MF 790 Stochastic Calculus
Final Exam. December 15, 2021
SOLUTIONS

This is the final exam. There are 4 questions for a total 100 points. Each question may contain multiple parts. You have between 8:30 and 10:30 AM to complete the exam

The exam is closed book, notes, cheat sheets, calculator, smart phone and smart watch.

You must upload your answers to Questrom Tools by 10:45 AM. If you type your answers or write them on a note-taking software program, upload your file to Questrom tools. If you write your answers on paper, take a picture of each page you would like to submit and upload the picture file to Questrom tools.

Write your name on every page of your exam (i.e. on every sheet of paper that you turn in)!

If you are stuck on a problem, MOVE ON to other parts of the exam and come back later. Also if you unsure of the answer, write as much as you can so that you can receive partial credit. Blank answers will receive 0 points. Also, please explain your reasoning/provide a derivation for your answers. Answers with no explanation will also receive no credit. Good luck!

Formulas.

(1) The standard normal cumulative distribution function (cdf)

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy.$$

(2) The price of a call and put option (with maturity T and strike K) in the Black-Scholes model at $t \leq T$ given the stock price is x.

$$c(t,x) = xN(d_+(\tau,x)) - Ke^{-r\tau}N(d_-(\tau,x))$$
$$p(t,x) = Ke^{-r\tau}N(-d_-(\tau,x)) - xN(-d_+(\tau,x))$$

where $\tau = T - t$ and

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}} \left(\log\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau \right)$$

(3) The delta for a call option in the Black-Scholes model at $t \leq T$ and given the stock price is x

$$\partial_x c(t,x) = N(d_+(\tau,x))$$

Abbreviations.

(1) PDE: Partial Differential Equation.
(2) SDE: Stochastic Differential Equation.
(3) ZCB: Zero Coupon Bond.

Throughout, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is fixed, $W = \{W_t\}_{t\geq 0}$ is a Brownian motion, and for any question regarding contingent claim pricing, you may assume $\mathbb{F} = \mathbb{F}^W$ is generated by W.

1.(20 points). Let B be a second Brownian motion independent of W, and define the process X via

$$X_t := \frac{1}{2}(W_t - B_t)^2; \qquad t \geq 0.$$

(a) (**10 Points**) Identify a function $\Sigma(t,x)$ such that $d[X,X]_t = \Sigma(t, X_t)dt$.
(b) (**10 Points**) Let $g(t,x)$ be a smooth function (not identically zero) for $t \geq 0$ and $x > 0$. Set up a differential equation for g so that $t \to g(t, X_t)$ is a martingale. You do not have to solve this equation, you simply have to set it up.

Solution:

(a) Note that

$$X_t = \frac{1}{2}W_t^2 + \frac{1}{2}B_t^2 - W_tB_t.$$

Thus, by Itô and stochastic integration by parts

$$\begin{aligned} dX_t &= W_tdW_t + \frac{1}{2}dt + B_tdB_t + \frac{1}{2}dt - d(W_tB_t), \\ &= W_tdW_t + B_tdB_t + dt - W_tdB_t - B_tdW_t - d[W,B]_t, \\ &= (W_t - B_t)dW_t - (W_t - B_t)dB_t + dt, \end{aligned}$$

where we have used the independence of W, B to say that $d[W,B]_t = 0$. Again using $d[W,B]_t = 0$ we obtain

$$\begin{aligned} d[X,X]_t &= (W_t - B_t)^2(d[W,W]_t - 2d[W,B]_t + d[B,B]_t); \\ &= 2(W_t - B_t)^2dt; \\ &= 4X_tdt. \end{aligned}$$

Thus, we may take $\Sigma(t,x) = \Sigma(x) = 4x$.

(b) By Itô's formula we obtain

$$dg(t,X_t) = g_t(t,X_t)dt + g_x(t,X_t)dX_t + \frac{1}{2}g_{xx}(t,X_t)d[X,X]_t.$$

Plugging in gives

$$\begin{aligned} dg(t,X_t) &= g_t(t,X_t)dt + g_x(t,X_t)\left((W_t - B_t)dW_t - (W_t - B_t)dB_t + dt\right) \\ &\qquad + \frac{1}{2}g_{xx}(t,X_t)\Sigma(t,X_t)dt; \\ &= \left(g_t(t,X_t) + g_x(t,X_t) + 2X_tg_{xx}(t,X_t)\right)dt \\ &\qquad + g_x(t,X_t)\left((W_t - B_t)dW_t - (W_t - B_t)dB_t\right). \end{aligned}$$

We see that $g(t,X_t)$ will be a martingale if the dt terms vanish.To ensure this, we want g to satisfy

$$g_t(t,x) + g_x(t,x) + 2xg_{xx}(t,x) = 0; \qquad t \geq 0, x > 0.$$

and this is our differential equation.

2. (35 Points) Bond Pricing, Forward Measure and PDEs. Assume under risk neutral measure the money market process R satisfies the SDE

$$dR_t = \mu(t, R_t)dt + a(t, R_t)dW_t^{\mathbb{Q}},$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$ Brownian motion, and the functions μ, a are "nice" (i.e. smooth, bounded, $a > 0$, etc). Similarly, under forward measure $\widetilde{\mathbb{P}}$ we model the forward price process via

$$\frac{d\mathrm{For}_t}{\mathrm{For}_t} = \sigma(t)dW^{\widetilde{\mathbb{P}}},$$

where $W^{\widetilde{\mathbb{P}}}$ is a $\widetilde{\mathbb{P}}$ Brownian motion, and σ is a strictly positive (nice) function of time. Lastly, recall the discount process

$$D_t = e^{-\int_0^t R_u du}; \qquad t \leq T.$$

(a) (**10 Points**) For a ZCB with maturity T, the price at time $t \leq T$ is

$$B(t, T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}\big|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T R_u du}\big|\mathcal{F}_t\right].$$

We expect $B(t, T) = b(t, R_t)$ for a certain function b of (t, r). Write down (you DO NOT have to re-derive it, or do anything else) the PDE that b should solve.

(b) (**10 Points**) Recall that $\widetilde{\mathbb{P}}$ is defined through the Randon-Nikodym derivative

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}} = \widetilde{Z}_T := \frac{D_T}{B(0, T)}.$$

(i) (**5 Points**) Show the density $\widetilde{Z}_t := \mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_T\big|\mathcal{F}_t\right], t \leq T$ takes the form $\widetilde{Z}_t = D_t B(t, T)/B(0, T)$.

(ii) (**5 Points**) Show the time $t \leq T$ value of the claim is $V_t = B(t, T)\mathbb{E}^{\widetilde{\mathbb{P}}}\left[V_T\big|\mathcal{F}_t\right]$.

(c) (**15 Points**) For a given bounded function χ, assume we own a European option with payoff $V_T = \chi(S_T)$ at T. Similarly to part (a) we expect

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\left[V_T\big|\mathcal{F}_t\right] = f(t, \mathrm{For}_t), \tag{0.1}$$

for a certain function f of (t, y). Write down the PDE that f should solve, but unlike in part (a), verify that if f solves the PDE then (0.1) holds. Therefore, we see that

$$V_t = b(t, R_t)f(t, \mathrm{For}_t) =: v(t, R_t, \mathrm{For}_t).$$

so the price at t is a function of time, the current money market rate and the current forward price.

Solution

(a) As we saw in class, the PDE for b is

$$0 = b_t(t,r) + \mu(t,r)b_r(t,r) + \frac{1}{2}a^2(t,r)b_{rr}(t,r) - rb(t,r), \quad t < T, r \in \mathbb{R},$$
$$1 = b(T,r), \quad r \in \mathbb{R}.$$

(b) (i) We have

$$\widetilde{Z}_t = \mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_T \middle| \mathcal{F}_t\right] = \frac{D_t}{B(0,T)}\mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}\middle|\mathcal{F}_t\right] = \frac{D_t B(t,T)}{B(0,T)}.$$

(ii) Note that

$$\frac{\widetilde{Z}_T}{\widetilde{Z}_t} = \frac{D_T}{D_t B(t,T)}.$$

Using this we obtain

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}V_T\middle|\mathcal{F}_t\right] = B(t,T)\mathbb{E}^{\mathbb{Q}}\left[\frac{\widetilde{Z}_T}{\widetilde{Z}_t}V_T\middle|\mathcal{F}_t\right] = B(t,T)\mathbb{E}^{\widetilde{\mathbb{P}}}\left[V_T\middle|\mathcal{F}_t\right].$$

(c) The PDE for f is

$$0 = f_t(t,y) + \frac{1}{2}\sigma^2(t)f_{yy}(t,y); \quad t < T, y > 0,$$
$$\chi(y) = f(T,y); \quad y > 0.$$

Now, assume f solves this PDE. By Itô and $\mathrm{For}_T = S_T$ we have

$$\begin{aligned} V_T &= \chi(S_T) = \chi(\mathrm{For}_T) = f(T,\mathrm{For}_T), \\ &= f(t,\mathrm{For}_t) + \int_t^T\left(f_t(u,\mathrm{For}_u) + \frac{1}{2}\sigma^2(u)f_{yy}(u,\mathrm{For}_u)\right)du + M_T - M_t, \\ &= f(t,\mathrm{For}_t) + M_T - M_t, \end{aligned}$$

where M is the $\widetilde{\mathbb{P}}$ martingale $M_t = \int_0^t f_y(u,\mathrm{For}_u)\sigma(u)dW_u^{\widetilde{\mathbb{P}}}, t \le T$. This implies

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\left[V_T\middle|\mathcal{F}_t\right] = f(t,\mathrm{For}_t).$$

3. (20 points, 5 points each part) (True/False). Indicate whether or not each of the following statements is true or false. If it is true, explain why. If it is false, either explain why or give a counter example. Answers with no explanation get no credit!

(a) In the Black-Scholes model, if the initial price is less than the strike price ($S_0 < K$) then the time zero price of a call option with strike K is less than the time zero price of a put option with strike K.

(b) The minimum of two stopping times σ and τ is also a stopping time.

(c) A self-financing trading strategy in (1) a discrete dividend paying stock and (2) a money market with constant rate $r > 0$, experiences a jump in the wealth process at each dividend payment date.

(d) Let τ be the first time the Brownian motion W reaches the level 5. Then, for any time t, the expected value of W_t given $\tau > t$ (i.e. $\mathbb{E}\left[W_t 1_{\tau>t}\right]$ is negative.

Solution:

(a) FALSE. Put-call parity gives $S_T - K = (S_T - K)^+ - (K - S_T)^-$. As the discounted stock price is a martingale under $\mathbb{Q}$ this implies
$$S_0 - Ke^{-rT} = c(0, S_0) - p(0, S_0).$$
Thus, the put is worth more if and only if $K > S_0 e^{rT}$ not necessarily $K > S_0$.

(b) TRUE. Fix $t \geq 0$. We have
$$\{\sigma \wedge \tau > t\} = \{\sigma > t\} \cap \{\tau > t\} = (\{\sigma \leq t\})^c \bigcap (\{\tau \leq t\})^c .$$
By definition, each of $\{\tau \leq t\}, \{\sigma \leq t\}$ are in $\mathcal{F}_t$. As any sigma-algebra is closed under complementation and intersection, $\{\sigma \wedge \tau > t\} \in \mathcal{F}_t$ and hence $\{\sigma \wedge \tau \leq t\} \in \mathcal{F}_t$, ensuring it is a stopping time.

(c) FALSE. There is no jump in the wealth process because the dividend payments (scaled by the position size) are reinvested back into the portfolio, offsetting the jump in the stock price.

(d) TRUE. By the Optional Sampling Theorem, as $\sigma = \tau \wedge t$ is a bounded stopping time (for t fixed) we have
$$0 = \mathbb{E}\left[W_{\tau \wedge t}\right] = 5\mathbb{P}\left[\tau \leq t\right] + \mathbb{E}\left[W_t 1_{t > \tau}\right].$$
Now, $W_t \sim N(0, t)$ and hence $\mathbb{P}\left[\tau \leq t\right] \geq \mathbb{P}\left[W_t \geq 5\right] > 0$. But, this means $\mathbb{E}\left[W_t 1_{t>\tau}\right] < 0$.

4. (25 Points) Geometric Average Call Option. As in the Black-Scholes model, assume the stock S and discount process D evolve according to
$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t; \qquad \frac{dD_t}{D_t} = -r dt,$$
where $\mu, \sigma > 0$ and $r > 0$ are constant. For a maturity $T > 0$ and strike K, write $c(t, x; r, \sigma^2)$ as the value the call option if $S_t = x$. Here, we include r and σ^2 in the formula for c to be explicit about their dependence.

The *geometric* average of S over $[0, T]$ is
$$\bar{S}^g_T := e^{\frac{1}{T}\int_0^T \log(S_t) dt}.$$
Our goal is to price a call option on $\bar{S}^g_T$ with strike K. To do this, we write $V_T = \left(\bar{S}^g_T - K\right)^+$ and use the fact (you DO NOT have to show this) that under any probability measure $\widetilde{\mathbb{P}}$, if $W^{\widetilde{\mathbb{P}}}$ is a $\widetilde{\mathbb{P}}$ Brownian motion then
$$\int_0^T W^{\widetilde{\mathbb{P}}}_t dt \overset{\widetilde{\mathbb{P}}}{\sim} N\left(0, \frac{T^3}{3}\right).$$

(a) (**10 Points**) Find constants $\widetilde{r}, \widetilde{\sigma}^2$ such that under $\mathbb{Q}$, the random variable $\bar{S}^g_T$ has the same distribution as the random variable $\widetilde{S}_T$, where $\widetilde{S}$ is a geometric Brownian motion with parameters $\widetilde{r}, \widetilde{\sigma}^2$, and with initial value S_0.

(b) (**15 Points**) Given your answer in part (a), identify V_0, the price of V_T at time 0, in terms of the call price function c. If you cannot explicitly identify $\widetilde{r}, \widetilde{\sigma}^2$ in part (a), just assume part (a) holds for some $\widetilde{r}, \widetilde{\sigma}^2$ and identify V_0.

Solution:

(a) Under $\mathbb{Q}$, we know

$$S_t = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W_t^{\mathbb{Q}}}; \qquad t \leq T,$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$ Brownian motion. Therefore

$$\begin{aligned}
\frac{1}{T}\int_0^T \log(S_t)dt &= \frac{1}{T}\int_0^T \left(\log(S_0) + \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{\mathbb{Q}}\right) dt, \\
&= \log(S_0) + \left(r - \frac{1}{2}\sigma^2\right)\frac{T}{2} + \frac{\sigma}{T}\int_0^T W_t^{\mathbb{Q}} dt, \\
&\overset{\mathbb{Q}}{\sim} \log(S_0) + \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T + \frac{\sigma}{T}N\left(0, \frac{T^3}{3}\right), \\
&\overset{\mathbb{Q}}{\sim} \log(S_0) + \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T + \frac{\sigma}{\sqrt{3}}N\left(0, T\right),
\end{aligned}$$

where the third equality uses the hint and the fourth equality uses the scaling property of normal random variables. Now, if $\widetilde{S}$ is a geometric Brownian motion with parameters $\widetilde{r}$ and $\widetilde{\sigma}^2$ under $\mathbb{Q}$ which starts at S_0 then we have

$$\log\left(\widetilde{S}_T\right) \overset{\mathbb{Q}}{\sim} \log(S_0) + \left(\widetilde{r} - \frac{1}{2}\widetilde{\sigma}^2\right)T + \widetilde{\sigma}N(0, T).$$

So, we can match the distributions of $\bar{S}^g_T$ and $\widetilde{S}_T$ if we equate

$$\widetilde{r} - \frac{1}{2}\widetilde{\sigma}^2 = \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right); \qquad \widetilde{\sigma} = \frac{\sigma}{\sqrt{3}}.$$

This gives

$$\widetilde{\sigma} = \frac{\sigma}{\sqrt{3}}; \qquad \widetilde{r} = \frac{1}{2}\left(r - \frac{1}{2}\sigma^2 + \widetilde{\sigma}^2\right) = \frac{r}{2} - \frac{\sigma^2}{12}.$$

(b) We know that

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(\bar{S}^g_T - K\right)^+\right].$$

By part (a) we know that $\bar{S}^g_T$ is distributed under $\mathbb{Q}$ like a geometric Brownian motion $\widetilde{S}_T$ with parameters $\widetilde{r}, \widetilde{\sigma}^2$ and which starts at S_0. We thus have

$$\begin{aligned}
V_0 &= e^{(\widetilde{r}-r)T}\mathbb{E}^{\mathbb{Q}}\left[e^{-\widetilde{r}T}\left(\widetilde{S}_T-K\right)^+\right], \\
&= e^{(\widetilde{r}-r)T}c\left(0,S_0;\widetilde{r},\widetilde{\sigma}^2\right), \\
&= e^{-\left(\frac{r}{2}+\frac{\sigma^2}{12}\right)T}c\left(0,S_0;\frac{r}{2}-\frac{\sigma^2}{12},\frac{\sigma}{\sqrt{3}}\right).
\end{aligned}$$

Note that the second equality above holds because for any $a>0, b^2>0$:

$$c(t,S_0;a,b^2)=\mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-aT}\left(\widetilde{S}_T-K\right)^+\right],$$

where $\widetilde{\mathbb{P}}$ is any probability measure such that $\widetilde{S}$ is a geometric Brownian motion under $\widetilde{\mathbb{P}}$ with parameters (a,b^2) and starting value S_0.