MF 790 HW 5, Part 2 - SOLUTIONS

This assignment is due on Thursday, November 11th at 8 AM. Problems 1,2 are worth 15 points each, and problem 3 is worth 20 points, for a total of 50 points.

1. Density Processes for Coin Toss Space. Let Ω correspond to coin toss space with three tosses. I.e.

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

For $j = 1, ..., 3$ let $\mathcal{F}_j$ contain all the information generated by the first j tosses. Let $\mathbb{P}$ and $\mathbb{Q}$ be two measures on $(\Omega, \mathcal{F}_3)$ such that $\mathbb{P}$ corresponds to independent tosses of a fair coin, and $\mathbb{Q}$ corresponds to independent tosses of an unfair coin, where the likelihood of a head is $3/4$ and the likelihood of a tail is $1/4$.

For $j = 1, ..., 3$ and $\omega \in \Omega$ compute the Radon-Nikodym derivative $Z_j := \frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_j}$, which is an $\mathcal{F}_j$ measurable random variable such that $\mathbb{Q}[A_j] = \mathbb{E}\left[1_{A_j} Z_j\right]$ for all $A_j \in \mathcal{F}_j$. In particular, what are $Z_1(\omega), Z_2(\omega)$ and $Z_3(\omega)$ when $\omega = HHT$?

Solution As we discussed on slide 13 of lecture 07, when there are only a finite number of outcomes, the likelihood ratio is the ratio of the respective probabilities of each outcome. As for the probabilities we have

ω	$\mathbb{P}[\omega]$	$\mathbb{Q}[\omega]$
HHH	1/8	27/64
HHT	1/8	9/64
HTH	1/8	9/64
HTT	1/8	3/64
THH	1/8	9/64
THT	1/8	3/64
TTH	1/8	3/64
TTT	1/8	1/64

$(j = 1)$ Here, to be $\mathcal{F}_1$ measurable, Z_1 must only depend on the first toss. Thus, write $A_H = \{HHH, HHT, HTH, HTT\}$ and $A_T = \{THH, THT, TTH, TTT\}$. We then have

$$\omega \in A_H \Rightarrow Z_1(\omega) = \frac{\mathbb{Q}[A_H]}{\mathbb{P}[A_H]} = \frac{3/4}{1/2} = \frac{3}{2},$$
$$\omega \in A_T \Rightarrow Z_1(\omega) = \frac{\mathbb{Q}[A_T]}{\mathbb{P}[A_T]} = \frac{1/4}{1/2} = \frac{1}{2}.$$

$(j = 2)$ Here, to be $\mathcal{F}_2$ measurable, Z_2 must only depend on the first two tosses. Thus, write $A_{HH} = \{HHH, HHT\}$, $A_{HT} = \{HTH, HTT\}$, $A_{TH} = \{THH, THT\}$

and $A_{TT} = \{TTH, TTT\}$. We then have

$$\omega \in A_{HH} \Rightarrow Z_2(\omega) = \frac{\mathbb{Q}[A_{HH}]}{\mathbb{P}[A_{HH}]} = \frac{9/16}{1/4} = \frac{9}{4},$$
$$\omega \in A_{HT} \Rightarrow Z_2(\omega) = \frac{\mathbb{Q}[A_{HT}]}{\mathbb{P}[A_{HT}]} = \frac{3/16}{1/4} = \frac{3}{4},$$
$$\omega \in A_{TH} \Rightarrow Z_2(\omega) = \frac{\mathbb{Q}[A_{TH}]}{\mathbb{P}[A_{TH}]} = \frac{3/16}{1/4} = \frac{3}{4},$$
$$\omega \in A_{TT} \Rightarrow Z_2(\omega) = \frac{\mathbb{Q}[A_{TT}]}{\mathbb{P}[A_{TT}]} = \frac{1/16}{1/4} = \frac{1}{4}.$$

$(j = 3)$ Here, $Z_3(\omega) = \mathbb{Q}[\omega]/\mathbb{P}[\omega]$ for each $\omega \in \Omega$. This gives

$$Z_3(HHH) = \frac{27/64}{1/8} = \frac{27}{8};\ Z_3(HHT) = \frac{9/64}{1/8} = \frac{9}{8},$$
$$Z_3(HTH) = \frac{9/64}{1/8} = \frac{9}{8};\ Z_3(HTT) = \frac{3/64}{1/8} = \frac{3}{8},$$
$$Z_3(THH) = \frac{9/64}{1/8} = \frac{9}{8};\ Z_3(THT) = \frac{3/64}{1/8} = \frac{3}{8},$$
$$Z_3(TTH) = \frac{3/64}{1/8} = \frac{3}{8};\ Z_3(TTT) = \frac{1/64}{1/8} = \frac{1}{8}.$$

Thus, for example, if $\omega = HHT$ then $Z_1(\omega) = 3/2$, $Z_2(\omega) = 9/4$ and $Z_3(\omega) = 9/8$, so the process is evolving with time.

2. Call Prices in a Variant of the Black-Scholes Model with Dividend Payments. Consider a variant of the Black-Scholes model where we allow μ, σ and r to be non-random functions of time.

(a) (no dividends) First, assume no dividend payments so that S and D have dynamics

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t; \qquad \frac{dD_t}{D_t} = -r_t dt.$$

Identify constants κ, λ such that the time zero price c_0 of a call option with strike K expiring at T in this model takes the form

$$c_0 = c^{BS}(0, S_0; \kappa, \lambda),$$

where we write $c^{BS}(0, s; r, \sigma)$ as the Black-Scholes call option price for a given r, σ. **Hint:** look at slide 27 on lecture 04.

(b) (dividend flow). Now, assume a continuous dividend payment rate of $a = \{a_t\}$ where a is also non-random. This gives (S, D) dynamics

$$\frac{dS_t}{S_t} = (\mu_t - a_t)dt + \sigma_t dW_t; \qquad \frac{dD_t}{D_t} = -r_t dt.$$

As in part (a), find constants κ, λ as well as a scaling factor $\mathfrak{s}$ such that the time zero price c_0 of a call option with strike K expiring at T in this model takes the

form
$$c_0 = \mathfrak{s} \times c^{BS}(0, S_0; \kappa, \lambda).$$
Here, your constants κ, λ may or may not be different from part (a).

Solution:

(a) We have
$$\begin{aligned} c_0 &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u du}(S_T - K)^+\right], \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u du}\left(S_0 e^{\int_0^T \left(r_u - \frac{1}{2}\sigma_u^2\right)du + \int_0^T \sigma_u dW_u^{\mathbb{Q}}} - K\right)^+\right]. \end{aligned}$$
Set $\kappa = (1/T)\int_0^T r_u du$ (which is not random) so that
$$c_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-\kappa T}\left(S_0 e^{\kappa T - \frac{1}{2}\int_0^T \sigma_u^2 du + \int_0^T \sigma_u dW_u^{\mathbb{Q}}} - K\right)^+\right].$$
Next, set $\lambda = \sqrt{(1/T)\int_0^T \sigma_u^2 du}$. As per the hint $\int_0^T \sigma_u dW_u^{\mathbb{Q}} \overset{\mathbb{Q}}{\sim} N(0, \lambda^2 T)$. But, this is also the distribution of $\lambda W_T^{\mathbb{Q}}$ under $\mathbb{Q}$. Thus, as everything else is not random we have
$$c_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-\kappa T}\left(S_0 e^{\left(\kappa - \frac{1}{2}\lambda^2\right)T + \lambda W_T^{\mathbb{Q}}} - K\right)^+\right] = c^{BS}(0, S_0; \kappa, \lambda).$$

(b) Similarly as in part (a) we have
$$\begin{aligned} c_0 &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u du}(S_T - K)^+\right], \\ &= e^{-\int_0^T a_u du}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T (r_u - a_u)du}\left(S_0 e^{\int_0^T \left(r_u - a_u - \frac{1}{2}\sigma_u^2\right)du + \int_0^T \sigma_u dW_u^{\mathbb{Q}}} - K\right)^+\right]. \end{aligned}$$
From part (a) we know
$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T (r_u - a_u)du}\left(S_0 e^{\int_0^T \left(r_u - a_u - \frac{1}{2}\sigma_u^2\right)du + \int_0^T \sigma_u dW_u^{\mathbb{Q}}} - K\right)^+\right] = c^{BS}(0, S_0; \kappa, \lambda)$$
for
$$\kappa = \frac{1}{T}\int_0^T (r_u - a_u)du; \qquad \lambda^2 = \frac{1}{T}\int_0^T \sigma_u^2 du.$$
The result then follows with
$$\mathfrak{s} = e^{-\int_0^T a_u du}.$$