Lecture #7

MA 511, Introduction to Analysis

June 3, 2021

Rearrangements

Definition

Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a **rearrangement** of $\sum_{k=1}^{\infty} a_k$ if there exists a one-to-one onto function $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

- We were able to construct a rearrangement of the alternating series that converged to a limit different from that of the original series, because the alternating series converges conditionally!

Theorem

If a series converges absolutely, then any rearrangement of this series converges to the same limit.

- The situation for conditionally convergent series is especially pathological. If $\sum_{k=1}^{\infty} a_k$ converges conditionally, then for any $r \in \mathbb{R}$, there exists a rearrangement of $\sum_{k=1}^{\infty} a_k$ that converges to r.

Double summations of infinite series

- The added hypothesis of absolute convergence also comes to save the day for the non-commutative double summations we saw!

Theorem

Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array of real numbers. If

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges, then both $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ converge to the same value. Moreover, we have:

$$\lim_{n \to \infty} s_{nn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

where $s_{nn} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$.

Products of series

- What happens if we try to take the product of two series?

$$\left(\sum_{i=1}^{\infty} a_k\right)\left(\sum_{j=1}^{\infty} b_k\right) = (a_1 + a_2 + a_3 + \cdots)(b_1 + b_2 + b_3 + \cdots)$$
$$= a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 + a_3 b_1) + \cdots$$
$$= \sum_{k=2}^{\infty} d_k$$

where $d_k = a_1 b_{k-1} + a_2 b_{k-2} \cdots + a_{k-1} b_1$. This is called the **Cauchy product** of the two series.

Theorem

If $\sum_{i=1}^{\infty} |a_k| = A$ and $\sum_{j=1}^{\infty} |b_k| = B$, then we have:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i b_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_i b_j = \sum_{k=2}^{\infty} d_k = AB$$

Products of series (cont.)

- We can actually weaken the hypothesis of the previous theorem to only require that one of the series converge absolutely and the other just converge (possibly conditionally).
- What if both converge conditionally? Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$.
- Why the Cauchy product? Soon we will consider **power series** which are of the form $\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$, and when we multiply two power series together and collect terms with the same power of x we have:

$$\begin{aligned}(a_0 + a_1 x + a_2 x^2 + \cdots)&(b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \\ &= d_0 + d_1 x + d_2 x^2 + \cdots\end{aligned}$$

  where $d_k = a_0 b_k + a_1 d_{k-1} + \cdots a_k b_0$, which is exactly the Cauchy product of $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$.

The Cantor set

Definition

The Cantor set C is formed as follows:

$$
\begin{aligned}
C_0 &= [0,1] \\
C_1 &= C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
C_2 &= \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right) \\
&\vdots
\end{aligned}
$$

At each step we have a union of closed intervals and we remove the open middle third of each. If we continue this process inductively, we obtain sets C_n consisting of 2^n intervals each of length $\frac{1}{3^n}$. Finally, the Cantor set is:

$$C = \bigcap_{n=0}^{\infty} C_n$$

Properties of the Cantor set

- What does C look like? Does it contain any points besides the endpoints of these intervals?
- What is the "length" of C? The "length" is 0.
- What is the cardinality of C? The cardinality is $\operatorname{card} \mathbb{R}$.
- What is the "dimension" of C? The "dimension" is $\frac{\log 2}{\log 3} \approx 0.631$.

Open sets

- Recall that given $a \in \mathbb{R}$ and $\varepsilon > 0$, the ε-neighborhood of a is the set $V_\varepsilon(a) = (a - \varepsilon, a + \varepsilon)$.

Definition

A set $O \subseteq \mathbb{R}$ is **open** if for all points $a \in O$ there exists an ε-neighborhood $V_\varepsilon(a) \subseteq O$.

Example: $\mathbb{R}$ itself, the empty set $\varnothing$, and open intervals (a, b) are all open.

Theorem

i. *The union of an arbitrary collection of open sets is open.*
ii. *The intersection of a finite collection of open sets is open.*