Lecture #1

MA 511, Introduction to Analysis

May 24, 2021

The Irrationality of $\sqrt{2}$

- G.H. Hardy argues that mathematics "must be justified as art if it is to be justified at all."

Theorem

There is no rational number whose square is 2.

- What does this tell us about our conception of numbers?

$$\mathbb{N} = \{1, 2, 3, \dots\}$$
$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$$
$$\mathbb{Q} = \{\text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers with } q \neq 0\}$$
$$\mathbb{R} = ???$$

- How well can we approximate $\sqrt{2}$ with a rational number? How many "numbers" is $\mathbb{Q}$ "missing?"

Sets

- To write proofs and develop the theory of analysis, we need some agreed upon terminology and notation!

Definition

Intuitively, a **set** is any collection of objects, which are called the **elements** of the set. We write $x \in A$ if x is an element of A. Given two sets A and B, the **union** $A \cup B$ is defined by:

$$x \in A \cup B \text{ if } x \in A \text{ or } x \in B \text{ or both}$$

The **intersection** $A \cap B$ is defined by:

$$x \in A \cap B \text{ if } x \in A \text{ and } x \in B$$

Example: Natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, "set builder notation" $D = \{x \in \mathbb{N} : x < 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, empty set $\varnothing$.

Sets (cont.)

- How can we relate sets to each other?

Definition

A is a **subset** of B, or B contains A, written $A \subseteq B$, or $B \supseteq A$, if every element of A is an element of B. This is called the **inclusion** relationship, and we have:

$$A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A$$

Most of the sets we will consider will be subsets of the real numbers $\mathbb{R}$. For $A \subseteq \mathbb{R}$ the **complement** of A, written A^c, is defined by:

$$A^c = \{x \in \mathbb{R} : x \notin A\}$$

Example: De Morgan's Laws:

$$(A \cap B)^c = A^c \cup B^c \text{ and } (A \cup B)^c = A^c \cap B^c$$

Functions

Definition

Given two sets A and B a **function** f from A to B is a rule or mapping that takes each element $x \in A$ and associates it with a single element $f(x)$ of B. In this case, we write $f : A \to B$ and call A the **domain** of f. The **range** of f is not necessarily all of B, but refers to the subset:

$$\{y \in B : f(x) = y \text{ for some } x \in A\}$$

Example: $f(x) = x^2 + x + 1$, Dirichlet function, absolute value.

Proposition

The absolute value function satisfies:

$$|ab| = |a||b|$$

*and the **triangle inequality**:*

$$|a + b| \leq |a| + |b|$$

Logic and Proofs

- **Direct proof ($P \Rightarrow Q$):** assume the hypothesis and proceed by rigorously logical deductions to demonstrate the conclusion.
  - *Example:* Proof of De Morgan's Laws

- **Proof by contradiction (P and not $Q \Rightarrow$ contradiction):** assume the hypothesis, negate the conclusion and proceed by rigorously logical deductions until a contradiction arises.
  - *Example:* Proof of the irrationality of $\sqrt{2}$

- **Contrapositive proof (not $Q \Rightarrow$ not P):** negate the conclusion and proceed by rigorously logical deductions to demonstrate the negation of the hypothesis.

Induction

- **Proof by induction:** For a proposition $P(n)$, depending on a natural number n, show that $P(1)$ holds and that if $P(n)$ holds, then $P(n+1)$ holds. By the following proposition, this implies that $P(n)$ holds for all $n \in \mathbb{N}$.

Proposition

If $S \subseteq \mathbb{N}$ with the property that:

i. $1 \in S$
ii. *If $n \in S$, then $n+1 \in S$.*

then $S = \mathbb{N}$.
(We may sometimes replace $\mathbb{N}$ with $\mathbb{N} \cup \{0\}$ and $1 \in S$ with $0 \in S$.)

Example: Proof that:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$