# MF 790 Stochastic Calculus
# Practice Midterm Exam. Fall 2020
# SOLUTIONS

This is the practice midterm exam. There are 3 questions for a total 100 points. Each question may contain multiple parts. You have 2 hours to complete the exam.

**To get the most out of the exam** please take this exam as if it were the real exam! I.e. do not use notes, the class textbook, the internet, and especially do not look at the solutions ahead of time! Give yourself two hours to take the exam, and abide by this rule!

**1. (30 Points)** Let $W$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

(a) **(10 Points)** Compute the differential $dX_t$ of the process $X_t = e^{tW_t}$ for $t \geq 0$.

(b) **(10 Points)**Compute the differential of $dX_t$ of the process $X_t = \frac{1}{1+W_t^2}$ for $t \geq 0$.

(c) **(10 Points)** Let $f(x)$ be a function and $\lambda$ be a number such that $f(0) = 1$ and $f''(x) = \lambda f(x)$ (for example, $f(x) = \cos(x), \lambda = -1$). Show, for general pairs $(f, \lambda)$ that $\mathbb{E}\left[f(W_t)\right] = e^{(\lambda/2)t}$. **Hint:** Find a constant $\gamma$ such that $X_t = e^{\gamma t} f(W_t), t \geq 0$ is a Martingale.

**Solution:**

(a) Let $f(t,x) = e^{tx}$. We have that $f_t(t,x) = xf(t,x)$, $f_x(t,x) = tf(t,x)$ and $f_{xx}(t,x) = t^2 f(t,x)$. Therefore, by Itô 's formula

$$\begin{aligned}\frac{dX_t}{X_t} &= \frac{df(t,W_t)}{f(t,W_t)} = W_t dt + t dW_t + \frac{1}{2}t^2 dt \\ &= (W_t + \frac{1}{2}t^2)dt + t dW_t.\end{aligned}$$

(b) Let $f(x) = \frac{1}{1+x^2}$. We have

$$f_x(x) = -\frac{2x}{(1+x^2)^2}; \quad f_{xx}(x) = \frac{6x^2-2}{(1+x^2)^3}.$$

Therefore, by Itô 's formula

$$dX_t = -\frac{2W_t}{(1+W_t)^2}dW_t + \frac{3W_t^2 - 1}{(1+W_t^2)^3}dt.$$

(c) As per the hint, let $X_t = e^{\gamma t} f(W_t)$ or, that $X_t = g(t, W_t)$ with $g(t,x) = e^{\gamma t} f(x)$. We have that $g_t(t,x) = \gamma e^{\gamma t} f(x)$, $g_x(t,x) = e^{\gamma t} f'(x)$ and $g_{xx}(t,x) = e^{\gamma t} f''(x)$. Therefore, by Itô we know

$$\begin{aligned}dX_t = dg(t,W_t) &= e^{\gamma t}\left(\gamma f(W_t)dt + f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt\right) \\ &= e^{\gamma t}\left(\gamma f(W_t)dt + f'(W_t)dW_t + \frac{1}{2}\lambda f(W_t)dt\right)\end{aligned}$$

where the second equality follows by assumption upon $f$. Now, setting $\gamma = -(1/2)\lambda$ we have

$$dX_t = e^{-(1/2)\lambda t} f'(W_t)dW_t$$

or, since $f(0) = 1$

$$X_t = 1 + \int_0^t e^{-(1/2)\lambda u} f'(W_u)dW_u$$

As $X$ is a Martingale we know

$$\mathbb{E}\left[\int_0^t e^{-(1/2)\lambda t} f'(W_t)dW_t\right] = 0$$

Therefore

$$\mathbb{E}\left[X_t\right] = \mathbb{E}\left[e^{-(1/2)\lambda t} f(W_t)\right] = 1$$

or, equivalently, that

$$\mathbb{E}\left[f(W_t)\right] = e^{(1/2)\lambda t}$$

**2. (30 Points).** Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be associated to infinite coin toss space with independent tosses of a fair coin, so that for $j = 1, 2, \ldots$ the probability of a head on the $j^{th}$ toss is $1/2$. Let $Z_j(\omega) = 1$ if $\omega_j = H$ ($j^{th}$ toss a head) and $-1$ otherwise, and create the random walk by $M_0 = 0$ and

$$M_n = \sum_{j=1}^{n} Z_j; \qquad n = 1, 2, \ldots.$$

Let $\Delta = \{\Delta_j\}_{j=1,2\ldots}$ be an adapted stochastic process such that for some $K > 0$, $|\Delta_j(\omega)| \leq K$ for all $j, \omega$. Consider the discrete time stochastic integral of $\Delta$ with respect to $M$ given by $I_0 = 0$ and

$$I_n = \sum_{j=1}^{n} \Delta_{j-1}(M_j - M_{j-1}); \qquad n = 1, 2, \ldots.$$

(a) **(15 Points)** Show that $I$ is a Martingale.
(b) **(15 Points)** Prove the discrete time Itô isometry

$$\text{Var}\left[I_n\right] = \mathbb{E}\left[[I, I]_n\right].$$

**Hint:** For the Martingale property, it suffices to show $\mathbb{E}\left[M_n \middle| \mathcal{F}_{n-1}\right] = M_{n-1}$. I.e. you need only verify the martingale property one period at a time.

**Solution:**

(a) First, note that $I$ is adapted since the quantities in the sum for $I_n$ only depend on $m \leq n$, and $\Delta, M$ are adapted. Next,

$$I_n - I_{n-1} = \Delta_{n-1}(M_n - M_{n-1}).$$

Thus, by TOWK and the Martingale property of $M$:

$$\begin{aligned}
\mathbb{E}\left[I_n - I_{n-1} \middle| \mathcal{F}_{n-1}\right] &= \mathbb{E}\left[\Delta_{n-1}(M_n - M_{n-1}) \middle| \mathcal{F}_{n-1}\right]; \\
&= \Delta_{n-1}\mathbb{E}\left[M_n - M_{n-1} \middle| \mathcal{F}_{n-1}\right]; \\
&= 0,
\end{aligned}$$

proving the result.

(b) For a given $n$ we have

$$\begin{aligned} I_n^2 &= \sum_{j=1}^{n} I_j^2 - I_{j-1}^2; \\ &= \sum_{j=1}^{n} (I_j - I_{j-1})^2 + 2\sum_{j=1}^{n} I_{j-1}(I_j - I_{j-1}); \\ &= [I, I]_n + 2\sum_{j=1}^{n} I_{j-1}(I_j - I_{j-1}). \end{aligned}$$

But, we have by the Tower, TOWK, and Martingale properties that

$$\begin{aligned} \mathbb{E}\left[I_{j=1}(I_j - I_{j-1})\right] &= \mathbb{E}\left[\mathbb{E}\left[I_{j-1}(I_j - I_{j-1})\middle|\mathcal{F}_{j-1}\right]\right] \\ &= \mathbb{E}\left[I_{j-1}\mathbb{E}\left[I_j - I_{j-1}\middle|\mathcal{F}_{j-1}\right]\right] \\ &= 0, \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{Var}\left[I_n\right] &= \mathbb{E}\left[I_n^2\right] - \left(\mathbb{E}\left[I_n\right]\right)^2; \\ &= \mathbb{E}\left[[I, I]_n\right] - \left(\mathbb{E}\left[I_0\right]\right)^2; \\ &= \mathbb{E}\left[[I, I]_n\right], \end{aligned}$$

where the second equality follows since $I$ is a Martingale and the third equality holds because $I(0) = 0$.

**3 (40 points).** Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be given and let $W$ be a Brownian motion. Recall the Black-Scholes model, where the stock $S$ and money market $B$ evolve according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t; \qquad \frac{dB_t}{B_t} = r dt; \qquad t \geq 0.$$

Here, $\mu \in \mathbb{R}, \sigma > 0, r > 0$ are constant. Next, let $T > 0$, and denote by $c(t, s)$ the price, at time $t \leq T$ and given $S_t = s$, of a call option which pays $(S_T - K)^+$ at time $T$. As we saw in class, $c(t, s)$ is given by the formula

$$\begin{aligned} c(t, s; r, \sigma^2) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(\widetilde{S}_T - K)^+\middle|\widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right], \\ &= sN(d_+(T-t, s; r, \sigma^2)) - Ke^{-r(T-t)}N(d_-(T-t, s; r, \sigma^2)). \end{aligned}$$

In the first line above, $\widetilde{S}$ is geometric Brownian motion with parameters $r, \sigma^2$ under $\mathbb{Q}$. In the second line, $N$ is the cdf for a $N(0, 1)$ random variable, and

$$d_{\pm}(\tau, s; r, \sigma^2) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{s}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right].$$

Note that we have made explicit the dependence of the call price on the parameters $r$ and $\sigma^2$.

Now, consider a call option on $S_T^2$ with strike $K$. In other words, at time $T$ the option pays

$$\hat{c}(T, S_T) = (S_T^2 - K)^+.$$

(a) **(10 Points)** If $\widetilde{S} \sim GBM(r, \sigma^2)$ under $\mathbb{Q}$ find $\widetilde{r}, \widetilde{\sigma}^2$ such that $\widetilde{S}^2 \sim GBM(\widetilde{r}, \widetilde{\sigma}^2)$ under $\mathbb{Q}$.

(b) **(15 Points)** Starting with the formula (which is always true) that the value of the "square call" is

$$\hat{c}(t, s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\left(\widetilde{S}_T^2 - K\right)^+ |\widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right],$$

express $\hat{c}$ in terms of the "regular" call option price function $c$, but with the modified $\tilde{r}, \tilde{\sigma}^2$, as well as any other correction terms which you may need.

(c) **(15 Points)** Identify the optimal hedging strategy $\Delta = \{\Delta_t\}_{t \leq T}$. Be as explicit as possible in your answer.

**Solution:**

(a) By Itô we know

$$d\widetilde{S}_t^2 = 2\widetilde{S}_t d\widetilde{S}_t + d[\widetilde{S}, \widetilde{S}]_t = 2\widetilde{S}^2\left(\mu dt + \sigma d\widetilde{W}_t\right) + \widetilde{S}^2\sigma^2 dt.$$

Thus, $\widetilde{S}^2 \sim GBM(\widetilde{r}, \widetilde{\sigma}^2)$ under $\mathbb{Q}$ with $\widetilde{r} = 2r + \sigma^2$ and $\widetilde{\sigma} = 2\sigma$.

(b) We know that

$$\hat{c}(t, s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\left(\widetilde{S}_T^2 - K\right)^+ |\widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right],$$

Write $\widetilde{Y}_t := \widetilde{S}_t$ so that $\widetilde{Y} \sim GBM(\widetilde{r}, \widetilde{\sigma}^2)$ under $\mathbb{Q}$. We can write the above as

$$\hat{c}(t, s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\left(\widetilde{Y}_T - K\right)^+ |\widetilde{\mathcal{F}}_t, \widetilde{Y}_t = s^2\right],$$

This is very close to the regular call option formula (for a call option written on $\widetilde{Y}$) except that (i) we evaluating at $\widetilde{Y}_t = s^2$ rather than $\widetilde{Y}_t = y$ and (ii) $\widetilde{Y} \sim GBM(\widetilde{r}, \widetilde{\sigma}^2)$ under $\mathbb{Q}$ but we are discounting by $r$ and not $\widetilde{r}$. However, this tells us that

$$\begin{aligned}
\hat{c}(t, s) &= e^{-r(T-t)+\widetilde{r}(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{-\widetilde{r}(T-t)}\left(\widetilde{Y}_T - K\right)^+ |\widetilde{\mathcal{F}}_t, \widetilde{Y}_t = s^2\right], \\
&= e^{-r(T-t)+\widetilde{r}(T-t)}c(t, s^2; \widetilde{r}, \widetilde{\sigma}^2).
\end{aligned}$$

Plugging in $\widetilde{r} = 2r + \sigma^2$ and $\widetilde{\sigma} = 2\sigma$ we obtain

$$\hat{c}(t, s) = e^{(r+\sigma^2)(T-t)}c(t, x^2; 2r + \sigma^2, 4\sigma^2).$$

(c) By the general theory of risk neutral pricing we know that the hedging strategy $\Delta$ takes the form

$$\Delta_t = \hat{c}_s(t, S_t)$$

From part $(b)$ we know that

$$\hat{c}_s(t,s) = e^{(r+\sigma^2)(T-t)} 2s c_s(t, s^2; 2r+\sigma^2, 4\sigma^2).$$

Thus

$$\Delta_t = e^{(r+\sigma^2)(T-t)} 2S_t c_s(t, S_t^2; 2r+\sigma^2, 4\sigma^2).$$