MF 790 HW 5, PART 2

This assignment is due on Thursday, November 11th at 8 AM. Each problem is worth 25 points, for a total of 50 points.

**1. Density Processes for Coin Toss Space.** Let $\Omega$ correspond to coin toss space with three tosses. I.e.

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

For $j = 1, ..., 3$ let $\mathcal{F}_j$ contain all the information generated by the first $j$ tosses. Let $\mathbb{P}$ and $\mathbb{Q}$ be two measures on $(\Omega, \mathcal{F}_3)$ such that $\mathbb{P}$ corresponds to independent tosses of a fair coin, and $\mathbb{Q}$ corresponds to independent tosses of an unfair coin, where the likelihood of a head is 3/4 and the likelihood of a tail is 1/4.

For $j = 1, ..., 3$ and $\omega \in \Omega$ compute the Radon-Nikodym derivative $Z_j := \frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_j}$, which is an $\mathcal{F}_j$ measurable random variable such that $\mathbb{Q}[A_j] = \mathbb{E}\left[1_{A_j} Z_j\right]$ for all $A_j \in \mathcal{F}_j$. In particular, what are $Z_1(\omega), Z_2(\omega)$ and $Z_3(\omega)$ when $\omega = HHT$?

**2. Call Prices in a Variant of the Black-Scholes Model with Dividend Payments.** Consider a variant of the Black-Scholes model where we allow $\mu, \sigma$ and $r$ to be non-random functions of time.

(a) (no dividends) First, assume no dividend payments so that $S$ and $D$ have dynamics

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t; \qquad \frac{dD_t}{D_t} = -r_t dt.$$

Identify constants $\kappa, \lambda$ such that the time zero price $c_0$ of a call option with strike $K$ expiring at $T$ in this model takes the form

$$c_0 = c^{BS}(0, S_0; \kappa, \lambda),$$

where we write $c^{BS}(0, s; r, \sigma)$ as the Black-Scholes call option price for a given $r, \sigma$. **Hint:** look at slide 27 on lecture 04.

(b) (dividend flow). Now, assume a continuous dividend payment rate of $a = \{a_t\}$ where $a$ is also non-random. This gives $(S, D)$ dynamics

$$\frac{dS_t}{S_t} = (\mu_t - a_t)dt + \sigma_t dW_t; \qquad \frac{dD_t}{D_t} = -r_t dt.$$

As in part (a), find constants $\kappa, \lambda$ as well as a scaling factor $\mathfrak{s}$ such that the time zero price $c_0$ of a call option with strike $K$ expiring at $T$ in this model takes the form

$$c_0 = \mathfrak{s} \times c^{BS}(0, S_0; \kappa, \lambda).$$

Here, your constants $\kappa, \lambda$ may or may not be different from part (a).