MF 790 HW 6, Part 1

This assignment is due on Thursday, December 2nd at 8 AM. Problem 1 is worth 20 points, and problems 2,3 are worth 25 points each, for a total of 70 points.

1. Covariation of Independent Brownian Motions and Lévy's Theorem. Fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and let $W_1, ..., W_d$ be d independent Brownian motions.

(a) Recall the quadratic covariation of two processes X, Y is first defined on a partition $\Pi = \{0 = t_0 < t_1 < ... < t_n = t\}$ of $[0, t]$ by

$$[X, Y]_t^{\Pi} = \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right)\left(Y_{t_i} - Y_{t_{i-1}}\right).$$

Then, provided the limit exists, we set $[X, Y]_t = \lim_{\|\Pi\| \to 0} [X, Y]_t^{\Pi}$.

For $p \neq q$ and any partition Π show that
(i) $\mathbb{E}\left[[W^p, W^q]_t^{\Pi}\right] = 0$.
(ii) $\mathrm{Var}\left[[W^p, W^q]_t^{\Pi}\right] = \sum_{i=1}^{n}(t_i - t_{i-1})^2 \leq t \times \|\Pi\|$
Therefore, taking $\|\Pi\| \to 0$, we may conclude that $[W^p, W^q]_t = 0$ almost surely for each $t \geq 0$.

(b) Now, define the martingale $\widetilde{W}$ by

$$\widetilde{W}_t := \sum_{p=1}^{d} \int_0^t \Delta_u^p dW_u^p, \qquad t \geq 0.$$

Using the heuristic

$$\begin{aligned} d[\widetilde{W}, \widetilde{W}]_t \approx (d\widetilde{W}_t)^2 &= \left(\sum_{p=1}^{d} \Delta_u^p dW_u^p\right)\left(\sum_{q=1}^{d} \Delta_u^q dW_u^q\right) \\ &= \sum_{p,q=1}^{d} \Delta_u^p \Delta_u^q dW_u^p dW_u^q \approx \sum_{p,q=1}^{d} \Delta_u^p \Delta_u^q d[W^p W^q]_u, \end{aligned}$$

in conjunction with Lévy's theorem, identify a condition on the $\{\Delta^p\}_{p=1}^d$ such that $\widetilde{W}$ is a Brownian motion.

2. The CIR Process. In this exercise we will construct an explicit solution to the CIR SDE

$$dX_t = \kappa(\theta - X_t)dt + \xi\sqrt{X_t}d\widetilde{W}_t; \qquad X_0 = x > 0,$$

where $\widetilde{W}$ is a (to-be-determined) Brownian motion, when θ takes the special form

$$\theta = \frac{\xi^2 d}{4\kappa}, \tag{0.1}$$

for some integer $d = 2, 3, ...$

(a) For $p = 1, ..., d$ let Y^p be an OU process with dynamics

$$dY_t^p = -\frac{1}{2}\kappa Y_t^p dt + dW_t^p; \qquad Y_0^p = \sqrt{\frac{x}{\kappa\theta}}.$$

Next, for arbitrary $\theta > 0$, define the process X by

$$X_t := \frac{\kappa\theta}{d}\sum_{p=1}^{d}(Y_t^p)^2, \qquad t \geq 0.$$

Note that $X_0 = x$. Using Itô's formula, show that

$$dX_t = \kappa(\theta - X_t)dt + \xi\sqrt{X_t} \times \frac{2\kappa\theta}{d\xi\sqrt{X_t}}\sum_{p=1}^{d} Y_t^p dW_t^p.$$

Note: it can be shown for $d \geq 2$, that $X_t > 0$ with probability one for all t so there is no problem with $\sqrt{X_t}$ in the denominator.

(b) Specifying θ from (0.1), and using your result from problem 1, show that

$$\widetilde{W}_t := \sum_{p=1}^{d}\int_0^t \frac{2\kappa\theta}{d\xi\sqrt{X_u}} Y_u^p dW_u^p,$$

is a Brownian motion. Conclude that

$$dX_t = \kappa(\theta - X_t)dt + \xi\sqrt{X_t}d\widetilde{W}_t,$$

solves the CIR SDE.

(c) Using the explicit distribution for each Y_t^p, what is the distribution of X_t? Here, leave your answer in terms of θ from (0.1), plugging in $d = 4\kappa\theta/\xi^2$ wherever d appears.

3. On the CIR Interest Rate Model. Read Example 6.5.2 and do Exercise 6.4 on page 285 of the class textbook.