MF 790 HW 5, Part 1 - SOLUTIONS

This assignment is due on Thursday, November 11th at 8 AM. There is a reading assignment, and three problems. Problems 1,2 are worth 15 points each, and problem 3 is worth 20 points, for a total of 50 points.

0. Reading Assignment. Read carefully Chapters 5.2 and 5.3 on pages 210-224 of the class textbook. Here, the risk neutral pricing theory is extended to when the stock price process follows a generalized Geometric Brownian motion $S \sim \text{genGBM}(\alpha, \sigma^2)$ (α plays the role of μ) where α, σ are adapted processes. Also, the interest rate is no longer constant, but a time varying process R. The main results are the same, however, that using Itô's formula, Girsanov's theorem, Lévy's characterization and the Martingale representation theorem, it follows that the price of any contingent claim with $\mathcal{F}_T$ measurable payoff V_T at T is given at $t \leq T$ by

$$V_t = \frac{1}{D_t Z_t}\mathbb{E}\left[D_T Z_T V_T \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T R_u du} V_T \middle| \mathcal{F}_t\right].$$

Here, the discount process is $D_t = e^{-\int_0^t R_u du}$ and $Z \sim \text{genGBM}(0, \Theta^2)$ where

$$\Theta_t = \frac{\alpha_t - R_t}{\sigma_t}$$

is the market price of risk process. Lastly, under the risk neutral measure $\mathbb{Q}$, created by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$, it follows that $S \sim \text{genGBM}(R, \sigma^2)$ so that

$$\frac{dS_t}{S_t} = R_t dt + \sigma_t dW_t^{\mathbb{Q}}; \qquad W_t^{\mathbb{Q}} = W_t + \int_0^t \Theta_u du.$$

The arguments are essentially identical to what we did in class, but now allowing α, σ, R to be stochastic processes. Especially when we price fixed income instruments, it is vital we allow these quantities to be stochastic processes.

1. Implying the Risk-Neutral Distribution. Do Exercise 5.9 on page 255 of the class textbook (Vol II).

Solution: We have that

$$c(0,T,x,K) = e^{-rT}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy.$$

From this it follows that

$$\begin{aligned} c_K(0,T,x,K) &= -e^{-rT}(K-K)\tilde{p}(0,T,x,K) - e^{-rT}\int_K^\infty \tilde{p}(0,T,x,y)dy, \\ &= -e^{-rT}\int_K^\infty \tilde{p}(0,T,x,y)dy; \\ c_{KK}(0,T,x,K) &= e^{-rT}\tilde{p}(0,T,x,K). \end{aligned}$$

2. Hedging a Cash Flow. Do Exercise 5.11 on pages 256-257 of the class textbook (Vol II).

Solution: Note that

$$d(D_tX_t) = \Delta_t D_t(dS_t - R_tS_tdt) - D_tC_tdt.$$

This gives that

$$D_tX_t + \int_0^t D_sC_sds = X_0 + \int_0^t \Delta_s D_s(dS_s - R_sS_sds).$$

At time T we want $X_T = 0$. Plugging this in gives

$$\int_0^T D_tC_tdt = X_0 + \int_0^T \Delta_t D_t(dS_t - R_tS_tdt) = X_0 + \int_0^T \Delta_t D_t\sigma_t S_t d\widetilde{W}_t, \tag{0.1}$$

under risk neutral measure. To find the Δ, X_0 which achieves this we use the Martingale Representation Theorem. Specifically, we set

$$M_t = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\int_0^T D_sC_sds \big| \mathcal{F}_t\right].$$

Note that $M_T = \int_0^T D_tC_tdt$. By Martingale Representation there exists an $\mathbb{F}$ adapted process Γ such that

$$M_t = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[M_T\right] + \int_0^t \Gamma_s d\widetilde{W}_s. \tag{0.2}$$

By setting $\Delta_t = \Gamma_t/(D_t\sigma_tS_t)$ and $X_0 = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[M_T\right] = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\int_0^T D_tC_tdt\right]$ it follows from (0.1) and (0.2) that

$$X_0 + \int_0^T \Delta_t D_t\sigma_t S_t d\widetilde{W}_t = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[M_T\right] + \int_0^T \Gamma_t d\widetilde{W}_t = M_T = \int_0^T D_tC_tdt,$$

the desired result.

3. Forward Start Options (job interview question). Let $0 \le t_1 \le T$ be given. A *forward-start option* is a contract entered at time 0 that pays $(S_T - S_{t_1})^+$ at time T. In other words, it is a European call except that the strike is set "at the money" at time t_1 rather than being set at time 0. In this exercise, the underlying asset for the forward-start option is a geometric Brownian motion,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

where $W = \{W_t\}_{t\le T}$ is a Brownian motion under the physical measure $\mathbb{P}$. The interest rate $r > 0$ is constant. The price at time zero of the forward-start option is

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(S_T - S_{t_1}\right)^+\right].$$

Let $c(\tau, s; K)$ denote the Black-Scholes price of a European call on S when the initial stock price is s, the strike price is K, and the time to expiration is τ : i.e.

$$c(\tau, s; K) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\left(S_T - K\right)^+ \;\middle|\; \mathcal{F}_{T-\tau}, S_{T-\tau} = s\right] \tag{0.3}$$

(a) Show that for every positive number y, we have the scaling property

$$yc(\tau, s; K) = c(\tau, ys; yK).$$

(b) Show that at time zero the forward-start option has price $c(\tau, S(0); K)$ for appropriate values of τ and K. Determine the appropriate values of τ and K.

Solution:

(a) Using that $S \overset{\mathbb{Q}}{\sim} \mathrm{GBM}\left(r, \sigma^2\right)$ we know that

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^{\mathbb{Q}}$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$ Brownian motion. Using (1) $W_T^{\mathbb{Q}} - W_{T-\tau}^{\mathbb{Q}}$ is $\mathbb{Q}$ - independent of $\mathcal{F}_{T-\tau}$ and (2) that $W_T^{\mathbb{Q}} - W_{T-\tau}^{\mathbb{Q}} \sim \sqrt{\tau}N(0,1)$ under $\mathbb{Q}$, we have from (0.3) that

$$c(\tau, s; K) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(se^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}z} - K\right)^+ e^{-\frac{1}{2}z^2} dz \tag{0.4}$$

where we have used that

$$\frac{S_T}{S_{T-\tau}} = e^{(r-\frac{1}{2}\sigma^2)\tau + \sigma(W_T^{\mathbb{Q}} - W_{T-\tau}^{\mathbb{Q}})}$$

which implies that given $S_{T-\tau} = s$

$$S_T = se^{(r-\frac{1}{2}\sigma^2)\tau + \sigma(W_T^{\mathbb{Q}} - W_{T-\tau}^{\mathbb{Q}})} \sim se^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}Z}$$

where $Z \sim N(0,1)$ under $\mathbb{Q}$. Using (0.4) we have for $y > 0$

$$\begin{aligned}
c(\tau, ys; yK) &= \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(yse^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}z} - yK\right)^+ e^{-\frac{1}{2}z^2} dz \\
&= \frac{ye^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(se^{(r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}z} - K\right)^+ e^{-\frac{1}{2}z^2} dz \\
&= yc(\tau, x; K)
\end{aligned}$$

(b) The price at time 0 of the forward-start option is

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\big(S_T - S_{t_1}\big)^+\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-rt_1}S_{t_1}e^{-r(T-t_1)}\left(\frac{S_T}{S_{t_1}} - 1\right)^+\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-rt_1}S_{t_1}e^{-r(T-t_1)}\left(\frac{S_T}{S_{t_1}} - 1\right)^+ \big|\mathcal{F}_{t_1}\right]\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-rt_1}S_{t_1}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t_1)}\left(\frac{S_T}{S_{t_1}} - 1\right)^+ \big|\mathcal{F}_{t_1}\right]\right]
\end{aligned}
$$

But

$$
\frac{S_T}{S_{t_1}} = \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T - t_1) + \sigma\big(W_T^{\mathbb{Q}} - W_{t_1}^{\mathbb{Q}}\big)\right\} \tag{0.5}
$$

is independent of $\mathcal{F}_{t_1}$. Therefore,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t_1)}\left(\frac{S_T}{S_{t_1}} - 1\right)^+ \big|\mathcal{F}_{t_1}\right] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t_1)}\left(\frac{S(T)}{S(t_1)} - 1\right)^+\right] \\
&= c(T - t_1, 1, 1)
\end{aligned}
$$

The last equality follows from (0.5). Indeed, we see that with $Z \sim N(0,1)$

$$
\begin{aligned}
\frac{S_T}{S_{t_1}} &= \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T - t_1) + \sigma\big(W_T^{\mathbb{Q}} - W_{t_1}^{\mathbb{Q}}\big)\right\} \\
&\sim \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T - t_1) + \sigma\sqrt{T - t_1}Z\right\} \\
&\sim \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T - t_1) + \sigma W_{T-t_1}^{\mathbb{Q}}\right\} \\
&\sim S_{T-t_1} \text{ given } S_0 = 1.
\end{aligned}
$$

We thus have that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(S_T - S_{t_1})^+\right] &= c(T - t_1, 1, 1)\mathbb{E}^{\mathbb{Q}}\left[e^{rt_1}S_{t_1}\right] \\
&= c(T - t_1, 1, 1)S_0 \\
&= c(T - t_1, S_0, S_0)
\end{aligned}
$$

where the second equality follows since the discounted stock price is a martingale under $\mathbb{Q}$ and the third equality follows from part (i). Thus, the forward-start option has the same price as the at-the-money European call with $T - t_1$ time units until expiration.