MF 790 HW 1 Part 1

This assignment is due on Thursday, September 16th at 8 AM. Each problem is worth 10 points for a total of 50 points. Throughout, a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is given. Below, by a "generic" function, we mean a function which is smooth and bounded.

1. Expected Value of functions of a Random Variable. Let X be a random variable (rv).

(a) If $\mathbb{P}[X \geq 0] = 1$, show that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq t]\, dt$.

(b) Assume X is continuous with probability density function (pdf) f, and g is a non-negative generic function. Show that

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) f(x) dx.$$

Hint: Use that $\int\limits_\Omega \int\limits_0^\infty f(t,\omega) dt dP(\omega) = \int\limits_0^\infty \int\limits_\Omega f(t,\omega) d\mathbb{P}(\omega) dt$ for generic functions f.

2. Moment Generating Functions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be given and let X be a random variable. Recall the moment generating function M_X is defined by

$$M_X(t) = \mathbb{E}\left[e^{tX}\right]; \qquad t \in \mathbb{R}.$$

It is certainly possible that $M_X(t) = \infty$ for some $t \in \mathbb{R}$, but we say the moment generating function of X *existx* if there is a $\delta > 0$ such that $M_X(t) < \infty$ for $|t| < \delta$. Furthermore, if two random variables X and Y have the same moment generating function (provided it exists) then they have the same distribution.

(a) Compute M_X for when (i) $X \sim N(\mu, \sigma^2)$ is normally distributed with mean μ and variance σ^2 and (ii) $X \sim \text{Exp}(\lambda)$ is exponentially distributed with parameter $\lambda > 0$.

(b) For $X \sim N(\mu, \sigma^2)$ and $\alpha, \beta \in \mathbb{R}$, what is the distribution of $Y = \alpha X + \beta$?

(c) For $X \sim \text{Exp}(\lambda)$ and $\alpha > 0$ what is the distribution of $Y = \alpha X$?

3. Covariance, Correlation and Normal Random Variables. Let X and Y be two rvs. The covariance of X and Y is

$$\text{Cov}[X, Y] := \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

In this problem we will show that <u>if</u> X and Y are jointly normal then

$$\text{Cov}[X, Y] = 0 \quad \Longleftrightarrow \quad X \perp\!\!\!\perp Y.$$

However, if X and Y are <u>individually, but not jointly</u> normal then $\text{Cov}[X, Y] = 0$ does not necessarily imply that $X \perp\!\!\!\perp Y$. We will do this in the following steps.

(a) For any rvs X, Y (not necessarily normal) show $X \perp\!\!\!\perp Y \Rightarrow \text{Cov}[X, Y] = 0$.

(b) For any rvs X, Y assume the expected values are μ_X, μ_Y respectively. Then $\text{Cov}[X, Y] = \text{Cov}[X - \mu_X, Y - \mu_Y]$. Thus, we may assume without loss of generality that $\mu_X = \mu_Y = 0$.

(c) Now assume (X,Y) is jointly normal with *diagonal* covariance matrix Σ:

$$\Sigma = \begin{pmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{pmatrix}.$$

Show that $X \perp\!\!\!\perp Y$. Therefore, from part (a) we know that $\mathrm{Cov}\left[X,Y\right]=0$.

(d) Now, do not assume Σ is diagonal. Rather, that it takes the general form

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}.$$

for a given constant ρ. Show that

$$\mathrm{Cov}\left[X,Y\right] = \rho\sigma_X\sigma_Y.$$

Thus, if $\mathrm{Cov}\left[X,Y\right]=0$ then Σ is diagonal and hence by part (c), $X \perp\!\!\!\perp Y$. **Hint:** in the two dimensional integral $\int_{x,y} xyf(x,y)dxdy$ where f is the joint pdf of (X,Y), make a change of variables of the form $x = \widetilde{x} + \lambda y$ for a certain constant λ which makes $\widetilde{X}$ and Y independent.

(e) Lastly, $X \sim N(0,1)$ and $Z \perp\!\!\!\perp X$ be such that $\mathbb{P}\left[Z=1\right] = \mathbb{P}\left[Z=-1\right] = 1/2$. Show that $Y = ZX \sim N(0,1)$, $\mathrm{Cov}\left[X,Y\right]=0$ but X and Y are not independent. **Hint:** Note that for $t \in \mathbb{R}$

$$\mathbb{P}\left[ZX \le t\right] = \mathbb{P}\left[X \le t, Z=1\right] + \mathbb{P}\left[X \ge -t, Z=-1\right].$$

To show X,Y are not independent consider the function x^2.

4. Conditional Probability and Conditional Expectation. Let $B \in \mathcal{F}$ be such that $0 < \mathbb{P}\left[B\right] < 1$. For a given $A \in \mathcal{F}$, recall that the conditional probability of A given B, written $\mathbb{P}\left[A\middle|B\right]$, is given by the formula

$$\mathbb{P}\left[A\middle|B\right] := \frac{\mathbb{P}\left[A\cap B\right]}{\mathbb{P}\left[B\right]}.$$

In this exercise we will relate conditional probability with conditional expectation. To do so define the random variables

$$1_B(\omega) = \begin{cases} 1 & \omega \in B \\ 0 & \omega \notin B \end{cases}; \qquad 1_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}.$$

Recall that by construction, $\mathbb{E}\left[1_B\right] = \mathbb{P}\left[B\right]$ and $\mathbb{E}\left[1_A\right] = \mathbb{P}\left[A\right]$.

Show that

$$\mathbb{E}\left[1_A\middle|1_B\right](\omega) = \mathbb{P}\left[A\middle|B\right]1_B(\omega) + \mathbb{P}\left[A\middle|B^c\right]1_{B^c}(\omega).$$

Thus, the conditional expectation of 1_A given 1_B is the conditional probability of A given B on B, and the conditional probability of A given B^c on B^c. **Hint:** What is $\sigma(1_B)$?

5. Conditional Expectation for a Discrete Time Process. In this exercise we will do something strange: we will compute the conditional expected value of the stock price at time 1, given the stock price at time 3. The purpose is to ensure that *no matter how "unrealistic" the goal is* as long as we have a random variable and a sigma-algerba, we can compute conditional expectations.

Consider the three coin toss sample space

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

and let the probability measure $\mathbb{P}$ correspond to independent coin tosses with (unfair coin) probability $2/3$ for head. Consider the process S as shown Figure . Note that

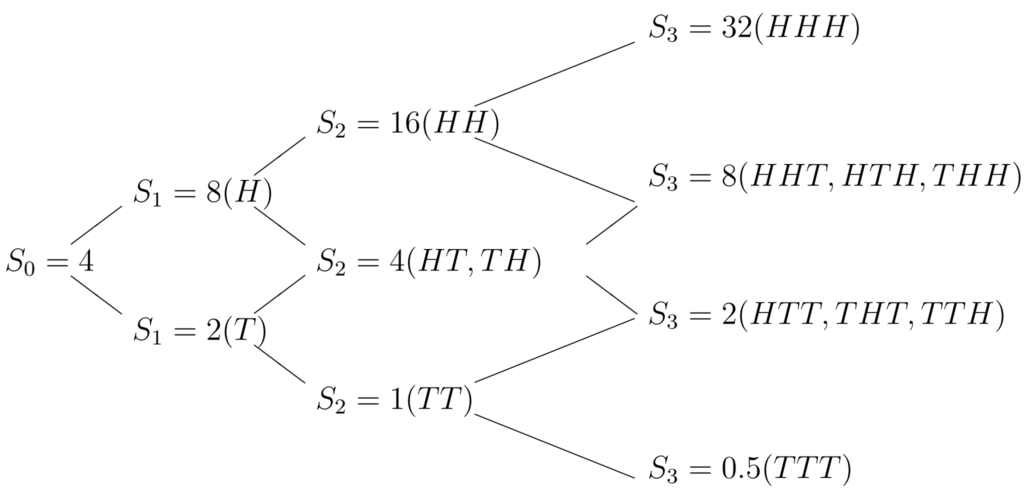

Figure 1. The process S

S_3 is constant on the sets

$$\begin{aligned}
\{S_3 = 32\} &= \{\omega : S_3(\omega) = 32\} = \{HHH\}, \\
\{S_3 = 8\} &= \{\omega : S_3(\omega) = 8\} = \{HHT, HTH, THH\}, \\
\{S_3 = 2\} &= \{\omega : S_3(\omega) = 2\} = \{HTT, THT, TTH\}, \\
\{S_3 = .5\} &= \{\omega : S_3(\omega) = 0.5\} = \{TTT\}.
\end{aligned}$$

As stated above, our goal is to compute $\mathbb{E}\left[S_1 \middle| S_3\right]$. Motivated by problem 3, we expect this quantity to take the form, for some constants $c_1, \dots, c_4$

$$\mathbb{E}\left[S_1 \middle| S_3\right](\omega) = \begin{cases} c_1 & \omega \in \{S_3 = 32\}; \\ c_2 & \omega \in \{S_3 = 8\} \\ c_3 & \omega \in \{S_3 = 2\} \\ c_4 & \omega \in \{S_3 = 0.50\}, \end{cases}$$

and we wish to determine c_1, c_2, c_3 and c_4. To do this, we use the partial averaging equation

$$\sum_{\omega \in \{S_3 = k\}} \mathbb{E}\left[S_1 \middle| S_3\right](\omega)\mathbb{P}\left[\{\omega\}\right] = \sum_{\omega \in \{S_3 = k\}} S_1(\omega)\mathbb{P}\left[\{\omega\}\right] \tag{0.1}$$

for $k = 32, 8, 2$ and $.5$.

(a) Use (0.1) for $k = 32$ to identify c_1.
(b) Use (0.1) for $k = 8$ to identify c_1.
(c) Use (0.1) for $k = 2$ to identify c_1.
(d) Use (0.1) for $k = .5$ to identify c_1.