MF 790 HW 4, Part 1

This assignment is due on Thursday, October 28th at 8 AM. Problems 1 and 2 are worth 20 points, and problem 3 is worth 10 points, for a total of 50 points.

1. Black-Scholes Formula for a Call Option. Do exercise 3.5 on page 118 of the class textbook.

2. Self Financing Trading Strategies. Do exercise 4.10 on pages 193-196 of the class textbook.

3. Black-Scholes for General European Options. Let $S \sim \text{GBM}(\mu, \sigma^2)$ with respect to some $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let $f(x) : (0, \infty) \mapsto \mathbb{R}$ be a bounded, smooth function. By repeating the argument in the case of a call option, show that if there exists a smooth function $c(t, s)$ satisfying the partial differential equation (PDE)

$$\begin{aligned} c_t(t,s) + rsc_s(t,s) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t,s) - rc(t,s) &= 0 \qquad t \in (0,T), s > 0 \\ c(T,s) &= f(s) \qquad s > 0 \end{aligned}$$

and such that for any probability measure $\mathbb{Q}$ (with filtration $\widetilde{\mathbb{F}}$ and Brownian motion $\widetilde{W}$) and $\tilde{S} \sim \text{GBM}(r, \sigma^2)$ under $\mathbb{Q}$ we have for all $t \geq 0$ that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^t e^{-2ru}\sigma^2 \tilde{S}_u^2 c_s(u, \tilde{S}_u)^2 du\right] < \infty,$$

(you are assuming this, you do not have to show this) then

(a) The value of the option $f(S_T)$ at time t given $S_t = s$ is

$$c(t,s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} f(\tilde{S}_T) \middle| \widetilde{\mathcal{F}}_t, \widetilde{S}_t = s\right].$$

(b) $c(t, s)$ admits the explicit form

$$c(t,s) = e^{-r(T-t)} \int_{-\infty}^{\infty} f(se^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}z}) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$