數值線性代數 HW4

b05502087 王竑睿

1 Let A be a 2 × 2 matrix of the form

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Find an orthogonal 2 × 2 matrix Q such that $T = Q^T AQ$ is an upper triangular matrix; derive Q step by step

[solution:]

- 先計算A的兩個特徵值λ_1, λ_2

$$\begin{aligned} & det(A - \lambda I) = 0 \\ \Rightarrow & (1-\lambda)(4-\lambda) - 6 = 0 \Rightarrow \lambda_1 = \frac{-\sqrt{33}+5}{2}, \quad \lambda_2 = \frac{\sqrt{33}+5}{2} \end{aligned}$$

- 求出A的特徵向量v_1, v_2

$$\begin{aligned} & (A - \lambda_1 I)v = 0 \\ \Rightarrow & v1 = \begin{pmatrix} \dfrac{-\sqrt{33}-3}{6} \\ 1 \end{pmatrix} \\ & (A - \lambda_2 I)v = 0 \\ \Rightarrow & v2 = \begin{pmatrix} \dfrac{\sqrt{33}-3}{6} \\ 1 \end{pmatrix} \end{aligned}$$

- 將兩個特徵向量作 gram schmidt 得到 e_1, e_2

$$e_1 = v_1 = \begin{pmatrix} \dfrac{-\sqrt{33}-3}{6} \\ 1 \end{pmatrix}$$

$$e_2 = v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{pmatrix} \dfrac{\sqrt{33}-3}{6} \\ 1 \end{pmatrix}$$

- 將e_1, e_2化為orthonormal 得到 q_1, q_2

$$\begin{aligned} q_1 &= \frac{e_1}{\|e_1\|_2} \\ &= \begin{pmatrix} -\dfrac{\sqrt{6}+\sqrt{22}}{2\sqrt{\sqrt{33}+13}} \\ \dfrac{\sqrt{6}}{\sqrt{\sqrt{33}+13}} \end{pmatrix} \end{aligned}$$

$$q_2 = \frac{e_2}{\|e_2\|_2} = \begin{pmatrix} \frac{\sqrt{187}(-33+13\sqrt{33})}{374\sqrt{13-\sqrt{33}}} \\ \frac{\sqrt{187}(\sqrt{33}+55)}{374\sqrt{13-\sqrt{33}}} \end{pmatrix}$$

- $Q = [q_1, q_2]$

$$Q = \begin{pmatrix} -\frac{\sqrt{6}+\sqrt{22}}{2\sqrt{\sqrt{33}+13}} & \frac{\sqrt{187}(-33+13\sqrt{33})}{374\sqrt{13-\sqrt{33}}} \\ \frac{\sqrt{6}}{\sqrt{\sqrt{33}+13}} & \frac{\sqrt{187}(\sqrt{33}+55)}{374\sqrt{13-\sqrt{33}}} \end{pmatrix}$$

使用 **matlab** 計算 Q^TAQ 的結果

```
>> Q = [e1',e2']

Q =

  -0.824564840132394   0.565767464968992
   0.565767464968992   0.824564840132394

>> Q'*A*Q

ans =

  -0.372281323269014   1.000000000000000
                   0   5.372281323269015
```

Q^TAQ確實為上三角矩陣

2 Let λ be a simple eigenvalue of A $\in R^{m\times m}$ with right eigenvector x and left eigenvector y (that is, Ax $=\lambda$x and yA $=$ yλ), normalized so that $\|x\|_2 = \|y\|_2 = 1$. Define the eigenvalue condition number by $\kappa(\lambda, A) = 1/|y * x|$.

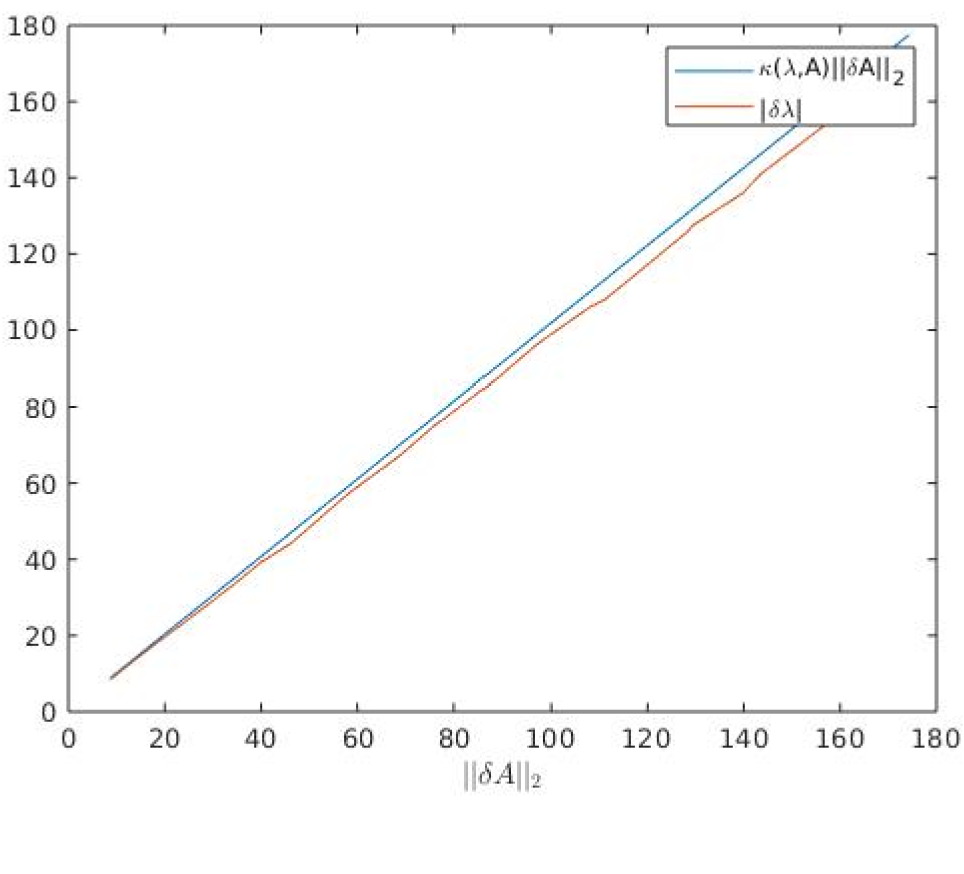
(a)

[solution:]

$$
\begin{aligned}
&(A+\delta A)(x+\delta x) = (\lambda+\delta\lambda)(x+\delta x)\\
\Rightarrow & Ax + A\delta x + \delta A x + \delta A\delta x = \lambda x + \lambda\delta x + \delta\lambda x + \delta\lambda\delta x\\
\Rightarrow & A\delta x + \delta A x = \lambda\delta x + \delta\lambda x + (\delta\lambda\delta x - \delta A\delta x) \quad (\text{因為}Ax = \lambda x)\\
\Rightarrow & (A-\lambda I)\delta x + \delta A x = \delta\lambda x + (\delta\lambda\delta x - \delta A\delta x)\\
\Rightarrow & y^*(A-\lambda I)\delta x + y^*\delta A x = \delta\lambda y^* x + y^*(\delta\lambda\delta x - \delta A\delta x)\\
\Rightarrow & 0 + y^*\delta A x = \delta\lambda y^* x + y^*(\delta\lambda\delta x - \delta A\delta x) \quad (\text{因為y是A的left eigenvector})\\
\Rightarrow & y^*\delta A x + y^*(\delta A\delta x - \delta\lambda\delta x) = \delta\lambda y^* x\\
\Rightarrow & \delta\lambda = \frac{y^*\delta A x}{y^* x} + \frac{y^*(\delta A\delta x - \delta\lambda\delta x)}{y^* x}\\
\Rightarrow & \delta\lambda = \frac{y^*\delta A x}{y^* x} + O(\|\delta A\|_2\|\delta x\|_2)\\
\Rightarrow & \delta\lambda = \frac{y^*\delta A x}{y^* x} + O(\|\delta A\|_2^2) \quad (\text{由power method } \|\delta x\|_2 = O(\|\delta A\|_2))
\end{aligned}
$$

Demonstrate the sensitivity of λ with a simple example for m = 16, for instance.

下圖為random產生16×16的矩陣所demo出的sensitivity of λ



(b)

[solution:]

- calculate the eigenvalue of the Jordan Block

$$
\begin{aligned}
&det(J - \lambda_h I) = 0 \\
\Rightarrow &(\lambda - \lambda_h)^m = 0 \\
\Rightarrow &\lambda_h = \lambda
\end{aligned}
$$

- calculate the right eigenvector of the Jordan Block

$$
\begin{aligned}
&(J - \lambda I)x = 0 \\
\Rightarrow &\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & \ddots & \ddots & 1 \\ & & \dots & 0 \end{bmatrix} x = 0 \\
\Rightarrow &x = span(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix})
\end{aligned}
$$

- calculate the left eigenvector of the Jordan Block

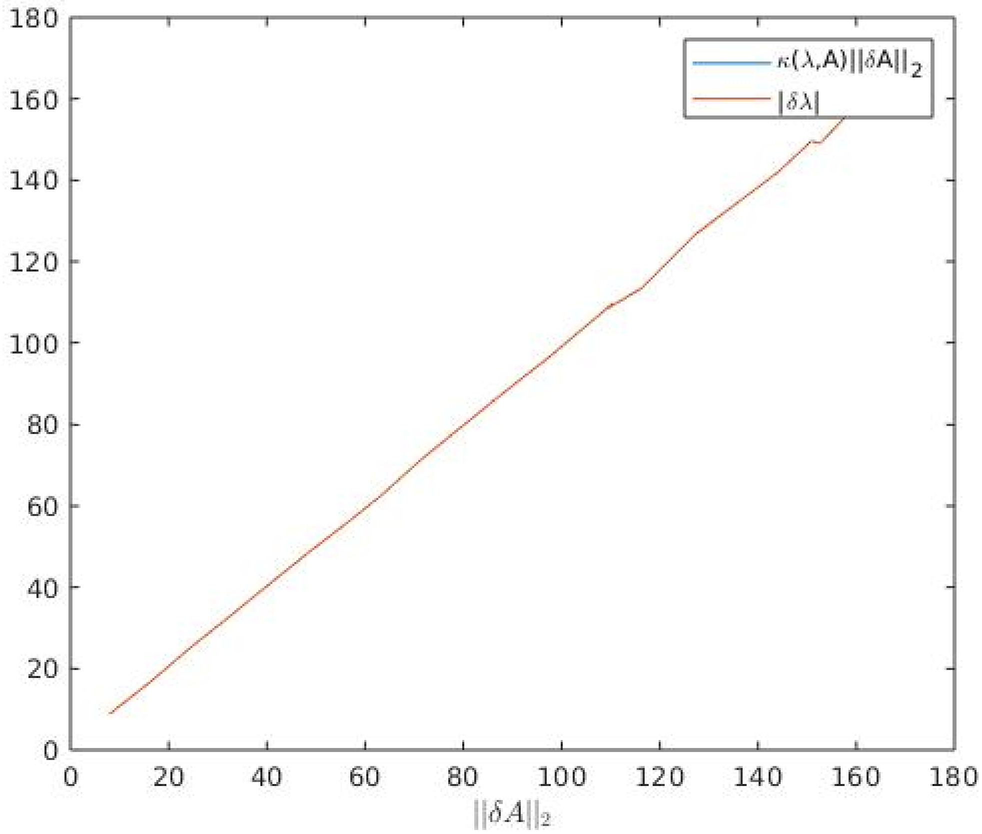
$$
\begin{aligned}
&y(J - \lambda I) = 0 \\
\Rightarrow &(J - \lambda I)^T y^T = 0 \\
\Rightarrow &\begin{bmatrix} 0 & & & \\ 1 & 0 & \ddots & \\ & \ddots & 0 & \\ & & 1 & 0 \end{bmatrix} y^T = 0 \\
\Rightarrow &y^T = span(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}) \\
\Rightarrow &y = span([0, 0, \dots, 1])
\end{aligned}
$$

- calculate the eigenvalue condition number

$$
\begin{aligned}
&\kappa(\lambda, J) \\
= &1/|y^* x| \\
= &\infty \quad (\text{因為 } y^* x = 0)
\end{aligned}
$$

Demonstrate the sensitivity of λ with a simple example for m = 16, for instance.

下圖為產生16×16的Jordan-Block矩陣所demo出的sensitivity of λ
由於$\kappa(\lambda, A)$為∞，所以$\kappa(\lambda, A)\|\delta A\|_2$未出現在圖中



此兩項Demo是利用以下code產生matrix，計算sensitivity，畫圖得到

```
function generate_matrix(m)
    J = zeros(m,m);
    la = 3;
    for i = 1:m
        for j = 1:m
            if(i==j)
                J(i,j)=la;
            elseif(i==(j-1))
                J(i,j)=1;
            end
        end
    end
    J
    Calculate_sense(J)
    A = rand(m,m);
    Calculate_sense(A);
end
function Calculate_sense(A)
    [lambda,x,y]=getxy(A);
    Y_delta_lambda = []
    Y_delta_A = []
    X_norm = []
    for len = 1:20
        delta_A = len*rand(16,16);
        A_plus_delta = A+delta_A;

        kappa = 1 / abs(y*x);
```

```
        delta_A_norm = norm(delta_A);
        X_norm = [X_norm, delta_A_norm];

        [lambda_2,x_2,y_2]=getxy(A_plus_delta);
        delta_lambda = abs(lambda_2-lambda);

        Y_delta_lambda = [Y_delta_lambda,delta_lambda];
        Y_delta_A = [Y_delta_A,kappa*delta_A_norm];
    end
    figure();
    plot(X_norm,Y_delta_A);
    hold on
    plot(X_norm,Y_delta_lambda)
    legend("\kappa(\lambda,A)||\deltaA||_2","|\delta\lambda|",'latex')
    xlabel("$||\delta A||_2$",'interpreter','latex');

    X_norm
    Y_delta_lambda
    Y_delta_A

end
function [lambda,x,y]=getxy(A)
    [Vr,Dr] = eig(A);
    [Vl,Dl] = eig(A');
    Dr(1,1);
    Vr(:,1);
    Dl(1,1);
    Vl(:,1);

    x = Vr(:,1);
    y = Vl(:,1)';
    lambda = Dr(1,1);
end
```

---

3 Let $\hat{T}$ and T be two m $\times$ m tridiagonal toeplitz matrices of the form

$$\hat{T} = \begin{bmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix}, T = \begin{bmatrix} a & b & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & b & a \end{bmatrix}$$

(a) Show that the eigenvalues and the corresponding eigenvectors are

$$\lambda_j = 2cos(\frac{j\pi}{m+1}), v_j = \begin{bmatrix} sin(\frac{j\pi}{m+1}) \\ sin(\frac{2j\pi}{m+1}) \\ \cdots \\ sin(\frac{mj\pi}{m+1}) \end{bmatrix}$$

,where $\hat{T}v_j = \lambda_j v_j$, for j = 1, 2,...,m.

[solution:]

- Calculate eigenvalues

$det(\hat{T} - \lambda I) = 0$

$\Rightarrow det(T^h) = 0$ (令 T^h 為 $\hat{T} - \lambda I$)

$det(T^h_m) = (-\lambda)det(T^h_{m-1}) + (-1)det(T^h_{m-2})$ (由行列式降階, T^h_m表示取T^h的左上角$m \times m$部份)

令$-\lambda = 2x$,

由chebyshev polynomial知$det(T^h_m)$是一個x的m次多項式，且m個root為

$x_k = cos(k\pi/(m+1)), k = 1, 2, ..., m$

轉換回λ形式的root

$\lambda_k = (-2)cos(k\pi/(m+1)), k = 1, 2, ..., m$

由於cos對角度有互補會差一個負號

$\lambda_k = (-2)cos(k\pi/(m+1)) = 2cos((m+1-k)\pi/(m+1)) = (-1)\lambda_{m+1-k}, k = 1, 2, ..., m$

因此，上述式子也可以去除負號

$\lambda_k = 2cos(k\pi/(m+1)), k = 1, 2, ..., m$

所以

$$\lambda_j = 2cos(j\pi/(m+1)), j = 1, 2, ..., m$$

- Calculate corresponding eigenvectors

若有$\lambda_j = 2cos(j\pi/(m+1)), j = 1, 2, ..., m$

令$\theta_j = \frac{j\pi}{m+1}$, v_{jk}為v_j的第k個element, 證明:$v_{jk} = sin(k\theta_j)$

使用數學歸納法

$k = 1$: we normalize the eigenvector with making its first element become $sin(\theta)$

$k = 2$:

從$\hat{T}v_j = \lambda_j v_j$, 可以得到$v_{j2} = \lambda_j v_{j1}$

$v_{j2} = 2cos(j\pi/(m+1))sin(\theta) = 2cos(\theta)sin(\theta) = sin(2\theta)$

假設$k = k' - 1, k' - 2$時, $v_{jk} = sin(k\theta_j)$皆成立,則

$k = k'$:

從$\hat{T}v_j = \lambda_j v_j$, 可以得到

$\Rightarrow v_{jk'} = \lambda_j v_{j(k'-1)} - v_{j(k'-2)}$

$\Rightarrow v_{jk'} = \lambda_j sin((k'-1)\theta) - sin((k'-2)\theta)$

$\Rightarrow v_{jk'} = 2cos(\theta)sin((k'-1)\theta) - sin((k'-2)\theta)$

$\Rightarrow v_{jk'} = sin(k'\theta)$ (由 chebyshev polynomial) $\Rightarrow$ k=k'時，亦成立

所以

$$v_j = \begin{bmatrix} sin(\theta_j) \\ sin(2\theta_j) \\ \cdots \\ sin(m\theta_j) \end{bmatrix} = \begin{bmatrix} sin(\frac{j\pi}{m+1}) \\ sin(\frac{2j\pi}{m+1}) \\ \cdots \\ sin(\frac{mj\pi}{m+1}) \end{bmatrix}$$

(b) What are the eigenvalues of T?

[solution:]

令$\lambda_j^{\hat{T}}$為$\hat{T}$的第j個eigenvalue

$$\text{由於 } T = a + b\hat{T}$$

因此

$$\lambda_j^{\hat{T}} = a + b\lambda_j$$

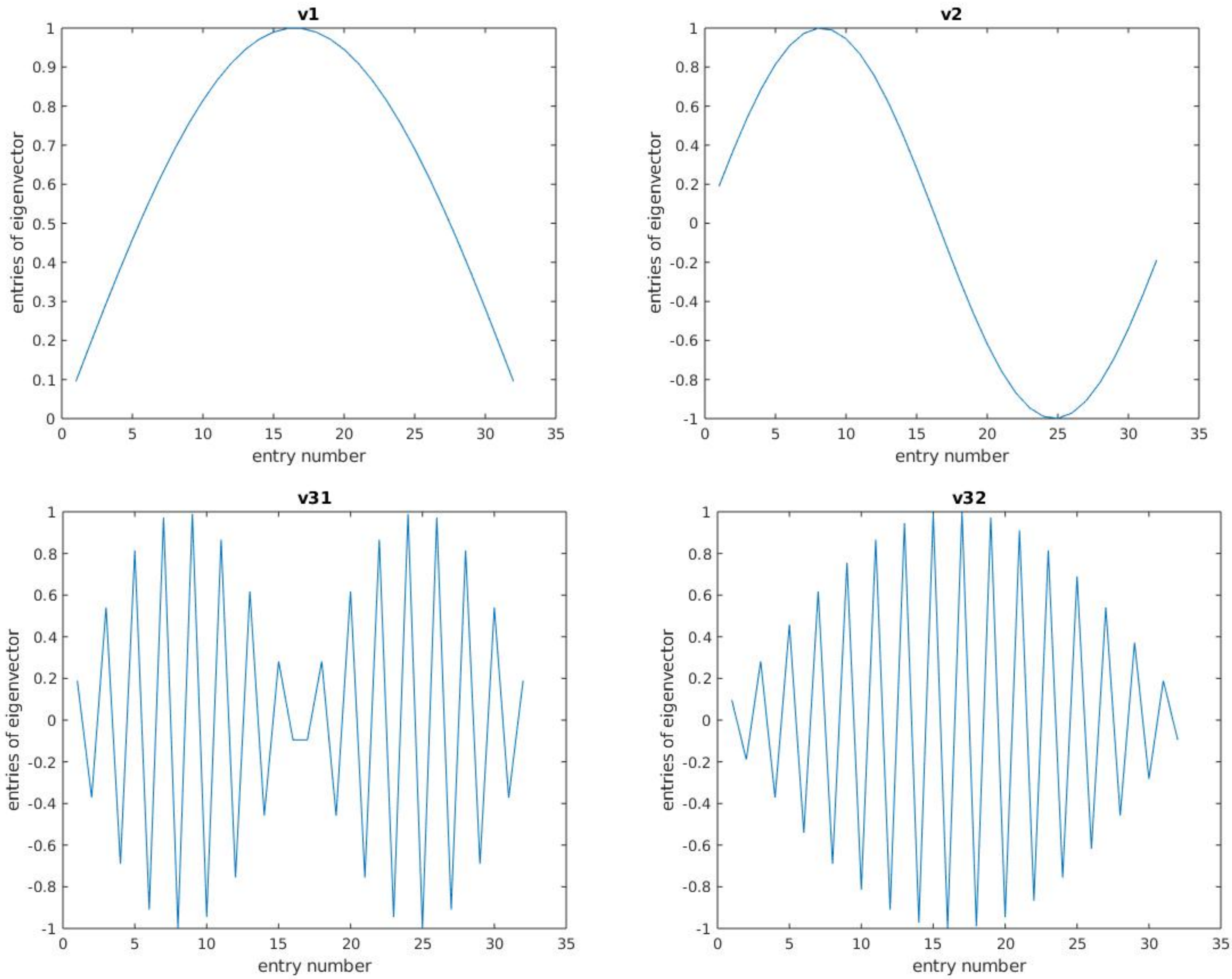
$$\Rightarrow \lambda_j^{\hat{T}} = a + 2bcos(j\pi/(m+1)), j = 1, 2, ..., m$$

所以

$$\lambda_j^{\hat{T}} = a + 2bcos(j\pi/(m+1)), j = 1, 2, ..., m$$

(c) For m = 32, plot the eigenvector of j = 1, 2, 31, 32.

[solution:]



4 Consider the matrix A written in Matlab as

```
A = diag(15:-1:1)+ones(15,1);
```

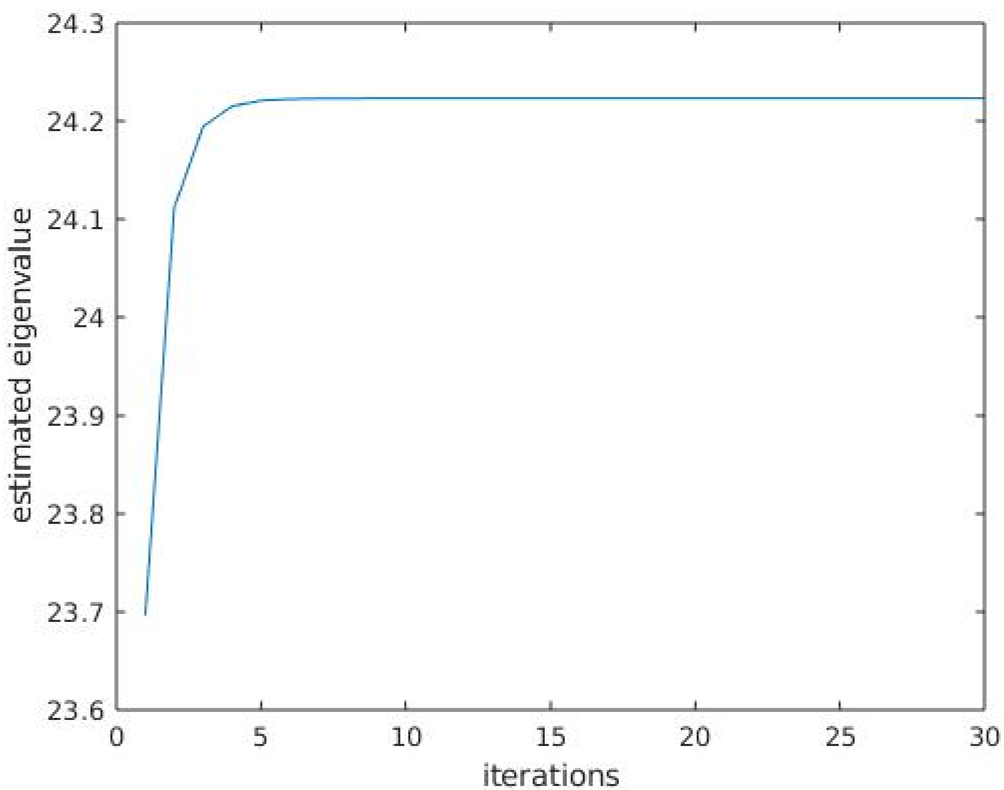
(a) Use the power iteration to find the largest eigenvalue of A. Comment on the rate of convergence of the method.

[solution:]

利用以下code，進行power Iteration

```
function EigenValue = powerIter(A)
    [m,n] = size(A);
    x = rand(n,1);
    iterTime = 30;
```

```
        Y = [];
        [V,D] = eig(A);
        for i=1:iterTime
            x = A*x;
            x = x/norm(x);
            x;
            EigenValue = (x'*A*x) / (x'*x);
            Y = [Y, EigenValue];
        end
        %eig(A)
        %EigenValue
        D(15,15);
        V(:,size(D,1));
        plot([1:iterTime],Y);
        xlabel("iterations")
        ylabel("estimated eigenvalue")
    end
```



最大的eigenvalue為:24.2231
大約花費了5個iteration達到收斂

(b) Use the inverse iteration to find the eigenvector that corresponds to a selected eigenvalue. Comment on the rate of convergence of the method.

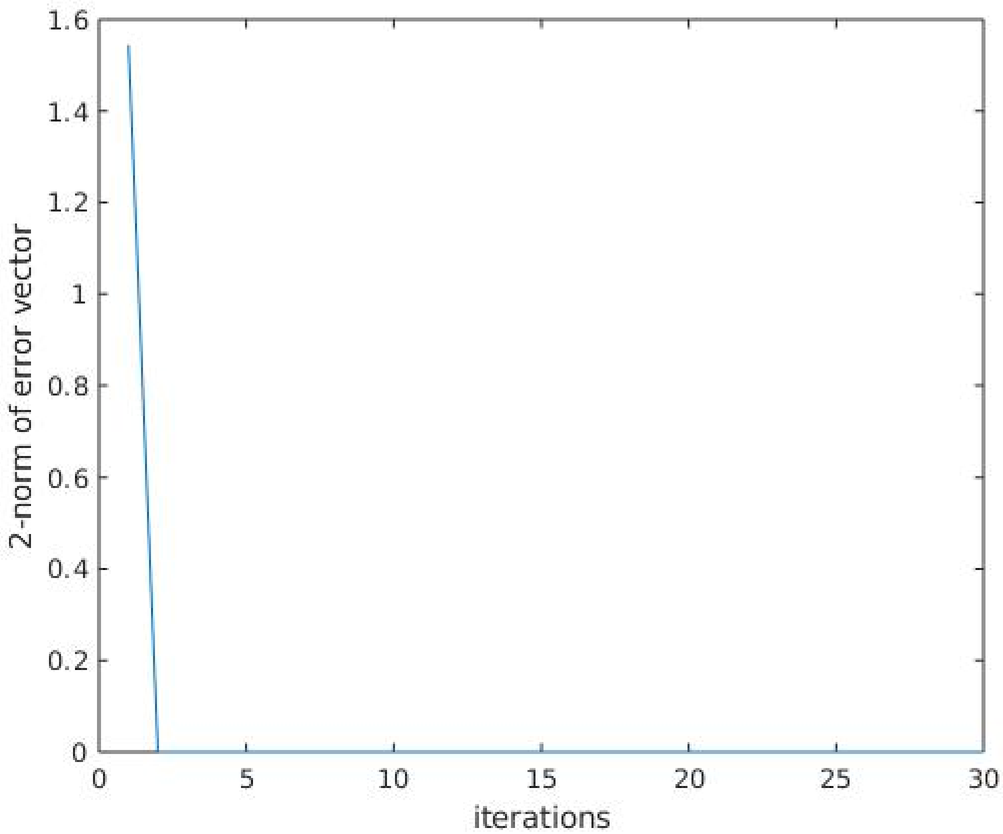
[solution:]

利用以下code，進行Inverse Iteration

```
    function EigenVector = InverseIter(A,mu)
        [m,n] = size(A);
```

```
    x = rand(n,1);
    iterTime = 30;

    [V,D] = eig(A);
    truth = V(:,size(D,1));
    Y = [];
    for i=1:iterTime
        err = norm(abs(x)-abs(truth),2);
        Y = [Y,err];
        x = inv(A-mu*eye(m))*x;
        x = x/norm(x);
    end
    EigenVector = x;
    plot([1:iterTime],Y);
    xlabel("iterations")
    ylabel("2-norm of error vector")
end
```



選取的eigenvalue為(a)中求出的24.2231。 對應的eigenvector為:

$$\begin{bmatrix} 0.4030 \\ 0.3636 \\ 0.3312 \\ 0.3041 \\ 0.2811 \\ 0.2614 \\ 0.2442 \\ 0.2291 \\ 0.2158 \\ 0.2040 \\ 0.1934 \\ 0.1838 \\ 0.1752 \\ 0.1673 \\ 0.1601 \end{bmatrix}$$

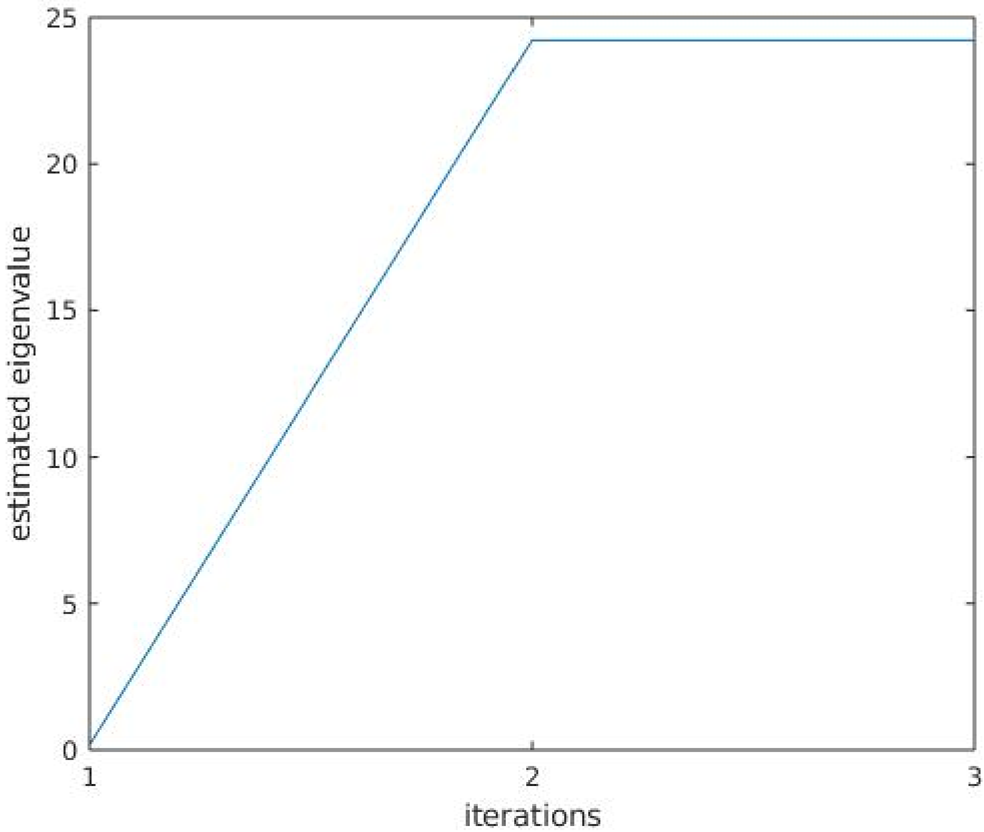
大約2次iterations

(c) Use the Rayleigh-quotient iteration to find the eigenvalue that corresponds to the eigenvector closest to the initial iteration vector. Comment on the rate of convergence of the method.

[solution:]

利用以下code，進行Rayleigh-quotient iteration

```
function eigenValue = Rayleigh(A,b)
    [m,n] = size(A);
    iterTime = 100;
    mu = rand(1,1)
    Y = [mu];
    ct = 1;
    for i=1:iterTime
        b = inv(A-mu*eye(m))*b;
        mu2 = (b'*A*b) / (b'*b)
        Y = [Y,mu2];
        ct = ct+1;
        if(norm(mu2-mu)<1e-10)
            break;
        end
        mu = mu2
    end
    eigenValue = mu;
    Y
    plot([1:ct],Y);
    xlabel("iterations")
    ylabel("estimated eigenvalue")
    set(gca,'XTick',[1:1:ct])
end
```



選取(b)所求出的eigenvector
最大的eigenvalue為:24.2231
幾乎是一次iteration就達到收斂

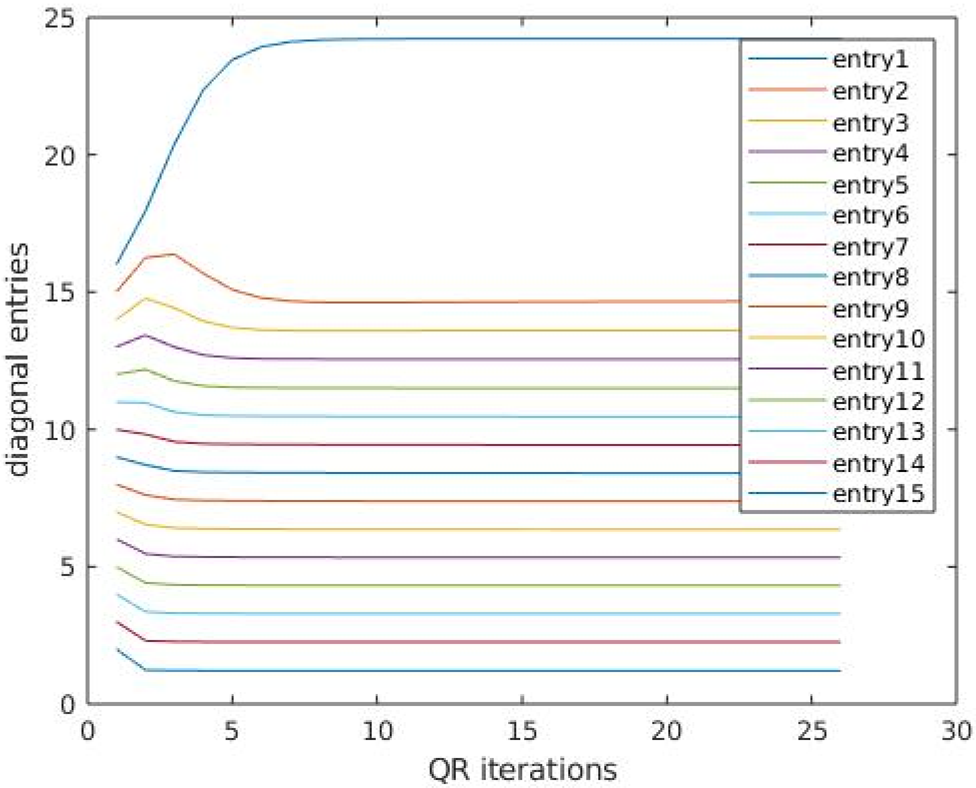
(d) Use the "pure" QR iteration to find all the eigenvalues of A. In this case, plot the diagonal entries of a matrix undergoing QR iteration.

[solution:]

利用以下code，進行QRalgo

```
function QRalgo(A,Thres,iterTime)
    %iterTime = 100;
    Toplot = [];
    it = 0;
    for i=1:iterTime
        i
        it = it + 1;
        D = diag(A);
        Toplot = [Toplot,D];
        [Q,R] = qr(A);
        A2 = Q'*A*Q;
        if(norm(A2-A)<Thres)
            break;
        end
        A = A2;
    end
    Toplot;
    figure();
```

```
    for i=1:15
        plot([1:it],Toplot(i,[1:it]))
        hold on
    end
    leg = string([1:15])
    for i=1:15
        leg(i) = "entry" + leg(i);
    end
    legend(leg);
    xlabel("QR iterations");
    ylabel("diagonal entries");
    ret = diag(A)
end
```



所有eigenvalue為:

$$\begin{bmatrix} 24.2231 \\ 14.6589 \\ 13.5912 \\ 12.5415 \\ 11.5020 \\ 10.4687 \\ 9.4392 \\ 8.4124 \\ 7.3872 \\ 6.3630 \\ 5.3390 \\ 4.3143 \\ 3.2878 \\ 2.2570 \\ 1.2147 \end{bmatrix}$$

(e) Use the QR iteration with Wilkinson's shift to find all the eigenvalues of A.

[solution:]

利用以下code，進行Wilkinson's shift

```
function ret=WilkShift(A, Thres, iterTime)
    %iterTime = 100;
    [m,n] = size(A);
    I = eye(m);
    for i=1:iterTime
        i
        s = A(m,n);
        [Q,R] = qr(A-s*I);
        A2 = R*Q + s*I;
        if(norm(A2-A)<Thres)
            break;
        end
        A = A2;
    end
    ret = diag(A);
end
```

所有eigenvalue為:

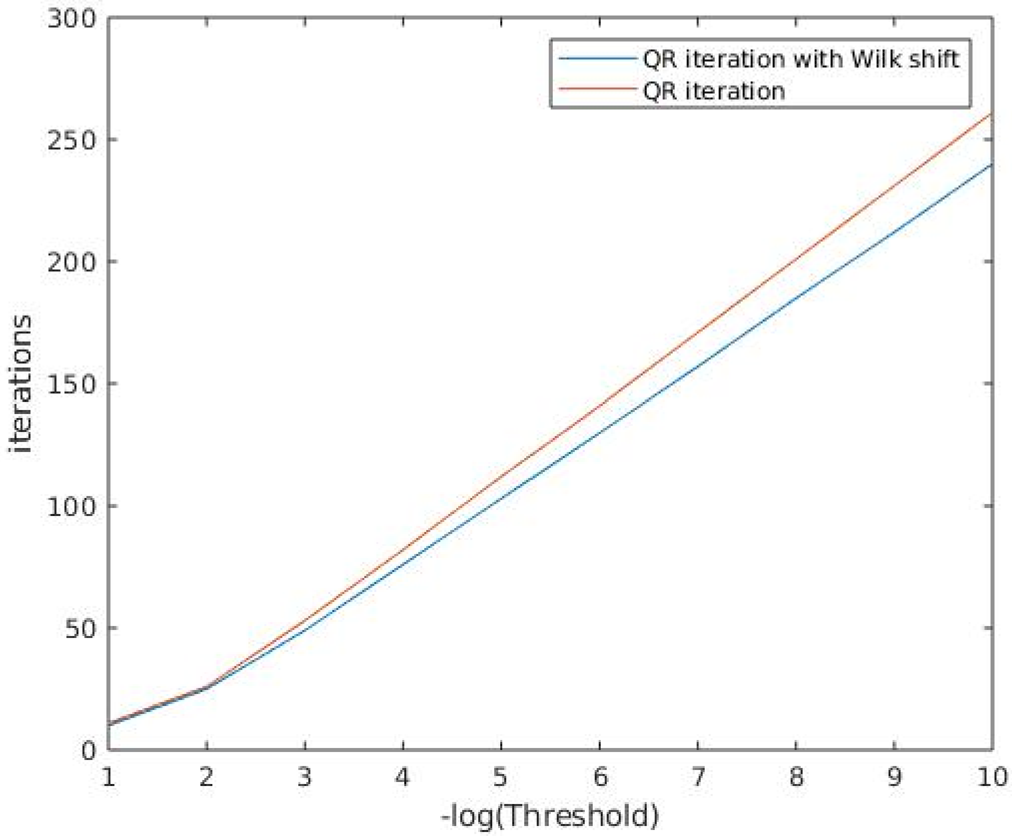
$$\begin{bmatrix} 24.2231 \\ 14.6599 \\ 13.5912 \\ 12.5413 \\ 11.5018 \\ 10.4685 \\ 9.4391 \\ 8.4123 \\ 7.3872 \\ 6.3630 \\ 5.3390 \\ 4.3143 \\ 3.2878 \\ 2.2570 \\ 1.2147 \end{bmatrix}$$

(f) Discuss the rate of convergence of the QR method considered in (d) and (e).

[solution:]

利用以下code，比較QR iteration以及QR iteration with Wilkinson's shift在不同threshold下需要的iteration數

```
function discuss(A)
    iterTime = 10^(9);
    Ywilk = [];
    YQR = [];
    for i = 1:10
        Thres = 10^(-i);
        [ret, TotalIter1]=WilkShift(A,Thres,iterTime);
        [ret, TotalIter2]=QRalgo(A,Thres,iterTime);
        Ywilk = [Ywilk,TotalIter1];
        YQR = [YQR, TotalIter2];
    end
    figure();
    plot([1:10],Ywilk);
    %plot([10:10:100],Ywilk);
    hold on;
    plot([1:10],YQR);
    legend("QR iteration with Wilk shift","QR iteration");
    ylabel("iterations");
    xlabel("$-log_{10}(Threshold)$",'Interpreter','latex');
end
```



- 由圖可以看出，同樣的Threshold下，QR iteration with Wilkinson's shift都能比QR iterations用更少的iterations數收斂