MF 790 HW 6, Part 2 - SOLUTIONS

This assignment is due on Thursday, December 2nd at 8 AM. Problem 1 is worth 30 points, for a total of 30 points.

1. Forwards and Futures Prices in the Hull-White Model. Recall in the Hull-White model with constant coefficients, we assume under $\mathbb{Q}$ that the money market rate process has dynamics

$$dR_t = \kappa(\theta - R_t)dt + adW_t^{\mathbb{Q}},$$

where $\kappa, a > 0$ and $\theta \in \mathbb{R}$. Note that have written a for the volatility. Next, let us assume that the risky asset S has $\mathbb{Q}$ dynamics

$$\frac{dS_t}{S_t} = R_t dt + \sigma dW_t^{\mathbb{Q}}.$$

where $\sigma > 0$.

(a) Using the bond-pricing result we derived in class, for a fixed $T > 0$ and for each $t \leq T$ identify the forward price For_t for a forward contract on S maturing at t. Be as explicit as possible in your answer.

(b) For $t \leq T$, identify the futures price Fut_t for a futures contract on S maturing at T. To do so, use the following steps

  (i) Prove the identity $\mathrm{Fut}_t = S_t \times \mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{\int_t^T R_u du} \big| \mathcal{F}_t\right]$ where

$$\frac{d\widehat{Z}_t}{\widehat{Z}_t} = \sigma dW_t^{\mathbb{Q}}, \widehat{Z}_0 = 1, \qquad \frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}} = \widehat{Z}_T.$$

  (ii) By *slightly* adjusting the bond pricing result we did in class for the Hull-White model (e.g. we have R instead of $-R$ and the coefficients κ, θ, a might change), explicitly identify

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{\int_0^T R_u du} \big| \mathcal{F}_t\right]$$

  and hence the futures price.

(c) How do the forward and futures prices compare? Is one always bigger than the other?

Solution

(a) From class we know that $B(t,T) = e^{-A(t)-C(t)R_t}$ where in the constant coefficient case

$$C(t) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right),$$

$$\begin{aligned} A(t) &= \left(\theta - \frac{a^2}{2\kappa^2}\right)(T-t) - \frac{1}{\kappa}\left(\theta - \frac{a^2}{\kappa^2}\right)\left(1 - e^{-\kappa(T-t)}\right) - \frac{a^2}{4\kappa^3}\left(1 - e^{-2\kappa(T-t)}\right). \\ &= \theta\left(T - t - \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\right) \\ &\quad - \frac{a^2}{2\kappa^2}\left(T - t - \frac{2}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) + \frac{1}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right)\right), \end{aligned}$$

where we write the last equality to isolate the dependence on θ. This yields the forward price

$$\mathrm{For}_t = \frac{S_t}{B(t,T)} = S_t e^{A(t)+C(t)R_t}.$$

(b) (i) Using that $S_T = S_t e^{\int_t^T R_u du}\widehat{Z}_T/\widehat{Z}_t$ we have

$$\mathrm{Fut}_t = \mathbb{E}^{\mathbb{Q}}\left[S_T \middle| \mathcal{F}_t\right] = S_t \mathbb{E}^{\mathbb{Q}}\left[e^{\int_t^T R_u du}\frac{\widehat{Z}_T}{\widehat{Z}_t}\middle|\mathcal{F}_t\right].$$

By the Bayes rule for conditional expectation under change of measure we know that a given $\mathcal{F}_T$ measurable random variable Y_T

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\left[Y_T \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[Y_T\frac{\widehat{Z}_T}{\widehat{Z}_t}\middle|\mathcal{F}_t\right].$$

Taking $Y_T = e^{\int_t^T R_u du}$ and using the definition of the futures price we see that

$$\mathrm{Fut}_t = S_t\mathbb{E}^{\mathbb{Q}}\left[e^{\int_t^T R_u du}\frac{\widehat{Z}_T}{\widehat{Z}_t}\middle|\mathcal{F}_t\right] = S_t\mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{\int_t^T R_u du}\middle|\mathcal{F}_t\right].$$

(ii) Note that under $\widetilde{\mathbb{P}}$ we have that $W_t^{\widetilde{\mathbb{P}}} = W_t^{\mathbb{Q}} - \sigma t, t \leq T$ is a Brownian motion. This means that R has $\widetilde{\mathbb{P}}$ dynamics

$$dR_t = \kappa(\theta - R_t)dt + a\left(dW_t^{\widetilde{\mathbb{P}}} + \sigma dt\right) = \kappa\left(\widetilde{\theta} - R_t\right)dt + a dW_t^{\widetilde{\mathbb{P}}}$$

where $\widetilde{\theta} = \theta + a\sigma/\kappa$. So, R is an OU process under $\widetilde{\mathbb{P}}$ as well. Next, the process $Y := -R$ has dynamics

$$dY_t = \kappa(-\widetilde{\theta} - Y_t)dt + a d(-W_t^{\widetilde{\mathbb{P}}}).$$

But, $-W^{\widetilde{\mathbb{P}}}$ is a $\widetilde{\mathbb{P}}$ Brownian motion. This gives

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{\int_t^T R_u du}\middle|\mathcal{F}_t\right] = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-\int_t^T Y_u du}\middle|\mathcal{F}_t\right] = e^{-\widetilde{A}(t)-\widetilde{C}(t)Y_t} = e^{-\widetilde{A}(t)+\widetilde{C}(t)R_t},$$

where $\widetilde{A}, \widetilde{C}$ are derived using the same formulas as in class, but now with $\kappa, -\widetilde{\theta}, a$ instead of κ, θ, a. Specifically, we have $\widetilde{C} = C$ from above, but

$$\widetilde{A}(t) = -\left(\theta + \frac{a\sigma}{\kappa}\right)\left(T - t - \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)\right)$$
$$- \frac{a^2}{2\kappa^2}\left(T - t - \frac{2}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) + \frac{1}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right)\right),$$

This yields the futures price

$$\mathrm{Fut}_t = S_t e^{-\widetilde{A}(t) + C(t)R_t}$$

(iii) We have that

$$\frac{\mathrm{For}_t}{\mathrm{Fut}_t} = e^{A(t) + \widetilde{A}(t)}.$$

But,

$$A(t) + \widetilde{A}(t) = -\frac{a\sigma}{\kappa^2}\left(\kappa(T-t) - \left(1 - e^{-\kappa(T-t)}\right)\right)$$
$$- \frac{a^2}{\kappa^3}\left(\kappa(T-t) - 2\left(1 - e^{-\kappa(T-t)}\right) + \frac{1}{2}\left(1 - e^{-2\kappa(T-t)}\right)\right).$$

For $x \geq 0$, consider the function

$$\varphi(x) := x - \left(1 - e^{-x}\right).$$

We have $\varphi(0) = 0$ and for $x > 0$, $\dot{\varphi}(x) = 1 - e^{-x} > 0$. This gives $\varphi(x) > 0$. Next, for $x \geq 0$, consider

$$\psi(x) := x - 2\left(1 - e^{-x}\right) + \frac{1}{2}\left(1 - e^{-2x}\right)$$

We have $\psi(0) = 0$ and for $x > 0$

$$\dot{\psi}(x) = 1 - 2e^{-x} + e^{-2x}; \qquad \ddot{\psi}(x) = 2e^{-x} - 2e^{-2x} > 0$$

This, ψ is convex on $(0, \infty)$ and since $\dot{\psi}(0) = 0$ we have that ψ is increasing on $(0, \infty)$ and hence $\psi(x) > 0$ for $x > 0$ as well. Therefore, as $a, \sigma, \kappa > 0$ we find that $A(t) + \widetilde{A}(t) < 0$ and hence the forward price is less than the futures price.