MF 790 HW 4, Part 1 - SOLUTIONS

This assignment is due on Thursday, October 28th at 8 AM. Problems 1 and 2 are worth 20 points, and problem 3 is worth 10 points, for a total of 50 points.

1. Black-Scholes Formula for a Call Option. Do exercise 3.5 on page 118 of the class textbook.

Solution: Since $S_t = S_0 e^{(r-\sigma^2/2)t+\sigma W_t}$ and W is a Brownian motion, we have

$$\begin{aligned}\mathbb{E}\left[e^{-rT}\left(S_T - K\right)^+\right] &= e^{-rT}\int_{-\infty}^{\infty}\left(S_0 e^{(r-\sigma^2/2)T+\sigma x} - K\right)^+ \frac{1}{\sqrt{2\pi T}}e^{-x^2/(2T)}dx,\\ &= e^{-rT}\int_{-\infty}^{\infty}\left(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}y} - K\right)^+ \frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy,\end{aligned}$$

where the second equality follows by setting $y = x/\sqrt{T}$. Note that

$$S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}y} \geq K \Longleftrightarrow y \geq \frac{1}{\sigma\sqrt{T}}\left(\log(K/S_0) - (r-\sigma^2/2)T\right) = -d_-(T,S_0).$$

This gives

$$\begin{aligned}\mathbb{E}\left[e^{-rT}\left(S_T - K\right)^+\right] = e^{-rT}&\int_{-d_-(T,S_0)}^{\infty} S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}y}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy\\ &- e^{-rT}K\int_{-d_-(T,S_0)}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy.\end{aligned}$$

Now,

$$e^{-rT}K\int_{-d_-(T,S_0)}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy = e^{-rT}K\left(1 - N(-d_-(T,S_0))\right) = e^{-rT}KN(d_-(T,S_0)).$$

Also, using the familiar "completing the square" argument it follows that

$$e^{-rT}\int_{-d_-(T,S_0)}^{\infty} S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{t}y}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy = S_0 N(\sigma\sqrt{T}+d_-(T,S_0)) = S_0 N(d_+(T,S_t))$$

which is the desired result.

2. Self Financing Trading Strategies. Do exercise 4.10 on pages 193-196 of the class textbook.

Solution:

i) Since $X_t = \Delta_t S_t + \Gamma_t M_t$ we have

$$\begin{aligned}dX_t &= \Delta_t dS_t + S_t d\Delta_t + d[\Delta,S]_t + \Gamma_t dM_t + M_t d\Gamma_t + d[\Gamma,M]_t\\ &= \Delta_t dS_t + r\Gamma_t M_t dt + (S_t d\Delta_t + d[\Delta,S]_t + M_t d\Gamma_t + d[\Gamma,M]_t)\\ &= \Delta_t dS_t + r(X_t - \Delta_t S_t)dt + (S_t d\Delta_t + d[\Delta,S]_t + M_t d\Gamma_t + d[\Gamma,M]_t)\end{aligned}$$

where in the second equality we used that $M_t = e^{rt}$ and in the third equality we used that $X_t = \Delta_t S_t + \Gamma_t M_t$. Now, we already know that

$$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$$

The only way these two equations are consistent is if

$$0 = S_t d\Delta_t + d[\Delta, S]_t + M_t d\Gamma_t + d[\Gamma, M]_t$$

But, this is precisely the self-financing condition.

ii) Since $\Gamma_t Mt = N_t$ we know that

$$\begin{aligned} dN_t &= \Gamma_t dM_t + M_t d\Gamma_t + d[M,\Gamma]_t = r\Gamma_t M_t dt + M_t d\Gamma_t + d[M,\Gamma]_t \\ &= rN_t dt + M_t d\Gamma_t + d[M,\Gamma]_t \end{aligned}$$

so that $dN_t - rN_t dt = M_t d\Gamma_t + d[M,\Gamma]_t$. It has already been shown that (here, we are omitting function arguments in c and it's derivatives)

$$dN_t = c_t dt + c_s dS_t + \frac{1}{2} c_{ss} d[S,S]_t - \Delta_t dS_t - S_t d\Delta_t - d[\Delta, S]_t$$

Using the self financing condition of part i) this becomes

$$\begin{aligned} dN_t &= c_t dt + c_s dS_t + \frac{1}{2} c_{ss} d[S,S]_t - \Delta_t dS_t + M_t d\Gamma_t + d[\Gamma, M]_t \\ &= c_t dt + c_s dS_t + \frac{1}{2} c_{ss} d[S,S]_t - \Delta_t dS_t + dN_t - rN_t dt \\ &= c_t dt + c_s \mu S_t dt + c_s \sigma S_t dW_t + \frac{1}{2} c_{ss} \sigma^2 S_t^2 dt - \Delta_t \mu S_t dt - \Delta_t \sigma S_t dW_t + dN_t - rN_t dt \\ &= dN_t + (c_t - c_s \mu S_t + \frac{1}{2} c_{ss} \sigma^2 S_t^2 - \Delta_t \mu S_t - rN_t) dt + \sigma S_t (c_s - \Delta_t) dW_t \end{aligned}$$

Since each of the dt and dW_t terms must vanish we get that $\Delta_t = c_s(t, S_t)$ and the Black-Scholes PDE.

3. Black-Scholes for General European Options. Let $S \sim \mathrm{GBM}(\mu, \sigma^2)$ with respect to some $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let $f(x) : (0, \infty) \mapsto \mathbb{R}$ be a bounded, smooth function. By repeating the argument in the case of a call option, show that if there exists a smooth function $c(t,s)$ satisfying the partial differential equation (PDE)

$$\begin{aligned} c_t(t,s) + rsc_s(t,s) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t,s) - rc(t,s) &= 0 \qquad t \in (0,T), s > 0 \\ c(T,s) &= f(s) \qquad s > 0 \end{aligned}$$

and such that for any probability measure $\mathbb{Q}$ (with filtration $\widetilde{\mathbb{F}}$ and Brownian motion $\widetilde{W}$) and $\tilde{S} \sim \mathrm{GBM}(r, \sigma^2)$ under $\mathbb{Q}$ we have for all $t \geq 0$ that

$$\mathbb{E}^{\mathbb{Q}} \left[\int_0^t e^{-2ru} \sigma^2 \tilde{S}_u^2 c_s(u, \tilde{S}_u)^2 du \right] < \infty,$$

(you are assuming this, you do not have to show this) then

(a) The value of the option $f(S_T)$ at time t given $S_t = s$ is

$$c(t,s) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} f(\tilde{S}_T) \middle| \widetilde{\mathcal{F}}_t, \widetilde{S}_t = s \right].$$

(b) $c(t,s)$ admits the explicit form

$$c(t,s) = e^{-r(T-t)} \int_{-\infty}^{\infty} f(se^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}z}) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

Solution:

(a) This follows from the exact same line of reasoning as in the class notes and textbook, except that everywhere one encounters $(\tilde{S}_T - K)^+$ one replaces it with $f(\tilde{S}_T)$.

(b) If $\tilde{S} \sim \mathrm{GBM}\,(r, \sigma^2)$ it follows that conditioned upon $\widetilde{F}_t$ and $\tilde{S}_t = s$ we have $\tilde{S}_T = se^{(r-\sigma^2/2)(T-t)+\sigma(\widetilde{W}_T - \widetilde{W}_t)}$. Since $\widetilde{W}_T - \widetilde{W}_t$ is independent of $\widetilde{\mathcal{F}}_t$ and $\widetilde{W}_T - \widetilde{W}_t \sim N(0, T-t)$ under $\mathbb{Q}$, the result follows.