MF 790 HW 1 Part 1- SOLUTIONS

This assignment is due on Thursday, September 16th at 8 AM. Each problem is worth 10 points for a total of 50 points. Throughout, a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is given. Below, by a "generic" function, we mean a function which is smooth and bounded.

1. Expected Value of functions of a Random Variable. Let X be a random variable (rv).

(a) If $\mathbb{P}[X \geq 0] = 1$, show that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq t]\, dt$.

(b) Assume X is continuous with probability density function (pdf) f, and g is a non-negative generic function. Show that

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) f(x) dx.$$

Hint: Use that $\int\limits_{\Omega} \int\limits_{0}^{\infty} f(t,\omega) dt dP(\omega) = \int\limits_{0}^{\infty} \int\limits_{\Omega} f(t,\omega) d\mathbb{P}(\omega) dt$ for generic functions f.

Solution:

(a) By definition,

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) dP(\omega).$$

We also have that

$$\begin{aligned}
\int_0^\infty \mathbb{P}[X \geq t]\, dt &= \int_0^\infty \int_{\Omega} 1_{X(\omega) \geq t} dP(\omega) dt \\
&= \int_{\Omega} \int_0^\infty 1_{t \leq X(\omega)} dt dP(\omega) \\
&= \int_{\Omega} X(\omega) dP(\omega),
\end{aligned}$$

where used the hint and that for $\alpha \in \mathbb{R}$, $\int_0^\infty 1_{t \leq \alpha} dt = \alpha$.

(b) Since g is non-negative we have from part (a) that

$$\begin{aligned}
\mathbb{E}[g(X)] &= \int_0^\infty \mathbb{P}[g(X) \geq t]\, dt \\
&= \int_0^\infty \int_{\mathbb{R}} 1_{t \leq g(x)} f(x) dx dt \\
&= \int_{\mathbb{R}} \int_0^\infty 1_{t \leq g(x)} f(x) dt dx \\
&= \int_{\mathbb{R}} g(x) f(x) dx,
\end{aligned}$$

where again we used the hint, as well as the fact that continuous random variables

$$\mathbb{P}[g(X) \geq t] = \mathbb{E}\left[1_{g(X) \geq t}\right] = \int_{\mathbb{R}} 1_{t \leq g(x)} f(x) dx.$$

2. Moment Generating Functions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be given and let X be a random variable. Recall the moment generating function M_X is defined by

$$M_X(t) = \mathbb{E}\left[e^{tX}\right]; \qquad t \in \mathbb{R}.$$

It is certainly possible that $M_X(t) = \infty$ for some $t \in \mathbb{R}$, but we say the moment generating function of X *existx* if there is a $\delta > 0$ such that $M_X(t) < \infty$ for $|t| < \delta$. Furthermore, if two random variables X and Y have the same moment generating function (provided it exists) then they have the same distribution.

(a) Compute M_X for when (i) $X \sim N(\mu, \sigma^2)$ is normally distributed with mean μ and variance σ^2 and (ii) $X \sim \text{Exp}(\lambda)$ is exponentially distributed with parameter $\lambda > 0$.
(b) For $X \sim N(\mu, \sigma^2)$ and $\alpha, \beta \in \mathbb{R}$, what is the distribution of $Y = \alpha X + \beta$?
(c) For $X \sim \text{Exp}(\lambda)$ and $\alpha > 0$ what is the distribution of $Y = \alpha X$?

Solution:

(a) For $X \sim N(\mu, \sigma^2)$ we have

$$M_X(t) = \int_{\mathbb{R}} \frac{1}{\sigma\sqrt{2\pi}} e^{tx - (x-\mu)^2/(2\sigma^2)} dx$$

If we can find ν and C such that

$$tx - \frac{1}{2\sigma^2}(x-\mu)^2 = -\frac{1}{2\sigma^2}(x-\nu)^2 + C, \tag{0.1}$$

then

$$M_X(t) = e^C \int_{\mathbb{R}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\nu)^2} dx = e^C.$$

To find ν, C we match the linear and constant terms (in x) in (0.1):

$$\text{(linear)} : xt + x\frac{\mu}{\sigma^2} = x\frac{\nu}{\sigma^2} \Longrightarrow \nu = \mu + \sigma^2 t;$$
$$\text{(constant)} : \; -\frac{\mu^2}{2\sigma^2} = C - \frac{1}{2\sigma^2}(\mu + \sigma^2 t)^2 \Longrightarrow C = \mu t + \frac{1}{2}\sigma^2 t^2.$$

Thus

$$M_X(t) = e^C = e^{\mu t + \frac{1}{2}\sigma^2 t^2}; \qquad t \in \mathbb{R}.$$

For $X \sim \text{Exp}(\lambda)$ and $t \in \mathbb{R}$ we have

$$M_X(t) = \int_0^\infty \lambda e^{tx - \lambda x} dx.$$

If $t \geq \lambda$ then $M_X(t) = \infty$. For $t < \lambda$ we have

$$M_X(t) = \frac{\lambda}{\lambda - t}\int_0^\infty (\lambda - t)e^{-(\lambda - t)x} dx = \frac{\lambda}{\lambda - t}.$$

(b) Using sing our result in part (a) we have for Y that

$$M_Y(t) = \mathbb{E}\left[e^{tY}\right] = \mathbb{E}\left[e^{\beta t + \alpha t X}\right] = e^{(\beta+\alpha\mu)t + (1/2)\alpha^2\sigma^2 t^2},$$

so that $Y \sim N(\beta + \alpha\mu, \alpha^2\sigma^2)$.

(c) We again compute the moment generating function of Y:

$$M_Y(t) = \mathbb{E}\left[e^{\alpha t X}\right] = M_X(\alpha t) = \begin{cases} \frac{\lambda}{\lambda - \alpha t} & t < \lambda/\alpha \\ \infty & else \end{cases}$$

Note that

$$\frac{\lambda}{\lambda - \alpha t} = \frac{\lambda/\alpha}{\lambda/\alpha - t},$$

so that we see that $Y \sim \operatorname{Exp}(\lambda/\alpha)$.

3. Covariance, Correlation and Normal Random Variables. Let X and Y be two rvs. The covariance of X and Y is

$$\operatorname{Cov}[X, Y] := \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

In this problem we will show that if X and Y are jointly normal then

$$\operatorname{Cov}[X, Y] = 0 \iff X \perp\!\!\!\perp Y.$$

However, if X and Y are individually, but not jointly normal then $\operatorname{Cov}[X, Y] = 0$ does not necessarily imply that $X \perp\!\!\!\perp Y$. We will do this in the following steps.

(a) For any rvs X, Y (not necessarily normal) show $X \perp\!\!\!\perp Y \Rightarrow \operatorname{Cov}[X, Y] = 0$.

(b) For any rvs X, Y assume the expected values are μ_X, μ_Y respectively. Then $\operatorname{Cov}[X, Y] = \operatorname{Cov}[X - \mu_X, Y - \mu_Y]$. Thus, we may assume without loss of generality that $\mu_X = \mu_Y = 0$.

(c) Now assume (X, Y) is jointly normal with *diagonal* covariance matrix Σ:

$$\Sigma = \begin{pmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{pmatrix}.$$

Show that $X \perp\!\!\!\perp Y$. Therefore, from part (a) we know that $\operatorname{Cov}[X, Y] = 0$.

(d) Now, do not assume Σ is diagonal. Rather, that it takes the general form

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}.$$

for a given constant ρ. Show that

$$\operatorname{Cov}[X, Y] = \rho\sigma_X\sigma_Y.$$

Thus, if $\operatorname{Cov}[X, Y] = 0$ then Σ is diagonal and hence by part (c), $X \perp\!\!\!\perp Y$. **Hint:** in the two dimensional integral $\int_{x,y} xyf(x, y)dxdy$ where f is the joint pdf of (X, Y), make a change of variables of the form $x = \widetilde{x} + \lambda y$ for a certain constant λ which makes $\widetilde{X}$ and Y independent.

(e) Lastly, $X \sim N(0,1)$ and $Z \perp\!\!\!\perp X$ be such that $\mathbb{P}[Z=1] = \mathbb{P}[Z=-1] = 1/2$. Show that $Y = ZX \sim N(0,1)$, $\mathrm{Cov}[X,Y] = 0$ but X and Y are not independent. **Hint:** Note that for $t \in \mathbb{R}$

$$\mathbb{P}[ZX \leq t] = \mathbb{P}[X \leq t, Z=1] + \mathbb{P}[X \geq -t, Z=-1].$$

To show X, Y are not independent consider the function x^2.

Solution:

(a) If X and Y are independent then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ which immediately gives that $\mathrm{Cov}[X,Y] = 0$.

(b) This follows because for any constants a, b

$$\begin{aligned}\mathbb{E}[X-a]\,Y - b - \mathbb{E}[X-a]\,\mathbb{E}[Y-b] &= (\mathbb{E}[XY] - a\mathbb{E}[Y] - b\mathbb{E}[X] + ab) \\ &- (\mathbb{E}[X]\,\mathbb{E}[Y] - a\mathbb{E}[Y] - b\mathbb{E}[X] + ab) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].\end{aligned}$$

(c) By definition (X,Y) has joint pdf

$$\begin{aligned}f(x,y) &= \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\frac{x^2}{\sigma_X^2} - \frac{1}{2}\frac{y^2}{\sigma_Y^2}}, \\ &= \left(\frac{1}{2\sqrt{\pi}\sigma_X} e^{-\frac{1}{2}\frac{x^2}{\sigma_X^2}}\right) \times \left(\frac{1}{2\sqrt{\pi}\sigma_Y} e^{-\frac{1}{2}\frac{y^2}{\sigma_Y^2}}\right).\end{aligned}$$

Therefore, the joint pdf is the product of the marginal pdf's, which are (respectively) those of $N(0,\sigma_X^2)$ and $N(0,\sigma_Y^2)$ rvs. This proves independence.

(d) As

$$\Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_X^2(1-\rho^2)} & -\frac{\rho}{\sigma_X\sigma_Y(1-\rho^2)} \\ -\frac{\rho}{\sigma_X\sigma_Y(1-\rho^2)} & \frac{1}{\sigma_Y^2(1-\rho^2)} \end{pmatrix},$$

we have

$$\mathbb{E}[XY] = \int_{x,y} xy \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_X^2(1-\rho^2)} + \frac{y^2}{\sigma_Y^2(1-\rho^2)} - 2\frac{xy}{\sigma_X\sigma_Y(1-\rho^2)}\right)} dxdy$$

Guided by the hint, it turns out that we want $\lambda = \sigma_X\rho/\sigma_Y$ (make sure to verify this!). This leaves, after making the change of variable and simplifying

$$\mathbb{E}[XY] = \int_{\widetilde{x},y} y\left(\widetilde{x} + \frac{\sigma_X\rho}{\sigma_Y}y\right) \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{\widetilde{x}^2}{\sigma_X^2(1-\rho^2)} + \frac{y^2}{\sigma_Y^2}\right)} d\widetilde{x}dy$$

But, the right hand side above corresponds to $\mathbb{E}\left[Y\left(\widetilde{X} + (\sigma_X\rho/\sigma_Y)Y\right)\right]$ where $\widetilde{X}\sigma N(0,\sigma_X^2(1-\rho^2))$, $Y \sim N(0,\sigma_Y^2)$ and where $\widetilde{X} \perp\!\!\!\perp Y$. From here, we immediately see that

$$\mathbb{E}[XY] = \sigma_X\sigma_Y\rho,$$

giving the result.

(e) As mentioned in the hint:

$$\mathbb{P}\left[ZX \le t\right] = \mathbb{P}\left[X \le t, Z = 1\right] + \mathbb{P}\left[X \ge -t, Z = -1\right].$$

For $f(x) = 1_{x \le t}$ and $g(z) = 1_{z=1}$ we have, since $X \perp\!\!\!\perp Z$:

$$\begin{aligned}\mathbb{P}\left[X \le t, Z = 1\right] &= \mathbb{E}\left[f(X)g(Z)\right] = \mathbb{E}\left[f(X)\right]\mathbb{E}\left[g(Z)\right],\\ &= \mathbb{P}\left[X \le t\right]\mathbb{P}\left[Z = 1\right],\\ &= \frac{1}{2}\int_{-\infty}^{t}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx.\end{aligned}$$

Similarly

$$\begin{aligned}\mathbb{P}\left[X \ge t, Z = -1\right] &= \mathbb{P}\left[X \ge -t\right]\mathbb{P}\left[Z = -1\right],\\ &= \mathbb{P}\left[X \le t\right]\mathbb{P}\left[Z = -1\right],\\ &= \frac{1}{2}\int_{-\infty}^{t}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx.\end{aligned}$$

where we used that X is symmetric. Thus

$$\mathbb{P}\left[ZX \le t\right] = \int_{-\infty}^{t}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx; \qquad t \in \mathbb{R},$$

which shows that $ZX \sim N(0,1)$. As for the covariance,

$$\begin{aligned}\operatorname{Cov}\left[Y, X\right] &= \mathbb{E}\left[ZX^2\right] - \mathbb{E}\left[ZX\right]\mathbb{E}\left[X\right],\\ &= \mathbb{E}\left[Z\right]\mathbb{E}\left[X^2\right] - \mathbb{E}\left[Z\right]\mathbb{E}\left[X\right]^2 = 0,\end{aligned}$$

since $\mathbb{E}\left[Z\right] = 0$. Lastly, to see that Y and X are not independent note that

$$\begin{aligned}&\mathbb{E}\left[Y^2X^2\right] = \mathbb{E}\left[Z^2X^4\right] = \mathbb{E}\left[Z^2\right]\mathbb{E}\left[X^4\right] = 3,\\ &\mathbb{E}\left[Y^2\right]\mathbb{E}\left[X^2\right] = 1\end{aligned}$$

so $\mathbb{E}\left[Y^2X^2\right] \ne \mathbb{E}\left[Y^2\right]\mathbb{E}\left[X^2\right]$.

4. Conditional Probability and Conditional Expectation. Let $B \in \mathcal{F}$ be such that $0 < \mathbb{P}\left[B\right] < 1$. For a given $A \in \mathcal{F}$, recall that the conditional probability of A given B, written $\mathbb{P}\left[A \middle| B\right]$, is given by the formula

$$\mathbb{P}\left[A \middle| B\right] := \frac{\mathbb{P}\left[A \cap B\right]}{\mathbb{P}\left[B\right]}.$$

In this exercise we will relate conditional probability with conditional expectation. To do so define the random variables

$$1_B(\omega) = \begin{cases}1 & \omega \in B\\ 0 & \omega \notin B\end{cases}; \qquad 1_A(\omega) = \begin{cases}1 & \omega \in A\\ 0 & \omega \notin A\end{cases}.$$

Recall that by construction, $\mathbb{E}\left[1_B\right] = \mathbb{P}\left[B\right]$ and $\mathbb{E}\left[1_A\right] = \mathbb{P}\left[A\right]$.

Show that

$$\mathbb{E}\left[1_A \middle| 1_B\right](\omega) = \mathbb{P}\left[A \middle| B\right]1_B(\omega) + \mathbb{P}\left[A \middle| B^c\right]1_{B^c}(\omega).$$

Thus, the conditional expectation of 1_A given 1_B is the conditional probability of A given B on B, and the conditional probability of A given B^c on B^c. **Hint:** What is $\sigma(1_B)$?

Solution: First, note that since 1_B takes constant values on B, B^c we have

$$\sigma(1_B) = \{B, B^c, \emptyset, \Omega\}.$$

Next, set

$$X = \mathbb{P}\left[A \middle| B\right] 1_B + \mathbb{P}\left[A \middle| B^c\right] 1_{B^c}.$$

To prove that $X = \mathbb{E}\left[1_A \middle| 1_B\right]$ we must verify i) that X is $\sigma(1_B)$ measurable, and ii) the partial averaging equation

$$\mathbb{E}\left[1_A 1_F\right] = \mathbb{E}\left[X 1_F\right]; \qquad F \in \sigma(1_B).$$

First, as X is constant on both B and B^c it is $\sigma(1_B)$ measurable. Next, regarding partial averaging, we have

$$\begin{aligned}
\mathbb{E}\left[1_A 1_B\right] = \mathbb{P}\left[A \cap B\right]; &\quad \mathbb{E}\left[X 1_B\right] = \mathbb{P}\left[A \middle| B\right] \mathbb{E}\left[1_B\right] = \mathbb{P}\left[A \cap B\right]; \\
\mathbb{E}\left[1_A 1_{B^c}\right] = \mathbb{P}\left[A \cap B^c\right]; &\quad \mathbb{E}\left[X 1_{B^c}\right] = \mathbb{P}\left[A \middle| B^c\right] \mathbb{E}\left[1_{B^c}\right] = \mathbb{P}\left[A \cap B^c\right]; \\
\mathbb{E}\left[1_A 1_\emptyset\right] = 0; &\quad \mathbb{E}\left[X 1_\emptyset\right] = 0; \\
\mathbb{E}\left[1_A 1_\Omega\right] = \mathbb{P}\left[A\right]; &\quad \mathbb{E}\left[X 1_\Omega\right] = \mathbb{E}\left[X\right] = \mathbb{P}\left[A \middle| B\right] \mathbb{E}\left[1_B\right] + \mathbb{P}\left[A \middle| B^c\right] \mathbb{E}\left[1_{B^c}\right] \\
&\quad \qquad = \mathbb{P}\left[A \cap B\right] + \mathbb{P}\left[A \cap B^c\right] = \mathbb{P}\left[A\right].
\end{aligned}$$

Above, we have repeatedly used that $1_B(\omega) 1_{B^c}(\omega) = 0$ for all ω. Thus, the partial averaging equation holds and hence $X = \mathbb{E}\left[1_A \middle| 1_B\right]$.

5. Conditional Expectation for a Discrete Time Process. In this exercise we will do something strange: we will compute the conditional expected value of the stock price at time 1, given the stock price at time 3. The purpose is to ensure that *no matter how "unrealistic" the goal is* as long as we have a random variable and a sigma-algerba, we can compute conditional expectations.

Consider the three coin toss sample space

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

and let the probability measure $\mathbb{P}$ correspond to independent coin tosses with (unfair coin) probability $2/3$ for head. Consider the process S as shown Figure . Note that S_3 is constant on the sets

$$\begin{aligned}
\{S_3 = 32\} &= \{\omega : S_3(\omega) = 32\} = \{HHH\}, \\
\{S_3 = 8\} &= \{\omega : S_3(\omega) = 8\} = \{HHT, HTH, THH\}, \\
\{S_3 = 2\} &= \{\omega : S_3(\omega) = 2\} = \{HTT, THT, TTH\}, \\
\{S_3 = .5\} &= \{\omega : S_3(\omega) = 0.5\} = \{TTT\}.
\end{aligned}$$

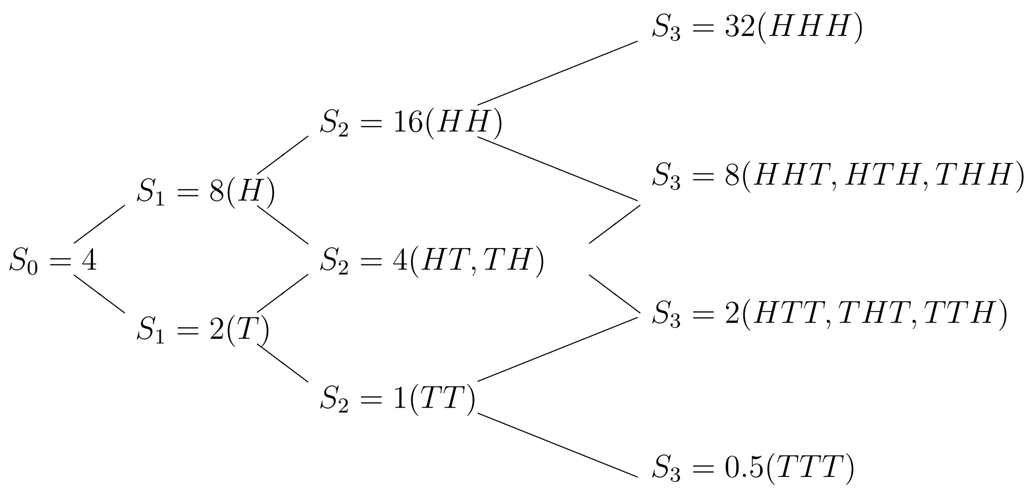

FIGURE 1. The process S

As stated above, our goal is to compute $\mathbb{E}\left[S_1 \middle| S_3\right]$. Motivated by problem 3, we expect this quantity to take the form, for some constants $c_1, \dots, c_4$

$$\mathbb{E}\left[S_1 \middle| S_3\right](\omega) = \begin{cases} c_1 & \omega \in \{S_3 = 32\}; \\ c_2 & \omega \in \{S_3 = 8\} \\ c_3 & \omega \in \{S_3 = 2\} \\ c_4 & \omega \in \{S_3 = 0.50\}, \end{cases}$$

and we wish to determine c_1, c_2, c_3 and c_4. To do this, we use the partial averaging equation

$$\sum_{\omega \in \{S_3 = k\}} \mathbb{E}\left[S_1 \middle| S_3\right](\omega)\mathbb{P}\left[\{\omega\}\right] = \sum_{\omega \in \{S_3 = k\}} S_1(\omega)\mathbb{P}\left[\{\omega\}\right] \tag{0.2}$$

for $k = 32, 8, 2$ and $.5$.

(a) Use (0.2) for $k = 32$ to identify c_1.
(b) Use (0.2) for $k = 8$ to identify c_1.
(c) Use (0.2) for $k = 2$ to identify c_1.
(d) Use (0.2) for $k = .5$ to identify c_1.

Solution

(a) The left hand side of (0.2) is $c_1\mathbb{P}\left[S_3 = 32\right] = c_1 \times 8/27$. The right hand side is $8 \times 8/27$ since $S_1(HHH) = 8$. Therefore $c_1 = 8$.
(b) The left hand side of (0.2) is $c_2\mathbb{P}\left[S_3 = 8\right] = c_2 \times 12/27$. The right hand side is $8 \times 8/27 + 2 \times 4/27 = 72/27$ since $S_1 = 8$ on $\{HHT, HTH\}$ and $S_1 = 2$ on $\{THH\}$. Therefore $c_2 = 6$.
(c) The left hand side of (0.2) $c_3\mathbb{P}S_3 = 2 = c_3 \times 6/27$. The right hand side is $8 \times 2/27 + 2 \times 4/27 = 24/27$ since $S_1 = 8$ on $\{HTT\}$ and $S_1 = 2$ on $\{TTH, THT\}$. Therefore $c_3 = 4$.

(d) The left hand side of (0.2) is $c_4\mathbb{P}\left[S_3 = .5\right] = c_4 \times 1/27$. The right hand side is $2 \times 1/27 = 2/27$ since $S_1 = 2$ on $\{TTT\}$. Therefore $c_4 = 2$.