MF 790 Stochastic Calculus
Midterm Exam. October 14, 2021
SOLUTIONS

This is the midterm exam. There are 4 questions for a total 100 points. Each question may contain multiple parts. You have between 8:15 AM and 10:15 AM to complete the exam

The exam is closed book, notes, cheat sheets, calculator, smart phone, smart watch. No internet searching for answers is allowed. You are on your honor to abide by these rules.

You must upload your answers to Questrom Tools by 10:30 AM. If you type your answers or write them on a note-taking software program, upload your file to Questrom tools. If you write your answers on paper, take a picture of each page you would like to submit and upload the picture file to Questrom tools.

Write your name on every page of your exam (i.e. on every sheet of paper that you turn in)!

If you are stuck on a problem, MOVE ON to other parts of the exam and come back later. Also if you unsure of the answer, write as much as you can so that you can receive partial credit. Blank answers will receive 0 points. Also, please explain your reasoning/provide a derivation for your answers. Answers with no explanation will also receive no credit. Good luck!

Throughout, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is fixed, and $W = \{W_t\}_{t\geq 0}$ is a Brownian motion.

1. (30 points, 15 points each).

(a) Obtain the Itô process decomposition for the process $X = \{X_t\}_{t\geq 0}$ defined by

$$X_t = e^{-t} W_t^2; \qquad t \geq 0.$$

(b) Now, consider the process $Y = \{Y_t\}_{t\geq 0}$ defined by

$$Y_t = W_t^2 + \int_0^t (W_u^3 - 1) du; \qquad t \geq 0.$$

Give the Itô process decomposition for Y. Is Y a Martingale?

Solution:

(a) $X_t = f(t, W_t)$ for $f(t,x) = e^{-t}x^2$. As $f_t(t,x) = -f(t,x)$, $f_x(t,x) = 2xe^{-t}$ and $f_{xx}(t,x) = 2e^{-t}$, by Itô

$$\begin{aligned} dX_t = df(t, W_t) &= \left(f_t(t, W_t) + \frac{1}{2} f_{xx}(t, W_t) \right) dt + f_x(t, W_t) dW_t, \\ &= e^{-t} \left(-W_t^2 + 1 \right) dt + 2e^{-t} W_t dW_t. \end{aligned}$$

(b) Note that $t \to \int_0^t (W_u^3 - 1) du$ is an Itô process. Thus,

$$\begin{aligned} dY_t &= d\left(W_t^2 + \int_0^t \left(W_u^3 - 1 \right) du \right), \\ &= 2W_t dW_t + dt + W_t^3 dt - dt, \\ &= 2W_t dW_t + W_t^3 dt. \end{aligned}$$

Since the dt terms are not zero, we know that Y is not a Martingale. As such, Y is a process with constant expectation but is not a Martingale.

2. True/False (20 points, 5 Points Each). Indicate whether or not each of the following statements is true or false. If it is true, explain why. If it is false, either explain why or give a counter example. Answers with no explanation get no credit!

(a) If W and B are independent Brownian motions then the average of W and B given by $X_t = (1/2)(W_t + B_t), t \geq 0$ is again a Brownian Motion.

(b) If X and Y are martingales then the average of X and Y given by $Z_t = (1/2)(X_t + Y_t), t \geq 0$ is again a martingale.

(c) If a process X has finite, non-zero quadratic variation : i.e. $0 < [X,X]_t < \infty$ with probability one for all $t \geq 0$, then X has infinite first variation : i.e. $FV(X) = \infty$ with probability one for all $t \geq 0$.

(d) Let $\Delta = (\Delta_t)_{t\geq 0}$ be an adapted stochastic process such that $\mathbb{E}\left[\int_0^t \Delta_u^2 du\right] < \infty$ for all t. Then, with $X_t = \int_0^t \Delta_u dW_u, t \geq 0$ it follows that $\lim_{t\uparrow\infty} [X,X]_t = \infty$.

Solution:

(a) FALSE. The distribution of $(1/2)(W_t + B_t)$ is $N(0, t/2)$ not $N(0, t)$ as it would have to be if X_t were a Brownian Motion.
(b) TRUE. This follows from the linearity of conditional expectation.
(c) TRUE. For any partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ we have

$$[X, X]_t^{\Pi}(\omega) \leq \left(\max_{i \leq n} |X_{t_i} - X_{t_{i-1}}| \times FV(X)_t^{\Pi} \right)(\omega).$$

As for each ω X is continuous, the first above goes to 0 as $\|\Pi\| \downarrow 0$ so if $0 < [X, X]_t(\omega) < \infty$ then it must be that $FV(X)_t(\omega) = \infty$.
(d) FALSE. Let $\Delta_t = e^{-t}$. Then with probability one

$$[X, X]_t = \int_0^t \Delta_u^2 du = \int_0^t e^{-2u} du = \frac{1}{2}\left(1 - e^{-2t}\right).$$

Thus, $\lim_{t \uparrow \infty} [X, X]_t = 1/2 < \infty$.

3. (20 points, 10 points each). Let $M = \{M_t\}_{t \geq 0}$ be a Martingale with continuous paths with $M_0 = 0$. Do NOT assume M is a Brownian motion. Consider the simple process $\Delta = \{\Delta_t\}_{t \geq 0}$ defined by

$$\Delta_t = \begin{cases} \Delta_0 & t \leq 1 \\ \Delta_1 & t > 1 \end{cases},$$

where Δ_0 is constant and Δ_1 is $\mathcal{F}_1$ measurable with $|\Delta_1|(\omega) \leq K$ for some $K > 0$ and all ω.

Define the stochastic integral I of Δ with respect to M by

$$I_t = \Delta_0 M_{t \wedge 1} + \Delta_1 \left(M_t - M_{t \wedge 1}\right); \qquad t \geq 0,$$

where $a \wedge b = \min\{a, b\}$.

It is clear (you DO NOT have to prove this) that the martingale property holds for I on $[0, 1]$.
(a) Does the Martingale property hold when $s \leq 1 < t$?
(b) Does the Martingale property hold when $1 < s < t$?

Solution:
(a) If $s \leq 1 < t$ then

$$\begin{aligned}
\mathbb{E}\left[I_t - I_s \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[\Delta_0 M_1 + \Delta_1 (M_t - M_1) - \Delta_0 M_s \middle| \mathcal{F}_s\right], \\
&= \Delta_0 \mathbb{E}\left[M_1 \middle| \mathcal{F}_s\right] + \mathbb{E}\left[\mathbb{E}\left[\Delta_1 (M_t - M_1) \middle| \mathcal{F}_1\right] \middle| \mathcal{F}_s\right] - \Delta_0 M_s, \\
&= \Delta_0 M_s + \mathbb{E}\left[\Delta_1 \mathbb{E}\left[M_t - M_1 \middle| \mathcal{F}_1\right] \middle| \mathcal{F}_s\right] - \Delta_0 M_s, \\
&= \mathbb{E}\left[\Delta_1 \times 0 \middle| \mathcal{F}_s\right], \\
&= 0.
\end{aligned}$$

(b) If $1 < s < t$ then

$$\begin{aligned}
\mathbb{E}\left[I_t - I_s \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[\Delta_0 M_1 + \Delta_1(M_t - M_1) - \Delta_0 M_1 - \Delta_1(M_s - M_1)\middle|\mathcal{F}_s\right],\\
&= \mathbb{E}\left[\Delta_1(M_t - M_s)\middle|\mathcal{F}_s\right],\\
&= \Delta_1\mathbb{E}\left[M_t - M_s\middle|\mathcal{F}_s\right],\\
&= 0.
\end{aligned}$$

4. (30 points, 10 points each). Recall the Black-Scholes model, where the stock S and money market B evolve according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t; \qquad \frac{dB_t}{B_t} = r dt; \qquad t \geq 0.$$

Here, $\mu \in \mathbb{R}, \sigma > 0, r > 0$ are constant.

Now, let $T > 0$, and denote by $c(t, s, K)$ and $p(t, s; K)$ the price, at time $t \leq T$ and given $S_t = s$, of a call and put option which respectively pay out $(S_T - K)^+$ and $(K - S_T)^+$ at time T. From class (and the HW) we know

$$\begin{aligned}
c(t, s; K) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(\widetilde{S}_T - K)^+ \middle| \widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right],\\
p(t, s; K) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(K - \widetilde{S}_T)^+ \middle| \widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right].
\end{aligned}$$

Here, $\widetilde{S}$ is geometric Brownian motion with parameters r, σ^2 under $\mathbb{Q}$. Lastly, write N as the cdf for a $N(0, 1)$ random variable, and recall

$$d_{\pm}(\tau, s, K) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{s}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right].$$

Note that we are making the dependence of c, p and $d_{\pm}$ explicit on K.

(a) Express the $\mathbb{Q}$ cdf of $\widetilde{S}_T$ given $\widetilde{\mathcal{F}_t}, \widetilde{S}_t = s$

$$\tau \to \mathbb{Q}\left[\widetilde{S}_T \leq \tau \middle| \widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right]$$

in terms of (one or both) $d_{\pm}$, as well as N.

(b) Write down (you DO NOT have to derive) the Black-Scholes PDE for the call option price c.

(c) Now, consider a "put on a call", a European option with time T payoff $v(T, s) = \left(L - (S_T - K)^+\right)^+$, where $K, L > 0$. Denoting the time t price given $S_t = s$ as $v(t, s)$, express v in terms of call and put option prices. Be explicit in terms of the strikes!

Solution:

(a) Given $\widetilde{S}_t = s$ we have $\widetilde{S}_T = s e^{(r-\sigma^2/2)(T-t)+\sigma(\widetilde{W}_T - \widetilde{W}_t)}$. This implies

$$\begin{aligned}
&\mathbb{Q}\left[\widetilde{S}_T \leq \tau \middle| \widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right]\\
&\quad = \mathbb{Q}\left[\widetilde{W}_T - \widetilde{W}_t \leq \frac{1}{\sigma}\left(\log\left(\frac{\tau}{s}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right) \middle| \widetilde{\mathcal{F}}_t\right].
\end{aligned}$$

As $\widetilde{W}_T - \widetilde{W}_t$ is independent of $\widetilde{\mathcal{F}}_t$ with $N(0, T-t)$ distribution under $\mathbb{Q}$ we obtain, writing $\widetilde{W}_T - \widetilde{W}_t = \sqrt{T-t}Z$ where $Z \sim N(0,1)$ under $\mathbb{Q}$

$$
\begin{aligned}
&\mathbb{Q}\left[\widetilde{S}_T \leq \tau \middle| \widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right] \\
&\qquad = \mathbb{Q}\left[Z \leq \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{\tau}{s}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)\right], \\
&\qquad = N\left(-d_-(T-t, s; \tau)\right) = 1 - N(d_-(T-t, s; \tau)).
\end{aligned}
$$

(b) The PDE is

$$
\begin{aligned}
0 &= c_t(t,s) + c_s(t,s)rs + \frac{1}{2}c_{ss}(t,s)\sigma^2 s^2 - rc(t,s), \\
(s-K)^+ &= c(T,s)
\end{aligned}
$$

(c) The payoff takes the form

$$
\left(L - (S_T - K)^+\right)^+ = \begin{cases} 0 & S_T \geq K + L \\ (L + K - S_T) & K + L > S_T \geq K \\ L & K > S_T \end{cases}
$$

Writing $L = K + L - S_T - (K - S_T)$ we see that

$$
\left(L - (S_T - K)^+\right)^+ = (L + K - S_T)^+ - (K - S_T)^+.
$$

therefore, the "put on a call" can be written as the difference between a put with strick $K + L$ and put with strike K. This gives the formula

$$
v(t,s) = p(t,s; L+K) - p(t,s;K).
$$