

Lecture #12

MA 511, Introduction to Analysis

June 10, 2021

Sets of discontinuity

Definition

Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$ the **set of discontinuities of f** , denoted by D_f , is the set of points where f fails to be continuous.

- For Dirichlet's function $g(x)$, we saw that $D_g = \mathbb{R}$. For the modified Dirichlet function $h(x)$, we saw that $D_h = \mathbb{R} \setminus \{0\}$. For Thomae's function $t(x)$, we saw that $D_t = \mathbb{Q}$.
- For an arbitrary subset $A \subseteq \mathbb{R}$, can we find a function f such that $D_f = A$?
- First, we will consider the possible D_f for a simpler class of functions: monotone functions.

Definition

A function $f : A \rightarrow \mathbb{R}$ is **increasing on A** if $f(x) \leq f(y)$ whenever $x < y$ and **decreasing** if $f(x) \geq f(y)$ whenever $x < y$ in A . A function is **monotone** if it is either increasing or decreasing.

Right-hand and left-hand limits

- We need to formalize what we mean when we say that a function approaches a limit “from the right” or “from the left”.

Definition (Right-hand limit)

Given a limit point c of a set A and a function $f : A \rightarrow \mathbb{R}$, we write $\lim_{x \rightarrow c^+} f(x) = L$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < x - c < \delta$. Equivalently, $\lim_{x \rightarrow c^+} f(x) = L$ if $\lim f(x_n) = L$ for all sequences (x_n) such that $x_n > c$ and $\lim(x_n) = c$.

Definition (Left-hand limit)

Given a limit point c of a set A and a function $f : A \rightarrow \mathbb{R}$, we write $\lim_{x \rightarrow c^-} f(x) = L$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $-\delta < x - c < 0$. Equivalently, $\lim_{x \rightarrow c^-} f(x) = L$ if $\lim f(x_n) = L$ for all sequences (x_n) such that $x_n < c$ and $\lim(x_n) = c$.

Types of discontinuities

Theorem

Given $f : A \rightarrow \mathbb{R}$ and a limit point c of A , $\lim_{x \rightarrow c} f(x) = L$ if and only if

$$\lim_{x \rightarrow c^+} f(x) = L \text{ and } \lim_{x \rightarrow c^-} f(x) = L.$$

Classification of discontinuities

Given $f : A \rightarrow \mathbb{R}$ and a point $c \in A$, if f is not continuous at c , then c falls into one of the following categories:

- i If $\lim_{x \rightarrow c} f(x)$ exists, but has a value different from $f(c)$, then f has a **removable discontinuity** at c .
- ii If $\lim_{x \rightarrow c} f(x)$ does not exist because $\lim_{x \rightarrow c^+} f(x) \neq \lim_{x \rightarrow c^-} f(x)$, then f has a **jump discontinuity** at c .
- iii If $\lim_{x \rightarrow c} f(x)$ does not exist for some other reason, then f has an **essential discontinuity** at c .

Characterizing sets of discontinuity

Theorem (D_f for monotone functions)

The only type of discontinuity that a monotone function can have is a jump discontinuity. Furthermore, D_f is either finite or countable.

Definition

Let f be defined on \mathbb{R} , and let $\alpha > 0$. The function f is **α -continuous at $x \in \mathbb{R}$** if there exists a $\delta > 0$ such that for all $y, z \in (x - \delta, x + \delta)$ it follows that $|f(y) - f(z)| < \alpha$. We also define D_f^α to be the set of points where f fails to be α -continuous.

Theorem (D_f for an arbitrary function)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary function. Then, D_f is an F_σ set.

- Since \mathbb{I} is not F_σ , there is no function $f : \mathbb{R} \rightarrow \mathbb{R}$ that has $D_f = \mathbb{I}$.
- How can we construct a function f such that D_f is a given finite, countable or F_σ set?