MF 790 Recommended HW 7, Part 1 - SOLUTIONS

This assignment is recommended to help you study for the final exam.

1. Spot dynamics for constant volatility forward prices and Hull & White money market rates. . Assume under $\mathbb{Q}$ the money market rate follows a Hull & White process

$$dR_t = \kappa(t)\left(\theta(t) - R_t\right)dt + a(t)dW_t^{\mathbb{Q}}, \qquad R_0 > 0,$$

and the forward price process has constant volatility σ

$$\frac{d\mathrm{For}_t}{\mathrm{For}_t} = \sigma dW_t^{\mathbb{Q}}.$$

In class we claimed that the spot price S has $\mathbb{Q}$ dynamics

$$\frac{dS_t}{S_t} = R_t dt + \left(\sigma - C(t)a(t)\right),$$

where $C(t) = \int_t^T e^{-\int_t^u \kappa(v)dv}du$. In this exercise you will show this rigorously.

(a) Using Itô's formula, the explicit bond price formula obtained in class for the Hull & White model, and the (general) fact that $t \to D_t B(t,T)$ is a $\mathbb{Q}$ martingale, show that

$$\frac{d(D_t B(t,T))}{D_t B(t,T)} = -C(t)a(t)dW_t^{\mathbb{Q}}.$$

(b) Using that $\widetilde{Z}_t = D_t B(t,T)/B(0,T)$ identify the process Γ from the slide 15 of the Forward Measure notes, and hence the dynamics of S under $\mathbb{Q}$.

Solution:

(a) We know $B(t,T) = e^{-A(t)-C(t)R_t}$ so that $D_t B(t,T) = e^{-\int_0^t R_u du - A(t) - C(t)R_t}$. This will enable us to compute the $\mathbb{Q}$ dynamics of $D_t B(t,T)$. However, we already know the dt vanish, because $t \to D_t B(t,T)$ is a $\mathbb{Q}$ martingale. In the exponent for $D_t B(t,T)$ the term in front of $dW_t^{\mathbb{Q}}$ for the dynamics is $-C(t)a(t)$. Therefore, we conclude

$$d(D_t B(t,T)) = D_t B(t,T)\left(-C(t)a(t)dW_t^{\mathbb{Q}}\right).$$

(b) We have

$$\frac{d\widetilde{Z}_t}{\widetilde{Z}_t} = \frac{d(D_t B(t,T))}{D_t B(t,T)} = -C(t)a(t)dW_t^{\mathbb{Q}},$$

which gives $\Gamma_t = -C(t)a(t)$.

2. Spot dynamics for constant volatility forward prices and money market rates satisfying a general SDE.. We now generalize problem 1 to when under $\mathbb{Q}$, the money market rates satisfies the SDE

$$dR_t = \mu(t, R_t)dt + a(t, R_t)dW_t^{\mathbb{Q}}.$$

Recall that using Feynman-Kač and the Markov property of R we showed that

$$B(t,T) = b(t, R_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T R_u du}\middle| R_t\right],$$

where the (deterministic) function $b(t,r)$ satisfies the PDE

$$\begin{aligned}0 &= b_t(t,r) + \frac{1}{2}a(t,r)^2 b_{rr}(t,r) + \mu(t,r)b_r(t,r) - rb(t,r),\\ 1 &= b(T,r)\end{aligned}$$

Using this fact, explicitly identify (in terms of b, it's derivatives as well as μ, a or σ) the volatility process σ_t in the dynamics

$$\frac{dS_t}{S_t} = R_t dt + \sigma_t dW_t^{\mathbb{Q}}.$$

Solution: By Itô and the PDE for b we know

$$\begin{aligned}d(D_t B(t,T)) &= d(D_t b(t,R_t))\\ &= -D_t b(t,R_t)R_t dt + D_t\Bigg(\bigg(b_t(t,R_t) + \frac{1}{2}a(t,R_t)^2 b_{rr}(t,R_t) + \mu(t,R_t)b_r(t,R_t)\bigg)dt\\ &\qquad + a(t,R_t)b_r(t,R_t)dW_t^{\mathbb{Q}}\Bigg),\\ &= -D_t b(t,R_t)R_t dt + D_t\left(R_t b(t,R_t)dt + a(t,R_t)b_r(t,R_t)dW_t^{\mathbb{Q}}\right),\\ &= D_t a(t,R_t)b_r(t,R_t)dW_t^{\mathbb{Q}},\\ &= D_t b(t,R_t)a(t,R_t)\frac{b_r(t,R_t)}{b(t,R_t)}dW_t^{\mathbb{Q}},\\ &= D_t B(t,T)a(t,R_t)\frac{b_r(t,R_t)}{b(t,R_t)}dW_t^{\mathbb{Q}}\end{aligned}$$

This gives

$$\Gamma_t = a(t,R_t)\frac{b_r(t,R_t)}{b(t,R_t)},$$

and hence

$$\sigma_t = \sigma + a(t,R_t)\frac{b_r(t,R_t)}{b(t,R_t)}.$$

3. Comparing forward contracts on call options and put options. Consider a model where under $\mathbb{Q}$ we have

$$\frac{dS_t}{S_t} = R_t dt + \sigma_t dW_t^{\mathbb{Q}}; \qquad \frac{dD_t}{D_t} = -R_t dt.$$

Next, consider two forward contracts. The first forward contract is on a call option with strike K (i.e. $C_T = (S_T - K)^+$), and the second forward contract is on a put option also with strike K (i.e. $P_T = (K - S_T)^+$). Writing C_t as the call price for $t \le T$ and P_t as the put price for $t \le T$, identify a condition upon the strike K and forward price For_t which will enable us to order $\mathrm{For}(C)_t$ and $\mathrm{For}(P)_t$.

Solution: By the general theory we know that

$$\text{For}(C)_t = \frac{C_t}{B(t,T)}; \qquad \text{For}(P)_t = \frac{P_t}{B(t,T)}.$$

Next, using put-call parity (i.e. that $S_T - K = C_T - P_T$) we see that

$$\begin{aligned} S_t - \frac{K}{B(t,T)} &= \mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}(S_T - K)\,\middle|\mathcal{F}_t\right], \\ &= C_t - B_t. \end{aligned}$$

This implies that

$$\text{For}_t - K = \text{For}(C)_t - \text{For}(P)_t,$$

so if $K < \text{For}_t$ the forward price for the call is higher, else the forward price of the put is higher.