Homework Solutions #1

CAS MA 511

Problem (1.2.1 (a))**.** Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show that $\sqrt{6}$ is irrational?

First of all, we prove the following useful lemma.

Lemma. *If p^2 is a multiple of 3, then p is a multiple of 3.*[1]

Proof of lemma. It suffices to prove the contrapositive, which says that if p is not a multiple of 3, then p^2 is not a multiple of 3. Assume that p is not a multiple of 3, then we there is some $k \in \mathbb{Z}$ such that:

$$\text{either } p = 3k + 1 \text{ or } p = 3k + 2$$

If $p = 3k+1$, then $p^2 = 9k^2+9k+1 = 3(3k^2+3k)+1$ which is not a multiple of 3. If If $p = 3k+2$, then $p^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$ which is not a multiple of 3. So, either way p^2 is not a multiple of 3. □

Proof that $\sqrt{3}$ is irrational. Assume, for contradiction that there are integers p and q such that $\left(\frac{p}{q}\right)^2 = 3$ and p and q share no common factor other than 1. Then, we have $p^2 = 3q^2$, so that by the above lemma, p is a multiple of 3. Moreover, $p = 3k$ for some $k \in \mathbb{Z}$, and we have $p^2 = 9k^2 = 3q^2$, so that $q^2 = 3k^2$. However, by applying the lemma again, this implies that q is also a multiple of 3, which contradicts that p and q share no common factor other than 1. Thus, no such p and q exist. □

A similar argument also works to show that $\sqrt{6}$ is irrational. We would have $p^2 = 3(2q^2) = 2(3q^2)$, so that p must be a multiple of 3 and a multiple of 2 and hence a multiple of 6. Then, $p = 6k$ implies that $q^2 = 6k^2$, so that q is also a multiple of 6 by the same logic.

Problem (1.2.1 (b))**.** Where does the proof of Theorem $1.1.1$ break down if we try to use it to prove $\sqrt{4}$ is irrational?
The argument breaks down for $\sqrt{4}$, because $p^2 = 4q^2$ does not imply that p is a multiple of 4, but only that p is a multiple of 2. For example if $p = 6$ which is not a multiple of 4, then $p^2 = 36$ which is. Letting $p = 2k$, we obtain $k^2 = q^2$. Then, k is a common multiple of p and q and hence equal to 1, so that $p = 2$ and $q = 1$ as expected.

Problem (1.2.5 (a))**.** Let A and B be subsets of R. If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

[1] This is not immediately obvious and does need to be proven.

Proof. Let $x \in (A \cap B)^c$ be arbitrary and fixed.[2] This means that $x \notin A \cap B$. $A \cap B$ consists of all elements in both A and B so $x \notin A$ or $x \notin B$. If $x \notin A$, then $x \in A^c$. This in turn means that $x \in A^c \cup B^c$. If $x \notin B$, then $x \in B^c$. This in turn means that $x \in A^c \cup B^c$. In all cases, we find that $x \in A^c \cup B^c$. x was an arbitrary element of $(A \cap B)^c$ so all elements of $(A \cap B)^c$ are in $x \in A^c \cup B^c$. We can conclude that $(A \cap B)^c \subseteq A^c \cup B^c$. □

Problem (1.2.5 (b))**.** Show $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

Proof. Let $x \in A^c \cup B^c$ be arbitrary and fixed. By definition, $x \in A^c$ or $x \in B^c$. If $x \in A^c$, then $x \notin A$. This means that $x \notin A \cap B$ so $x \in (A \cap B)^c$. If $x \in B^c$, then $x \notin B$. This means that $x \notin A \cap B$ so $x \in (A \cap B)^c$. In all cases, the desired result holds. We now know that $(A \cap B)^c \subseteq A^c \cup B^c$ and $(A \cap B)^c \supseteq A^c \cup B^c$. The only way for both to hold is if $(A \cap B)^c = A^c \cup B^c$. □

Problem (1.2.5 (c))**.** Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Proof. Let $x \in (A \cup B)^c$ be arbitrary and fixed. This means that $x \notin A \cup B$. If it were true that $x \in A$ or that $x \in B$, then we would have $x \in A \cup B$. Therefore, $x \notin A$ and $x \notin B$. So, $x \in A^c$ and $x \in B^c$. We finally conclude that $x \in A^c \cap B^c$ and so $(A \cup B)^c \subseteq A^c \cap B^c$.
Let $x \in A^c \cap B^c$ be arbitrary and fixed. $x \in A^c$ and $x \in B^c$ by definition. So, $x \notin A$ and $x \notin B$. All elements of $A \cup B$ are in A or in B so $x \notin A \cup B$. This means that $x \in (A \cup B)^c$ and so $(A \cup B)^c \supseteq A^c \cap B^c$. These sets contain each other so we can conclude that $(A \cup B)^c = A^c \cap B^c$. □

Problem (1.2.8 (a))**.** Here are two important definitions related to a function $f : A \to B$. The function f is *one-to-one* if $a_1 \neq a_2$ in A implies $f(a_1) \neq f(a_2)$ in B. The function f is *onto* if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$. Give an example of the following:
$f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto.
Claim: $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$ is 1-1 but not onto.

Proof. Let $x, y \in \mathbb{N}$ be arbitrary and fixed elements such that $x \neq y$. Without loss of generality, we can let y be the larger value and write $y = x + b$ for some $b \in \mathbb{N}$.[3] $f(y) = f(x + b) = 2(x + b) = 2x + 2v \neq 2x = f(x)$ so f is 1-1.
There is no $n \in \mathbb{N}$ such that $f(n) = 1$ because this would imply that $n = \frac{1}{2}$, which is not in $\mathbb{N}$. f is not onto. □

Problem (1.2.8 (b))**.** $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1.

Claim: $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \begin{cases} n - 1 & n \geq 2 \\ 1 & n = 1 \end{cases}$ is onto but not 1-1.

Proof. Let $x \in \mathbb{N}$ be arbitrary and fixed. $x + 1 \in \mathbb{N}$ because $\mathbb{N}$ is closed under addition. Furthermore, $x + 1 \geq 2$ so $f(x + 1) = (x + 1) - 1 = x$. f is therefore an onto function.
$f(1) = 1$ and $f(2) = 2 - 1 = 1$ but $1 \neq 2$. This means f is not 1-1. □

Problem (1.2.8 (c))**.** $f : \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

Claim: $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = \begin{cases} \frac{n-1}{2} & n \text{ odd} \\ -\frac{n}{2} & n \text{ even} \end{cases}$ is 1-1 and onto.

[2] These are important properties that should be stated. Arbitrary means x is not a "special" element. Fixed means x is not changing. These properties together let us conclude the statement is true for all elements.

[3] It is convenient to label the larger and smaller values this way, but it does not invalidate our conclusions to do this.

Proof. Let $x, y \in \mathbb{N}$ be arbitrary and fixed elements such that $x \neq y$. Without loss of generality, we can let y be the larger value and write $y = x + b$ for some $b \in \mathbb{N}$. If x and y have different parity (even- or odd-ness), then clearly $f(x) \neq f(y)$ because one value is ≤ -1 and the other is ≥ 0. If x and y are both even, then we have $f(y) = f(x+b) = -\frac{x+b}{2} = -\frac{x}{2} - \frac{b}{2} \neq -\frac{x}{2} = f(x)$. If x and y are both odd, then we have $f(y) = f(x+b) = \frac{(x+b)-1}{2} = \frac{x-1}{2} + \frac{b}{2} \neq \frac{x-1}{2} = f(x)$. We therefore conclude that f is 1-1.
Let $x \in \mathbb{Z}$ be arbitrary and fixed. If $x \leq -1$, then define $y = -2x$. $x \leq -1$ means that $-x \geq 1$ so $-x \in \mathbb{N}$. $\mathbb{N}$ is closed under multiplication so $y \in \mathbb{N}$. By the way we have defined y, it is clear that it is even. We then compute that $f(y) = f(-2x) = -\frac{(-2x)}{2} = x$. So f hits all $x \in \mathbb{Z}$ with $x \leq -1$. If $x \geq 0$, define $y = 2x + 1$. If $x = 0$, then $y = 1$ so $y \in \mathbb{N}$. In the other cases, we use the fact that $\mathbb{N}$ is closed under addition and multiplication to conclude $y \in \mathbb{N}$. It is clear by the definition of y that it is odd. We now compute $f(y) = f(2x+1) = \frac{(2x+1)-1}{2} = x$. So f also hits all $x \in \mathbb{Z}$ with $x \geq 0$ so f is onto. □

Problem (1.2.12 (a))**.** Let $y_1 = 6$ and for each $n \in \mathbb{N}$ define $y_{n+1} = \frac{2y_n - 6}{3}$. Use induction to prove that the sequence satisfies $y_n > -6$ for all $n \in \mathbb{N}$.

Proof. First, we note that $y_1 > -6$. Now, assume for the sake of induction that $y_n > -6$. We can compute the following

$$\begin{aligned} y_{n+1} &= \frac{2y_n - 6}{3} && (1) \\ &> \frac{2(-6) - 6}{3} && (2) \\ &> \frac{-18}{3} && (3) \\ &> -6 && (4) \end{aligned}$$

$y_n > 6$ implies that $y_{n+1} > 6$ and $y_1 > -6$ so we conclude via induction that $y_n > -6$ for all $n \in \mathbb{N}$. □

Problem (1.2.12 (b))**.** Use another induction argument to show that the sequence $(y_1, y_2, y_3 ...)$ is decreasing.[4]

Proof. We begin by computing $y_{n+1} - y_n$ for $n \in \mathbb{N}$ with $n \geq 2$.

$$\begin{aligned} y_{n+1} - y_n &= \frac{2y_n - 6}{3} - \frac{2y_{n-1} - 6}{3} && (5) \\ &= \frac{2y_n - 6}{3} + \frac{-2y_{n-1} + 6}{3} && (6) \\ &= \frac{2y_n - 2y_{n-1} - 6 + 6}{3} && (7) \\ &= \frac{2}{3}(y_n - y_{n-1}) && (8) \end{aligned}$$

If we assume for the sake of induction that $y_n - y_{n-1} < 0$, our work above shows that $y_{n+1} - y_n < 0$. The case for $n = 1$ requires us to compute $y_2 = \frac{2(6)-6}{3} = 2$. This shows that $y_2 - y_1 = -4$. These facts together suffice to show that y_n is a decreasing sequence via induction. □

Problem (1.3.2 (a))**.** Give an example of each of the following, or state that the request is impossible:

[4]We can also prove it by rewriting $y_{n+1} - y_n$ in terms of y_n and using the previous result.

A set B with $\inf(B) \geq \sup(B)$.
Claim: sets satisfying this property are exactly the sets of the form $\{a\}$ (these are called singleton sets).[5]

Proof. The set of upper bounds of $\{a\}$ is $U = \{x \in \mathbb{R} : x \geq a\}$. Clearly, $a \in U$. If $b \in U$ is such that $b \leq a$ then we have that $a \leq b$ and $b \leq a$. The only way that this holds is if $b = a$. Thus, $a = \sup\{a\}$. We can repeat a similar argument with the set of lower bounds, $L = \{x \in \mathbb{R} : x \leq a\}$, to show that $a = \inf\{a\}$. So, sets of the form $\{a\}$ have $\inf\{a\} \geq \sup\{a\}$.
Let B be a set not of this form. Either $B = \emptyset$ or B has at least 2 elements. If $B = \emptyset$, then neither $\inf$ nor $\sup$ exist (the sets of upper and lower bounds are both $\mathbb{R}$ which has no least or greatest element). If B has at least 2 elements, then we can choose $a, b \in B$ with $a < b$. We find that $\inf B \leq a < b \leq \sup B$. This means that $\inf B < \sup B$ so we have proven no other sets can have this property. □

Problem (1.3.2 (b))**.** A finite set that contains its infimum but not its supremum.
This is not possible. Claim: Finite sets contain both their infimum and supremum.

Proof. Assume for the sake of contradiction that A is a finite set with $s = \sup A$ and $s \notin A$. $s \geq a$ for all $a \in A$ and $s \notin A$ means that $s > a$ for all $a \in A$. Given $a_1 \in A$, define $b_1 = \frac{s+a_1}{2}$. Consider the following inequality: $a = \frac{a+a}{2} < \frac{s+a}{2} < \frac{s+s}{2} = s$. This shows that $a_1 < b_1 < s$. If b_1 is an upper bound for A, then s is not a supremum and we have a contradiction. If b_1 is not an upper bound, then there exists $a_2 \in A$ with $a_2 > b_1$. We finish our proof with by an induction argument. We define $b_n = \frac{s+a_n}{2}$ and use the same algebra to conclude $a_n < b_n < s$. If any b_n is an upper bound we again get a contradiction. If b_n is not an upper bound then we can define a_{n+1} to be an element of A with $a_{n+1} > b_n$. Such an element always exists since b_n is not an upper bound. Our induction argument shows that we can continue this construction infinitely. However, this requires an infinite number of unique elements of A, contradicting our assumption that A is finite. In all cases we reach a contradiction so we conclude all finite sets include their supremum. A similar argument shows they also contain their infimum. □

Problem (1.3.2 (c))**.** A bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.
Claim: $A = \mathbb{Q} \cap (0, 1]$ contains it's supremum but not infimum.

Proof. By the way we have defined A, it is clear that 0 is a lower bound and 1 is an upper bound. $1 \in A$ and it is an upper bound so all other upper bounds must be larger and $\sup A = 1$. 0 is $\inf A$ as a result of the Archimedean property.[6] □

Problem (1.3.5 (a))**.** As in example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.
If $c \geq 0$, show that $\sup(cA) = c \sup(A)$.

Proof. Let $s = \sup A$. If $c = 0$, then $cA = \{0\}$ and $cs = 0$. Using the same argument as 1.3.2 (a), we see that $cs = \sup cA$ in this case. If $c > 0$, more work is needed. $s \geq a$ for all $a \in A$. Because $c > 0$, this immediately shows $cs \geq ca$ for all $a \in cA$. This means cs is indeed an upper bound of cA. Assume for the sake of contradiction that $\sup cA \neq cs$. This means that there is some $b < cs$ that is an upper bound of cA. This means $b \geq ca$ for all $a \in A$. Again, because $c > 0$, we get that $\frac{b}{c} \geq a$ for all $a \in A$ and so $\frac{b}{c}$ is an upper bound for A. $b < cs$ also shows us that $\frac{b}{c} < s$. But this means that $s \neq \sup A$ and we have a contradiction. □

[5] This is a stronger claim that we are asked to show. Only the first part is necessary for statement in the problem
[6] We didn't learn this until the next section and that is why you are not asked to prove your set work in this problem.

Problem (1.3.5 (b))**.** Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.
Claim: For $c < 0$, $\sup cA = c \inf A$.

Proof. We follow a very similar argument as above for the proof. The key difference comes from the fact that $c < 0$. This causes all inequalities to be reversed when we multiply or divide. This corrects for the difference in inequality in the definitions of infimum and supremum. $\square$

Problem (1.3.7)**.** Prove that if a is an upper bound for A, and if a is also an element of A, then it must be that $\sup(A) = a$.

Proof. Assume for the sake of contradiction that $\sup A \neq a$. a is an upper bound of A so it must be that there is some other upper bound, $b < a$. However, $a \in A$ so $b < a$ means b is not a valid upper bound. This is a contradiction so $\sup A = a$. $\square$

Problem (1.4.1 (a))**.** Recall that $\mathbb{I}$ stands for the set of irrational numbers. Show that if $a, b \in \mathbb{Q}$ then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

Proof. Let $a = \frac{p}{q}$ and $b = \frac{x}{y}$ for $p, x \in \mathbb{Z}$ and $q, y \in \mathbb{N}$. We compute that $ab = \frac{p}{q}\frac{x}{y} = \frac{px}{qy}$ and $a + b = \frac{p}{q} + \frac{x}{y} = \frac{py}{qy} + \frac{qx}{qy} = \frac{py+qx}{qy}$. $\mathbb{Z}$ and $\mathbb{N}$ are closed under addition and multiplication so $px, py, qx \in \mathbb{Z}$ and $qy \in \mathbb{N}$. This means that $ab, a + b \in \mathbb{Q}$. $\square$

Problem (1.4.1 (b))**.** Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

Proof. Assume for the sake of contradiction that $a + t \in \mathbb{Q}$. Then consider $a + t + (-a) = t$. $-a, a + t \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under addition so $t \in \mathbb{Q}$. This contradicts the statement that $t \in \mathbb{I}$ so we must conclude that $a + t \in \mathbb{I}$. Assume now that $at \in \mathbb{Q}$ and $a \neq 0$. Consider $at\frac{1}{a} = t$. $\frac{1}{a}, at \in \mathbb{Q}$ and $\mathbb{Q}$ is closed under multiplication so $t \in \mathbb{Q}$. This contradicts the statement that $t \in \mathbb{I}$ so we must also conclude that $at \in \mathbb{I}$ for $a \neq 0$. $\square$

Problem (1.4.1 (c))**.** Part (A) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers a and t, what can we say about $s + t$ and st?
$\mathbb{I}$ is not closed under addition or multiplication. Very little can be said about $s + t$ or at if $s, t \in \mathbb{I}$.

Proof. Let $s = \sqrt{2}$ and $t = -\sqrt{2}$. We compute that $s+t = \sqrt{2}+-\sqrt{2} = 0$ and $st = \sqrt{2}(-\sqrt{2}) = -2$. $0, -2 \in \mathbb{Q}$ so $\mathbb{I}$ is not closed under either operation. If $r = \sqrt{3}$, then $sr = \sqrt{2}\sqrt{3} = \sqrt{6}$ which we know is in $\mathbb{I}$. $s+r = \sqrt{2}+\sqrt{3}$ is also in $\mathbb{I}$ for a more complicated reason. If it were in $\mathbb{Q}$, then so would $(s+r)^2$ by closure of $\mathbb{Q}$ under multiplication. $(s+r)^2 = (\sqrt{2}+\sqrt{3})^2 = 2+2\sqrt{6}+3 = 5+2\sqrt{6}$. We know that $2\sqrt{6} \in \mathbb{I}$ by part b and therefore that $5+2\sqrt{6} \in \mathbb{I}$ also by part b. This is a contradiction to the closure of $\mathbb{Q}$ under multiplication so $s + r \in \mathbb{I}$. We have shown examples of combinations of irrationals via addition and multiplication that yield either rationals or irrationals. $\square$

Problem (1.4.4)**.** Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup(T) = b$.

Proof. b is an upper bound for the set $[a, b]$ so no elements of $[a, b]$ are larger than b. $T \subset [a, b]$ so it is certainly true that T does not contain any elements larger than b. So, b is also an upper bound for T. Let $\varepsilon > 0$ be arbitrary and consider $(b - \varepsilon, b)$. $\mathbb{Q}$ is dense in $\mathbb{R}$ so we have an element in this interval which is also in T. This holds for all ε so we can conclude that $\sup T = b$. $\square$

Problem (1.4.6 (a))**.** Recall that a set B is *dense* in $\mathbb{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are sense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.
The set of all rational numbers $\frac{p}{q}$ with $q \leq 10$.
Claim: This set is not dense in $\mathbb{R}$.

Proof. Consider the interval $(\frac{1}{100}, \frac{9}{100})$. 0 is a less than the entire interval so we only need to consider the possibility that $p > 0$ and $1 \leq q \leq 10$ gives a rational in the interval. Considering $p = 1$ and checking each q values shows that all of these terms are greater than the entire interval and all other values of p are even larger so we have no elements in this interval. $\square$

Problem (1.4.6 (b))**.** The set of all rational numbers $\frac{p}{q}$ with q a power of 2.
Claim: This set is dense in $\mathbb{R}$.

Proof. Let $a < b$ be arbitrary and fixed. We know $b - a > 0$. We now consider $\frac{1}{b-a} \in \mathbb{R}$. $\mathbb{N}$ is unbounded in $\mathbb{R}$ so we have $n \in \mathbb{N}$ with $n > \frac{1}{b-a}$, $2^n > n$ for all $n \in \mathbb{N}$ so $2^n > \frac{1}{b-a}$. We can then conclude that $\frac{1}{2^n} < b - a$. We can therefore find some $p \in \mathbb{N}$ such that $a < \frac{p}{2^n} < b$. This holds for all $a < b$ so we conclude that our set is dense in $\mathbb{R}$. $\square$

Problem (1.4.6 (c))**.** The set of all rational numbers $\frac{p}{q}$ with $10|p| \geq q$.
Claim: This set is not dense in $\mathbb{R}$.

Proof. Rearranging the inequality which defines our set gives $|\frac{p}{q}| \geq 10$. Therefore, there are no elements in the interval $(-10, 10)$ so the set is not dense in $\mathbb{R}$. $\square$

Problem (1.5.1)**.** Finish the following proof for Theorem 1.5.7.
Assume B is a countable set. Thus, there exists $f : \mathbb{N} \to B$, which is 1-1 and onto. Let $A \subseteq B$ be an infinite subset of B. We must show that A is countable.
Let $n_1 = \min\{n \in \mathbb{N} : f(n) \in A\}$. As a start to a definition of $g : \mathbb{N} \to A$, set $g(1) = f(n_1)$. Show how to inductively continue this process to produce a 1-1 function g from $\mathbb{N}$ onto A.

Proof. We inductively define g and n_k as follows: $g(k) = f(n_k)$, $n_k = \min\{n \in \mathbb{N} : f(n) \in A, n > n_{k-1}\}$ for all $k \in \mathbb{N}$. We need to show that this definition is valid and that it is both 1-1 and onto.[7]
f is onto B so all elements $b \in B$ have some $n \in \mathbb{N}$ with $f(n) = b$. All $a \in A$ are also in B so they have $n \in \mathbb{N}$ with $f(n) = a$. So f is onto A. A is infinite and f is onto so $\{n \in \mathbb{N} : f(n) \in A\}$ is infinite. $\{n \in \mathbb{N} : f(n) \in A, n > n_{k-1}\} = \{n \in \mathbb{N} : f(n) \in A,\} \setminus \bigcup_{i=1}^{k-1}\{n_i\}$. $\bigcup_{i=1}^{k-1}\{n_i\}$ is finite and $\{n \in \mathbb{N} : f(n) \in A\}$ is infinite so $\{n \in \mathbb{N} : f(n) \in A, n > n_{k-1}\}$ is infinite.[8] Every set from which n_k is defined is infinite so they are obviously not $\emptyset$. $\{n \in \mathbb{N} : f(n) \in A, n > n_{k-1}\} \subseteq \mathbb{N}$ so the set is bounded below and therefore has a well-defined infimum. Finally, for $x \neq y$ in $\mathbb{N}$, $|x - y| \geq 1$. Similar to the statement on sumprema, infima must be arbitrarily close to elements in the set. For $0 < \varepsilon < 1$ there must be exactly 1 element in $(\inf\{n \in \mathbb{N} : f(n) \in A, n > n_{k-1}\},$ $\inf\{n \in \mathbb{N} : f(n) \in A, n > n_{k-1}\} + \varepsilon)$ for all ε. The only way that this is possible is if this element is distance 0 from $\inf\{n \in \mathbb{N} : f(n) \in A, n > n_{k-1}\}$. This means that $\{n \in \mathbb{N} : f(n) \in A, n > n_{k-1}\}$ contains its infimum and so the minimum is well-defined. All n_k are therefore properly defined.
Let $k < j$ be in $\mathbb{N}$. By the way we have defined our n_i, we immediately see that
$n_k \notin \{n \in \mathbb{N} : f(n) \in A, n > n_{j-1}\}$ and $n_j \in \{n \in \mathbb{N} : f(n) \in A, n > n_{j-1}\}$ so $n_k \neq n_j$. $g(k) =$

[7] It is probably reasonable to accept the fact that n_k are well-defined without proof but we will do it anyway.
[8] A more thorough proof would invoke the closure of $\mathbb{N}$ under addition and our natural identification between finite sets and $\mathbb{N}$ to justify this.

$f(n_k)$ and $g(j) = f(n_j)$ so $g(k) = g(j)$ if and only if $f(n_k) = f(n_j$. However, f is 1-1 and $n_k \neq n_j$ so this is not the case. It follows that g is also 1-1.
Let $a \in A$ be arbitrary and fixed. We previously showed that f is onto A so there is some $n' \in \mathbb{N}$ such that $f(n') = a$. Assume for the sake of contradiction that $n' \neq n_k$ for any $k \in \mathbb{N}$. $f(n') = a$ so $n' \in \{n \in \mathbb{N} : f(n) \in A\}$ and $n' \neq n_1$ so $n' > n_1$. This means that $n' \in \{n \in \mathbb{N} : f(n) \in A, n > n_1\}$. Additionally, if $n' \in \{n \in \mathbb{N} : f(n) \in A, n > n_{k-1}\}$ and $n' \neq n_k$ then $n' > n_k$. This means that $n' \in \{n \in \mathbb{N} : f(n) \in A, n > n_k\}$. By induction, we conclude that $n' > n_k$ for all $k \in \mathbb{N}$. This means that n' is an upper bound for $\mathbb{N}$ in $\mathbb{R}$ so we get a contradiction. This means that $n' = n_k$ for some $k \in \mathbb{N}$. This holds for all $a \in A$ so g is onto. g is a 1-1 and onto function from $\mathbb{N}$ to A so A is countable. This assumed that A was not finite. Clearly we can produce finite sets from an infinite set (singleton sets for example) so we must conclude that subsets of countable sets are countable or finite (the actual phrasing of the theorem).. □

Problem (1.5.4 (a))**.** Show $(a, b) \sim \mathbb{R}$ for any interval (a, b).

Proof. Let $f : (a, b) \to \mathbb{R}$ be defined by $f(x) = \frac{x - \frac{a+b}{2}}{(x-a)(x-b)}$. In principle we can prove this is 1-1 and onto with elementary algebra and the condition that $a < x < b$ but we will not prove that here. The vertical and horizontal line tests suffice for convincing ourselves. □

Problem (1.5.4 (b))**.** Show that an unbounded interval like $(a, \infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

Proof. Let $f : (a, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{(x-a+1)(x-a-1)}{x-a}$. Again, we claim that this function is 1-1 and onto and that these properties can be shown with elementary algebra. We also claim this map is also 1-1 and onto if the domain is changed to be $(-\infty, a)$. □

Problem (1.5.4 (c))**.** Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0, 1) \sim (0, 1)$ by exhibiting a 1-1, onto function between the two sets.

Proof. Let $f : [0, 1) \to (0, 1)$ be defined by $f(x) = \begin{cases} x & x \notin \left\{1 - \frac{1}{a^n} : n \in \mathbb{N}\right\} \\ \frac{1}{2} & x = 0 \\ 1 - \frac{1}{2^{n+1}} & x = 1 - \frac{1}{2^n} \end{cases}$. Clearly, for $x \notin \{0\} \cup \left\{1 - \frac{1}{a^n} : n \in \mathbb{N}\right\}$, the function is 1-1 and onto and only produces values in the same set. For $x \in \{0\} \cup \left\{1 - \frac{1}{a^n} : n \in \mathbb{N}\right\}$, we produce elements in $\left\{1 - \frac{1}{a^n} : n \in \mathbb{N}\right\}$ so if we can show the function is 1-1 and onto with this restricted domain, we will have shown that f is 1-1 and onto for our intervals. For $y \in \left\{1 - \frac{1}{a^n} : n \in \mathbb{N}\right\}$ arbitrary and fixed, we can write $y = 1 - \frac{1}{2^m}$ for some $m \in \mathbb{N}$. Let $x \in \{0\} \cup \left\{1 - \frac{1}{a^n} : n \in \mathbb{N}\right\}$ be $1 - \frac{1}{2^{m-1}}$ if $m \geq 2$ and 0 if $m = 1$. $f(x) = 1 - \frac{1}{2^{(m-1)+1}} = 1 - \frac{1}{2^m} = y$ if $m \geq 2$ and $f(x) = \frac{1}{2} = 1 - \frac{1}{2^1} = y$ if $m = 1$. In all cases $f(x) = y$ so we have found $x \in \{0\} \cup \left\{1 - \frac{1}{a^n} : n \in \mathbb{N}\right\}$ with $f(x) = y$ for arbitrary $y \in \left\{1 - \frac{1}{a^n} : n \in \mathbb{N}\right\}$. This means f is onto on the restricted and full domain. If $x \neq y$ are in $\left\{1 - \frac{1}{a^n} : n \in \mathbb{N}\right\}$, then $x = 1 - \frac{1}{2^n}$ and $y = 1 - \frac{1}{2^m}$ for $n \neq m$ in $\mathbb{N}$. $f(x) - f(y) = 1 - \frac{1}{2^{n+1}} - 1 + \frac{1}{2^{m+1}} = \frac{1}{2^{m+1}}(1 - \frac{1}{2^{n-m}})$. Since $n \neq m$, $1 - \frac{1}{2^{n-m}} \neq 0$ and $f(x) \neq f(y)$. $f(1 - \frac{1}{2^n}) = 1 - \frac{1}{2^{n+1}} > \frac{1}{2}$ so no $x \in \left\{1 - \frac{1}{a^n} : n \in \mathbb{N}\right\}$ satisfies $f(x) = f(0)$.With this, we conclude that f is 1-1 on the restricted domain and therefore also 1-1 on the original domain. □

Problem (1.5.5 (a))**.** Why is $A \sim A$ for every set A?

Proof. Define $f : A \to A$ by $f(x) = x$. f is 1-1 and onto for obvious reasons so we have our desired function. $\square$

Problem (1.5.5 (b))**.** Given sets A and B, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

Proof. Let $f : A \to B$ be a 1-1 and onto function guaranteed by the definition of $\sim$. We will define $g : B \to A$ which is also 1-1 and onto, showing that $B \sim A$. For all $b \in B$ define $g(b) = a$ for the unique $a \in A$ such that $f(a) = b$. This is a valid definition because f is 1-1 and onto. Because f is onto, there is at least 1 $a \in A$ such that $f(a) = b$. This means there is at least 1 possible target for $g(b)$. Assume for the sake of contradiction that a is not unique. This means there is $a' \in A$ with $f(a') = b$. But f is 1-1 so $a' = a$ and we get a contradiction. This means there is exactly 1 target for $g(b)$ so the function is well-defined. For $b, b' \in B$, $g(b) = a$ for the unique value satisfying $f(a) = b$ and $g(b') = a'$ for the unique value satisfying $f(a') = b'$. If $b \neq b'$, then $f(a) \neq f(a')$. f is 1-1 so $a \neq a'$ and we see $g(b) \neq g(b')$. g is a 1-1 function. For $a \in A$ arbitrary and fixed, consider $f(a) \in B$. $g(f(b)) = a$ as it is the unique point with $f(a) = f(a)$. This holds for all $a \in A$ so g is onto. $\square$

Problem (1.5.5 (c))**.** For three sets A, B, and C, show that $A \sim B$ and $B \sim C$ implies $A \sim C$. These three properties are what is meant by saying that $\sim$ is an *equivalence relation*.

Proof. Let $f : A \to B$ and $g : B \to C$ be 1-1 and onto functions guaranteed by the definition of $\sim$. We claim that $h : A \to C$ defined by $h(x) = g(f(x)$ is 1-1 and onto. Let $a, a' \in A$ be arbitrary and fixed. $h(a) = g(f(a))$ and $h(a') = g(f(a'))$. If $a \neq a'$, then $f(a) \neq f(a')$ because f is 1-1. This also means that $g(f(a)) \neq g(f(a'))$ because g is 1-1. So, $h(a) \neq h(a')$ and h is 1-1. Let $c \in C$ be arbitrary and fixed. g is onto so there is $b \in B$ such that $g(b) = c$. f is onto so there is $a \in A$ such that $f(a) = b$. This a satisfies $h(a) = g(f(a)) = g(b) = c$ so h is onto. $\square$

Problem (2.2.1)**.** What happens if we reverse the order of the quantifiers in Definition 2.2.3?
Definition: A sequence (x_n) *verconges* to x if *there exists* and $\varepsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \varepsilon$.
Gave an example of a vercongent sequence. Is there an example of a vercongent sequence that is divergent? Can a sequence verconge to two different values? What exactly is being described in this strange definition?

Proof. Consider the sequences $(x_n) = 0$ and $(y_n) = (-1)^n$. Taking $\varepsilon > 0$, we see that (x_n) verconges to all $x \in (-\varepsilon, \varepsilon)$ by noting $|x - x_n| = |x| < \varepsilon$ for all $n \in \mathbb{N}$. (x_n) is constant so it also converges. The example (y_n) does not converge, but for $\varepsilon > 1$, a similar calculation shows that for all $n \in \mathbb{N}$, $|y - y_n| < \varepsilon - 1$ for $y \in (1 - \varepsilon, \varepsilon - 1)$. In reality, the condition that a sequence verconges is simply the condition that it is bounded. Any bound can be used to trivially show the sequence verconges to 0 and verconging to x means that $M = |x| + \varepsilon$ is a bound for any ε satisfying the definition. $\square$

Problem (2.2.4 (a))**.** Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.
A sequence that with an infinite number of ones that does not converge to one.

Proof. Let $(x_n) = (-1)^n$. All even terms are 1 so there are an infinite number of them. The triangle inequality shows that it does not converge to any limit.

$$\begin{aligned} |x - x_{2n}| + |x - x_{2m-1}| &\geq |x_{2n} - x_{2m-1}| && (9)\\ &\geq |1 + 1| && (10)\\ &\geq 2 && (11) \end{aligned}$$

So, even if $|x - x_{2n}| < \varepsilon$ as $n \to \infty$, $|x - x_{2m-1}| > 2 - \varepsilon$. For $\varepsilon < 1$ this shows $|x - x_{2m-1}| > \varepsilon$ so the sequence does not converge. □

Problem (2.2.4 (b))**.** A sequence with an infinite number of ones that converges to a limit not equal to one.

Proof. This is not possible. Let (y_n) be a sequence with an infinite number of ones. Let $y \neq 1$ be arbitrary and fixed, then consider $\varepsilon = \frac{|1-y|}{2}$. $|1 - y| > \varepsilon$ so all $n \in \mathbb{N}$ such that $y_n = 1$ violate the convergence criterion. There is no $N \in \mathbb{N}$ for which these terms stop being an issue because, if there were such an N, there would be at most N ones in the sequence. This is a $\varepsilon > 0$ for which the convergence criterion cannot be satisfied so the sequence cannot converge to y if $y \neq 1$. □

Problem (2.2.4 (c))**.** A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Proof. Define the sequence $(z_n) = \begin{cases} 0 & n \in \left\{\frac{n(n+1)}{2} + n : n \in \mathbb{N}\right\} \\ 1 & \text{else} \end{cases}$. For similar reasons as part a, the sequence diverges. There are always terms which are distance 1 apart in the tail of the sequence. To find a sequence of m ones in a row, consider $\frac{(m-1)m}{2} + m \leq n \leq \frac{m(m+1)}{2} + m - 1$. $\frac{(m-1)m}{2} + m$ is 1 more than the zero corresponding to $n = m - 1$ and $\frac{m(m+1)}{2} + m - 1$ is 1 less than the zero corresponding to $n = m$ so these are the beginning and end of a string of consecutive ones. $\frac{m(m+1)}{2} + m - 1 - \left(\frac{(m-1)m}{2} + m\right) + 1 = \frac{m(m-1)}{2} + \frac{m(2)}{2} + m - 1 - \frac{(m-1)m}{2} - m + 1 = m$. The length of this string is therefore m as we desired. This works for all $m \in \mathbb{N}$ so we have a sequence with the desired properties. □

Problem (2.3.2 (a))**.** Using only Definition 2.2.3, prove that if $(x_n) \to 2$ then:
$\left(\frac{2x_n-1}{3}\right) \to 1$.

Proof. Let $\varepsilon > 0$ be arbitrary and fixed. Let $N \in \mathbb{N}$ be such that $|x_n - 2| < \frac{3}{2}\varepsilon$ for all $n > N$. The convergence of (x_n) to 2 guarantees such an N to exist. An algebraic manipulation gives the following

$$|x_n - 2| < \frac{3}{2}\varepsilon \tag{12}$$
$$\frac{2}{3}|x_n - 2| < \varepsilon \tag{13}$$
$$|\frac{2}{3}(x_n - 2)| < \varepsilon \tag{14}$$
$$|\frac{2x_n}{3} - \frac{4}{3}| < \varepsilon \tag{15}$$
$$|\frac{2x_n}{3} - \frac{1}{3} - 1| < \varepsilon \tag{16}$$
$$|\frac{2x_n - 1}{3} - 1| < \varepsilon \tag{17}$$

Thus, we have found $N \in \mathbb{N}$ such that $n > N$ implies $|\frac{2x_n-1}{3} - 1| < \varepsilon$ for a given ε. This construction works for all $\varepsilon > 0$ so we have proven convergence. □

Problem (2.3.2 (b))**.** $\left(\frac{1}{x_n}\right) \to \frac{1}{2}$.

Proof. Let $0 < \varepsilon < 1$ be arbitrary and fixed. Let $N \in \mathbb{N}$ be such that $|x_n - 2| < 2\varepsilon$ for all $n > N$. The convergence of (x_n) to 2 guarantees such an N to exist. $\varepsilon < 1$ implies that $|x_n| > 1$. In turn this tells us that $\frac{1}{|x_n|}$. An algebraic manipulation gives the following

$$|x_n - 2| < 2\varepsilon \tag{18}$$

$$\frac{1}{2}|x_n - 2| < \varepsilon \tag{19}$$

$$\frac{1}{2}\frac{1}{|x_n|}|x_n - 2| < \varepsilon \tag{20}$$

$$|\frac{x_n - 2}{2x_n}| < \varepsilon \tag{21}$$

$$|\frac{x_n}{2x_n} - \frac{2}{2x_n}| < \varepsilon \tag{22}$$

$$|\frac{1}{2} - \frac{1}{x_n}| < \varepsilon \tag{23}$$

Thus, we have found $N \in \mathbb{N}$ such that $n > N$ implies $|\frac{1}{2} - \frac{1}{x_n}| < \varepsilon$ for a given $0 < \varepsilon < 1$. This construction works for all $0 < \varepsilon < 1$. For all $\varepsilon \geq 1$, we can choose the same N that we get for $\varepsilon = \frac{1}{2}$ and the result still follows.[9] Thus, we can find our desired N for any $\varepsilon > 0$ and have proven convergence. □

Problem (2.3.5)**.** Let (x_n) and (y_n) be given, and define (z_n) to be the "shuffled" sequence $(z_n) = (x_1, y_1, x_2, y_2, x_3, y_3, ..., x_n, y_n, ...)$. Prove that (z_n) is convergent if and only if (x_n) and $y_n)$ are both convergent with $\lim x_n = \lim y_n$.

Proof. Suppose $\lim x_n = \lim y_n = L$ and let $\varepsilon > 0$ be arbitrary. By our definition of convergence, we can choose $N_x, N_y \in \mathbb{N}$ such that $n > N_x$ implies $|x_n - L| < \varepsilon$ and $n > N_y$ implies $|y_n - L| < \varepsilon$. Define $N = \max\{2N_x, 2N_y\}$. Note that $z_{2n} = y_n$ and $z_{2n-1} = x_n$ for all $n \in \mathbb{N}$. For all $n > N$, z_n is either $x_{\frac{n+1}{2}}$ or $y_{\frac{n}{2}}$. If $z_n = x_{\frac{n+1}{2}}$, then we use $n > 2N_x$ to compute $\frac{n}{2} > N_x$ and conclude that $\frac{n+1}{2} > N_x$. This means that $|x_{\frac{n+1}{2}} - L| < \varepsilon$ and so $|z_n - L| < \varepsilon$. If $z_n = y_{\frac{n}{2}}$, then we use $n > 2N_y$ to conclude $\frac{n}{2} > N_y$. This means that $|y_{\frac{n}{2}} - L| < \varepsilon$ and so $|z_n - L| < \varepsilon$. These cases cover all z_n for $n > N$ so we have shown $\lim z_n = L$.
Suppose $\lim z_n = L$ and let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $n > N$ implies $|z_n - L| < \varepsilon$. We claim that this N is a valid choice to show convergence for both (x_n) and (y_n). Using our previous observation about the position of x_n and y_n as terms in z_n, we see that $x_n = z_m$ for some $m \geq n$ and $y_n = z_k$ for some $k \geq n$. This means that $|x_n - L| = |z_m - L| < \varepsilon$ and $|y_n - L| = |z_k - L| < \varepsilon$. So, we conclude that $\lim x_n = \lim y_n = L$. □

Problem (2.3.12 (a))**.** A typical task in analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n \to a$ and determine the validity of each claim. Try to produce a counterexample for any that are false.
If every a_n is an upper bound for a set B, then a is also an upper bound for B.

Proof. This is true. Assume for the sake of contradiction that a is not an upper bound for B. Let $b \in B$ be a fixed point such that $a < b$. Let $\varepsilon = b - a$. By the definition of convergence, there is some $N \in \mathbb{N}$ such that $n > N$ implies $|a_n - a| < \varepsilon$. For any such n, we see that $a_n < a + \varepsilon = a + b - a = b$. This means that a_n is not an upper bound for B, a contradiction to our hypothesis. □

[9] This is a common trick. For larger ε, we can choose the same N that was used for smaller ε. We do still need to note this for the proof's sake.

Problem (2.3.12 (b)). If every a_n is in the complement of the interval $(0,1)$, than a is also in the complement of $(0,1)$.

Proof. This is true. Assume for the sake of contradiction that $a \in (0,1)$. Let $\varepsilon = \min\{a, 1-a\}$. By the definition of convergence, there is some $N \in \mathbb{N}$ such that $n > N$ implies $|a_n - a| < \varepsilon$. For any such n, we see that $a - \varepsilon < a_n < a + \varepsilon$. Using the our definition of ε, we compute the following:

$$a - \varepsilon < a_n < a + \varepsilon \tag{24}$$
$$a - \min\{a, 1-a\} < a_n < a + \min\{a, 1-a\} \tag{25}$$
$$a - a \leq a - \min\{a, 1-a\} << a_n < a + \min\{a, 1-a\} \leq a + 1 - a \tag{26}$$
$$0 < a_n < 1 \tag{27}$$

This contradicts the hypothesis that $a_n \notin (0,1)$ so we conclude that $a \notin (0,1)$. □

Problem (2.3.12 (c)). If every a_n is rational, then a is rational.

Proof. This is false. We will construct a sequence (a_n) consisting of rational numbers which converges to $x \in \mathbb{R}$ for arbitrary $x > 0$.[10] For $x < 0$, we can take the sequence $-(a_n)$ where (a_n) is the corresponding sequence converging to $-x$. Obviously, for $x = 0$ we can take the sequence which is identically 0. Thus, if we can construct sequences for arbitrary $x > 0$, we can do it for any $x \in \mathbb{R}$. Let $a_1 = 0$ and for $n \in \mathbb{N}$ with $n \geq 2$ define a_n to be $\frac{p}{2^k}$ with $k \in \mathbb{N}$ is the smallest value such that $\left\{\frac{m}{2^k} : m \in \mathbb{N}\right\}$ has at least 1 element in (a_{n-1}, x) and with $p \in \mathbb{N}$ the largest value such that $\frac{p}{2^k} < x$. The Archimedean property guarantees that we can always find such k and p. The way we have defined our sequence, it is clearly increasing and bounded so it does converge. For all $\varepsilon > 0$, there is a $\frac{p}{2^n}$ in $(x - \varepsilon, x)$ due to the Archimedean property. By the way we have defined a_n, it is clear that the corresponding sequence of k values is increasing and so unbounded. This allows us to conclude that we will have a $k \geq n$ and therefore a term in our sequence in $(x - \varepsilon, x)$. This hold for all $\varepsilon > 0$ so this sequence converges to x. Thus, we can construct a sequence of rationals which converges to any possible $x \in \mathbb{R}$. Taking x to be any irrational shows our desired result. □

[10] This is a stronger statement than is asked, but any construction for any specific irrational could be easily modified to show this result for all $x \in \mathbb{R}$.