Linear Discriminants

CSci 5525: Advanced Machine Learning

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Announcements

- HW0 due today (by 11:59 pm CDT)
- HW1 posted today (due in 1 week 9/19)
- Project proposals due next week (9/21)

Problem

Suppose you work at a fruit company and you want to design a system which can determine whether a piece of fruit is good or bad. Let's say you have data from the past month which consists of the mass and label such as 'good' or 'bad' for each piece of fruit. For example:

Mass (g)	Label
70.2	Good
93.2	Good
40.9	Bad
82.3	Good
68.1	Bad
87.6	Bad
96.8	Good

How would you design the system?

Classification

- Dataset: $\mathcal{D} = \{(\text{Mass}_i, \text{Good/Bad}_i)\}_{i=1}^n = \{(\mathbf{x}_i, y_i)\}_{i=1}^n$
- $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^p$, $y \in \mathcal{Y}$ (discrete set)
- Mostly focus on binary classification $\mathcal{Y} = \{0, 1\}$
- Goal: find prediction function $f : \mathcal{X} \to \mathcal{Y}$

Linear Classification

- In this lecture we consider linear predictors f parameterized by weight vector $\mathbf{w} \in \mathbb{R}^p$

$$\hat{y} = f(\mathbf{x}) = \text{sign}(\mathbf{w}^\top \mathbf{x})$$

- Natural loss function is 0-1 loss:

$$\ell(y, \hat{y}) = \mathbb{1}[y \neq \hat{y}]$$

ERM for Linear Classification

- Given iid data $(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)$ the ERM problem is

$$\text{argmin}_{w \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[y_i \neq \hat{y}]$$

- Question: Is it always possible to minimize empirical risk down to 0?

Linearly Separable

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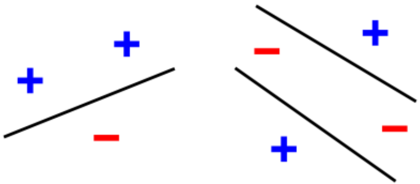


Figure: Illustration of linear separability from Wikipedia.

Feature Transformation/Representation

- Enrich linear regression/classification by transforming features $\mathbf{x}$ into $\phi(\mathbf{x})$
- Predict with transformed features: $f(\mathbf{x}) = \mathbf{w}^\top \phi(\mathbf{x})$
- Examples:

$$x \in \mathbb{R}, \phi(x) = \ln(1 + x)$$
$$\mathbf{x} \in \mathbb{R}^p, \phi(\mathbf{x}) = (1, x(1), \ldots, x(p), x(1)^2, \ldots, x(p)^2, x(1)x(2), \ldots, x(p-1)x(p))$$
$$x \in \mathbb{R}, \phi(x) = (1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots)$$

- Feature transformation could turn a linearly inseparable dataset into a linearly separable one

XOR Example

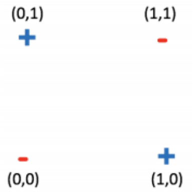


Figure: XOR dataset is not linearly separable.

- Consider the following feature transformation

$$\phi(\mathbf{x}) = (1, x_1, x_2, x_1 x_2)$$

XOR Example

- Using the previous feature transformation, we can learn the following predictor

$$f(\mathbf{x}) = -1 + 2x_1 + 2x_2 - 3.5x_1x_2$$

- Predictor is linear in $\phi(\mathbf{x})$ and perfectly classifies XOR dataset

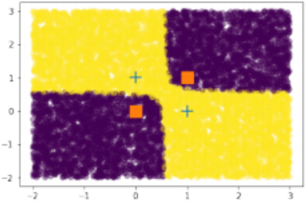


Figure: Nonlinear decision boundary of linear mapping f.

Hardness of ERM

- ERM optimization problem:

$$\text{argmin}_{w \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[y_i \neq \text{sign}(\mathbf{w}^\top \mathbf{x}_i)]$$

- This problem is NP-Hard (think about why)

Hardness of ERM

- To obtain efficient algorithms, replace 0-1 loss with other *surrogate loss* function (that is convex)
- Hinge Loss:

$$L(f, \mathbf{x}, y) = \max(0, 1 - yf(\mathbf{x})) = \begin{cases} 1 - yf(\mathbf{x}) & \text{if } yf(\mathbf{x}) < 1, \\ 0 & \text{otherwise.} \end{cases}$$

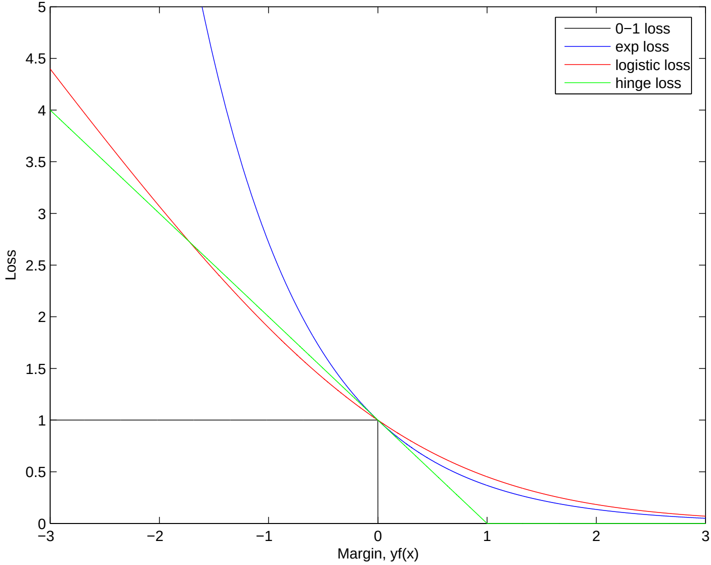
- Exponential Loss:

$$L(f, \mathbf{x}, y) = \exp(-yf(\mathbf{x}))$$

- Logistic Loss:

$$L(f, \mathbf{x}, y) = \log(1 + \exp(-yf(\mathbf{x})))$$

Loss Functions



Discriminant Functions

- One of the simplest representation for a 2-class problem

$$f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + w_0$$

- Class assignment based on $\text{sign}(f(\mathbf{x}))$
  - If $f(\mathbf{x}) \geq 0$, $\text{sign}(f(\mathbf{x})) = +1$, then $\mathbf{x} \in C_1$, otherwise $\mathbf{x} \in C_2$
- $\mathbf{w}$ is orthogonal to the decision boundary
- With $\tilde{\mathbf{w}} = (\mathbf{w}, w_0)$ and $\tilde{\mathbf{x}} = (\mathbf{x}, 1)$, we have

$$f(\mathbf{x}) = \tilde{\mathbf{w}}^\top \tilde{\mathbf{x}}$$

- At times, we will ignore the offset term w_0 w.l.o.g.
  (without loss of generality)

Least Squares for Multiclass Classification

- Consider a training dataset $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^{n}$ for a K-class problem
  - $\mathbf{y}_i$ encodes the class membership, say $\mathbf{y}_i^\top = (0, 1, 0, 0)$
  - Y : $n \times K$ matrix with rows $\mathbf{y}_i^\top$
  - X : $n \times p$ matrix with rows $\mathbf{x}_i^\top$
  - W : $p \times K$ matrix with columns $\mathbf{w}_k$
  - Goal:

$$\mathbf{w}_k^\top \mathbf{x}_i = \mathbf{x}_i^\top \mathbf{w}_k = X_{i,:}\mathbf{w}_k \approx Y_{ik}$$

- The sum-of-squares error to be minimized over W is

$$E(W) = \frac{1}{2}\sum_{k=1}^{K}\sum_{i=1}^{n} \| Y_{ik} - X_{i,:}\mathbf{w}_k \|^2 = \frac{1}{2}\,\mathrm{Tr}\left\{(Y - XW)^\top (Y - XW)\right\}$$

Least Squares for Classification (cont.)

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- The problem has a closed form solution

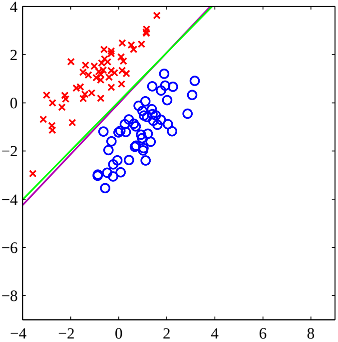
$$W = (X^\top X)^{-1} X^\top Y = X^\dagger Y$$

- $X^\dagger$ is pseudoinverse of X
- Solving each problem separately: $\mathbf{w}_k = X^\dagger \mathbf{y}_k$
- The discriminant function has the following form

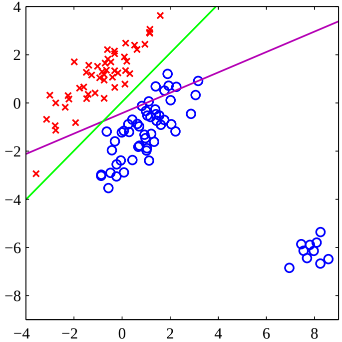
$$f(\mathbf{x}) = W^\top \mathbf{x} = Y^\top (X^\dagger)^\top \mathbf{x}$$

Least Squares is Noise Sensitive

Magenta curve is least squares
Green curve is logistic regression

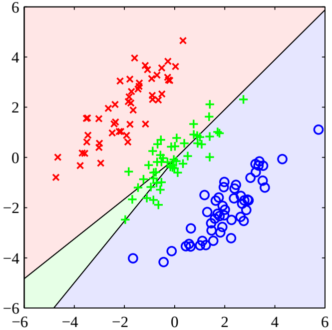


Clean dataset

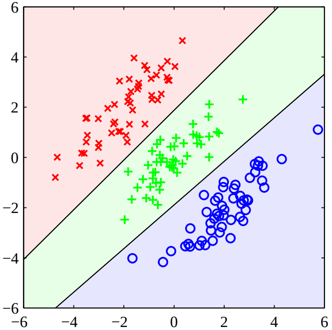


Noisy dataset

Least Squares for Multiclass Problems



Least Squares



Logistic Regression

Classification by Projection

- Classify after dimensionality reduction
  - Project p dimensional data $\mathbf{x}$ to 1 dimensions: $\mathbf{w}^\top \mathbf{x}$
  - Make sure class separation is maximized
- If $\mathbf{m}_1, \mathbf{m}_2$ are the means of the two classes

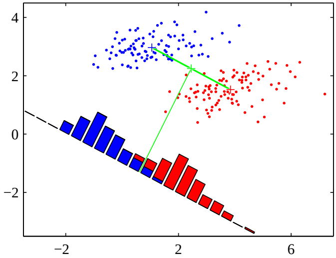
$$\max_{\|\mathbf{w}\|^2=1} \mathbf{w}^\top(\mathbf{m}_2 - \mathbf{m}_1)$$

- Performing the optimization (using 'Langrange multipliers')

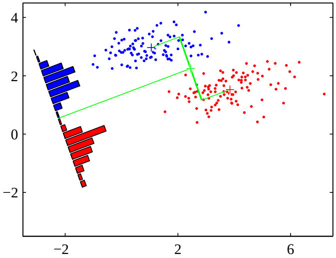
$$\mathbf{w} \propto (\mathbf{m}_2 - \mathbf{m}_1)$$

- May be problematic if data has non-diagonal covariance

Classification by Projection (cont.)



Classification by Projection



Fisher's Linear Discriminant

Fisher's Linear Discriminant

- Desirable to have low within class variance

$$\sigma_k^2 = \sum_{\mathbf{x}_i \in C_k} \|\mathbf{w}^\top (\mathbf{x}_i - \mathbf{m}_k)\|^2$$

- Between-class and within-class covariance matrices

$$S_B = (\mathbf{m}_2 - \mathbf{m}_1)(\mathbf{m}_2 - \mathbf{m}_1)^\top$$
$$S_w = \sum_{\mathbf{x}_i \in C_1} (\mathbf{x}_i - \mathbf{m}_1)(\mathbf{x}_i - \mathbf{m}_1)^\top + \sum_{\mathbf{x}_i \in C_2} (\mathbf{x}_i - \mathbf{m}_2)(\mathbf{x}_i - \mathbf{m}_2)^\top$$

- Fisher's criterion: Ratio of between-class and within-class variance

$$J(\mathbf{w}) = \frac{\|\mathbf{w}^\top (\mathbf{m}_2 - \mathbf{m}_1)\|^2}{\sigma_1^2 + \sigma_2^2} = \frac{\mathbf{w}^\top S_B \mathbf{w}}{\mathbf{w}^\top S_w \mathbf{w}}$$

Fisher's Linear Discriminant (cont.)

- Fisher's criterion is

$$J(\mathbf{w}) = \frac{\mathbf{w}^\top S_B \mathbf{w}}{\mathbf{w}^\top S_w \mathbf{w}}$$

- A 'direct calculation' gives

$$\mathbf{w} \propto S_w^{-1}(\mathbf{m}_2 - \mathbf{m}_1)$$

- A linear discriminant can be constructed using $\mathbf{w}$
  - Construct the projected version of the data $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$
  - Choose a threshold $\lambda \in \mathbb{R}$ to form linear discriminant $f(\mathbf{x}) \geq \lambda$
  - Predict class based on value of $f(\mathbf{x}) \geq \lambda$
- Extension to multiclass: Project to $(K-1)$ dimensions
- Need to train a classifier in the low dimensional representation