MF 790 HW 2 Part 1 - SOLUTIONS

This assignment is due on Thursday, September 30th at 8 AM. Problems 1 and 2 are worth 15 points each. Problems 3 and 4 are worth 10 points each.

1. Aspects of Brownian Motion. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be given and assume W is a Brownian Motion with respect to $\mathbb{F}$.

(a) Show that the process $\{X_t := W_t^2 - t\}_{t\geq 0}$ is a martingale.

(b) Show that $\{X_t := W_t^3\}_{t\geq 0}$ has constant expectation in time but is not a martingale. **Hint:** Expand $(W_t - W_s)^3 = W_t^3 - 3W_t^2 W_s + 3W_t W_s^2 - W_s^3$ and use part (a).

(c) (analog of $\mathbb{E}\left[S_1 \middle| S_3\right]$ for Brownian motion) For $s < t$, compute $\mathbb{E}\left[W(s) \middle| W(t)\right]$ **Hint:** Write $W_s = (W_s - cW_t) + cW_t$ for some constant c. Find c so that $W_s - cW_t$ and W_t are independent.

Solution

(a) X is clearly adapted to $\mathbb{F}$. Furthermore,

$$\mathbb{E}\left[|X_t\right] = \mathbb{E}\left[|W_t^2 - t|\right] \leq 2\mathbb{E}\left[W_t^2\right] + 2t = 4t < \infty,$$

so the conditional expectation of X_t given $\mathcal{F}_s$ is well defined for any $s < t$. Lastly, for $s < t$, using the independence and normality of increments

$$\begin{aligned}
\mathbb{E}\left[X_t \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[W_t^2 - t \middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[(W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2 - t \middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[(W_t - W_s)^2\right] + 2W_s\mathbb{E}\left[W_t - W_s\right] + W_s^2 - t \\
&= (t - s) + W_s^2 - t \\
&= W_s^2 - s = X_s,
\end{aligned}$$

and the martingale property follows.

(b) Since $W_t \sim N(0, t)$, $\mathbb{E}\left[X_t\right] = 0$ for all t. As for the non-Martingale claim, using the hint:

$$\begin{aligned}
\mathbb{E}\left[X_t \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[W_t^3 \middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[(W_t - W_s)^3 + 3W_t^2 W_s - 3W_t W_s^2 + W_s^3 \middle| \mathcal{F}_s\right], \\
&= \mathbb{E}\left[(W_t - W_s)^3\right] + 3W_s\mathbb{E}\left[W_t^2 \middle| \mathcal{F}_s\right] - 3W_s^2\mathbb{E}\left[W_t \middle| \mathcal{F}_s\right] + W_s^3, \\
&= 3W_s\mathbb{E}\left[W_t^2 - t \middle| \mathcal{F}_s\right] + 3W_s t - 3W_s^3 + W_s^3, \\
&= 3W_s(W_s^2 - s) + 3W_s t - 3W_s^3 + W_s^3, \\
&= W_s^3 + 3W_s(t - s) = X_t + 3W_s(t - s).
\end{aligned}$$

Since W_s is not identically 0, X is not a martingale. In the above equalities we have used

(i) Line (2) : $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$, linearity of conditional expectation, TOWK since W_s is $\mathcal{F}_s$ measurable.

(ii) Line (3) : subtracting and adding $3W_s t$.

(iii) Line (4) : $\mathbb{E}[X^3] = 0$ for $X \sim N(0, t-s)$, part (a) which showed that $W_t^2 - t$ was a Martingale and the fact that W is a Martingale.

(c) Using the hint, note that

$$\mathrm{Cov}[W_s - cW_t, cW_t] = \mathrm{Cov}[W_s, cW_t] - c^2\mathrm{Cov}[W_t, W_t] = cs - c^2 t$$

and so for $c = s/t$ we have that $\mathrm{Cov}[W_s - cW_t, cW_t] = 0$ which in turn implies that $W_s - cW_t$ and cW_t are independent (since they are jointly normally distributed). Thus

$$\mathbb{E}\left[W_s \middle| W_t\right] = \mathbb{E}\left[(W_s - cW_t) + cW_t \middle| W_t\right] = \mathbb{E}[W_s - cW_t] + cW_t = cW_t.$$

Thus, plugging in $c = s/t$ we find $\mathbb{E}\left[W_s \middle| W_t\right] = (s/t)W_t$. This is very interesting, because to estimate the average of W_s given the later W_t, we simply linearly interpolate between 0 (where $W_0 = 0$) and t (where $W_t = W_t$).

2. Brownian Motion Squared is Markov. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be given and assume W is a Brownian Motion with respect to $\mathbb{F}$. Show that the process $\{X_t := W_t^2\}_{t\geq 0}$ is Markov. **Warning and Hint:** As W is Markov we know

$$\mathbb{E}\left[g(X_t)\middle|\mathcal{F}_s\right] = \mathbb{E}\left[g(W_t^2)\middle|\mathcal{F}_s\right] = h(W_s),$$

for some function h. This does NOT imply X is Markov. To show that X is Markov, you must show that h is such that we can write $h(W_s) = \widetilde{h}(X_s)$ for some function $\widetilde{h}$.

Solution As in the hint, we know for $s < t$ the Markov property for W implies

$$\mathbb{E}\left[g(X_t)\middle|\mathcal{F}_s\right] = \mathbb{E}\left[g(W_t^2)\middle|\mathcal{F}_s\right] = h(W_s)$$

where (as shown in lecture)

$$h(y) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} g((x+y)^2) e^{-\frac{x^2}{2(t-s)}}\, dx$$

If we can show that h is even (i.e. $h(y) = h(-y)$) then $h(y) = h(|y|) = h(\sqrt{y^2})$ and so $h(W_t) = h(\sqrt{W_t^2}) = h(\sqrt{X_t}) =: \widetilde{h}(X_t)$ and hence X is Markov. Now, to show that h is even, note that

$$\begin{aligned}
h(y) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} g((x+y)^2) e^{-\frac{x^2}{2(t-s)}}\, dx \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} g((-x-y)^2) e^{-\frac{x^2}{2(t-s)}}\, dx \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t-s)}} g((z-y)^2) e^{-\frac{z^2}{2(t-s)}}\, dx \quad (z = -x) \\
&= h(-y)
\end{aligned}$$

The Markov property follows.

3. Other Variations of Brownian Motion. Do Exercise 3.4 on page 117 of the class textbook (Vol. II).

Solution

(i) Let Π be the partition $0 = t_0 < t_1 < ... < t_n = T$ for a fixed $T > 0$. As per the hint we have that

$$\sum_{j=1}^{n}(W_{t_j} - W_{t_{j-1}})^2(\omega) \leq \max_{j=1,\dots,n} \left|W_{t_j} - W_{t_{j-1}}\right|(\omega) \sum_{j=1}^{n} \left|W_{t_j} - W_{t_{j-1}}\right|(\omega). \tag{0.1}$$

Now, as $\|\Pi\| \downarrow 0$ we have with probability one that

$$\sum_{j=1}^{n} \left(W_{t_j} - W_{t_{j-1}}\right)^2(\omega) \to T,$$
$$\max_{j=1,\dots,n} \left|W_{t_j} - W_{t_{j-1}}\right|(\omega) \to 0.$$

The latter fact follows because the paths of W are almost surely continuous and hence uniformly continuous on the compact interval $[0, T]$. Thus if for some $K = K(\omega)$ we have, as $\|\Pi\| \downarrow 0$ that

$$\sum_{j=1}^{n} \left|W_{t_j} - W_{t_{j-1}}\right|(\omega) \leq K,$$

then the inequality in (0.1) is violated. Therefore the first order variation of Brownian Motion is infinite for all $T > 0$ with probability one.

(ii) In a very similar manner to (0.1) we have for any partition Π taking the form $0 = t_0 < t_1 < ... < t_n = T$ that

$$\sum_{j=1}^{n} \left|W_{t_j} - W_{t_{j-1}}\right|^3(\omega) \leq \max_{j=1,\dots,n} \left|W_{t_j} - W_{t_{j-1}}\right|(\omega) \sum_{j=1}^{n} \left(W_{t_j} - W_{t_{j-1}}\right)^2(\omega). \tag{0.2}$$

Again, as $\|\Pi\| \downarrow 0$ we have with probability one that

$$\sum_{j=1}^{n} \left(W_{t_j} - W_{t_{j-1}}\right)^2(\omega) \to T,$$
$$\max_{j=1,\dots,n} \left|W_{t_j} - W_{t_{j-1}}\right|(\omega) \to 0.$$

Thus, (0.2) implies that as $\|\Pi\| \downarrow 0$ with probability one:

$$\sum_{j=1}^{n} \left|W_{t_j} - W_{t_{j-1}}\right|^3(\omega) \to 0.$$

4. A "Normal" Random Walk. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be given. Let $\{Z_j\}_{j=1,2,\ldots}$ be independent identically distributed (iid) $N(\mu, \sigma^2)$ random variables where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Similarly to the random walk discussed in class, define the discrete time stochastic process $X = \{X_n\}_{n=0,1,\ldots}$ by

$$X_0(\omega) = 0; \qquad X_n(\omega) = \sum_{j=1}^{n} Z_j(\omega); \quad n = 1, 2, \ldots.$$

Thus, X is a random walk which, at each time, moves according to an independent normal random variable. Lastly, define the filtration $\mathbb{F} = \{\mathcal{F}_n\}_{n=0,1,\ldots}$ by

$$\mathcal{F}_0 = \{\Omega, \emptyset\}; \qquad \mathcal{F}_n = \sigma(Z_1, \ldots, Z_n); \quad n = 1, 2, \ldots.$$

(a) For each n, identify the distribution of the quadratic variation process $[X, X]_n$.
(b) Show with probability one that

$$\lim_{n \uparrow \infty} \frac{[X, X]_n}{n}(\omega)$$

exists and identify the limit. Is this limit random? How does it compare to the "regular" random walk?

Solution

(a) We have that

$$(X_j - X_{j-1})^2(\omega) = Z_j^2(\omega).$$

This gives

$$[X, X]_n(\omega) = \sum_{j=1}^{n} Z_j^2(\omega).$$

As $\left\{Z_j^2\right\}_{j=1,2,\ldots}$ are iid $N(\mu, \sigma^2)$ random variables it follows that $[X, X]_n$ is σ^2 times a non-central chi-square random variable with n degrees of freedom and non-centrality parameter $\lambda = n\mu^2$.

(b) Since $\mathbb{E}\left[Z_1^2\right] = \sigma^2$, $\mathrm{Var}\left[Z_1^2\right] < \infty$ it follows by the Strong Law of Large Numbers that with probability one

$$\lim_{n \uparrow \infty} \frac{[X, X]_n(\omega)}{n} = \lim_{n \uparrow \infty} \frac{1}{n} \sum_{j=1}^{n} Z_j^2(\omega) = \mu^2 + \sigma^2.$$

The limit is not-random and will differ from the regular random walk if $\mu^2 + \sigma^2 \neq 1$.