Lecture #13

MA 511, Introduction to Analysis

June 14, 2021

Derivatives

Definition (Derivative)

Let $A \subseteq \mathbb{R}$ be an interval, $c \in A$, and $g : A \to \mathbb{R}$ be a function. The derivative of g at c is defined as

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

if this limit exists (we say g is differentiable at c). g is differentiable on A if it is differentiable everywhere in A.

- We can define the derivative for any domain, but the most interesting results come from the case where the domain is an interval
- It is tempting to think of derivatives as the rules we learn in calculus, but only exceptionally nice functions actually fit these forms

Properties of Differentiable Functions

Theorem

Let $A \subseteq \mathbb{R}$ be an interval, $c \in A$, and $g : A \to \mathbb{R}$ be a function. If g is differentiable at c, then it is also continuous at c.

Theorem (Algebraic Differentiability Theorem)

Let $A \subseteq \mathbb{R}$ be an interval, $c \in A$, and $f, g : A \to \mathbb{R}$ be differentiable at c.

- i $(f+g)'(c) = f'(c) + g'(c)$
- ii $(kf)'(c) = kf'(c)$ *for all* $k \in \mathbb{R}$
- iii $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$
- iv $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$ *if* $g(c) \neq 0$

- Most derivative rules follow from basic algebra and the ALT for functional limits

The Chain Rule

Theorem (Chain Rule)

Let $A, B \subseteq \mathbb{R}$ be intervals, $f : A \to \mathbb{R}$, and $g : B \to \mathbb{R}$ be such that $f(A) \subseteq B$. Furthermore, let f be differentiable at $c \in A$ and g be differentiable at $f(c) \in B$. The function $g \circ f : A \to \mathbb{R}$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))\, f'(c)$$

- Depending on the definitions used for exponential functions, trigonometric functions, and irrational powers of x, we may be able to verify their values with just these tools
- Try to prove the power rule for integer powers

Intermediate Value Property of Derivatives

Theorem (Interior Extremum Theorem)

Let $f : (a, b) \to \mathbb{R}$ be differentiable on the entire domain. If f has a maximum (or minimum) at $c \in (a, b)$, then $f'(c) = 0$

Theorem (Darboux's Theorem)

If $f : [a, b] \to \mathbb{R}$ is differentiable on the entire domain and satisfies $\min\{f'(a), f'(b)\} < \alpha < \max\{f'(a), f'(b)\}$ for some $\alpha \in \mathbb{R}$, then there is $c \in (a, b)$ such that $f'(c) = \alpha$

Mean Value Theorem

Theorem (Rolle's Theorem)

Let $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b). *If* $f(a) = f(b)$, *then there is some* $c \in (a, b)$ *such that* $f'(c) = 0$

Theorem (Mean Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b). *There is some* $c \in (a, b)$ *such that* $f'(c) = \frac{f(b)-f(a)}{b-a}$

- As obvious as it seems and as simple as the proof is, the MVT will be crucial for many future proofs.

Mean Value Theorem Corollaries

Corollary

Let $A \subseteq \mathbb{R}$ be an interval and $g : A \to \mathbb{R}$ be differentiable. If $g'(c) = 0$ for all $c \in A$, then $g(x) = k$ for all $x \in A$ and some constant $k \in \mathbb{R}$.

Corollary

Let $A \subseteq \mathbb{R}$ be an interval and $f, g : A \to \mathbb{R}$ be differentiable. If $f'(c) = g'(c)$ for all $c \in A$, then $f(x) = g(x) + k$ for all $x \in A$ and some constant $k \in \mathbb{R}$.

- Functions we intuitively know to be the most simple have the simplest derivatives
- Derivatives contain almost all information about differentiable functions

Generalized Mean Value Theorem

Theorem (Generalized Mean Value Theorem)

If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there is a point $c \in (a, b)$ such that

$$(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c)$$

If $g'(x) \neq 0$ for all $x \in (a, b)$, then the expression can be written as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

- This is the version of the MVT we will use to prove L'Hopital's Rule and other results in the coming weeks.