MF 790 HW 2 Part 1

This assignment is due on Thursday, September 30th at 8 AM. Problems 1 and 2 are worth 15 points each. Problems 3 and 4 are worth 10 points each.

1. Aspects of Brownian Motion. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be given and assume W is a Brownian Motion with respect to $\mathbb{F}$.

(a) Show that the process $\{X_t := W_t^2 - t\}_{t\geq 0}$ is a martingale.

(b) Show that $\{X_t := W_t^3\}_{t\geq 0}$ has constant expectation in time but is not a martingale. **Hint:** Expand $(W_t - W_s)^3 = W_t^3 - 3W_t^2 W_s + 3W_t W_s^2 - W_s^3$ and use part (a).

(c) (analog of $\mathbb{E}\left[S_1 \middle| S_3\right]$ for Brownian motion) For $s < t$, compute $\mathbb{E}\left[W(s) \middle| W(t)\right]$ **Hint:** Write $W_s = (W_s - cW_t) + cW_t$ for some constant c. Find c so that $W_s - cW_t$ and W_t are independent.

2. Brownian Motion Squared is Markov. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be given and assume W is a Brownian Motion with respect to $\mathbb{F}$. Show that the process $\{X_t := W_t^2\}_{t\geq 0}$ is Markov. **Warning and Hint:** As W is Markov we know

$$\mathbb{E}\left[g(X_t) \middle| \mathcal{F}_s\right] = \mathbb{E}\left[g(W_t^2) \middle| \mathcal{F}_s\right] = h(W_s),$$

for some function h. This does NOT imply X is Markov. To show that X is Markov, you must show that h is such that we can write $h(W_s) = \widetilde{h}(X_s)$ for some function $\widetilde{h}$.

3. Other Variations of Brownian Motion. Do Exercise 3.4 on page 117 of the class textbook (Vol. II).

4. A "Normal" Random Walk. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be given. Let $\{Z_j\}_{j=1,2,\ldots}$ be independent identically distributed (iid) $N(\mu, \sigma^2)$ random variables where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Similarly to the random walk discussed in class, define the discrete time stochastic process $X = \{X_n\}_{n=0,1,\ldots}$ by

$$X_0(\omega) = 0; \qquad X_n(\omega) = \sum_{j=1}^{n} Z_j(\omega); \quad n = 1, 2, \ldots.$$

Thus, X is a random walk which, at each time, moves according to an independent normal random variable. Lastly, define the filtration $\mathbb{F} = \{\mathcal{F}_n\}_{n=0,1,\ldots}$ by

$$\mathcal{F}_0 = \{\Omega, \emptyset\}; \qquad \mathcal{F}_n = \sigma(Z_1, \ldots, Z_n); \quad n = 1, 2, \ldots.$$

(a) For each n, identify the distribution of the quadratic variation process $[X, X]_n$.

(b) Show with probability one that

$$\lim_{n\uparrow\infty} \frac{[X, X]_n}{n}(\omega)$$

exists and identify the limit. Is this limit random? How does it compare to the "regular" random walk?