MF 790 Stochastic Calculus
Practice Final Exam. Fall, 2021

This is the practice final exam. There are 4 questions for a total 100 points. Each question may contain multiple parts.

To get the most out of the exam please take this exam as if it were the real exam! I.e. do not use notes, the class textbook, the internet, and especially do not look at the solutions ahead of time! Give yourself two hours to take the exam, and abide by this rule!

Formulas.

(1) The standard normal cumulative distribution function (cdf)

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy.$$

(2) The price of a call and put option (with maturity T and strike K) in the Black-Scholes model at $t \leq T$ given the stock price is x.

$$c(t, x) = xN(d_+(\tau, x)) - Ke^{-r\tau} N(d_-(\tau, x))$$
$$p(t, x) = Ke^{-r\tau} N(-d_-(\tau, x)) - xN(-d_+(\tau, x))$$

where $\tau = T - t$ and

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}} \left(\log\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right) \tau \right)$$

(3) The delta for a call option in the Black-Scholes model at $t \leq T$ and given the stock price is x

$$\partial_x c(t, x) = N(d_+(\tau, x))$$

Throughout, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is fixed, $W = \{W_t\}_{t \geq 0}$ is a Brownian motion, and unless otherwise noted, $\mathbb{F} = \mathbb{F}^W$ is generated by W.

1 (20 Points). Fix $T = 1$ and define

$$Z_1 := W_1^2.$$

As $W_1^2 \geq 0$ and $\mathbb{E}\left[W_1^2\right] = 1$ we may define a new measure $\widehat{\mathbb{P}}$ by $d\widehat{\mathbb{P}}/d\mathbb{P} = Z_1$.

(a) (**10 Points**) With $Z_t := \mathbb{E}\left[Z_T \middle| \mathcal{F}_t\right]$ show that $Z_t = W_t^2 + 1 - t$.

(b) (**10 Points**) Identify the process ϕ such that $\widehat{W_t} := W_t - \int_0^t \phi_u du$ is a Brownian motion under $\widehat{\mathbb{P}}$.

2 (25 points, 5 points each part) (True/False). Indicate whether or not each of the following statements is true or false. If it is true, explain why. If it is false, either explain why or give a counter example. Answers with no explanation get no credit!

(a) If the money market rate process R is non negative, then, because the forward contract is riskier, the forward price always exceeds the futures price.

(b) For a stock paying a continuous non-negative dividend rate, the discounted price process is a supermartingale under $\mathbb{Q}$.

(c) In the Black-Scholes model, given the same strike K, maturity T and current price $S_t = x, t < T$, the gamma for a put option is less than the gamma for a call option.

(d) Let W and B be two independent Brownian motions and set $M_t = B_t W_t$ for $t \geq 0$. Then there is no (smooth) function f such that $Y_t = f(M_t)$ is a Brownian motion.

(e) If σ and τ are two stopping times such that for each ω $\sigma(\omega) \leq \tau(\omega)$. Then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

3 (30 Points). In the Black-Scholes model, fix a time $T > 0$ and consider a "power call" which is a claim with payoff $V_T = (S_T^p - K)^+$ for a fixed constant $p > 1$. Our goal is to price and hedge this claim.

(a) **(10 points).** Identify $\tilde{r}$ and $\tilde{\sigma}$ so that under risk neutral measure, S^p is a geometric Brownian motion with parameters $\tilde{r}, \tilde{\sigma}^2$.

(b) **(10 points).** Evaluate the value V_t of the claim V_T at time t for $t \leq T$. Express your answer in terms of the Black-Scholes call price.

(c) **(10 points).** Explicitly identify the initial capital and hedging strategy which replicates V_T.

Hint: Use that $t \to e^{-rt} V_t$ is a $\mathbb{Q}$ martingale.

4 (25 Points). Assume that under risk neutral measure $\mathbb{Q}$, the money market rate follows a CIR process

$$dR_t = \kappa(\theta - R_t)dt + \xi \sqrt{R_t} dW_t^{\mathbb{Q}}.$$

(a) (**15 Points**) The price of a T maturity zero coupon bond is $B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T R_u du}\middle|\mathcal{F}_t\right]$. Derive (do not just write down) a partial differential equation (PDE), such that if the function $b = b(t,r)$ solves the PDE, then $B(t,T) = b(t,R_t)$.

(b) (**10 Points**) Guessing $b(t,r) = e^{-A(t)-C(t)r}$, derive (do not just write down) the ordinary differential equations that A, C should solve. There is no need to solve the equations.