Neural networks

Autoencoder - definition

UNSUPERVISED LEARNING

Topics: unsupervised learning

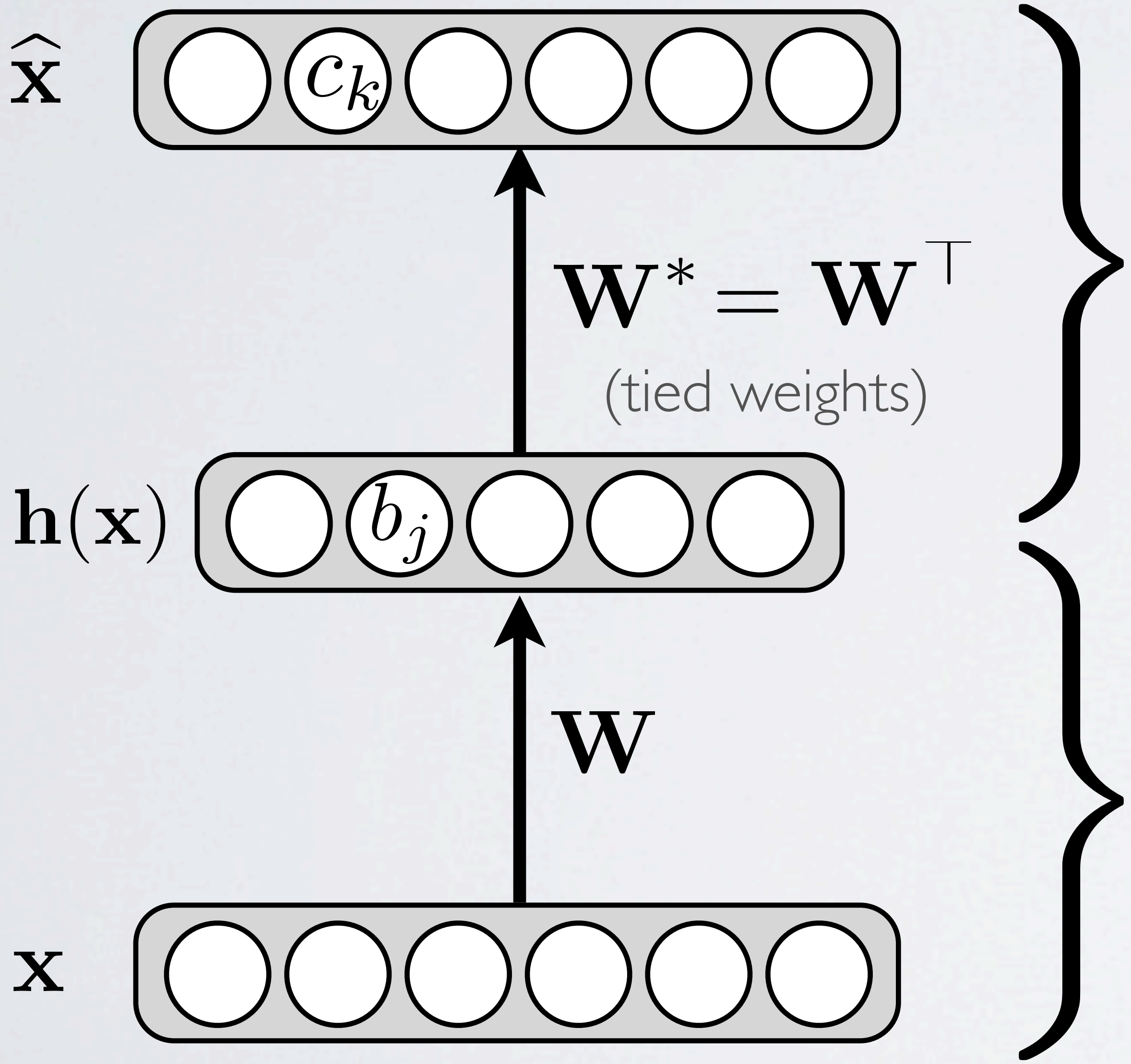
- Unsupervised learning: only use the inputs $\mathbf{x}^{(t)}$ for learning
  - automatically extract meaningful features for your data
  - leverage the availability of unlabeled data
  - add a data-dependent regularizer to trainings

- We will see 3 neural networks for unsupervised learning
  - restricted Boltzmann machines
  - **autoencoders**
  - sparse coding model

AUTOENCODER

Topics: autoencoder, encoder, decoder, tied weights

- Feed-forward neural network trained to reproduce its input at the output layer



Decoder

$$\begin{aligned}\widehat{\mathbf{x}} &= o(\widehat{\mathbf{a}}(\mathbf{x})) \\ &= \underbrace{\mathrm{sigm}}_{\text{for binary inputs}}(\mathbf{c} + \mathbf{W}^*\mathbf{h}(\mathbf{x}))\end{aligned}$$

Encoder

$$\begin{aligned}\mathbf{h}(\mathbf{x}) &= g(\mathbf{a}(\mathbf{x})) \\ &= \mathrm{sigm}(\mathbf{b} + \mathbf{W}\mathbf{x})\end{aligned}$$

Neural networks

Autoencoder - loss function

AUTOENCODER

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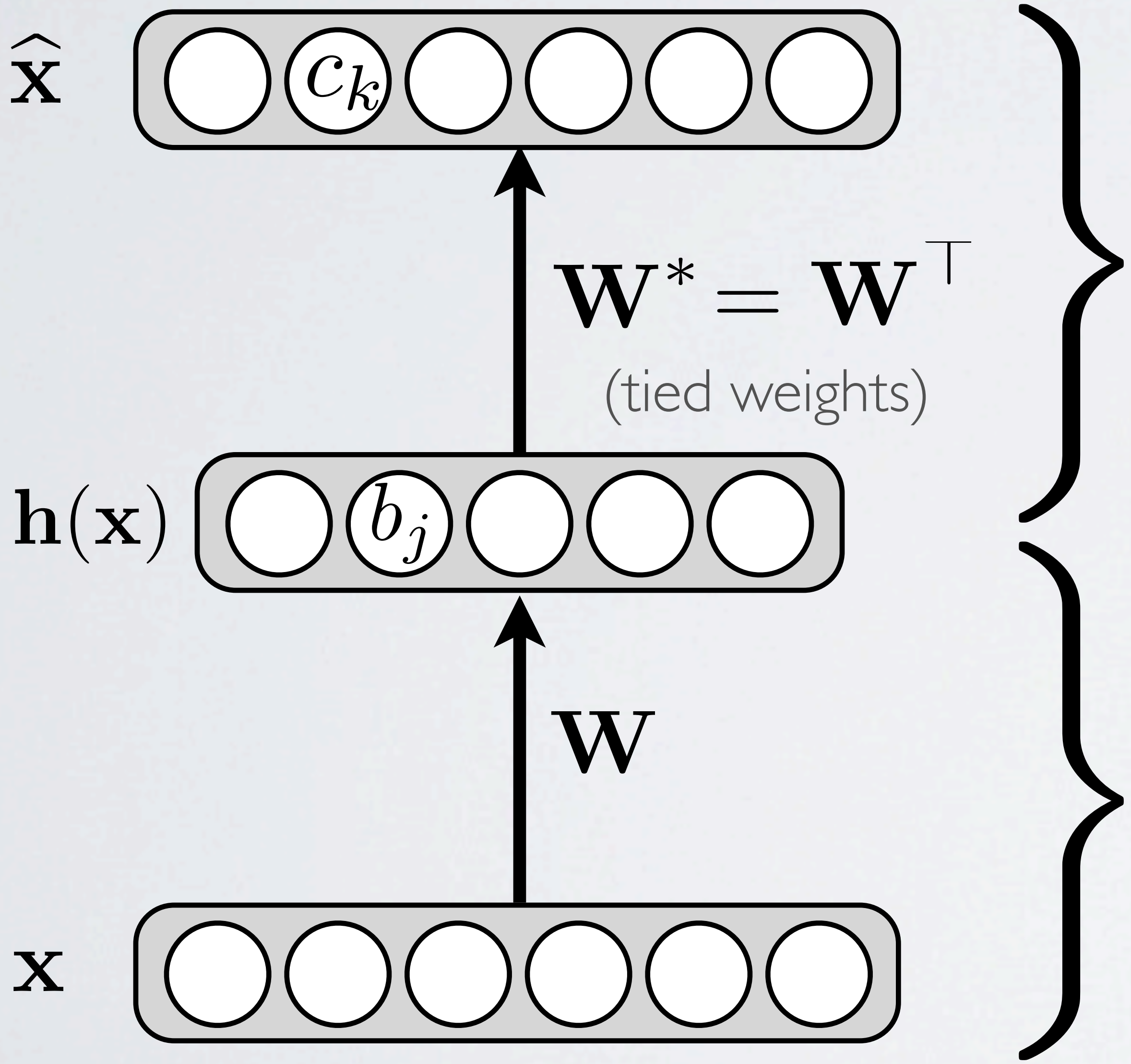

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AUTOENCODER

Topics: loss function

$$f(\mathbf{x}) \equiv \widehat{\mathbf{x}}$$

- For binary inputs:

$$l(f(\mathbf{x})) = -\sum_k \left(x_k \log(\widehat{x}_k) + (1 - x_k) \log(1 - \widehat{x}_k)\right)$$

  - cross-entropy (more precisely: sum of Bernoulli cross-entropies)

- For real-valued inputs:

$$l(f(\mathbf{x})) = \frac{1}{2} \sum_k (\widehat{x}_k - x_k)^2$$

  - sum of squared differences (squared euclidean distance)
  - we use a linear activation function at the output

AUTOENCODER

Topics: loss function gradient

$$f(\mathbf{x}) \equiv \widehat{\mathbf{x}}$$

- For both cases, the gradient $\nabla_{\widehat{\mathbf{a}}(\mathbf{x}^{(t)})} l(f(\mathbf{x}^{(t)}))$ has a very simple form:

$$\nabla_{\widehat{\mathbf{a}}(\mathbf{x}^{(t)})} l(f(\mathbf{x}^{(t)})) = \widehat{\mathbf{x}}^{(t)} - \mathbf{x}^{(t)}$$

- Parameter gradients are obtained by backpropagating the gradient $\nabla_{\widehat{\mathbf{a}}(\mathbf{x}^{(t)})} l(f(\mathbf{x}^{(t)}))$ like in a regular network
  - **important**: when using tied weights ($\mathbf{W}^* = \mathbf{W}^\top$), $\nabla_{\mathbf{W}} l(f(\mathbf{x}^{(t)}))$ is the sum of two gradients !
    - this is because $\mathbf{W}$ is present in the encoder **and** in the decoder

AUTOENCODER

Topics: adaptation to the type of input

$$f(\mathbf{x}) \equiv \widehat{\mathbf{x}}$$

- Recipe to adapt an autoencoder to a new type of input
  - choose a joint distribution $p(\mathbf{x}|\boldsymbol{\mu})$ over the inputs
    - $\boldsymbol{\mu}$ is the vector of parameters of that distribution
  - choose the relationship between $\boldsymbol{\mu}$ and the hidden layer $\mathbf{h}(\mathbf{x})$
  - use $l(f(\mathbf{x})) = -\log p(\mathbf{x}|\boldsymbol{\mu})$ as the loss function
- Example: we get the sum of squared distance by
  - choosing a Gaussian distribution with mean $\boldsymbol{\mu}$ and identity covariance for $p(\mathbf{x}|\boldsymbol{\mu}) = \frac{1}{(2\pi)^{D/2}} \exp(-\frac{1}{2}\sum_k (x_k - \mu_k)^2)$
  - choosing $\boldsymbol{\mu} = \mathbf{c} + \mathbf{W}^* \mathbf{h}(\mathbf{x})$

Neural networks

Autoencoder - example

AUTOENCODER

Topics: autoencoder, encoder, decoder, tied weights

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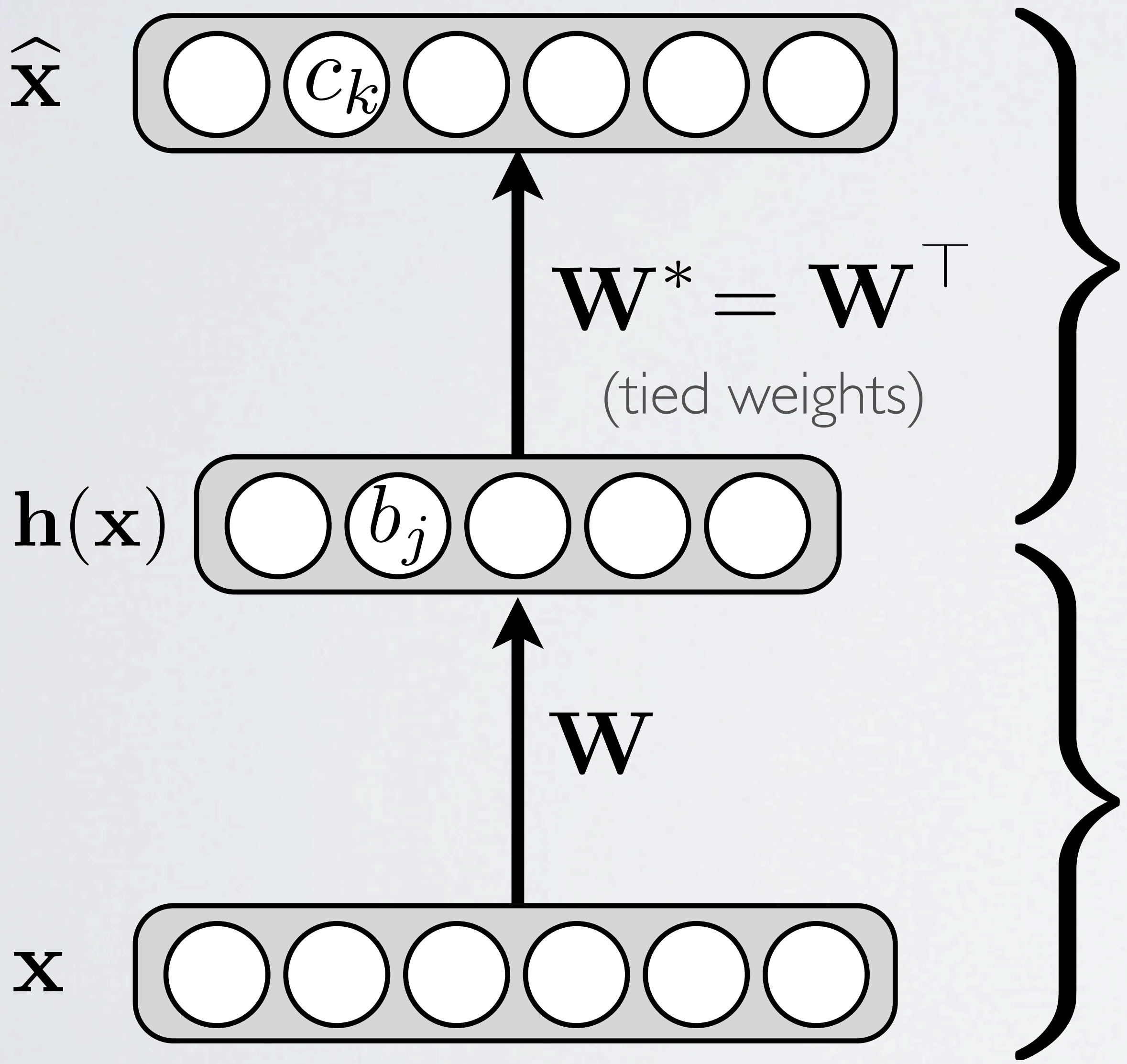

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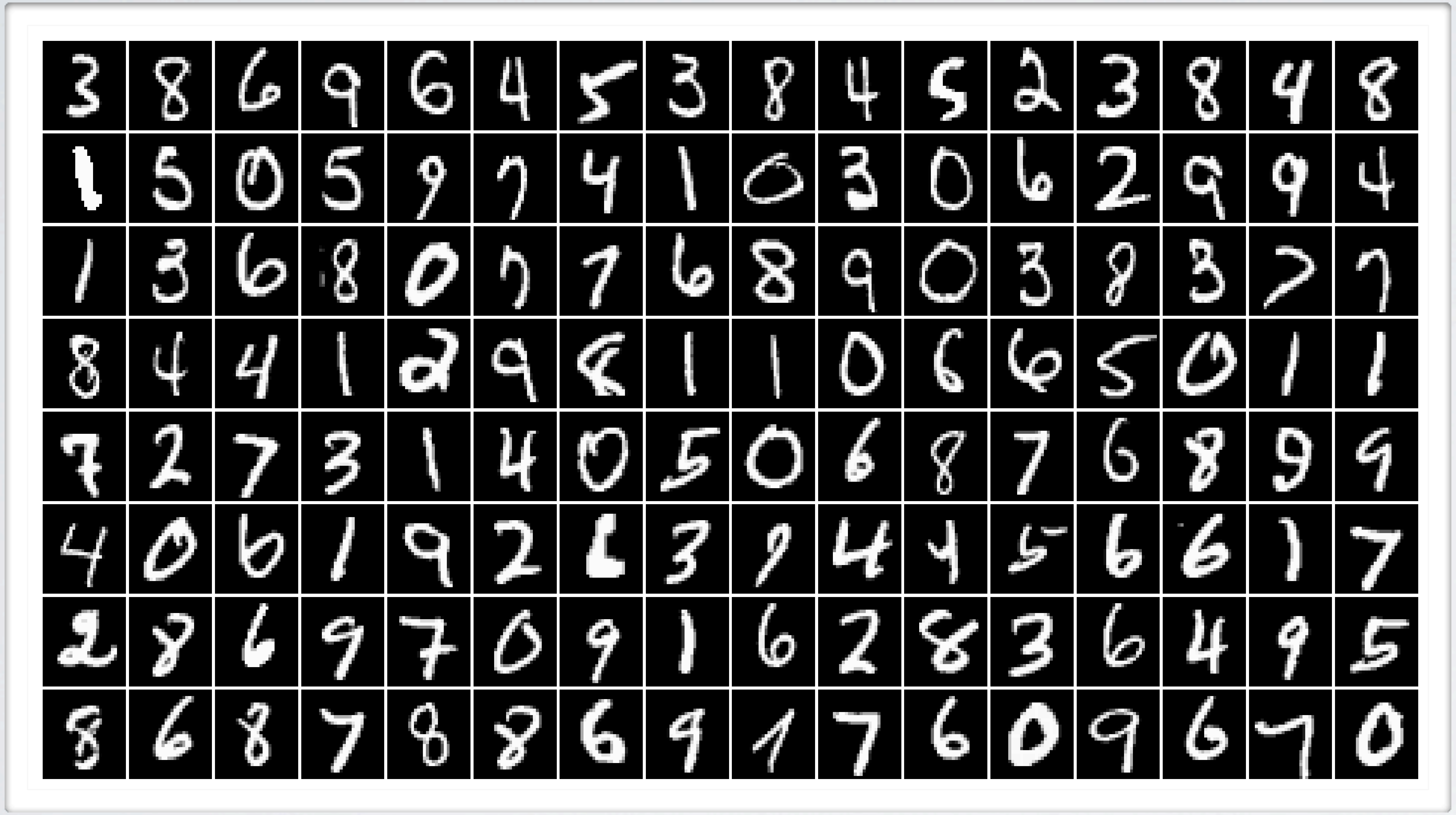
\end{aligned}
$$

Encoder

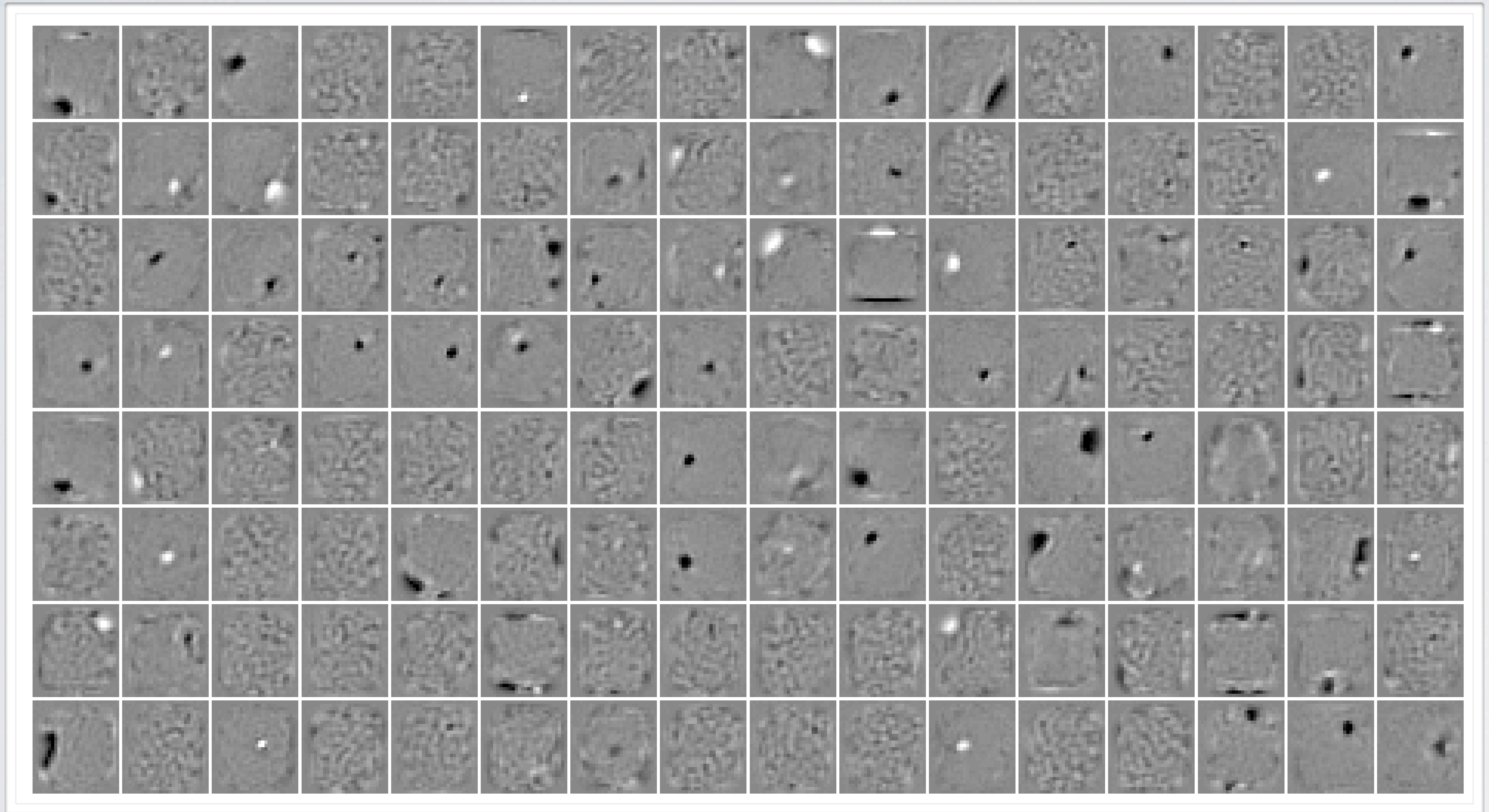
$$
\begin{aligned}
\mathbf{h}(\mathbf{x}) &= g(\mathbf{a}(\mathbf{x})) \\
&= \mathrm{sigm}(\mathbf{b} + \mathbf{W}\mathbf{x})
\end{aligned}
$$

EXAMPLE OF DATA SET: MNIST



FILTERS (AUTOENCODER)

(Larochelle et al., JMLR2009)



Neural networks

Autoencoder - linear autoencoder

AUTOENCODER

Topics: autoencoder, encoder, decoder, tied weights

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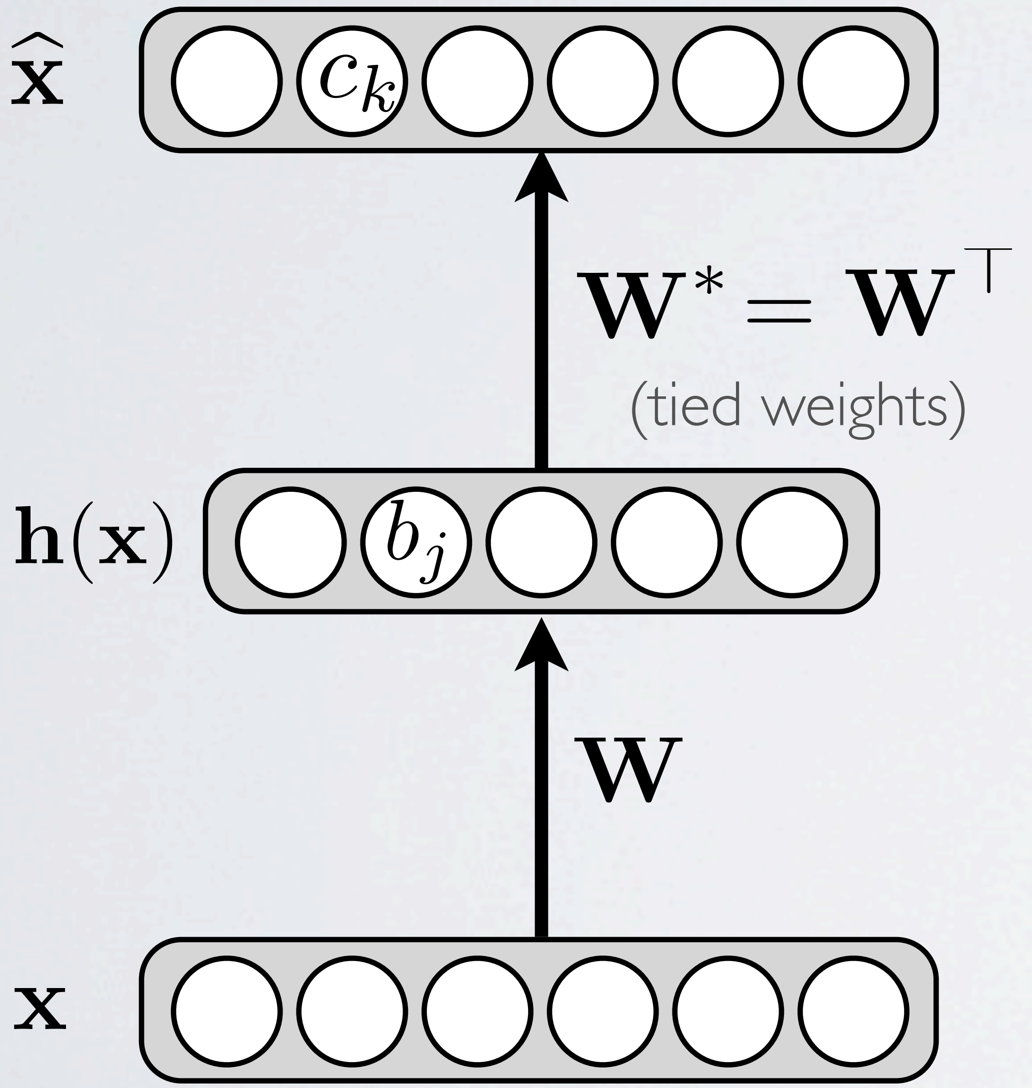

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AUTOENCODER

Topics: optimality of a linear autoencoder

- To do the proof, we need the following theorem:
  - let $\mathbf{A}$ be any matrix, with singular value decomposition $\mathbf{A} = \mathbf{U}\,\Sigma\,\mathbf{V}^\top$
    - Σ is a diagonal matrix
    - $\mathbf{U}$, $\mathbf{V}$ are orthonormal matrices (columns/rows are orthonormal vectors)
  - let $\mathbf{U}_{\cdot,\leq k}\,\Sigma_{\leq k,\leq k}\,\mathbf{V}^\top_{\cdot,\leq k}$ be the decomposition where we keep only the k largest singular values
  - then, the matrix $\mathbf{B}$ of rank k that is closest to $\mathbf{A}$:

$$\boxed{\mathbf{B}^* = \underset{\mathbf{B}\ \text{s.t. rank}(\mathbf{B})=k}{\arg\min}\ ||\mathbf{A}-\mathbf{B}||_F}$$

is $\mathbf{B}^* = \mathbf{U}_{\cdot,\leq k}\,\Sigma_{\leq k,\leq k}\,\mathbf{V}^\top_{\cdot,\leq k}$

$$\min_{\theta} \sum_{t} \frac{1}{2} \sum_{i} (x_i^{(t)} - \widehat{x}_i^{(t)})^2$$

$$\min_{\theta} \sum_t \frac{1}{2} \sum_i (x_i^{(t)} - \underbrace{\widehat{x}_i^{(t)}}_{\substack{\text{based on} \\ \text{linear encoder}}})^2$$

$$\min_\theta \sum_t \frac{1}{2} \sum_i (x_i^{(t)} - \underbrace{\widehat{x}_i^{(t)}}_{\text{based on linear encoder}})^2 \quad \geq \quad \min_{\mathbf{W}^*, \mathbf{h}(\mathbf{X})} \frac{1}{2} ||\mathbf{X} - \mathbf{W}^* \mathbf{h}(\mathbf{X})||_F^2$$

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$$\underset{\mathbf{W}^*, \mathbf{h}(\mathbf{X})}{\arg\min} \frac{1}{2} || \mathbf{X} - \mathbf{W}^* \mathbf{h}(\mathbf{X}) ||_F^2 \quad = \quad \left(\mathbf{W}^* \leftarrow \mathbf{U}_{\cdot, \leq k} \, \Sigma_{\leq k, \leq k}, \;\; \mathbf{h}(\mathbf{X}) \leftarrow \mathbf{V}_{\cdot, \leq k}^{\top} \right)$$

based on previous theorem, where $\mathbf{X} = \mathbf{U} \, \Sigma \, \mathbf{V}^\top$
and k is the hidden layer size

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Let's show $\mathbf{h}(\mathbf{X})$ is a linear encoder:

$$\min_{\theta} \sum_t \frac{1}{2} \sum_i (x_i^{(t)} - \underbrace{\widehat{x}_i^{(t)}}_{\text{based on linear encoder}})^2 \quad \geq \quad \min_{\mathbf{W}^*, \mathbf{h}(\mathbf{X})} \frac{1}{2} ||\overbrace{\mathbf{X}}^{\text{matrix where columns are } \mathbf{x}^{(t)}} - \mathbf{W}^* \underbrace{\mathbf{h}(\mathbf{X})}_{\text{matrix of all hidden layers (could be any encoder)}}||_F^2$$

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← multiplying by identity

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&= \mathbf{V}_{\cdot, \leq k}^{\top} (\mathbf{V}\, \Sigma^{\top}\, \mathbf{U}^{\top}\, \mathbf{U}\, \Sigma\, \mathbf{V}^{\top})^{-1} (\mathbf{V}\, \Sigma^{\top}\, \mathbf{U}^{\top}\, \mathbf{X}) && \longleftarrow \text{replace with SVD}
\end{aligned}$$

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\end{aligned}$$

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&= \mathbf{V}_{\cdot, \leq k}^{\top}\, \mathbf{V}\, (\Sigma^{\top}\, \Sigma)^{-1}\, \mathbf{V}^{\top}\, \mathbf{V}\, \Sigma^{\top}\, \mathbf{U}^{\top}\, \mathbf{X} && \longleftarrow \mathbf{V}(\Sigma^{\top}\Sigma)^{-1}\mathbf{V}^{\top}\, \mathbf{V}\Sigma^{\top}\Sigma\mathbf{V}^{\top} = \mathbf{I}
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&= \mathbf{V}_{\cdot, \leq k}^\top\, \mathbf{V}\, (\Sigma^\top\,\Sigma)^{-1}\, \Sigma^\top\, \mathbf{U}^\top\, \mathbf{X} && \longleftarrow \mathbf{V}^\top\, \mathbf{V} = \mathbf{I} \text{ (orthonormal)}
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&= \mathbf{V}_{\cdot, \leq k}^\top \, \mathbf{V} \, (\Sigma^\top \, \Sigma)^{-1} \, \mathbf{V}^\top \, \mathbf{V} \, \Sigma^\top \, \mathbf{U}^\top \, \mathbf{X} && \longleftarrow \mathbf{V}(\Sigma^\top \Sigma)^{-1} \mathbf{V}^\top \mathbf{V} \Sigma^\top \Sigma \mathbf{V}^\top = \mathbf{I} \\
&= \mathbf{V}_{\cdot, \leq k}^\top \, \mathbf{V} \, (\Sigma^\top \, \Sigma)^{-1} \, \Sigma^\top \, \mathbf{U}^\top \, \mathbf{X} && \longleftarrow \mathbf{V}^\top \mathbf{V} = \mathbf{I} \text{ (orthonormal)} \\
&= \mathrm{I}_{\leq k, \cdot} \, (\Sigma^\top \, \Sigma)^{-1} \, \Sigma^\top \, \mathbf{U}^\top \, \mathbf{X} && \longleftarrow \text{idem}
\end{aligned}$$

$$\min_{\theta} \sum_t \frac{1}{2} \sum_i (x_i^{(t)} - \underbrace{\widehat{x}_i^{(t)}}_{\text{based on linear encoder}})^2 \quad \geq \quad \min_{\mathbf{W}^*, \mathbf{h}(\mathbf{X})} \frac{1}{2} ||\overbrace{\mathbf{X}}^{\text{matrix where columns are } \mathbf{x}^{(t)}} - \mathbf{W}^* \underbrace{\mathbf{h}(\mathbf{X})}_{\text{matrix of all hidden layers (could be any encoder)}}||_F^2$$

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based on previous theorem, where $\mathbf{X} = \mathbf{U}\, \Sigma\, \mathbf{V}^\top$
and k is the hidden layer size

Let's show $\mathbf{h}(\mathbf{X})$ is a linear encoder:

$$
\begin{aligned}
\mathbf{h}(\mathbf{X}) &= \mathbf{V}_{\cdot, \leq k}^\top \\
&= \mathbf{V}_{\cdot, \leq k}^\top\, (\mathbf{X}^\top\, \mathbf{X})^{-1}\, (\mathbf{X}^\top\, \mathbf{X}) && \leftarrow \text{multiplying by identity} \\
&= \mathbf{V}_{\cdot, \leq k}^\top\, (\mathbf{V}\, \Sigma^\top\, \mathbf{U}^\top\, \mathbf{U}\, \Sigma\, \mathbf{V}^\top)^{-1}\, (\mathbf{V}\, \Sigma^\top\, \mathbf{U}^\top\, \mathbf{X}) && \leftarrow \text{replace with SVD} \\
&= \mathbf{V}_{\cdot, \leq k}^\top\, (\mathbf{V}\, \Sigma^\top\, \Sigma\, \mathbf{V}^\top)^{-1}\, \mathbf{V}\, \Sigma^\top\, \mathbf{U}^\top\, \mathbf{X} && \leftarrow \mathbf{U}^\top \mathbf{U} = \mathbf{I} \text{ (orthonormal)} \\
&= \mathbf{V}_{\cdot, \leq k}^\top\, \mathbf{V}\, (\Sigma^\top\, \Sigma)^{-1}\, \mathbf{V}^\top\, \mathbf{V}\, \Sigma^\top\, \mathbf{U}^\top\, \mathbf{X} && \leftarrow \mathbf{V}(\Sigma^\top\Sigma)^{-1}\mathbf{V}^\top\, \mathbf{V}\Sigma^\top\Sigma\mathbf{V}^\top = \mathbf{I} \\
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&= \mathrm{I}_{\leq k, \cdot}\, (\Sigma^\top\, \Sigma)^{-1}\, \Sigma^\top\, \mathbf{U}^\top\, \mathbf{X} && \leftarrow \text{idem} \\
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\end{aligned}
$$

$$\min_{\theta} \sum_t \frac{1}{2} \sum_i (x_i^{(t)} - \underbrace{\widehat{x}_i^{(t)}}_{\text{based on linear encoder}})^2 \quad \geq \quad \min_{\mathbf{W}^*, \mathbf{h}(\mathbf{X})} \frac{1}{2} ||\overbrace{\mathbf{X}}^{\text{matrix where columns are } \mathbf{x}^{(t)}} - \mathbf{W}^* \underbrace{\mathbf{h}(\mathbf{X})}_{\substack{\text{matrix of all hidden layers} \\ \text{(could be any encoder)}}}||_F^2$$

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based on previous theorem, where $\mathbf{X} = \mathbf{U}\, \Sigma\, \mathbf{V}^\top$ and k is the hidden layer size

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&= \mathbf{V}^\top_{\cdot, \leq k} (\mathbf{V}\, \Sigma^\top\, \mathbf{U}^\top\, \mathbf{U}\, \Sigma\, \mathbf{V}^\top)^{-1} (\mathbf{V}\, \Sigma^\top\, \mathbf{U}^\top\, \mathbf{X}) && \longleftarrow \text{replace with SVD} \\
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\end{aligned}$$

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&= \mathrm{I}_{\leq k, \cdot}\, \Sigma^{-1}\, \mathbf{U}^\top\, \mathbf{X} \\
&= \Sigma_{\leq k, \leq k}^{-1}\, (\mathbf{U}_{\cdot, \leq k})^\top\, \mathbf{X} && \longleftarrow \text{multiplying by } \mathbf{I}_{\leq k, \cdot} \text{ selects the } k \text{ first rows}
\end{aligned}$$

$$\min_{\theta} \sum_t \frac{1}{2} \sum_i (x_i^{(t)} - \underbrace{\widehat{x}_i^{(t)}}_{\text{based on linear encoder}})^2 \quad \geq \quad \min_{\mathbf{W}^*, \mathbf{h}(\mathbf{X})} \frac{1}{2} || \overbrace{\mathbf{X}}^{\text{matrix where columns are } \mathbf{x}^{(t)}} - \mathbf{W}^* \underbrace{\mathbf{h}(\mathbf{X})}_{\substack{\text{matrix of all hidden layers} \\ \text{(could be any encoder)}}} ||_F^2$$

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&= \underbrace{\Sigma_{\leq k, \leq k}^{-1} \, (\mathbf{U}_{\cdot, \leq k})^{\top}}_{\text{this is a linear encoder}} \, \mathbf{X} && \longleftarrow \text{multiplying by } \mathbf{I}_{\leq k, \cdot} \text{ selects the } k \text{ first rows}
\end{aligned}$$

AUTOENCODER

Topics: optimality of a linear autoencoder

- So an optimal pair of encoder and decoder is

$$\mathbf{h}(\mathbf{x}) = \underbrace{\left(\Sigma^{-1}_{\leq k, \leq k}\,(\mathbf{U}_{\cdot,\leq k})^\top\right)}_{\mathbf{W}}\mathbf{x} \qquad \widehat{\mathbf{x}} = \underbrace{(\mathbf{U}_{\cdot,\leq k}\;\Sigma_{\leq k,\leq k})}_{\mathbf{W}^*}\mathbf{h}(\mathbf{x})$$

  - for the sum of squared difference error
  - for an autoencoder with a linear decoder
  - where optimality means "has the lowest training reconstruction error"
- If inputs are normalized as follows: $\mathbf{x}^{(t)} \leftarrow \frac{1}{\sqrt{T}}\left(\mathbf{x}^{(t)} - \frac{1}{T}\sum_{t'=1}^{T}\mathbf{x}^{(t')}\right)$
  - encoder corresponds to Principal Component Analysis (PCA)
    - singular values and (left) vectors = the eigenvalues/vectors of covariance matrix

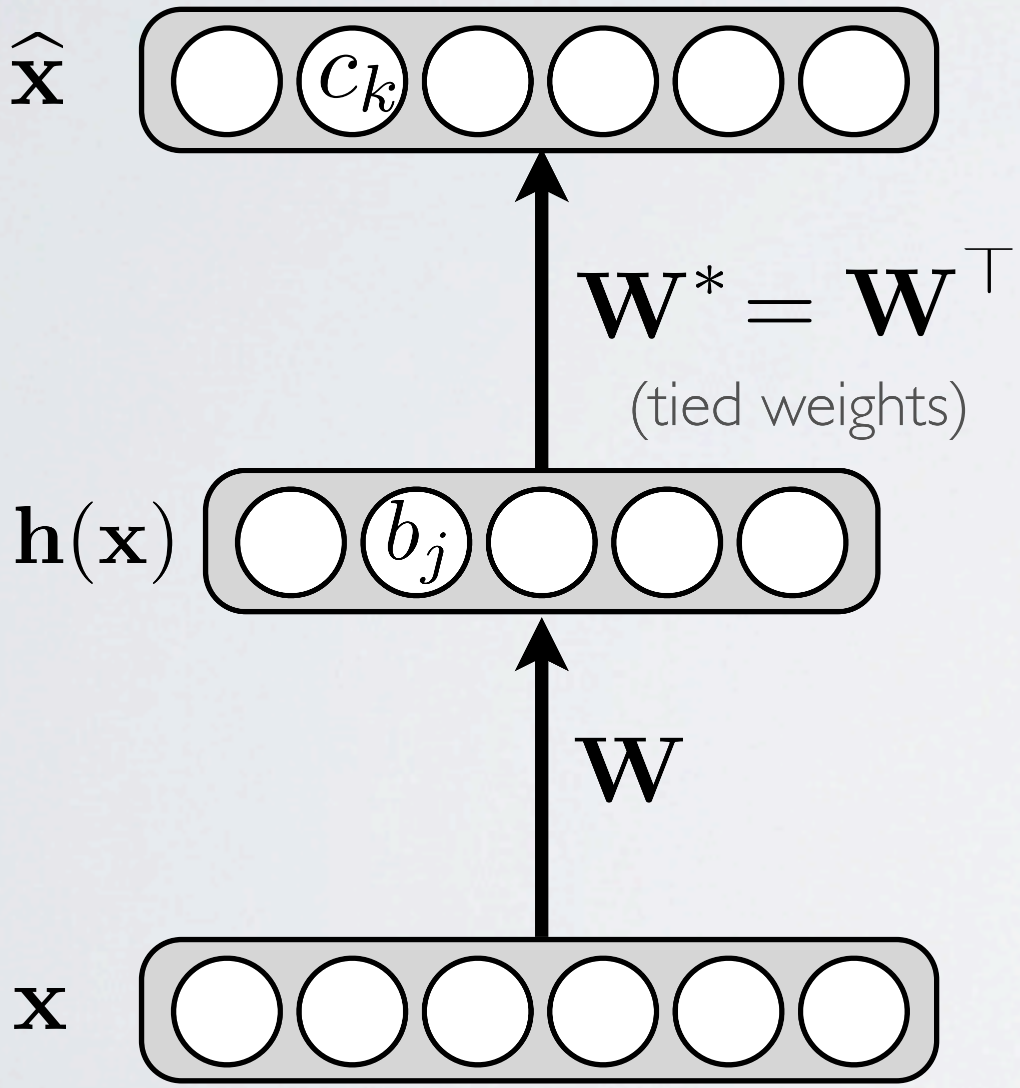
Neural networks

Autoencoder - undercomplete vs. overcomplete hidden layer

AUTOENCODER

Topics: autoencoder, encoder, decoder, tied weights

- Feed-forward neural network trained to reproduce its input at the output layer



Decoder

$$
\begin{aligned}
\widehat{\mathbf{x}} &= o(\widehat{\mathbf{a}}(\mathbf{x})) \\
&= \underbrace{\mathrm{sigm}}_{\text{for binary inputs}}(\mathbf{c} + \mathbf{W}^* \mathbf{h}(\mathbf{x}))
\end{aligned}
$$

Encoder

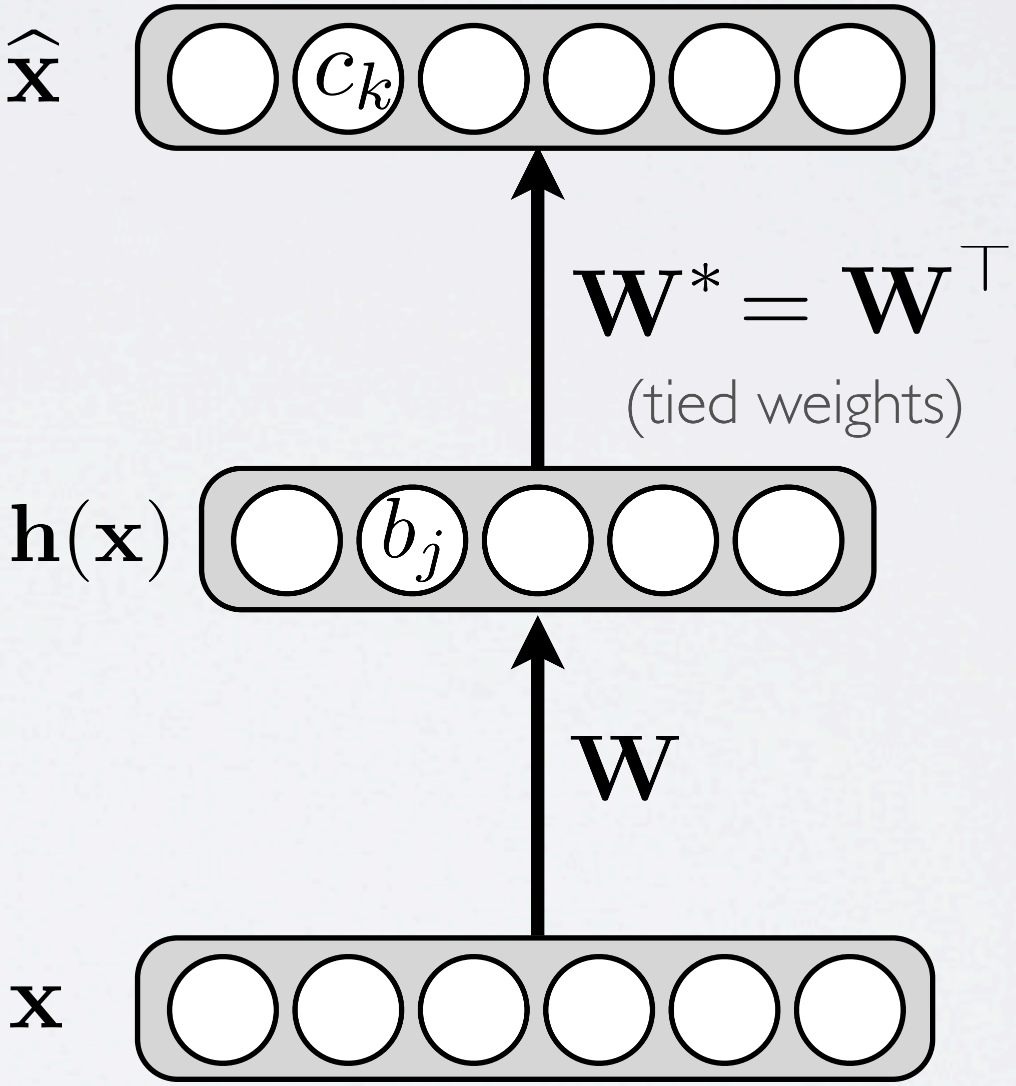
$$
\begin{aligned}
\mathbf{h}(\mathbf{x}) &= g(\mathbf{a}(\mathbf{x})) \\
&= \mathrm{sigm}(\mathbf{b} + \mathbf{W}\mathbf{x})
\end{aligned}
$$

UNDERCOMPLETE HIDDEN LAYER

Topics: undercomplete representation

- Hidden layer is undercomplete if smaller than the input layer
  - hidden layer "compresses" the input
  - will compress well only for the training distribution
- Hidden units will be
  - good features for the training distribution

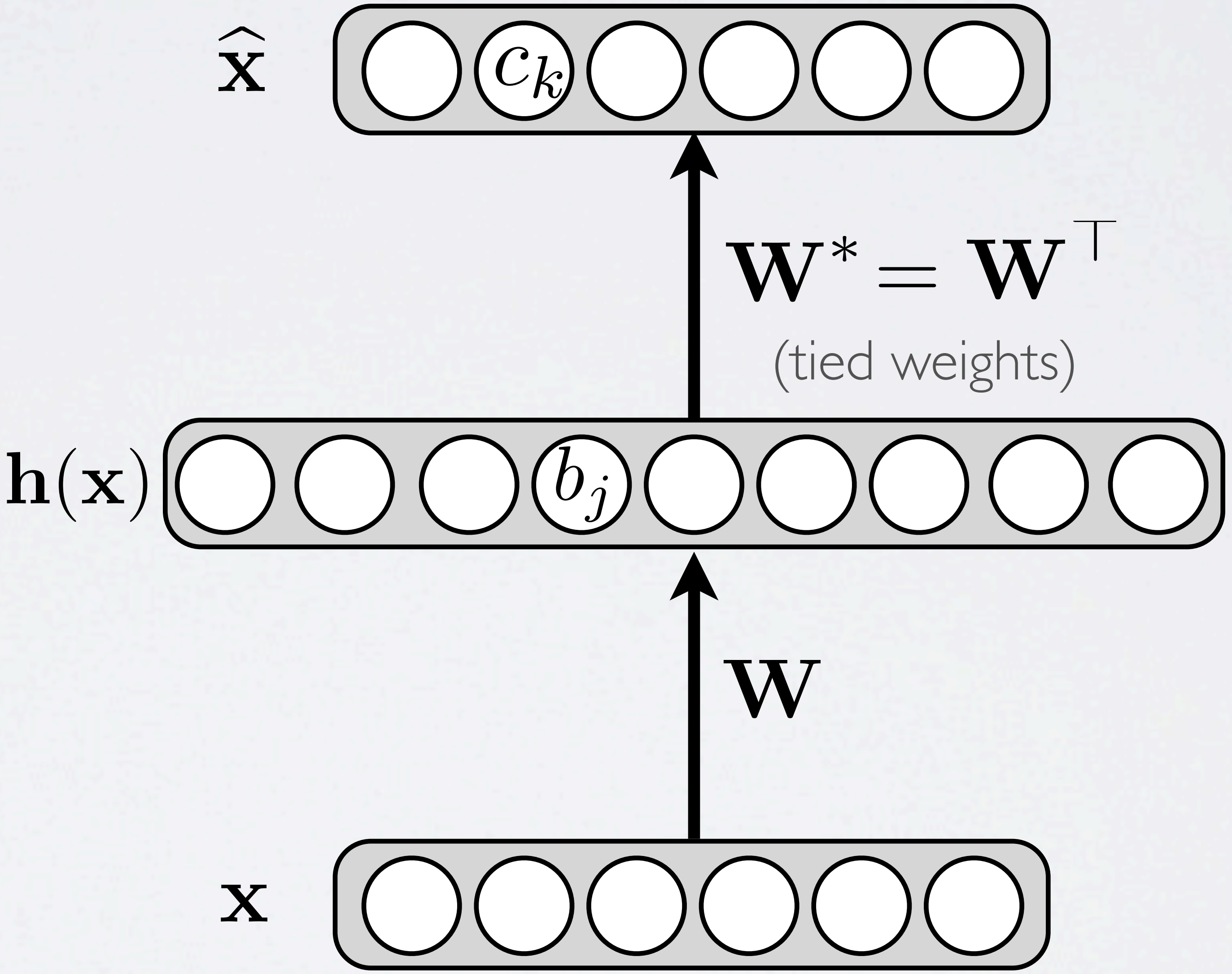
  - but bad for other types of input



OVERCOMPLETE HIDDEN LAYER

Topics: overcomplete representation

- Hidden layer is overcomplete if greater than the input layer
  - no compression in hidden layer
  - each hidden unit could copy a different input component
- No guarantee that the hidden units will extract meaningful structure



Neural networks

Autoencoder - denoising autoencoder

OVERCOMPLETE HIDDEN LAYER

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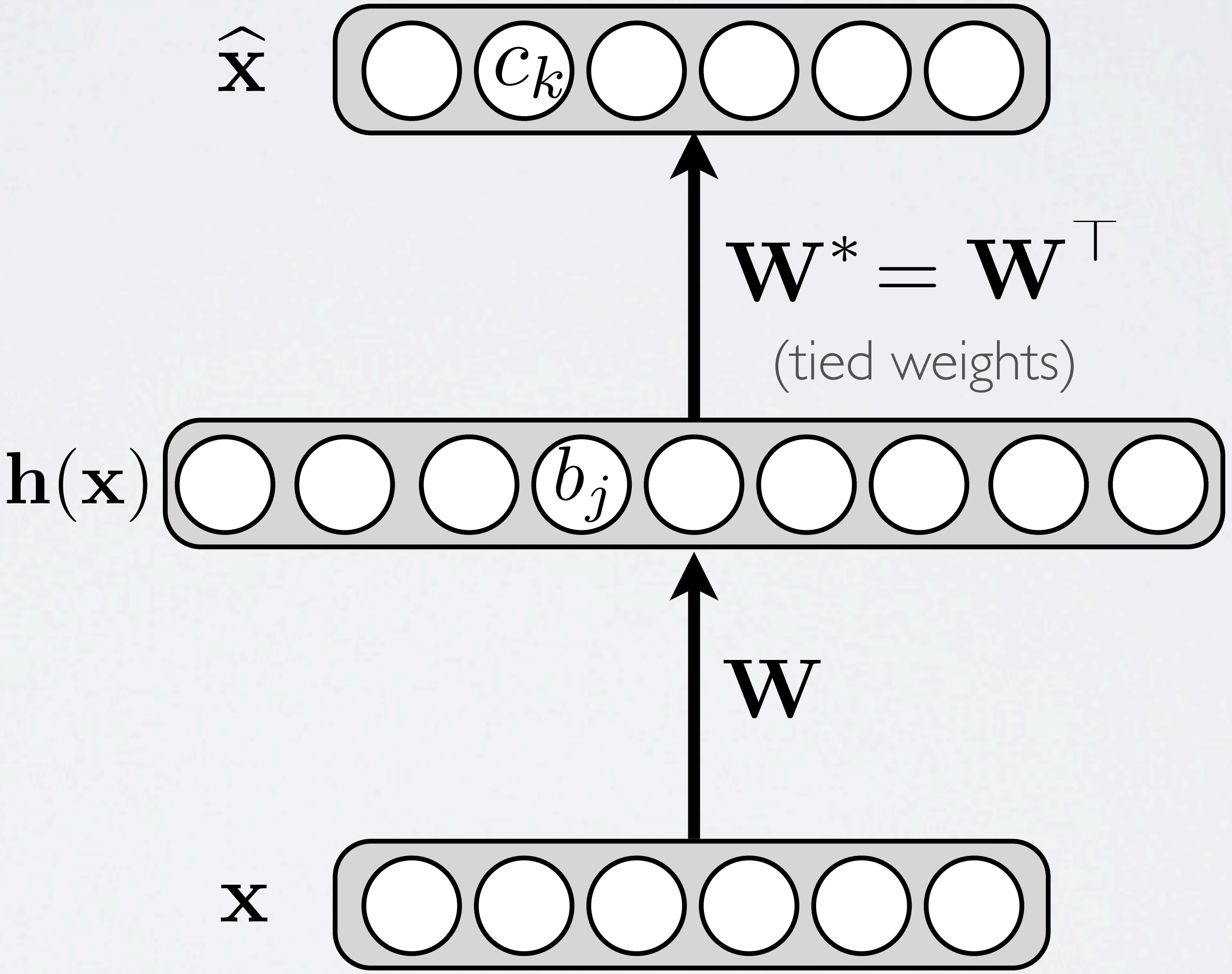
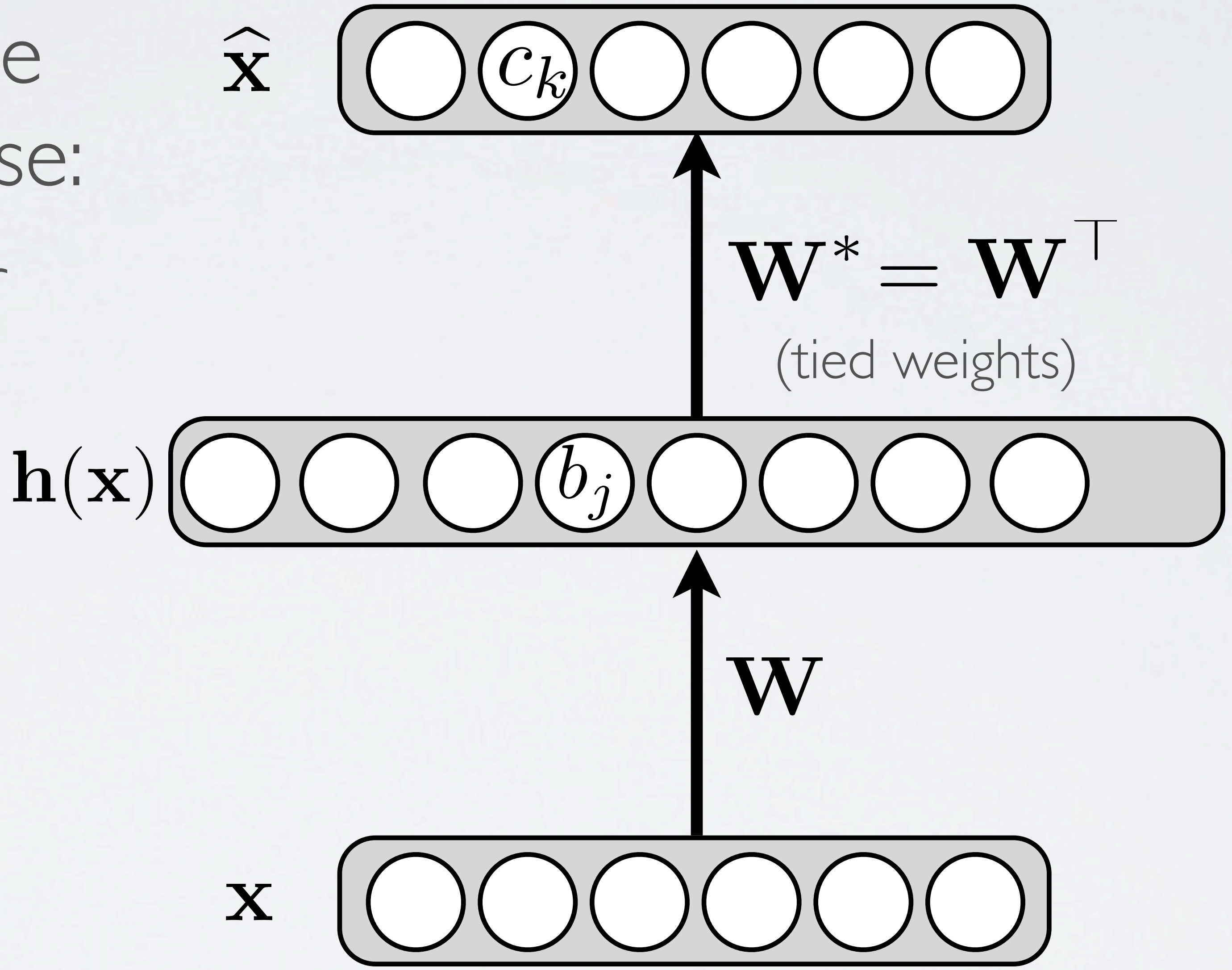


DENOISING AUTOENCODER

Topics: denoising autoencoder

- Idea: representation should be robust to introduction of noise:
  - random assignment of subset of inputs to 0, with probability ν
  - Gaussian additive noise
- Reconstruction $\widehat{\mathbf{x}}$ computed from the corrupted input $\widetilde{\mathbf{x}}$
- Loss function compares $\widehat{\mathbf{x}}$ reconstruction with the **noiseless input** $\mathbf{x}$



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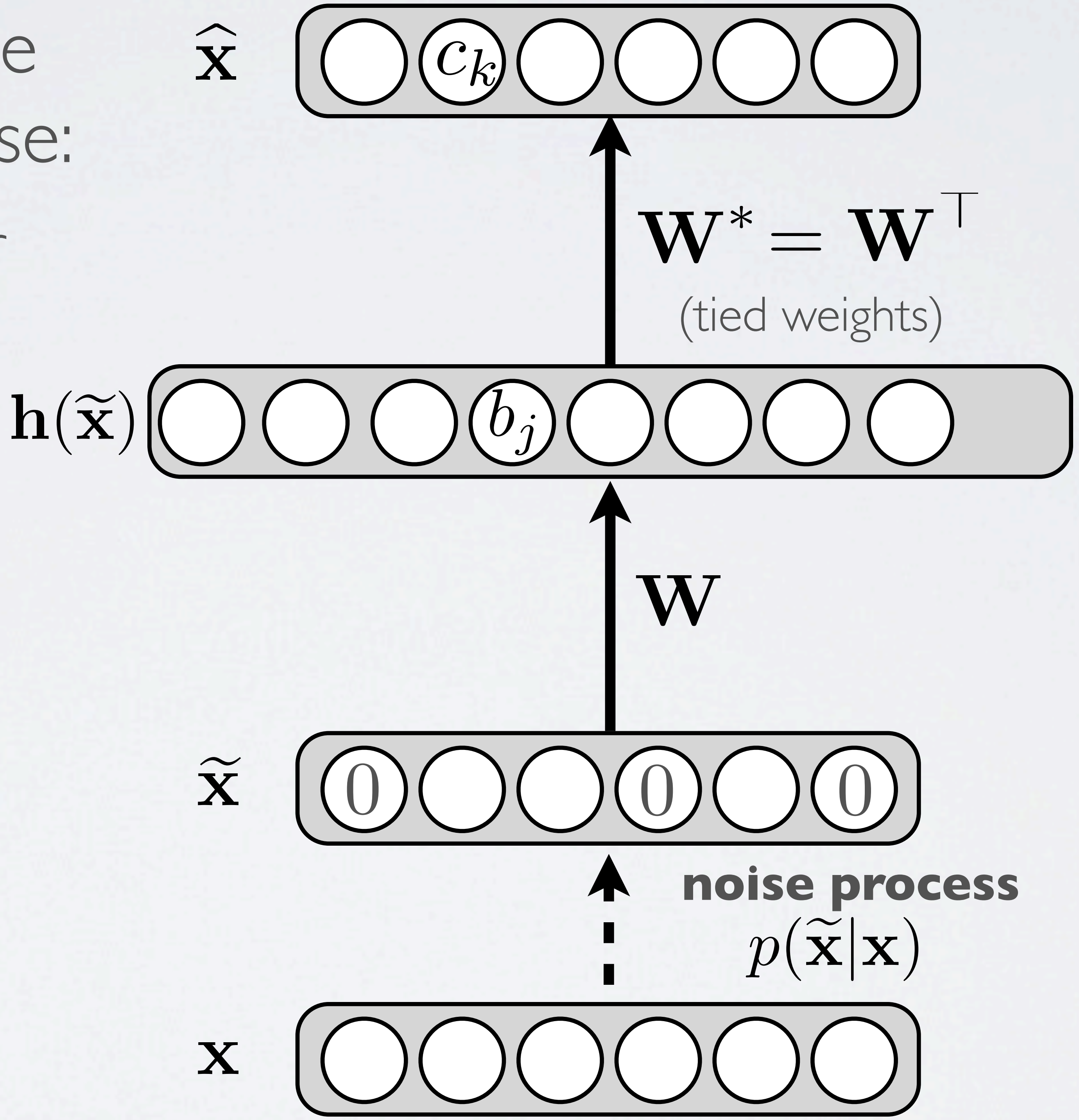

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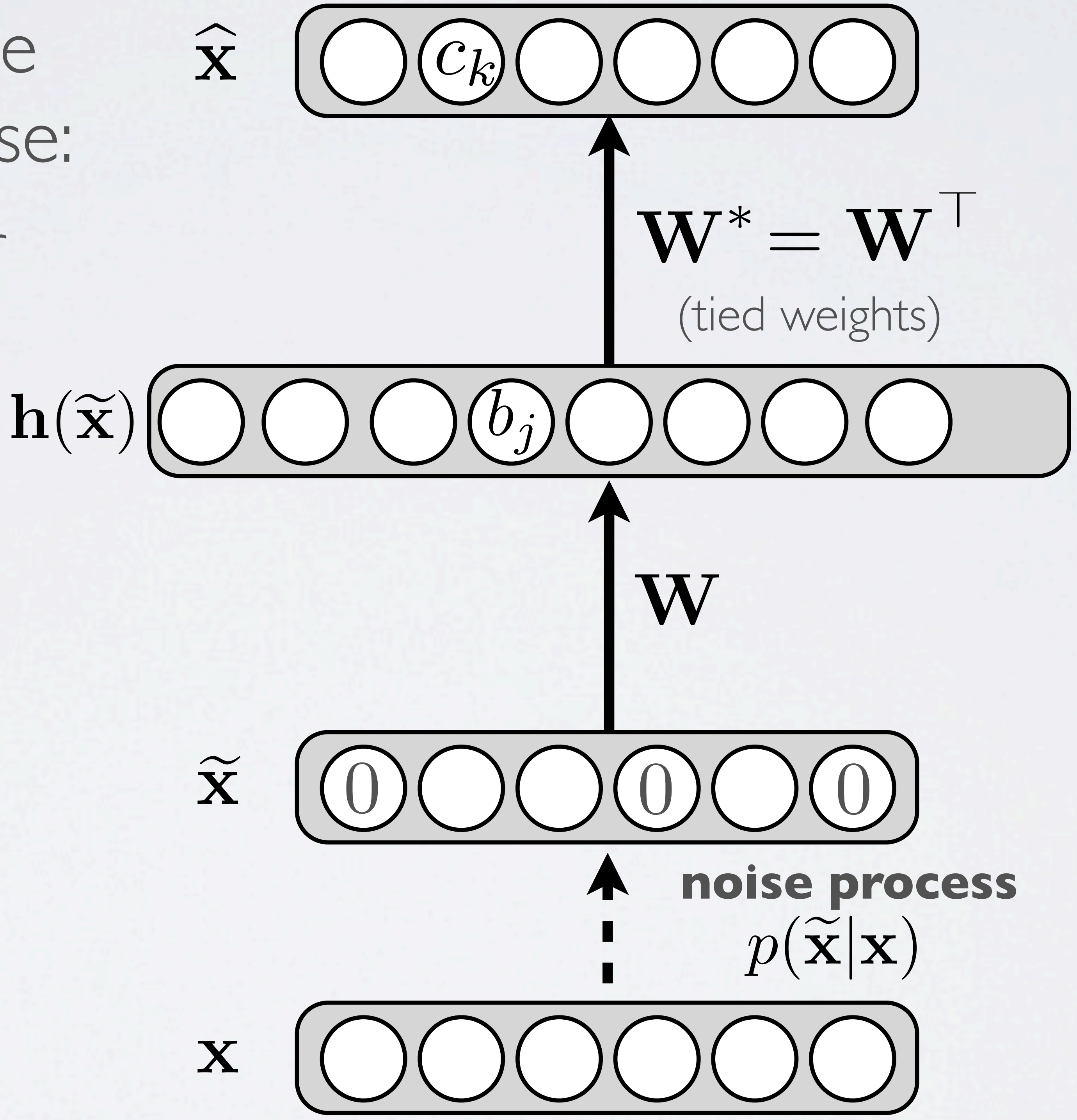
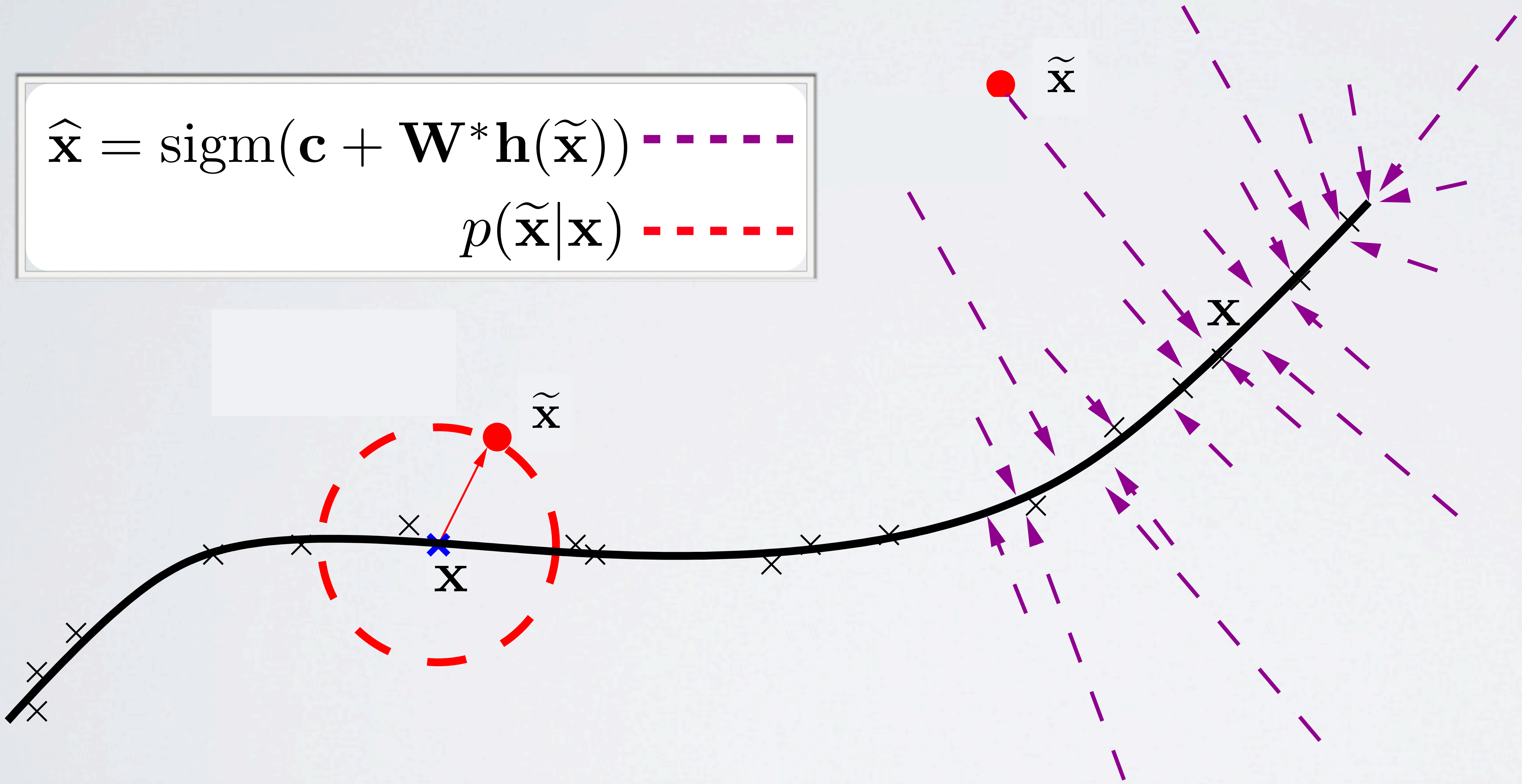


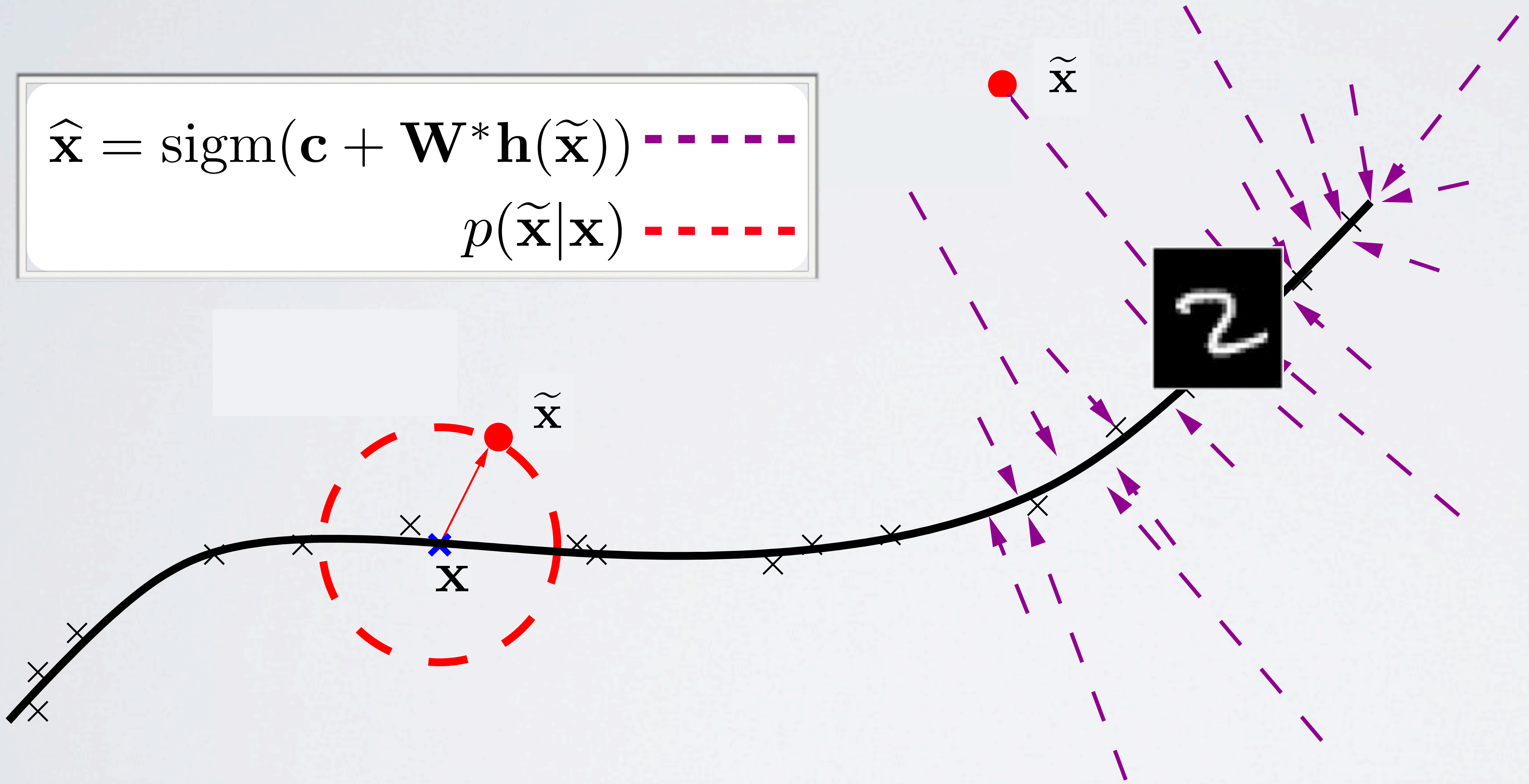
DENOISING AUTOENCODER



DENOISING AUTOENCODER

$$\widehat{\mathbf{x}} = \mathrm{sigm}(\mathbf{c} + \mathbf{W}^{*}\mathbf{h}(\widetilde{\mathbf{x}}))$$

$$p(\widetilde{\mathbf{x}}|\mathbf{x})$$



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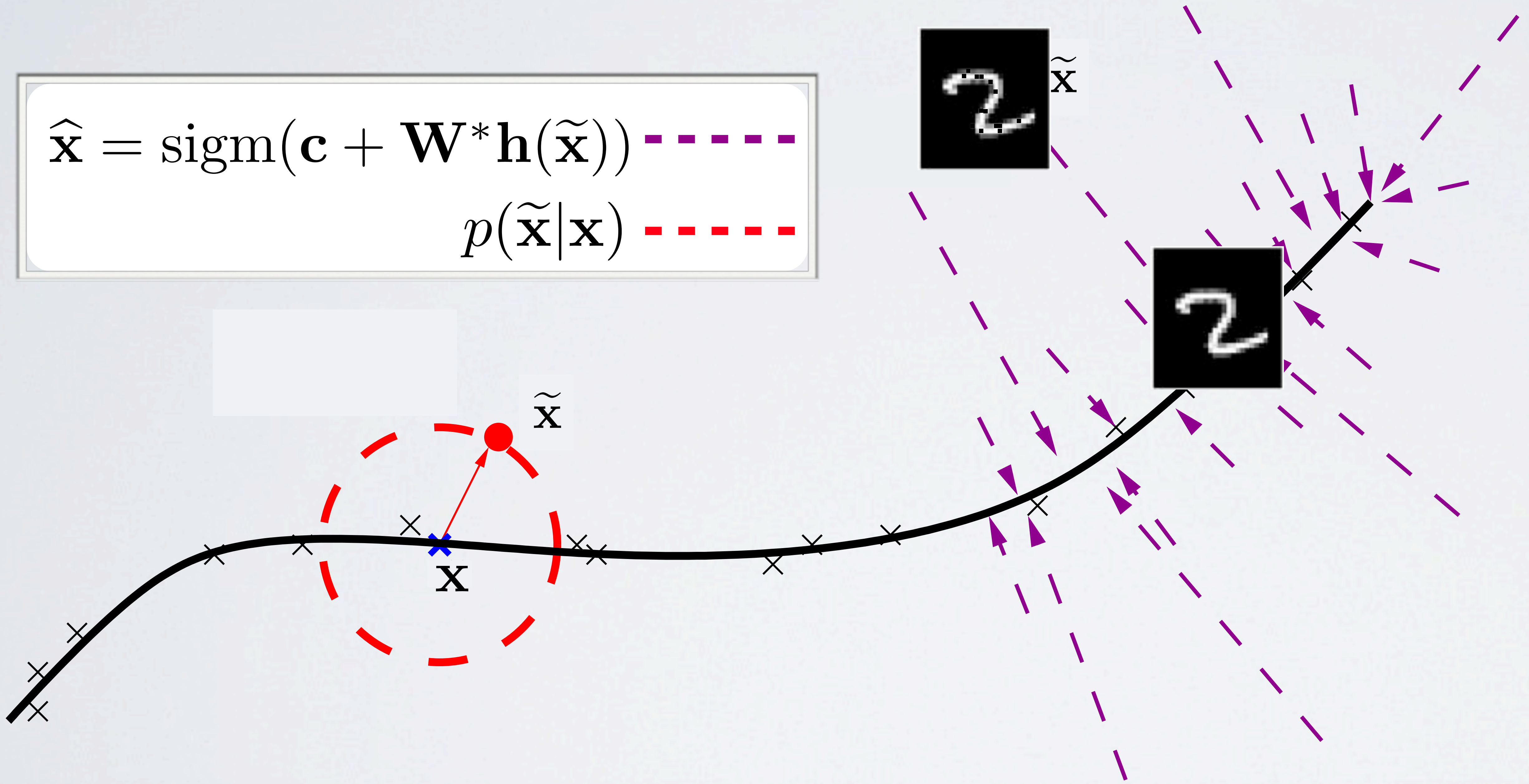

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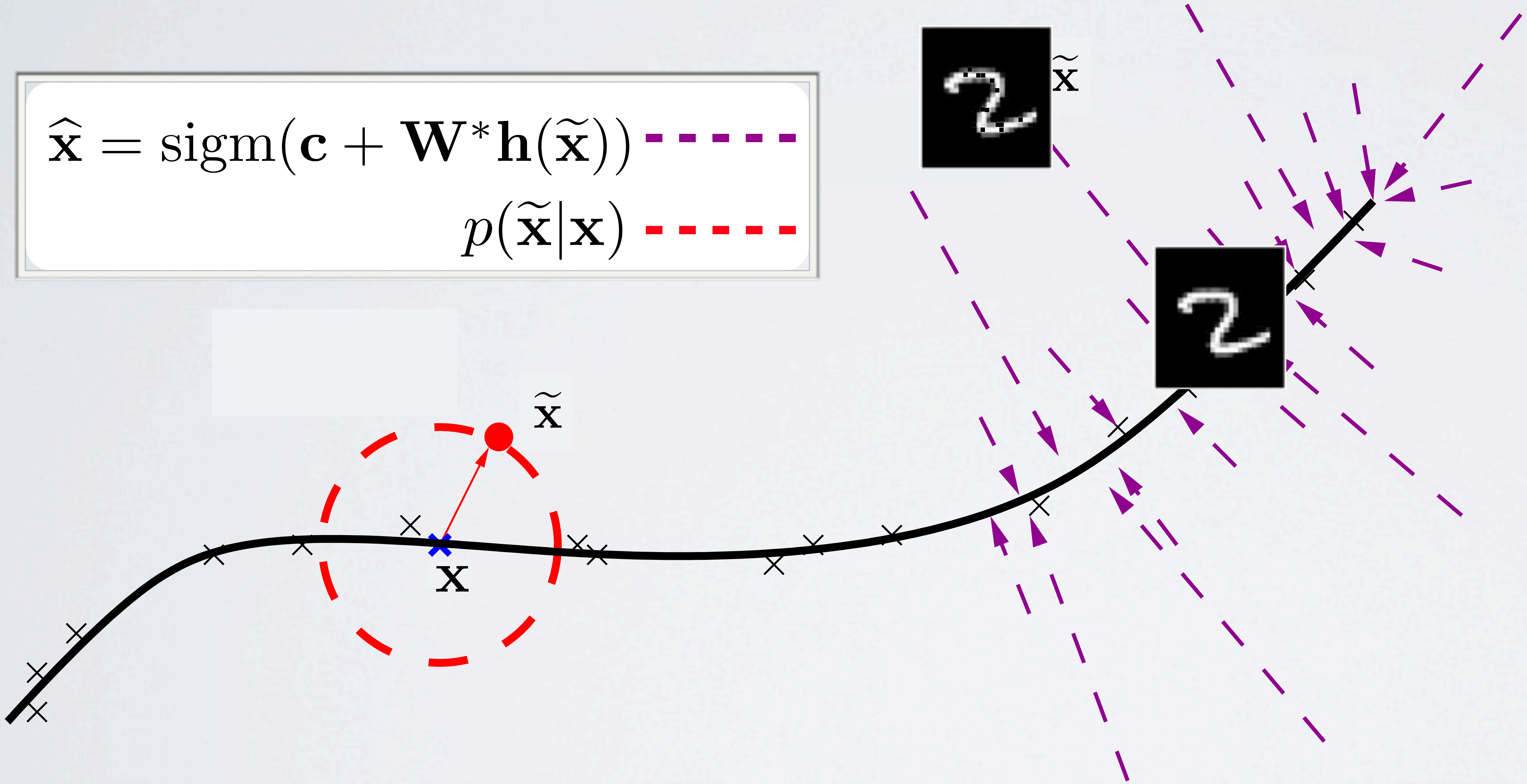
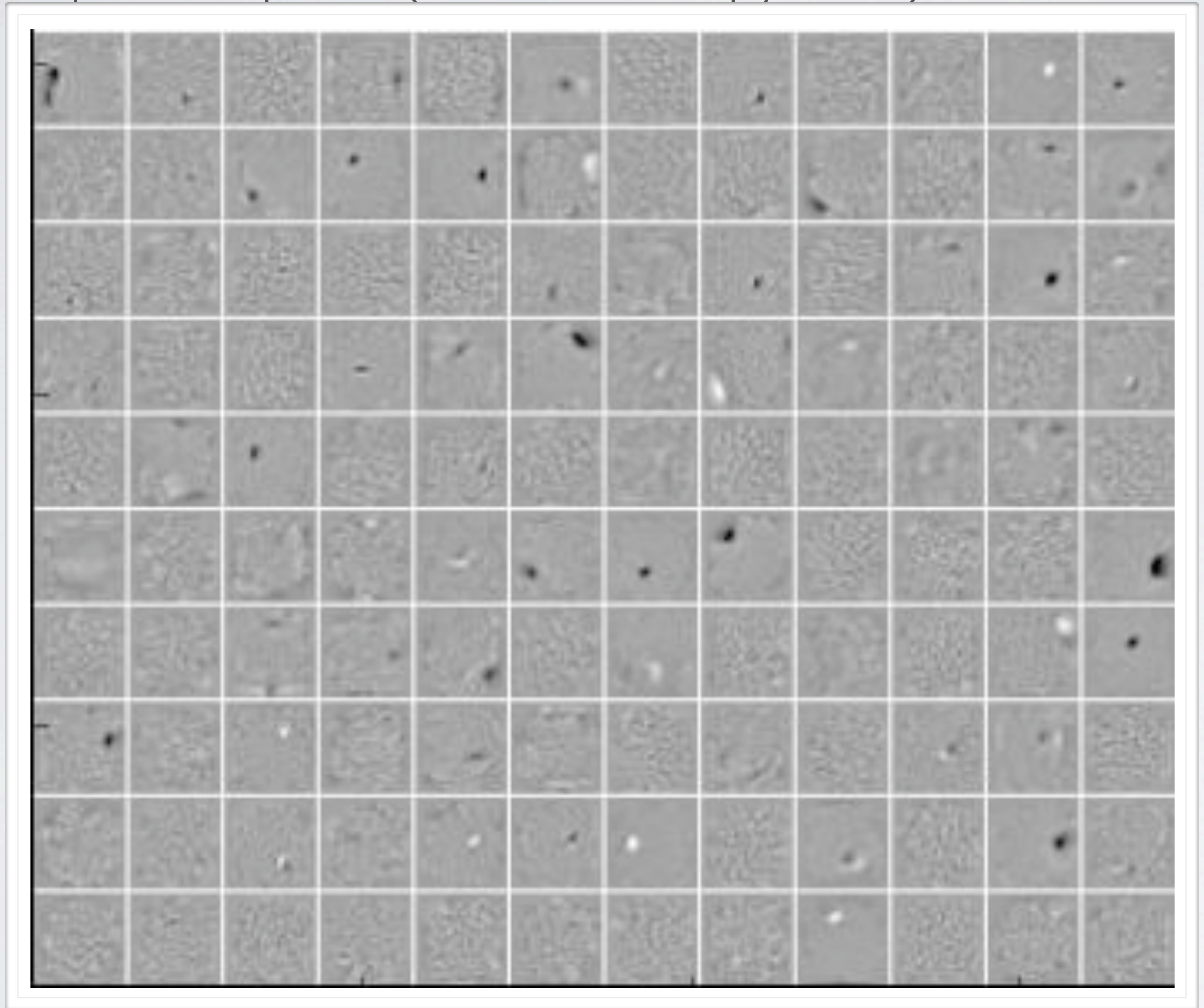


FILTERS (DENOISING AUTOENCODER)

(Vincent, Larochelle, Bengio and Manzagol, ICML 2008)

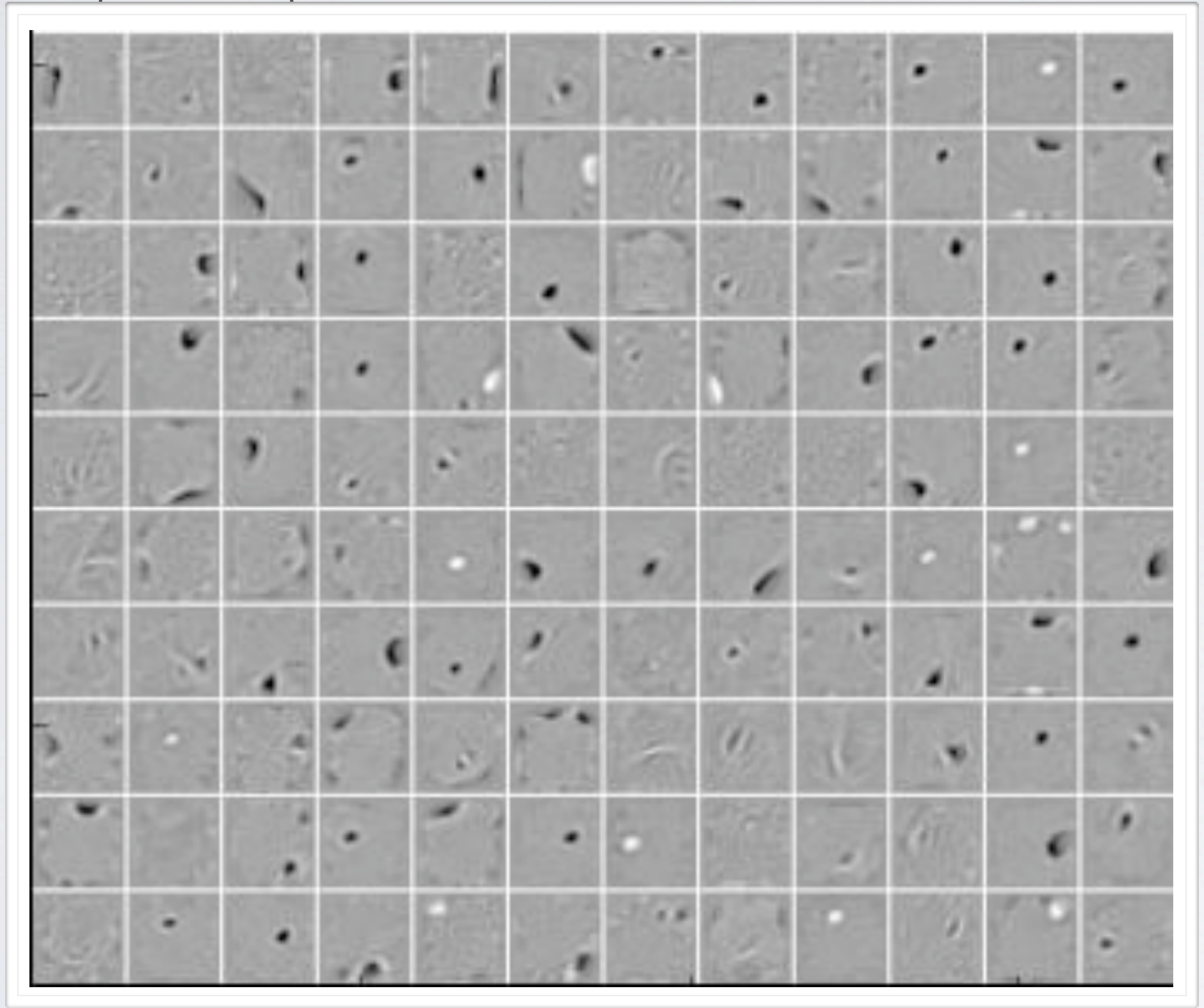
- No corrupted inputs (cross-entropy loss)



FILTERS (DENOISING AUTOENCODER)

(Vincent, Larochelle, Bengio and Manzagol, ICML 2008)

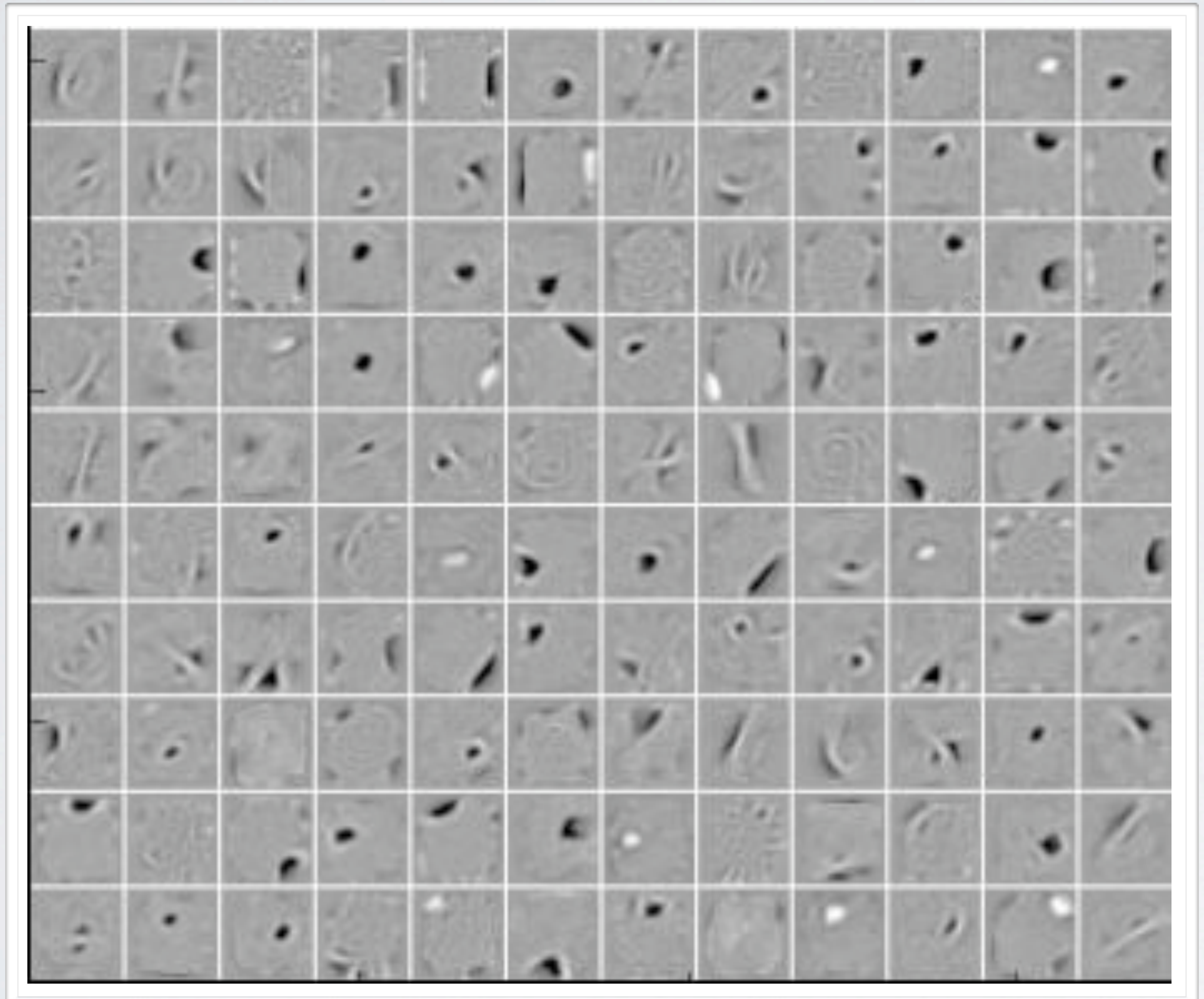
- 25% corrupted inputs



FILTERS (DENOISING AUTOENCODER)

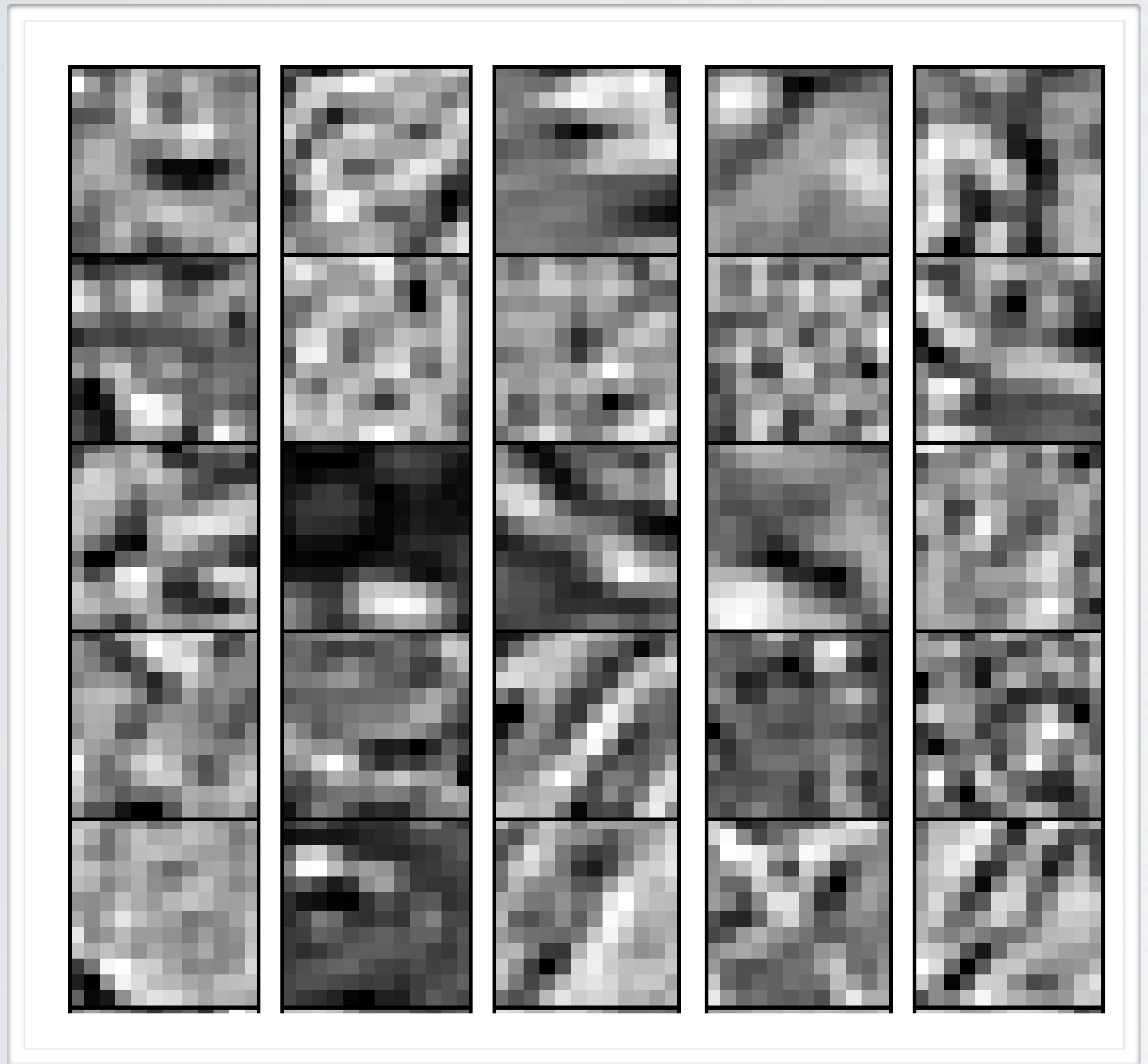
(Vincent, Larochelle, Bengio and Manzagol, ICML 2008)

- 50% corrupted inputs

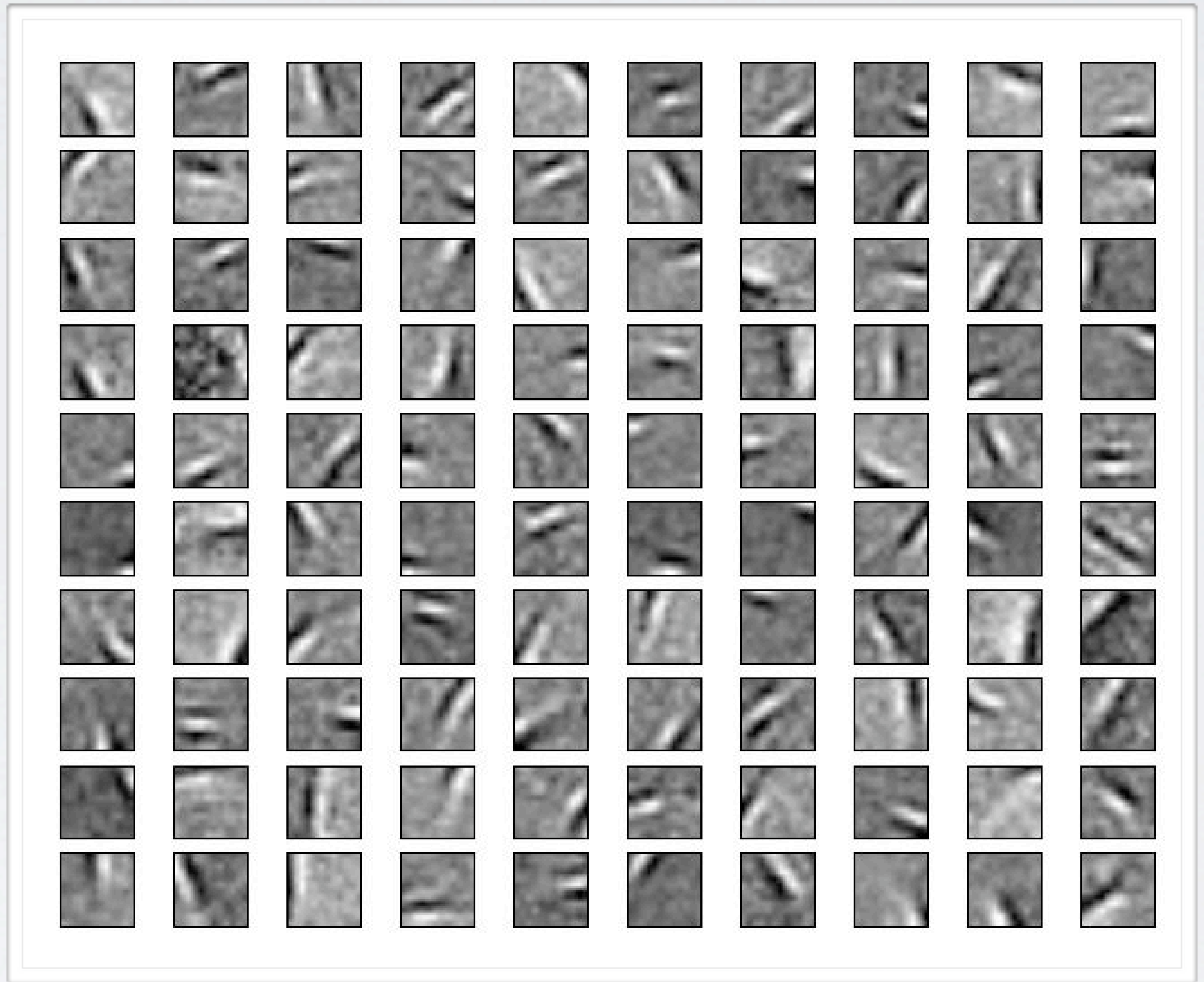


SQUARED ERROR LOSS

- Training on natural image patches, with squared-difference loss
  - PCA is not the best solution



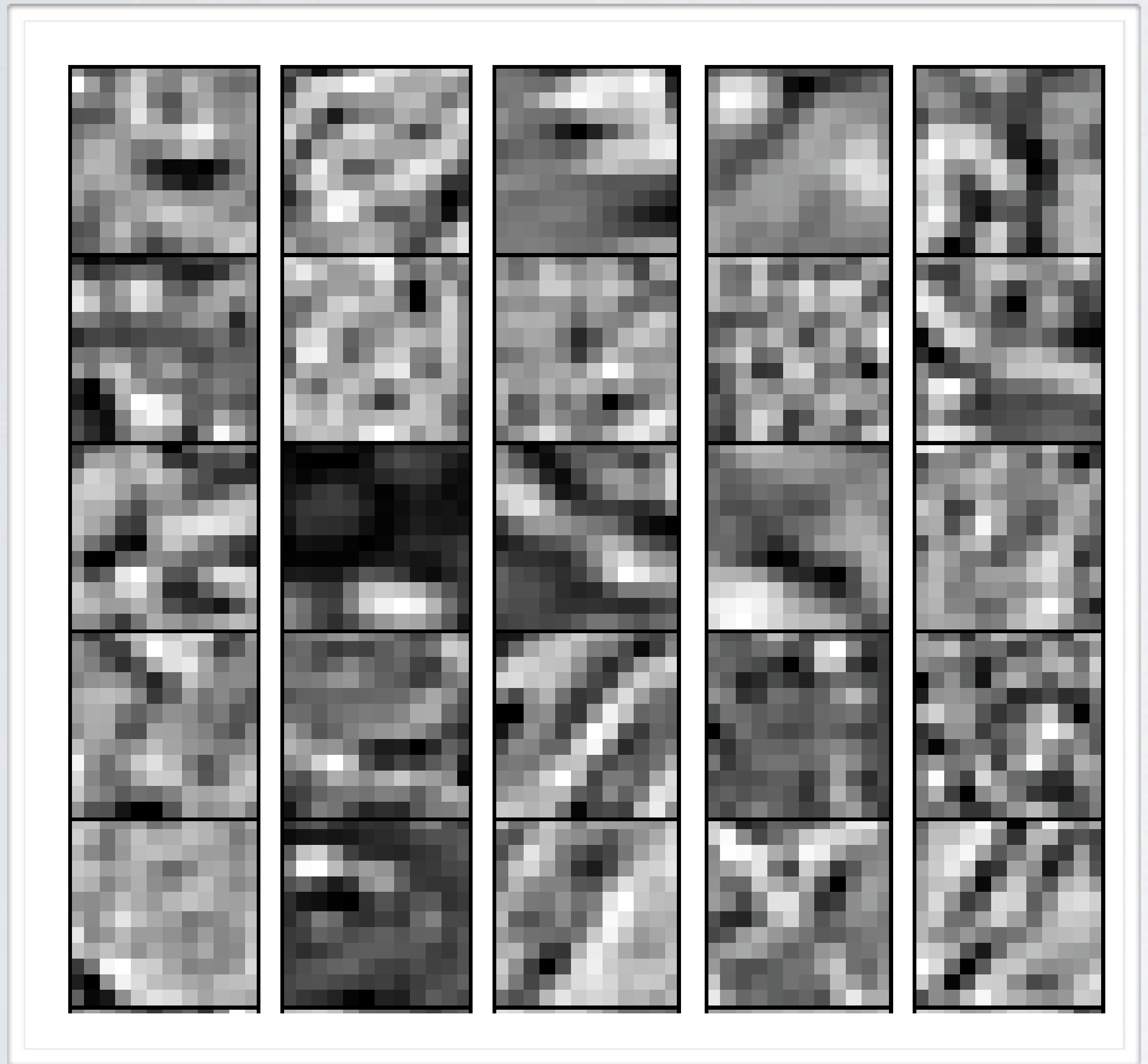
Data



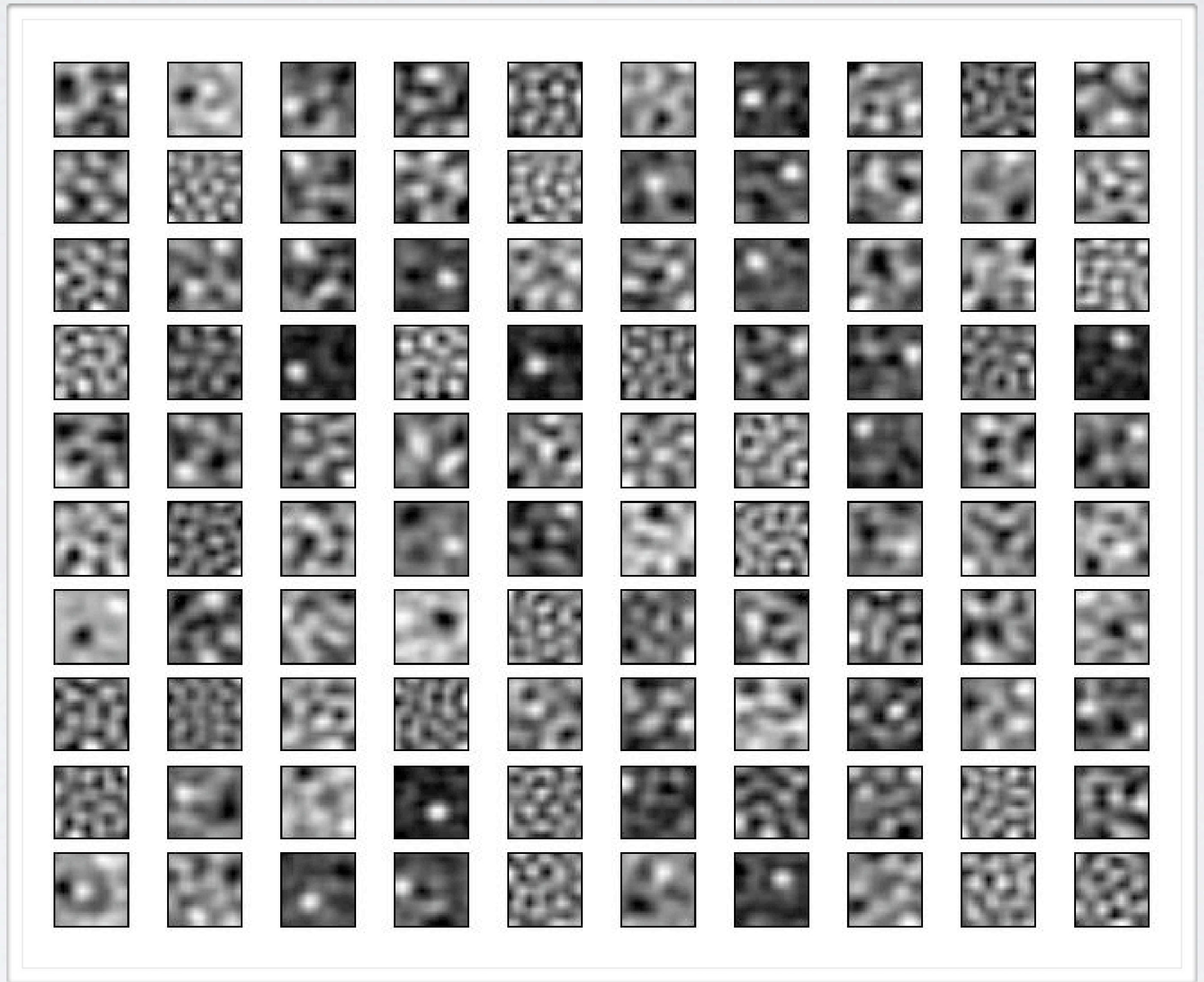
Filters

SQUARED ERROR LOSS

- Training on natural image patches, with squared-difference loss
  - Not equivalent to weight decay



Data



Filters

Neural networks

Autoencoder - contractive autoencoder

OVERCOMPLETE HIDDEN LAYER

Topics: overcomplete representation

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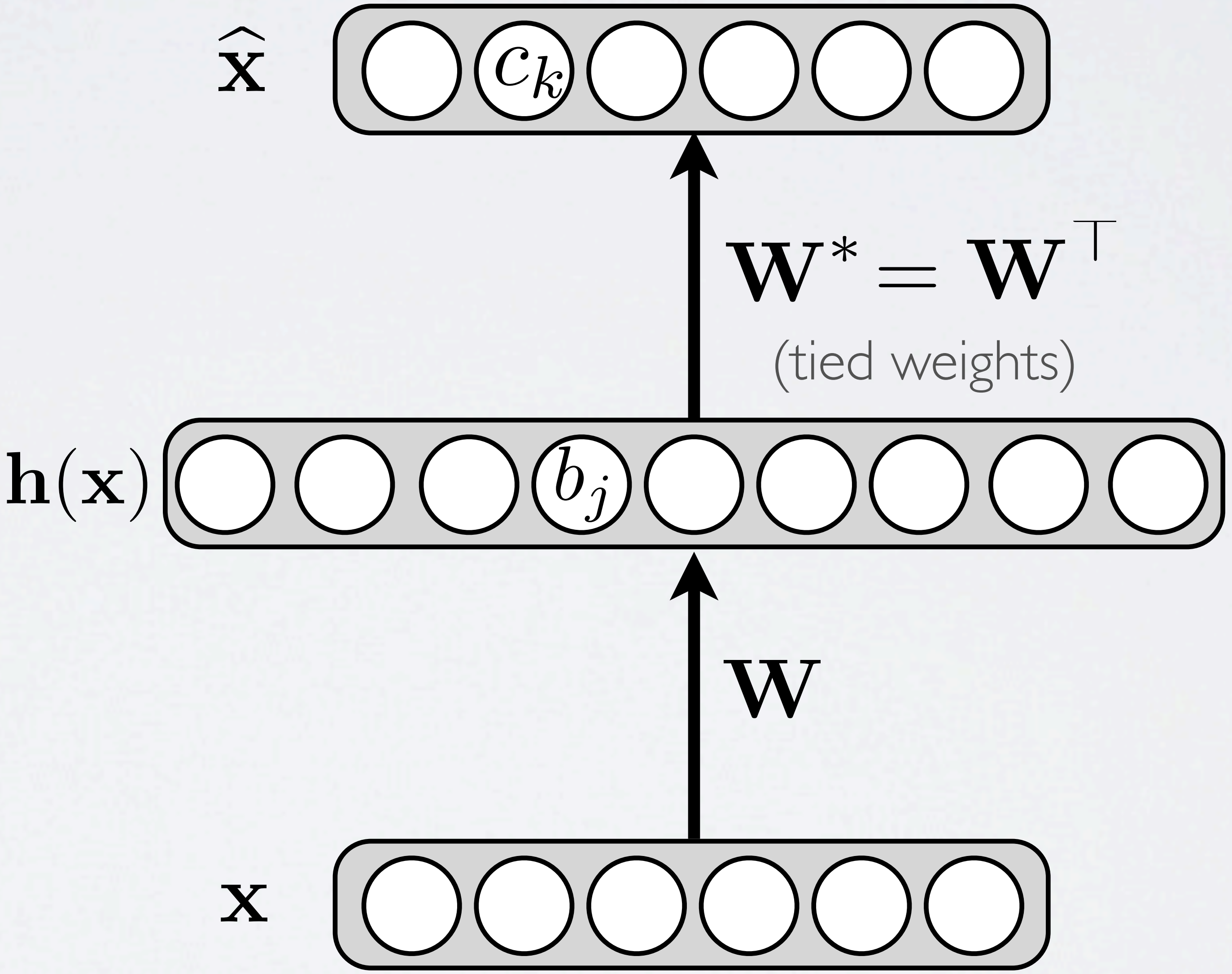
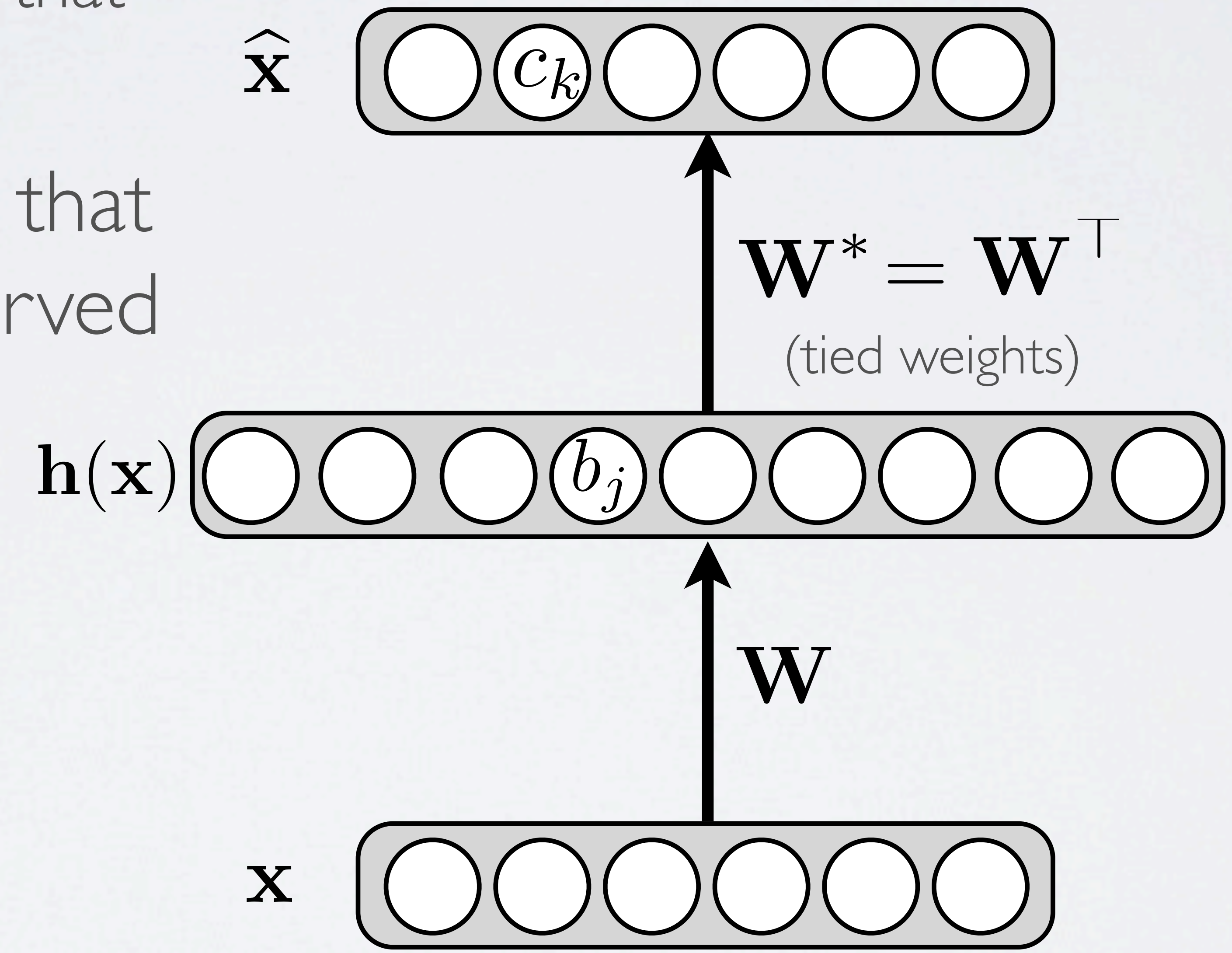


CONTRACTIVE AUTOENCODER

Topics: contractive autoencoder

- Alternative approach to avoid uninteresting solutions
  - add an explicit term in the loss that penalizes that solution
- We wish to extract features that **only** reflect variations observed in the training set
  - we'd like to be invariant to the other variations



CONTRACTIVE AUTOENCODER

Topics: contractive autoencoder

- New loss function:

$$\underbrace{l(f(\mathbf{x}^{(t)}))}_{\substack{\text{autoencoder}\\\text{reconstruction}}} + \lambda \underbrace{||\nabla_{\mathbf{x}^{(t)}} \mathbf{h}(\mathbf{x}^{(t)})||_F^2}_{\substack{\text{Jacobian of}\\\text{encoder}}}$$

  - where, for binary observations:

$$l(f(\mathbf{x}^{(t)})) = -\sum_k \left(x_k^{(t)} \log(\widehat{x}_k^{(t)}) + (1 - x_k^{(t)}) \log(1 - \widehat{x}_k^{(t)}) \right)$$

$$||\nabla_{\mathbf{x}^{(t)}} \mathbf{h}(\mathbf{x}^{(t)})||_F^2 = \sum_j \sum_k \left(\frac{\partial h(\mathbf{x}^{(t)})_j}{\partial x_k^{(t)}} \right)^2$$

CONTRACTIVE AUTOENCODER

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$$\underbrace{l(f(\mathbf{x}^{(t)}))}_{\text{autoencoder reconstruction}} + \lambda \underbrace{||\nabla_{\mathbf{x}^{(t)}} \mathbf{h}(\mathbf{x}^{(t)})||_F^2}_{\text{Jacobian of encoder}}$$

  - where, for binary observations:

$$l(f(\mathbf{x}^{(t)})) = -\sum_k \left(x_k^{(t)} \log(\widehat{x}_k^{(t)}) + (1 - x_k^{(t)}) \log(1 - \widehat{x}_k^{(t)}) \right)$$

encoder keeps good information

$$||\nabla_{\mathbf{x}^{(t)}} \mathbf{h}(\mathbf{x}^{(t)})||_F^2 = \sum_j \sum_k \left(\frac{\partial h(\mathbf{x}^{(t)})_j}{\partial x_k^{(t)}} \right)^2$$

encoder throws away all information

CONTRACTIVE AUTOENCODER

Topics: contractive autoencoder

- New loss function:

$$\underbrace{l(f(\mathbf{x}^{(t)}))}_{\text{autoencoder reconstruction}} + \lambda \underbrace{||\nabla_{\mathbf{x}^{(t)}} \mathbf{h}(\mathbf{x}^{(t)})||_F^2}_{\text{Jacobian of encoder}}$$

  - where, for binary observations:

$$l(f(\mathbf{x}^{(t)})) = -\sum_k \left(x_k^{(t)} \log(\widehat{x}_k^{(t)}) + (1 - x_k^{(t)}) \log(1 - \widehat{x}_k^{(t)}) \right)$$

encoder keeps good information

$$||\nabla_{\mathbf{x}^{(t)}} \mathbf{h}(\mathbf{x}^{(t)})||_F^2 = \sum_j \sum_k \left(\frac{\partial h(\mathbf{x}^{(t)})_j}{\partial x_k^{(t)}} \right)^2$$

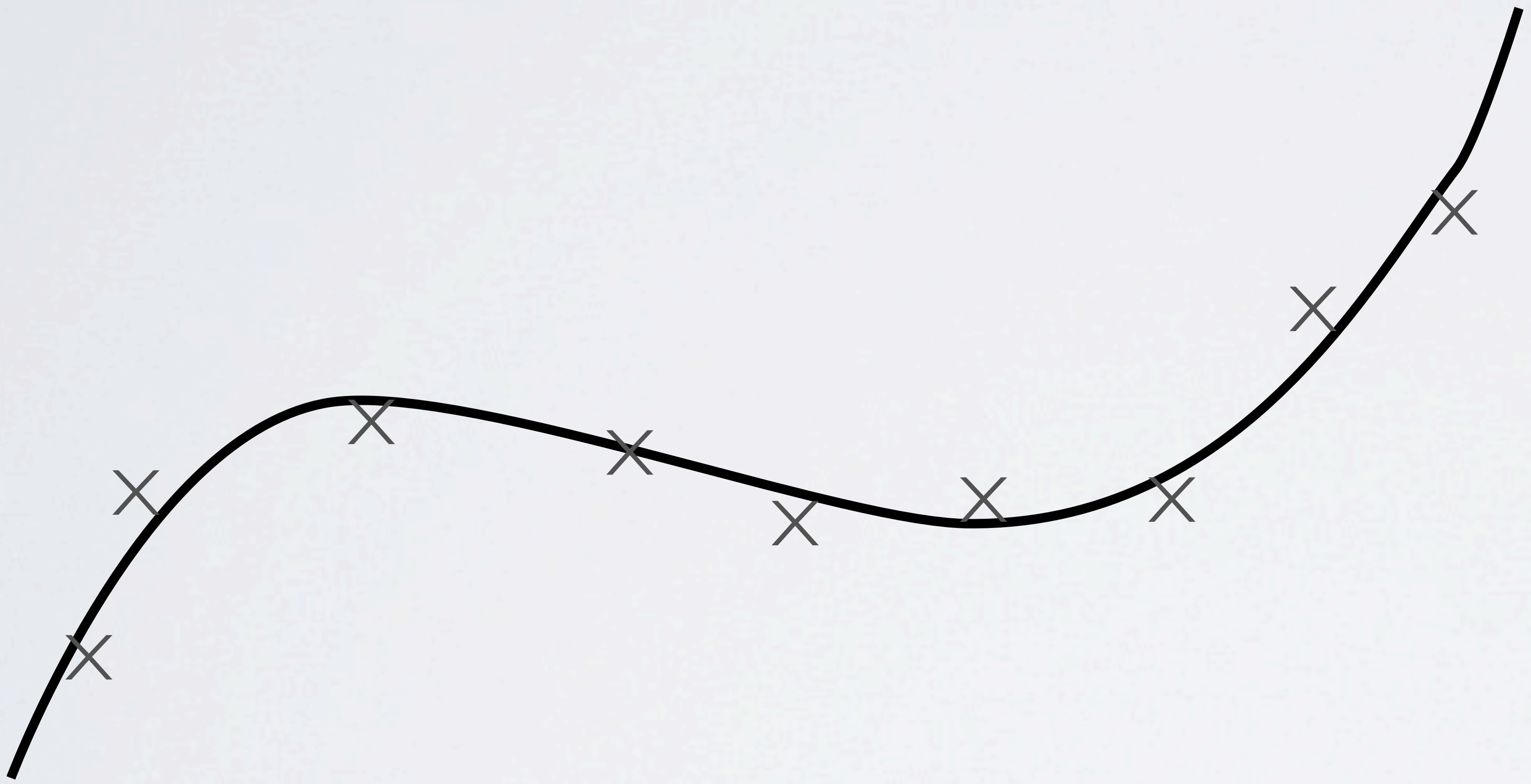
encoder throws away all information

encoder keeps **only** good information

CONTRACTIVE AUTOENCODER

Topics: contractive autoencoder

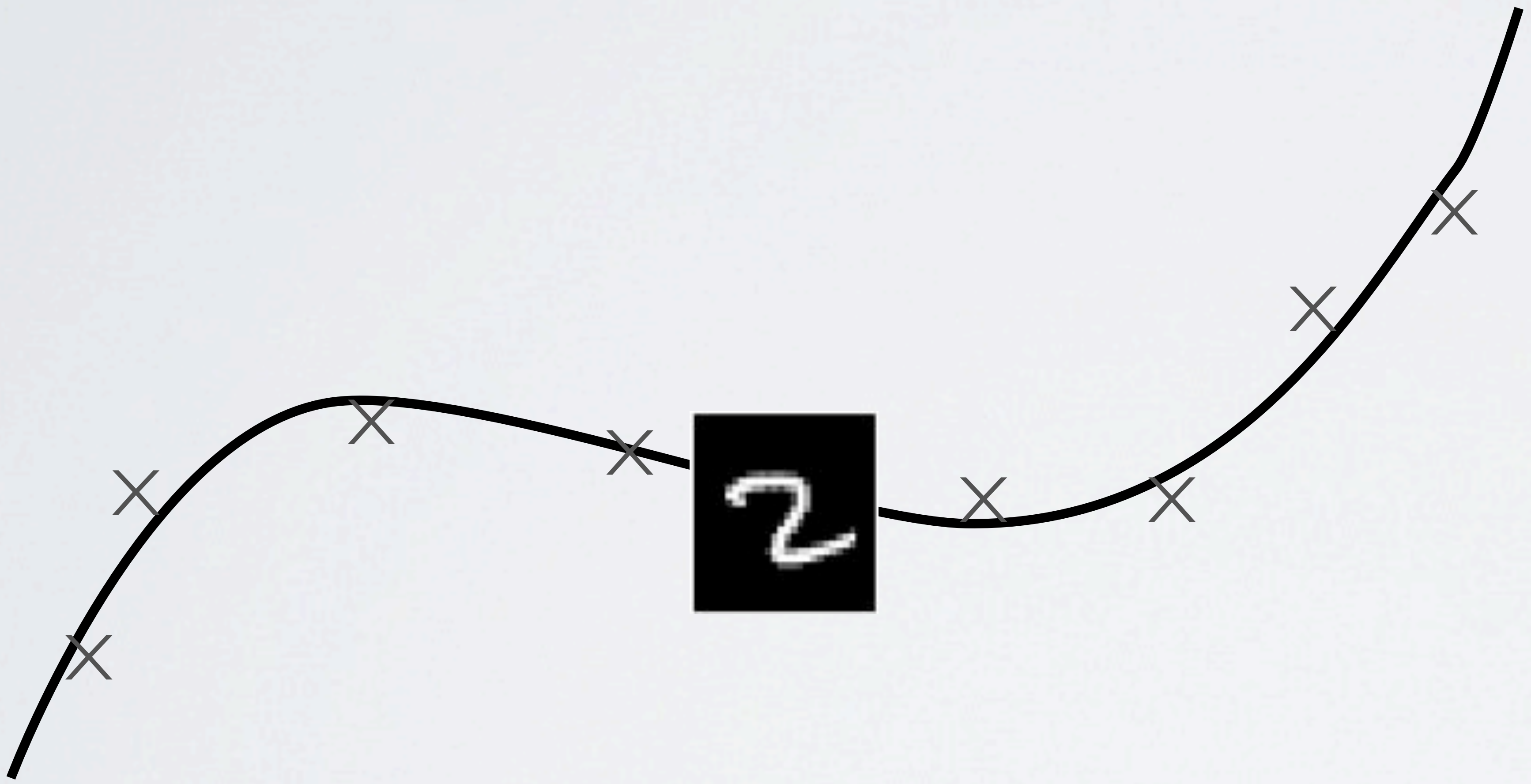
- Illustration:



CONTRACTIVE AUTOENCODER

Topics: contractive autoencoder

- Illustration:



CONTRACTIVE AUTOENCODER

Topics: contractive autoencoder

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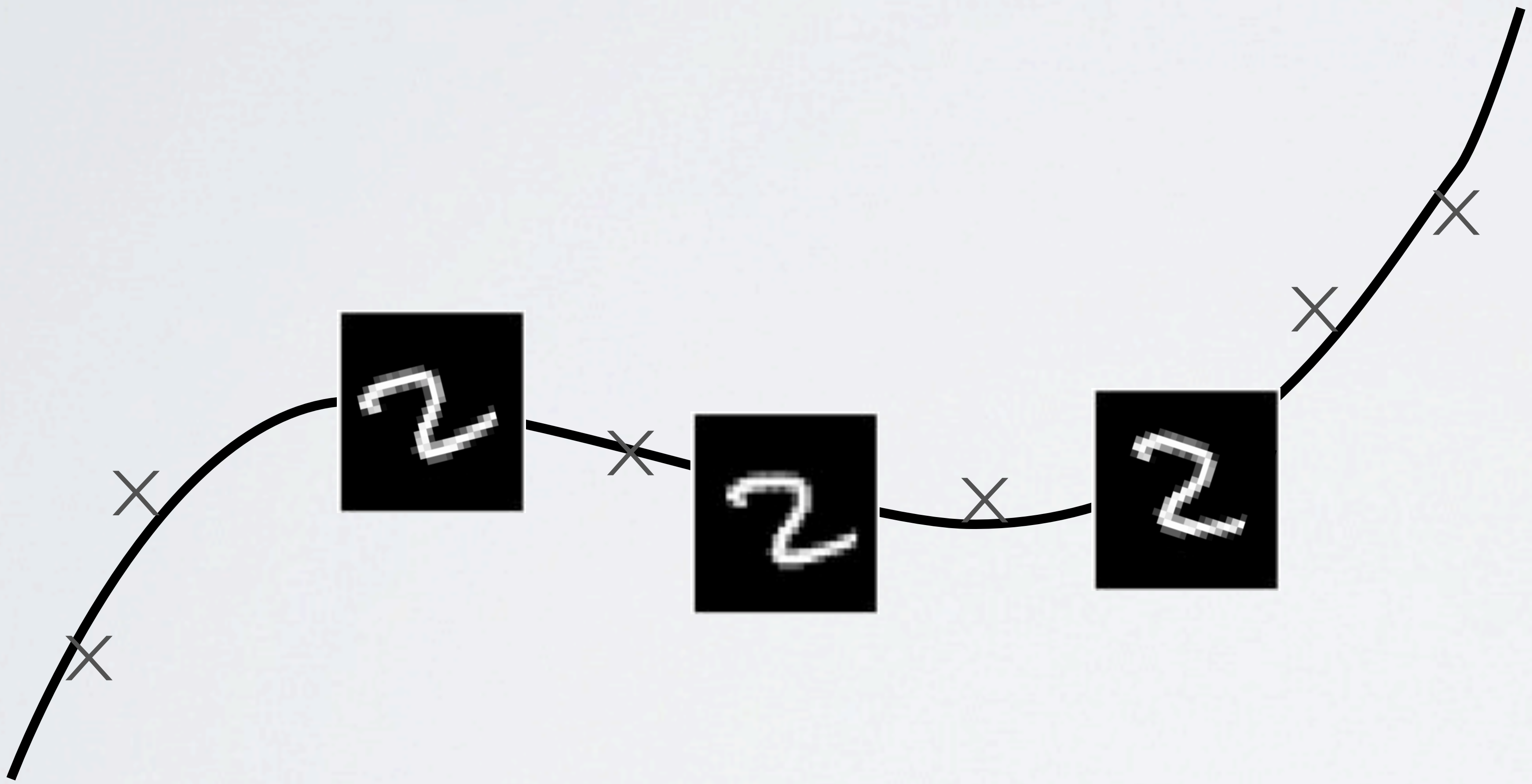

CONTRACTIVE AUTOENCODER

Topics: contractive autoencoder

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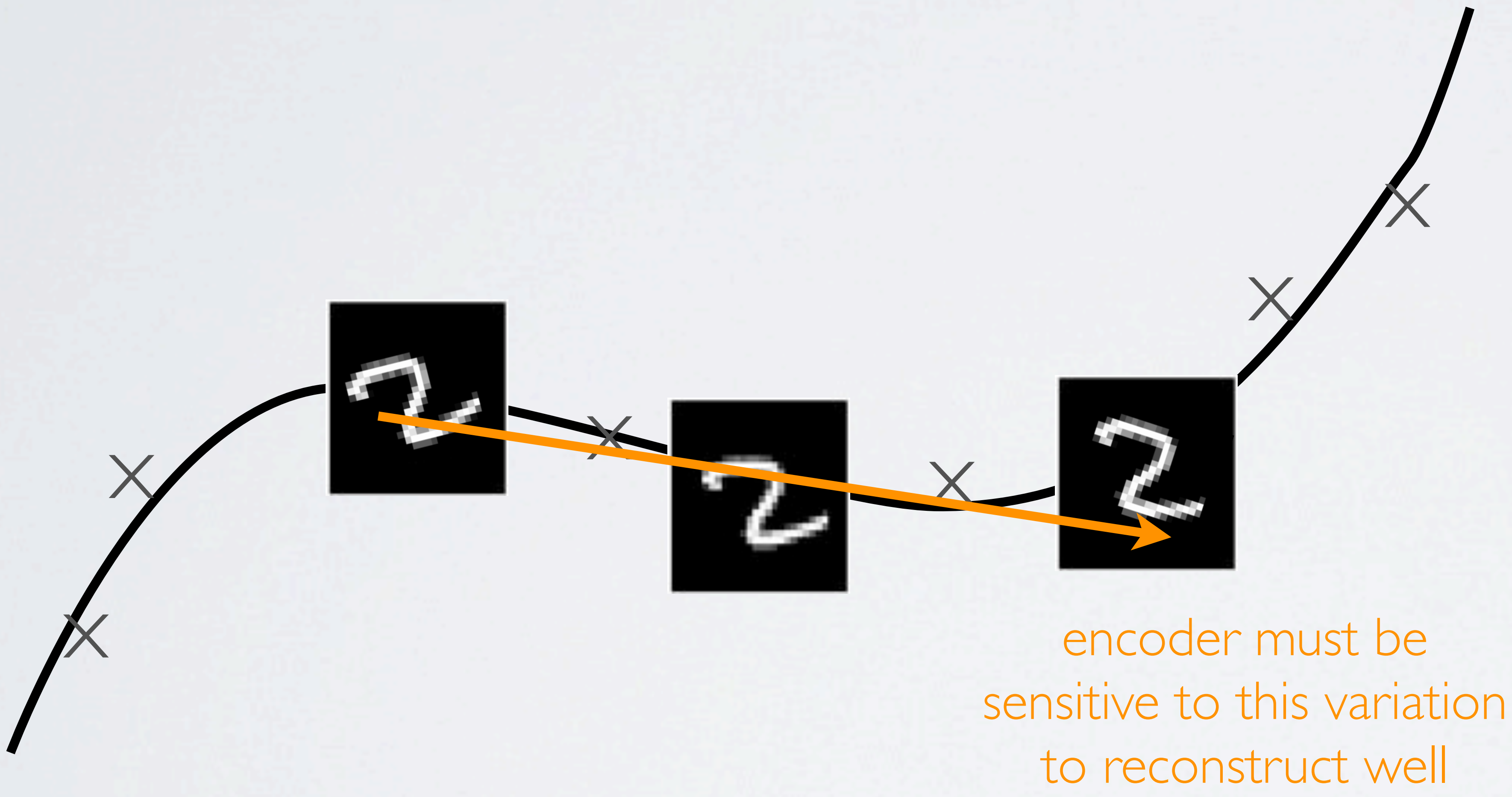
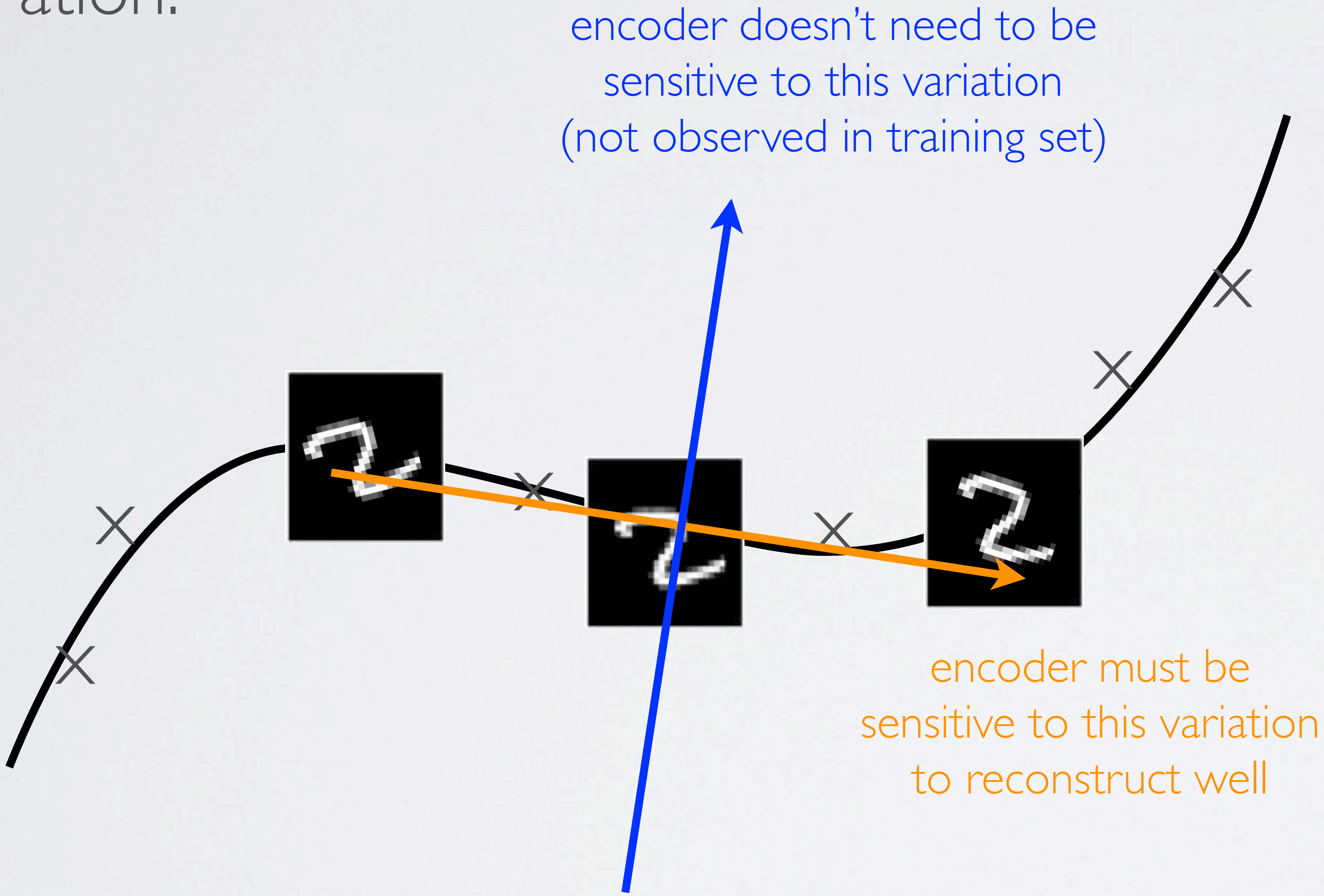


CONTRACTIVE AUTOENCODER

Topics: contractive autoencoder

- Illustration:



WHICH AUTOENCODER ?

Topics: denoising autoencoder, contractive autoencoder

- Both the denoising and contractive autoencoder perform well
  - Advantage of denoising autoencoder: simpler to implement
    - requires adding one or two lines of code to regular autoencoder
    - no need to compute Jacobian of hidden layer
  - Advantage of contractive autoencoder: gradient is deterministic
    - can use second order optimizers (conjugate gradient, LBFGS, etc.)
    - might be more stable than denoising autoencoder, which uses a sampled gradient
- To learn more on contractive autoencoders:
  - Contractive Auto-Encoders: Explicit Invariance During Feature Extraction.
    Salah Rifai, Pascal Vincent, Xavier Muller, Xavier Glorot et Yoshua Bengio, 2011.