MF 790 HW 2 Part 2 - SOLUTIONS

This assignment is due on Thursday, September 30th at 8 AM. POINTS Problems 1 and 4 are worth 10 points each. Problems 2 and 3 are worth 15 points each.

1. Continuous Martingales have Rough Paths. In this exercise we will see that all non-constant, Martingales with continuous sample paths have very rough sample paths. To see this, fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and let $M = (M_t)_{t\geq 0}$ is a continuous time Martingale with continuous paths. Assume $M_0(\omega) = 0$ for all ω and $\mathbb{E}\left[M_t^2\right] < \infty$ for all $t \geq 0$.

Let $t > 0$ and $\Pi = \{0 = t_0 < t_1 < ... < t_n = t\}$ be any partition of $[0, t]$. Show that

$$\mathbb{E}\left[[M, M]_t^{\Pi}\right] = \mathbb{E}\left[M_t^2\right].$$

Therefore (you do NOT have to prove the following statement) taking $\|\Pi\| \downarrow 0$ it follows that $\mathbb{E}\left[[M, M]_t\right] = \mathbb{E}\left[M_t^2\right] > 0$, where this last inequality follows as M_t is non-constant. Thus (at least on average), $[M, M]_t > 0$ and hence, as we saw in class and in the last homework, the first variation is infinite.

Solution: For the partition Π we have

$$[M, M]_t^{\Pi} = \sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})^2,$$

and hence

$$\begin{aligned}
\mathbb{E}\left[[M, M]_t^{\Pi}\right] &= \sum_{i=1}^{n} \mathbb{E}\left[(M_{t_i} - M_{t_{i-1}})^2\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[M_{t_i}^2\right] - 2\mathbb{E}\left[M_{t_i} M_{t_{i-1}}\right] + \mathbb{E}\left[M_{t_{i-1}}^2\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[M_{t_i}^2\right] - 2\mathbb{E}\left[\mathbb{E}\left[M_{t_i} M_{t_{i-1}} \middle| \mathcal{F}_{t_{i-1}}\right]\right] + \mathbb{E}\left[M_{t_{i-1}}^2\right] \text{ (Tower)} \\
&= \sum_{i=1}^{n} \mathbb{E}\left[M_{t_i}^2\right] - 2\mathbb{E}\left[M_{t_{i-1}} \mathbb{E}\left[M_{t_i} \middle| \mathcal{F}_{t_{i-1}}\right]\right] + \mathbb{E}\left[M_{t_{i-1}}^2\right] \text{ (TOWK)} \\
&= \sum_{i=1}^{n} \mathbb{E}\left[M_{t_i}^2\right] - \mathbb{E}\left[M_{t_{i-1}}^2\right] \text{ (martingale)} \\
&= \mathbb{E}\left[M_t^2\right] \; (M_0 = 0, \text{ telescoping sum})
\end{aligned}$$

giving the result.

2. Example of an Integral for a Simple Integrand. Do Exercise 4.3 on page 190 of the class textbook (Vol. II).

Solution

(i) FALSE. $I(t) - I(s) = W_s(W_t - W_s)$ which is not independent of $\mathcal{F}_s$ since W_s is $\mathcal{F}_s$ measurable.

(ii) FALSE. As per the hint:

$$\mathbb{E}\left[(I(t)-I(s))^4\right]=\mathbb{E}\left[W_s^4\right]\mathbb{E}\left[(W_t-W_s)^4\right]=9s^2(t-s)^2.$$

But,

$$\operatorname{Var}\left[I(t)-I(s)\right]=\mathbb{E}\left[W_s^2(W_t-W_s)^2\right]=s(t-s).$$

So, $I(t)-I(s)$ is not normally distributed.

(iii) TRUE.

$$\mathbb{E}\left[I(t)\middle|\mathcal{F}_s\right]=\Delta(0)W_s+W_s\mathbb{E}\left[W_t-W_s\right]=\Delta(0)W_s=I(s).$$

(iv) TRUE. Using the properties of Brownian Motion:

$$\begin{aligned}\mathbb{E}\left[I(t)^2\middle|\mathcal{F}_s\right]&=\Delta(0)^2\mathbb{E}\left[W_s^2\middle|\mathcal{F}_s\right]+2\Delta(0)\mathbb{E}\left[W_s^2(W_t-W_s)\middle|\mathcal{F}_s\right]\\&\qquad+\mathbb{E}\left[W_s^2(W_t-W_s)^2\middle|\mathcal{F}_s\right],\\&=\Delta(0)^2W_s^2+(t-s)W_s^2\\ \mathbb{E}\left[\int_0^t\Delta^2(u)du\middle|\mathcal{F}_s\right]&=\Delta(0)^2s+\mathbb{E}\left[W_s^2(t-s)\middle|\mathcal{F}_s\right]=\Delta(0)s+(t-s)W_s^2.\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}\left[I^2(t)-\int_0^t\Delta^2(u)du\middle|\mathcal{F}_s\right]&=\Delta(0)^2W_s^2+(t-s)W_s^2-\Delta(0)^2s-(t-s)W_s^2,\\&=\Delta(0)^2W_s^2-\Delta(0)^2s,\\&=I_s^2-\int_0^s\Delta(u)^2du.\end{aligned}$$

3. Integral for Simple, Non-random Integrands. Do exercise 4.2 on page 189 − 190 of the class textbook. Note : as mentioned in part (i) of the problem, it suffices to prove part (i) assuming that s and t are on the partition.

Solution

(i) Let $0\le s<t\le T$ and assume s and t are on the partition (e.g. $s=t_i, t=t_j$). We have

$$I(t)-I(s)=\sum_{l=i+1}^{j}\Delta_{t_{l-1}}\left(W_{t_l}-W_{t_{l-1}}\right)$$

Since Δ is non-random, $\Delta_{t_{l-1}}$ is independent of $\mathcal{F}_s$ for $l=i+1,\dots,j$. By the independent increment property $W_{t_l}-W_{t_{l-1}}$ is also independent of $\mathcal{F}_s$ for $l=i+1,\dots,j$. Therefore, $I(t)-I(s)$ is independent of $\mathcal{F}_s$.

(ii) Using the above formula and the independent, normal increment property, we have

$$I(t)-I(s)\sim N\left(0,\sum_{l=i+1}^{j}\Delta_{t_{l-1}}^2(t_l-t_{l-1})=\int_s^t\Delta_u^2du\right)$$

(iii) By (i) and (ii), $\mathbb{E}\left[I(t)-I(s)\middle|\mathcal{F}_s\right]=\mathbb{E}\left[I(t)-I(s)\right]=0$ and the martingale property follows.

(iv) Note that

$$I(t)^2 - I(s)^2 = \sum_{k,l=i+1}^{j} \Delta_{t_{k-1}} \Delta_{t_{l-1}} (W_{t_l} - W_{t_{l-1}})(W_{t_k} - W_{t_{k-1}})$$

Using the independent, normally distributed increment property it follows that $I(t)^2 - I(s)^2$ is independent of $\mathcal{F}_s$ and

$$\begin{aligned}
\mathbb{E}\left[I(t)^2 - I(s)^2 | \mathcal{F}_s\right] &= \mathbb{E}\left[I(t)^2 - I(s)^2\right] \\
&= \sum_{l=i+1}^{j} \Delta_{t_{l-1}}^2 (t_l - t_{l-1}) \\
&= \int_s^t \Delta_u^2 du
\end{aligned}$$

Since Δ is non-random, this gives that $I(t)^2 - \int_0^t \Delta_u^2 du$ is a Martingale.

4. Simple Approximation for $\int W dW$. Let W be a Brownian Motion, $T > 0$ and $\Pi : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0, T]$. Consider the simple process Δ^n given by

$$\Delta_t^n = \sum_{i=1}^{n} W_{t_{i-1}} 1_{(t_{i-1}, t_i]}(t)$$

Show that

$$\mathbb{E}\left[\int_0^T \left(\Delta_t^n - W_t\right)^2 dt\right] = \frac{1}{2} \sum_{i=1}^{n} (t_i - t_{i-1})^2$$

and hence

$$\lim_{\|\Pi\| \to 0} \mathbb{E}\left[\int_0^T \left(\Delta_t^n - W_t\right)^2 dt\right] = 0$$

Thus, the stochastic integral $\int W_t dW_t$ is well defined as the limit of the integrals for the simple processes Δ^n.

Solution Write W_i for W_{t_i}. Note that for $t_{i-1} < t \leq t_i$ we have that

$$\Delta_t^n - W_t = W_{i-1} - W_t$$

This gives that

$$\int_0^T (\Delta_t^n - W_t)^2 dt = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (\Delta_t^n - W_t)^2 dt = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (W_t - W_{i-1})^2 dt$$

Thus

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \left(\Delta_t^n - W_t\right)^2 dt\right] \\
&\qquad = \mathbb{E}\left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} (W_t - W_{i-1})^2 dt\right] = \sum_{i=1}^n \mathbb{E}\left[\int_{t_{i-1}}^{t_i} (W_t - W_{i-1})^2 dt\right] \\
&\qquad = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mathbb{E}\left[(W_t - W_{i-1})^2\right] dt = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (t - t_{i-1}) dt \\
&\qquad = \sum_{i=1}^n \frac{1}{2}(t_i - t_{i-1})^2
\end{aligned}
$$

Since $(1/2)\sum_{i=1}^n (t_i - t_i - 1)^2 \leq \frac{1}{2}\|\Pi\| T$ the result follows as $\|\Pi\| \downarrow 0$.