(1) (3 marks) Let K be a number field of degree n. Show that it is not possible to find elements $\beta_1, \cdots, \beta_n \in \mathcal{O}_K$ such that $D(\beta_1, \cdots, \beta_n) = -D(K)$.

(2) (9 marks) Let $K = \mathbb{Q}(2^{1/3})$. As $\mathbb{Q}$-vector space, K has basis $\{1, 2^{1/3}, 2^{2/3}\}$. Let $\alpha = a + b2^{1/3} + c2^{2/3}$ be some element from K thus $a, b, c \in \mathbb{Q}$.

- Find the transformation matrix m_α obtained from the linear transform defined by the multiplication of α w.r.t the above basis. From m_α, compute the $\mathrm{Trace}_{K/\mathbb{Q}}$ and $\mathrm{Norm}_{K/\mathbb{Q}}$ of α.
- Let $\beta = 2^{1/3} - 2^{2/3} \in K$.
  - Find the minimal polynomial for β and compute the trace and norm for β from its minimal polynomial.
  - Find all K-conjugates of β and determine the number fields generated by these conjugates. Explain that the discriminants for these conjugate number fields are equivalent.

(3) (3 marks) Let $K = \mathbb{Q}(\sqrt{2})$ and ideal $I = (5 + \sqrt{2}, 7 + 2\sqrt{2})$. Compute $D(K)$ and $D(I)$.

(4) (9 marks) Given an irreducible polynomial $f(x) = x^3 - x^2 - 2x - 8$ and let α be a root of $f(x)$. Let $K = \mathbb{Q}(\alpha)$ be the number field generated by α.

- Let $\beta = \frac{1}{2}(\alpha^2 + \alpha)$. Show that β is an algebraic integer.
- Compute $D(\alpha)$.
- Deduce $D(1, \alpha, \beta)$ from $D(\alpha)$ and then explain why $\{1, \alpha, \beta\}$ is an integral basis for K.

(5) (6 marks) Let $K = \mathbb{Q}(\sqrt{6})$ and $P = (2, \sqrt{6})$ be an ideal in $\mathcal{O}_K$.

- Show that $P = 2\mathbb{Z} + \sqrt{6}\mathbb{Z}$.
- Define the set $\tilde{P} = \{\alpha \in K \mid \alpha P \subseteq \mathcal{O}_K\}$. Show that $\tilde{P} = \frac{1}{2}P$.
- Find some element $\alpha \in \tilde{P}$ such that $\alpha P \nsubseteq P$.