Dividend Paying Stocks

MF 790 Stochastic Calculus

Outline

- Stocks paying a dividend rate.
  - Asset price and wealth process dynamics.
  - The pricing formula.
- Stocks paying discrete dividends.
  - Asset price and wealth process dynamics.
  - The pricing formula.

If no dividends,

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$$

$$\frac{dD_t}{D_t} = -r_t dt$$

i) a continuous dividend stream.

- assume S pays a continuous dividend with rate $a = \{a_t\}$ process.

If I hold Δ_t shares over $[t, t+dt]$, then I get the dividend $\Delta_t S_t a_t dt$

a: proportional rate

Dynamics.

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t - a_t dt$$

Wealth processes.

key insight: we receive the dividend, and we <u>reinvest</u> it, b/c our wealth process is self-financing.

$X_t = \Delta_t S_t + (X_t - \Delta_t S_t)$

$X_{t+\Delta} = \Delta_t S_{t+\Delta} + (X_t - \Delta_t S_t) r dt + \underbrace{\Delta_t S_t a_t dt}_{\text{div payment}}$

$X_t = \Delta_t S_t + (X_t - \Delta_t S_t)$

$X_{t+\Delta} = \Delta_t S_{t+\Delta} + (X_t - \Delta_t S_t) r dt + \Delta_t S_t a_t dt$

$$\begin{aligned} dX_t &= \Delta_t dS_t + (X_t - \Delta_t S_t) r dt + \Delta_t S_t a_t dt \\ &= \Delta_t S_t \mu_t dt + \Delta_t S_t \sigma_t dW_t - \Delta_t S_t a_t dt \\ &\quad + (X_t - \Delta_t S_t) r_t dt + \Delta_t S_t a_t dt \\ &= r_t X_t dt + \Delta_t S_t \sigma_t (dW_t + \theta_t dt) \end{aligned}$$

— same as before!

$$dX_t = r_t X_t dt + \Delta_t S_t \sigma_t (dW_t + \theta_t dt)$$

$= r_t X_t dt + \Delta_t S_t \sigma_t dW_t^Q$

1) discounted wealth processes are martingales $d(D_t X_t) = D_t \Delta_t S_t \sigma_t dW_t^Q$

2) discounted stock prices are <u>not</u> Q mart

$$\frac{dS_t}{S_t} = (\mu_t - a_t)\,dt + \sigma_t dW_t$$
$$= (r_t - a_t)\,dt + \sigma_t dW_t^Q$$

Consider when $X_0 = S_0$, $\Delta \equiv 1$

- this is not just owning a stock
- this is owning a stock and reinvesting the dividends

$$D_t X_t = D_t S_t + \underbrace{\int_0^t D_u a_u S_u du}_{\text{cash flow stream from the dividend}}$$

$a \geq 0 \Rightarrow DS$ is Q supermart.

The pricing formula is the same as before.

$$V_t = \mathbb{E}^Q[D_T/D_t V_T \mid \mathcal{F}_t]$$

$$M_t^Q = \mathbb{E}^Q[D_T V_T \mid \mathcal{F}_t] = M_0^Q + \int_0^t \Gamma_u dW_u^Q$$

$$d(D_t X_t^\Delta) = D_t \Delta_t S_t \sigma_t dW_t^Q \qquad \Delta_t = \frac{\Gamma_t}{D_t S_t \sigma_t} \qquad X_0 = M_0^Q$$

$$\Rightarrow \cancel{D_T} X_T^\Delta = \cancel{D_T} V_T.$$

Q: What changes?

Consider a European option with payoff $V_T = f(S_T)$

$$V_t = \mathbb{E}^Q[e^{-\int_t^T r_u du} f(S_T) \mid \mathcal{F}_t]$$

$$\frac{dS_t}{S_t} = (r_t - a_t)\,dt + \sigma_t dW_t^Q$$

this changes things

B-S with dividends

$$\frac{dS_t}{S_t} = (r-a)\,dt + \sigma dW_t^Q$$

$$V_t = \mathbb{E}^Q[e^{-r(T-t)}(S_T - K)^+ \mid \mathcal{F}_t]$$
$$= e^{-a(T-t)} \mathbb{E}^Q[e^{-(r-a)(T-t)}(S_T - K)^+ \mid \mathcal{F}_t]$$

$$= e^{-a(T-t)} C^{BS}(t, S_t ; \tilde{r} = r-a)$$

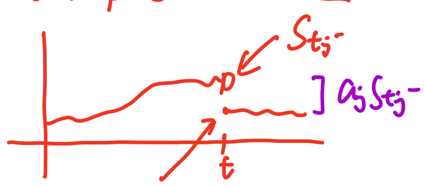
Discrete Dividends

$$0 < t_1 < t_2 < \cdots < t_n < T$$

at each t_j a lump payment of $a_j S_{t_j}$ is made as the dividend.

a_j : $\mathcal{F}_{t_j}$ - mbl . $a_j < 1$ (stock price after to

$S_{t_j^-}$: stock price right before the payment.



$$S_{t_j} = S_{t_j^-} - a_j S_{t_j^-}$$
$$= (1-a_j) S_{t_j^-}$$

jump in stock price

X: price that can jump

X_{t^-} : value before jump at t

X_t : value after . . . .

dynamics in between dividend payment times $t \in (t_{j-1}, t_j)$

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$$

at t_j,

$$S_{t_j} = (1-a_j) S_{t_j^-}$$

Wealth process dynamics

$$t \in (t_{j-1}, t_j)$$

$$dX_t = rX_t + \Delta_t S_t \sigma_t dW_t^{Q}$$

at t_j

— loss of $a_j S_{t_j} \Delta_{t_j}$; due to stock price jump

– gain of $a_j S_{t_j} \Delta_{t_j}$ due to the dividend payment

– no jump !

Throughout , $[0, T] \to dX_t = r_t X_t + \Delta_t S_t \sigma_t dW_t^{Q}$

throughout implies the pricing formula is the same

$$V_t = \mathbb{E}^{Q}[e^{-\int_t^T r_u du} V_T \mid \mathcal{F}_t]$$

But, the actual value might change due to the dividends.

– especially for European options.

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dW_t^{Q} \qquad t \in (t_{j-1}, t_j)$$

$$S_{t_j} = (1-a_j) S_{t_j -}$$

$$Y_t \triangleq \int_0^t (r_u - \tfrac{1}{2}\sigma_u^2) du + \int_0^t \sigma_u dW_u^{Q}$$

$$S_{t_j -} = S_{t_{j-1}} e^{Y_{t_j} - Y_{t_{j-1}}}$$

$$S_{t_j} = S_{t_{j-1}} e^{Y_{t_j} - Y_{t_{j-1}}} (1-a_j)$$

$$\frac{S_T}{S_0} = \frac{S_T}{S_{t_n}} \frac{S_{t_n}}{S_{t_{n-1}}} \cdots \frac{S_{t_1}}{S_0}$$

$$= e^{Y_T - Y_{t_n}} (1-a_n) e^{Y_{t_n} - Y_{t_{n-1}}} (1-a_{n-1}) e^{Y_{t_{n-1}} - Y_{t_{n-2}}} \cdots (1-a_1) e^{Y_{t_1} - \overset{=0}{Y_{t_0}}}$$

$$= e^{Y_T} \prod_{i=1}^{n} (1-a_i)$$

can be random

$$S_T = S_0 e^{\int_0^T (r_u - \frac{1}{2}\sigma_u^2) du + \int_0^T \sigma_u dW_u^{Q}} \prod_{i=1}^{n} (1-a_i)$$

Technically, the $\{a_j\}$ can be random

(a_j to be $\mathcal{F}_{t_j}$ mbl)

But , if we assume they are constant.

$$S_t = \hat{S}_0 e^{\int_0^T (r_u - \frac{1}{2}\sigma_u^2) du + \int_0^T \sigma_u dW_u^{Q}}$$

$$\hat{S}_0 = S_0 \prod_{i=1}^{n} (1-a_j)$$

– We only adjust the initial stock price

E.g. pricing a call in B-S with dividends

$$S_T = S_0 \prod_{j=1}^{n}(1-a_j)\, e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T^Q}$$

Call price at $0<t<T$, if $S_t = S$ is the same!

$$S_T = S_t\, e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T^Q - W_t^Q)}$$

at 0, we must plug in $\hat{S}_0$. Call price at $0 = C^{BS}(0, \hat{S}_0)$

Conditioning on $S_t = S$ incorporates the dividends

$$\hat{S}_0\, e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t^Q} = S$$

$$W_t^Q = \frac{1}{\sigma}\left(\log(S/\hat{S}_0) - (r-\tfrac{1}{2}\sigma^2)t\right)$$

But, once we done this,

$$\text{price} = C^{BS}(t, S)$$

Model

- Fixed $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. W is a Brownian motion.
  - Here, $\mathbb{F} = \mathbb{F}^W$ (so we can use martingale representation).
- Take adapted processes μ, σ, r.
  - Risky asset $S \sim \mathrm{genGBM}(\mu, \sigma)$. $\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$.
    - In the absence of dividends.
  - Discount process $D \sim \mathrm{genGBM}(r, 0)$. $\frac{dD_t}{D_t} = -r_t dt$.
    - μ, σ, r such that (S, D) are well defined.

Continuous Dividends

- Assume the stock pays a continuous dividend rate a.
  - What this means: if I hold Δ_t shares over $[t, t + dt]$, then I receive a dividend payment of $\Delta_t a_t S_t dt$.
    - a: adapted process.
  - This does not really apply to a single stock, but is not too bad an assumption for, e.g., mutual funds.
- Stock dynamics: $\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t - a_t dt$.
  - Dividend payment reduces the stock price.

Wealth Dynamics

- Assume over $[t, t+dt]$ we hold Δ_t shares of S.
  - Denote by X the (self-financing) wealth process.
  - $X_t = \Delta_t S_t + (X_t - \Delta_t S_t)$.
  - $X_{t+dt} = \underbrace{\Delta_t S_{t+\Delta}}_{\text{stock}} + \underbrace{\Delta_t S_t a_t dt}_{\text{dividend}} + \underbrace{(X_t - \Delta_t S_t) r_t dt}_{\text{money market}}$.
- Same dynamics as without dividends!

$$\begin{aligned} dX_t &= \Delta_t dS_t + \Delta_t S_t a_t dt + (X_t - \Delta_t S_t) r dt, \\ &= \Delta_t S_t \left((\mu_t - a_t) dt + \sigma_t dW_t \right) + \Delta_t S_t a_t + (X_t - \Delta_t S_t) r dt, \\ &= r_t X_t dt + \Delta_t \sigma_t S_t \left(dW_t + \Theta_t dt \right), \\ \Theta_t &= \frac{\mu_t - r_t}{\sigma_t}. \end{aligned}$$

Wealth Dynamics

- $dX_t = r_t X_t dt + \Delta_t S_t \left(dW_t + \Theta_t dt \right)$.
  - Same dynamics because drop stock price $(-S_t a_t dt)$ is exactly offset by the dividend payment $(S_t a_t dt)$.
- Define the risk neutral measure $\mathbb{Q}$ through
  - $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = Z_T^{\Theta} = e^{-\int_0^T \Theta_u dW_u - \frac{1}{2}\int_0^T \Theta_u^2 du}$.
  - Girsanov: $W_t^{\mathbb{Q}} = W_t + \int_0^t \Theta_u du, t \leq T$ is a $\mathbb{Q}$ B.M..
- $dX_t = r_t X_t dt + \Delta_t S_t dW_t^{\mathbb{Q}}$.
  - Discounted wealth processes are still martingales under $\mathbb{Q}$.

Stock Dynamics

- What about the stock?
  - $\frac{dS_t}{S_t} = (\mu_t - a_t)dt + \sigma_t dW_t = (r_t - a_t)dt + \sigma_t dW_t^{\mathbb{Q}}$.
    - Discounted stock price is NOT a martingale under $\mathbb{Q}$!
    - If $a \geq 0$, discounted stock price is $\mathbb{Q}$ super-martingale.
    - We must reinvest the dividends to obtain a martingale.
  - I.e. for $\Delta \equiv 1$ and $X_0 = S_0$, as dividends are reinvested
    - $D_t S_t \neq D_t X_t = D_t S_t + \int_0^t D_u S_u a_u du$.

$$\frac{dS_t}{S_t} = (r_t - a_t)dt + \sigma_t dW_t^{\mathbb{Q}}.$$

- This matters when we price European options.
- Consider a call option for constant μ, σ, r, a.
  - As the discounted wealth process dynamics are the same, and martingale representation still holds, the pricing formula is the same.

$$\begin{aligned}
V_t &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ \middle| \mathcal{F}_t\right], \qquad S \overset{\mathbb{Q}}{\sim} \mathrm{GBM}\left(r - a, \sigma^2\right), \\
&= e^{-a(T-t)}\mathbb{E}^{\mathbb{Q}}\left[e^{-(r-a)(T-t)}(S_T - K)^+ \middle| \mathcal{F}_t\right], \\
&= e^{-a(T-t)}c(t, S_t; r - a).
\end{aligned}$$

  - $c(t, s; r - a)$: BS call price for a money market rate of $r - a$.

Discrete Dividends

- Consider when dividends are paid discretely.
  - Fix times $0 < t_1 < t_2 < \cdots < t_n < T$.
  - At t_j a dividend is paid of $a_j S_{t_j-}$ for a $\mathcal{F}_{t_j}$ mbl rv a_j.
    - S_{t_j-}: stock price right before the dividend payment.
  - Stock price drops from S_{t_j-} to $S_{t_j} = S_{t_j-}(1 - a_j)$.
  - If we hold Δ_{t_j} shares in S at t_j, we receive $\Delta_{t_j} a_j S_{t_j-}$.
  - In between dividend times, $S \sim \mathrm{genGBM}\,(\mu, \sigma^2)$.
    - $t \in (t_{j-1}, t_j)$ implies $\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t = r_t dt + \sigma_t dW_t^{\mathbb{Q}}$.

Wealth Dynamics

- As before, because we obtain the dividend, the wealth dynamics are the same.
- Between dividend times
  - $dX_t = \Delta_t dS_t + (X_t - \Delta_t S_t) r_t dt = r_t X_t dt + \Delta_t S_t \sigma_t dW_t^{\mathbb{Q}}$.
- At dividend time t_j.
  - Loss due to drop in stock price: $\Delta_t a_j S_{t_j-}$.
  - Gain due to dividend payment: $\Delta_t a_j S_{t_j-}$.
    - Exact offset - no jump in wealth.
- Thus, $dX_t = r_t X_t + \Delta_t S_t \sigma_t dW_t^{\mathbb{Q}}$ throughout.
  - Discounted wealth processes are again martingales under $\mathbb{Q}$.

Replication and Pricing

- Because $dX_t = r_t X_t + \Delta_t S_t \sigma_t dW_t^{\mathbb{Q}}$ throughout.
  - The pricing formula is again the same.
  - For a payoff V_T, $V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du} V_T \middle| \mathcal{F}_t\right]$.
- For European options with payoff $V_T = f(S_T)$ we need to know what S_T looks like under $\mathbb{Q}$.

S_T under $\mathbb{Q}$

- Write $Y_t := \int_0^t \left(r_u - \frac{1}{2}\sigma_u^2\right) du + \int_0^t \sigma_u dW_u^{\mathbb{Q}}$.
- $S \overset{\mathbb{Q}}{\sim} \mathrm{genGBM}\left(r, \sigma^2\right)$ in $\left(t_j, t_{j+1}\right)$ implies
  - $S_{t_{j+1}} = (1 - a_{j+1})S_{t_{j+1}-} = (1 - a_{j+1})S_{t_j} e^{Y_{t_{j+1}} - Y_{t_j}}$.

$$
\begin{aligned}
\frac{S_T}{S_0} &= \frac{S_T}{S_{t_n}} \times \frac{S_{t_n}}{S_{t_{n-1}}} \times \cdots \times \frac{S_{t_1}}{S_0}, \\
&= e^{Y_T - Y_{t_n}} \times (1 - a_n)e^{Y_{t_n} - Y_{t_{n-1}}} \times (1 - a_{n-1})e^{Y_{t_{n-1}} - Y_{t_{n-2}}} \times \cdots \times e^{Y_{t_1}}, \\
&= e^{Y_T} \prod_{j=1}^{n} (1 - a_{t_j}).
\end{aligned}
$$

S_T under $\mathbb{Q}$

- We just showed
  - $S_T = S_0 e^{\int_0^T \left(r_u - \frac{1}{2}\sigma_u^2\right)du + \int_0^T \sigma_u dW_u^{\mathbb{Q}}} \prod_{j=1}^n (1 - a_j)$.
- This works assuming a_j is $\mathcal{F}_{t_j}$ mbl, $a_j < 1, \forall j$.
- If we specify to when the $\left\{a_{t_j}\right\}$ are non-random constants.
  - $S \overset{\mathbb{Q}}{\sim} \mathrm{genGBM}\left(r, \sigma^2\right)$ with initial price
    - $\widehat{S}_0 := S_0 \prod_{j=1}^n (1 - a_{t_j})$.

Option Pricing in the Black-Scholes Model

- Assume r, σ are constant, in addition to the $\{a_j\}$.
  - $S_T = \widehat{S}_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T^{\mathbb{Q}}}$
  - Black-Scholes model, but with the adjusted starting price.
  - For times $0 < t < T$ and given $S_t = s$, option prices are as before.
  - At time $t = 0$ we just have to plug in the adjusted starting price.
    - E.g. call price is $c(0, \widehat{S}_0)$.