

## I. PROOF OF LEMMA 1

**Lemma 1** If  $\mathbf{W}_S$  has an upper bound, then we have

$$\|\mathbf{W}_S \mathbf{W}_A \mathbf{x} - \mathbf{W}_S \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2 \leq c\sigma^2 \quad (1)$$

for some universal constant  $c$  independent of  $\sigma$ .

Proof: From Eqn. (1), we have

$$\begin{aligned} & \|\mathbf{W}_S \mathbf{W}_A \mathbf{x} - \mathbf{W}_S \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2 \\ & \leq \|\mathbf{W}_S\|_2^2 \|\mathbf{W}_A \mathbf{x} - \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2. \end{aligned} \quad (2)$$

A convolutional layer with RealSN constraint [1] is utilized to generate the synthesis sparsifying transform, which makes  $\|\mathbf{W}_S\|_2^2$  have an upper bound. Defined the upper bound is  $\alpha$ , and the upper bound of Eqn. (2) can be further determined by

$$\begin{aligned} & \|\mathbf{W}_S \mathbf{W}_A \mathbf{x} - \mathbf{W}_S \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2 \\ & \leq \alpha \|\mathbf{W}_A \mathbf{x} - \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2. \end{aligned} \quad (3)$$

Let  $(\mathbf{W}_A \mathbf{x})_i$  represent the  $i$ -th element of  $\mathbf{W}_A \mathbf{x}$  and  $\beta_i$  denote the  $i$ -th element of  $\mathbf{b}$ . The soft thresholding operator  $\text{Soft}[(\mathbf{W}_A \mathbf{x})_i, \beta_i]$  is defined as

$$\text{Soft}[(\mathbf{W}_A \mathbf{x})_i, \beta_i] = \begin{cases} (\mathbf{W}_A \mathbf{x})_i + \beta_i, & (\mathbf{W}_A \mathbf{x})_i < -\beta_i \\ 0, & |(\mathbf{W}_A \mathbf{x})_i| \leq \beta_i \\ (\mathbf{W}_A \mathbf{x})_i - \beta_i, & (\mathbf{W}_A \mathbf{x})_i > \beta_i, \end{cases} \quad (4)$$

then one of the following situations will happen.

**CASE 1:** Any element of  $\mathbf{W}_A \mathbf{x}$  satisfies  $(\mathbf{W}_A \mathbf{x})_i < -\beta_i$ , then  $\text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b}) = \mathbf{W}_A \mathbf{x} + \mathbf{b}$ . Therefore, we have

$$\begin{aligned} & \|\mathbf{W}_S \mathbf{W}_A \mathbf{x} - \mathbf{W}_S \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2 \\ & \leq \alpha \|\mathbf{W}_A \mathbf{x} - \mathbf{W}_A \mathbf{x} - \mathbf{b}\|_2^2 \\ & = \alpha \|\mathbf{b}\|_2^2 \end{aligned} \quad (5)$$

**CASE 2:** Any element of  $\mathbf{W}_A \mathbf{x}$  satisfies  $|(\mathbf{W}_A \mathbf{x})_i| \leq \beta_i$ , then  $\text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b}) = 0$ . Therefore, we have

$$\begin{aligned} & \|\mathbf{W}_S \mathbf{W}_A \mathbf{x} - \mathbf{W}_S \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2 \\ & \leq \alpha \|\mathbf{W}_A \mathbf{x}\|_2^2. \end{aligned} \quad (6)$$

Since  $|(\mathbf{W}_A \mathbf{x})_i| \leq \beta_i$ , then  $\|\mathbf{W}_A \mathbf{x}\|_2^2 \leq \|\mathbf{b}\|_2^2$ , it follows that

$$\begin{aligned} & \|\mathbf{W}_S \mathbf{W}_A \mathbf{x} - \mathbf{W}_S \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2 \\ & \leq \alpha \|\mathbf{b}\|_2^2. \end{aligned} \quad (7)$$

**CASE 3:** Any element of  $\mathbf{W}_A \mathbf{x}$  satisfies  $(\mathbf{W}_A \mathbf{x})_i > \beta_i$ , then  $\text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b}) = \mathbf{W}_A \mathbf{x} - \mathbf{b}$ . Therefore, we have

$$\begin{aligned} & \|\mathbf{W}_S \mathbf{W}_A \mathbf{x} - \mathbf{W}_S \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2 \\ & \leq \alpha \|\mathbf{W}_A \mathbf{x} - \mathbf{W}_A \mathbf{x} + \mathbf{b}\|_2^2 \\ & = \alpha \|\mathbf{b}\|_2^2. \end{aligned} \quad (8)$$

**CASE 4:** CASE 4 is a union of the CASE 1, CASE 2 and CASE 3. Therefore, as long as we find the upper bound of  $\|\mathbf{W}_S \mathbf{W}_A \mathbf{x} - \mathbf{W}_S \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2$  under the CASE 1-3, the upper bound of that under the CASE 4 will be determined. Hence, when CASE 4 happens, the upper bound of  $\|\mathbf{W}_S \mathbf{W}_A \mathbf{x} - \mathbf{W}_S \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2$  is the maximum upper

bound of the first three situations. Based on Eqn. (5), Eqn. (7) and Eqn. (8), we have

$$\|\mathbf{W}_S \mathbf{W}_A \mathbf{x} - \mathbf{W}_S \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2 \leq \alpha \|\mathbf{b}\|_2^2 \quad (9)$$

Recall the definition of threshold vectors  $\mathbf{b} = \mathbf{c} \otimes \mathbf{s}$ , and each element of proportional constant vectors  $\mathbf{c}$  has a limited range  $c_i \in [c_{\min}, c_{\max}]$ . Let  $\beta_{\max} = c_{\max} \cdot \sigma$  denote the maximum element of  $\mathbf{b}$ . Thus,

$$\|\mathbf{W}_S \mathbf{W}_A \mathbf{x} - \mathbf{W}_S \text{Soft}(\mathbf{W}_A \mathbf{x}, \mathbf{b})\|_2^2 \leq \alpha c_{\max}^2 \sigma^2 \quad (10)$$

## II. PROOF OF LEMMA 2

**Lemma 2** (The bounded gradient of data fidelity function) Consider the sampling model  $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$  and  $\mathbf{y} = \mathbf{F}_u \mathbf{x} + \mathbf{n}$ , where  $\|\mathbf{n}\|_2 \leq \epsilon$ , then the functions  $f(\mathbf{x}) = \frac{1}{2} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_2^2$  and  $f(\mathbf{x}) = \frac{1}{2} \|\mathbf{y} - \mathbf{F}_u \mathbf{x}\|_2^2$  have the bounded gradient, respectively.

Proof:

(In the SCI task) The gradient of  $f(\mathbf{x}) = \frac{1}{2} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_2^2$  in the SCI formulation is

$$\nabla f(\mathbf{x}) = \mathbf{H}^\top (\mathbf{H}\mathbf{x} - \mathbf{y}) \quad (11)$$

thus, we have

$$\begin{aligned} \|\nabla f(\mathbf{x})\|_2 & \leq \|\mathbf{H}^\top \mathbf{H}\mathbf{x}\|_2 + \|\mathbf{H}^\top \mathbf{y}\|_2 \\ & \leq \|\mathbf{H}^\top\|_2 (\|\mathbf{H}\mathbf{x}\|_2 + \|\mathbf{y}\|_2). \end{aligned} \quad (12)$$

Since the values in the sensing matrix  $\mathbf{H}$  and the pixel values in the image  $\mathbf{x}$  are normalized into  $[0, 1]$ , we obtain

$$\|\mathbf{H}\mathbf{x}\|_2 \leq \|\mathbf{H}\|_2 \|\mathbf{x}\|_2 \leq \sqrt{\lambda_1 N}. \quad (13)$$

where  $\lambda_1$  denotes the maximum eigenvalue of  $\mathbf{H}^\top \mathbf{H}$ , and  $N$  represents the pixel values of the image. In Eqn. (13),  $\|\mathbf{y}\|_2 = \|\mathbf{H}\mathbf{x} + \mathbf{n}\|_2$  and  $\|\mathbf{n}\|_2 \leq \epsilon$ , here,  $\mathbf{x}$  is the image whose pixel values are normalized into  $[0, 1]$ . Therefore,

$$\|\mathbf{y}\|_2 = \|\mathbf{H}\mathbf{x} + \mathbf{n}\|_2 \leq \|\mathbf{H}\mathbf{x}\|_2 + \|\mathbf{n}\|_2 \leq \sqrt{\lambda_1 N} + \epsilon. \quad (14)$$

Let  $\lambda_2$  denote the maximum eigenvalue of  $\mathbf{H}\mathbf{H}^\top$ , then we have

$$\|\mathbf{H}^\top\|_2 \leq \sqrt{\lambda_2}. \quad (15)$$

Substituting Eqn. (13) and Eqn. (14) Eqn. (15) into Eqn. (12), we get

$$\begin{aligned} \|\nabla f(\mathbf{x})\|_2 & \leq \|\mathbf{H}^\top\|_2 (\|\mathbf{H}\mathbf{x}\|_2 + \|\mathbf{y}\|_2) \\ & \leq \sqrt{\lambda_2} (2\sqrt{\lambda_1 N} + \epsilon). \end{aligned} \quad (16)$$

This completes the proof of **Lemma 2** in the SCI task.

(In the CSMRI task) See [2] for detailed proof.

## III. PROOF OF THEOREM 2

**Theorem 2** (Convergence of the proposed algorithms) Based on **Lemma 2** and **Theorem 1**, as well as the value of  $\sigma$  satisfies the condition of sequentially diminishing as the unfolding stage  $t$  increases in the proposed DUN-BSTNet algorithm, then both  $\mathbf{x}^t$  and  $\mathbf{v}^t$  will converge, respectively.

Proof: Following [3], we define  $\boldsymbol{\theta}^t = (\mathbf{x}^t, \mathbf{v}^t)$ , and let  $\Theta$  be the domain of  $\boldsymbol{\theta}^t$  for all stages  $t$ . On the domain  $\Theta$ , we define a distance function  $D: \Theta \times \Theta \rightarrow \mathbb{R}$  such that

$$D(\boldsymbol{\theta}^t, \boldsymbol{\theta}^{t-1}) = \|\mathbf{x}^t - \mathbf{x}^{t-1}\|_2 + \|\mathbf{v}^t - \mathbf{v}^{t-1}\|_2. \quad (17)$$

(In the SCI task) Consider the proposed formulation

$$\mathbf{x}^t = \arg \min_{\mathbf{x}} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_2^2 + \gamma \|\mathbf{x} - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2^2, \quad (18)$$

based on the first order optimality of Eqn. (18), we have

$$\nabla f(\mathbf{x}) + \gamma (\mathbf{x} - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)) = 0. \quad (19)$$

We assume that the optimal solution is  $\mathbf{x}^t$  at the  $t$ -th iteration. Substituting this solution into the above equation yields

$$\mathbf{x}^t - \mathcal{D}(\mathbf{x}^{t-1}, \sigma) = -\frac{\nabla f(\mathbf{x}^t)}{\gamma^{t-1}}. \quad (20)$$

At the  $t$ -th iteration, the value of  $\gamma$  for updating image is  $\gamma^{t-1}$ . Using the **Lemma 2**, we have

$$\|\mathbf{x}^t - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2 \leq \frac{\sqrt{\lambda_2} (2\sqrt{\lambda_1 N} + \epsilon)}{\gamma^{t-1}}. \quad (21)$$

Based on this bound and the bounded property of BSTNet, we obtain

$$\begin{aligned} & \|\mathbf{x}^t - \mathbf{x}^{t-1}\|_2 \\ &= \|\mathbf{x}^t - \mathcal{D}(\mathbf{x}^{t-1}, \sigma) + \mathcal{D}(\mathbf{x}^{t-1}, \sigma) - \mathbf{x}^{t-1}\|_2 \\ &\leq \|\mathbf{x}^t - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2 + \|\mathcal{D}(\mathbf{x}^{t-1}, \sigma) - \mathbf{x}^{t-1}\|_2 \\ &\leq \frac{\sqrt{\lambda_2} (2\sqrt{\lambda_1 N} + \epsilon)}{\gamma^{t-1}} + C\sigma^{t-1}. \end{aligned} \quad (22)$$

Based on this, we have

$$\begin{aligned} & \|\mathbf{v}^t - \mathbf{v}^{t-1}\|_2 \\ &= \|\mathcal{D}_\sigma(\mathbf{x}^t) - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2 \\ &= \|\mathcal{D}_\sigma(\mathbf{x}^t) - \mathbf{x}^t \\ &\quad + \mathbf{x}^t - \mathbf{x}^{t-1} + \mathbf{x}^{t-1} - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2 \\ &\leq \|\mathcal{D}_\sigma(\mathbf{x}^t) - \mathbf{x}^t\|_2 \\ &\quad + \|\mathbf{x}^t - \mathbf{x}^{t-1}\|_2 + \|\mathbf{x}^{t-1} - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2. \end{aligned} \quad (23)$$

The inequality holds true due to the triangle inequality. Based on the bounded property of BSTNet and (22), we have

$$\begin{aligned} & \|\mathbf{v}^t - \mathbf{v}^{t-1}\|_2 \\ &\leq \|\mathcal{D}_\sigma(\mathbf{x}^t) - \mathbf{x}^t\|_2 + \|\mathbf{x}^t - \mathbf{x}^{t-1}\|_2 \\ &\quad + \|\mathbf{x}^{t-1} - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2 \\ &\leq C\sigma^t + \frac{\sqrt{\lambda_2} (2\sqrt{\lambda_1 N} + \epsilon)}{\gamma^{t-1}} + C\sigma^{t-1} + C\sigma^{t-1} \\ &\leq 3C\sigma^{t-1} + \frac{\sqrt{\lambda_2} (2\sqrt{\lambda_1 N} + \epsilon)}{\gamma^{t-1}}. \end{aligned} \quad (24)$$

Using the distance function defined in Eqn. (17), it is easily to see from Eqn. (22) and Eqn. (24) that

$$\begin{aligned} D(\boldsymbol{\theta}^t, \boldsymbol{\theta}^{t-1}) &\leq \frac{\sqrt{\lambda_2} (2\sqrt{\lambda_1 N} + \epsilon)}{\gamma^{t-1}} + C\sigma^{t-1} \\ &\quad + 3C\sigma^{t-1} + \frac{\sqrt{\lambda_2} (2\sqrt{\lambda_1 N} + \epsilon)}{\gamma^{t-1}} \\ &= \frac{\sqrt{\lambda_2} (4\sqrt{\lambda_1 N} + 2\epsilon)}{\gamma^{t-1}} + 4C\sigma^{t-1}, \end{aligned} \quad (25)$$

Since  $\frac{1}{1+\gamma^t} < 1$  and  $\sigma$  diminishes grandually, when  $t \rightarrow \infty$ ,  $\gamma^t \rightarrow \infty$  and  $\sigma \rightarrow 0$ , then  $\|\mathbf{x}^t - \mathbf{x}^{t-1}\|_2 \rightarrow 0$ ,  $\|\mathbf{v}^t - \mathbf{v}^{t-1}\|_2 \rightarrow 0$ , and  $D(\boldsymbol{\theta}^t, \boldsymbol{\theta}^{t-1}) \rightarrow 0$ .

(In the CSMRI task) Consider the proposed formulation

$$\mathbf{x}^t = \arg \min_{\mathbf{x}} \|\mathbf{y} - \mathbf{F}_u \mathbf{x}\|_2^2 + \gamma \|\mathbf{x} - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2^2, \quad (26)$$

based on the first order optimality of Eqn. (26), we have

$$\nabla f(\mathbf{x}) + \gamma (\mathbf{x} - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)) = 0. \quad (27)$$

We assume that the optimal solution is  $\mathbf{x}^t$  at the  $t$ -th iteration. Substituting this solution into the above equation yields

$$\mathbf{x}^t - \mathcal{D}(\mathbf{x}^{t-1}, \sigma) = -\frac{\nabla f(\mathbf{x}^t)}{\gamma^{t-1}}. \quad (28)$$

At the  $t$ -th iteration, the value of  $\gamma$  for updating image is  $\gamma^{t-1}$ . Using the **Lemma 2**, we have

$$\|\mathbf{x}^t - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2 \leq \frac{(2\sqrt{N} + \epsilon)}{\gamma^{t-1}}. \quad (29)$$

Based on this bound and the bounded property of BSTNet, we obtain

$$\begin{aligned} & \|\mathbf{x}^t - \mathbf{x}^{t-1}\|_2 \\ &= \|\mathbf{x}^t - \mathcal{D}(\mathbf{x}^{t-1}, \sigma) + \mathcal{D}(\mathbf{x}^{t-1}, \sigma) - \mathbf{x}^{t-1}\|_2 \\ &\leq \|\mathbf{x}^t - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2 + \|\mathcal{D}(\mathbf{x}^{t-1}, \sigma) - \mathbf{x}^{t-1}\|_2 \\ &\leq \frac{(2\sqrt{N} + \epsilon)}{\gamma^{t-1}} + C\sigma^{t-1}. \end{aligned} \quad (30)$$

Based on this, we have

$$\begin{aligned} & \|\mathbf{v}^t - \mathbf{v}^{t-1}\|_2 \\ &= \|\mathcal{D}_\sigma(\mathbf{x}^t) - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2 \\ &= \|\mathcal{D}_\sigma(\mathbf{x}^t) - \mathbf{x}^t \\ &\quad + \mathbf{x}^t - \mathbf{x}^{t-1} + \mathbf{x}^{t-1} - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2 \\ &\leq \|\mathcal{D}_\sigma(\mathbf{x}^t) - \mathbf{x}^t\|_2 \\ &\quad + \|\mathbf{x}^t - \mathbf{x}^{t-1}\|_2 + \|\mathbf{x}^{t-1} - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2. \end{aligned} \quad (31)$$

The inequality holds true due to the triangle inequality. By utilizing the bounded property of BSTNet and (30), we have

$$\begin{aligned} & \|\mathbf{v}^t - \mathbf{v}^{t-1}\|_2 \\ &\leq \|\mathcal{D}_\sigma(\mathbf{x}^t) - \mathbf{x}^t\|_2 + \|\mathbf{x}^t - \mathbf{x}^{t-1}\|_2 \\ &\quad + \|\mathbf{x}^{t-1} - \mathcal{D}(\mathbf{x}^{t-1}, \sigma)\|_2 \\ &\leq C\sigma^t + \frac{(2\sqrt{N} + \epsilon)}{\gamma^{t-1}} + C\sigma^{t-1} + C\sigma^{t-1} \\ &\leq 3C\sigma^{t-1} + \frac{(2\sqrt{N} + \epsilon)}{\gamma^{t-1}}. \end{aligned} \quad (32)$$

Using the distance function defined in Eqn. (17), it is easily to see from Eqn. (30) and Eqn. (32) that

$$\begin{aligned} D(\boldsymbol{\theta}^t, \boldsymbol{\theta}^{t-1}) &\leq \frac{(2\sqrt{N} + \epsilon)}{\gamma^{t-1}} + C\sigma^{t-1} \\ &\quad + 3C\sigma^{t-1} + \frac{(2\sqrt{N} + \epsilon)}{\gamma^{t-1}} \\ &= \frac{(4\sqrt{N} + 2\epsilon)}{\gamma^{t-1}} + 4C\sigma^{t-1}, \end{aligned} \quad (33)$$

Since  $\gamma^t = \eta\gamma^{t-1} = \eta^{t-1}\gamma^0$  and  $\sigma$  diminishes grandually, where  $\eta > 1$ , when  $t \rightarrow \infty$ ,  $\gamma^t \rightarrow \infty$  and  $\sigma \rightarrow 0$ , then  $\|\mathbf{x}^t - \mathbf{x}^{t-1}\|_2 \rightarrow 0$ ,  $\|\mathbf{v}^t - \mathbf{v}^{t-1}\|_2 \rightarrow 0$ , and  $D(\boldsymbol{\theta}^t, \boldsymbol{\theta}^{t-1}) \rightarrow 0$ .

This completes the proof of **Theorem 2** in different tasks.

#### IV. THE VISUALIZATIONS OF 10 SCENES



Fig. 1: The RGB images on 10 scenes of KAIST dataset.

#### REFERENCES

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