MF 790 HW 1 Part 2

This assignment is due on Thursday, September 16th at 8 AM. Problems 1 and 2 are 20 points each, and problem 3 is 10 points, for a total of 50 points.

1. Markov versus Martingale - Discrete Examples. We independently toss a fair coin twice. As such, the set of possible outcomes is

$$\Omega = \{HH, HT, TH, TT\}$$

Let $\mathcal{F}$ denote the set of all subsets of Ω. Construct the two period filtration $\mathbb{F}$ via

$$\mathcal{F}_0 = \{\emptyset, \Omega\}; \quad \mathcal{F}_1 = \{\emptyset, \Omega, A_H, A_T\}; \quad \mathcal{F}_2 = \mathcal{F},$$

where $A_H = \{HH, HT\}$ and $A_T = \{TH, TT\}$. Adapted to $\mathbb{F}$ we have the processes X and Y shown in Figure 1. Indicate if each of the following statements is true or false, and provide reasoning for your answer.

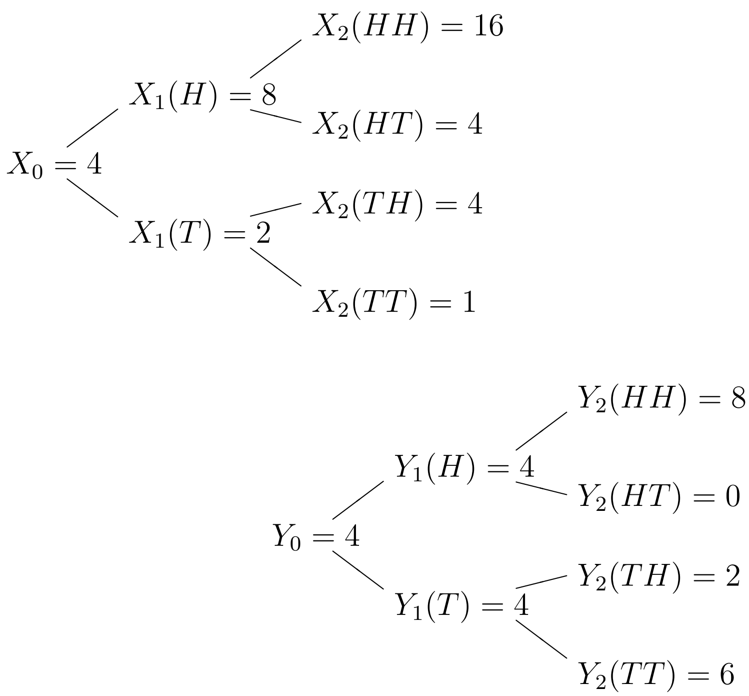

Figure 1. The process X and Y

(a) X is a martingale. If X is not a martingale, is there a way to adjust the probability of a heads (still assuming independent tosses) such that X is a martingale?
(b) X is Markov.
(c) Y is a martingale. If Y is not a martingale, is there a way to adjust the probability of a heads (still assuming independent tosses) such that Y is a martingale?
(d) Y is Markov.

(e) Now, assume the tosses are NOT independent coming from a fair coin. Rather, assume the generic form

$$\mathbb{P}\left[HH\right] = p_1, \quad \mathbb{P}\left[HT\right] = p_2, \quad \mathbb{P}\left[TH\right] = p_3, \quad \mathbb{P}\left[TT\right] = 1 - p_1 - p_2 - p_3,$$

for $0 < p_1, p_2, p_3 < 1$ such that $p_1 + p_2 + p_3 < 1$. Show that there are an *uncountable* number of (p_1, p_2, p_3) such that Y is a Martingale.

2. Additional Properties of Conditional Expectation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be given and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub σ-algebra of $\mathcal{F}$. Let X be a random variable such that $\mathbb{E}\left[X^2\right] < \infty$.

Let Y be *any* $\mathcal{G}$ measurable random variable such that $\mathbb{E}\left[Y^2\right] < \infty$. Show that

$$\mathrm{Var}\left[X - \mathbb{E}\left[X \middle| \mathcal{G}\right]\right] \leq \mathrm{Var}\left[X - Y\right].$$

Thus, $\mathbb{E}\left[X \middle| \mathcal{G}\right]$ is the $\mathcal{G}$ measurable random variable which is "closest" to X amongst all $\mathcal{G}$ measurable random variables Y in that $\mathrm{Var}\left[X - Y\right]$ is minimized at $Y = \mathbb{E}\left[X \middle| \mathcal{G}\right]$. **Hint:** Let $Z = Y - \mathbb{E}\left[X \middle| \mathcal{G}\right]$ and use TOWK evaluate

$$\mathrm{Var}\left[X - \mathbb{E}\left[X \middle| \mathcal{G}\right] - Z\right].$$

3. Functions of a Random Walk. (*Stochastic Calculus for Finance I*, Exercise 2.4(*ii*)) Let M_n be the random walk described in class : i.e. $M_0 = 0$ and $M_n = \sum_{j=1}^{n} Z_j$ where $(Z_j)_{j=1,2,\ldots}$ are i.i.d random variables taking the values ± 1 with equal probability. Next, define the process S by

$$S_n = e^{\sigma M_n} \cosh(\sigma)^{-n}, \qquad n = 0, 1, \ldots.$$

Show that S is a martingale. **Hint:** it suffices to prove the one period result. $\mathbb{E}\left[S_{n+1} \middle| \mathcal{F}_n\right] = S_n$