MF 790 HW 3

This assignment is due on Thursday, October 7th at 8 AM. The first problem is worth 20 points, while problems 2 and 3 are worth 15 points each.

1. Itô Integrals Using Midpoint, Right Endpoint. As seen in class, for a fixed $t > 0$ if we approximate $\int_0^t W_u dW_u$ by

$$\sum_{i=1}^{n} W_{t_{i-1}} \left(W_{t_i} - W_{t_{i-1}}\right) \tag{0.1}$$

for a given partition Π, then taking $\|\Pi\| \downarrow 0$ we obtain

$$\int_0^t W_u dW_u = \frac{1}{2}\left(W_t^2 - t\right), t \geq 0,$$

which (and this can be shown directly too) shows that $t \to W_t^2 - t$ is a martingale.

In this exercise we will see what happens if, instead of evaluating W at the left side of the interval in (0.1), we evaluate it at the midpoint or the right side of the interval. To make calculations easier we will assume, for each $t \geq 0$ that the partition is $0 < t/n < 2t/n < ... < nt/n = t$ and see what happens when $n \uparrow \infty$.

(a) (right hand side) Show

$$\lim_{n\uparrow\infty} \sum_{i=1}^{n} W_{t_i} \left(W_{t_i} - W_{t_{i-1}}\right) = \frac{1}{2}\left(W_t^2 + t\right)$$

Is $t \to W_t^2 + t$ a martingale?

(b) (midpoint) Write $W_i := W_{(t_i+t_{i-1})/2}$. Show

$$\lim_{n\uparrow\infty} \sum_{i=1}^{n} W_i \left(W_{t_i} - W_{t_{i-1}}\right) = \frac{1}{2}W_t^2$$

Is $t \to W_t^2$ a martingale?

2. Practice with Itô's formula : Finding Martingales Associated to GBM. Let $\mu \in \mathbb{R}$ and $\sigma > 0$ be constants. Recall that we say S is a Geometric Brownian Motion (GBM) with drift μ and volatility σ^2 if

$$S_t = S_0 e^{(\mu-\sigma^2/2)t+\sigma W_t}; \qquad t \geq 0,$$

where S_0 is a positive constant and W is a Brownian Motion. We write $S \sim \text{GBM}(\mu, \sigma^2)$.

(a) Give the Itô process decomposition for $f(S_t)$ when $f(x) = x^p, p \in \mathbb{R}$. Is $S^p \sim \text{GBM}(\alpha, \theta^2)$ for some α, θ? For a fixed μ, σ^2, is there a value of p for which S^p is a martingale?

(b) Give the Itô process decomposition for $f(S_t)$ when $f(x) = \log(x)$. Is there a value of μ, σ^2 for which $\log(S)$ is a Martingale?

(c) Give the Itô process decomposition for $f(t, S_t)$ when $f(t, x) = e^{-\lambda t} x^p$ for $\lambda, p \in \mathbb{R}$. Find $\hat{\lambda}$ so that for any given μ, σ and p, $t \to e^{-\hat{\lambda} t} S_t^p$ is a martingale. With this $\hat{\lambda}$ compute $\mathbb{E}\left[S_t^p\right]$ without using the p.d.f. for S_t.

Hint: For a given Itô process X with decomposition $dX_t = \Theta_t dt + \Delta_t dW_t$ if $\Theta_t(\omega) = 0$ for (t, ω) then X is a Martingale.

3. Derivation of Itô 's formula for general f''. Do exercise 4.14 on page 198−199 of the class textbook.