MF 790 HW 2 Part 2

This assignment is due on Thursday, September 30th at 8 AM. POINTS Problems 1 and 4 are worth 10 points each. Problems 2 and 3 are worth 15 points each.

1. Continuous Martingales have Rough Paths. In this exercise we will see that all non-constant, Martingales with continuous sample paths have very rough sample paths. To see this, fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and let $M = (M_t)_{t\geq 0}$ is a continuous time Martingale with continuous paths. Assume $M_0(\omega) = 0$ for all ω and $\mathbb{E}\left[M_t^2\right] < \infty$ for all $t \geq 0$.

Let $t > 0$ and $\Pi = \{0 = t_0 < t_1 < ... < t_n = t\}$ be any partition of $[0, t]$. Show that

$$\mathbb{E}\left[[M, M]_t^{\Pi}\right] = \mathbb{E}\left[M_t^2\right].$$

Therefore (you do NOT have to prove the following statement) taking $\|\Pi\| \downarrow 0$ it follows that $\mathbb{E}\left[[M, M]_t\right] = \mathbb{E}\left[M_t^2\right] > 0$, where this last inequality follows as M_t is non-constant. Thus (at least on average), $[M, M]_t > 0$ and hence, as we saw in class and in the last homework, the first variation is infinite.

2. Example of an Integral for a Simple Integrand. Do Exercise 4.3 on page 190 of the class textbook.

3. Integral for Simple, Non-random Integrands. Do exercise 4.2 on page $189-190$ of the class textbook. Note : as mentioned in part (i) of the problem, it suffices to prove part (i) assuming that s and t are on the partition.

4. Simple Approximation for $\int W dW$. Let W be a Brownian Motion, $T > 0$ and $\Pi : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0, T]$. Consider the simple process Δ^n given by

$$\Delta_t^n = \sum_{i=1}^{n} W_{t_{i-1}} 1_{(t_{i-1}, t_i]}(t)$$

Show that

$$\mathbb{E}\left[\int_0^T \left(\Delta_t^n - W_t\right)^2 dt\right] = \frac{1}{2}\sum_{i=1}^{n}(t_i - t_{i-1})^2$$

and hence

$$\lim_{\|\Pi\|\to 0} \mathbb{E}\left[\int_0^T \left(\Delta_t^n - W_t\right)^2 dt\right] = 0$$

Thus, the stochastic integral $\int W_t dW_t$ is well defined as the limit of the integrals for the simple processes Δ^n.