MF 790 HW 3 - SOLUTIONS

This assignment is due on Thursday, October 7th at 8 AM. The first problem is worth 20 points, while problems 2 and 3 are worth 15 points each.

1. Itô Integrals Using Midpoint, Right Endpoint. As seen in class, for a fixed $t > 0$ if we approximate $\int_0^t W_u dW_u$ by

$$\sum_{i=1}^{n} W_{t_{i-1}} \left(W_{t_i} - W_{t_{i-1}} \right) \tag{0.1}$$

for a given partition Π, then taking $\|\Pi\| \downarrow 0$ we obtain

$$\int_0^t W_u dW_u = \frac{1}{2} \left(W_t^2 - t \right), t \geq 0,$$

which (and this can be shown directly too) shows that $t \to W_t^2 - t$ is a martingale.

In this exercise we will see what happens if, instead of evaluating W at the left side of the interval in (0.1), we evaluate it at the midpoint or the right side of the interval. To make calculations easier we will assume, for each $t \geq 0$ that the partition is $0 < t/n < 2t/n < ... < nt/n = t$ and see what happens when $n \uparrow \infty$.

(a) (right hand side) Show

$$\lim_{n\uparrow\infty} \sum_{i=1}^{n} W_{t_i} \left(W_{t_i} - W_{t_{i-1}} \right) = \frac{1}{2} \left(W_t^2 + t \right)$$

Is $t \to W_t^2 + t$ a martingale?

(b) (midpoint) Write $W_i := W_{(t_i+t_{i-1})/2}$. Show

$$\lim_{n\uparrow\infty} \sum_{i=1}^{n} W_i \left(W_{t_i} - W_{t_{i-1}} \right) = \frac{1}{2} W_t^2$$

Is $t \to W_t^2$ a martingale?

Solution

(a) Note that

$$\begin{aligned}
\sum_{i=1}^{n} W_{t_i} \left(W_{t_i} - W_{t_{i-1}} \right) &= \sum_{i=1}^{n} \left(W_{t_i} - W_{t_{i-1}} \right)^2 + \sum_{i=1}^{n} W_{t_{i-1}} \left(W_{t_i} - W_{t_{i-1}} \right) \\
&\overset{n\uparrow\infty}{\longrightarrow} t + \int_0^t W_u dW_u, \\
&= t + \frac{1}{2}(W_t^2 - t) = \frac{1}{2}(W_t^2 + t).
\end{aligned}$$

$t \to W_t^2 + t$ is not a martingale. Indeed, using that $t \to W_t^2 - t$ is martingale we find

$$\mathbb{E}\left[W_t^2 + t \middle| \mathcal{F}_s \right] = 2t + \mathbb{E}\left[W_t^2 - t \middle| \mathcal{F}_s \right] = 2(t-s) + W_s^2 + s > W_s^2 + s,$$

so that in fact, it is a sub-martingale.

(b) Note that

$$W_i\left(W_{t_i}-W_{t_{i-1}}\right)=\left(W_i-W_{t_{i-1}}\right)\left(W_{t_i}-W_{t_{i-1}}\right)+W_{t_{i-1}}\left(W_{t_i}-W_{t_{i-1}}\right)$$
$$=\frac{1}{2}(W_{t_i}-W_{t_{i-1}})^2+W_{t_{i-1}}\left(W_{t_i}-W_{t_{i-1}}\right)$$

so that

$$\sum_{i=1}^{n}W_i\left(W_{t_i}-W_{t_{i-1}}\right)=\frac{1}{2}\sum_{i=1}^{n}\left(W_{t_i}-W_{t_{i-1}}\right)^2+\sum_{i=1}^{n}W_{t_{i-1}}\left(W_{t_i}-W_{t_{i-1}}\right)$$

The first term goes to $t/2$ and the second term goes to $\int_0^t W_u dW_u=\frac{1}{2}(W_t^2-t)$. This gives the first result. As for the martingale statement, note that

$$\mathbb{E}\left[W_t^2\middle|\mathcal{F}_s\right]=t+\mathbb{E}\left[W_t^2-t\middle|\mathcal{F}_s\right]=(t-s)+W_s^2,$$

so the process is not a martingale.

2. Practice with Itô's formula : Finding Martingales Associated to GBM. Let $\mu\in\mathbb{R}$ and $\sigma>0$ be constants. Recall that we say S is a Geometric Brownian Motion (GBM) with drift μ and volatility σ^2 if

$$S_t=S_0e^{(\mu-\sigma^2/2)t+\sigma W_t};\qquad t\geq 0,$$

where S_0 is a positive constant and W is a Brownian Motion. We write $S\sim\text{GBM}(\mu,\sigma^2)$.

(a) Give the Itô process decomposition for $f(S_t)$ when $f(x)=x^p, p\in\mathbb{R}$. Is $S^p\sim\text{GBM}(\alpha,\theta^2)$ for some α,θ? For a fixed μ,σ^2, is there a value of p for which S^p is a martingale?

(b) Give the Itô process decomposition for $f(S_t)$ when $f(x)=\log(x)$. Is there a value of μ,σ^2 for which $\log(S)$ is a Martingale?

(c) Give the Itô process decomposition for $f(t,S_t)$ when $f(t,x)=e^{-\lambda t}x^p$ for $\lambda,p\in\mathbb{R}$. Find $\hat{\lambda}$ so that for any given μ,σ and p, $t\to e^{-\hat{\lambda}t}S_t^p$ is a martingale. With this $\hat{\lambda}$ compute $\mathbb{E}\left[S_t^p\right]$ without using the p.d.f. for S_t.

Hint: For a given Itô process X with decomposition $dX_t=\Theta_t dt+\Delta_t dW_t$ if $\Theta_t(\omega)=0$ for (t,ω) then X is a Martingale.

Solution:

(a) We have $\dot{f}(x)=(p/x)f(x)$, $\ddot{f}(x)=(p(p-1)/x^2)f(x)$ so that

$$dS_t^p=\frac{p}{S_t}S_t^p dS_t+\frac{1}{2}\frac{p(p-1)}{S_t^2}S^p d[S,S]_t,$$
$$=pS_t^p\mu dt+pS_t^p\sigma dW_t+\frac{1}{2}p(p-1)S_t^p\sigma^2 dt,$$
$$=S_t^p\left(p\mu+\frac{1}{2}p(p-1)\sigma^2\right)dt+S_t^p\sigma dW_t.$$

Therefore, $S^p \sim \text{GBM}\left(p(\mu + (1/2)(p-1)\sigma^2), \sigma^2\right)$. Furthermore, S^p will be a martingale when the dt terms vanish, or equivalently, when

$$p = 1 - \frac{2\mu}{\sigma^2}.$$

(b) Since $\dot{\log}(x) = 1/x$, $\ddot{\log}(x) = -1/x^2$:

$$\begin{aligned} d\log(S)_t &= \frac{dS_t}{S_t} - \frac{1}{2}\frac{d[S,S]_t}{S_t^2}, \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t. \end{aligned}$$

Thus, for $\mu = (1/2)\sigma^2$ we have that $\log(S)$ is a martingale.

(c) For the given $f(t,x)$, since $x > 0$

$$\begin{aligned} f_t(t,x) &= -\lambda f(t,x); \qquad f_x(t,x) = \frac{p}{x}f(t,x); \\ f_{xx}(t,x) &= \frac{p(p-1)}{x^2}f(t,x). \end{aligned}$$

Using the time-dependent Itô 's formula gives

$$\begin{aligned} df(t,S_t) &= -\lambda f(t,S_t)dt + \frac{p}{S_t}f(t,S_t)dS_t + \frac{1}{2}\frac{p(p-1)}{S_t^2}d[S,S]_t \\ &= f(t,S_t)\left(\left(-\lambda + \mu p + \frac{1}{2}p(p-1)\sigma^2\right)dt + \sigma p dW_t\right) \end{aligned}$$

We thus get a martingale if $\hat{\lambda} = \mu p + (1/2)p(p-1)\sigma^2$. Since martingales have constant expectations we know that

$$1 = \mathbb{E}\left[e^{-\hat{\lambda}t}S_t^p\right] \implies \mathbb{E}\left[S_t^p\right] = e^{\hat{\lambda}t} = e^{\left(p\mu + \frac{1}{2}p(p-1)\sigma^2\right)t}$$

3. Derivation of Itô 's formula for general f''. Do exercise 4.14 on page 198−199 of the class textbook.

Solution:

(i) Z_j is clearly $\mathcal{F}_{t_{j+1}}$ measurable since f'' is continuous and W is adapted. Furthermore,

$$\mathbb{E}\left[Z_j \middle| \mathcal{F}_t\right] = f''(W_{t_j})\left(\mathbb{E}\left[(W_{t_{j+1}} - W_{t_j})^2\right] - (t_{j+1} - t_j)\right) = 0,$$

since W is adapted and $W_{t_{j+1}} - W_{t_j}$ is independent of $\mathcal{F}_{t_j}$ and normally distributed with mean 0 and variance $t_{j+1} - t_j$. Using the same properties of W

yields

$$\begin{aligned}
\mathbb{E}\left[Z_j^2\middle|\mathcal{F}_t\right] &= f''(W_{t_j})^2\mathbb{E}\left[\left((W_{t_{j+1}}-W_{t_j})^2-(t_{j+1}-t_j)\right)^2\right],\\
&= f''(W_{t_j})^2\mathbb{E}\left[(W_{t_{j+1}}-W_{t_j})^4-2(W_{t_{j+1}}-W_{t_j})^2(t_{j+1}-t_j)+(t_{j+1}-t_j)^2\right],\\
&= f''(W_{t_j})^2\left(3(t_{j+1}-t_j)^2-2(t_{j+1}-t_j)^2+(t_{j+1}-t_j)^2\right),\\
&= 2f''(W_{t_j})^2(t_{j+1}-t_j)^2.
\end{aligned}$$

(ii) Taking out what is known yields

$$\mathbb{E}\left[\sum_{j=0}^{n-1} Z_j\right] = \sum_{j=0}^{n-1}\mathbb{E}\left[Z_j\right] = \sum_{j=0}^{n-1}\mathbb{E}\left[\mathbb{E}\left[Z_j\middle|\mathcal{F}_{t_j}\right]\right] = 0.$$

(iii) Assuming $\mathbb{E}\left[\int_0^T f''(W_t)^2dt\right] < \infty$ and using part (ii) gives

$$\begin{aligned}
\operatorname{Var}\left[\sum_{j=0}^{n-1} Z_j\right] &= \mathbb{E}\left[\left(\sum_{j=0}^{n-1} Z_j\right)^2\right]\\
&= \sum_{j=0}^{n-1}\mathbb{E}\left[Z_j^2\right] + 2\sum_{j=0}^{n-1}\sum_{k=j+1}^{n-1}\mathbb{E}\left[Z_jZ_k\right].
\end{aligned}$$

Now, by part (i) for $j<k$

$$\begin{aligned}
\mathbb{E}\left[Z_j^2\right] &= \mathbb{E}\left[\mathbb{E}\left[Z_j^2\middle|\mathcal{F}_{t_j}\right]\right] = 2(t_{j+1}-t_j)^2\mathbb{E}\left[f''(W_{t_j})^2\right];\\
\mathbb{E}\left[Z_jZ_k\right] &= \mathbb{E}\left[Z_j\mathbb{E}\left[Z_k\middle|\mathcal{F}_{t_k}\right]\right] = 0.
\end{aligned}$$

This yields

$$\begin{aligned}
\operatorname{Var}\left[\sum_{j=0}^{n-1} Z_j\right] &= 2\mathbb{E}\left[\sum_{j=0}^{n-1} f''(W_{t_j})^2(t_{j+1}-t_j)^2\right],\\
&\le 2\|\Pi\|\mathbb{E}\left[\sum_{j=0}^{n-1} f''(W_{t_j})^2(t_{j+1}-t_j)\right].
\end{aligned}$$

As $\|\Pi\|\downarrow 0$, the variance goes to 0 since

$$\mathbb{E}\left[\sum_{j=0}^{n-1} f''(W_{t_j})^2(t_{j+1}-t_j)\right] \to \mathbb{E}\left[\int_0^T f''(W_t)^2dt\right] < \infty.$$