leq \omega' \leq \omega$.

For each link l, there are at most $\frac{T}{\omega}$ active intervals in $\mathcal{S}_l \cap [0,T]$. For a subset $\mathcal{S} \subset \mathbb{R}$, let $\lambda_T(\mathcal{S}) = \lambda(\mathcal{S} \cap [0,T])$. Within one period $[0,T]$,

$$0 \leq \lambda_T(\mathcal{S}_l) - \lambda_T(\mathcal{S}'_l) \leq \frac{T}{\omega} \cdot 2\delta.$$

By the link rate definition,

$$0 \leq R_{\mathcal{S}}(l) - R_{\mathcal{S}'}(l) = \frac{\lambda_T(\mathcal{S}_l) - \lambda_T(\mathcal{S}'_l)}{T} \leq \frac{2\delta}{\omega}.$$

Therefore, $R(l) - R_{\mathcal{S}'}(l) < R_{\mathcal{S}}(l) + \epsilon - R_{\mathcal{S}'}(l) \leq \frac{2\delta}{\omega} + \epsilon$.

We conclude that for a rate vector $R \in \widetilde{\mathcal{R}}(\omega, D)$, there exists $R'_n \in \widetilde{\mathcal{R}}(\omega', D')$ and integer $n \geq 1$, such that $R(l) - R'_n(l) < \frac{2\delta}{\omega} + \frac{1}{n}$ for all $l \in \mathcal{L}$. Since the sequence $\{R'_n\}_{n=1}^{\infty}$ is bounded by $R_{\mathcal{S}}$, by the Bolzano–Weierstrass theorem, it has a convergent subsequence $\{R_{n_i}\}_{i=1}^{\infty}$ with $R'(l) := \lim_{i \to \infty} R_{n_i}(l)$ for all $l \in \mathcal{L}$. Therefore,

$$\begin{aligned}
R(l) - R'(l) &= R(l) - \lim_{i \to \infty} R_{n_i}(l) \\
&< \frac{2\delta}{\omega} + \lim_{i \to \infty} \frac{1}{n_i} = \frac{2\delta}{\omega}.
\end{aligned}$$

Since $\widetilde{\mathcal{R}}(\omega', D')$ is closed, the limit point $R' \in \widetilde{\mathcal{R}}(\omega', D')$. □

Proof of Corollary 3. For $R \in \mathbb{R}^{|\mathcal{L}|}$ and for the p-norm with $1 \leq p \leq \infty$, $\|R\|_\infty \leq \|R\|_p \leq |\mathcal{L}|^{1/p} \|R\|_\infty$. As $\|D - D'\|_\infty \leq \|D - D'\|_p = \delta$, by Theorem 2, for any $\omega > 2\delta$ and $R \in \widetilde{\mathcal{R}}(\omega, D)$, there exists $\omega - 2\delta \leq \omega' \leq \omega$ and $R' \in \widetilde{\mathcal{R}}(\omega', D')$, such that $\|R - R'\|_\infty < 2\delta/\omega$. Hence, $\|R - R'\|_p \leq |\mathcal{L}|^{1/p} \|R - R'\|_\infty < 2|\mathcal{L}|^{1/p}\delta/\omega$. □

APPENDIX B
PROOFS OF SECTION III

Proof of Lemma 4. Assume that $\mathcal{S}'$ is not collision-free, then there exist $l, l' \in \mathcal{L}$, such that $\lambda(\mathcal{S}'_{l'} \cap (\mathcal{S}'_l + D(l,l'))) > 0$. In $\mathcal{S}'_{l'}$ and $\mathcal{S}'_l$, boundaries of all intervals are aligned at $k\Delta$ with some integer k. Thus the overlap has length of at least Δ: There exist intervals $\Theta \in \mathcal{S}_l$ and $\Theta' \in \mathcal{S}_{l'}$, such that

$$\begin{aligned}
\lambda((\Theta' - t_{l'}) \cap (\Theta - t_l + D(l,l'))) &\geq \Delta \\
\lambda((\Theta' - t_{l'} + t_l) \cap (\Theta + D(l,l'))) &\geq \Delta \\
\lambda(\Theta' \cap (\Theta + D(l,l'))) &\geq \Delta - (t_{l'} - t_l) \\
&> 0,
\end{aligned}$$

which contradicts to the fact that $\lambda((\mathcal{S}_l + D(l,l')) \cap \mathcal{S}_{l'}) = 0$. Therefore, the shifted schedule $\mathcal{S}'$ is collision-free.

Besides, shifting will not impact lengths of intervals: For all $l \in \mathcal{L}$, $\lambda(\mathcal{S}'_l) = \lambda(\mathcal{S}_l - \delta_l) = \lambda(\mathcal{S}_l)$. Hence, $R_{\mathcal{S}}(l) = \lim_{T\to\infty} \frac{\lambda(\mathcal{S}_l \cap [0,T])}{T} = \lim_{T\to\infty} \frac{\lambda(\mathcal{S}'_l \cap [-t_l, T-t_l])}{T} = R_{\mathcal{S}'}(l)$. ☐

Proof of Lemma 5. Suppose S ($\mathcal{S}$) is not collision-free, then there exist $l \in \mathcal{L}$ and $l' \in \mathcal{I}(l)$, such that $\lambda((\mathcal{S}_l + D(l,l')) \cap \mathcal{S}_{l'}) \geq \Delta$. In the overlap of $\mathcal{S}_l + D(l,l')$ and $\mathcal{S}_{l'}$, there exists a timeslot $[u\Delta, (u+1)\Delta]$. Therefore, we have $S(l', u + \frac{D(l,l')}{\Delta}) = S(l,u) = 1$.

Conclusively, if S is not collision-free, then there exist links l, l' and integer u, such that $S(l,u) + S(l', u + \frac{D(l,l')}{\Delta}) \geq 2$, which is the contrapositive of statement in Lemma 5. ☐

Proof of Theorem 6. For an integer delay matrix D, $\mathcal{R}(1, kD) = \mathcal{R}(1/k, D)$ and $\mathcal{R}(1, D/k) = \mathcal{R}(k, D)$ hold by (3) with $\Delta = 1/k$ and k respectively. Besides, since all schedules with timeslot size k is indeed a special case of schedules with timeslot size 1, $\mathcal{R}(k, D) \subseteq \mathcal{R}(1, D)$. Similarly, $\mathcal{R}(1, D) \subseteq \mathcal{R}(1/k, D)$. Then it only remains to prove $\mathcal{R}(1, kD) \subseteq \mathcal{R}(1, D)$.

For any rate vector $R \in \mathcal{R}(1, kD)$ and $\epsilon > 0$, there exists a collision-free schedule S with period T for $\mathcal{N}_k = (\mathcal{L}, \mathcal{I}, kD)$, such that $R_S(l) > R(l) - \epsilon$ for all $l \in \mathcal{L}$. Then we define schedules $S_0, S_1, \ldots, S_{k-1}$ for $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$ by

$$S_i(l, u) = S(l, ku + i).$$

For each i, S_i also has period T and is collision free for $(\mathcal{L}, \mathcal{I}, D)$ since for all $l \in \mathcal{L}$ and $l' \in \mathcal{I}(l)$,

$$\begin{aligned}
&S_i(l,u) + S_i(l', u + D(l,l')) \\
&= S(l, ku+i) + S(l', ku + i + kD(l,l')) < 2.
\end{aligned}$$

Therefore, $R_{S_i} \in \mathcal{R}(1, D)$ for all i.

Moreover,

$$\begin{aligned}
R_S(l) &= \frac{1}{kT} \sum_{u'=0}^{kT-1} S(l, u') \\
&= \frac{1}{kT} \sum_{i=0}^{k-1} \sum_{u=0}^{T-1} S(l, ku+i) \\
&= \sum_{i=0}^{k-1} \frac{1}{k} \left(\frac{1}{T} \sum_{u=0}^{T-1} S_i(l,u) \right) \\
&= \sum_{i=0}^{k-1} \frac{1}{k} R_{S_i}(l) = \frac{\sum_{i=0}^{k-1} R_{S_i}(l)}{k},
\end{aligned}$$

i.e., $R_S = \frac{R_{S_0} + \ldots + R_{S_{k-1}}}{k}$. Since $R_{S_i} \in \mathcal{R}(1, D)$, by convexity of $\mathcal{R}(1, D)$, R_S and R are in $\mathcal{R}(1, D)$. Consequently, $\mathcal{R}(1, kD) \subseteq \mathcal{R}(1, D)$. ☐

Proof of Theorem 7. First, since any discrete schedule with timeslot 1 is indeed an ω-schedule,

$$\widetilde{\mathcal{R}}(\omega, D) \supseteq \mathcal{R}(1, D).$$

Then we prove $\widetilde{\mathcal{R}}(\omega, D) \subseteq \mathcal{R}(1, D)$. For a rate vector $R \in \widetilde{\mathcal{R}}(\omega, D)$ and fixed $\epsilon > 0$, there exists a collision-free ω-schedule $\mathcal{S}$, such that $R_{\mathcal{S}}(l) > R(l) - \epsilon/4$ for all $l \in \mathcal{L}$. For link $l \in \mathcal{L}$, $\mathcal{S}_l$ is formed by the disjoint active intervals $\Theta_{l,i} = [t_{l,i}, s_{l,i}]$, $i \in \mathbb{Z}$, and $\mathbb{R} \setminus \mathcal{S}_l$ is formed by the disjoint inactive intervals $\Phi_{l,i} = (s_{l,i}, t_{l,i+1})$, $i \in \mathbb{Z}$, where $\Phi_{l,i}$ is adjacent from the right to $\Theta_{l,i}$.

Since a real number is a Dedekind cut in $\mathbb{Q}$ [22], if $\lambda(\Phi_{l,i}), \lambda(\Theta_{l,i}) \in \mathbb{R} \setminus \mathbb{Q}$, $[\lambda(\Phi_{l,i}), \infty) \cap \mathbb{Q}$ has no smallest element and $(-\infty, \lambda(\Theta_{l,i})] \cap \mathbb{Q}$ has no largest element. Let $\delta_{l,-1} = 0$. For each $i \geq 0$, there exist positive integers $p_{l,i}, q_{l,i}, m_{l,i}, n_{l,i}$ with $\gcd(p_{l,i}, q_{l,i}) = \gcd(m_{l,i}, n_{l,i}) = 1$, such that

$$0 \leq \lambda(\Theta_{l,i}) - \delta_{l,i-1} - \frac{p_{l,i}}{q_{l,i}} =: \iota_{l,i} < \epsilon\omega/8, \tag{7}$$

$$0 \leq \frac{m_{l,i}}{n_{l,i}} - \lambda(\Phi_{l,i}) - \iota_{l,i} =: \delta_{l,i} < \epsilon\omega/8. \tag{8}$$

Define

$$\begin{aligned}
\Theta'_{l,i} &= [t_{l,i} + \delta_{l,i-1}, s_{l,i} - \iota_{l,i}], \\
\Phi'_{l,i} &= (s_{l,i} - \iota_{l,i}, t_{l,i+1} + \delta_{l,i}),
\end{aligned}$$

and $\mathcal{S}'_l = \{\Theta'_{l,0}, \Theta'_{l,1}, \ldots\}$. Note that

$$\begin{aligned}
[t_{l,0}, \infty) \setminus (\cup_{i\geq 0} \Theta'_{l,i}) &= [t_{l,0}, \infty) \setminus (\cup_{i\geq 0} [t_{l,i} + \delta_{l,i-1}, s_{l,i} - \iota_{l,i}]) \\
&= \cup_{i\geq 0} (s_{l,i} - \iota_{l,i}, t_{l,i+1} + \delta_{l,i}) \\
&= \cup_{i\geq 0} \Phi'_{l,i}
\end{aligned}$$

Then $[t_{l,0}, \infty) \setminus \mathcal{S}'_l = \{\Phi'_{l,0}, \Phi'_{l,1}, \ldots\}$. By (7) and (8),

$$\lambda(\Theta'_{l,i}) = \lambda(\Theta_{l,i}) - \delta_{l,i-1} - \iota_{l,i} = p_{l,i}/q_{l,i}, \tag{9}$$

$$\lambda(\Phi'_{l,i}) = \lambda(\Phi_{l,i}) + \iota_{l,i} + \delta_{l,i} = m_{l,i}/n_{l,i}, \tag{10}$$

indicating that all intervals $\Theta'_{l,i}$ and $\Phi'_{l,i}$ are of rational length. Moreover,

$$0 \leq \lambda(\Theta_{l,i}) - \lambda(\Theta'_{l,i}) = \delta_{l,i-1} + \iota_{l,i} < \epsilon\omega/4.$$

As $\lambda(\Theta_{l,i}) \geq \omega$, within $[t_{l,0}, t+t_{l,0}]$ for any $t > 0$, there are no more than t/ω active intervals in $\mathcal{S}_l$ and $\mathcal{S}'_l$. Since $\mathcal{S}'_l \subseteq \mathcal{S}_l$ and $\mathcal{S}_l$ is a collision-free schedule for network $\mathcal{N}$, $\mathcal{S}'_l$ is also a collision-free schedule for network $\mathcal{N}$. Comparing the rate vectors of $\mathcal{S}_l$ and $\mathcal{S}'_l$,

$$\begin{aligned}
&R_{\mathcal{S}}(l) - R_{\mathcal{S}'}(l) \\
&= \lim_{t\to\infty} \frac{\lambda(\mathcal{S}_l \cap [t_{l,0}, t+t_{l,0}])}{t} - \lim_{t\to\infty} \frac{\lambda(\mathcal{S}'_l \cap [t_{l,0}, t+t_{l,0}])}{t} \\
&= \lim_{t\to\infty} \frac{\lambda((\mathcal{S}_l \setminus \mathcal{S}'_l) \cap [t_{l,0}, t+t_{l,0}])}{T} \\
&< \lim_{t\to\infty} \frac{\omega\epsilon}{4t}\frac{t}{\omega} \\
&= \epsilon/4.
\end{aligned}$$

Therefore,

$$R(l) - R_{\mathcal{S}'}(l) < R_{\mathcal{S}} + \epsilon/4 - R_{\mathcal{S}'}(l) < \epsilon/2. \tag{11}$$

Note that $\mathcal{S}'_l$ may not be periodic. However, by the similar arguments in Proposition 1, we can define a schedule $\mathcal{S}^T$ with a sufficiently large period $T \in \mathbb{Q}$ and $T > 0$ satisfying

$$R_{\mathcal{S}^T}(l) \geq R_{\mathcal{S}'}(l) - \epsilon/2 \tag{12}$$

as follows: Let

$$D^* = \max_{l\in\mathcal{L}} \max_{l'\in\mathcal{I}(l)} |D(l,l')|.$$

$\mathcal{S}^T$ is defined by

$$\begin{aligned}
&\mathcal{S}^T_l \cap [t_{l,0}, T-D^*+t_{l,0}) = \mathcal{S}'_l \cap [t_{l,0}, T-D^*+t_{l,0}), \\
&\lambda(\mathcal{S}^T_l \cap [T-D^*+t_{l,0}, T+t_{l,0}]) = 0.
\end{aligned}$$

There exists a positive integer k, $k \leq (T-D^*)/\omega$, such that $(T-D^*+t_{l,0}) \in (\Theta'_{l,k} \cup \Phi'_{l,k})$. Consider two cases:

- If $(T-D^*+t_{l,0}) \in \Theta'_{l,k}$, Then

$$\begin{aligned}
\mathcal{S}^T_l &= \{\Theta'_{l,0}, \Theta'_{l,1}, \ldots, \Theta'_{l,k-1}, \Theta''_{l,k}\}, \\
[t_{l,0}, T+t_{l,0}] \setminus \mathcal{S}^T_l &= \{\Phi'_{l,0}, \Phi'_{l,1}, \ldots, \Phi'_{l,k-1}, \Phi''_{l,k}\},
\end{aligned}$$

  where

$$\begin{aligned}
\Theta''_{l,k} &= \Theta'_{l,k} \cap (\Theta'_{l,k} - (T-D^*+t_{l,0})), \\
\Phi''_{l,k} &= [T-D^*+t_{l,0}, T+t_{l,0}].
\end{aligned}$$

- If $(T-D^*+t_{l,0}) \in \Phi'_{l,k}$, then

$$\begin{aligned}
\mathcal{S}^T_l &= \{\Theta'_{l,0}, \Theta'_{l,1}, \ldots, \Theta'_{l,k}\}, \\
[t_{l,0}, T+t_{l,0}] \setminus \mathcal{S}^T_l &= \{\Phi'_{l,0}, \Phi'_{l,1}, \ldots, \Phi'_{l,k-1}, \Phi''_{l,k}\},
\end{aligned}$$

  where

$$\Phi''_{l,k} = \Phi'_{l,k} \cup [T-D^*+t_{l,0}, T+t_{l,0}].$$

Since $\lambda(\Theta'_{l,i}) = p_{l,i}/q_{l.i}$, $\lambda(\Phi'_{l,i}) = m_{l,i}/n_{l,i}$ (ref. (9) and (10)), and $T-D^*$ are rational, all intervals in $\mathcal{S}^T_l$ and $\mathbb{R} \setminus \mathcal{S}^T_l$ are of rational lengths. With a slight abuse of notations, we find $p_{l,k+1}, q_{l,k+1}, m_{l,k+1}, n_{l,k+1}$ with $\gcd(p_{l,k+1}, q_{l,k+1}) = \gcd(m_{l,k+1}, n_{l,k+1}) = 1$ such that

$$\begin{aligned}
\frac{p_{l,k+1}}{q_{l,k+1}} &= \lambda(\Theta'_{l,k} \cap (\Theta'_{l,k} - (T-D^*+t_{l,0}))), \text{ and} \\
\frac{m_{l,k+1}}{n_{l,k+1}} &= \lambda(\Phi'_{l,k} \cup [T-D^*+t_{l,0}, T+t_{l,0}]).
\end{aligned}$$

Let M be the least common multiple of all denominators $q_{l,i}$ and $n_{l,i}$, $l \in \mathcal{L}$ and $i = 0, 1, \ldots, k+1$. Then all $p_{l,i}/q_{l,i}$ and $m_{l,i}/n_{l,i}$ are multiples of $1/M$, i.e., all intervals in $\mathcal{S}^T_l$ and $[t_{l,0}, \infty) \setminus \mathcal{S}^T_l$ can be slotted into timeslots with length $1/M$. Then we define a periodic discrete schedule S for $u \in \mathbb{N}$ according to $\mathcal{S}^T_l$: for $u = 0, 1, \ldots, MT-1$,

$$\begin{aligned}
&S(l,u) = 1, \quad \text{if } [u/M + t_{l,0}, (u+1)/M + t_{l,0}] \subseteq \mathcal{S}^T_l, \\
&S(l,u) = 0, \quad \text{otherwise}.
\end{aligned}$$

S has period MT.

Assume that S is not collision-free, then there exist l, l', u with $l' \in \mathcal{I}(l)$ and $S(l,u) + S(l', u + MD(l,l')) = 2$, indicating that

$$\begin{aligned}
&[u/M + t_{l,0}, (u+1)/M + t_{l,0}] \in \mathcal{S}^T_l, \text{ and} \\
&[u/M + D(l,l') + t_{l,0}, (u+1)/M + D(l,l') + t_{l,0}] \in \mathcal{S}^T_{l'}.
\end{aligned}$$

Hence $\lambda((\mathcal{S}^T_l + D(l,l')) \cap \mathcal{S}^T_{l'}) \geq 1/M > 0$, which is contradict to the fact that $\mathcal{S}^T_l \subseteq \mathcal{S}'_l$ is collision-free. Therefore, the discrete schedule S is collision-free.

Meanwhile,

$$\begin{aligned}
R_S(l) &= \frac{\sum_{u=0}^{MT-1} S(l,u)}{MT} \\
&= \frac{\sum_{u=0}^{MT-1} \lambda([u/M + t_{l,0}, (u+1)/M + t_{l,0}])}{T} \\
&= \frac{\lambda(\mathcal{S}^T_l \cap [t_{l,0}, T+t_{l,0}])}{T} \\
&= R_{\mathcal{S}^T}(l).
\end{aligned}$$

Therefore, combine with (11) and (12), $R_S(l) = R_{\mathcal{S}^T}(l) > R_{\mathcal{S}'}(l) - \epsilon/2 > R(l) - \epsilon$ for fixed $\epsilon > 0$. Hence, $R \in \mathcal{R}(\frac{1}{M}, D) = \mathcal{R}(1, D)$, indicating that $\widetilde{\mathcal{R}}(\omega, D) \subseteq \mathcal{R}(1, D)$. Since $\widetilde{\mathcal{R}}(\omega, D) \supseteq \mathcal{R}(1, D)$, we have $\widetilde{\mathcal{R}}(\omega, D) = \mathcal{R}(1, D)$. □

Proof of Corollary 8. By Theorem 7, we have

$$\mathcal{R}(\Delta, D) = \mathcal{R}(1, D') = \widetilde{\mathcal{R}}(\omega', D') = \widetilde{\mathcal{R}}(\omega, D),$$

where $0 \leq \omega' = \omega/\Delta \leq 1$. For rational delay matrix D, $\Delta = 1/q$, hence

$$\mathcal{R}(\frac{1}{q}, D) = \widetilde{\mathcal{R}}(\omega, D),$$

where $0 < \omega \leq \frac{1}{q}$. □

Appendix C
Comparison Between Rounding and Dirichlet's Approximation

Consider a network $\mathcal{N} = (\mathcal{L}, \mathcal{I}, D)$. Suppose the accuracy of rounding is 10^{-n} with positive integer n, then $\|D - D'\|_\infty \leq 5 \times 10^{-(n+1)}$ where D' is the approximation of D. Hence $10^n D'$ is an integer matrix. For any rate vector $R \in \mathcal{R}(1, 10^n D') = \widetilde{\mathcal{R}}(1, 10^n D') = \widetilde{\mathcal{R}}(10^{-n}, D')$, by Theorem 2, there exists $R' \in \widetilde{\mathcal{R}}(\omega, D)$ with $\omega \in (0, 10^{-n}]$, such that $\|R - R'\|_\infty < 2\|D - D'\|_\infty / 10^{-n} \leq 1$.

With Dirichlet's theorem, by Theorem 7, the rate vector difference is bounded by $2/(qQ\omega)$ with $\omega \in [2/qQ, (2 + Q)/(qQ)]$. We can choose $\omega = (2 + Q)/(qQ)$ to obtain $\|R - R'\|_\infty < 2/(qQ\omega) = 2/(2+Q)$, which can be controlled by choice of Q. Compared with rounding, Dirichlet's theorem provides us with more flexibility of upper bound of rate vector difference.