Forwards and Futures

MF 790 Stochastic Calculus

Outline

- Forwards
- Futures

Forwards + Futures.

"The usual" model

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$$

$$\frac{dD_t}{D_t} = -R_t dt$$

μ, σ, R : process

$\theta_t = \frac{\mu_t - R_t}{\sigma_t}$: mkt price of risk process.

$$\frac{dQ}{dP} = Z_T = e^{-\int_0^T \theta_u dW_u - \frac{1}{2}\int}$$

under risk neutral measure Q,

$$\frac{dS_t}{S_t} = R_t dt + \sigma_t dW_t^Q$$

Time $t<T$ price of ZCB maturing at T is

$$B(t,T) = \mathbb{E}^Q[e^{-\int_t^T R_u du} | \mathcal{F}_t]$$

$$= \mathbb{E}^Q[\frac{D_T}{D_t} | \mathcal{F}_t]$$

Fix $t<T$. A forward contract initiated at t is a agreement (made at t) to purchase S at T for a pre-determined price K_t. K_t must be s.t.

K_t is known at t ($\mathcal{F}_t$-mbl)

There is no exchange of money at t.

We identify K_t via risk-neutral pricing.

Cash flow of $S_T - K_t$ at T

$$\Rightarrow V_t = \mathbb{E}^Q[\frac{D_T}{D_t}(S_T - K_t) | \mathcal{F}_t] \overset{?}{=} 0$$

$$\mathbb{E}^Q[\frac{D_T}{D_t} S_T | \mathcal{F}_t] = \frac{1}{D_t} \mathbb{E}^Q[D_T S_T | \mathcal{F}_t] = D_t S_t / D_t = S_t$$

$$\mathbb{E}^Q[\frac{D_T}{D_t}(S_T - K_t) | \mathcal{F}_t] \overset{?}{=} 0$$

$$= S_t - K_t \underbrace{\mathbb{E}^Q[D_T/D_t | \mathcal{F}_t]}_{B(t,T)} = S_t - K_t B(t,T)$$

$$\Rightarrow K_t = For_t = \frac{S_t}{B(t,T)}$$

Intuitively, we put today's stock price worth of money into the money market account.

$$For_t = \frac{S_t}{B(t,T)}$$

Constant rate model $\quad R_t \equiv r$

$$\begin{aligned} B(t,T) &= E^{Q}[e^{-r(T-t)} \mid \mathcal{F}_t] \\ &= e^{-r(T-t)} \end{aligned}$$

$$For_t = S_t e^{-r(T-t)}$$

Model

- Fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let W be a B.M. and $\mathbb{F} = \mathbb{F}^W$.
- Dynamics for the asset S and discount process D
  - $\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$.
  - $\frac{dD_t}{D_t} = -R_t dt$
    - $\mu, \sigma > 0, R$: processes such that S, D are well defined.
- Let $\Theta := \frac{\mu - r}{\sigma}$ and define the risk neutral measure $\mathbb{Q}$ on $\mathcal{F}_T$ ($T > 0$ is fixed throughout) by
  - $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ where $\frac{dZ_t}{Z_t} = -\Theta_t dW_t, \; Z_0 = 1$.
- Under $\mathbb{Q}$
  - $W_t^{\mathbb{Q}} = W_t + \int_0^t \Theta_u du, t \leq T$ is a Brownian motion.
  - $\frac{dS_t}{S_t} = R_t dt + \sigma_t dW_t^{\mathbb{Q}}$.

Zero Coupon Bond Price

- For any claim with time T payoff V_T ($\mathcal{F}_T$ mbl), the price at $t \leq T$ is
  - $V_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t} V_T \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T R_u du} V_T \middle| \mathcal{F}_t\right].$
- In particular, the time t price of a zero coupon bond (ZCB) which pays \$1 at T is
  - $B(t, T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t} \middle| \mathcal{F}_t\right].$

Forward Contract

- Let $t \leq T$. A forward contract on S, initiated at t, is an agreement made at time t, to buy S at T for a price K_t.
  - K_t must be known at t ($\mathcal{F}_t$ mbl).
  - K_t is determined precisely so no money is exchanged at t.
    - It costs nothing to enter the contract.
- Original motivation for the contract.
  - Producers (or sellers) of goods wanted to "lock-in" prices ahead of time, in order to avoid future adverse price movements.

Pricing a Forward Contract

- How can we identify the forward price K_t?
- Two questions
  - (1) What is the cash flow? Answer: $S_T - K_t$.
  - (2) When is the cash flow? Answer: at T.
- Thus, the time t price is
  - $V_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}(S_T - K_t)\,\middle|\mathcal{F}_t\right]$.

$\mathcal{F}_t$ mbl K_t so $V_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}(S_T - K_t)\middle|\mathcal{F}_t\right] = 0$?

- As the discounted stock price is a $\mathbb{Q}$ martingale
  - $\mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}S_T\middle|\mathcal{F}_t\right] = \frac{1}{D_t}\mathbb{E}^{\mathbb{Q}}\left[D_T S_T\middle|\mathcal{F}_t\right] = \frac{1}{D_t}D_t S_t = S_t$.
- As K_t must be $\mathcal{F}_t$ mbl
  - $\mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}K_t\middle|\mathcal{F}_t\right] = K_t\mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}\middle|\mathcal{F}_t\right] = K_t B(t,T)$.
- Thus, we see that $V_t = 0$ provided
  - $K_t = \mathrm{For}_t := \frac{S_t}{B(t,T)}$.

$$K_t = \mathrm{For}_t := \frac{S_t}{B(t,T)}$$

- Example: Black-Scholes.
  - $S_t = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W_t^{\mathbb{Q}}}$.
  - $B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\middle|\mathcal{F}_t\right] = e^{-r(T-t)}$.
  - $\mathrm{For}_t = S_t e^{r(T-t)}$.

Futures Contract.

It is similar to a forward contract in that it is an agreement to purchase S at T.

It differs to mitigate two risks.

1) Counterparty default

— trading on an exchange

2) Terminal cash flow risk $(S_T >>> k_e)$

— Settle price changes daily

— Marking to margin

Let us first price a futures contract in discrete time.

$$0 = t_0 < t_1 < \cdots < t_n = T$$

$$(t_k, t_{k+1}) = \text{"day"}$$

We will assume R is constant over the day.

$$\frac{D_{t_{k+1}}}{D_{t_k}} = e^{-R_{t_k}(t_{k+1} - t_k)}$$

$\uparrow$ Rate is picked at the beginning of the day

For now, let $\{Fut_{t_k}\}_{k=0}^{n}$ be the TBD futures price process.

1) $Fut_T = Fut_{t_n} = S_T = S_{t_n}$

— no arbitrage

As with the forward, we identify the futures price $\{Fut_{t_k}, k < n\}$ using the principle that it costs nothing to enter the contract at ....

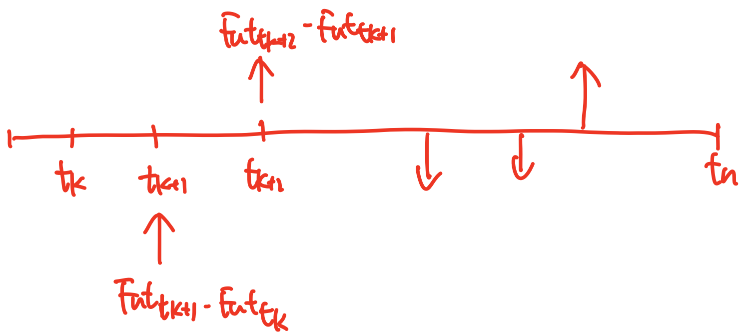
$$0 = \mathbb{E}^{\mathbb{Q}}[\text{Discounted Cash flows between } t_k \text{ and } t_n \mid \mathcal{F}_{t_k}]$$

daily flow: futures price change

(t_k, t_{k+1}): payment of $Fut_{t_{k+1}} - Fut_{t_k}$ at t_{k+1}.

$$0 = \mathbb{E}^{Q}\Big[\sum_{j=k}^{n-1} D_{t_k, t_{j+1}} \left(Fut_{t_{j+1}} - Fut_{t_j}\right) \Big| \mathcal{F}_{t_k}\Big]$$

discount factor between t_k and t_{j+1}



$$D_{t_k, t_{j+1}} = e^{-\sum_{\ell=k}^{j} R(t_\ell)(t_{\ell+1} - t_\ell)}$$

$\rightarrow$: $D_{t_k, t_{j+1}}$ is $\mathcal{F}_{t_j}$ mbl

We use "cost nothing to enter at any time" condition to identify $\{Fut_{t_k}, k<n\}$ $\quad Fut_{t_n} = S_{t_n}$.

Consider $t_k = t_{n-1}$

$$0 = \mathbb{E}^{Q}\left[D_{t_{n-1}, t_n}\left(Fut_{t_n} - Fut_{t_{n-1}}\right) \middle| \mathcal{F}_{t_{n-1}}\right]$$
$$= \mathbb{E}^{Q}\left[e^{-R(t_{n-1})(t_n - t_{n-1})}\left(S_{t_n} - Fut_{t_{n-1}}\right) \middle| \mathcal{F}_{t_{n-1}}\right]$$
$$\overset{?}{=} e^{-R(t_{n-1})(t_n - t_{n-1})}\left[\mathbb{E}^{Q}\left[S_{t_n} \middle| \mathcal{F}_{t_{n-1}}\right] - Fut_{t_{n-1}}\right]$$

$$\Rightarrow Fut_{t_{n-1}} = \mathbb{E}^{Q}\left[S_{t_n} \middle| \mathcal{F}_{t_{n-1}}\right]$$

Consider t_{n-2},

$$0 \overset{?}{=} \mathbb{E}^{Q}\Big[\underbrace{e^{-R(t_{n-2})(t_{n-1} - t_{n-2})}\left(Fut_{t_{n-1}} - Fut_{t_{n-2}}\right)}_{①} + \underbrace{e^{-R(t_{n-1})(t_n - t_{n-1})}\left(Fut_{t_n} - Fut_{t_{n-1}}\right)}_{②} \Big| \mathcal{F}_{t_{n-2}}\Big]$$

$$② = \mathbb{E}^{Q}\left[\mathbb{E}^{Q}\left[e^{-R(t_{n-1})(t_n - t_{n-1})}\left(S_{t_n} - \mathbb{E}^{Q}\left[S_{t_n} \middle| \mathcal{F}_{t_{n-1}}\right]\right) \middle| \mathcal{F}_{t_{n-1}}\right] \middle| \mathcal{F}_{t_{n-2}}\right] = 0$$

$$① = e^{-R(t_{n-2})(t_{n-1} - t_{n-2})}\left(\mathbb{E}^{Q}\left[\mathbb{E}^{Q}\left[S_{t_n} \middle| \mathcal{F}_{t_{n-1}}\right] \middle| \mathcal{F}_{t_{n-2}}\right] - Fut_{t_{n-2}}\right)$$

$\Rightarrow Fut_{t_{n-2}} = \mathbb{E}^{Q}[Fut_{t_{n-1}} | \mathcal{F}_{t_{n-2}}]$

$$\cdot\ Fut_{t_{n-2}} = \mathbb{E}^{Q}[Fut_{t_{n-1}} \mid \mathcal{F}_{t_{n-2}}]$$

$$= \mathbb{E}^{Q}[\mathbb{E}^{Q}[S_{t_n} \mid \mathcal{F}_{t_{n-1}}] \mid \mathcal{F}_{t_{n-2}}]$$

$$= \mathbb{E}^{Q}[S_{t_n} \mid \mathcal{F}_{t_{n-2}}]$$

Let's Guess $\quad Fut_{t_k} = \mathbb{E}^{Q}[S_{t_n} \mid \mathcal{F}_{t_k}] \qquad k=0,\dots,n$

1) $Fut_{t_n} = S_{t_n}$ ✓

2) $Fut_{t_{n-1}} = \mathbb{E}[S_{t_n} \mid \mathcal{F}_{t_{n-1}}]$ ✓

3) $Fut_{t_{n-2}} = \mathbb{E}[S_{t_n} \mid \mathcal{F}_{t_{n-2}}]$ ✓

$$0 = \mathbb{E}^{Q}\Big[\sum_{j=k}^{n-1} D_{t_k,t_{j+1}} (Fut_{t_{j+1}} - Fut_{t_j}) \mid \mathcal{F}_{t_k}\Big]$$

$$\mathbb{E}^{Q}[D_{t_k,t_{j+1}}(Fut_{t_{j+1}}, Fut_{t_j}) \mid \mathcal{F}_{t_k}]$$

$$= \mathbb{E}^{Q}[\underbrace{\mathbb{E}^{Q}[D_{t_k,t_{j+1}}(Fut_{t_{j+1}} - Fut_{t_j}) \mid \mathcal{F}_{t_j}]}_{①} \mid \mathcal{F}_{t_k}]$$

$$① = D_{t_k,t_{j+1}}\big(\underbrace{\mathbb{E}^{Q}[\mathbb{E}^{Q}[S_{t_n} \mid \mathcal{F}_{t_{j+1}}] \mid \mathcal{F}_{t_j}]}_{\mathbb{E}^{Q}[S_{t_n} \mid \mathcal{F}_{t_j}]\ \{\text{TOWER}\}} - \mathbb{E}^{Q}[S_{t_n} \mid \mathcal{F}_{t_j}]\big)$$

$= 0$. ✓

Discrete time $\qquad 0 = t_0 < t_1 < t_2 < \dots < t_n = T$

$$\Rightarrow Fut_{t_k} = \mathbb{E}^{Q}[S_{t_n} \mid \mathcal{F}_{t_k}] \leftarrow (Q\text{-mart})$$

In continuous time, we conjecture that

$$Fut_t = \mathbb{E}^{Q}[S_T \mid \mathcal{F}_t]$$

$Fut_T = S_T$ ✓

Q: what is the continuous time analog of "it costs nothing to enter the contract"?

$$0 = \mathbb{E}^{Q}\Big[\sum_{j=k}^{n-1} D_{t_k,t_{j+1}} (Fut_{t_{j+1}} - Fut_{t_j}) \mid \mathcal{F}_{t_k}\Big]$$

Analogous formula.

$$0 = \mathbb{E}^{Q}\Big[\int_t^T D_{t,u}\, d(Fut_u) \mid \mathcal{F}_t\Big] = \mathbb{E}^{Q}\Big[\int_t^T \frac{D_u}{D_t}\, d(Fut_u) \mid \mathcal{F}_t\Big]$$

– Does this hold?

$\underline{Fut_u = \mathbb{E}^Q[S_T \mid \mathcal{F}_u]}$

$\Rightarrow$ Fut is a Q-mart

$\mathbb{E}^Q[Fut_t \mid \mathcal{F}_s] = \mathbb{E}^Q[\mathbb{E}^Q[S_T \mid \mathcal{F}_t] \mid \mathcal{F}_s]$

$= \mathbb{E}^Q[S_T \mid \mathcal{F}_s] = Fut_s$

Mart. Repre.

$$Fut_u = \mathbb{E}^Q[S_T] + \int_0^u \Gamma_v \, dW_v^Q$$

$$0 \overset{!}{=} \mathbb{E}^Q\left[\int_t^T \frac{D_u}{D_t} d(Fut_u) \,\Big|\, \mathcal{F}_t\right]$$

$(d\,Fut_u = \Gamma_u \, dW_u^Q)$

$$= \frac{1}{D_t} \mathbb{E}^Q\left[\int_t^T D_u \Gamma_u \, dW_u^Q \,\Big|\, \mathcal{F}_t\right]$$

$$= \frac{1}{D_t} \mathbb{E}^Q[M_T - M_t \mid \mathcal{F}_t]$$

$$M_u \triangleq \int_0^u D_u \Gamma_u \, dW_u^Q$$

$= 0$

Thus, $Fut_t = \mathbb{E}^Q[S_T \mid \mathcal{F}_t]$

$$For_t = \frac{S_t}{B(t,T)} = \frac{\frac{1}{D_t}\mathbb{E}^Q[D_T S_T \mid \mathcal{F}_t]}{\frac{1}{D_t}\mathbb{E}^Q[D_T \mid \mathcal{F}_t]} = \frac{\mathbb{E}^Q[D_T S_T \mid \mathcal{F}_t]}{\mathbb{E}^Q[D_T \mid \mathcal{F}_t]}$$

$$S_t = \frac{1}{D_t}\mathbb{E}^Q[D_T S_T \mid \mathcal{F}_t]$$

We also know it cost nothing to enter into a <u>dynamic</u> strategy in the contract.

Δ: trading strategy.

(Discrete time)

$$\text{flow at } t_{j+1} = \Delta_{t_j}(\text{Fut}_{t_{j+1}} - \text{Fut}_{t_j})$$

$$0 \stackrel{?}{=} \mathbb{E}^{Q}\left[\int_t^T D_u / D_t \, \Delta_u \, d(\text{Fut}_u) \mid \mathcal{F}_t\right]$$

$$d\,\text{Fut}_u = P_u \, dW_u^Q$$

$$= \frac{1}{D_t}\mathbb{E}^{Q}\left[\int_t^T D_u \Delta_u P_u \, dW_u^Q \mid \mathcal{F}_t\right]$$

$$= \frac{1}{D_t}\mathbb{E}^{Q}\left[M_T^{\Delta} - M_t^{\Delta} \mid \mathcal{F}_t\right]$$

$$M_u^{\Delta} \triangleq \int_0^u D_u \Delta_u P_u \, dW_u^Q$$

$$= 0$$

$$\text{Fut}_t = \mathbb{E}^{Q}[S_T \mid \mathcal{F}_t] \qquad \text{For}_t = \frac{S_t}{B(t,T)} = \frac{\frac{1}{D_t}\mathbb{E}^{Q}[D_T S_T \mid \mathcal{F}_t]}{\frac{1}{D_t}\mathbb{E}^{Q}[D_T \mid \mathcal{F}_t]} = \frac{\mathbb{E}^{Q}[D_T S_T \mid \mathcal{F}_t]}{\mathbb{E}^{Q}[D_T \mid \mathcal{F}_t]}$$

$$\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\text{Cov}^{Q}(X,Y \mid G) = \mathbb{E}^{Q}[XY \mid G] - \mathbb{E}^{Q}[X \mid G]\,\mathbb{E}^{Q}[Y \mid G]$$

$$X = D_t, \quad Y = S_T, \quad G = \mathcal{F}_t$$

$$= \mathbb{E}^{Q}[D_T \mid \mathcal{F}_t](\text{for}_t - \text{Fut}_t)$$

$$S_T = \frac{1}{D_T} e^{\int_u^t \sigma_u dW_u^Q - \frac{1}{2}\int_u^t \sigma_u^2 du}$$

$$\text{Cov}^{Q}(D_T, S_T \mid \mathcal{F}_t) = \mathbb{E}^{Q}[D_T \mid \mathcal{F}_t](\text{for}_t - \text{Fut}_t)$$

We expect For > Fut $\Rightarrow$ positive Q correlation between D_T, S_T

For = Fut $\Rightarrow$ zero ...

For < Fut $\Rightarrow$ neg ...

If $R_t \equiv r$ is constant

$D_T = e^{-rT}$ is constant

$\Rightarrow \text{Cov}^{Q}(D_T, S_T \mid \mathcal{F}_t) = 0 \Rightarrow \text{For}_t = \text{Fut}_t$

— need a richer interest rate environment to see a price difference.

Futures Contract

- A futures contract on S, like a forward, is an agreement to buy S at time T for a certain value K.
- Futures, however, are traded over an exchange and
  - Price changes are settled daily through a margin account, which one must open to trade on the exchange.
    - This is called "marking to margin".
  - Gains and losses are realized during the lifetime of the futures contract rather than at the end like, as with forwards.
- Motivation for the contract: reduce *counter party credit risk* by trading on an exchange, settling daily.

Pricing a Futures Contract

- What does it mean to "settle prices changes daily through a margin account"?
  - How do we price the futures contract?
- First, assume discrete time $0 = t_0 < t_1 < ... < t_n = T$.
  - $[t_k, t_{k+1})$ is a "day" for $k = 0, ..., n-1$.
  - R is constant over the day: $\frac{D_{t_{k+1}}}{D_{t_k}} = e^{-R_{t_k}(t_{k+1}-t_k)}$.
  - Fut_{t_k}: (tbd) price of the futures contract at t_k.
    - No arbitrage implies $\mathrm{Fut}_{t_n} = S_T$.
  - The gain/loss over day k is $\mathrm{Fut}_{t_{k+1}} - \mathrm{Fut}_{t_k}$.
    - Realized at t_{k+1} and put into (deducted from) the margin account.

Pricing a Futures Contract: Discrete Time

- To determine Fut_{t_k}, we use the same principle as with the forward contract
  - At each t_k it costs nothing to enter the contract..
- Along with $\mathrm{Fut}_T = S_T$, this will determine Fut_{t_k}. Indeed, for each k
  - $0 = \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=k}^{n-1} D_{k,j+1} \underbrace{\left(\mathrm{Fut}_{t_{j+1}} - \mathrm{Fut}_{t_j}\right)}_{\text{gain over day j, realized at } t_{j+1}} \Big| \mathcal{F}_{t_k}\right].$
  - $D_{k,j+1} = e^{-\sum_{\ell=k}^{j} R(t_\ell)(t_{\ell+1}-t_\ell)}$: discrete discount factor between t_k and t_{j+1}, which is $\mathcal{F}_{t_j}$ mbl.

$$0 = \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=k}^{n-1} D_{k,j+1}\left(\mathrm{Fut}_{t_{j+1}} - \mathrm{Fut}_{t_j}\right)\middle|\mathcal{F}_{t_k}\right]. \quad \dagger$$

- Claim: $\mathrm{Fut}_{t_k} = \mathbb{E}^{\mathbb{Q}}\left[S_T \middle| \mathcal{F}_{t_k}\right]$ enforces $\dagger$.
  - Proof of claim: for $j = k, \ldots, n-1$, as $D_{k,j+1}$ is $\mathcal{F}_{t_j}$ mbl

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[D_{k,j+1}\left(\mathrm{Fut}_{t_{j+1}} - \mathrm{Fut}_{t_j}\right)\middle|\mathcal{F}_{t_k}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[D_{k,j+1}\left(\mathrm{Fut}_{t_{j+1}} - \mathrm{Fut}_{t_j}\right)\middle|\mathcal{F}_{t_j}\right]\middle|\mathcal{F}_{t_k}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[D_{k,j+1}\left(\mathbb{E}^{\mathbb{Q}}\left[\mathrm{Fut}_{t_{j+1}}\middle|\mathcal{F}_{t_j}\right] - \mathrm{Fut}_{t_j}\right)\middle|\mathcal{F}_{t_k}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[D_{k,j+1}\left(\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[S_T\middle|\mathcal{F}_{t_{j+1}}\right]\middle|\mathcal{F}_{t_j}\right] - \mathbb{E}^{\mathbb{Q}}\left[S_T\middle|\mathcal{F}_{t_j}\right]\right)\middle|\mathcal{F}_{t_k}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[D_{k,j+1}\left(\mathbb{E}^{\mathbb{Q}}\left[S_T\middle|\mathcal{F}_{t_j}\right] - \mathbb{E}^{\mathbb{Q}}\left[S_T\middle|\mathcal{F}_{t_j}\right]\right)\middle|\mathcal{F}_{t_k}\right] \\
&\quad = 0.
\end{aligned}$$

    - TOWER, TOWK, Definition of Fut_t, TOWER.

Futures Price in Continuous Time

- We just showed in the discrete case that
  - $\mathrm{Fut}_{t_k} = \mathbb{E}^{\mathbb{Q}}\left[S_T \middle| \mathcal{F}_{t_k}\right]$, $k = 0, \ldots, n$.
  - $0 = t_0 < t_1 < \ldots < t_n = T$.
- This suggests in continuous time we *should* have
  - $\mathrm{Fut}_t = \mathbb{E}^{\mathbb{Q}}\left[S_T \middle| \mathcal{F}_t\right]$, $t \leq T$.
- Clearly, $\mathrm{Fut}_T = S_T$. But, what is the continuous time analog of
  - "It costs nothing to enter the futures contract". I.e.
    - $0 = \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=k}^{n-1} D_{k,j+1}\left(\mathrm{Fut}_{t_{j+1}} - \mathrm{Fut}_{t_j}\right) \middle| \mathcal{F}_{t_k}\right]$.
    - For $k = 0, \ldots, n-1$.

$$0 = \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=k}^{n-1} D_{k,j+1}\left(\mathrm{Fut}_{t_{j+1}} - \mathrm{Fut}_{t_j}\right)\middle|\mathcal{F}_{t_k}\right]. \quad \dagger$$

- The continuous time analog of $\dagger$ is, for $t \leq T$
  - $0 = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \frac{D_u}{D_t} d\left(\mathrm{Fut}_u\right)\middle|\mathcal{F}_t\right] \ \ddagger$
- But, $\ddagger$ holds if $\mathrm{Fut}_t = \mathbb{E}^{\mathbb{Q}}\left[S_T \middle| \mathcal{F}_t\right]$.
  - We know $t \to \mathrm{Fut}_t$ is a $\mathbb{Q}$ martingale.
  - By mart. rep. $\mathrm{Fut}_t = \mathbb{E}^{\mathbb{Q}}\left[S_T\right] + \int_0^t \Gamma_u dW_u^{\mathbb{Q}}$ for some Γ.
  - Define the $\mathbb{Q}$ martingale $t \to M_t := \int_0^t D_u \Gamma_u dW_u^{\mathbb{Q}}$.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \frac{D_u}{D_t} d\left(\mathrm{Fut}_u\right)\middle|\mathcal{F}_t\right] &= \frac{1}{D_t}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T D_u \Gamma_u dW_u^{\mathbb{Q}} \middle| \mathcal{F}_t\right] \\
&= \frac{1}{D_t}\mathbb{E}^{\mathbb{Q}}\left[M_T - M_t \middle| \mathcal{F}_t\right] = 0.
\end{aligned}$$

$$\mathrm{Fut}_T = S_T, \quad 0 = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \tfrac{D_u}{D_t} d\left(\mathrm{Fut}_u\right) \middle| \mathcal{F}_t\right], t \leq T \quad \dagger$$

- $\mathrm{Fut}_t = \mathbb{E}^{\mathbb{Q}}\left[S_T \middle| \mathcal{F}_t\right]$ verifies $\dagger$.
- It costs nothing to enter into a futures contract at any $t \leq T$.
  - But, this was just to enter into <u>one</u> futures contract.
- In fact, it costs nothing to enter into *any* dynamic trading strategy in the futures contract.
  - Let Δ be a process and define the $\mathbb{Q}$ martingale $t \to M_t^{\Delta} := D_t \Delta_t \Gamma_t dW_t^{\mathbb{Q}}$.
  - $\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \frac{D_u}{D_t} \Delta_u d\left(\mathrm{Fut}_u\right) \middle| \mathcal{F}_t\right] = \frac{1}{D_t}\mathbb{E}^{\mathbb{Q}}\left[M_T^{\Delta} - M_t^{\Delta} \middle| \mathcal{F}_t\right] = 0.$

Forward Price versus Futures Price

- The forward and futures respective prices are
  - $\mathrm{For}_t = \frac{S_t}{B(t,T)} = \frac{\mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}S_T\middle|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}\middle|\mathcal{F}_t\right]} = \frac{\mathbb{E}^{\mathbb{Q}}\left[D_T S_T\middle|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[D_T\middle|\mathcal{F}_t\right]}.$
  - $\mathrm{Fut}_t = \mathbb{E}^{\mathbb{Q}}\left[S_T\middle|\mathcal{F}_t\right].$
- The conditional covariance of r.v. X, Y under $\mathbb{Q}$ given a σ-algebra $\mathcal{G}$ is
  - $\mathrm{Cov}^{\mathbb{Q}}\left(X, Y\middle|\mathcal{G}\right) := \mathbb{E}^{\mathbb{Q}}\left[XY\middle|\mathcal{G}\right] - \mathbb{E}^{\mathbb{Q}}\left[X\middle|\mathcal{G}\right]\mathbb{E}^{\mathbb{Q}}\left[Y\middle|\mathcal{G}\right].$

$$\mathrm{For}_t = \frac{\mathbb{E}^{\mathbb{Q}}\left[D_T S_T \middle| \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[D_T \middle| \mathcal{F}_t\right]} \quad \mathrm{Fut}_t = \mathbb{E}^{\mathbb{Q}}\left[S_T \middle| \mathcal{F}_t\right]$$

- Therefore, we see that
  - $$\begin{aligned} \mathrm{Cov}^{\mathbb{Q}}\left(D_T, S_T \middle| \mathcal{F}_t\right) &= \mathbb{E}^{\mathbb{Q}}\left[D_T S_T \middle| \mathcal{F}_t\right] - \mathbb{E}^{\mathbb{Q}}\left[D_T \middle| \mathcal{F}_t\right] \mathbb{E}^{\mathbb{Q}}\left[S_T \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[D_T \middle| \mathcal{F}_t\right] \left(\mathrm{For}_t - \mathrm{Fut}_t\right) \end{aligned}$$
- This lets us order the forward and futures price based on the $\mathbb{Q}$ - cond. covar. of D_T, S_T given $\mathcal{F}_t$.
  - Positive: $\Rightarrow \mathrm{For}_t > \mathrm{Fut}_t$.
  - Negative: $\Rightarrow \mathrm{For}_t < \mathrm{Fut}_t$.
  - Zero: $\Rightarrow \mathrm{For}_t = \mathrm{Fut}_t$.
- E.g. constant money market rate so $D_T = e^{-rT}$.
  - Zero covar, so $\mathrm{For}_t = \mathrm{Fut}_t = S_t e^{r(T-t)}$.
- We need a richer model for the interest rate to separate forward and futures prices.