MF 790 Recommended HW 7, Part 2 - SOLUTIONS

This assignment is recommended to help you study for the final exam.

1. Drift-less geometric Brownian might not increase by any fixed amount? Let W be a Brownian motion, and consider the process $M \sim \mathrm{GBM}\left(0, \sigma^2\right), \sigma > 0$ which has dynamics

$$\frac{dM_t}{M_t} = \sigma dW_t; \qquad M_0 = 1.$$

Using the optional sampling theorem, show for any $\varepsilon > 0$ that if we set τ_ε as the hitting time to $1+\varepsilon$

$$\tau_\varepsilon(\omega) \;:=\; \inf\left\{t \geq 0 \mid M_t(\omega) = 1+\varepsilon\right\},$$

then $\mathbb{P}\left[\tau_\varepsilon = \infty\right] > 0$. This result seems remarkable, given how much a Brownian motion varies, but it is true. There is non-zero likelihood a drift-less geometric Brownian motion reaches any fixed level above it's starting point.

Warning: we only proved optional sampling for bounded stopping times.

Hint: you may use without proof that $\mathbb{P}\left[\tau_\varepsilon < \infty\right] = \lim_n \mathbb{P}\left[\tau_\varepsilon \leq n\right]$.

Solution Using optional sampling on the bounded stopping time $\tau_\varepsilon \wedge n$, the martingale property for M implies

$$1 = \mathbb{E}\left[M_{\tau_\varepsilon \wedge n}\right] = (1+\varepsilon)\mathbb{P}\left[\tau_\varepsilon \leq n\right] + \mathbb{E}\left[M_n 1_{\tau_\varepsilon > n}\right] \geq (1+\varepsilon)\mathbb{P}\left[\tau_\varepsilon \leq n\right].$$

This gives

$$\mathbb{P}\left[\tau_\varepsilon < \infty\right] = \lim_n \mathbb{P}\left[\tau_\varepsilon \leq n\right] \leq \frac{1}{1+\varepsilon},$$

which in turn implies

$$\mathbb{P}\left[\tau_\varepsilon = \infty\right] \geq \frac{\varepsilon}{1+\varepsilon}.$$

2. Non negative martingales get stuck at 0**.** Let M be a non-negative martingale with continuous paths starting at 1, and let τ be the hitting time to 0 of M. Now, it is certainly possible that $\tau = \infty$ with probability one (for example, this is the case for M from problem 1). However, in this exercise we will show that if $\mathbb{P}\left[\tau < \infty\right] > 0$ then M gets stuck at 0 once it hits zero.

This exercise has important implications for the stock price process S in an arbitrage free model. Indeed, as S is a non-negative $\mathbb{Q}$ martingale, the result implies that if S hits 0, it must stay there. Try to think from an arbitrage perspective why it is "obvious" that we need S, if it can hit zero, to stay there.

Show the result in the following steps

(a) Fix a $t > 0$ and define the bounded stopping times $\tau_n = \tau \wedge n$ and $\sigma_n = (\tau + t) \wedge n$. Show that $\{\tau \leq n\} \in \mathcal{F}_{\tau_n}$.

(b) Assume the following result (you do not have to prove this)

$$\mathbb{E}\left[M_{\tau+t}1_{\tau<\infty}\right] = \lim_{n} \mathbb{E}\left[M_{\sigma_n}1_{\tau\leq n}\right].$$

Using part (a) and optional sampling, show that $\mathbb{E}\left[M_{\tau+t}1_{\tau<\infty}\right] = 0$.

(c) Argue why part (b) gives the result.

Solution:

(a) We must verify that $\{\tau \leq n\} \cap \{\tau \wedge n \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$. This follows because τ is stopping time and hence

$$\{\tau \leq n\} \cap \{\tau \wedge n \leq t\} = \{\tau \leq t \wedge n\} = \begin{cases} \{\tau \leq t\} \in \mathcal{F}_t & t \leq n \\ \{\tau \leq n\} \in \mathcal{F}_n \subseteq \mathcal{F}_t & t > n \end{cases}.$$

(b) Given the limiting result we have, using TOWER, TOWK, part (a) and the optional sampling theorem (which we can apply because the stopping times are bounded)

$$\begin{aligned} \mathbb{E}\left[M_{\tau+t}1_{\tau<\infty}\right] &= \lim_{n} \mathbb{E}\left[M_{\sigma_n}1_{\tau\leq n}\right], \\ &= \lim_{n} \mathbb{E}\left[1_{\tau\leq n}\mathbb{E}\left[M_{\sigma_n}\middle|\mathcal{F}_{\tau_n}\right]\right], \\ &= \lim_{n} \mathbb{E}\left[1_{\tau\leq n}M_{\tau_n}\right], \\ &= \lim_{n} \mathbb{E}\left[1_{\tau\leq n}M_{\tau}\right] = 0. \end{aligned}$$

(c) The random variable $X := M_{\tau+t}1_{\tau<\infty}$ satisfies (i) $\mathbb{P}[X \geq 0] = 1$ and (ii) $\mathbb{E}[X] = 0$. This is only possible if $\mathbb{P}[X = 0] = 1$. Thus, have that almost surely

$$M_{\tau+t}1_{\tau<\infty} = 0,$$

which implies for any $t > 0$ that $M_{\tau+t} = 0$ with probability one, on the set $\{\tau < \infty\}$.