MF 790 Stochastic Calculus
Practice Final Exam. Fall, 2021
SOLUTIONS

This is the practice final exam. There are 4 questions for a total 100 points. Each question may contain multiple parts.

To get the most out of the exam please take this exam as if it were the real exam! I.e. do not use notes, the class textbook, the internet, and especially do not look at the solutions ahead of time! Give yourself two hours to take the exam, and abide by this rule!

Formulas.

(1) The standard normal cumulative distribution function (cdf)

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy.$$

(2) The price of a call and put option (with maturity T and strike K) in the Black-Scholes model at $t \leq T$ given the stock price is x.

$$c(t, x) = xN(d_+(\tau, x)) - Ke^{-r\tau} N(d_-(\tau, x))$$
$$p(t, x) = Ke^{-r\tau} N(-d_-(\tau, x)) - xN(-d_+(\tau, x))$$

where $\tau = T - t$ and

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}} \left(\log\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right) \tau \right)$$

(3) The delta for a call option in the Black-Scholes model at $t \leq T$ and given the stock price is x

$$\partial_x c(t, x) = N(d_+(\tau, x))$$

Throughout, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is fixed, $W = \{W_t\}_{t\geq 0}$ is a Brownian motion, and unless otherwise noted, $\mathbb{F} = \mathbb{F}^W$ is generated by W.

1 (20 Points). Fix $T = 1$ and define

$$Z_1 := W_1^2.$$

As $W_1^2 \geq 0$ and $\mathbb{E}\left[W_1^2\right] = 1$ we may define a new measure $\widehat{\mathbb{P}}$ by $d\widehat{\mathbb{P}}/d\mathbb{P} = Z_1$.

(a) (**10 Points**) With $Z_t := \mathbb{E}\left[Z_T \middle| \mathcal{F}_t\right]$ show that $Z_t = W_t^2 + 1 - t$.

(b) (**10 Points**) Identify the process ϕ such that $\widehat{W}_t := W_t - \int_0^t \phi_u du$ is a Brownian motion under $\widehat{\mathbb{P}}$.

Solution

(a) By Itô's formula, $d(W_t^2) = 2W_t dW_t + dt$ so that $t \to W_t^2 - t$ is a Martingale. This implies that

$$Z_t = \mathbb{E}\left[W_1^2 \middle| \mathcal{F}_t\right] = 1 + \mathbb{E}\left[W_1^2 - 1 \middle| \mathcal{F}_t\right] = 1 + W_t^2 - t.$$

(b) Z has the dynamics

$$dZ_t = 2W_t dW_t + dt - dt = 2W_t dt = \frac{2W_t}{W_t^2 + 1 - t} Z_t dW_t.$$

Girsanov's theorem then implies

$$\phi_t = \frac{2W_t}{W_t^2 + 1 - t}$$

2 (25 points, 5 points each part) (True/False). Indicate whether or not each of the following statements is true or false. If it is true, explain why. If it is false, either explain why or give a counter example. Answers with no explanation get no credit!

(a) If the money market rate process R is non negative, then, because the forward contract is riskier, the forward price always exceeds the futures price.

(b) For a stock paying a continuous non-negative dividend rate, the discounted price process is a supermartingale under $\mathbb{Q}$.

(c) In the Black-Scholes model, given the same strike K, maturity T and current price $S_t = x, t < T$, the gamma for a put option is less than the gamma for a call option.

(d) Let W and B be two independent Brownian motions and set $M_t = B_t W_t$ for $t \geq 0$. Then there is no (smooth) function f such that $Y_t = f(M_t)$ is a Brownian motion.

(e) If σ and τ are two stopping times such that for each ω $\sigma(\omega) \leq \tau(\omega)$. Then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

Solution:

(a) FALSE. The forwards and futures price generally satisfy the relationship

$$\mathrm{Cov}^{\mathbb{Q}}\left(D_T, S_T\right) = \mathbb{E}^{\mathbb{Q}}\left[D_T \middle| \mathcal{F}_t\right]\left(\mathrm{For}_t - \mathrm{Fut}_t\right),$$

and so their ranking depends on how S_T and D_T vary with one another under $\mathbb{Q}$. This relationship can be negative, positive or 0. In fact, when interest rates are constant the forward and futures price agree.

(b) TRUE. The $\mathbb{Q}$ dynamics of DS are

$$\frac{d(D_tS_t)}{D_tS_t} = -a_t dt + \sigma_t dW_t^{\mathbb{Q}},$$

giving the super-martingale property.

(c) FALSE. The identity $x - K = (x-K)^+ - (K-x)^+$ shows that

$$C(t,x) - P(t,s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)\big|S_t = x\right] = x - Ke^{-r(T-t)}.$$

as the discounted stock price is a $\mathbb{Q}$ martingale. This shows the gamma of the call and put option coincide.

(d) TRUE. The dynamics of M are

$$dM_t = W_t dB_t + B_t dW_t + d[W,B]_t = W_t dB_t + B_t dW_t.$$

as W and B are independent. Thus, for any smooth function f we have

$$\begin{aligned} df(M_t) &= \dot{f}(M_t)dM_t + \frac{1}{2}\ddot{f}(M_t)d[M,M]_t, \\ &= \dot{f}(M_t)\left(W_t dB_t + B_t dW_t\right) + \frac{1}{2}\ddot{f}(M_t)\left(W_t^2 + B_t^2\right)dt. \end{aligned}$$

As $W_t^2 + B_t^2 > 0$ with probability one for all t, the obtain a martingale we must have $\ddot{f}(x) = 0$ or that $f(x) = mx + b$ for constants m, b. But, for this specification we have

$$df(M_t) = m\left(W_t dB_t + B_t dW_t\right) \Longrightarrow d[f(M), f(M)]_t = m^2(W_t^2 + B_t^2)dt.$$

There is no way we can make the right most term dt, as we would need to ensure $f(M)$ is a Brownian motion.

(e) TRUE. Let $A \in \mathcal{F}_\sigma$ so that we know $A \cap \{\sigma \le \tau\} \in \mathcal{F}_t$ for all $t \ge 0$. Since $\sigma \le \tau$ implies $\{\tau \le t\} \subseteq \{\sigma \le t\}$ we deduce

$$\begin{aligned} A \cap \{\tau \le t\} &= A \cap \{\tau \le t\} \cap \{\sigma \le t\} \\ &= \underbrace{A \cap \{\sigma \le t\}}_{\in \mathcal{F}_t} \bigcap \underbrace{\tau \le t}_{\in \mathcal{F}_t}, \\ &\in \mathcal{F}_t. \end{aligned}$$

Here, we used the defining property of τ being a stopping time as well. As this holds for every t we conclude $A \in \mathcal{F}_\tau$.

3 (30 Points). In the Black-Scholes model, fix a time $T > 0$ and consider a "power call" which is a claim with payoff $V_T = (S_T^p - K)^+$ for a fixed constant $p > 1$. Our goal is to price and hedge this claim.

(a) **(10 points).** Identify $\tilde{r}$ and $\tilde{\sigma}$ so that under risk neutral measure, S^p is a geometric Brownian motion with parameters $\tilde{r}, \tilde{\sigma}^2$.

(b) **(10 points).** Evaluate the value V_t of the claim V_T at time t for $t \le T$. Express your answer in terms of the Black-Scholes call price.

(c) **(10 points).** Explicitly identify the initial capital and hedging strategy which replicates V_T.

Hint: Use that $t \to e^{-rt}V_t$ is a $\mathbb{Q}$ martingale.

Solution:

(a) Under risk neutral measure $\mathbb{Q}$, S is a geometric Brownian motion with parameters (r, σ^2). Thus, using Itô 's formula we have

$$\begin{aligned} d(S_t)^p &= p(S_t)^{p-1}dS_t + \frac{1}{2}p(p-1)(S_t)^{p-2}d[S,S]_t, \\ &= \left(r + \frac{1}{2}p(p-1)\sigma^2\right) S_t^p dt p S_t^p \sigma dW_t^{\mathbb{Q}}. \end{aligned}$$

Therefore, with $\tilde{r} = r + \frac{1}{2}p(p-1)\sigma^2$ and $\tilde{\sigma} = p\sigma$, S^p is a geometric Brownian motion with parameters $\tilde{\mu}$ and $\tilde{\sigma}^2$.

(b) Using part (a) we compute

$$\begin{aligned} V_t &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}V_T \middle| \mathcal{F}_t\right] = e^{(\tilde{r}-r)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{-\tilde{r}(T-t)}\left(S_T^p - K\right)^+ \middle| \mathcal{F}_t\right], \\ &= e^{(\tilde{r}-r)(T-t)}c(t, S_t^p; \tilde{r}, \tilde{\sigma}^2) \end{aligned}$$

(c) We know the discounted value process is a Martingale. Thus, using Itô's formula and the a priori fact the dt terms will vanish gives

$$d(e^{-rt}V_t) = e^{(\tilde{r}-r)T-\tilde{r}t}\tilde{\sigma}S_t^p c_x(t, S_t^p; \tilde{r}, \tilde{\sigma}^2)dW_t^{\mathbb{Q}}.$$

Next, we know for any trading strategy Δ that the discounted wealth process $t \to e^{-rt}X_t$ is also a martingale under $\mathbb{Q}$ with dynamics

$$dX_t = e^{-rt}\Delta_t \sigma S_t dW_t^{\mathbb{Q}}.$$

Therefore, to replicate we must set

$$\begin{aligned} \Delta_t &= \frac{e^{rt}}{\sigma S_t}\left(e^{(\tilde{r}-r)T-\tilde{r}t}\tilde{\sigma}S_t^p c_x(t, S_t^p; \tilde{r}, \tilde{\sigma}^2)\right), \\ &= p e^{\frac{1}{2}p(p-1)\sigma^2(T-t)} S_t^{p-1} c_x(t, S_t^p; \tilde{r}, \tilde{\sigma}^2) \end{aligned}$$

The initial capital is $V_0 = e^{\frac{1}{2}\sigma^2 p(p-1)T}c(0, S_0^p; \tilde{r}, \tilde{\sigma}^2)$.

4 (25 Points). Assume that under risk neutral measure $\mathbb{Q}$, the money market rate follows a CIR process

$$dR_t = \kappa(\theta - R_t)dt + \xi\sqrt{R_t}dW_t^{\mathbb{Q}}.$$

(a) (**15 Points**) The price of a T maturity zero coupon bond is $B(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T R_u du} \middle| \mathcal{F}_t\right]$. Derive (do not just write down) a partial differential equation (PDE), such that if the function $b = b(t,r)$ solves the PDE, then $B(t,T) = b(t, R_t)$.

(b) (**10 Points**) Guessing $b(t,r) = e^{-A(t)-C(t)r}$, derive (do not just write down) the ordinary differential equations that A, C should solve. There is no need to solve the equations.

Solution:

(a) By Itô's formula the process $Y_t := D_t b(t, R_t)$ has dynamics

$$dY_t = D_t\Bigg(\left(-R_t b(t,R_t) + b_t(t,R_t) + b_r(t,R_t)\kappa(\theta - R_t) + \frac{1}{2}b_{rr}(t,R_t)\xi^2 R_t\right)dt$$
$$+ b_r(t,R_t)\xi\sqrt{R_t}dW_t^{\mathbb{Q}}\Bigg).$$

If b is such that Y is a martingale, then

$$Y_t = \mathbb{E}^{\mathbb{Q}}\left[Y_T\middle|\mathcal{F}_t\right] \quad \Leftrightarrow \quad e^{-\int_0^t R_u du}b(t,R_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T R_u du}b(T,R_T)\middle|\mathcal{F}_t\right].$$

If additionally, $b(T,r) = 1$ then by TOWK we may conclude

$$b(t,R_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T R_u du}\middle|\mathcal{F}_t\right] = B(r,T).$$

The martingale property will follow if the dt terms in the dynamics for Y vanish for $t \le T$. This will be the case (and the terminal condition will hold as well) if b solves the PDE

$$0 = b_t(t,r) + b_r(t,r)\kappa(\theta - r) + \frac{1}{2}b_{rr}(t,r)\xi^2 r - rb(t,r), 0 \le t < T, r > 0$$
$$1 = b(T,r), r > 0$$

(b) For $b(t,r) = e^{-A(t)-C(t)r}$ we have

$$b_t(t,r) = b(t,r)\left(-\dot{A}(t) - \dot{C}(t)r\right), b_r(t,r) = -C(t)b(t,r), b_{rr}(t,r) = C(t)^2 b(t,r).$$

Plugging this into the first line of the PDE (and dividing out by $b(t,r)$) yields

$$-\dot{A}(t) - r\dot{C}(t) - C(t)\kappa(\theta - r) + \frac{1}{2}C(t)^2\xi^2 r - r = 0.$$

Setting the "r" terms and "constant" terms to 0 we have two equations

$$0 = \dot{A}(t) + \kappa\theta C(t); \qquad 0 = \dot{C}(t) - \kappa C(t) - \frac{1}{2}\xi^2 C(t)^2 + 1$$

These give the ODEs, along with the terminal conditions $A(T) = C(T) = 0$ which we need to make $b(T,r) = 1$.