

# Homework Solutions #3

CAS MA 511

**Problem (3.4.1).** If  $P$  is a perfect set and  $K$  is compact, is the intersection  $P \cap K$  always compact? Always perfect?

*Proof.* Since  $P$  is closed and  $K$  is closed,  $P \cap K$  is closed. Furthermore, since  $K$  is bounded and  $P \cap K \subseteq K$ ,  $P \cap K$  is bounded and hence compact. If  $P$  is nonempty and  $p \in P$  and if  $K = \{p\}$ , then  $P \cap K = \{p\}$  has an isolated point and hence is not perfect. So,  $P \cap K$  must be closed, but is not necessarily perfect.  $\square$

**Problem (3.4.4).** Repeat the Cantor construction from Section 3.1 starting with the interval  $[0, 1]$ . This time however, remove the open middle fourth from each component.

- (a) Is the resulting set compact? Perfect?
- (b) Using the algorithms from Section 3.1, compute the length and dimension of this Cantor-like set.

*Proof.* For (a), let  $\tilde{C}_0 = [0, 1]$  and let  $\tilde{C}_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$  be the result of removing the open middle fourth of  $\tilde{C}_0$ . Continuing this way we obtain a sequence of sets  $\tilde{C}_n$ . Each  $\tilde{C}_n$  is closed and hence the countable union

$$\tilde{C} = \bigcap_{n=1}^{\infty} \tilde{C}_n$$

is closed. Furthermore, we have  $\tilde{C} \subseteq [0, 1]$ , so  $\tilde{C}$  is bounded and hence compact.

Now, let  $x \in \tilde{C}$ . Then, for  $n \in \mathbb{N}$ , since  $x \in \tilde{C}$ , we have  $x \in \tilde{C}_n$  for each  $n$ . Since  $x \in \tilde{C}_n$ , which is the finite union of  $2^{n-1}$  closed intervals of length  $\frac{1}{4} \left(\frac{3}{8}\right)^{n-1}$ , it must lie in one such interval. Let  $x_n$  be one of the endpoints of that interval such that  $x_n \neq x$ . Thus, we obtain a sequence  $(x_n)$  with  $x_n \neq x$  such that  $0 \leq |x_n - x| \leq \frac{1}{4} \left(\frac{3}{8}\right)^{n-1}$ . Now, since  $\frac{1}{4} \left(\frac{3}{8}\right)^{n-1} \rightarrow 0$ , by the order limit theorem  $|x_n - x| \rightarrow 0$  and hence  $(x_n) \rightarrow x$ . Thus,  $x$  is a limit point and since  $x$  was arbitrary this implies that  $\tilde{C}$  has no isolated points and is therefore perfect.

For (b), at each step we remove  $2^{n-1}$  open intervals whose length is  $\frac{1}{4} \left(\frac{3}{8}\right)^{n-1}$ , so the remaining length is:

$$1 - \frac{1}{4} - 2 \frac{1}{4} \frac{3}{8} - 4 \frac{1}{4} \left(\frac{3}{8}\right)^2 - \dots = 1 - \sum_{n=1}^{\infty} 2^{n-1} \left(\frac{3}{8}\right)^{n-1} \frac{1}{4} = 1 - \frac{\frac{1}{4}}{1 - \frac{3}{4}} = 1 - 1 = 0$$

So, the length of  $\tilde{C}$  is 0. Now, if we scale every real number by a factor of  $\frac{8}{3}$ , then we get 2 copies of  $\tilde{C}$ , so the dimension of  $\tilde{C}$  is  $\frac{\log 2}{\log \frac{8}{3}} \approx 0.707$ .  $\square$

**Problem (3.5.2).** Replace each \_\_\_\_\_ with the word *finite* or *countable*, depending on which is more appropriate.

- (a) The \_\_\_\_\_ union of  $F_\sigma$  sets is an  $F_\sigma$  set.  
 (b) The \_\_\_\_\_ intersection of  $F_\sigma$  sets is an  $F_\sigma$  set.  
 (c) The \_\_\_\_\_ union of  $G_\delta$  sets is an  $G_\delta$  set.  
 (d) The \_\_\_\_\_ intersection of  $G_\delta$  sets is an  $G_\delta$  set.

*Proof.* For (a), the countable union of  $F_\sigma$  sets is an  $F_\sigma$  set. Let  $A_n = \bigcup_{m=1}^{\infty} A_{n,m}$  be an  $F_\sigma$  set. Then, we have:

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m} = \bigcup_{\{(n,m): n,m \in \mathbb{N}\}} A_{n,m}$$

So, we just need to show that  $\{(n,m) : n,m \in \mathbb{N}\}$  is countable. We can lay out the elements of this set as follows:

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & \cdots \\ (2,1) & (2,2) & (2,3) & \cdots \\ (3,1) & (3,2) & (3,3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Now, just as we did for  $\mathbb{Q}$  we can snake back and forth starting at the top left to form a bijection between this set and  $\mathbb{N}$ . The only difference is that we do not need to skip any pairs, since while  $(1,1)$  and  $(2,2)$  would correspond to the same fraction it need not be the case that  $A_{1,1} = A_{2,2}$ . Another way to state this fact is that  $F_{\sigma\sigma} = F_\sigma$ .

For (b), the finite intersection of  $F_\sigma$  sets is an  $F_\sigma$  set. If we show that the intersection of two  $F_\sigma$  sets is  $F_\sigma$ , then it will hold for any finite intersection by induction. If  $A$  and  $B$  are  $F_\sigma$ , then we have:

$$A \cap B = \left( \bigcup_{n=1}^{\infty} A_n \right) \cap \left( \bigcup_{m=1}^{\infty} B_m \right) = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_n \cap B_m$$

Thus,  $A \cap B$  is indeed  $F_\sigma$ . If we took a countable intersection and tried to write it as a union it would require an uncountable index set, which is a bit tricky. Instead, we will show that the countable intersection of  $F_\sigma$  sets need not be  $F_\sigma$  using a proof by contradiction.

Suppose for contradiction that the countable intersection of  $F_\sigma$  sets is  $F_\sigma$ . Now, let  $A$  be a  $G_\delta$  set, that is  $A$  is the countable intersection of open sets. Since open sets are  $F_\sigma$ , it follows that  $A$  is the countable intersection of  $F_\sigma$  sets and hence is  $F_\sigma$  by our induction. So, we have shown that any  $G_\delta$  set is  $F_\sigma$ , however this is a contradiction since the set of irrationals  $\mathbb{I}$  is a  $G_\delta$  set, but not an  $F_\sigma$  set. Thus, the countable intersection of  $F_\sigma$  sets need not be  $F_\sigma$ .

For (c), the finite union of  $G_\delta$  sets is an  $F_\sigma$  set. This follows from (a) and De Morgan's laws.

For (d), the countable intersection of  $G_\delta$  sets is an  $F_\sigma$  set. This follows from (b) and De Morgan's laws.  $\square$

**Problem (3.5.10).** Prove that the set of real numbers  $\mathbb{R}$  cannot be written as the countable union of nowhere-dense sets.

*Proof.* Suppose for contradiction that  $\mathbb{R}$  can be written as the countable union of nowhere-dense sets. That is, there are nowhere-dense sets  $E_1, E_2, E_3, \dots$  such that  $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}$ . Then, for each  $E_n$ , we have  $E_n \subseteq \overline{E_n} \subseteq \mathbb{R}$ . Thus, we have:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \overline{E_n} \subseteq \mathbb{R}$$

However, this implies that  $\mathbb{R} = \bigcup_{n=1}^{\infty} \overline{E}_n$ . Then, taking complements, we have:

$$\emptyset = \mathbb{R}^c = \left( \bigcup_{n=1}^{\infty} \overline{E}_n \right)^c = \bigcap_{n=1}^{\infty} \overline{E}_n^c$$

Now, by exercise 3.5.8, each  $\overline{E}_n^c$  is dense in  $\mathbb{R}$ . Thus, by theorem 3.5.2, the above intersection should be non-empty, so we have a contradiction, and we are forced to conclude that there are no such sets  $E_1, E_2, E_3, \dots$ .  $\square$

**Problem (4.2.6).** Decide if the following claims are true or false, and give short justifications for each conclusion.

- (a) If a particular  $\delta$  has been constructed as a suitable response to a particular  $\varepsilon$  challenge, then any smaller positive  $\delta$  will also suffice.
- (b) If  $\lim_{x \rightarrow a} f(x) = L$  and  $a$  happens to be in the domain of  $f$ , then  $L = f(a)$ .
- (c) If  $\lim_{x \rightarrow a} f(x) = L$ , then  $\lim_{x \rightarrow a} 3[f(x) - 2]^2 = 3(L - 2)^2$ .
- (d) If  $\lim_{x \rightarrow a} f(x) = 0$ , then  $\lim_{x \rightarrow a} f(x)g(x) = 0$ , for any function  $g$  (with domain equal to the domain of  $f$ ).

*Proof.* For (a), the claim is true. Suppose that given  $\varepsilon > 0$ , we have some  $\delta > 0$  such that whenever  $0 < |x - c| < \delta$  it follows that  $|f(x) - L| < \varepsilon$ . Now consider  $0 < \delta' < \delta$ . Then, whenever we have  $0 < |x - c| < \delta'$  it follows that  $0 < |x - c| < \delta$  and hence  $|f(x) - L| < \varepsilon$ .

For (b), the claim is false. Let  $f(x)$  be as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then,  $\lim_{x \rightarrow 0} f(x) = 0$ , but  $f(0) = 1$ .

For (c), the claim is true. If  $\lim_{x \rightarrow a} f(x) = L$ , then by the algebraic limit theorem for functional limits it follows that:

$$\begin{aligned} \lim_{x \rightarrow a} 3[f(x) - 2]^2 &= 3 \left( \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} 2 \right) \left( \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} 2 \right) \\ &= 3(L - 2)(L - 2) \\ &= 3(L - 2)^2 \end{aligned}$$

For (d), the claim is false. Let  $f(x) = x$  on  $(0, 1)$  and let  $g(x) = \frac{1}{x}$  on the same domain. Then,  $\lim_{x \rightarrow 0} f(x) = 0$ , but we have:

$$\lim_{x \rightarrow 0} f(x)g(x) = \lim_{x \rightarrow 0} 1 = 1 \neq 0$$

(Notice that  $g$  is unbounded! As the next exercise shows, this claim would be true if  $g$  were required to be bounded.)  $\square$

**Problem (4.2.7).** Let  $g : A \rightarrow \mathbb{R}$  and assume that  $f$  is a bounded function on  $A$ . Show that if  $\lim_{x \rightarrow c} g(x) = 0$ , then  $\lim_{x \rightarrow c} g(x)f(x) = 0$  as well.

*Proof.* Let  $g$  and  $f$  be real-valued functions on  $A$  with  $f$  bounded by  $M > 0$ . Then, assume that  $\lim_{x \rightarrow c} g(x) = 0$ . Let  $\varepsilon > 0$ . Now, since  $\lim_{x \rightarrow c} g(x) = 0$  there exists a  $\delta > 0$  such that for  $0 < |x - c| < \delta$  we have  $|g(x)| < \frac{\varepsilon}{M}$ . Then, for  $0 < |x - c| < \delta$  we have:

$$|g(x)f(x)| = |g(x)||f(x)| \leq |g(x)|M < \frac{\varepsilon}{M}M = \varepsilon$$

Thus,  $\lim_{x \rightarrow c} g(x)f(x) = 0$ . □

**Problem (4.3.6).** Provide an example of each or explain why the request is impossible.

- (a) Two functions  $f$  and  $g$ , neither of which is continuous at 0 but such that  $f(x)g(x)$  and  $f(x) + g(x)$  are continuous at 0.
- (b) A function  $f(x)$  continuous at 0 and  $g(x)$  not continuous at 0 such that  $f(x) + g(x)$  is continuous at 0.
- (c) A function  $f(x)$  continuous at 0 and  $g(x)$  not continuous at 0 such that  $f(x)g(x)$  is continuous at 0.
- (d) A function  $f(x)$  not continuous at 0 such that  $f(x) + \frac{1}{f(x)}$  is continuous at 0.
- (e) A function  $f(x)$  not continuous at 0 such that  $f(x)^3$  is continuous at 0.

*Proof.* For (a), let  $f(x)$  be Dirichlet's function and let  $g(x)$  be as follows:

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then, neither  $f(x)$  nor  $g(x)$  is continuous at 0, but  $f(x)g(x) = 0$  and  $f(x) + g(x) = 1$  are both constant and hence continuous at 0.

For (b), let  $f(x)$  be continuous at 0, and let  $g(x)$  not be continuous at 0. Now, suppose  $f(x) + g(x)$  is continuous at 0. However, then since the difference of continuous functions is continuous, we have that  $f(x) - (f(x) + g(x)) = -g(x)$  is continuous at 0, which is a contradiction and hence the request is impossible.

For (c), we can let  $f(x) = 0$  and let  $g(x)$  be Dirichlet's function. Then,  $f(x)$  is continuous everywhere and hence at 0, while  $g(x)$  is nowhere continuous and hence not at 0. But,  $f(x)g(x) = 0$ , which is continuous at 0.

For (d), let  $f(x)$  be as follows:

$$g(x) = \begin{cases} \frac{1}{2} & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$$

Then,  $f(x)$  is not continuous at 0, but  $f(x) + \frac{1}{f(x)} = \frac{5}{2}$  is constant and hence continuous at 0.

For (e), let  $f(x)$  be not continuous at 0. Now, suppose that  $f(x)^3$  is continuous at 0. Then, since  $\sqrt[3]{x}$  is continuous on  $\mathbb{R}$  and the composition of functions is continuous, we have that  $\sqrt[3]{f(x)^3} = f(x)$  is continuous at 0, which is a contradiction and hence the request is impossible. □

**Problem (4.3.13).** Let  $f$  be a function defined on all of  $\mathbb{R}$  that satisfies the additive condition  $f(x + y) = f(x) + f(y)$  for all  $x, y \in \mathbb{R}$ .

- (a) Show that  $f(0) = 0$  and that  $f(-x) = -f(x)$  for all  $x \in \mathbb{R}$ .

- (b) Let  $k = f(1)$ , Show that  $f(n) = kn$  for all  $n \in \mathbb{N}$ , and then prove that  $f(z) = kz$  for all  $x \in \mathbb{Z}$ . Now prove that  $f(r) = kr$  for any rational number  $r$ .
- (c) Show that if  $f$  is continuous at  $x = 0$ , then  $f$  is continuous at every point in  $\mathbb{R}$  and conclude that  $f(x) = kx$  for all  $x \in \mathbb{R}$ . Thus, any additive function that is continuous at  $x = 0$  must necessarily be a linear function through the origin.

*Proof.* For (a), if we let  $x = y = 0$ , then we have  $f(0) = f(x + y) = f(x) + f(y) = f(0) + f(0)$ . Subtracting  $f(0)$  on both sides, we obtain  $f(0) = 0$ . Furthermore, if  $y = -x$ , then we have  $0 = f(0) = f(x - x) = f(x) + f(-x)$  and hence  $f(-x) = -f(x)$ .

For (b), we first note that  $f(n) = kn$  holds when  $k = 1$ . Now suppose that  $f(n) = kn$  holds for  $n$ . Then, we have:

$$f(n + 1) = f(n) + f(1) = kn + k = (k + 1)n$$

Thus, the formula holds for  $n + 1$ , whenever it holds for  $n$ , and since it holds for  $n = 1$ , by induction it holds for all  $n \in \mathbb{N}$ . Note that by (a),  $f(0) = 0$  so that it also holds for 0. Now, consider  $z \in \mathbb{Z}$  with  $z < 0$ . We must have  $z = -n$ , for some  $n \in \mathbb{N}$  so that:

$$f(z) = f(-n) = -f(n) = -kn = zk$$

So, it holds for all  $z \in \mathbb{Z}$ . Finally, consider  $r \in \mathbb{Q}$ . Then  $r = \frac{p}{q}$ , where  $p, q \in \mathbb{Z}$  (and  $q \neq 0$ ). So, we have:

$$kp = f(p) = f\left(q \frac{p}{q}\right) = qf\left(\frac{p}{q}\right) = qf(r)$$

Thus, dividing by  $q$ , we obtain  $f(r) = k\frac{p}{q} = kr$ .

For (c), assume that  $f$  is continuous at 0. Then, consider some other point  $c \in \mathbb{R}$ , and let  $\varepsilon > 0$ . Then, since  $f$  is continuous at 0 we can find a  $\delta > 0$  such that  $|x| < \delta$  implies  $|f(x)| < \varepsilon$ . Now, consider  $|x - c| < \delta$ . This implies that  $|f(x - c)| < \varepsilon$  and hence  $|f(x) - f(c)| < \varepsilon$ , by part (a) and our assumption that  $f$  is additive. Thus,  $f$  is continuous at  $c$  and hence continuous on  $\mathbb{R}$ .

Now, consider some irrational  $i \in \mathbb{I}$ , take a sequences of rationals  $r_n \rightarrow i$ . Then, since  $f$  is continuous on  $\mathbb{R}$  we must have that  $f(r_n) \rightarrow f(i)$ . By part (c),  $f(r_n) = kr_n \rightarrow ki$  and furthermore  $f(i) = ki$ . Thus,  $f(x) = kx$  for all  $x \in \mathbb{R}$ .  $\square$

**Problem (4.4.7).** Prove that  $f(x) = \sqrt{x}$  is uniformly continuous on  $[0, \infty)$ .

*Proof.* Given  $\varepsilon > 0$ , let  $\delta = \varepsilon^2$  and consider  $x$  and  $y$  with  $|x - y| < \delta$ . Now, either  $\sqrt{x} + \sqrt{y} < \varepsilon$  or  $\sqrt{x} + \sqrt{y} \geq \varepsilon$ . If  $\sqrt{x} + \sqrt{y} < \varepsilon$ , then, we have:

$$|\sqrt{x} - \sqrt{y}| \leq |\sqrt{x}| + |\sqrt{y}| = \sqrt{x} + \sqrt{y} < \varepsilon$$

If on the other hand,  $\sqrt{x} + \sqrt{y} \geq \varepsilon$ , then we have:

$$|\sqrt{x} - \sqrt{y}| = |\sqrt{x} - \sqrt{y}| \cdot \frac{|\sqrt{x} + \sqrt{y}|}{|\sqrt{x} + \sqrt{y}|} = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x - y|}{\varepsilon} < \frac{\delta}{\varepsilon} = \varepsilon$$

So, either way  $|\sqrt{x} - \sqrt{y}| < \varepsilon$ , whenever  $|x - y| < \delta$ , and hence  $\sqrt{x}$  is uniformly continuous.  $\square$

**Problem (4.4.11).** Show that  $g$  is continuous if and only if  $g^{-1}(O)$  is open whenever  $O \subseteq \mathbb{R}$  is an open set.

*Proof.* First, assume that  $g$  is continuous. Let  $O$  be open and let  $x \in g^{-1}(O)$ . Then,  $g(x) \in O$  and since  $O$  is open there exists a  $\varepsilon > 0$  such that  $V_\varepsilon(g(x)) \subseteq O$ . Now, since  $g$  is continuous, there exists  $\delta > 0$  such that whenever  $y \in V_\delta(x)$  it follows that  $g(y) \in V_\varepsilon(g(x))$ . In particular, this implies that  $g(V_\delta(x)) \subseteq V_\varepsilon(g(x)) \subseteq O$  and hence  $V_\delta(x) \subseteq g^{-1}(O)$ . Thus, given  $x \in g^{-1}(O)$ , we can find an open neighborhood  $V_\delta(x)$  in  $g^{-1}(O)$  and hence  $g^{-1}(O)$  is open.

Next, assume that whenever  $O$  is open it follows that  $g^{-1}(O)$  is open. Now, let  $\varepsilon > 0$ . Then, since  $V_\varepsilon(g(x))$  is open,  $g^{-1}(V_\varepsilon(g(x)))$  is also open. Since  $g^{-1}(V_\varepsilon(g(x)))$  is open and  $x \in g^{-1}(V_\varepsilon(g(x)))$ , there is some  $V_\delta(x) \subset g^{-1}(V_\varepsilon(g(x)))$ , which shows that  $g$  is continuous by the topological definition.  $\square$

**Problem (4.5.2).** Provide an example of each of the following, or explain why the request is impossible.

- (a) A continuous function defined on an open interval with range equal to a closed interval.
- (b) A continuous function defined on a closed interval with range equal to an open interval.
- (c) A continuous function defined on an open interval with range equal to an unbounded closed set different from  $\mathbb{R}$ .
- (d) A continuous function defined on all of  $\mathbb{R}$  with range equal to  $\mathbb{Q}$ .

*Proof.* For (a), let  $f$  be a function on  $(0, 1)$  defined by  $f(x) = 0$ . Then, the range  $f((0, 1)) = [0, 0]$  is a closed interval.

For (b), the request is impossible since closed intervals are compact and the image of a compact set is compact and hence the range must be compact.

For (c), let  $f$  be a function on  $(0, 1)$  defined by  $f(x) = \left(\frac{1}{x}\right) \left(\frac{1}{1-x}\right)$ . Then, the range of  $f$  is clearly unbounded. Furthermore, on  $(0, 1)$ , if  $m(x) = \min\{x, 1-x\}$ , then  $f(x) \geq \frac{1}{m(x)^2}$ . In particular, the minimum value of  $f(x)$  occurs when  $x = 1-x$ , that is  $x = \frac{1}{2}$ . Since  $\frac{1}{2} \in (0, 1)$ , we have that the range  $f((0, 1)) = [4, \infty)$  is closed.

For (d), the request is impossible since the image of connected sets must be connected and  $\mathbb{R}$  is connected while  $\mathbb{Q}$  is not.  $\square$

**Problem (4.5.7).** Let  $f$  be a continuous function on the closed interval  $[0, 1]$  with range also contained in  $[0, 1]$ . Prove that  $f$  must have a fixed point; that is, show  $f(x) = x$  for at least one value of  $x \in [0, 1]$ .

*Proof.* Let  $f$  be a continuous function on  $[0, 1]$ . Then, since  $f$  is continuous and  $[0, 1]$  is compact and connected, we must have that  $f([0, 1])$  is compact and connected. In particular, there must be  $a, b \in \mathbb{R}$  such that  $f([0, 1]) = [a, b]$ . Since  $f([0, 1]) \subseteq [0, 1]$ , we also have that  $a, b \in [0, 1]$ .

Now, consider the function  $g(x) = f(x) - x$  on  $[0, 1]$ . We have  $g(0) = a$  and  $g(1) = b - 1$ . If either  $a = 0$  or  $b = 1$ , then we have a fixed point. So, we just need to consider the case when  $a > 0$  and  $b < 1$ . Then, the range of  $g(x) = [b - 1, a]$  contains 0 in its interior. So, by the IVT there exists  $c \in [0, 1]$  such that  $g(c) = 0$ . Finally,  $g(c) = f(c) - c = 0$  implies that  $f(c) = c$  so that  $f$  has a fixed point for this case as well.  $\square$

**Problem (4.6.1).** Using modifications of Dirichlet's and Thomae's functions, construct a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  so that:

- (a)  $D_f = \mathbb{Z}^c$   
(b)  $D_f = \{x : 0 < x \leq 1\}$

*Proof.* For (a), consider the following function:

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

For any  $x \in \mathbb{Z}^c$ , we have  $\sin(\pi x) \neq 0$ . Furthermore, any rational sequence converging to  $x$  will converge to 0 and hence  $f$  is not continuous at  $x$ . Now, for  $x \in \mathbb{Z}$ , we have  $\sin(\pi x) = 0$ , and furthermore, since the modified Dirichlet function is continuous at 0, it follows that  $f(x)$  is continuous at  $x$ .

For (b), consider the following function:

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \text{ and } 0 \leq x \leq 1 \\ 0 & \text{if } x \notin \mathbb{Q} \text{ or } x < 0 \text{ or } x > 1 \end{cases}$$

For  $x < 0$  and  $x > 1$ ,  $f(x)$  is constant and hence continuous. On the interval  $[0, 1]$ , the function agrees with the modified Dirichlet function. This along with the fact that  $f(x) = 0$  for  $x \leq 0$  implies that  $f(x)$  is continuous at 0. The left hand limit is clearly 0 and the right hand limit is 0 since the modified Dirichlet function is continuous at 0.  $\square$

**Problem (4.6.2).** Given a countable set  $A = \{a_1, a_2, a_3, \dots\}$ , define  $f(a_n) = \frac{1}{n}$  and  $f(x) = 0$  for all  $x \notin A$ . Find  $D_f$ .

*Proof.* Suppose  $x \in A$ , then for  $n \in \mathbb{N}$  consider  $(x - \frac{1}{n}, x + \frac{1}{n})$ . Since  $A$  has only countably many points and any interval in  $\mathbb{R}$  has uncountably many points there must be some  $x_n \in (x - \frac{1}{n}, x + \frac{1}{n})$  that is not in  $A$ . Now, since  $\frac{1}{n} \rightarrow 0$ , it follows that  $x_n \rightarrow x$ . Furthermore, since  $x_n \notin A$  we have that  $f(x_n) = 0$  for all  $n$  and hence  $f(x_n) \rightarrow 0$ . However, since  $x \in A$ , we have  $f(x) \neq 0$  and hence  $f$  is not continuous at  $x$ .

Now, consider  $x \notin A$  and consider an arbitrary sequence  $(x_n) \rightarrow x$ . If  $(x_n)$  contains finitely many points of  $A$ , then  $f(x_n)$  will be eventually 0 and hence converge to 0. On the other hand, suppose that  $(x_n)$  contains infinitely many points of  $A$ . Let  $\varepsilon > 0$  and let  $N_1 \in \mathbb{N}$  be large enough such that  $\varepsilon > \frac{1}{N_1}$ . Now, suppose that  $n_i$  is the index of the  $i$ -th element of  $A$  as an element of  $(x_n)$  (assuming that  $x_{n_i} \in A$ ). For example, if  $a_3 \in (x_n)$ , then there is some  $n_3$  such that  $x_{n_3} = a_3$ . Then, consider:

$$N_2 = \max\{n_i : i < N_1\}$$

So, for  $n \geq N_2$ , if  $x_n \in A$  it follows that the index of  $x_n$  in  $A$  is great than  $N_1$  and hence  $f(x_n) \leq \frac{1}{N_1} < \varepsilon$ . If  $x_n \notin A$ , then  $f(x_n) = 0 < \varepsilon$ . So, either way  $f(x_n) < \varepsilon$  and hence  $f(x_n) \rightarrow 0$ . Since  $x_n$  was arbitrary, it follows that  $\lim_{y \rightarrow x} f(y) = 0 = f(x)$  and hence  $f$  is continuous at  $x$ . In summary, we have  $D_f = A$ , that is  $f$  is continuous on  $A^c$ .  $\square$