# Stopping Times and Optional Sampling

MF 790 Stochastic Calculus

# Outline

- Stopping Times
- Stopping Time $\sigma$-algebra.
- Stopped Process.
- Optional Sampling Theorem

# Stopping Times : Motivation

- When investing, we often make decisions at times which are not deterministic.
  - E.g.: we sell the stock when the share price reaches 100.
  - E.g.: we refinance our mortgage when mortgage rates fall to a new 5 year low.
- Making decisions at "random times" is natural.
  - What sounds more reasonable?
    - "I will sell the stock on Friday, July 15th 2022 at 3:00 PM."
    - "I will sell the stock when the price reaches a six month high."
- We must understand how to use these random times in our models.

# Random Times

- Fix $(\Omega, \mathcal{F})$. A <u>random time</u> $\tau$ is a non-negative random variable which might take the value $\infty$.
  - $\tau : \Omega \to [0, \infty]$ is $\mathcal{F}$ measurable.
- We interpret $\tau(\omega)$ as the time that some event occurs in scenario $\omega$.
  - $\tau(\omega) = \infty$ corresponds to the event never occurring.
- Examples ($W$ is Brownian motion)
  - $\tau(\omega) = \text{inf}\,[t \mid W_t(\omega) = 2]$.
  - $\tau(\omega) = \text{sup}\left[t \mid e^{W_t(\omega) - t} = 3\right]$.

# Stopping Times

- Suppose we have an information flow $\mathbb{F}$, and fix a (non-random) time $t \geq 0$.
- Question: can we know if $\tau$ occurred by $t$ using our information at $t$?
  - E.g. $\mathbb{F} = \mathbb{F}^W$ and $\tau(\omega) = \inf\left[t \mid W_t(\omega) = 2\right]$.
    - At $t$, do we know if $W$ has reached 2 or not? YES.
  - E.g. $\mathbb{F} = \mathbb{F}^W$ and $\tau(\omega) = \sup\left[t \mid e^{W_t(\omega)-t} = 3\right]$.
    - At $t$, do we know that at no later time $u$, $W_u = u + \log(3)$? NO.
- It is NOT always possible to know if a random time has occurred by $t$ given our information at $t$.

# Stopping Times

- A random time $\tau$ is a $\underline{\mathbb{F}\text{ - stopping time}}$ if for each $t$ we know if $\tau$ has occurred by $t$, given our info at $t$.
  - Mathematically: $\{\omega \mid \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.
  - We say $\tau$ is a "stopping time" if the filtration is clear.
- Example: a non-random time. $\tau(\omega) = t_0$ for all $\omega$.
  - $\{\tau \leq t\} = \begin{cases} \Omega & t_0 \leq t \\ \emptyset & t_0 > t \end{cases}$, and $\Omega, \emptyset$ are in every $\sigma$-algebra.

# Hitting Times

- Fix $(\Omega, \mathcal{F}), \mathbb{F}$ and let $X$ be an adapted process with continuous sample paths.

- A hitting time for $X$ is a random time of the form

  - $\tau(\omega) = \inf [t \mid X_t(\omega) = \lambda]$ for some level $\lambda$.

- Examples:

  - First time a Brownian motion reaches $-2$.

  - First time a geometric Brownian motion reaches 1.

# Hitting Times

- Claim: for adapted $X$ with continuous paths, hitting times are stopping times.
  - For $\tau = \inf[t \mid X_t = \lambda]$ we have $\{\tau \leq t\} \in \mathcal{F}_t, t \geq 0$.
- Proof:
  - $\{\tau \leq 0\} = \{\tau = 0\} = \{X_0 = \lambda\} \in \mathcal{F}_0$ as $X$ is adapted.
  - Set $Q$ as the rational numbers. For $t > 0$ we claim
    - $\{\tau \leq t\} = \bigcap_{n=1}^{\infty} \bigcup_{q \in Q, q \leq t} \{\lambda - \frac{1}{n} < X_q < \lambda + \frac{1}{n}\}$.
  - If we believe this, then $\{\tau \leq t\} \in \mathcal{F}_t$, and we are done.
    - By adaptivity $\{\lambda - \frac{1}{n} < X_q < \lambda + \frac{1}{n}\} \in \mathcal{F}_q \subseteq \mathcal{F}_t$ as $q \leq t$.
    - As the rational numbers are countable, $\{\tau \leq t\}$ is a countable intersection of a countable union of events in $\mathcal{F}_t$, hence in $\mathcal{F}_t$.

$$\{\tau \leq t\} = A := \bigcap_{n=1}^{\infty} \bigcup_{q \in Q, q \leq t} \{\lambda - \tfrac{1}{n} < X_q < \lambda + \tfrac{1}{n}\}$$

- Assume $\omega \in \{\tau \leq t\}$. By continuity of $X$ and the density of the rationals.
  - There is some $s \leq t$ such that $X_s(\omega) = \lambda$.
  - For each $n$, there are $0 \leq a_n < s < b_n \leq t$ such that $\lambda - \frac{1}{n} < X_r(\omega) < \lambda + \frac{1}{n}$ for all $r \in (a_n, b_n)$.
  - There is some rational $q_n \in (a_n, b_n)$.
    - Thus, $\forall\ n, \exists\ q_n$ s.t. $\omega \in \{\lambda - \frac{1}{n} < X_{q_n} < \lambda + \frac{1}{n}\}$
- Thus, $\omega \in A$ and $\{\tau \leq t\} \subseteq A$.

$$\{\tau \le t\} = A := \bigcap_{n=1}^{\infty} \bigcup_{q \in Q, q \le t} \{\lambda - \tfrac{1}{n} < X_q < \lambda + \tfrac{1}{n}\}$$

- Assume $\omega \in A$, and for each $n$ take a rational $q_n$ such that $\lambda - \frac{1}{n} < X_{q_n}(\omega) < \lambda + \frac{1}{n}$.
  - $\{q_n\}_n$ is a collection of times in $[0, t]$ which is compact. Thus, (up to a subsequence) $q_n \to q \in [0, T]$.
  - By continuity $X_{q_n}(\omega) \to X_q(\omega)$.
  - This implies $X_q(\omega) = \lambda$ for some $q \le T$.
- Thus, $\omega \in \{\tau \le t\}$ and $A \subseteq \{\tau \le t\}$.

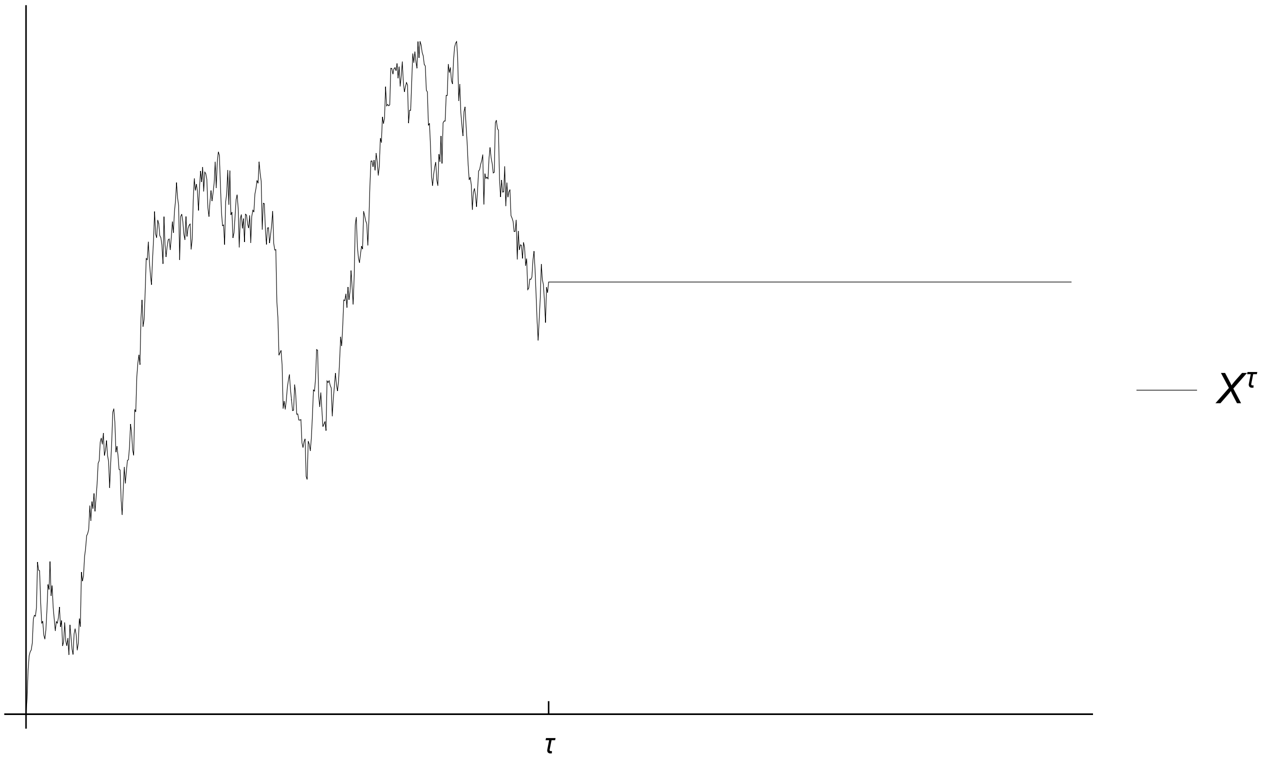
# Stopping Time $\sigma$-algebra

- The $\sigma$ - algebra associated to a stopping time $\tau$ is
  - $\mathcal{F}_\tau := \{A \in \mathcal{F} \mid A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall\, t \geq 0\}$.
- Qualitatively,
  - $A \in \mathcal{F}_\tau$ if and only if, when we restrict ourselves from $\Omega$ to $A$, it is still possible, for every $t \geq 0$, to know if $\tau$ occurred by $t$ given our information $\mathcal{F}_t$ at $t$.
- By definition, $\Omega \in \mathcal{F}_\tau$ and trivially $\emptyset \in \mathcal{F}_\tau$.
  - One can show $\mathcal{F}_\tau$ is closed under complementation and countable unions as well.
  - $\mathcal{F}_\tau$ is a $\sigma$-algebra.

# The Stopped Process

- Let $X$ be adapted, and $\tau$ a stopping time.
  - Not necessarily a hitting time of $X$.
- $X_\tau$ is the random variable $X_{\tau(\omega)}(\omega)$.
  - I.e. $X_\tau = X_1(\omega)$ if $\tau(\omega) = 1$. $X_\tau = X_3(\omega)$ if $\tau(\omega) = 3$.
- The <u>stopped process</u> $X^\tau = \{X_t^\tau\}_{t \geq 0}$ is defined by
  - $X_t^\tau(\omega) := X_{t \wedge \tau(\omega)}(\omega)$.
    - Notation: $a \wedge b = \min[a, b]$.
  - I.e. $X^\tau = X$ up until $\tau$, $X^\tau$ is "stuck" at $X_\tau$ afterwards.

# The Stopped Process



# Optional Sampling Theorem (OST)

- Recall the martingale property.
  - $M$ is martingale if $\mathbb{E}\left[M_t \middle| \mathcal{F}_s\right] = M_s$ with probability one for all $0 \leq s \leq t$.
- The OST says that we may replace $s \leq t$ above with two (bounded) stopping times $\sigma \leq \tau$.
  - We say a stopping time $\tau$ is <u>bounded</u> if there is some $K > 0$ such that $\mathbb{P}[\tau \leq K] = 1$.

# Optional Sampling Theorem (OST)

- Theorem: let $M$ be a martingale, and $\sigma, \tau$ be two bounded stopping times s.t. $\mathbb{P}[\sigma \leq \tau] = 1$. Then
  - $\mathbb{E}\left[M_\tau \middle| \mathcal{F}_\sigma\right] = M_\sigma$ with probability one.
- Idea behind the result.
  - Assume $\sigma(\omega) = \begin{cases} 1 & \omega \in E \\ 2 & \omega \in E^c \end{cases}$, $\tau(\omega) = \begin{cases} 2 & \omega \in F \\ 3 & \omega \in F^c \end{cases}$.
    - $\sigma, \tau$ stopping times provided $E \in \mathcal{F}_1$ and $F \in \mathcal{F}_2$.
  - Claim: $M_\sigma$ is $\mathcal{F}_\sigma$ measurable.
    - $\{M_\sigma \leq \lambda\} \cap \{\sigma \leq t\} = \begin{cases} \emptyset \in \mathcal{F}_t & 0 \leq t < 1 \\ \{M_1 \leq \lambda\} \cap E \in \mathcal{F}_1 \subseteq \mathcal{F}_t & 1 \leq t < 2 \\ \{M_2 \leq \lambda\} \cap E^c \in \mathcal{F}_2 \subseteq \mathcal{F}_t & 2 \leq t \end{cases}$.

## $\mathbb{E}\left[M_\tau \middle| \mathcal{F}_\sigma\right] = M_\sigma$, $M_\sigma$ is $\mathcal{F}_\sigma$ mbl

- We need only verify $\mathbb{E}\left[M_\tau 1_A\right] = \mathbb{E}\left[M_\sigma 1_A\right]$ for $A \in \mathcal{F}_\sigma$ .

$$
\begin{aligned}
\mathbb{E}\left[M_\tau 1_A\right] &= \mathbb{E}\left[(M_2 1_F + M_3 1_{F^c}) 1_A (1_{\sigma=1} + 1_{\sigma=2})\right], \\
&\overset{(1)}{=} \mathbb{E}\left[M_2 1_F 1_A (1_{\sigma=1} + 1_{\sigma=2}) + 1_{F^c} 1_A (1_{\sigma=1} + 1_{\sigma=2}) \mathbb{E}\left[M_3 \middle| \mathcal{F}_2\right]\right], \\
&\overset{(2)}{=} \mathbb{E}\left[M_2 1_A 1_{\sigma=1} + M_2 1_A 1_{\sigma=2}\right], \\
&\overset{(3)}{=} \mathbb{E}\left[M_1 1_A 1_{\sigma=1} + M_2 1_A 1_{\sigma=2}\right], \\
&= \mathbb{E}\left[M_\sigma 1_A\right]
\end{aligned}
$$

- (1): $F^c \in \mathcal{F}_2, \underbrace{A \cap \{\sigma = 1\} \in \mathcal{F}_1}_{\text{b/c } A \in \mathcal{F}_\sigma}, \underbrace{A \cap \{\sigma = 2\}}_{\text{b/c } A \in \mathcal{F}_\sigma} \in \mathcal{F}_2$, TOWER, TOWK.
- (2): $M$ is a martingale, collecting terms.
- (3): $A \cap \{\sigma = 1\} \in \mathcal{F}_1$, TOWER/TOWK, $M$ is a martingale.

# OST Notes

- We proved OST when $\sigma, \tau$ take 2 values.
  - Extension to when $\sigma, \tau$ take a finite number of values follows arguments very similar to what we did.
    - But, notationally more complex.
- Extension to general bounded stopping times is very technical.
- Extension to unbounded stopping times requires additional assumptions.
  - E.g. if $\tau = \infty$ what is $M_\tau$?

# Consequences of OST

- Consequences of OST
  - For a martingale $M$ and bounded stopping time $\tau$, the stopped process $M^\tau$ is a Martingale.
    - OST gives $\mathbb{E}\left[M_t^\tau \middle| \mathcal{F}_{\tau\wedge s}\right] = \mathbb{E}\left[M_{\tau\wedge t} \middle| \mathcal{F}_{\tau\wedge s}\right] = M_{\tau\wedge s} = M_s^\tau$.
    - One can use this to extend to $\mathbb{E}\left[M_t^\tau \middle| \mathcal{F}_s\right] = M_s^\tau$.
  - For a martingale $M$ and bounded stopping time $\tau$ we have
    - $\mathbb{E}\left[M_\tau\right] = M_0$ (OST with $\sigma \equiv 0$)
    - Application: $\mathbb{E}^{\mathbb{Q}}\left[D_{\tau\wedge T} V_{\tau\wedge T}\right] = V_0$ for any stopping time.