# MF 790 Stochastic Calculus
# Practice Midterm Exam. Fall 2020

This is the practice midterm exam. There are 3 questions for a total 100 points. Each question may contain multiple parts. You have 2 hours to complete the exam.

**To get the most out of the exam** please take this exam as if it were the real exam! I.e. do not use notes, the class textbook, the internet, and especially do not look at the solutions ahead of time! Give yourself two hours to take the exam, and abide by this rule!

**1. (30 Points)** Let $W$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

(a) **(10 Points)** Compute the differential $dX_t$ of the process $X_t = e^{tW_t}$ for $t \geq 0$.

(b) **(10 Points)** Compute the differential of $dX_t$ of the process $X_t = \frac{1}{1+W_t^2}$ for $t \geq 0$.

(c) **(10 Points)** Let $f(x)$ be a function and $\lambda$ be a number such that $f(0) = 1$ and $f''(x) = \lambda f(x)$ (for example, $f(x) = \cos(x), \lambda = -1$). Show, for general pairs $(f, \lambda)$ that $\mathbb{E}\left[f(W_t)\right] = e^{(\lambda/2)t}$. **Hint:** Find a constant $\gamma$ such that $X_t = e^{\gamma t} f(W_t), t \geq 0$ is a Martingale.

**2. (30 Points).** Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be associated to infinite coin toss space with independent tosses of a fair coin, so that for $j = 1, 2, \ldots$ the probability of a head on the $j^{th}$ toss is 1/2. Let $Z_j(\omega) = 1$ if $\omega_j = H$ ($j^{th}$ toss a head) and $-1$ otherwise, and create the random walk by $M_0 = 0$ and

$$M_n = \sum_{j=1}^{n} Z_j; \qquad n = 1, 2, \ldots.$$

Let $\Delta = \{\Delta_j\}_{j=1,2\ldots}$ be an adapted stochastic process such that for some $K > 0$, $|\Delta_j(\omega)| \leq K$ for all $j, \omega$. Consider the discrete time stochastic integral of $\Delta$ with respect to $M$ given by $I_0 = 0$ and

$$I_n = \sum_{j=1}^{n} \Delta_{j-1}(M_j - M_{j-1}); \qquad n = 1, 2, \ldots.$$

(a) **(15 Points)** Show that $I$ is a Martingale.

(b) **(15 Points)** Prove the discrete time Itô isometry

$$\mathrm{Var}\left[I_n\right] = \mathbb{E}\left[[I, I]_n\right].$$

**Hint:** For the Martingale property, it suffices to show $\mathbb{E}\left[M_n \middle| \mathcal{F}_{n-1}\right] = M_{n-1}$. I.e. you need only verify the martingale property one period at a time.

**3 (40 points).** Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be given and let $W$ be a Brownian motion. Recall the Black-Scholes model, where the stock $S$ and money market $B$ evolve according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t; \qquad \frac{dB_t}{B_t} = r dt; \qquad t \geq 0.$$

Here, $\mu \in \mathbb{R}, \sigma > 0, r > 0$ are constant. Next, let $T > 0$, and denote by $c(t, s)$ the price, at time $t \leq T$ and given $S_t = s$, of a call option which pays $(S_T - K)^+$ at time $T$. As we saw in class, $c(t, s)$ is given by the formula

$$\begin{aligned} c(t, s; r, \sigma^2) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(\widetilde{S}_T - K)^+ \middle| \widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right], \\ &= sN(d_+(T-t, s; r, \sigma^2)) - Ke^{-r(T-t)}N(d_-(T-t, s; r, \sigma^2)). \end{aligned}$$

In the first line above, $\widetilde{S}$ is geometric Brownian motion with parameters $r, \sigma^2$ under $\mathbb{Q}$. In the second line, $N$ is the cdf for a $N(0,1)$ random variable, and

$$d_{\pm}(\tau, s; r, \sigma^2) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{s}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right].$$

Note that we have made explicit the dependence of the call price on the parameters $r$ and $\sigma^2$.

Now, consider a call option on $S_T^2$ with strike $K$. In other words, at time $T$ the option pays

$$\hat{c}(T, S_T) = (S_T^2 - K)^+.$$

(a) **(10 Points)** If $\widetilde{S} \sim GBM(r, \sigma^2)$ under $\mathbb{Q}$ find $\widetilde{r}, \widetilde{\sigma}^2$ such that $\widetilde{S}^2 \sim GBM(\widetilde{r}, \widetilde{\sigma}^2)$ under $\mathbb{Q}$.

(b) **(15 Points)** Starting with the formula (which is always true) that the value of the "square call" is

$$\hat{c}(t, s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\left(\widetilde{S}_T^2 - K\right)^+ |\widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right],$$

express $\hat{c}$ in terms of the "regular" call option price function $c$, but with the modified $\tilde{r}, \tilde{\sigma}^2$, as well as any other correction terms which you may need.

(c) **(15 Points)** Identify the optimal hedging strategy $\Delta = \{\Delta_t\}_{t \leq T}$. Be as explicit as possible in your answer.