MF 790 HW 4, Part 2 - SOLUTIONS

This assignment is due on Thursday, October 28th at 8 AM. Problems 1 is worth 20 points, and problems 2, 3 are worth 15 points each, for a total of 50 points.

1. Computation of the Greeks. Do Exercise 4.9 parts (i) through (v) on page 192 respectively of the class textbook.

Solution: To be consistent with the book, throughout, we use "x" instead of "s" for the function argument.

(i) Set $\tau = T - t$. Note that $d_-(\tau, x) = d_+(\tau, x) - \sigma\sqrt{\tau}$. This gives (suppressing the arguments for d_- and d_+):

$$\begin{aligned}
Ke^{-r\tau}N'(d_-) &= K\frac{e^{-r\tau}}{\sqrt{2\pi}}e^{-\frac{1}{2}d_-^2},\\
&= K\frac{e^{-r\tau}}{\sqrt{2\pi}}e^{-\frac{1}{2}d_+^2+d_+\sigma\sqrt{\tau}-\frac{1}{2}\sigma^2\tau},\\
&= xN'(d_+)\left(\frac{K}{x}e^{-r\tau}e^{\log\left(\frac{x}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)\tau-\frac{1}{2}\sigma^2\tau}\right),\\
&= xN'(d_+).
\end{aligned}$$

(ii) Using part (i) (again, suppressing the function arguments):

$$\begin{aligned}
c_x &= N(d_+) + xN'(d_+)(d_+)_x - Ke^{-r(T-t)}N'(d_-)(d_-)_x,\\
&= N(d_+) + xN'(d_+)\left((d_+)_x - (d_-)_x\right) = N(d_+).
\end{aligned}$$

since $(d_-)_x = (d_+)_x$.

(iii) Using part (i) (again, suppressing the function arguments):

$$\begin{aligned}
c_t &= -xN'(d_+)(d_+)_t - rKe^{-r(T-t)}N(d_-) + Ke^{-r(T-t)}N'(d_-)(d_-)_t,\\
&= -xN'(d_+)\left(d_+ - d_-\right)_t - rKe^{-r(T-t)}N(d_-),\\
&= -\frac{\sigma x}{2\sqrt{T-t}}N'(d_+) - rKe^{-r(T-t)}N(d_-).
\end{aligned}$$

since $(d_+ - d_-)_t = \sigma/(2\sqrt{T-t})$.

(iv) Note that by part (ii), $c_{xx} = N'(d_+)/(x\sigma\sqrt{T-t})$. Plugging everything in gives

$$\begin{aligned}
&c_t + \frac{1}{2}\sigma^2x^2c_{xx} + rxc_x - rc\\
&= -\frac{\sigma x}{2\sqrt{T-t}}N'(d_+) - rKe^{-r(T-t)}N(d_-) + \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) + rxN(d_+)\\
&\qquad - rxN(d_+) + rKe^{-r(T-t)}N(d_-),\\
&= 0.
\end{aligned}$$

(v) With $\tau = T - t$, $t \uparrow T$ is the same as $\tau \downarrow 0$. Then:

$$\lim_{\tau\downarrow 0} d_\pm(x,\tau) = \lim_{\tau\downarrow 0} \frac{\log\left(\frac{x}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = \begin{cases} \infty & x > K \\ 0 & x = K \\ -\infty & x < K \end{cases},$$

since $\sigma\sqrt{\tau}, (r \pm \sigma^2/2)\tau \to 0$ as $\tau \downarrow 0$. Thus, since $N(\infty) = 1, N(0) = 1/2$ and $N(-\infty) = 0$:

$$\lim_{t\uparrow T} c(t,x) = \begin{cases} x - K & x > K \\ 0 & x \leq K \end{cases}.$$

2. Vega and Implied Volatility. Continuing the previous exercise, recall the price of the call option at $t \leq T$ given $S_t = s$ is

$$\begin{aligned} c(t,s) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(\widetilde{S}_T - K)^+ \big| \widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right] \\ &= xN(d_+(T-t,s)) - Ke^{-r(T-t)}N(d_-(T-t,s)). \end{aligned}$$

where under $\mathbb{Q}$, $\widetilde{S} \sim \text{GBM}\,(r,\sigma^2)$. Thinking of c as a function of the volatility σ, in this exercise we will investigate the *vega* $c_\sigma(t,s;\sigma)$.

(a) Show for all (t,s) that $(s - Ke^{-r(T-t)})^+ \leq c(t,s) \leq s$. **Hint**: use the expected value representation for $c(t,s)$ along with $(a-b) \leq (a-b)^+ \leq a$ for all reals $a, b > 0$.

(b) In class we saw "$c(t,s)$ = huge formula" but in reality the formula is not that bad. Indeed, show that

$$c_\sigma(t,s;\sigma) = K\sqrt{T-t}e^{-r(T-t)}\dot{N}\left(d_-(T-t,s)\right)$$

Conclude that $c_\sigma(t,s;\sigma) > 0$ and hence the call option price is strictly increasing in the volatility. **Hint**: use the result from part (i) of Exercise 4.9 in the class textbook above.

(c) Show that $\lim_{\sigma\to 0} c(t,s;\sigma) = (s - Ke^{-r(T-t)})^+$ and $\lim_{\sigma\to\infty} c(t,s;\sigma) = s$.

Based upon the above results we see for $t \leq T, S_t = s$ that given any "market" call price c^{mkt} lying within the "reasonable" range

$$c^{\text{mkt}} \in \left(\left(s - Ke^{-r(T-t)}\right)^+, s\right)$$

there is a unique volatility $\widehat{\sigma} = \widehat{\sigma}(t,s)$ such that

$$c^{\text{mkt}} = c(t,s;\widehat{\sigma}).$$

This volatility is called the "Black-Scholes implied volatility", and is widely used when quoting options prices. It is also what gives rise to the "implied volatility surface".

Solution

(a) As per the hint we have

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(\widetilde{S}_T - K)\big|\widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right] \leq c(t,s) \leq \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\widetilde{S}_T\big|\widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right].$$

At time t we can hedge the payoff $S_T - K$ being long one share of the stock and short $Ke^{-r(T-t)}$ in the money market. This costs us $S_t - Ke^{-r(T-t)}$. Thus, by our pricing convention and the previous results on pricing options with payoffs of the form $f(S_T)$ we see that

$$s - Ke^{-r(T-t)} = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(\widetilde{S}_T - K)\middle|\widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right].$$

This gives $c(t,s) \geq s - Ke^{-r(T-t)}$ but of course, $c(t,s) \geq 0$. Therefore $c(t,s) \geq (s - Ke^{-r(T-t)})^+$. Similarly, at t we can hedge the payoff S_T by being long one share of the stock. This gives

$$s = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\widetilde{S}_T\middle|\widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right],$$

and the result follows.

(b) Write $\tau = T - t$. Using the hint we have $s\dot{N}(d_+(\tau,s)) = Ke^{-r\tau}\dot{N}(d_-(\tau,s))$. Next, let us write

$$d_\pm(\tau,s,\sigma) = \frac{\log\left(\frac{s}{K}\right) + r\tau}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}.$$

to highlight the dependence on σ. This gives, suppressing the function arguments in $d_\pm$,

$$\begin{aligned}
c_\sigma(t,s;\sigma) &= s\dot{N}(d_+)\left(\frac{1}{2}\sqrt{\tau} - \frac{\log\left(\frac{s}{K}\right) + r\tau}{\sigma^2\sqrt{\tau}}\right) \\
&\qquad + Ke^{-r\tau}\dot{N}(d_-)\left(\frac{1}{2}\sqrt{\tau} + \frac{\log\left(\frac{s}{K}\right) + r\tau}{\sigma^2\sqrt{\tau}}\right), \\
&= Ke^{-r\tau}\dot{N}(d_-)\left(\frac{1}{2}\sqrt{\tau} - \frac{\log\left(\frac{s}{K}\right) + r\tau}{\sigma^2\sqrt{\tau}}\right) \\
&\qquad + Ke^{-r\tau}\dot{N}(d_-)\left(\frac{1}{2}\sqrt{\tau} + \frac{\log\left(\frac{s}{K}\right) + r\tau}{\sigma^2\sqrt{\tau}}\right), \\
&= Ke^{-r\tau}\dot{N}(d_-)\sqrt{\tau}.
\end{aligned}$$

(c) Again, write $\tau = T - t$. Direct calculation shows that, suppressing function arguments in $d_\pm$.

$$\lim_{\sigma\to\infty} N(d_+) = 1; \quad \lim_{\sigma\to\infty} N(d_-) = 0; \quad \lim_{\sigma\to 0} N(d_\pm) = \begin{cases} 1 & \log\left(\frac{s}{K}\right) + r\tau > 0 \\ \frac{1}{2} & \log\left(\frac{s}{K}\right) + r\tau = 0 \\ 0 & \log\left(\frac{s}{K}\right) + r\tau < 0 \end{cases}$$

This immediately gives

$$\lim_{\sigma\to\infty} c(t,s;\sigma) = x.$$

As for $\sigma \to 0$, first note that $\log(s/K) + r\tau \geq, \leq, = 0$ is the same as $s - Ke^{-r\tau} \geq, \leq, = 0$. This gives

$$\begin{aligned}\lim_{\sigma \to 0} c(t, s; \sigma) &= \left(s - Ke^{-r(T-t)}\right) 1_{s-Ke^{-r(T-t)}>0} + \frac{1}{2}\left(s - Ke^{-r(T-t)}\right) 1_{s-Ke^{-r(T-t)}=0} \\ &\quad + 0 \times 1_{s-Ke^{-r(T-t)}>0} \\ &= \left(s - Ke^{-r(T-t)}\right)^+ .\end{aligned}$$

3. Self-financing Trading Without Re-balancing (taken from Professor Shreve).

A summer quant intern is assigned the task of monitoring the effectiveness of a delta-hedging strategy for a long call position, where the call has expiry T and strike K. Given $t < T$ and $S_t = s$, we denote by $c(t, s; \sigma)$ the call price in the Black-Scholes model. We include σ to highlight the dependence on σ.

Assume the call expires n days from now, and set $t_0 = 0$, $t_n = T$ and t_j the time of market opening on day j. At time t_j, the stock price is S_{t_j}. The market price of the call is observed, yielding implied volatility σ_{t_j}. The delta-hedge

$$\Delta_{t_j} = c_s\left(t_j, S_{t_j}; \sigma_{t_j}\right)$$

is computed, and a short position in the stock of size Δ_{t_j} is taken. The portfolio holding the long call and the short stock position thus has opening of the day value

$$c\left(t_j, S_{t_j}; \sigma_{t_j}\right) - \Delta_{t_j} S_{t_j}.$$

The value of the portfolio at the close of the day is

$$c\left(t_{j+1}, S_{(t_{j+1})-}; \sigma_{t_j}\right) - \Delta_{t_j} S_{(t_{j+1})-},$$

where $S_{(t_{j+1})-}$ denotes the closing price of the stock on day j. The profit (normally called P&L for "profit and loss") on day j is thus

$$P_j = c\left(t_{j+1}, S_{(t_{j+1})-}; \sigma_{t_j}\right) - c\left(t_j, S_{t_j}; \sigma_{t_j}\right) - \Delta_{t_j}\left(S_{(t_{j+1})-} - S_{t_j}\right).$$

The intern is asked to monitor the daily P&L over the lifetime of the option and observe if $\sum_{j=0}^{n-1} P_j$ is approximately zero.

In this problem, we ask if there is any reason to expect $\sum_{j=0}^{n-1} P_j \approx 0$. We make the simplifying assumption that the implied volatilities all the take the same value, i.e. $\sigma_{t_j} = \sigma > 0$ for all j. We also assume the closing stock price on day j is the opening price on day $j + 1$. These assumptions are not satisfied in real markets. However, understanding whether $\sum_{j=0}^{n-1} P_j$ would be approximately zero under these idealized conditions provides insight into what to expect in real markets.

(a) Show that in the Black-Scholes model, $\sum_{j=0}^{n-1} P_j$ is approximately equal to an expression involving the call price at the final time, the call price at the initial time, and an integral with respect to the stock price.

(b) Show that the expression you obtained in (a) is not approximately zero, but rather is equal to a certain integral with respect to time. (Hint: Use Itô 's formula.)

(c) Conclude from your answer in (b) that $\sum_{j=0}^{n-1} P_j$ is approximately equal to

$$\sum_{j=0}^{n-1} r\big[c\big(t_j, S(t_j)\big) - \Delta(t_j)S(t_j)\big](t_{j+1} - t_j).$$

Hint Use the fact that $c(t, s; \sigma)$ satisfies the Black-Scholes partial differential equation.)

Note The expression obtained in (c) represents the earnings that would accrue if the daily portfolio value returned the risk-free rate r. This is the core of the Black-Scholes argument; the long call position together with the value of the hedge should return the risk-free rate on the net value of the long call and short stock position.

Solution

(a) We compute

$$\begin{aligned}
\sum_{j=0}^{n-1} P_j &= \sum_{j=0}^{n-1} \left(c\left(t_{j+1}, S_{(t_{j+1})-}; \sigma\right) - c\left(t_j, S_{t_j}; \sigma\right)\right) - \sum_{j=0}^{n-1} \Delta_{t_j}\left(S_{(t_{j+1})-} - S_{t_j}\right) \\
&= c\left(T, S_T; \sigma\right) - c\left(0, S_0; \sigma\right) - \sum_{j=0}^{n-1} \Delta_{t_j}\left(S_{(t_{j+1})-} - S_{t_j}\right) \\
&\approx c\left(T, S_T; \sigma\right) - c\left(0, S_0; \sigma\right) - \int_0^T \Delta_t dS_t,
\end{aligned}$$

where Δ_t appearing in the above integrand is $c_s(t, S_t)$.

(b) According to the Itô's formula

$$\begin{aligned}
dc(t, S_t; \sigma) &= c_t(t, S_t; \sigma)dt + c_s(t, S_t; \sigma)dS_t + \frac{1}{2}c_{ss}(t, S_t; \sigma)d[S, S]_t \\
&= c_t(t, S_t; \sigma)dt + \Delta_t dS_t + \frac{1}{2}\sigma^2 S_t^2 c_{ss}(t, S_t; \sigma)dt.
\end{aligned}$$

Integrating from $t = 0$ to $t = T$, we obtain

$$c(T, S_T; \sigma) - c(0, S_0; \sigma) - \int_0^T \Delta_t dS_t = \int_0^T \left(c_t(t, S_t; \sigma) + \frac{1}{2}\sigma^2 S_t^2 c_{ss}(t, S_t; \sigma)\right) dt.$$

(iii) According to the Black-Scholes partial differential equation,

$$c_t(t, s; \sigma) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t, s; \sigma) = rc(t, s; \sigma) - rsc_s(t, s; \sigma),$$

and from (a) and (b) we have

$$\begin{aligned}
\sum_{j=0}^{n-1} P_j &\approx \int_0^T r\left(c(t, S_t; \sigma) - S_t c_s(t, S_t; \sigma)\right) dt \\
&= \int_0^T r\left(c(t, S_t; \sigma) - \Delta_t S_t\right) dt \\
&\approx \sum_{j=0}^{n-1} r\left(c(t_j, S_{t_j}; \sigma) - \Delta_{t_j} S_{t_j}\right)(t_{j+1} - t_j).
\end{aligned}$$