

I. SUPPLEMENTARY MATERIAL

A. Proof of Lemma 1

According to the paradigm compatibility, we can construct the following inequality:

$$\begin{aligned}
& \|\mathbf{W}^T \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) - \mathbf{W}^T \text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon)\|_2^2 \\
&= \|\mathbf{W}^T [\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon)]\|_2^2 \\
&\leq \|\mathbf{W}^T\|_2^2 \|\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon)\|_2^2 \quad (1) \\
&= \lambda_{\max}(\mathbf{W}\mathbf{W}^T) \|\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon)\|_2^2 \\
&= \lambda_{\max}(\mathbf{W}\mathbf{W}^T) \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2
\end{aligned}$$

Let $(\mathbf{W}\mathbf{x})_i$ represent the i -th element of $\mathbf{W}\mathbf{x}$. The soft thresholding operator $\text{soft}(\mathbf{W}\mathbf{x}, \varepsilon)$ is defined as:

$$\text{soft}(\mathbf{W}\mathbf{x}, \varepsilon) = \begin{cases} (\mathbf{W}\mathbf{x})_i + \varepsilon_i, & (\mathbf{W}\mathbf{x})_i < -\varepsilon_i \\ 0, & |(\mathbf{W}\mathbf{x})_i| \leq \varepsilon_i \\ (\mathbf{W}\mathbf{x})_i - \varepsilon_i, & (\mathbf{W}\mathbf{x})_i > \varepsilon_i \end{cases} \quad (2)$$

We divide our discussion of Eqn. (1) into the following scenarios:

(i) $(\mathbf{W}\mathbf{x}_1)_i < (\mathbf{W}\mathbf{x}_2)_i$

a. if $(\mathbf{W}\mathbf{x}_1)_i < (\mathbf{W}\mathbf{x}_2)_i < -\varepsilon_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = \mathbf{W}\mathbf{x}_1 + \varepsilon$, $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = \mathbf{W}\mathbf{x}_2 + \varepsilon$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|(\mathbf{W}\mathbf{x}_2 + \varepsilon) - (\mathbf{W}\mathbf{x}_1 + \varepsilon)\|_2^2 \\
&= \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1\|_2^2 \quad (3) \\
&\leq \|\mathbf{W}\|_2^2 \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2 \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2
\end{aligned}$$

b. if $(\mathbf{W}\mathbf{x}_1)_i < -\varepsilon_i \leq (\mathbf{W}\mathbf{x}_2)_i \leq \varepsilon_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = \mathbf{W}\mathbf{x}_1 + \varepsilon$, $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = 0$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|0 - (\mathbf{W}\mathbf{x}_1 + \varepsilon)\|_2^2 \\
&= \|-\mathbf{W}\mathbf{x}_1 - \varepsilon\|_2^2 \\
&= \|-\varepsilon - \mathbf{W}\mathbf{x}_1\|_2^2 \quad (4) \\
&\leq \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1\|_2^2 \\
&\leq \|\mathbf{W}\|_2^2 \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2 \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2
\end{aligned}$$

c. if $(\mathbf{W}\mathbf{x}_1)_i < -\varepsilon_i < \varepsilon_i < (\mathbf{W}\mathbf{x}_2)_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = \mathbf{W}\mathbf{x}_1 + \varepsilon$, $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = \mathbf{W}\mathbf{x}_2 - \varepsilon$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|(\mathbf{W}\mathbf{x}_2 - \varepsilon) - (\mathbf{W}\mathbf{x}_1 + \varepsilon)\|_2^2 \\
&= \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1 - 2\varepsilon\|_2^2 \quad (5) \\
&\leq \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1\|_2^2 \\
&\leq \|\mathbf{W}\|_2^2 \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2 \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2
\end{aligned}$$

d. if $-\varepsilon_i \leq (\mathbf{W}\mathbf{x}_1)_i < (\mathbf{W}\mathbf{x}_2)_i \leq \varepsilon_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = 0$, $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = 0$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|0 - 0\|_2^2 \quad (6) \\
&= 0
\end{aligned}$$

e. if $-\varepsilon_i \leq (\mathbf{W}\mathbf{x}_1)_i \leq \varepsilon_i < (\mathbf{W}\mathbf{x}_2)_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = 0$, $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = \mathbf{W}\mathbf{x}_2 - \varepsilon$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|(\mathbf{W}\mathbf{x}_2 - \varepsilon) - 0\|_2^2 \\
&\leq \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1\|_2^2 \quad (7) \\
&\leq \|\mathbf{W}\|_2^2 \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2 \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2
\end{aligned}$$

f. if $\varepsilon_i < (\mathbf{W}\mathbf{x}_1)_i < (\mathbf{W}\mathbf{x}_2)_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = \mathbf{W}\mathbf{x}_1 - \varepsilon$, $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = \mathbf{W}\mathbf{x}_2 - \varepsilon$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|(\mathbf{W}\mathbf{x}_2 - \varepsilon) - (\mathbf{W}\mathbf{x}_1 - \varepsilon)\|_2^2 \\
&= \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1\|_2^2 \quad (8) \\
&\leq \|\mathbf{W}\|_2^2 \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2 \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2
\end{aligned}$$

(ii) $(\mathbf{W}\mathbf{x}_2)_i < (\mathbf{W}\mathbf{x}_1)_i$

a. if $(\mathbf{W}\mathbf{x}_2)_i < (\mathbf{W}\mathbf{x}_1)_i < -\varepsilon_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = \mathbf{W}\mathbf{x}_2 + \varepsilon$, $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = \mathbf{W}\mathbf{x}_1 + \varepsilon$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|(\mathbf{W}\mathbf{x}_2 + \varepsilon) - (\mathbf{W}\mathbf{x}_1 + \varepsilon)\|_2^2 \\
&= \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1\|_2^2 \quad (9) \\
&\leq \|\mathbf{W}\|_2^2 \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2 \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2
\end{aligned}$$

b. if $(\mathbf{W}\mathbf{x}_2)_i < -\varepsilon_i \leq (\mathbf{W}\mathbf{x}_1)_i \leq \varepsilon_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = \mathbf{W}\mathbf{x}_2 + \varepsilon$, $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = 0$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|(\mathbf{W}\mathbf{x}_2 + \varepsilon) - 0\|_2^2 \\
&= \|\mathbf{W}\mathbf{x}_2 + \varepsilon\|_2^2 \\
&= \|\mathbf{W}\mathbf{x}_2 - (-\varepsilon)\|_2^2 \quad (10) \\
&\leq \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1\|_2^2 \\
&\leq \|\mathbf{W}\|_2^2 \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2 \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2
\end{aligned}$$

c. if $(\mathbf{W}\mathbf{x}_2)_i < -\varepsilon_i < \varepsilon_i < (\mathbf{W}\mathbf{x}_1)_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = \mathbf{W}\mathbf{x}_2 + \varepsilon$, $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = \mathbf{W}\mathbf{x}_1 - \varepsilon$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|(\mathbf{W}\mathbf{x}_2 + \varepsilon) - (\mathbf{W}\mathbf{x}_1 - \varepsilon)\|_2^2 \\
&= \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1 + 2\varepsilon\|_2^2 \quad (11) \\
&\leq \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1\|_2^2 \\
&\leq \|\mathbf{W}\|_2^2 \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2 \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2
\end{aligned}$$

d. if $-\varepsilon_i \leq (\mathbf{W}\mathbf{x}_2)_i < (\mathbf{W}\mathbf{x}_1)_i \leq \varepsilon_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = 0$, $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = 0$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|0 - 0\|_2^2 \quad (12) \\
&= 0
\end{aligned}$$

e. if $-\varepsilon_i \leq (\mathbf{W}\mathbf{x}_2)_i \leq \varepsilon_i < (\mathbf{W}\mathbf{x}_1)_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = 0$, $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = \mathbf{W}\mathbf{x}_1 - \varepsilon$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|0 - (\mathbf{W}\mathbf{x}_1 - \varepsilon)\|_2^2 \\
&= \|\varepsilon - \mathbf{W}\mathbf{x}_1\|_2^2 \\
&\leq \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1\|_2^2 \\
&\leq \|\mathbf{W}\|_2^2 \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2 \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2
\end{aligned} \tag{13}$$

f. if $\varepsilon_i < (\mathbf{W}\mathbf{x}_2)_i < (\mathbf{W}\mathbf{x}_1)_i$, then $\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) = \mathbf{W}\mathbf{x}_2 - \varepsilon$, $\text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon) = \mathbf{W}\mathbf{x}_1 - \varepsilon$:

$$\begin{aligned}
& \|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2 \\
&= \|(\mathbf{W}\mathbf{x}_2 - \varepsilon) - (\mathbf{W}\mathbf{x}_1 - \varepsilon)\|_2^2 \\
&= \|\mathbf{W}\mathbf{x}_2 - \mathbf{W}\mathbf{x}_1\|_2^2 \\
&\leq \|\mathbf{W}\|_2^2 \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2 \\
&= \|\mathbf{x}_2 - \mathbf{x}_1\|_2^2
\end{aligned} \tag{14}$$

From the above inequalities, it can be seen that the upper bound of $\|\text{soft}(\mathbf{W}\mathbf{x}_2, \varepsilon) - \text{soft}(\mathbf{W}\mathbf{x}_1, \varepsilon)\|_2^2$ is either 0 or $\|\mathbf{x}_2 - \mathbf{x}_1\|_2^2$. The $\mathbf{W}^T \text{soft}(\mathbf{W}\mathbf{x}, \varepsilon)$ is $\lambda_{\max}(\mathbf{W}\mathbf{W})^T$ -Lipschitz.

B. Images Reconstructed by SR Algorithms and CSMRI Algorithms

To further evaluate the effectiveness of our constructed algorithm, we supplement the comparison results of different algorithms for SR task and CSMRI task, as shown in Fig. 1 - Fig. 4.

The reconstruction results of some of the SR methods on the Set12 dataset are presented in Fig. 1 and Fig. 2. From Fig. 1, it can be observed that the VDSR, PAN, and SRMD methods exhibit noticeable blur and artifacts on the facial region. Although the RCAN and USRNet demonstrate promising performance in face reconstruction, they still suffer from some loss of fine image textures. In contrast, our proposed method consistently restores fine details. As depicted in Fig. 2, USRNet and RCAN exhibit superior performance compared to VDSR, PAN, and SRMD in the case of motion blur kernel. While USRNet can restore sharper edges than RCAN, both models fail to reproduce realistic textures. As anticipated, GDUNet surpasses USRNet in terms of visual quality.

Fig. 3 and Fig. 4 display the reconstruction results of some of the CSMRI algorithms. From the magnified details in Fig. 3, it is evident that both IDPCNN and GDUNet exhibit enhanced line sharpness compared to other algorithms. Furthermore, the difference images indicate that GDUNet exhibits a closer resemblance to the original image than IDPCNN, suggesting superior restoration results on smooth areas. As observed in Fig. 4, the reconstructions produced by the DLMRI and NLR algorithms lack intricate details. While algorithms like BM3D-MRI can alleviate some artifacts, these reconstructions still exhibit noticeable noise. In comparison to the aforementioned algorithms, the ADMM-CSNet and IDPCNN algorithms demonstrate superior reconstruction performance. The proposed GDUNet not only effectively retains most of the details but also notably reduces noise.

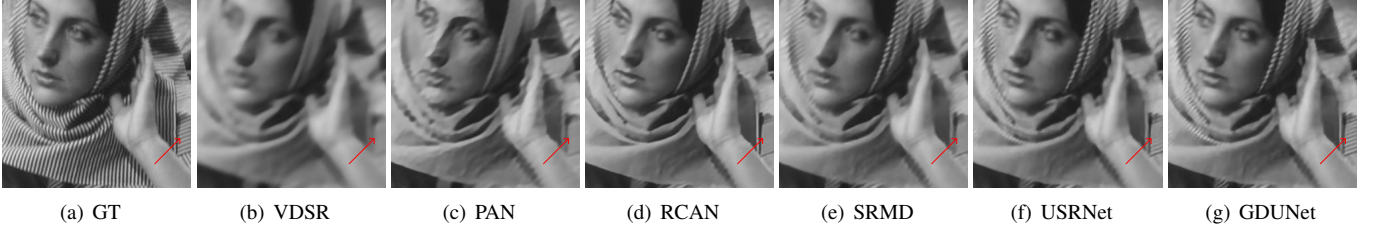


Fig. 1. Reconstructed SR results of an LR synthetic image in the case of anisotropic Gaussian kernel. To facilitate observation, the specific image segment are selected. From left to right, and top to bottom: the original image (a), and the images recovered by (b) VDSR (PSNR = 24.38 dB, SSIM = 0.7141); (c) PAN (PSNR = 26.01 dB, SSIM = 0.7866); (d) RCAN (PSNR = 27.94 dB, SSIM = 0.8448); (e) SRMD (PSNR = 27.51 dB, SSIM = 0.8248); (f) USRNet (PSNR = 29.29 dB, SSIM = 0.8592); (g) GDUNet (PSNR = 29.41 dB, SSIM = 0.8629).

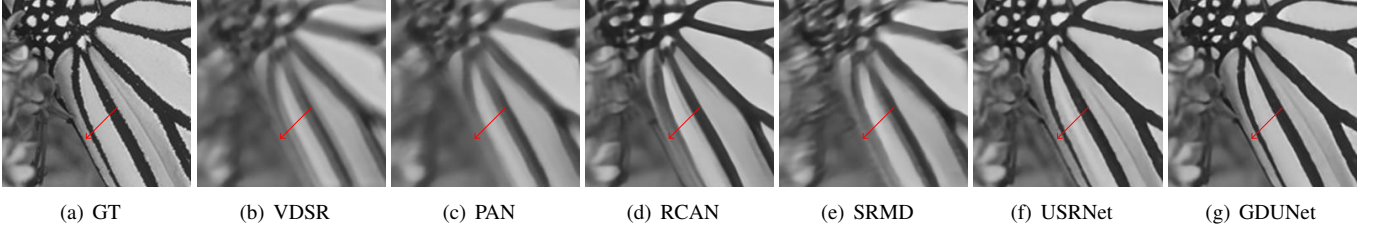


Fig. 2. Reconstructed SR results of an LR synthetic image in the case of motion blur kernel. To facilitate observation, the specific image segment are selected. From left to right, and top to bottom: the original image (a), and the images recovered by (b) VDSR (PSNR = 18.50 dB, SSIM = 0.5520); (c) PAN (PSNR = 18.21 dB, SSIM = 0.5513); (d) RCAN (PSNR = 21.78 dB, SSIM = 0.7463); (e) SRMD (PSNR = 18.43 dB, SSIM = 0.5871); (f) USRNet (PSNR = 30.44 dB, SSIM = 0.9292); (g) GDUNet (PSNR = 31.90 dB, SSIM = 0.9460).

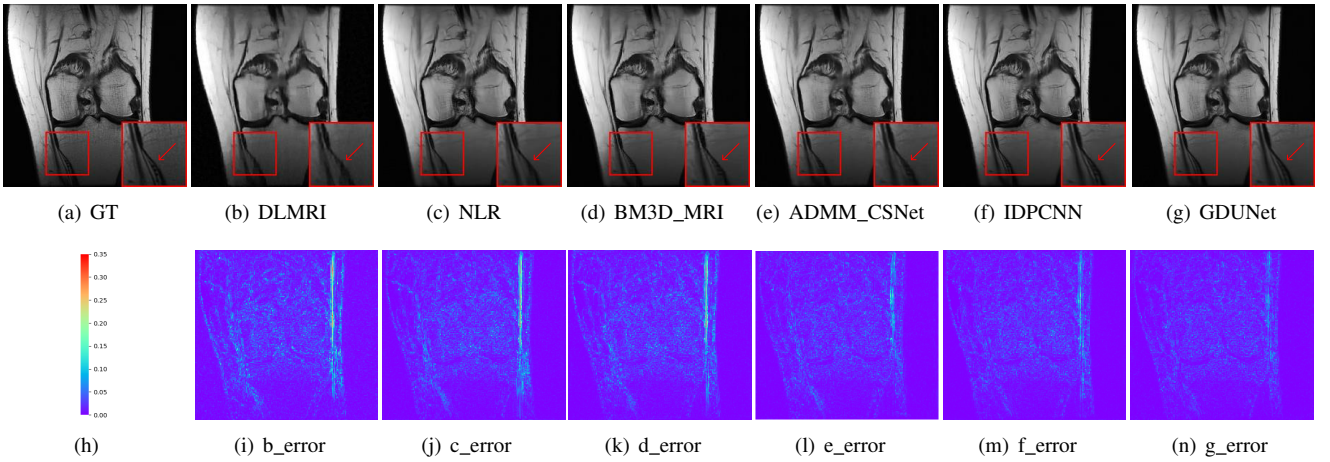


Fig. 3. At the case of sampling ratio 0.2, the knee image recovered by the CSMRI algorithms under the Pseudo-radial sampling mode. Below each reconstructed image, we calculate its difference from the original image. From left to right, and top to bottom: the original image (a), and the images recovered by (b) DLMRI (PSNR = 31.10 dB, SSIM = 0.8705); (c) NLR (PSNR = 32.08 dB, SSIM = 0.9020); (d) BM3D_MRI (PSNR = 32.23 dB, SSIM = 0.9038); (e) ADMM_CSNet (PSNR = 34.14 dB, SSIM = 0.9282); (f) IDPCNN (PSNR = 34.29 dB, SSIM = 0.9320); (g) GDUNet (PSNR = 35.19 dB, SSIM = 0.9412).

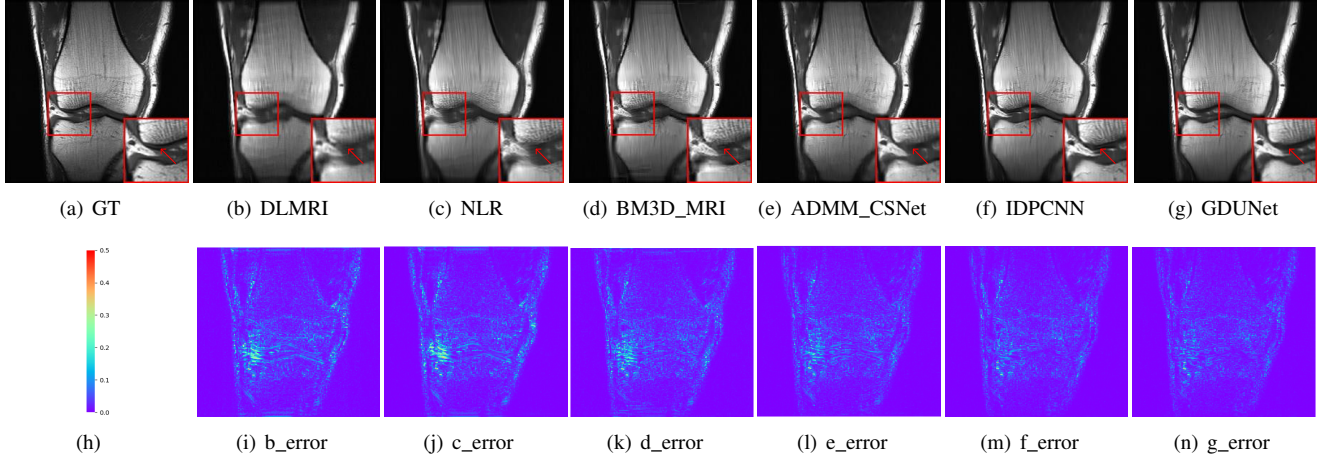


Fig. 4. At the case of sampling ratio 0.2, the knee image recovered by the CSMRI algorithms under the Cartesian sampling mode. Below each reconstructed image, we calculate its difference from the original image. From left to right, and top to bottom: the original image (a), and the images recovered by (b) DLMRI (PSNR = 28.77 dB, SSIM = 0.8563); (c) NLR (PSNR = 28.92 dB, SSIM = 0.8750); (d) BM3D_MRI (PSNR = 29.71 dB, SSIM = 0.8783); (e) ADMM_CSNet (PSNR = 30.97 dB, SSIM = 0.9042); (f) IDPCNN (PSNR = 31.60 dB, SSIM = 0.9115); (g) GDUNet (PSNR = 32.08 dB, SSIM = 0.9183).