MF 796
Computational Methods in Mathematical Finance
Take-Home Midterm

Professors Chris Kelliher & Eugene Sorets

April 5, 2021

Name (please print): ______________________________

Signature: ______________________________

Please work alone. You may refer to standard books and you may reuse the code *from your own homeworks.* Each problem has more than one solution. Please provide only one and explain all your steps and choices.

Please upload this exam as you normally upload a homework, together with the code you write by 9am Tuesday, April 6.

In the code, please mark clearly which (sub)problem each piece of code refers to.

We will be on Zoom (our usual class link) between 7:30am and 9am Monday in case there are questions about the exam. You are not required to attend this zoom session, but we do ask that you do your best to concentrate your questions to this session.

Should residual questions arise after the question period, please email both of us simultaneously at sorets@bu.edu and cmk44@bu.edu and we will do our best to respond in a timely manner. To give us proper time to respond, all questions should be submitted prior to 5pm ET.

Enjoy and good luck!

Eugene and Chris.

Problem 1 (20 points) The SDE for the CEV model for $\beta > 0$ is:

$$dS_t = r\, S_t dt + \sigma S_t^{\beta}\, dW_t$$

(a) (5 Points) Write the PDE that would result from using the CEV model.

(b) (3 Points) What are the parameters in the model and what is their interpretation?

(c) (2 Points) What parameters would lead this model to match the dynamics of the Black-Scholes model?

(d) (6 Points) Write a discretization of this equation which would lead to an implicit scheme.

(e) (4 Points) Discuss how you select the time and space grid.

Problem 2 (20 points) Let X be a random variable that denotes the default time of issuer A. Suppose that X is exponentially distributed:

$$P(X \leq t) = \begin{cases} 1 - e^{-\lambda_1 t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{1}$$

for some positive λ_1. Similarly, let Y denote the default time of issuer B and have distribution

$$P(Y \leq t) = \begin{cases} 1 - e^{-\lambda_2 t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2}$$

for $\lambda_2 > 0$. Now let Z denote the time until either issuer A or issuer B defaults, whichever happens first. Assume further that X and Y are independent. Then the CDF of Z is:

$$P(Z \leq t) = \begin{cases} 1 - e^{-(\lambda_1 + \lambda_2) t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{3}$$

(a) (4 Points) Under the assumption that X and Y are independent, show that CDF of Z in (3) is correct.

(b) (4 Points) Using the inverse transform technique, show how we can transform $\hat{U}$, a uniformly distributed random number between 0 and 1, to a random number with the CDF in (3).

(c) (4 Points) Let FTD (first to default) be the instrument that pays \$1 if the first default (among issuers A and B) occurs before the specified insurance period ends and \$0 otherwise.

If you knew λ_1 and λ_2 and could not solve the problem analytically, what method would you use to compute the price of the FTD numerically? Provide pseudocode for your solution.

(d) (4 Points) What happens to the price of the FTD when we increase λ_1? Explain your answer.

(e) (4 Points) Now suppose X and Y are no longer uncorrelated and have correlation ρ. What happens to the price of FTD when we increase ρ and hold all other parameters constant (including λ_1 and λ_2)? Explain your answer.

Problem 3 (20 points) Consider the following risk-neutral pricing formula for a European Digital Call:

$$c_0 = \mathbb{E}\left[e^{-rT} 1_{\{S_T > K\}}\right]$$

(a) (8 Points) Derive the pricing formula that can be used to price European Digital Call options using the characteristic function of a stochastic process.

NOTE: Your answer should have a single integral and be a function of the characteristic function and K.

(b) (6 Points) Describe two approaches for calculating the final integral and the trade-offs between them.

(c) (6 Points) Discuss how you would use this pricing formula to create an approximation for an American Digital put option. The payoff for an American Digital put option can be written as:

$$p_0 = \mathbb{E}\left[e^{-rT} 1_{\{M_T < K\}}\right]$$

where M_T is the minimum value for the asset over the observation period.

Comment on what factors would go into the accuracy of your approximation and whether the estimate would be reliable.

Problem 4 (20 points) Consider the following risk neutral pricing formula for a European put option:

$$p_0 = \mathbb{E}\left[(K - S)^+\right]$$

For purposes of this problem you may assume that interest rates are equal to zero and thus ignore discounting terms.

(a) (5 Points) Construct a butterfly centered on some strike, K^*, using only put options. Please include the strike of each option as well as the units and whether you should be long or short each put.

(b) (5 Points) Using the Breeden-Litzenberger technique, derive the relationship between the risk-neutral density of S and put options.

(c) (5 points) Why must put option prices be convex with respect to strike in the absence of arbitrage? Suppose you noticed that this condition was violated. Construct a riskless portfolio that would take advantage of this.

(d) (5 points) Describe the structure that you would trade if you believe that the market was mispricing the *Cumulative* Distribution Function of S. You should assume only European Calls and Puts are traded in your answer.

Problem 5 (20 points) For this problem please refer to the file `DataForProblem5.csv` in the same folder as this test.

Each column refers to a security and each row represent a day in history.

Each cell contains that day's return of that security in decimals.

We also view each day as a possible state of the world and assume that all these states are equally likely.

Also assume that your total wealth is $1 and no shorting is allowed.

Justify each of your answers by setting up the optimization problem you are planning to solve, explaining why you think this choice is the right one, and which method you are going to use to solve it.

(a) (4 Points) Using the history of the returns of the instruments Sec1 through Sec10 in `DataForProblem5.csv`, construct the fully-invested minimum variance portfolio.

(b) (4 Points) Find the fully-invested optimal portfolio, assuming your risk aversion coefficient is 0.5.

(c) (4 Points) Find the portfolio that would maximize the expected return.

(d) (4 Points) Find the fully-invested portfolio that would track the benchmark B1 best. Explain your choice of definition of "best" and compute the portfolio.

(e) (4 Points) Do you think the portfolio in (b) will be stable under small changes to expected returns? Please explain why or why not.