MF 790 HW 5, Part 1

This assignment is due on Thursday, November 11th at 8 AM. There is a reading assignment, and three problems. Problems 1,2 are worth 15 points each, and problem 3 is worth 20 points, for a total of 50 points.

0. Reading Assignment. Read carefully Chapters 5.2 and 5.3 on pages 210-224 of the class textbook. Here, the risk neutral pricing theory is extended to when the stock price process follows a generalized Geometric Brownian motion $S \sim \text{genGBM}\left(\alpha, \sigma^2\right)$ (α plays the role of μ) where α, σ are adapted processes. Also, the interest rate is no longer constant, but a time varying process R. The main results are the same, however, that using Itô's formula, Girsanov's theorem, Lévy's characterization and the Martingale representation theorem, it follows that the price of any contingent claim with $\mathcal{F}_T$ measurable payoff V_T at T is given at $t \leq T$ by

$$V_t = \frac{1}{D_t Z_t}\mathbb{E}\left[D_T Z_T V_T \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T R_u du} V_T \middle| \mathcal{F}_t\right].$$

Here, the discount process is $D_t = e^{-\int_0^t R_u du}$ and $Z \sim \text{genGBM}\left(0, \Theta^2\right)$ where

$$\Theta_t = \frac{\alpha_t - R_t}{\sigma_t}$$

is the market price of risk process. Lastly, under the risk neutral measure $\mathbb{Q}$, created by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$, it follows that $S \sim \text{genGBM}\left(R, \sigma^2\right)$ so that

$$\frac{dS_t}{S_t} = R_t dt + \sigma_t dW_t^{\mathbb{Q}}; \qquad W_t^{\mathbb{Q}} = W_t + \int_0^t \Theta_u du.$$

The arguments are essentially identical to what we did in class, but now allowing α, σ, R to be stochastic processes. Especially when we price fixed income instruments, it is vital we allow these quantities to be stochastic processes.

1. Implying the Risk-Neutral Distribution. Do Exercise 5.9 on page 255 of the class textbook (Vol II).

2. Hedging a Cash Flow. Do Exercise 5.11 on pages 256-257 of the class textbook (Vol II).

3. Forward Start Options (job interview question). Let $0 \leq t_1 \leq T$ be given. A *forward-start option* is a contract entered at time 0 that pays $(S_T - S_{t_1})^+$ at time T. In other words, it is a European call except that the strike is set "at the money" at time t_1 rather than being set at time 0. In this exercise, the underlying asset for the forward-start option is a geometric Brownian motion,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

where $W = \{W_t\}_{t \leq T}$ is a Brownian motion under the physical measure $\mathbb{P}$. The interest rate $r > 0$ is constant. The price at time zero of the forward-start option is

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(S_T - S_{t_1}\right)^+\right].$$

Let $c(\tau, s; K)$ denote the Black-Scholes price of a European call on S when the initial stock price is s, the strike price is K, and the time to expiration is τ : i.e.

$$c(\tau, s; K) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\left(S_T - K\right)^+ \,\middle|\, \mathcal{F}_{T-\tau}, S_{T-\tau} = s\right] \tag{0.1}$$

(a) Show that for every positive number y, we have the scaling property

$$yc(\tau, s; K) = c(\tau, ys; yK).$$

(b) Show that at time zero the forward-start option has price $c(\tau, S(0); K)$ for appropriate values of τ and K. Determine the appropriate values of τ and K.