MF 790 HW 1 PART 2 - SOLUTIONS

This assignment is due on Thursday, September 16th at 8 AM. Problems 1 and 2 are 20 points each, and problem 3 is 10 points, for a total of 50 points.

1. Markov versus Martingale - Discrete Examples. We independently toss a fair coin twice. As such, the set of possible outcomes is

$$\Omega = \{HH, HT, TH, TT\}$$

Let $\mathcal{F}$ denote the set of all subsets of Ω. Construct the two period filtration $\mathbb{F}$ via

$$\mathcal{F}_0 = \{\emptyset, \Omega\}; \quad \mathcal{F}_1 = \{\emptyset, \Omega, A_H, A_T\}; \quad \mathcal{F}_2 = \mathcal{F},$$

where $A_H = \{HH, HT\}$ and $A_T = \{TH, TT\}$. Adapted to $\mathbb{F}$ we have the processes X and Y shown in Figure 1. Indicate if each of the following statements is true or false, and provide reasoning for your answer.

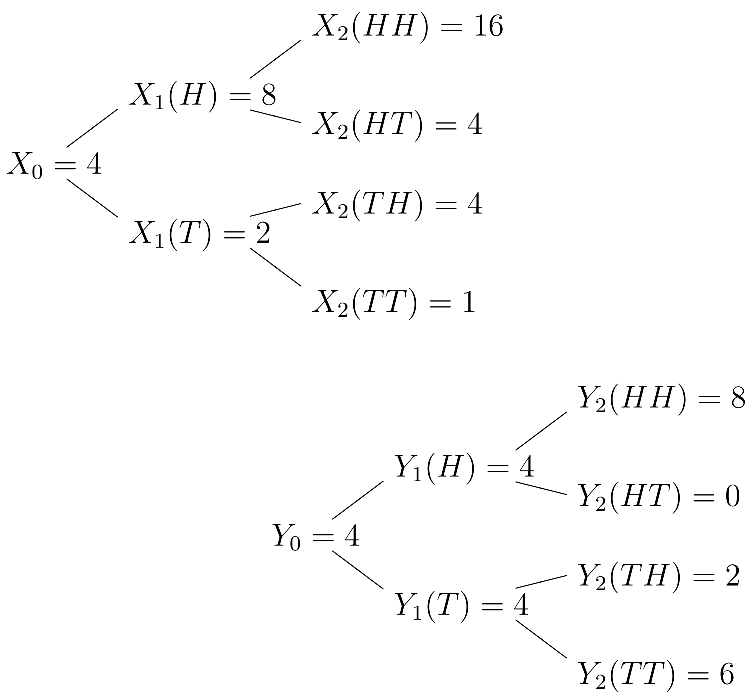

FIGURE 1. The process X and Y

(a) X is a martingale. If X is not a martingale, is there a way to adjust the probability of a heads (still assuming independent tosses) such that X is a martingale?
(b) X is Markov.
(c) Y is a martingale. If Y is not a martingale, is there a way to adjust the probability of a heads (still assuming independent tosses) such that Y is a martingale?
(d) Y is Markov.

(e) Now, assume the tosses are NOT independent coming from a fair coin. Rather, assume the generic form

$$\mathbb{P}[HH] = p_1, \quad \mathbb{P}[HT] = p_2, \quad \mathbb{P}[TH] = p_3, \quad \mathbb{P}[TT] = 1 - p_1 - p_2 - p_3,$$

for $0 < p_1, p_2, p_3 < 1$ such that $p_1+p_2+p_3 < 1$. Show that there are an *uncountable* number of (p_1, p_2, p_3) such that Y is a Martingale.

Solution :

(a) **False** Note that $\mathbb{E}[X_1] = 5 \neq 4 = X_0$. Now, assume the probability of a head is $p \in (0,1)$ and observe that $X_{n+1} = 2X_n$ if the $(n+1)^{st}$ toss is a head, and $.5X_n$ if it is a tail. In this case, we have for $n = 0, 1$

$$\mathbb{E}\left[X_{n+1} \middle| \mathcal{F}_n\right] = p2X_n + (1-p).5X_n = \frac{1}{2}(3p+1)X_n.$$

Thus, if $p = 1/3$ then X will be a martingale.

(b) **True** From part (a) we find

$$\mathbb{E}\left[g(X_{n+1}) \middle| \mathcal{F}_n\right] = \frac{1}{2}g(2X_n) + \frac{1}{2}g(.5X_n) =: h(X_n).$$

(c) **True** This holds because

$$\begin{aligned}
\mathbb{E}[Y_1] &= \frac{1}{2}4 + \frac{1}{2}4 = 4 = Y_0, \\
\mathbb{E}\left[Y_2 \middle| \mathcal{F}_1\right](H) &= \frac{1}{2}8 + \frac{1}{2}0 = 4 = Y_1(H), \\
\mathbb{E}\left[Y_2 \middle| F_2\right](T) &= \frac{1}{2}2 + \frac{1}{2}6 = 4 = Y_1(T).
\end{aligned}$$

(d) **False**. Note that

$$\mathbb{E}\left[g(Y_2) \middle| \mathcal{F}_1\right](H) = \frac{1}{2}g(8) + \frac{1}{2}g(0),$$

but

$$\mathbb{E}\left[g(Y_2) \middle| \mathcal{F}_1\right](T) = \frac{1}{2}g(2) + \frac{1}{2}g(6).$$

Thus, if $g(x) = x^2$, then

$$\frac{1}{2}g(8) + \frac{1}{2}g(0) = 32,$$

but

$$\frac{1}{2}g(2) + \frac{1}{2}g(6) = 20.$$

Thus, when $f(x) = x^2$, knowing that $Y_1 = 4$ without knowing if the first toss was a head or tail does not give enough information to uniquely identify $\mathbb{E}\left[f(Y_2) \middle| \mathcal{F}_1\right]$.

(e) As $Y_1(H) = Y_1(T) = Y_0$ it is clear $\mathbb{E}\left[Y_1\middle|\mathcal{F}_0\right] = Y_0$ for any allowable choice of p_1, p_2, p_3. Now, assume that Y is a martingale. This implies $Y_1 = \mathbb{E}\left[Y_2\middle|\mathcal{F}_1\right]$. Using partial averaging this means

$$\begin{aligned} 4(p_1+p_2) &= \mathbb{E}\left[Y_1 A_H\right] = \mathbb{E}\left[\mathbb{E}\left[Y_2\middle|\mathcal{F}_1\right] A_H\right] = \mathbb{E}\left[Y_2 A_H\right] = 8p_1 + 0p_2, \\ 4(p_3 + 1 - p_1 - p_2 - p_3) &= \mathbb{E}\left[Y_1 A_T\right] = \mathbb{E}\left[\mathbb{E}\left[Y_2\middle|\mathcal{F}_1\right] A_T\right] = \mathbb{E}\left[Y_2 A_T\right] \\ &= 2p_3 + 6(1 - p_1 - p_2 - p_3). \end{aligned}$$

This first equation implies $p_2 = p_1$. Plugging this into the second equation implies $p_3 = 1/2 - p_1$. As we need $p_1, p_2, p_3 > 0$ and $p_1 + p_2 + p_3 < 1$ we see that for any $0 < p_1 < 1/2$ we can set $p_2 = p_1$, $p_3 = (1/2) - p_1$ and $p_1 + p_2 + p_3 = p_1 + 1/2 < 1$. This gives the uncountable family.

2. Additional Properties of Conditional Expectation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be given and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub σ-algebra of $\mathcal{F}$. Let X be a random variable such that $\mathbb{E}\left[X^2\right] < \infty$.

Let Y be *any* $\mathcal{G}$ measurable random variable such that $\mathbb{E}\left[Y^2\right] < \infty$. Show that

$$\mathrm{Var}\left[X - \mathbb{E}\left[X\middle|\mathcal{G}\right]\right] \le \mathrm{Var}\left[X - Y\right].$$

Thus, $\mathbb{E}\left[X\middle|\mathcal{G}\right]$ is the $\mathcal{G}$ measurable random variable which is "closest" to X amongst all $\mathcal{G}$ measurable random variables Y in that $\mathrm{Var}\left[X - Y\right]$ is minimized at $Y = \mathbb{E}\left[X\middle|\mathcal{G}\right]$. **Hint:** Let $Z = Y - \mathbb{E}\left[X\middle|\mathcal{G}\right]$ and use TOWK evaluate

$$\mathrm{Var}\left[X - \mathbb{E}\left[X\middle|\mathcal{G}\right] - Z\right].$$

SOLUTION: Using the hint, set $Z = Y - \mathbb{E}\left[X\middle|G\right]$. Note that Z is still $\mathcal{G}$ measurable and $\mathbb{E}\left[Z^2\right] < \infty$ (by Jensen's inequality). Thus, minimizing over Y is the same as minimizing over Z. Furthermore, we have

(0.1)

$$\begin{aligned} \mathrm{Var}\left[X - Y\right] &= \mathrm{Var}\left[X - \mathbb{E}\left[X\middle|\mathcal{G}\right] - Z\right], \\ &= \mathbb{E}\left[(X - \mathbb{E}\left[X\middle|\mathcal{G}\right] - Z)^2\right] - \left(\mathbb{E}\left[X - \mathbb{E}\left[X\middle|\mathcal{G}\right] - Z\right]\right)^2, \\ &= \mathbb{E}\left[(X - \mathbb{E}\left[X\middle|\mathcal{G}\right])^2\right] - 2\mathbb{E}\left[(X - \mathbb{E}\left[X\middle|\mathcal{G}\right])Z\right] + \mathbb{E}\left[Z^2\right] - \mathbb{E}\left[Z\right]^2, \\ &= \mathbb{E}\left[(X - \mathbb{E}\left[X\middle|\mathcal{G}\right])^2\right] - 2\mathbb{E}\left[(X - \mathbb{E}\left[X\middle|\mathcal{G}\right])Z\right] + \mathrm{Var}\left[Z\right]. \end{aligned}$$

where the second to last equality follows because the linearity of conditional expectation implies

$$\mathbb{E}\left[X - \mathbb{E}\left[X\middle|\mathcal{G}\right]\middle|\mathcal{G}\right] = \mathbb{E}\left[X\middle|\mathcal{G}\right] - \mathbb{E}\left[X\middle|\mathcal{G}\right] = 0,$$

and hence by the tower property

$$\mathbb{E}\left[X - \mathbb{E}\left[X\middle|\mathcal{G}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[X - \mathbb{E}\left[X\middle|\mathcal{G}\right]\middle|\mathcal{G}\right]\right] = 0.$$

Now

$$\begin{aligned}
\mathbb{E}\left[(X-\mathbb{E}\left[X\middle|\mathcal{G}\right])Z\right] &= \mathbb{E}\left[\mathbb{E}\left[(X-\mathbb{E}\left[X\middle|\mathcal{G}\right])Z\middle|\mathcal{G}\right]\right] \quad \text{(Tower)},\\
&= \mathbb{E}\left[Z\times\mathbb{E}\left[X-\mathbb{E}\left[X\middle|\mathcal{G}\right]\middle|\mathcal{G}\right]\right] \quad \text{(TOWK)},\\
&= 0
\end{aligned}$$

since $\mathbb{E}\left[X-\mathbb{E}\left[X\middle|\mathcal{G}\right]\middle|\mathcal{G}\right]=0$. Thus, from (0.1) we have

$$\mathrm{Var}\left[X-Y\right]=\mathbb{E}\left[(X-\mathbb{E}\left[X\middle|\mathcal{G}\right])^2\right]+\mathrm{Var}\left[Z\right].$$

Since $\mathrm{Var}\left[Z\right]\geq 0$ for all Z we see that

$$\mathrm{Var}\left[X-Y\right]\geq\mathbb{E}\left[(X-\mathbb{E}\left[X\middle|\mathcal{G}\right])^2\right]=\mathrm{Var}\left[X-\mathbb{E}\left[X\middle|\mathcal{G}\right]\right].$$

This gives the result.

3. Functions of a Random Walk. (*Stochastic Calculus for Finance I*, Exercise 2.4(*ii*)) Let M_n be the random walk described in class : i.e. $M_0=0$ and $M_n=\sum_{j=1}^n Z_j$ where $(Z_j)_{j=1,2,\dots}$ are i.i.d random variables taking the values ± 1 with equal probability. Next, define the process S by

$$S_n=e^{\sigma M_n}\cosh(\sigma)^{-n}, \qquad n=0,1,\dots.$$

Show that S is a martingale. **Hint:** it suffices to prove the one period result. $\mathbb{E}\left[S_{n+1}\middle|\mathcal{F}_n\right]=S_n$

Solution: We have for $n=0,1,\dots$ that

$$\begin{aligned}
\mathbb{E}\left[S_{n+1}\middle|\mathcal{F}_n\right] &= \cosh(\sigma)^{-(n+1)}\mathbb{E}\left[e^{\sigma M_{n+1}}\middle|\mathcal{F}_n\right]\\
&= \cosh(\sigma)^{-(n+1)}\mathbb{E}\left[e^{\sigma(M_{n+1}-M_n+M_n)}\middle|\mathcal{F}_n\right]\\
&= \cosh(\sigma)^{-(n+1)}e^{\sigma M_n}\mathbb{E}\left[e^{\sigma Z_{n+1}}\right]\\
&= S_n\cosh(\sigma)^{-1}\left(\frac{e^{\sigma}}{2}+\frac{e^{-\sigma}}{2}\right)\\
&= S_n
\end{aligned}$$

where the third equality follows by the independence of $\{X_j\}_{j=1,\dots}$, the fourth equality follows by the definition of S_n and the fifth equality by the definition of cosh.