MF 790 HW 6, Part 1 - SOLUTIONS

This assignment is due on Thursday, December 2nd at 8 AM. Problem 1 is worth 20 points, and problems 2,3 are worth 25 points each, for a total of 70 points.

1. Covariation of Independent Brownian Motions and Lévy's Theorem. Fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and let $W_1, ..., W_d$ be d independent Brownian motions.

(a) Recall the quadratic covariation of two processes X, Y is first defined on a partition $\Pi = \{0 = t_0 < t_1 < ... < t_n = t\}$ of $[0, t]$ by

$$[X, Y]_t^{\Pi} = \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right)\left(Y_{t_i} - Y_{t_{i-1}}\right).$$

Then, provided the limit exists, we set $[X, Y]_t = \lim_{\|\Pi\| \to 0} [X, Y]_t^{\Pi}$.

For $p \neq q$ and any partition Π show that

(i) $\mathbb{E}\left[[W^p, W^q]_t^{\Pi}\right] = 0$.

(ii) $\mathrm{Var}\left[[W^p, W^q]_t^{\Pi}\right] = \sum_{i=1}^{n}(t_i - t_{i-1})^2 \leq t \times \|\Pi\|$

Therefore, taking $\|\Pi\| \to 0$, we may conclude that $[W^p, W^q]_t = 0$ almost surely for each $t \geq 0$.

(b) Now, define the martingale $\widetilde{W}$ by

$$\widetilde{W}_t := \sum_{p=1}^{d} \int_0^t \Delta_u^p dW_u^p, \qquad t \geq 0.$$

Using the heuristic

$$d[\widetilde{W}, \widetilde{W}]_t \approx (d\widetilde{W}_t)^2 = \left(\sum_{p=1}^{d} \Delta_u^p dW_u^p\right)\left(\sum_{q=1}^{d} \Delta_u^q dW_u^q\right)$$
$$= \sum_{p,q=1}^{d} \Delta_u^p \Delta_u^q dW_u^p dW_u^q \approx \sum_{p,q=1}^{d} \Delta_u^p \Delta_u^q d[W^p W^q]_u,$$

in conjunction with Lévy's theorem, identify a condition on the $\{\Delta^p\}_{p=1}^d$ such that $\widetilde{W}$ is a Brownian motion.

Solution

(a) For part (i) we have

$$\mathbb{E}\left[[W^p, W^q]_t^{\Pi}\right] = \mathbb{E}\left[\sum_{i=1}^{n}(W_{t_i}^p - W_{t_{i-1}}^p)(W_{t_i}^q - W_{t_{i-1}}^q)\right].$$

The result follows since for each i, by independence of W^p, W^q and the distributional properties of Brownian motion, we may conclude

$$\mathbb{E}\left[(W_{t_i}^p - W_{t_{i-1}}^p)(W_{t_i}^q - W_{t_{i-1}}^q)\right] = \mathbb{E}\left[(W_{t_i}^p - W_{t_{i-1}}^p)\right] \times \mathbb{E}\left[(W_{t_i}^q - W_{t_{i-1}}^q)\right] = 0.$$

For part (ii) we have

$$\begin{aligned}
\operatorname{Var}\left[[W^p, W^q]_t^\Pi\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n}(W_{t_i}^p - W_{t_{i-1}}^p)(W_{t_i}^q - W_{t_{i-1}}^q)\right)^2\right], \\
&= \sum_{i=1}^{n} \mathbb{E}\left[(W_{t_i}^p - W_{t_{i-1}}^p)^2(W_{t_i}^q - W_{t_{i-1}}^q)^2\right] \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \mathbb{E}\left[(W_{t_i}^p - W_{t_{i-1}}^p)(W_{t_i}^q - W_{t_{i-1}}^q)(W_{t_j}^p - W_{t_{j-1}}^p)(W_{t_j}^q - W_{t_{j-1}}^q)\right].
\end{aligned}$$

For the diagonal terms, using the independence of W^p, W^q and the distributional properties of Brownian motion, we know

$$\begin{aligned}
&\mathbb{E}\left[(W_{t_i}^p - W_{t_{i-1}}^p)^2(W_{t_i}^q - W_{t_{i-1}}^q)^2\right] = \mathbb{E}\left[(W_{t_i}^p - W_{t_{i-1}}^p)^2\right] \times \mathbb{E}\left[(W_{t_i}^q - W_{t_{i-1}}^q)^2\right] \\
&\qquad = (t_i - t_{i-1})^2.
\end{aligned}$$

For the off-diagonal terms, note that $j \geq i+1$ implies that $t_{j-1} \geq t_i$. Then, by independence of W^p, W^q

$$\begin{aligned}
&\mathbb{E}\left[(W_{t_i}^p - W_{t_{i-1}}^p)(W_{t_i}^q - W_{t_{i-1}}^q)(W_{t_j}^p - W_{t_{j-1}}^p)(W_{t_j}^q - W_{t_{j-1}}^q)\right] \\
&\quad = \mathbb{E}\left[(W_{t_i}^p - W_{t_{i-1}}^p)(W_{t_j}^p - W_{t_{j-1}}^p)\right] \mathbb{E}\left[(W_{t_i}^q - W_{t_{i-1}}^q)(W_{t_j}^q - W_{t_{j-1}}^q)\right]
\end{aligned}$$

Using the martingale property for W^p and TOWER at $\mathcal{F}_{t_{j-1}}$

$$\begin{aligned}
&\mathbb{E}\left[(W_{t_i}^p - W_{t_{i-1}}^p)(W_{t_j}^p - W_{t_{j-1}}^p)\right] \\
&\quad \mathbb{E}\left[(W_{t_i}^p - W_{t_{i-1}}^p)\mathbb{E}\left[(W_{t_j}^p - W_{t_{j-1}}^p)\middle|\mathcal{F}_{t_{j-1}}\right]\right] = 0.
\end{aligned}$$

It thus follows that

$$\operatorname{Var}\left[[W^p, W^q]_t^\Pi\right] = \sum_{i=1}^{n}(t_i - t_{i-1})^2 \leq \|\Pi\| t,$$

giving the result.

(b) From part (a) we see that

$$d[\widetilde{W}, \widetilde{W}]_t = \sum_{p,q=1}^{d} \Delta_u^p \Delta_u^q d[W^p W^q]_u = \sum_{p=1}^{d} (\Delta_u^p)^2 du.$$

As $\widetilde{W}$ is a continuous martingale, by Lévy we know that $\widetilde{W}$ will be a Brownian motion provided $[\widetilde{W}, \widetilde{W}]_t = t$ with probability one for all $t \geq 0$. The condition we therefore need is

$$1 = \sum_{p=1}^{d} (\Delta_u^p)^2,$$

with probability one for all $t \geq 0$.

2. The CIR Process. In this exercise we will construct an explicit solution to the CIR SDE

$$dX_t = \kappa(\theta - X_t)dt + \xi\sqrt{X_t}d\widetilde{W}_t; \qquad X_0 = x > 0,$$

where $\widetilde{W}$ is a (to-be-determined) Brownian motion, when θ takes the special form

$$\theta = \frac{\xi^2 d}{4\kappa}, \tag{0.1}$$

for some integer $d = 2, 3, ...$

(a) For $p = 1, ..., d$ let Y^p be an OU process with dynamics

$$dY_t^p = -\frac{1}{2}\kappa Y_t^p dt + dW_t^p; \qquad Y_0^p = \sqrt{\frac{x}{\kappa\theta}}.$$

Next, for arbitrary $\theta > 0$, define the process X by

$$X_t := \frac{\kappa\theta}{d}\sum_{p=1}^{d}(Y_t^p)^2, \qquad t \geq 0.$$

Note that $X_0 = x$. Using Itô's formula, show that

$$dX_t = \kappa(\theta - X_t)dt + \xi\sqrt{X_t} \times \frac{2\kappa\theta}{d\xi\sqrt{X_t}}\sum_{p=1}^{d} Y_t^p dW_t^p.$$

Note: it can be shown for $d \geq 2$, that $X_t > 0$ with probability one for all t so there is no problem with $\sqrt{X_t}$ in the denominator.

(b) Specifying θ from (0.1), and using your result from problem 1, show that

$$\widetilde{W}_t := \sum_{p=1}^{d}\int_0^t \frac{2\kappa\theta}{d\xi\sqrt{X_u}} Y_u^p dW_u^p,$$

is a Brownian motion. Conclude that

$$dX_t = \kappa(\theta - X_t)dt + \xi\sqrt{X_t}d\widetilde{W}_t,$$

solves the CIR SDE.

(c) Using the explicit distribution for each Y_t^p, what is the distribution of X_t? Here, leave your answer in terms of θ from (0.1), plugging in $d = 4\kappa\theta/\xi^2$ wherever d appears.

Solution:

(a) By Itô and the definition of $\widetilde{W}$ we know

$$\begin{aligned} dX_t &= \frac{\kappa\theta}{d}\sum_{p=1}^{d}\left(2Y_t^p\left(-\frac{1}{2}\kappa Y_t^p dt + dW_t^p\right) + dt\right), \\ &= \left(\kappa\theta - \frac{\kappa^2\theta}{d}\sum_{p=1}^{d}(Y_t^p)^2\right)dt + \frac{2\kappa\theta}{d}\sum_{p=1}^{d}Y_t^p dW_t^p, \\ &= \kappa(\theta - X_t)dt + \xi\sqrt{X_t}\frac{2\kappa\theta}{d\xi\sqrt{X_t}}\sum_{p=1}^{d}Y_t^p W_t^p. \end{aligned}$$

(b) We have $\widetilde{W}_t = \sum_{p=1}^{d}\int_0^t \Delta_u^p dW_u^p$ for

$$\Delta_u^p := \frac{2\kappa\theta}{d\xi\sqrt{X_u}}Y_u^p.$$

From problem 1 part (b) we know $\widetilde{W}$ will be a Brownian motion provided with probability one for all $t \geq 0$ we have

$$1 = \sum_{p=1}^{d}(\Delta_t^p)^2 = \frac{4\kappa^2\theta^2}{d^2\xi^2 X_t}\sum_{p=1}^{d}(Y_t^p)^2 = \frac{4\kappa\theta}{d\xi^2 X_t}X_t = \frac{4\kappa\theta}{d\xi^2}.$$

Therefore, since θ satisfies (0.1) we obtain the result.

(c) We know for each $t \geq 0, p = 1, ..., d$ that $Y_t^p \sim N(\nu_t, \zeta_t^2)$ where

$$\mu_t = \sqrt{\frac{x}{\kappa\theta}}e^{-\frac{1}{2}\kappa t}; \qquad \zeta_t^2 = \frac{1}{\kappa}\left(1 - e^{-\kappa t}\right).$$

Therefore, we see that

$$X_t = \frac{\kappa\theta}{d}\sum_{p=1}^{d}\left(N(\nu_t, \zeta_t^2)^p\right)^2 = \frac{\kappa\theta\zeta_t^2}{d}\sum_{p=1}^{d}\left(N\left(\frac{\nu_t}{\zeta_t}, 1\right)^p\right)^2.$$

Since the normals are independent we see that X_t is $\kappa\theta\zeta_t^2/d$ times a non-central chi-square distribution with d degrees of freedom and non-centrality parameter

$$\lambda = d\frac{\nu_t^2}{\zeta_t^2}$$

Plugging back in for μ_t, ζ_t and replacing d with $4\kappa\theta/\xi^2$ throughout we see that

$$\frac{\kappa\theta\zeta_t^2}{d} = \frac{\xi^2(1 - e^{-\kappa t})}{4\kappa}; \quad d = \frac{4\kappa\theta}{\xi^2}; \qquad \lambda = \frac{4\kappa x e^{-\kappa t}}{\xi^2\left(1 - e^{-\kappa t}\right)}.$$

Remarkably, the same distributional conclusions hold for arbitrary θ, not just θ from (0.1).

3. On the CIR Interest Rate Model. Read Example 6.5.2 and do Exercise 6.4 on page 285 of the class textbook.

Solution:

(i) Since $\varphi'(t) = -(1/2)\sigma^2 C(t,T)\varphi(t)$ the formula for (6.9.8) is clear. As for (6.9.9) we have

$$C'(t,T) = -\frac{2\varphi''(t)}{\sigma^2\varphi(t)} + \frac{2(\varphi'(t))^2}{\sigma^2\varphi(t)^2} = -\frac{2\varphi''(t)}{\sigma^2\varphi(t)} + \frac{\sigma^2}{2}\left(\frac{2\varphi'(t)}{\sigma^2\varphi(t)}\right)^2,$$

from which the result follows.

(ii) Using the result from (i) and (6.5.14) we see that

$$0 = -C'(t,T) + bC(t,T) + \frac{1}{2}\sigma^2 C(t,T)^2 - 1 = -\frac{2b\varphi'(t)}{\sigma^2\varphi(t)} + \frac{2\varphi''(t)}{\sigma^2\varphi(t)} - 1.$$

This gives the result, when multiplying by $2\sigma^2\varphi(t)$.

(iii) The solutions to the characteristic equation are

$$\lambda_\pm = \frac{1}{2}b \pm \gamma.$$

This means

$$\varphi(t) = a_1 e^{\left(\frac{1}{2}b+\gamma\right)t} + a_2 e^{\left(\frac{1}{2}b-\gamma\right)t} = \widetilde{a}_1 e^{-\left(\frac{1}{2}b+\gamma\right)(T-t)} + \widetilde{a}_2 e^{-\left(\frac{1}{2}b-\gamma\right)(T-t)},$$

if we set

$$a_1 = \widetilde{a}_1 e^{-\left(\frac{1}{2}b+\gamma\right)T}; \qquad a_2 = \widetilde{a}_2 e^{-\left(\frac{1}{2}b-\gamma\right)T}.$$

As $\widetilde{a}_1, \widetilde{a}_2$ are general, and since $\gamma > b/2$ the result follows by setting

$$\widetilde{a}_1 = \frac{c_1}{\frac{1}{2}b+\gamma}; \qquad \widetilde{a}_2 = -\frac{c_2}{\frac{1}{2}b-\gamma},$$

for to-be-determined c_1, c_2.

(iv) (6.9.12) follows by taking a derivative. Since $C(T,T) = 0$ we know form (i) that $\varphi'(T) = 0$ as well. Then $c_1 = c_2$ is immediate.

(v) The first equality follows from (iv) and obtaining a common denominator. For the second equality, use $b^2/4 - \gamma^2 = -\sigma^2/2$ and the definition of cosh and sinh. The formula for $\varphi'(t)$ follows immediately from (6.9.12), $c_1 = c_2$ and the definition of sinh. Lastly, (6.5.16) is clear given part (i).

(vi) Since $A(T,T) = 0$ we have

$$0 = A(T,T) = A(t,T) + \int_t^T A(u,T)du = A(t,T) + \frac{2a}{\sigma^2}\int_t^T \frac{\varphi'(u)}{\varphi(u)}du$$
$$= A(t,T) + \frac{2a}{\sigma^2}\log\left(\frac{\varphi(T)}{\varphi(t)}\right).$$

From part (v), along with $\sinh(0) = 0, \cosh(0) = 1$ we see that

$$\frac{\varphi(T)}{\varphi(t)} = \frac{2\gamma e^{\frac{1}{2}b(T-t)}}{2\gamma\cosh(\gamma(T-t)) + b\sinh(\gamma(T-t))}.$$

The result then follows.