I. Proof of Theorem 1

Theorem 1 (Bound of the proposed BSTNet) For any input $\boldsymbol{x} \in \mathbb{R}^n$, the BSTNet based on **Lemma 1** is bounded such that

$$||\mathrm{BSTNet}_\sigma(\boldsymbol{x}) - \boldsymbol{x}||_2^2 \leqslant M\sigma^2 \tag{1}$$

for some universal constant M independent of σ.

Proof: Let $\boldsymbol{x}_0$ denote the input $\boldsymbol{x}$, then the output of the k-th stage in BSTNet can be denoted as

$$\boldsymbol{x}_k = \beta(\boldsymbol{x}_0 + \alpha \boldsymbol{W}_S \boldsymbol{z}_{k-1}). \tag{2}$$

Denote $\boldsymbol{W}_S \boldsymbol{z}$ by $\mathcal{D}_w(\boldsymbol{z})$, then the image after the first stage of denoising can be written as

$$\boldsymbol{x}_1 = \beta\left[\boldsymbol{x}_0 + \alpha \mathcal{D}_w(\boldsymbol{x}_0)\right]. \tag{3}$$

Since the input of each stage is determined by the output of the previous stage, we have

$$\begin{gathered}\boldsymbol{x}_2 = \beta\left[\boldsymbol{x}_0 + \alpha \mathcal{D}_w(\boldsymbol{x}_1)\right], \\ \dots \\ \boldsymbol{x}_k = \beta\left[\boldsymbol{x}_0 + \alpha \mathcal{D}_w(\boldsymbol{x}_{k-1})\right].\end{gathered} \tag{4}$$

Furthermore, the difference between the input image $\boldsymbol{x}_0$ and the output image $\mathrm{BSTNet}_\sigma(\boldsymbol{x}_0)$ denoted as $\boldsymbol{x}_k$ satisfies

$$\begin{aligned}&\|\boldsymbol{x}_0 - \boldsymbol{x}_k\|_2^2 \\ =&||\boldsymbol{x}_0 - \beta\boldsymbol{x}_0 - \beta\alpha\mathcal{D}_w(\boldsymbol{x}_{k-1}) \\ &- \beta\alpha\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x}_{k-1} + \beta\alpha\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x}_{k-1}||_2^2 \\ \leq& \|\boldsymbol{x}_0 - \beta\boldsymbol{x}_0 - \beta\alpha\boldsymbol{W}_S\boldsymbol{W}\boldsymbol{x}_{k-1}\|_2^2 \\ &+ \beta\alpha\left\|\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x}_{k-1} - \mathcal{D}_w(\boldsymbol{x}_{k-1})\right\|_2^2.\end{aligned} \tag{5}$$

Based on **Lemma 1**, Eqn. (5) can be rewritten as

$$\begin{aligned}&\|\boldsymbol{x}_0 - \boldsymbol{x}_k\|_2^2 \\ \leq& \|\boldsymbol{x}_0 - \beta\boldsymbol{x}_0 - \beta\alpha\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x}_{k-1}\|_2^2 + \beta\alpha c_1\sigma^2 \\ \leq& \left\|\boldsymbol{x}_0 - \frac{\boldsymbol{x}_0}{\boldsymbol{I} + \alpha\boldsymbol{W}_S\boldsymbol{W}} - \frac{\alpha\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x}_{k-1}}{\boldsymbol{I} + \alpha\boldsymbol{W}_S\boldsymbol{W}_A}\right\|_2^2 + \beta\alpha c_1\sigma^2 \\ \leq& \left\|\frac{\alpha\boldsymbol{W}_S\boldsymbol{W}_A}{\boldsymbol{I} + \alpha\boldsymbol{W}_S\boldsymbol{W}_A}\right\|_2^2 \|\boldsymbol{x}_0 - \boldsymbol{x}_{k-1}\|_2^2 + \beta\alpha c_1\sigma^2 \\ \leq& \beta\alpha\left\|\boldsymbol{W}_S\boldsymbol{W}_A\right\|_2^2 \|\boldsymbol{x}_0 - \boldsymbol{x}_{k-1}\|_2^2 + \beta\alpha c_1\sigma^2.\end{aligned} \tag{6}$$

In the same way, we can get

$$\begin{aligned}&\|\boldsymbol{x}_0 - \boldsymbol{x}_{k-1}\|_2^2 \\ =& \|\boldsymbol{x}_0 - \beta\boldsymbol{x}_0 - \beta\alpha\mathcal{D}_w(\boldsymbol{x}_{k-2})\|_2^2 \\ =&||\boldsymbol{x}_0 - \beta\boldsymbol{x}_0 - \beta\alpha\mathcal{D}_w(\boldsymbol{x}_{k-2}) \\ &- \beta\alpha\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x}_{k-2} + \beta\alpha\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x}_{k-2}||_2^2 \\ \leq& \|\boldsymbol{x}_0 - \beta\boldsymbol{x}_0 - \beta\alpha\boldsymbol{W}_S\boldsymbol{W}\boldsymbol{x}_{k-2}\|_2^2 \\ &+ \beta\alpha\left\|\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x}_{k-2} - \mathcal{D}_w(\boldsymbol{x}_{k-2})\right\|_2^2 \\ \leq& \|\boldsymbol{x}_0 - \beta\boldsymbol{x}_0 - \beta\alpha\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x}_{k-2}\|_2^2 + \beta\alpha c_1\sigma^2 \\ \leq& \left\|\boldsymbol{x}_0 - \frac{\boldsymbol{x}_0}{\boldsymbol{I} + \alpha\boldsymbol{W}_S\boldsymbol{W}} - \frac{\alpha\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x}_{k-2}}{\boldsymbol{I} + \alpha\boldsymbol{W}_S\boldsymbol{W}_A}\right\|_2^2 + \beta\alpha c_1\sigma^2 \\ \leq& \left\|\frac{\alpha\boldsymbol{W}_S\boldsymbol{W}_A}{\boldsymbol{I} + \alpha\boldsymbol{W}_S\boldsymbol{W}_A}\right\|_2^2 \|\boldsymbol{x}_0 - \boldsymbol{x}_{k-2}\|_2^2 + \beta\alpha c_1\sigma^2 \\ \leq& \beta\alpha\left\|\boldsymbol{W}_S\boldsymbol{W}_A\right\|_2^2 \|\boldsymbol{x}_0 - \boldsymbol{x}_{k-2}\|_2^2 + \beta\alpha c_1\sigma^2.\end{aligned} \tag{7}$$

Following this way, we can obtain $\|\boldsymbol{x}_0 - \boldsymbol{x}_{k-2}\|_2^2$, ..., and $\|\boldsymbol{x}_0 - \boldsymbol{x}_1\|_2^2$ in turn. The results of each stage are substituted into Eqn. (6), we have

$$\begin{aligned}&\|\boldsymbol{x}_0 - \boldsymbol{x}_k\|_2^2 \\ \leq& \beta^k\alpha^k\left(\|\boldsymbol{W}_S\boldsymbol{W}_A\|_2^2\right)^{k-1} c_k\sigma^2 \\ &+\beta^{k-1}\alpha^{k-1}\left(\|\boldsymbol{W}_S\boldsymbol{W}_A\|_2^2\right)^{k-2} c_{k-1}\sigma^2 \\ &+\dots \\ &+\beta^2\alpha^2\left(\|\boldsymbol{W}_S\boldsymbol{W}_A\|_2^2\right)^{1} c_2\sigma^2 \\ &+\beta\alpha c_1\sigma^2.\end{aligned} \tag{8}$$

Since β, α, and $\|\boldsymbol{W}_S\boldsymbol{W}_A\|_2^2$ are all constants, Eqn. (8) can be further expressed as

$$\begin{aligned}&\|\boldsymbol{x}_k - \boldsymbol{x}_0\|_2^2 \\ \leq& c_k\sigma^2 + c_{k-1}\sigma^2 + \dots + c_1\sigma^2 \\ =&(c_k + c_{k-1} + \dots + c_1)\sigma^2 \\ =&M\sigma^2\end{aligned} \tag{9}$$

where M denotes any constant that is not zero, then it means that BSTNet satisfies the bounded property.

II. Proof of Lemma 1

Lemma 1 If $\boldsymbol{W}_S$ has an upper bound, then we have

$$\|\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_S Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})\|_2^2 \leq c\sigma^2 \tag{10}$$

for some universal constant c independent of σ.

Proof: From Eqn. (10), we have

$$\begin{aligned}&||\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_S Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2 \\ \leq& ||\boldsymbol{W}_S||_2^2 \, ||\boldsymbol{W}_A\boldsymbol{x} - Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2.\end{aligned} \tag{11}$$

A convolutional layer with RealSN constraint [1] is utilized to generate the synthesis sparsifying transform, which makes $\|\boldsymbol{W}_S\|_2^2$ have an upper bound. Defined the upper bound is α, and the upper bound of Eqn. (11) can be further determined by

$$\begin{aligned}&||\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_S Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2 \\ \leq& \alpha\, ||\boldsymbol{W}_A\boldsymbol{x} - Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2.\end{aligned} \tag{12}$$

Let $(\boldsymbol{W}_A\boldsymbol{x})_i$ represent the i-th element of $\boldsymbol{W}_A\boldsymbol{x}$ and β_i denote the i-th element of $\boldsymbol{b}$. The soft thresholding operator $Soft[(\boldsymbol{W}_A\boldsymbol{x})_i, \beta_i]$ is defined as

$$Soft[(\boldsymbol{W}_A\boldsymbol{x})_i, \beta_i] = \begin{cases}(\boldsymbol{W}_A\boldsymbol{x})_i + \beta_i, & (\boldsymbol{W}_A\boldsymbol{x})_i < -\beta_i \\ 0, & |(\boldsymbol{W}_A\boldsymbol{x})_i| \leq \beta_i \\ (\boldsymbol{W}_A\boldsymbol{x})_i - \beta_i, & (\boldsymbol{W}_A\boldsymbol{x})_i > \beta_i,\end{cases} \tag{13}$$

then one of the following situations will happen.

CASE 1: Any element of $\boldsymbol{W}_A\boldsymbol{x}$ satisfies $(\boldsymbol{W}_A\boldsymbol{x})_i < -\beta_i$, then $Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b}) = \boldsymbol{W}_A\boldsymbol{x} + \boldsymbol{b}$. Therefore, we have

$$\begin{aligned}&||\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_S Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2 \\ \leq& \alpha\, ||\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{b}||_2^2 \\ =& \alpha\, ||\boldsymbol{b}||_2^2\end{aligned} \tag{14}$$

CASE 2: Any element of $\boldsymbol{W}_A\boldsymbol{x}$ satisfies $|(\boldsymbol{W}_A\boldsymbol{x})_i| \leq \beta_i$, then $Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b}) = 0$. Therefore, we have

$$\begin{aligned} & ||\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_S Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2 \\ & \leq \alpha\, ||\boldsymbol{W}_A\boldsymbol{x}||_2^2 . \end{aligned} \tag{15}$$

Since $|(\boldsymbol{W}_A\boldsymbol{x})_i| \leq \beta_i$, then $||\boldsymbol{W}_A\boldsymbol{x}||_2^2 \leq ||\boldsymbol{b}||_2^2$, it follows that

$$\begin{aligned} & ||\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_S Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2 \\ & \leq \alpha\, ||\boldsymbol{b}||_2^2 . \end{aligned} \tag{16}$$

CASE 3: Any element of $\boldsymbol{W}_A\boldsymbol{x}$ satisfies $(\boldsymbol{W}_A\boldsymbol{x})_i > \beta_i$, then $Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b}) = \boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{b}$. Therefore, we have

$$\begin{aligned} & ||\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_S Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2 \\ & \leq \alpha\, ||\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_A\boldsymbol{x} + \boldsymbol{b}||_2^2 \\ & = \alpha\, ||\boldsymbol{b}||_2^2 . \end{aligned} \tag{17}$$

CASE 4: CASE 4 is a union of the **CASE 1**, **CASE 2** and **CASE 3**. Therefore, as long as we find the upper bound of $||\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_S Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2$ under the **CASE 1-3**, the upper bound of that under the **CASE 4** will be determined. Hence, when **CASE 4** happens, the upper bound of $||\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_S Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2$ is the maximum upper bound of the first three situations. Based on Eqn. (14), Eqn. (16) and Eqn. (17), we have

$$||\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_S Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2 \leq \alpha\, ||\boldsymbol{b}||_2^2 \tag{18}$$

Recall the definition of threshold vectors $\boldsymbol{b} = \boldsymbol{c} \otimes \boldsymbol{s}$, and each element of proportional constant vectors $\boldsymbol{c}$ has a limited range $c_i \in [c_{min}, c_{max}]$. Let $\beta_{max} = c_{max} \cdot \sigma$ denote the maximum element of $\boldsymbol{b}$. Thus,

$$||\boldsymbol{W}_S\boldsymbol{W}_A\boldsymbol{x} - \boldsymbol{W}_S Soft(\boldsymbol{W}_A\boldsymbol{x}, \boldsymbol{b})||_2^2 \leq \alpha c_{max}^2 \sigma^2 \tag{19}$$

III. Proof of Lemma 2

Lemma 2 (The bounded gradient of data fidelity function) Consider the sampling model $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}$ and $\boldsymbol{y} = \boldsymbol{F}_u\boldsymbol{x} + \boldsymbol{n}$, where $||\boldsymbol{n}||_2 \leq \epsilon$, then the functions $f(\boldsymbol{x}) = \frac{1}{2}||\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}||_2^2$ and $f(\boldsymbol{x}) = \frac{1}{2}||\boldsymbol{y} - \boldsymbol{F}_u\boldsymbol{x}||_2^2$ have the bounded gradient, respectively.

Proof:

(In the SCI task) The gradient of $f(\boldsymbol{x}) = \frac{1}{2}||\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}||_2^2$ in the SCI formulation is

$$\nabla f(\boldsymbol{x}) = \boldsymbol{H}^\top(\boldsymbol{H}\boldsymbol{x} - \boldsymbol{y}) \tag{20}$$

thus, we have

$$\begin{aligned} \|\nabla f(\boldsymbol{x})\|_2 &\leq \left\|\boldsymbol{H}^\top\boldsymbol{H}\boldsymbol{x}\right\|_2 + \left\|\boldsymbol{H}^\top\boldsymbol{y}\right\|_2 \\ &\leq \left\|\boldsymbol{H}^\top\right\|_2 (\|\boldsymbol{H}\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2) . \end{aligned} \tag{21}$$

Since the values in the sensing matrix $\boldsymbol{H}$ and the pixel values in the image $\boldsymbol{x}$ are normolized into $[0, 1]$, we obtain

$$\|\boldsymbol{H}\boldsymbol{x}\|_2 \leq \|\boldsymbol{H}\|_2 \|\boldsymbol{x}\|_2 \leq \sqrt{\lambda_1 N}. \tag{22}$$

where λ_1 denotes the maximum eigenvalue of $\boldsymbol{H}^\top\boldsymbol{H}$, and N represents the pixel values of the image. In Eqn. (22), $\|\boldsymbol{y}\|_2 = \|\boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}\|_2$ and $\|\boldsymbol{n}\|_2 \leq \epsilon$, here, $\boldsymbol{x}$ is the image whose pixel values are normalized into $[0, 1]$. Therefore,

$$\|\boldsymbol{y}\|_2 = \|\boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}\|_2 \leq \|\boldsymbol{H}\boldsymbol{x}\|_2 + \|\boldsymbol{n}\|_2 \leq \sqrt{\lambda_1 N} + \epsilon. \tag{23}$$

Let λ_2 denote the maximum eigenvalue of $\boldsymbol{H}\boldsymbol{H}^\top$, then we have

$$\left\|\boldsymbol{H}^\top\right\|_2 \leq \sqrt{\lambda_2}. \tag{24}$$

Substituting Eqn. (22) and Eqn. (23) Eqn. (24) into Eqn. (21), we get

$$\begin{aligned} \|\nabla f(\boldsymbol{x})\|_2 &\leq \left\|\boldsymbol{H}^\top\right\|_2 (\|\boldsymbol{H}\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2) \\ &\leq \sqrt{\lambda_2}\left(2\sqrt{\lambda_1 N} + \epsilon\right). \end{aligned} \tag{25}$$

This completes the proof of **Lemma 2** in the SCI task.

(In the CSMRI task) See [2] for detailed proof.

IV. Proof of Theorem 2

Theorem 2 (Convergence of the proposed algorithms) Based on **Lemma 2** and **Theorem 1**, as well as the value of σ satisfies the condition of sequentially diminishing as the unfolding stage t increases in the proposed DUN-BSTNet algorithm, then both $\boldsymbol{x}^t$ and $\boldsymbol{v}^t$ will converge, respectively.

Proof: Following [3], we define $\boldsymbol{\theta}^t = (\boldsymbol{x}^t, \boldsymbol{v}^t)$, and let $\boldsymbol{\Theta}$ be the domain of $\boldsymbol{\theta}^t$ for all stages t. On the domain $\boldsymbol{\Theta}$, we define a distance function D: $\boldsymbol{\Theta} \times \boldsymbol{\Theta} \to \mathbb{R}$ such that

$$D(\boldsymbol{\theta}^t, \boldsymbol{\theta}^{t-1}) = \left|\left|\boldsymbol{x}^t - \boldsymbol{x}^{t-1}\right|\right|_2 + \left|\left|\boldsymbol{v}^t - \boldsymbol{v}^{t-1}\right|\right|_2 . \tag{26}$$

(In the SCI task) Consider the proposed formulation

$$\boldsymbol{x}^t = \arg\min_{\boldsymbol{x}} \|\boldsymbol{y} - \boldsymbol{H}\boldsymbol{x}\|_2^2 + \gamma \left\|\boldsymbol{x} - \mathcal{D}\left(\boldsymbol{x}^{t-1}, \sigma\right)\right\|_2^2, \tag{27}$$

based on the first order optimality of Eqn. (27), we have

$$\nabla f(\boldsymbol{x}) + \gamma\left(\boldsymbol{x} - \mathcal{D}\left(\boldsymbol{x}^{t-1}, \sigma\right)\right) = 0. \tag{28}$$

We assume that the optimal solution is $\boldsymbol{x}^t$ at the $t-$th iteration. Substituting this solution into the above equation yields

$$\boldsymbol{x}^t - \mathcal{D}\left(\boldsymbol{x}^{t-1}, \sigma\right) = -\frac{\nabla f(\boldsymbol{x}^t)}{\gamma^{t-1}}. \tag{29}$$

At the $t-$th iteration, the value of γ for updating image is γ^{t-1}. Using the **Lemma 2**, we have

$$\left\|\boldsymbol{x}^t - \mathcal{D}\left(\boldsymbol{x}^{t-1}, \sigma\right)\right\|_2 \leq \frac{\sqrt{\lambda_2}\left(2\sqrt{\lambda_1 N} + \epsilon\right)}{\gamma^{t-1}}. \tag{30}$$

Based on this bound and the bounded property of BSTNet, we obtain

$$\begin{aligned} & \left\|\boldsymbol{x}^t - \boldsymbol{x}^{t-1}\right\|_2 \\ & = \left\|\boldsymbol{x}^t - \mathcal{D}\left(\boldsymbol{x}^{t-1}, \sigma\right) + \mathcal{D}\left(\boldsymbol{x}^{t-1}, \sigma\right) - \boldsymbol{x}^{t-1}\right\|_2 \\ & \leq \left\|\boldsymbol{x}^{(t)} - \mathcal{D}\left(\boldsymbol{x}^{t-1}, \sigma\right)||_2 + ||\mathcal{D}\left(\boldsymbol{x}^{t-1}, \sigma\right) - \boldsymbol{x}^{t-1}\right\|_2 \\ & \leq \frac{\sqrt{\lambda_2}\left(2\sqrt{\lambda_1 N} + \epsilon\right)}{\gamma^{t-1}} + C\sigma^{t-1}. \end{aligned} \tag{31}$$

Based on this, we have

$$\begin{aligned} & \left|\left|\boldsymbol{v}^t - \boldsymbol{v}^{t-1}\right|\right|_2 \\ & = \left|\left|\mathcal{D}_\sigma\left(\boldsymbol{x}^t\right) - \mathcal{D}(\boldsymbol{x}^{t-1}, \sigma)\right|\right|_2 \\ & = ||\mathcal{D}_\sigma\left(\boldsymbol{x}^t\right) - \boldsymbol{x}^t \\ & \quad + \boldsymbol{x}^t - \boldsymbol{x}^{t-1} + \boldsymbol{x}^{t-1} - \mathcal{D}(\boldsymbol{x}^{t-1}, \sigma)||_2 \\ & \leq \left|\left|\mathcal{D}_\sigma\left(\boldsymbol{x}^t\right) - \boldsymbol{x}^t\right|\right|_2 \\ & \quad + \left|\left|\boldsymbol{x}^t - \boldsymbol{x}^{t-1}\right|\right|_2 + \left|\left|\boldsymbol{x}^{t-1} - \mathcal{D}(\boldsymbol{x}^{t-1}, \sigma)\right|\right|_2 . \end{aligned} \tag{32}$$

The inequality holds true due to the triangle inequality. Based on the bounded property of BSTNet and (31), we have

$$\begin{aligned}
&\left|\left|\boldsymbol{v}^t-\boldsymbol{v}^{t-1}\right|\right|_2\\
\leq&\left|\left|\mathcal{D}_\sigma\left(\boldsymbol{x}^t\right)-\boldsymbol{x}^t\right|\right|_2+\left|\left|\boldsymbol{x}^t-\boldsymbol{x}^{t-1}\right|\right|_2\\
&+\left|\left|\boldsymbol{x}^{t-1}-\mathcal{D}\left(\boldsymbol{x}^{t-1},\sigma\right)\right|\right|_2\\
\leq&C\sigma^t+\frac{\sqrt{\lambda_2}\left(2\sqrt{\lambda_1 N}+\epsilon\right)}{\gamma^{t-1}}+C\sigma^{t-1}+C\sigma^{t-1}\\
\leq&3C\sigma^{t-1}+\frac{\sqrt{\lambda_2}\left(2\sqrt{\lambda_1 N}+\epsilon\right)}{\gamma^{t-1}}.
\end{aligned} \tag{33}$$

Using the distance function defined in Eqn. (26), it is easily to see from Eqn. (31) and Eqn. (33) that

$$\begin{aligned}
D\left(\boldsymbol{\theta}^t,\boldsymbol{\theta}^{t-1}\right)\leq&\frac{\sqrt{\lambda_2}\left(2\sqrt{\lambda_1 N}+\epsilon\right)}{\gamma^{t-1}}+C\sigma^{t-1}\\
&+3C\sigma^{t-1}+\frac{\sqrt{\lambda_2}\left(2\sqrt{\lambda_1 N}+\epsilon\right)}{\gamma^{t-1}}\\
=&\frac{\sqrt{\lambda_2}\left(4\sqrt{\lambda_1 N}+2\epsilon\right)}{\gamma^{t-1}}+4C\sigma^{t-1},
\end{aligned} \tag{34}$$

Since $\frac{1}{1+\gamma^t}<1$ and σ diminishes grandually, when $t\to\infty$, $\gamma^t\to\infty$ and $\sigma\to 0$, then $\left\|\boldsymbol{x}^t-\boldsymbol{x}^{t-1}\right\|_2\to 0$, $\left\|\boldsymbol{v}^t-\boldsymbol{v}^{t-1}\right\|_2\to 0$, and $D\left(\boldsymbol{\theta}^t,\boldsymbol{\theta}^{t-1}\right)\to 0$.

(In the CSMRI task) Consider the proposed formulation

$$\boldsymbol{x}^t=\arg\min_{\boldsymbol{x}}\|\boldsymbol{y}-\boldsymbol{F}_u\boldsymbol{x}\|_2^2+\gamma\left\|\boldsymbol{x}-\mathcal{D}\left(\boldsymbol{x}^{t-1},\sigma\right)\right\|_2^2, \tag{35}$$

based on the first order optimality of Eqn. (35), we have

$$\nabla f(\boldsymbol{x})+\gamma\left(\boldsymbol{x}-\mathcal{D}\left(\boldsymbol{x}^{t-1},\sigma\right)\right)=0. \tag{36}$$

We assume that the optimal solution is $\boldsymbol{x}^t$ at the $t-$th iteration. Substituting this solution into the above equation yields

$$\boldsymbol{x}^t-\mathcal{D}\left(\boldsymbol{x}^{t-1},\sigma\right)=-\frac{\nabla f(\boldsymbol{x}^t)}{\gamma^{t-1}}. \tag{37}$$

At the $t-$th iteration, the value of γ for updating image is γ^{t-1}. Using the **Lemma 2**, we have

$$\left\|\boldsymbol{x}^t-\mathcal{D}\left(\boldsymbol{x}^{t-1},\sigma\right)\right\|_2\leq\frac{\left(2\sqrt{N}+\epsilon\right)}{\gamma^{t-1}}. \tag{38}$$

Based on this bound and the bounded property of BSTNet, we obtain

$$\begin{aligned}
&\left\|\boldsymbol{x}^t-\boldsymbol{x}^{t-1}\right\|_2\\
=&\left\|\boldsymbol{x}^t-\mathcal{D}\left(\boldsymbol{x}^{t-1},\sigma\right)+\mathcal{D}\left(\boldsymbol{x}^{t-1},\sigma\right)-\boldsymbol{x}^{t-1}\right\|_2\\
\leq&\left\|\boldsymbol{x}^{(t)}-\mathcal{D}\left(\boldsymbol{x}^{t-1},\sigma\right)||_2+||\mathcal{D}\left(\boldsymbol{x}^{t-1},\sigma\right)-\boldsymbol{x}^{t-1}\right\|_2\\
\leq&\frac{\left(2\sqrt{N}+\epsilon\right)}{\gamma^{t-1}}+C\sigma^{t-1}.
\end{aligned} \tag{39}$$

Based on this, we have

$$\begin{aligned}
&\left|\left|\boldsymbol{v}^t-\boldsymbol{v}^{t-1}\right|\right|_2\\
=&\left|\left|\mathcal{D}_\sigma\left(\boldsymbol{x}^t\right)-\mathcal{D}(\boldsymbol{x}^{t-1},\sigma)\right|\right|_2\\
=&||\mathcal{D}_\sigma\left(\boldsymbol{x}^t\right)-\boldsymbol{x}^t\\
&+\boldsymbol{x}^t-\boldsymbol{x}^{t-1}+\boldsymbol{x}^{t-1}-\mathcal{D}(\boldsymbol{x}^{t-1},\sigma)||_2\\
\leq&\left|\left|\mathcal{D}_\sigma\left(\boldsymbol{x}^t\right)-\boldsymbol{x}^t\right|\right|_2\\
&+\left|\left|\boldsymbol{x}^t-\boldsymbol{x}^{t-1}\right|\right|_2+\left|\left|\boldsymbol{x}^{t-1}-\mathcal{D}(\boldsymbol{x}^{t-1},\sigma)\right|\right|_2.
\end{aligned} \tag{40}$$

The inequality holds true due to the triangle inequality. By utilizing the bounded property of BSTNet and (39), we have

$$\begin{aligned}
&\left|\left|\boldsymbol{v}^t-\boldsymbol{v}^{t-1}\right|\right|_2\\
\leq&\left|\left|\mathcal{D}_\sigma\left(\boldsymbol{x}^t\right)-\boldsymbol{x}^t\right|\right|_2+\left|\left|\boldsymbol{x}^t-\boldsymbol{x}^{t-1}\right|\right|_2\\
&+\left|\left|\boldsymbol{x}^{t-1}-\mathcal{D}\left(\boldsymbol{x}^{t-1},\sigma\right)\right|\right|_2\\
\leq&C\sigma^t+\frac{\left(2\sqrt{N}+\epsilon\right)}{\gamma^{t-1}}+C\sigma^{t-1}+C\sigma^{t-1}\\
\leq&3C\sigma^{t-1}+\frac{\left(2\sqrt{N}+\epsilon\right)}{\gamma^{t-1}}.
\end{aligned} \tag{41}$$

Using the distance function defined in Eqn. (26), it is easily to see from Eqn. (39) and Eqn. (41) that

$$\begin{aligned}
D\left(\boldsymbol{\theta}^t,\boldsymbol{\theta}^{t-1}\right)\leq&\frac{\left(2\sqrt{N}+\epsilon\right)}{\gamma^{t-1}}+C\sigma^{t-1}\\
&+3C\sigma^{t-1}+\frac{\left(2\sqrt{N}+\epsilon\right)}{\gamma^{t-1}}\\
=&\frac{\left(4\sqrt{N}+2\epsilon\right)}{\gamma^{t-1}}+4C\sigma^{t-1},
\end{aligned} \tag{42}$$

Since $\gamma^t=\eta\gamma^{t-1}=\eta^{t-1}\gamma^0$ and σ diminishes grandually, where $\eta>1$, when $t\to\infty$, $\gamma^t\to\infty$ and $\sigma\to 0$, then $\left\|\boldsymbol{x}^t-\boldsymbol{x}^{t-1}\right\|_2\to 0$, $\left\|\boldsymbol{v}^t-\boldsymbol{v}^{t-1}\right\|_2\to 0$, and $D\left(\boldsymbol{\theta}^t,\boldsymbol{\theta}^{t-1}\right)\to 0$.

This completes the proof of **Theorem 2** in different tasks.

V. The visualizations of 10 scenes

The visualizations of 10 scenes are shown as Fig. 1.

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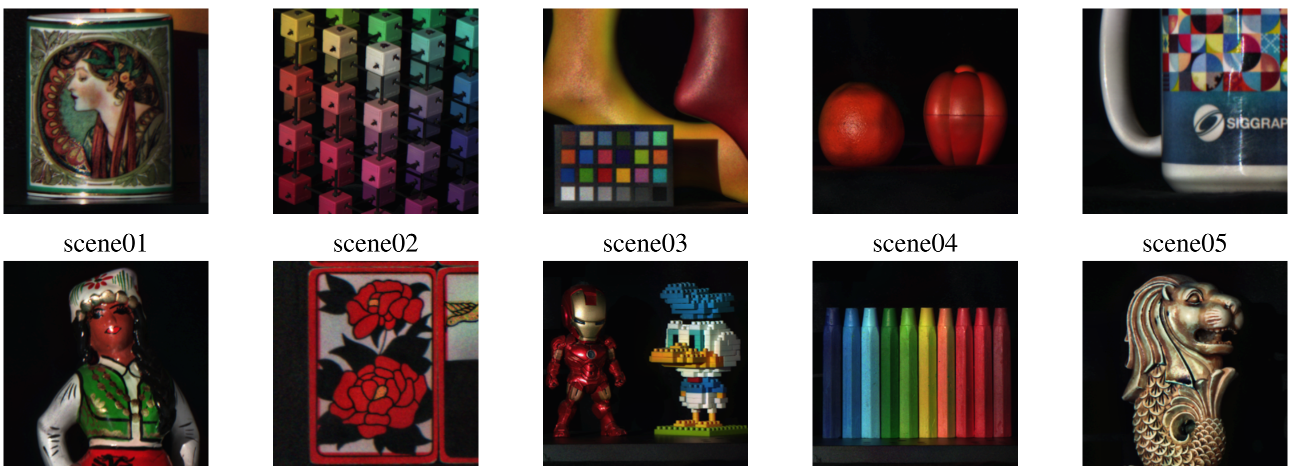

scene01 scene02 scene03 scene04 scene05

scene06 scene07 scene08 scene09 scene10

Fig. 1: The RGB images on 10 scenes of KAIST dataset.