Forward Measure

MF 790 Stochastic Calculus

Outline

- Forward Contracts on Contingent Claims
- Forward Measure
- Modelling under Forward Measure

Model

- Fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let W be a B.M. and $\mathbb{F} = \mathbb{F}^W$.
- Dynamics for the asset S and discount process D
  - $\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t, \qquad \frac{dD_t}{D_t} = -R_t dt.$
    - $\mu, \sigma > 0$, R: processes such that S, D are well defined.
- Set $\Theta = (\mu - r)/\sigma$, and define the risk neutral measure $\mathbb{Q}$ on $\mathcal{F}_T$ ($T > 0$ fixed) by
  - $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ where $\frac{dZ_t}{Z_t} = -\Theta_t dW_t,\ Z_0 = 1.$
- Under $\mathbb{Q}$
  - $W_t^{\mathbb{Q}} = W_t + \int_0^t \Theta_u du, t \leq T$ is a Brownian motion.
  - $\frac{dS_t}{S_t} = R_t dt + \sigma_t dW_t^{\mathbb{Q}}.$

Contingent Claim Pricing Review

- Let V_T be a $\mathcal{F}_T$ mbl r.v. (”contingent claim”).
- For $t \leq T$ the price of the claim is
  - $V_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t} V_T \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T R_u du} V_T \middle| \mathcal{F}_t\right]$
  - IMPORTANT NOTE: $D_t V_t = \mathbb{E}^{\mathbb{Q}}\left[D_T V_T \middle| F_t\right]$ for $t \leq T$.
    - Discounted value process is a $\mathbb{Q}$ martingale for any V_T.
    - This generalizes the “discounted stock price is a $\mathbb{Q}$ martingale” statement we saw before.

Forward Contract on a Contingent Claim

- Now, assume at $t \leq T$ we enter into a forward contract on V_T.
- Agreement to buy V_T at T for $\mathrm{For}(V)_t$.
  - $\mathrm{For}(V)_t$ must be known at t.
    - $\mathrm{For}(V)_t$ is $\mathcal{F}_t$ mbl.
  - $\mathrm{For}(V)_t$ s.t. it costs nothing to enter into the agreement.
    - $0 = \mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}\left(V_T - \mathrm{For}(V)_t\right)\middle|\mathcal{F}_t\right]$.
- This gives the forward price
  - $\mathrm{For}(V)_t = \dfrac{\mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}V_T\middle|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t}\middle|\mathcal{F}_t\right]} = \dfrac{V_t}{B(t,T)}$.
    - Generalizes $\mathrm{For}_t = \frac{S_t}{B(t,T)}$ to arbitrary claims.

Forward Measure

- Define $\widetilde{Z}_T := \frac{D_T}{B(0,T)} = \frac{D_T}{\mathbb{E}^{\mathbb{Q}}[D_T]}$ so that
  - $\mathbb{Q}\left[\widetilde{Z}_T > 0\right] = 1, \qquad \mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_T\right] = \frac{\mathbb{E}^{\mathbb{Q}}[D_T]}{\mathbb{E}^{\mathbb{Q}}[D_T]} = 1.$
  - We may define a new measure on $\mathcal{F}_T$ using $\widetilde{Z}_T$.
- The *forward measure* is defined by
  - $\widetilde{\mathbb{P}}[A] := \mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_T 1_A\right] = \frac{\mathbb{E}^{\mathbb{Q}}[1_A D_T]}{B(0,T)}, \ A \in \mathcal{F}_T.$
  - $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}} = \widetilde{Z}_T = \frac{D_T}{B(0,T)}.$

Why Forward Measure?

- Why do we care about/study the forward measure?
- The forward measure has some very nice properties.
  - Forward price processes are martingales under forward measure.
  - Modeling forward prices under forward measure (as opposed to "spot" prices under risk neutral measure) allows us to separate the claim payoff from the discount factor.

Prices under Forward Measure

- The density process between $\widetilde{\mathbb{P}}$ and $\mathbb{Q}$ over $[0, T]$ is
  - $\widetilde{Z}_t = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_T \middle| \mathcal{F}_t\right], t \leq T.$
- We can also express this as
  - $\widetilde{Z}_t = \frac{1}{B(0,T)}\mathbb{E}^{\mathbb{Q}}\left[D_T \middle| \mathcal{F}_t\right] = \frac{D_t}{B(0,T)}\mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t} \middle| \mathcal{F}_t\right] = \frac{D_t B(t,T)}{B(0,T)}.$
  - This implies $\frac{\widetilde{Z}_T}{\widetilde{Z}_t} = \frac{D_T/B(0,T)}{D_t B(t,T)/B(0,T)} = \frac{D_T}{D_t B(t,T)}.$
- Thus, for any claim with $\mathcal{F}_T$ mbl payoff V_T
  - $\mathbb{E}^{\widetilde{\mathbb{P}}}\left[V_T \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{\widetilde{Z}_T}{\widetilde{Z}_t} V_T \middle| \mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t} V_T \middle| \mathcal{F}_t\right]}{B(t,T)} = \frac{V_t}{B(t,T)}.$

Forward Prices under Forward Measure

- We just showed $\mathbb{E}^{\widetilde{\mathbb{P}}}\left[V_T \middle| \mathcal{F}_t\right] = \frac{V_t}{B(t,T)}$.
  - But $\frac{V_t}{B(t,T)} = \mathrm{For}(V)_t$, so $\mathrm{For}(V)_t = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[V_T \middle| \mathcal{F}_t\right]$.
- The forward price process is a martingale under forward measure!
- When $V_T = S_T$ we have $\mathrm{For}_t = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[S_T \middle| \mathcal{F}_t\right]$.
  - Compare to $\mathrm{Fut}_t = \mathbb{E}^{\mathbb{Q}}\left[S_T \middle| \mathcal{F}_t\right]$.

Modelling under Forward Measure

- We now have two ways to express the contingent claim price at $t \leq T$.
  - (1) $V_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{D_t} V_T \middle| \mathcal{F}_t\right]$.
  - (2) $V_t = B(t,T)\mathrm{For}(V)_t = B(t,T)\mathbb{E}^{\widetilde{\mathbb{P}}}\left[V_T \middle| \mathcal{F}_t\right]$.
- (1) requires us to *jointly* model D and V under $\mathbb{Q}$.
  - Might be hard when rates are stochastic.
- (2) allows us to separately model D and V
  - We specify a model for D under $\mathbb{Q}$, and a model for $\mathrm{For}(V)$ under $\widetilde{\mathbb{P}}$.
    - Common industry practice to model forward prices under forward measure.

Application: Constant Vol. Forward Prices

- Modelling inputs
  - A general model for the rate R under $\mathbb{Q}$.
    - E.g. Hull & White, CIR, etc..
    - This will determine D, $B(\cdot, T)$ and hence $\widetilde{\mathbb{P}}$.
  - A $\mathrm{GBM}\,(0, \sigma^2)$ model for the forward price process under $\widetilde{\mathbb{P}}$.
    - $\frac{d\mathrm{For}_t}{\mathrm{For}_t} = \sigma dW_t^{\widetilde{\mathbb{P}}}$ (recall: $t \to \mathrm{For}_t$ must be a $\widetilde{\mathbb{P}}$ martingale).
- We will discuss later how to obtain both $W^{\widetilde{\mathbb{P}}}$, and the spot price dynamics under $\mathbb{Q}$.

Call Option Pricing

- Goal: price a call option on S: $V_T = (S_T - K)^+$.
  - Note: this is an option on S_T, not For_T.
- Claim: $V_t = S_t N(d_+(t)) - KB(t,T)N(d_-(t))$
  - N: $N(0,1)$ cdf., $d_\pm(t) = \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{\mathrm{For}_t}{K}\right) \pm \frac{1}{2}\sigma^2(T-t)\right)$.
- Similar to the "classical" Black-Scholes formula.
  - We now allow for random interest rates.
  - Differing assumption: constant vol. for forward prices.
- If $R \equiv r > 0$ is constant, we obtain the classical Black-Scholes formula.

Proof of $V_t = S_t N(d_+(t)) - KB(t,T)N(d_-(t))$

- We have already shown
  - $V_t = B(t,T)\mathbb{E}^{\widetilde{\mathbb{P}}}\left[(S_T - K)^+ \middle| \mathcal{F}_t\right]$.
- But, $S_T = \mathrm{For}_T = \mathrm{For}_t e^{-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T^{\widetilde{\mathbb{P}}} - W_t^{\widetilde{\mathbb{P}}})}$.
  - First equality by def., second by $\mathrm{For} \overset{\widetilde{\mathbb{P}}}{\sim} \mathrm{GBM}\,(0,\sigma^2)$.
- This gives
  - $V_t = B(t,T)\mathbb{E}^{\widetilde{\mathbb{P}}}\left[\left(\mathrm{For}_t e^{-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T^{\widetilde{\mathbb{P}}} - W_t^{\widetilde{\mathbb{P}}})} - K\right)^+ \middle| \mathcal{F}_t\right]$.

$$V_t = B(t,T)\mathbb{E}^{\widetilde{\mathbb{P}}}\left[\left(\mathrm{For}_t e^{-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T^{\widetilde{\mathbb{P}}}-W_t^{\widetilde{\mathbb{P}}})} - K\right)^+ \middle| \mathcal{F}_t\right]$$

- For_t is known at t, so on $\{\mathrm{For}_t = x\}$.
  - $V_t = B(t,T)\mathbb{E}^{\widetilde{\mathbb{P}}}\left[\left(xe^{-\frac{1}{2}\sigma^2(T-t)+\sigma(W_T^{\widetilde{\mathbb{P}}}-W_t^{\widetilde{\mathbb{P}}})} - K\right)^+ \middle| \mathcal{F}_t, \mathrm{For}_t = x\right]$
    $= B(t,T)\int_{\mathbb{R}}(xe^{-\frac{1}{2}\sigma^2(T-t)+\sigma z} - K)^+ \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{z^2}{2(T-t)}}\,dz$
    - $W_T^{\widetilde{\mathbb{P}}} - W_t^{\widetilde{\mathbb{P}}} \overset{\widetilde{\mathbb{P}}}{\sim} N(0, T-t)$, $W_T^{\widetilde{\mathbb{P}}} - W_t^{\widetilde{\mathbb{P}}} \perp\!\!\!\perp \mathcal{F}_t$.
- But,
  - $C^{BS}(t,x;r\equiv 0) = \int_{\mathbb{R}}(xe^{-\frac{1}{2}\sigma^2(T-t)+\sigma z} - K)^+ \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{z^2}{2(T-t)}}\,dz$.
- Thus, $V_t = B(t,T)C^{BS}(t,\mathrm{For}_t; r\equiv 0)$.
  - Claim follows by plugging in for $C^{BS}(t,x;0)$.

Dynamics under Forward Measure

- We assumed $\mathrm{For} \overset{\widetilde{\mathbb{P}}}{\sim} \mathrm{GBM}\left(0, \sigma^2\right)$.
  - How do we obtain the $\widetilde{\mathbb{P}}$ Brownian motion $W^{\widetilde{\mathbb{P}}}$?
  - What does this imply about the dynamics of S?
- For the density process $\widetilde{Z}$ between $\widetilde{\mathbb{P}}$ and $\mathbb{Q}$
  - $t \to \widetilde{Z}_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{D_T}{B(0,T)}\middle|\mathcal{F}_t\right]$ is a $\mathbb{Q}$ mart..
- By mart. rep. there is a process Γ such that
  - $\frac{d\widetilde{Z}_t}{\widetilde{Z}_t} = \Gamma_t dW_t^{\mathbb{Q}}$.
- Girsanov's theorem then implies
  - $W_t^{\widetilde{\mathbb{P}}} := W_t^{\mathbb{Q}} - \int_0^t \Gamma_u du, t \leq T$ is a $\widetilde{\mathbb{P}}$ Brownian motion.

Dynamics under Forward Measure

- What about the dynamics of S?
  - Recall that $\widetilde{Z}_t = \frac{D_t}{B(0,T)} \mathbb{E}^{\mathbb{Q}} \left[\frac{D_T}{D_t} \middle| \mathcal{F}_t \right] = \frac{D_t B(t,T)}{B(0,T)}$.
- This gives $S_t = B(t,T)\mathrm{For}_t = \frac{B(0,T)}{D_t} \widetilde{Z}_t \mathrm{For}_t$
- To compute the dynamics under $\mathbb{Q}$ we use Itô and
  - $d\left(\frac{1}{D_t}\right) = \frac{R_t}{D_t} dt, \qquad d\widetilde{Z}_t = \Gamma_t \widetilde{Z}_t dW_t^{\mathbb{Q}}$.
  - $d\mathrm{For}_t = \mathrm{For}_t \sigma dW_t^{\widetilde{\mathbb{P}}} = \mathrm{For}_t \sigma \left(dW_t^{\mathbb{Q}} - \Gamma_t dt \right)$.
  - $d\left[\widetilde{Z}, \mathrm{For}\right]_t = \widetilde{Z}_t \mathrm{For}_t \Gamma_t \sigma dt$.

Dynamics under Forward Measure

- We then have

$$
\begin{aligned}
dS_t &= R_t S_t dt + B(0,T)\frac{1}{D_t} d\left(\widetilde{Z}_t \mathrm{For}_t\right), \\
&= R_t S_t dt + B(0,T)\frac{1}{D_t}\Bigg(\widetilde{Z}_t \mathrm{For}_t \sigma \left(dW_t^{\mathbb{Q}} - \Gamma_t dt\right) + \mathrm{For}_t \widetilde{Z}_t \Gamma_t dW_t^{\mathbb{Q}} \\
&\qquad\qquad + \widetilde{Z}_t \mathrm{For}_t \Gamma_t \sigma dt\Bigg), \\
&= R_t S_t dt + S_t \left(\sigma + \Gamma_t\right) dW_t^{\mathbb{Q}}.
\end{aligned}
$$

- Volatility is now $\sigma + \Gamma_t$ where $d\widetilde{Z}_t = \widetilde{Z}_t \Gamma_t dW_t^{\mathbb{Q}}$.
  - $\widetilde{Z}_t = \frac{D_t B(t,T)}{B(0,T)}$.

$$\frac{dS_t}{S_t} = R_t dt + (\sigma + \Gamma_t)\, dW_t^{\mathbb{Q}}, \quad d\widetilde{Z}_t = \widetilde{Z}_t \Gamma_t dW_t^{\mathbb{Q}}, \quad \widetilde{Z}_t = \frac{D_t B(t,T)}{B(0,T)}$$

- Example: Hull & White model for R under $\mathbb{Q}$.
  - $dR_t = \kappa(t)\left(\theta(t) - R_t\right) dt + a(t) dW_t^{\mathbb{Q}}$.
  - $B(t, T) = e^{-A(t) - C(t) R_t}$ where
    - $C(t) = \int_t^T e^{-\int_t^u \kappa(v) dv} du$.
    - $A(t) = \int_t^T \left(\kappa(u)\theta(u)C(u) - \frac{1}{2} a(u)^2 C(u)^2\right) du$.
  - Using Feynman-Kač and Itô one can show
    - $\Gamma_t = -C(t)a(t)$.
  - $\frac{dS_t}{S_t} = R_t dt + \left(\sigma - C(t)a(t)\right) dW_t^{\mathbb{Q}}$.