數值線性代數 HW3

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1 Consider the basic iterative method

$$Mx_{k+1} = Nx_k + b$$

1.1 (a) Show that the spectral radius of $G = M^{-1}N$ approximately satisfies

$$\rho(G) \approx \frac{x_{k+1} - x_k}{x_k - x_{k-1}}$$

$$\begin{aligned}
x_{k+1} &= Gx_k + M^{-1}b \\
x_k &= Gx_{k-1} + M^{-1}b \\
\Rightarrow x_{k+1} - x_k &= G(x_k - x_{k-1}) \\
\Rightarrow y_k &= Gy_{k-1} \quad (令 y_k = x_{k+1} - x_k)
\end{aligned}$$

藉由 power method,
知道 y_{k-1} 會趨近於dominant eigenvector y

$$\begin{aligned}
\Rightarrow y_k &= Gy_{k-1} \approx Gy = \lambda y \approx \lambda y_{k-1} \\
\Rightarrow \frac{\|y_k\|}{\|y_{k-1}\|} &\approx |\lambda| = \rho(G) \\
\Rightarrow \rho(G) &\approx \frac{\|x_{k+1} - x_k\|}{\|x_k - x_{k-1}\|}
\end{aligned}$$

1.2 (b) Show that if $\rho(M^{-1}N)$ is known, an estimate for the error is given by

$$\|x_k - x\|_2 \leq \frac{\rho(G)}{1 - \rho(G)} \|x_k - x_{k-1}\|_2$$

先證明 $\|e_{k+1}\|_2 \leq \rho(G)\|e_k\|_2$

$$\begin{aligned}
x_{k+1} &= M^{-1}(b + Nx_k) \\
\Rightarrow x_{k+1} &= x_k + M^{-1}(b - Ax_k) \quad \text{(A=M-N)} \\
\Rightarrow e_{k+1} &= x - x_{k+1} = (x - x_k) - M^{-1}(b - Ax_k) \\
\Rightarrow e_{k+1} &= (x - x_k) - M^{-1}A(x - x_k) \\
\Rightarrow e_{k+1} &= (I - M^{-1}A)(x - x_k) \\
\Rightarrow e_{k+1} &= (I - M^{-1}(M - N))e_k \\
\Rightarrow e_{k+1} &= M^{-1}Ne_k \\
\Rightarrow \|e_{k+1}\|_2 &= \left\|M^{-1}Ne_k\right\|_2 \leq \left\|M^{-1}N\right\|_2 \|e_k\|_2 \\
\Rightarrow \|e_{k+1}\|_2 &\leq \rho(M^{-1}N)\|e_k\|_2 \quad \text{(matrix spectral radius is less than norm)}
\end{aligned}$$

由題目，令$G = M^{-1}N$

$$
\begin{aligned}
&\|e_k\|_2 \le \rho(G)\|e_{k-1}\|_2 \\
\Rightarrow&\|x - x_k\|_2 \le \rho(G)\|x - x_{k-1}\|_2 \\
\Rightarrow&\|x - x_k\|_2 \le \rho(G)(\|x - x_k\|_2 + \|x_k - x_{k-1}\|_2) \quad \text{(三角不等式)} \\
\Rightarrow&\|x - x_k\|_2 \le \frac{\rho(G)}{1-\rho(G)}(\|x_k - x_{k-1}\|_2) \\
\Rightarrow&\|x_k - x\|_2 \le \frac{\rho(G)}{1-\rho(G)}(\|x_k - x_{k-1}\|_2)
\end{aligned}
$$

2 Consider a 500 x 500 sparse matrix A constructed as described in Trefethen and Bau's book on P. 300.

利用以下code,產生A,b

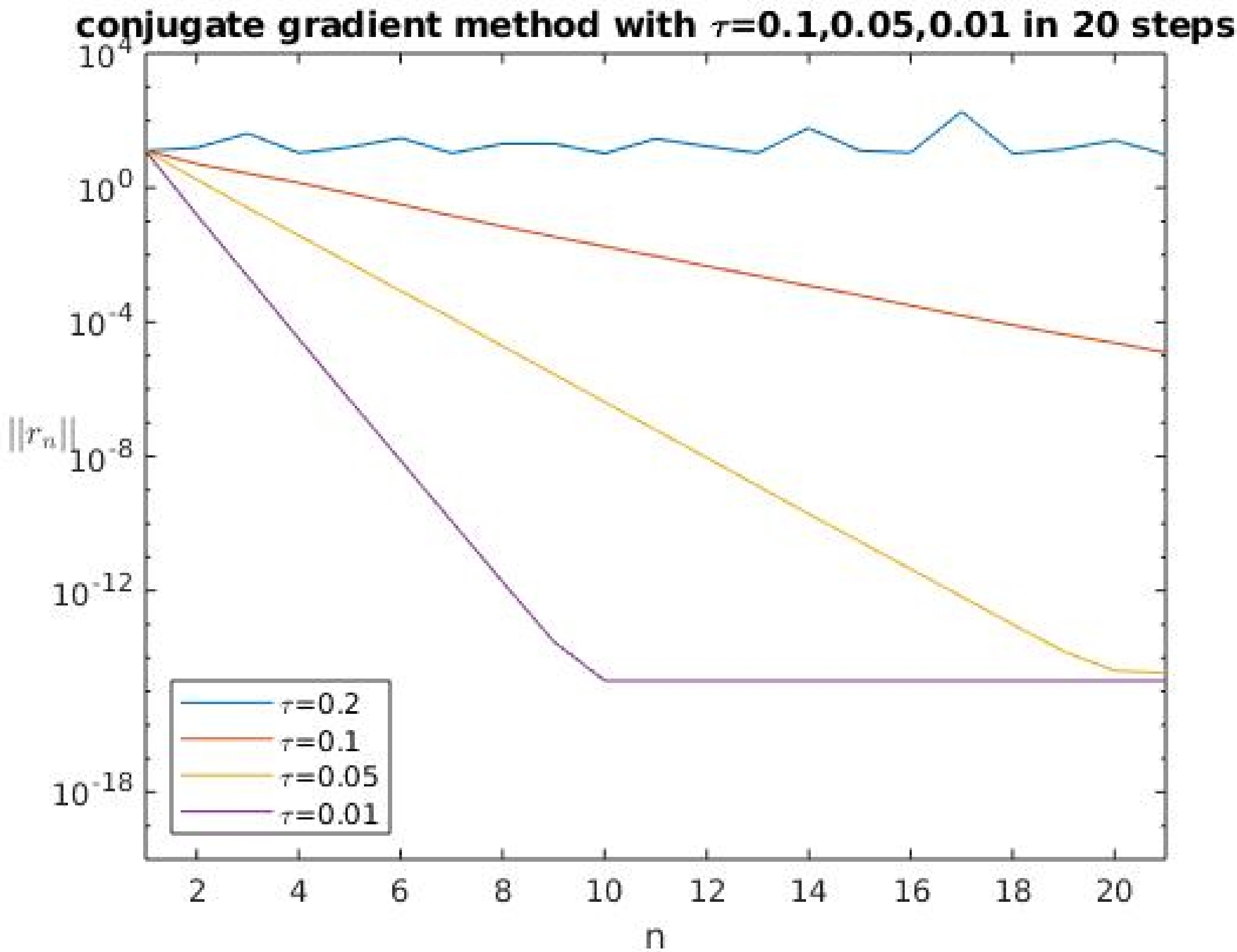
```
function [A,b] = genA(m,n,t)
    % 1 at diagonal
    % random [-1,1] at each off-diagonal (symmtric)
    % each off-diagonal if abs()>t become zero
    A = zeros(m,n);
    for i = 1:m
        for j = 1:i
            if(i==j)
                A(i,j) = 1;
            else
                putIn = (-1+2*rand(1,1));
                if(abs(putIn)>t)
                    A(i,j) = 0;
                    A(j,i) = 0;
                else
                    A(i,j) = putIn;
                    A(j,i) = putIn;
                end
            end
        end
    end
    b = -1+2*rand(m,1);
    %b = rand(m,1);
end
```

2.1 (a) Reproduce Fig.38.1 (the CG convergence curves for this matrix) shown at P.

利用以下code,進行conjugate gradient method

```
function [X,Y,x,r]=CG(A,b,N)
    [m,n] = size(A);
    Alpha = zeros(1,N+1); %20 iteration
    Beta = zeros(1,N+1); %20 iteration
    r = zeros(m,N+1); %20 iteration residual
```

```
    p = zeros(m,N+1); %20 iteration
    x = zeros(n,N+1); %20 iteration

    x(:,1)=zeros(n,1);
    r(:,1)=b; %m
    p(:,1)=r(:,1);

    X = [1:N+1];
    Y = zeros(1,N+1);
    Y(1,1) = norm(r(:,1),2);
    for i=2:N+1
        Aup = r(:,i-1)'*r(:,i-1);
        Adown = p(:,i-1)'*A*p(:,i-1);
        Alpha(1,i) = Aup/Adown;

        x(:,i) = x(:,i-1)+Alpha(1,i)*p(:,i-1);

        r(:,i) = r(:,i-1)-Alpha(1,i)*A*p(:,i-1);

        Bup = r(:,i)'*r(:,i);
        Bdown = r(:,i-1)'*r(:,i-1);
        Beta(1,i) = Bup/Bdown;

        p(:,i) = r(:,i)+Beta(1,i)*p(:,i-1);
        norm(r(:,i),2)

        Y(1,i) = norm(A*x(:,i)-b,2);
    end
    %norm(r(:,N+1),2)
end
```

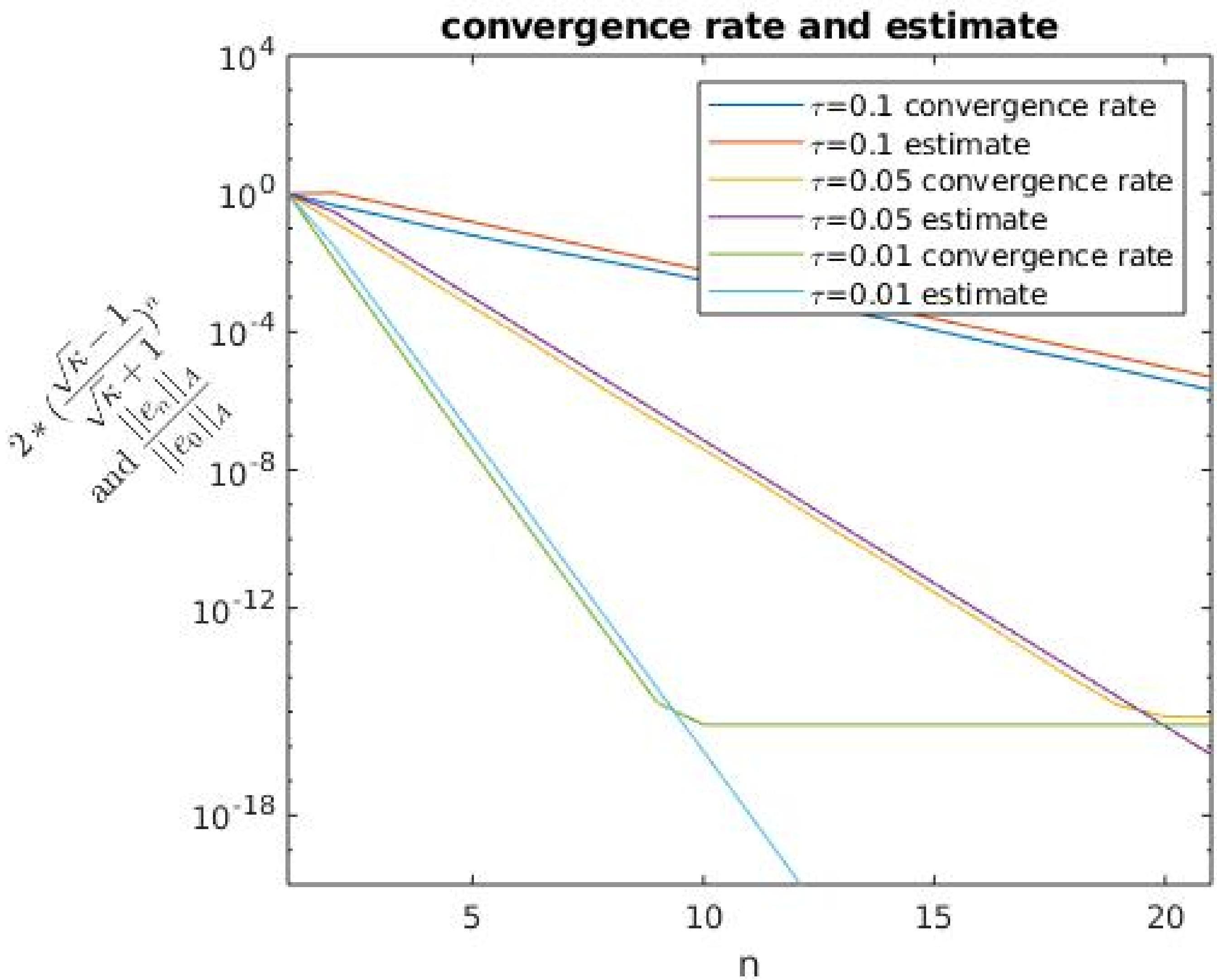


2.2 (b) Produce a plot for $\tau = 0.01$, 0.05, 0.1 indicating how closely the above estimates match the actual convergence rate

利用以下code, 計算 convergence rate 以及 error estimate

```
function [X,Y,Y2]=condition(A,b,N,xCG)
    [m,n] = size(A);
    %kappa = norm(inv(A),2)*norm(A,2);
    %using eigenvalue to get kappa
    [V,D]=eig(A);
    lambda_max = max(max(diag(D)));
    lambda_min = min(min(diag(D)));
    kappa = lambda_max/lambda_min;

    xT = A\b;
    %norm(A*xT-b,2)
    e = zeros(m,N+1);

    e(:,1) = xT-xCG(:,1);
    e1A = sqrt(e(:,1)'*A*e(:,1))

    X = [1:N+1];
```

```
    Y = zeros(1,N+1);
    Y2 = zeros(1,N+1);
    Y(1,1) = 1;
    Y2(1,1) = 1;
    for i=2:N+1
        e(:,i) = xT-xCG(:,i);
        eiA = sqrt(e(:,i)'*A*e(:,i));
        ratio = eiA / e1A;
        bound = 2*(((sqrt(kappa)-1)/(sqrt(kappa)+1))^(i-1));
        %[ratio,bound]
        Y(1,i) = ratio;
        Y2(1,i) = bound;
    end
end
```

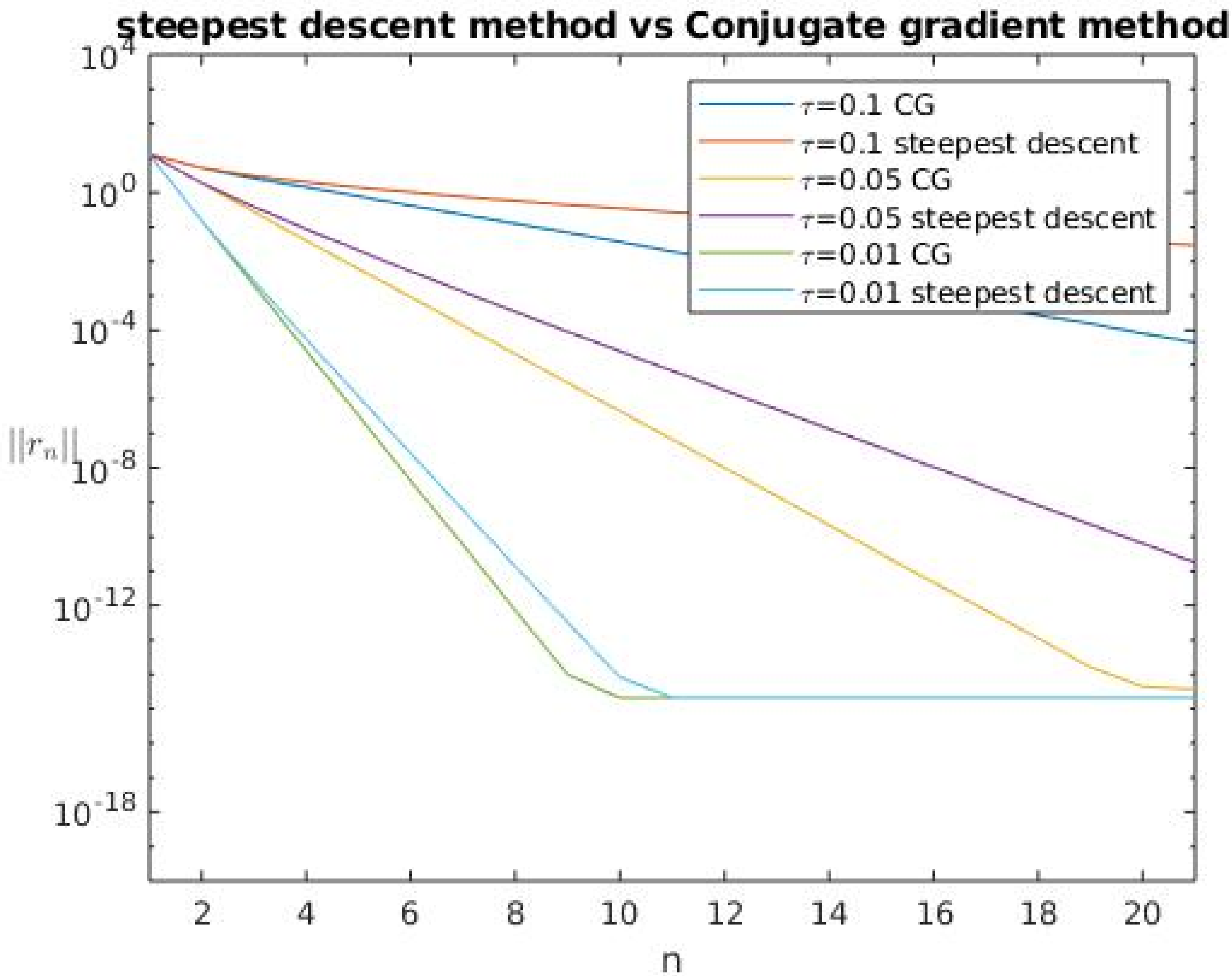


- 由圖可知，在收斂前，conjugate-gradient method的convergence rate確實會以此estimate為upper bound且兩者的值相當接近

2.3 (c) Use the method of steepest descent to solve this linear system again and compare results with those obtained using CG.

利用以下code, 進行steepest descent method

```
function [X,Y,x,r]=steepest(A,b,N)
    [m,n] = size(A);
    Alpha = zeros(1,N+1); %20 iteration
    r = zeros(m,N+1); %20 iteration residual
    x = zeros(n,N+1); %20 iteration

    x(:,1)=zeros(n,1);
    r(:,1)=b; %m

    X = [1:N+1];
    Y = zeros(1,N+1);
    Y(1,1) = norm(r(:,1),2);
    for i=2:N+1
        Aup = r(:,i-1)'*r(:,i-1);
        Adown = r(:,i-1)'*A*r(:,i-1);
        Alpha(1,i) = Aup/Adown;

        x(:,i) = x(:,i-1)+Alpha(1,i)*r(:,i-1);
        r(:,i) = r(:,i-1)-Alpha(1,i)*(A*r(:,i-1)); %steepest

        Y(1,i) = norm(A*x(:,i)-b,2);
    end
end
```



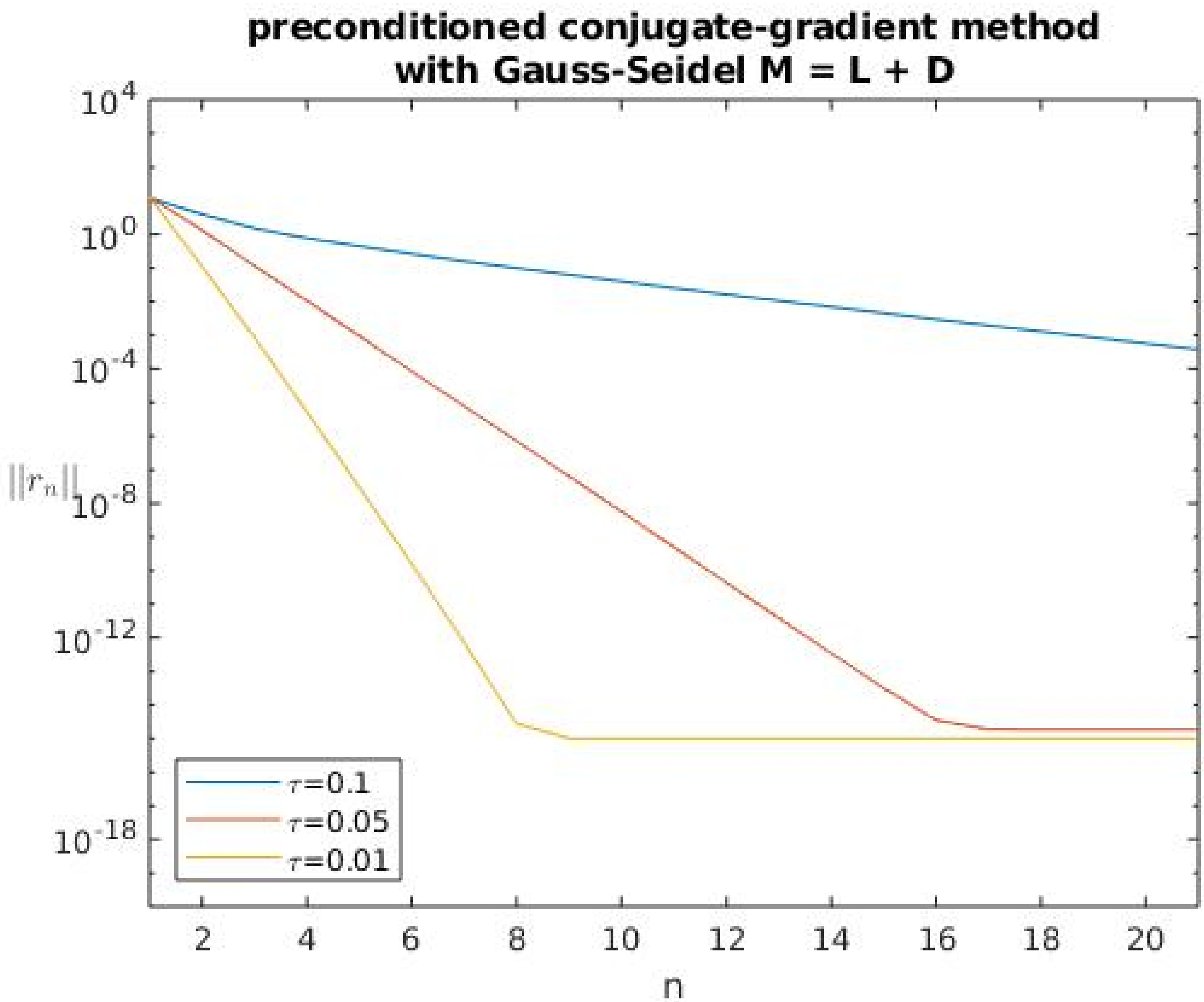
- 由圖可知steepest descent method的收斂速度較conjugate gradient method慢

2.4 (d) Redo this problem also using the preconditioned conjugate-gradient method with the gauss-Seidel preconditioner M = D+L. Comments on your results.

利用以下code, 進行 preconditioned conjugate-gradient method with gauss-Seidel preconditioner

```
%% Solution of x in Ax=b using Gauss Seidel Method
function [X,Y,x]=Precondlib(A,b,tol,maxIt)
    [m,n] = size(A);
    x=zeros(n,1);
    n=size(x,1);
    normVal=Inf;
    iter = 1;
    Y = [norm(A*x(:)-b,2)];
    while (normVal>tol || iter<=maxIt)
        x_old=x;
        for i=1:n
            sigma=0;
            for j=1:i-1
                sigma=sigma+A(i,j)*x(j);
```

```
            end
            for  j=i+1:n
                 sigma=sigma+A(i,j)*x_old(j);
            end
            x(i)=(1/A(i,i))*(b(i)-sigma);
        end
        normVal=norm(x_old-x);
        iter = iter + 1;
        Y=[Y,norm(A*x(:)-b,2)];
    end
    X = [1:iter];
end
```



- 由圖可知，preconditioned conjugate-gradient method的收斂速度較快
- $\tau = 0.01$時，可利用preconditioned matrix提前2個iteration收斂
- $\tau = 0.05$時，可利用preconditioned matrix提前4個iteration收斂