A

Construction of the Real Numbers

In Chapter 1 we have described the sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ of natural numbers, integers, and rational numbers, respectively, as follows:

$$\mathbb{N} = \{1, 2, 3, \dots\}, \quad \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\},$$

and

$$\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0\right\}.$$

The set $\mathbb{R}$ of "real numbers" was then introduced as a set containing $\mathbb{Q}$ and the "irrational numbers" in such a way that the elements of $\mathbb{R}$ are in one-to-one correspondence with the points on the "number line". But $\mathbb{R}$ defies a simplistic description such as that given above for $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$. Thus, while we can conceive easily what a rational number is, it is a little harder to say precisely what a real number is. For this reason, we made an assumption that there exists a set $\mathbb{R}$ containing $\mathbb{Q}$ that satisfies the three sets of properties given in Section 1.1, namely the Algebraic Properties A1-A5, the Order Properties O1-O2, and the Completeness Property. The main aim of this appendix is to show that such a set $\mathbb{R}$ does indeed exist and is essentially unique. The approach that we shall take is due to Cantor, and uses Cauchy sequences of rational numbers. In what follows, we shall assume familiarity with the set $\mathbb{Q}$ and the usual algebraic operations on $\mathbb{Q}$ as well as the usual order relation that permits us to talk of the subset $\mathbb{Q}^+$ of positive rational numbers in such a way that the properties A1-A5 and O1-O2 in Section 1.1 are satisfied if we replace $\mathbb{R}$ by $\mathbb{Q}$ throughout. This appendix is divided into three sections, which are organized as follows. In the first section below, we discuss some preliminaries about equivalence relations and equivalence classes. Then in the next section, we outline a construction of $\mathbb{R}$ using Cauchy sequences of rational numbers. The "uniqueness" of $\mathbb{R}$ is formally established in the last section.

A.1 Equivalence Relations

The notion of an equivalence relation is basic to much of mathematics, and it will be useful in our formal construction of $\mathbb{R}$ from $\mathbb{Q}$. The most basic equivalence relation on any set is that of equality denoted by $=$. Fundamental properties of this relation motivate the following definition.

S. R. Ghorpade, B. V. Limaye, *A Course in Calculus and Real Analysis*,
Undergraduate Texts in Mathematics, https://doi.org/10.1007/978-3-030-01400-1

Let S be a set. A **relation** on S is a subset of $S \times S$. If $\sim$ is a relation on S and $a, b \in S$, then we usually write $a \sim b$ to indicate that the ordered pair (a, b) is an element of the subset $\sim$ of $S \times S$. A relation $\sim$ on S is called an **equivalence relation** if (i) $\sim$ is **reflexive**, that is, $a \sim a$ for all $a \in S$, (ii) $\sim$ is **symmetric**, that is, $b \sim a$ whenever $a, b \in S$ satisfy $a \sim b$, and (iii) $\sim$ is **transitive**, that is, $a \sim c$ whenever $a, b, c \in S$ satisfy $a \sim b$ and $b \sim c$.

If $\sim$ is an equivalence relation on a set S and if $a \in S$, then the set $\{x \in S : x \sim a\}$ is called the **equivalence class** of a and is denoted[1] by $[a]$; in general, a subset E of S is called an **equivalence class** (with respect to $\sim$) if $E = [a]$ for some $a \in S$. A key fact about equivalence relations is the following result, which basically says that an equivalence relation on a set partitions the set into disjoint equivalence classes.

Proposition A.1. *Let S be a set and let $\sim$ be an equivalence relation on S. Then any two equivalence classes (with respect to $\sim$) are either disjoint or identical. Consequently, if $\mathcal{E}$ denotes the collection of distinct equivalence classes with respect to $\sim$, then*

$$S = \bigcup_{E \in \mathcal{E}} E,$$

where the union is disjoint.

Proof. Let $a, b \in S$ and suppose the equivalence classes $[a]$ and $[b]$ are not disjoint, that is, there exists $c \in [a] \cap [b]$. Then $c \sim a$ and $c \sim b$. Hence using the fact that $\sim$ is an equivalence relation, we see that for every $x \in S$,

$$x \in [a] \iff x \sim a \iff x \sim c \iff x \sim b \iff x \in [b].$$

This shows that $[a] = [b]$. Thus any two equivalence classes are either disjoint or identical. Finally, since $a \in [a]$ for each $a \in S$, we obtain $S = \bigcup_{a \in S} [a]$. □

We give several examples of equivalence relations and corresponding equivalence classes below. The detailed verification of the assertions made in these examples is left to the reader.

Examples A.2. (i) On the set $\mathbb{N}$, define a relation $\sim$ by

$$m \sim n \iff m \text{ and } n \text{ have the same } \textbf{parity}\text{, that is, } (-1)^m = (-1)^n.$$

Then $\sim$ is an equivalence relation. There are exactly two equivalence classes with respect to $\sim$, namely the set of odd positive integers and the set of even positive integers.

[1] When $S \subseteq \mathbb{Q}$, the notation $[a]$ for the equivalence class of an element a of S conflicts with the notation used in the text for the integer part of a. To avoid any possible confusion, we shall always use in this appendix the notation $\lfloor a \rfloor$ for the integer part of a.

(ii) On the set $\mathbb{N} \times \mathbb{N}$, define a relation $\sim$ by

$$(a,b) \sim (c,d) \iff a+d = b+c \quad \text{for } (a,b),(c,d) \in \mathbb{N} \times \mathbb{N}.$$

Then $\sim$ is an equivalence relation, and the equivalence classes with respect to $\sim$ are in one-to-one correspondence with the set $\mathbb{Z}$ of all integers.

(iii) Let $S = \{(m,n) : m,n \in \mathbb{Z} \text{ and } n \neq 0\}$. The relation $\sim$ on S defined by

$$(a,b) \sim (c,d) \iff ad = bc \quad \text{for } (a,b),(c,d) \in S$$

is an equivalence relation on S, and the equivalence classes with respect to $\sim$ are in one-to-one correspondence with the set $\mathbb{Q}$ of all rational numbers.

(iv) Let $n \in \mathbb{N}$. Consider the relation $\sim$ on $\mathbb{Z}$ defined by

$$a \sim b \iff a - b \text{ is divisible by } n \quad \text{for } a,b \in \mathbb{Z}.$$

Then $\sim$ is an equivalence relation, called **congruence modulo** n. There are exactly n distinct equivalence classes with respect to $\sim$ given by $C_0, C_1, \ldots, C_{n-1}$, where for $0 \leq i < n$, the set C_i consists of integers that leave remainder i when divided by n. These equivalence classes are known as **residue classes modulo** n, and the set $\{C_0, C_1, \ldots, C_{n-1}\}$ of all residue classes modulo n is sometimes denoted by $\mathbb{Z}/n\mathbb{Z}$.

We remark that examples (ii) and (iii) above can be used to formally construct $\mathbb{Z}$ from $\mathbb{N}$, and to construct $\mathbb{Q}$ from $\mathbb{Z}$. For an axiomatic treatment of $\mathbb{N}$, we refer to the book of Landau [54].

A.2 Cauchy Sequences of Rational Numbers

We shall now define the notion of a Cauchy sequence in $\mathbb{Q}$. This is completely analogous to the notion discussed in Chapter 2, except that we will refrain from using real numbers anywhere. In particular, ϵ will denote a positive rational number, that is, $\epsilon \in \mathbb{Q}^+$. Note that since it is well understood what positive rational numbers are, the notion of the absolute value of a rational number is well-defined and satisfies the basic properties given in Proposition 1.8.

A **sequence** in $\mathbb{Q}$ is a function from $\mathbb{N}$ to $\mathbb{Q}$. We usually write (a_n) to denote the sequence $\mathbf{a} : \mathbb{N} \to \mathbb{Q}$ defined by $\mathbf{a}(n) := a_n$ for $n \in \mathbb{N}$. The rational number a_n is called the n**th term** of the sequence (a_n). A sequence (a_n) of rational numbers is said to be

1. **bounded above** if there exists $\alpha \in \mathbb{Q}$ such that $a_n \leq \alpha$ for all $n \in \mathbb{N}$,
2. **bounded below** if there exists $\beta \in \mathbb{Q}$ such that $a_n \geq \beta$ for all $n \in \mathbb{N}$,
3. **bounded** if it is bounded above as well as bounded below,
4. **Cauchy** if for every $\epsilon \in \mathbb{Q}^+$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $m,n \in \mathbb{N}$ with $m,n \geq n_0$.

We shall also say that a sequence (c_n) of rational numbers is **null**, and write $c_n \to 0$, if for every $\epsilon \in \mathbb{Q}^+$, there is $n_0 \in \mathbb{N}$ such that $|c_n| < \epsilon$ for all $n \geq n_0$.

Examples A.3. (i) Let (a_n) be the sequence in $\mathbb{Q}$ defined by $a_n := 1/n$ for $n \in \mathbb{N}$. Then (a_n) is a null sequence. Indeed, given any $\epsilon \in \mathbb{Q}^+$, say $\epsilon = p/q$, where $p, q \in \mathbb{N}$, the positive integer $n_0 := q + 1$ satisfies $n_0 > 1/\epsilon$. Hence $|a_n| < \epsilon$ for all $n \geq n_0$.

(ii) Let (a_n) be the sequence in $\mathbb{Q}$ defined by $a_n := (n-1)/n$ for $n \in \mathbb{N}$. Then (a_n) is a Cauchy sequence. To see this, let $\epsilon = p/q \in \mathbb{Q}^+$ be given, where $p, q \in \mathbb{N}$. Now the positive integer $n_0 := 2(q+1)$ satisfies $n_0 > 2/\epsilon$. Hence

$$|a_n - a_m| = \left|\frac{n-1}{n} - \frac{m-1}{m}\right| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{2}{n_0} < \epsilon \quad \text{for all } m, n \geq n_0.$$

Note that although (a_n) is a Cauchy sequence, it is not a null sequence. In fact, $|a_n| \geq 1/2$ for all $n \geq 2$.

Proposition A.4. (i) *Every Cauchy sequence of rational numbers is bounded.*

(ii) *Every null sequence of rational numbers is Cauchy.*

(iii) *Let (a_n) be a Cauchy sequence of rational numbers that is not a null sequence. Then there exist $\epsilon_0 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ such that $|a_n - a_{n_0}| < \epsilon_0$ and $|a_n| \geq \epsilon_0$ for all $n \geq n_0$.*

Proof. (i) Let (a_n) be a Cauchy sequence. Then there exists $k \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $m, n \geq k$. Consequently,

$$|a_n| \leq \alpha \text{ for all } n \in \mathbb{N}, \quad \text{where} \quad \alpha := \max\{|a_1|, \ldots, |a_{k-1}|, |a_k| + 1\}.$$

Hence (a_n) is bounded.

(ii) Let (c_n) be a null sequence. Given any $\epsilon \in \mathbb{Q}^+$, there exists $n_0 \in \mathbb{N}$ such that $|c_n| < \epsilon/2$ for all $n \geq n_0$. Then

$$|c_n - c_m| \leq |c_n| + |c_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } n \geq n_0.$$

Hence (c_n) is Cauchy.

(iii) Since (a_n) is not a null sequence, there exists $\epsilon \in \mathbb{Q}^+$ such that for every $k \in \mathbb{N}$, there exists $n_1 \geq k$ satisfying $|a_{n_1}| \geq \epsilon$. Also, since (a_n) is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $|a_m - a_n| < \epsilon/2$ for all $m, n \geq n_0$. Let $k := n_0$, and find $n_1 \geq n_0$ such that $|a_{n_1}| \geq \epsilon$. Then

$$\epsilon \leq |a_{n_1}| \leq |a_{n_1} - a_n| + |a_n| \leq \frac{\epsilon}{2} + |a_n|, \text{ and hence } |a_n| \geq \frac{\epsilon}{2} \text{ for all } n \geq n_0.$$

Thus $\epsilon_0 := \epsilon/2 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ have the desired property. □

Now let us define

$$\mathscr{C} := \text{the set of all Cauchy sequences of rational numbers.}$$

Further, consider the relation $\sim$ on $\mathscr{C}$ defined by

$$(a_n) \sim (b_n) \iff (a_n - b_n) \text{ is a null sequence}, \quad \text{where } (a_n), (b_n) \in \mathscr{C}.$$

Proposition A.5. *The relation $\sim$ is an equivalence relation on $\mathscr{C}$.*

Proof. Clearly, $\sim$ is reflexive and symmetric. Suppose $(a_n), (b_n), (c_n) \in \mathscr{C}$ are such that $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$. Given any $\epsilon \in \mathbb{Q}^+$, the number $\epsilon/2$ is also in $\mathbb{Q}^+$. Hence there exist $n_1, n_2 \in \mathbb{N}$ such that

$$|a_n - b_n| < \frac{\epsilon}{2} \text{ for all } n \geq n_1 \quad \text{and} \quad |b_n - c_n| < \frac{\epsilon}{2} \text{ for all } n \geq n_2.$$

Now if $n_0 = \max\{n_1, n_2\}$, then for each $n \geq n_0$,

$$|a_n - c_n| = |(a_n - b_n) + (b_n - c_n)| \leq |a_n - b_n| + |b_n - c_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $(a_n) \sim (c_n)$. Thus $\sim$ is transitive as well. □

We are now ready to define a model for $\mathbb{R}$ that we seek to construct. Let

$$\mathcal{R} := \text{the set of all equivalence classes of elements of } \mathscr{C} \text{ with respect to } \sim.$$

As in the previous section, the equivalence class of a Cauchy sequence (a_n) in $\mathscr{C}$ with respect to $\sim$ will be denoted by $[(a_n)]$. Given any $r \in \mathbb{Q}$, the constant sequence (r), that is, the sequence (r_n) with $r_n = r$ for all $n \in \mathbb{N}$, is clearly Cauchy. We will denote by $\mathcal{Q}$ the subset of $\mathcal{R}$ consisting of the equivalence classes of constant sequences of rational numbers. It is clear that the map from $\mathbb{Q}$ to $\mathcal{Q}$ given by $r \longmapsto [(r)]$ is one-one and onto. Thus we can, and will, identify $\mathcal{Q}$ with $\mathbb{Q}$. In particular, the equivalence classes of the constant sequences (0) and (1) will be denoted simply by 0 and 1, respectively.

We now define addition and multiplication on the set $\mathcal{R}$ as follows.

$$[(a_n)] + [(b_n)] = [(a_n + b_n)] \quad \text{and} \quad [(a_n)] \cdot [(b_n)] = [(a_n b_n)] \quad \text{for } (a_n), (b_n) \in \mathscr{C}.$$

Proposition A.6. *The operations of addition and multiplication on $\mathcal{R}$ are well-defined and satisfy the following algebraic properties:*

A1. $a + (b + c) = (a + b) + c$ *and* $a(bc) = (ab)c$ *for all* $a, b, c \in \mathcal{R}$.
A2. $a + b = b + a$ *and* $ab = ba$ *for all* $a, b \in \mathcal{R}$.
A3. $a + 0 = a$ *and* $a \cdot 1 = a$ *for all* $a \in \mathcal{R}$.
A4. *Let* $a \in \mathcal{R}$. *Then there exists* $a' \in \mathcal{R}$ *such that* $a + a' = 0$. *Further, if* $a \neq 0$, *then there exists* $a^* \in \mathcal{R}$ *such that* $aa^* = 1$.
A5. $a(b + c) = ab + ac$ *for all* $a, b, c \in \mathcal{R}$.

Proof. To show that the operations of addition and multiplication on $\mathcal{R}$ are well-defined, it suffices to show that for all (a_n), (a'_n), (b_n), $(b'_n) \in \mathscr{C}$,

$$(a_n) \sim (a'_n) \text{ and } (b_n) \sim (b'_n) \Longrightarrow (a_n + b_n) \sim (a'_n + b'_n) \text{ and } (a_n b_n) \sim (a'_n b'_n).$$

The assertion $(a_n + b_n) \sim (a'_n + b'_n)$ follows from the definition, since $|(a_n + b_n) - (a'_n + b'_n)| \leq |a_n - a'_n| + |b_n - b'_n|$. To see that $(a_n b_n) \sim (a'_n b'_n)$, we use part (i) of Proposition A.4 and obtain $\alpha', \beta \in \mathbb{Q}^+$ such that $|a'_n| \le \alpha'$ and $|b_n| \le \beta$ for all $n \in \mathbb{N}$, so that

$$|a_n b_n - a'_n b'_n| = |(a_n - a'_n)b_n + a'_n(b_n - b'_n)| \le \beta|a_n - a'_n| + \alpha'|b_n - b'_n|.$$

Now since $a_n - a'_n \to 0$ and $b_n - b'_n \to 0$, it is readily seen that $a_n b_n - a'_n b'_n \to 0$.

Having established that addition and multiplication on $\mathcal{R}$ are well-defined, we see that properties A1, A2, A3, and A5 are immediate consequences of the definition and the corresponding properties of rational numbers.

Moreover, for every $a = [(a_n)] \in \mathcal{R}$, the element $a' := [(-a_n)]$ is clearly in $\mathcal{R}$ and it satisfies $a + a' = 0$. Finally, suppose $a = [(a_n)] \in \mathcal{R}$ is such that $a \ne 0$. Then by part (iii) of Proposition A.4, there exist $\epsilon_0 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ such that $|a_n| \ge \epsilon_0$ for all $n \ge n_0$. In particular, $a_n \ne 0$ for all $n \ge n_0$. Define the sequence (a_n^*) in $\mathbb{Q}$ by $a_n^* := 1$ for $1 \le n < n_0$ and $a_n^* = 1/a_n$ for $n \ge n_0$. Then $|a_n^* - a_m^*| \le (1/\epsilon_0^2)|a_n - a_m|$ for all $n, m \ge n_0$. Since (a_n) is a Cauchy sequence, this implies that (a_n^*) is also a Cauchy sequence. Moreover, $a_n a_n^* - 1 = 0$ for all $n \ge n_0$, and so $(a_n a_n^*) \sim (1)$. This proves A4. $\square$

As noted in Section 1.1, several simple properties (such as, $a \cdot 0 = 0$ for all $a \in \mathcal{R}$) are formal consequences of properties A1–A5 proved in Proposition A.6, and these will now be tacitly assumed; also uniqueness of the additive inverse $a' \in \mathcal{R}$ for $a \in \mathcal{R}$, and of the multiplicative inverse $a^* \in \mathcal{R}$ for $a \in \mathcal{R}$ with $a \ne 0$ as in A4, is a formal consequence of Proposition A.6, and we will adopt the usual notation $-a$ for a', and $1/a$ or a^{-1} for a^*. We remark also that as a consequence of Proposition A.6, the addition and multiplication on $\mathcal{R}$ are compatible with the usual addition and multiplication on $\mathbb{Q}$ when $\mathbb{Q}$ is identified with the subset $\mathcal{Q}$ of $\mathcal{R}$ as before.

Now let us turn to order properties. We shall say that a sequence $(a_n) \in \mathscr{C}$ is **positive** if it satisfies the following property:

There exist $r \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ such that $a_n \ge r$ for all $n \ge n_0$.

Note that if $(a'_n) \in \mathscr{C}$ is such that $(a'_n) \sim (a_n)$ and (a_n) satisfies the above property, then so does (a'_n). Indeed, since $(a'_n) \sim (a_n)$, there exists $n_1 \in \mathbb{N}$ such that $|a'_n - a_n| < r/2$ for all $n \ge n_1$. Now if we let $r' := r/2$ and $n_2 := \max\{n_0, n_1\}$, then we obtain

$$a'_n > a_n - \frac{r}{2} \ge r - \frac{r}{2} = r' \text{ for all } n \ge n_2.$$

With this in view, we define $\mathcal{R}^+$ to be the set of all equivalence classes of positive sequences in $\mathscr{C}$. It is clear that $\mathcal{R}^+$ is a well-defined subset of $\mathcal{R}$.

Proposition A.7. *The set $\mathcal{R}^+$ satisfies the following order properties:*

O1. *Given any $a \in \mathcal{R}$, exactly one of the following statements is true:*

$$a \in \mathcal{R}^+; \qquad a = 0; \qquad -a \in \mathcal{R}^+.$$

O2. *If $a, b \in \mathcal{R}^+$, then $a + b \in \mathcal{R}^+$ and $ab \in \mathcal{R}^+$.*

Proof. Let $(a_n) \in \mathscr{C}$ be such that $[(a_n)] \neq 0$. By part (iii) of Proposition A.4, there exist $\epsilon_0 \in \mathbb{Q}^+$ and $n_0 \in \mathbb{N}$ such that $|a_n - a_{n_0}| < \epsilon_0$ and $|a_n| \geq \epsilon_0$ for all $n \geq n_0$. In particular, $a_{n_0} \neq 0$. Now if $a_{n_0} > 0$, then the above inequalities imply $a_n = a_{n_0} + (a_n - a_{n_0}) > \epsilon_0 - \epsilon_0 = 0$ for all $n \geq n_0$, and consequently, $a_n \geq \epsilon_0$ for all $n \geq n_0$. Likewise, if $a_{n_0} < 0$, then $-a_n \geq \epsilon_0$ for all $n \geq n_0$. Thus $[(a_n)] \in \mathcal{R}^+$ or $-[(a_n)] = [(-a_n)] \in \mathcal{R}^+$. This proves O1.

Next, if $(a_n), (b_n) \in \mathscr{C}$ are such that $[(a_n)], [(b_n)] \in \mathcal{R}^+$, then there exist $r_1, r_2 \in \mathbb{Q}^+$ and $n_1, n_2 \in \mathbb{N}$ such that

$$a_n > r_1 \text{ for all } n \geq n_1 \quad \text{and} \quad b_n > r_2 \text{ for all } n \geq n_2.$$

Now if we let $n_0 = \max\{n_1, n_2\}$, then we clearly have

$$a_n + b_n > r_1 + r_2 \text{ for all } n \geq n_0 \quad \text{and} \quad a_n b_n > r_1 r_2 \text{ for all } n \geq n_0.$$

Since $r_1 + r_2,\ r_1 r_2 \in \mathbb{Q}^+$, we obtain $[(a_n)] + [(b_n)] \in \mathcal{R}^+$ and $[(a_n)][(b_n)] \in \mathcal{R}^+$. This proves O2. $\square$

Using the set $\mathcal{R}^+$, we can define an order relation on $\mathcal{R}$ exactly as in Section 1.1, namely, for all $a, b \in \mathcal{R}$, we write $a < b$ or $b > a$ if $b - a \in \mathcal{R}^+$. Moreover, we shall write $a \leq b$ or $b \geq a$ to mean that either $a < b$ or $a = b$. The usual properties of this order relation, as listed in (i), (ii), and (iii) on page 3 (with $\mathbb{R}$ replaced by $\mathcal{R}$), and also the fact that $1 > 0$ are formal consequences of O1 and O2, and will thus be tacitly assumed. Moreover, the notions of a subset of $\mathcal{R}$ being bounded above, bounded below, or bounded as well as the notions of upper bound, lower bound, supremum, and infimum for subsets of $\mathcal{R}$ can now be defined exactly as they were defined for subsets of $\mathbb{R}$ in Chapter 1. Note also that the order relation on $\mathcal{R}$ that we have just defined is compatible with the known order relation on the set $\mathbb{Q}$, that is, if $r, s \in \mathbb{Q}$ and if $[(r)], [(s)]$ are the corresponding elements of $\mathcal{Q}$, then $r < s$ if and only if $[(r)] < [(s)]$.

We shall now proceed to prove that the set $\mathcal{R}$, which we have constructed from $\mathbb{Q}$, has the completeness property. As a preliminary step, we will first show that $\mathcal{R}$ has the archimedean property. It may be recalled that in Proposition 1.3, the archimedean property of $\mathbb{R}$ was deduced from the assumption that $\mathbb{R}$ has the completeness property. Here we will give a direct proof to show that $\mathcal{R}$ has the archimedean property, and later, use it to derive the completeness property of $\mathcal{R}$.

Proposition A.8. *Given any $a \in \mathcal{R}$, there is some $k \in \mathbb{N}$ such that $k > a$.*

Proof. First note that if $a \in \mathcal{Q}$, then a corresponds to a unique rational number p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Hence $k := |p| + 1$ clearly satisfies $k > a$. Now suppose $a = [(a_n)]$ is an arbitrary element of $\mathcal{R}$, where (a_n) is a Cauchy sequence of rational numbers. Then for $\epsilon = 1/2$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < 1/2$ for all $n, m \geq n_0$. Since the archimedean property holds for rational numbers, there exists $\ell \in \mathbb{N}$ such that $|a_{n_0}| < \ell$. Hence for each $n \geq n_0$, we obtain

$$|a_n| \leq |a_n - a_{n_0}| + |a_{n_0}| < \frac{1}{2} + \ell \quad \text{and hence} \quad |(1+\ell) - a_n| \geq (1+\ell) - |a_n| > \frac{1}{2}.$$

It follows that $[(a_n)] < [(1+\ell)]$, that is, $k := 1 + \ell \in \mathbb{N}$ satisfies $k > a$. □

The archimedean property proved in Proposition A.8 enables us to define the integer part of every $x \in \mathcal{R}$ exactly as in the paragraph following Proposition 1.3 of Chapter 1, and this, in turn, permits us to deduce that between any two elements of $\mathcal{R}$, there is a rational number. It is important to note that this result uses only the algebraic and order properties together with the archimedean property (Propositions A.6, A.7, and A.8).

Proposition A.9. *Given any $a, b \in \mathcal{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. The proof is identical to that of Proposition 1.6 and hence omitted. A more general result (Proposition A.12) is proved in the next section. □

Corollary A.10. *Let $(r_n) \in \mathscr{C}$ and let $a = [(r_n)]$ be the corresponding element of $\mathcal{R}$. If there exists $n_0 \in \mathbb{N}$ such that $r_n \geq 0$ for all $n \geq n_0$, then $a \geq 0$. More generally, if there exist $\alpha, \beta \in \mathcal{R}$ and $n_0 \in \mathbb{N}$ such that $\beta \leq r_n \leq \alpha$ for all $n \geq n_0$, then $\beta \leq a \leq \alpha$.*

Proof. Suppose there exists $n_0 \in \mathbb{N}$ such that $r_n \geq 0$ for all $n \geq n_0$. Let, if possible, $a < 0$. Then by Proposition A.9, there exists $s \in \mathbb{Q}$ such that $a < s < 0$. Now $-s > 0$ and $r_n - s \geq -s$ for all $n \geq n_0$. So it follows from the definition of $\mathcal{R}^+$ that $a - s > 0$, which is a contradiction. This proves that $a \geq 0$.

Next suppose $\alpha, \beta \in \mathcal{R}$ and $n_0 \in \mathbb{N}$ are such that $\beta \leq r_n \leq \alpha$ for all $n \geq n_0$. By Proposition A.9, there exist $\alpha_n, \beta_n \in \mathbb{Q}$ such that $\beta - \frac{1}{n} < \beta_n < \beta$ and $\alpha < \alpha_n < \alpha + \frac{1}{n}$ for each $n \in \mathbb{N}$. This implies that $(\alpha_n), (\beta_n) \in \mathscr{C}$. Moreover, $\alpha = [(\alpha_n)]$ and $\beta = [(\beta_n)]$. (Verify!) Now applying the first assertion in the corollary to $(r_n - \beta_n)$ and $(\alpha_n - r_n)$, we obtain $\beta \leq a \leq \alpha$. □

We are now ready to prove that the set $\mathcal{R}$ has the completeness property.

Proposition A.11. *Every nonempty subset of $\mathcal{R}$ that is bounded above has a supremum.*

Proof. Let $\mathcal{S}$ be a nonempty subset of $\mathcal{R}$ that is bounded above. Since $\mathcal{S}$ is nonempty, there is some $a_0 \in \mathcal{S}$, and since $\mathcal{S}$ is bounded above, there is some $\alpha_0 \in \mathcal{R}$ such that α_0 is an upper bound of $\mathcal{S}$, that is, $a \leq \alpha_0$ for all $a \in \mathcal{S}$. Now let $\beta_1 := (a_0 + \alpha_0)/2$. If β_1 is an upper bound of $\mathcal{S}$, we let $a_1 := a_0$ and $\alpha_1 := \beta_1$, whereas if β_1 is not an upper bound of $\mathcal{S}$, then there exists $b \in \mathcal{S}$

such that $\beta_1 < b$, and in this case, we let $a_1 := b$ and $\alpha_1 := \alpha_0$. In any case, $a_0 \le a_1$ and $\alpha_0 \ge \alpha_1$, and moreover,

$$a_1 \in \mathcal{S}, \quad \alpha_1 \text{ is an upper bound of } \mathcal{S}, \quad \text{and} \quad 0 \le \alpha_1 - a_1 \le \frac{\alpha_0 - a_0}{2}.$$

Next, we replace (a_0, α_0) by (a_1, α_1) and proceed as before. In general, given $n \in \mathbb{N}$ and $a_i \in \mathcal{S}$ and upper bounds α_i of $\mathcal{S}$ with $0 \le (\alpha_i - a_i) \le (\alpha_0 - a_0)/2^i$ for $0 \le i \le n-1$ and with $a_0 \le a_1 \le \cdots \le a_{n-1}$ and $\alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_{n-1}$, we choose $a_n \in \mathcal{S}$ and an upper bound α_n of $\mathcal{S}$ as follows. Let $\beta_n := (a_{n-1} + \alpha_{n-1})/2$. If β_n is an upper bound of $\mathcal{S}$, we let $a_n := a_{n-1}$ and $\alpha_n := \beta_n$, whereas if β_n is not an upper bound of $\mathcal{S}$, then there exists $b \in \mathcal{S}$ such that $\beta_n < b$, and in this case, we let $a_n := b$ and $\alpha_n := \alpha_{n-1}$. In any case, $a_{n-1} \le a_n$ and $\alpha_{n-1} \ge \alpha_n$, and moreover,

$$a_n \in \mathcal{S}, \quad \alpha_n \text{ is an upper bound of } \mathcal{S}, \quad \text{and} \quad 0 \le \alpha_n - a_n \le \frac{\alpha_0 - a_0}{2^n}.$$

Note that if $a_n = \alpha_n$ for some $n \ge 0$, then clearly α_n is the supremum of $\mathcal{S}$.

Now suppose $a_n < \alpha_n$ for all $n \ge 0$. By Proposition A.9, for each $n \in \mathbb{N}$, there exists $r_n \in \mathbb{Q}$ such that $a_n < r_n < \alpha_n$. We claim that (r_n) is a Cauchy sequence. To see this, let $\epsilon \in \mathbb{Q}^+$ be given. Applying Proposition A.8 to $a = (\alpha_0 - a_0)/\epsilon$, we see that there exists $k \in \mathbb{N}$ such that

$$\frac{\alpha_0 - a_0}{k} < \epsilon \quad \text{and hence} \quad \frac{\alpha_0 - a_0}{2^k} < \epsilon,$$

where the last inequality follows by noting that $2^j \ge j$ for all $j \in \mathbb{N}$, as can be seen easily by induction on j. Now given any $m, n \in \mathbb{N}$ with $m \ge n \ge k$, since $a_m < r_m < \alpha_m$, $a_n < r_n < \alpha_n$, and $a_m \ge a_n$, we obtain

$$r_n - r_m < \alpha_n - r_m < \alpha_n - a_m \le \alpha_n - a_n.$$

In a similar way, since $\alpha_m \le \alpha_n$, we obtain

$$r_n - r_m > a_n - r_m > a_n - \alpha_m \ge a_n - \alpha_n.$$

It follows that

$$|r_n - r_m| < \alpha_n - a_n \le \frac{\alpha_0 - a_0}{2^n} \le \frac{\alpha_0 - a_0}{2^k} < \epsilon \quad \text{for all } m, n \ge k.$$

Thus $(r_n) \in \mathscr{C}$, and so $\alpha := [(r_n)] \in \mathcal{R}$. We shall now show that α is the supremum of $\mathcal{S}$. To this end, let us first observe that for every fixed $m \in \mathbb{N}$, the inequalities $a_m \le a_n < r_n < \alpha_n \le \alpha_m$ hold for all $n \ge m$, and so by Corollary A.10, we see that $a_m \le \alpha \le \alpha_m$.

Now suppose, if possible, α is not an upper bound of $\mathcal{S}$. Then there is $a \in \mathcal{S}$ such that $a > \alpha$. By Proposition A.9, there exists $\delta \in \mathbb{Q}^+$ such that $\delta < a - \alpha$. Further, by Proposition A.8, there is $m \in \mathbb{N}$ such that $m > (\alpha_0 - a_0)/\delta$, and so

$$0 \le \alpha_m - a_m \le \frac{\alpha_0 - a_0}{2^m} \le \frac{\alpha_0 - a_0}{m} < \delta.$$

Hence $\alpha_m < a_m + \delta \le \alpha + \delta < a$. But this contradicts the fact that α_m is an upper bound of $\mathcal{S}$. Hence α is an upper bound of $\mathcal{S}$.

Next, suppose α is not the least upper bound of $\mathcal{S}$. Then there exists $\beta \in \mathcal{R}$ with $\beta < \alpha$ such that β is an upper bound of $\mathcal{S}$. Again, choose $\delta \in \mathbb{Q}^+$ such that $0 < \delta < \alpha - \beta$ and $m \in \mathbb{N}$ such that $0 \le \alpha_m - a_m < \delta$. Then $a_m > \alpha_m - \delta \ge \alpha - \delta > \beta$, which is a contradiction, since $a_m \in \mathcal{S}$ and β is an upper bound of $\mathcal{S}$. It follows that α is the supremum of $\mathcal{S}$. $\square$

A.3 Uniqueness of a Complete Ordered Field

In the previous section, we have shown that the set $\mathcal{R}$ possesses all the properties that were postulated for $\mathbb{R}$ in Chapter 1. In other words, we have established the existence of the set of all real numbers. We will now prove its "uniqueness". First, we introduce some useful terminology.

By a **field** we shall mean a set F that has operations of addition and multiplication (that is, maps from $F \times F$ to F that associate elements $a + b$ and ab of F to $(a, b) \in F \times F$) and has distinct elements 0_F and 1_F in it such that the five algebraic properties A1–A5 in Proposition A.6 are satisfied with $\mathcal{R}$ replaced throughout by F, and with 0 and 1 replaced by 0_F and 1_F. It is easy to see that in a field F, elements 0_F and 1_F satisfying A3 are unique, and these are sometimes called the additive identity and the multiplicative identity of F, respectively. Note that $\mathbb{Q}$ and $\mathcal{R}$ are examples of fields. A special case of Example A.2 (iv), namely the set $\mathbb{Z}/p\mathbb{Z}$ of residue classes modulo a prime number p, is a field having only finitely many elements.

If a field F contains a subset F^+ satisfying the two order properties O1–O2 with $\mathcal{R}^+$ replaced throughout by F^+, then F is called an **ordered field**; in this case, for every $a, b \in F$, we write $a < b$ or $b > a$ if $b - a \in F^+$; also, we write $a \le b$ or $b \ge a$ if either $a < b$ or $a = b$. The notions of boundedness, supremum, etc. are defined for subsets of an ordered field in exactly the same way as in the case of $\mathbb{R}$. Note that $\mathbb{Q}$ and $\mathcal{R}$ are ordered fields, but $\mathbb{Z}/p\mathbb{Z}$ is not. In fact, an ordered field F cannot be finite. Indeed, $1_F > 0_F$ (because otherwise $-1_F > 0_F$, and so $1_F = (-1_F)(-1_F) > 0_F$, which would be a contradiction). Hence for every $n \in \mathbb{N}$, if we let $n_F := 1_F + \cdots + 1_F$ (n times), then $n_F > 0_F$; moreover, $0_F < 1_F < 2_F < \cdots$, and so F contains infinitely many elements. Furthermore, in an ordered field F, for every $m \in \mathbb{Z}$ with $m < 0$, we let m_F denote the additive inverse of $(-m)_F$, that is, the unique element of F satisfying $(-m)_F + m_F = 0_F$. For every $r = m/n$ in $\mathbb{Q}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, we let $r_F := (m_F)(n_F)^{-1}$. It is easily seen that $r \longmapsto r_F$ gives a well-defined, one-one map of $\mathbb{Q} \to F$, which preserves algebraic and order operations, that is, for all $r, s \in \mathbb{Q}$,

$$(r + s)_F = r_F + s_F, \quad (rs)_F = r_F s_F, \quad \text{and} \quad r < s \Longrightarrow r_F < s_F.$$

Thus F contains a copy of $\mathbb{Q}$, namely $\mathbb{Q}_F := \{r_F : r \in \mathbb{Q}\}$. In an ordered field F, the **absolute value** of an element can be defined as in the case of $\mathbb{Q}$ or $\mathbb{R}$. Thus for every $a \in F$, we let $|a| := a$ if $a \geq 0_F$ and $|a| := -a$ if $a < 0_F$. It is easily seen that $|ab| = |a||b|$ and $|a + b| \leq |a| + |b|$ for all $a, b \in F$.

Let F be an ordered field. We say that F is **archimedean** if for every $a \in F$, there exists $n \in \mathbb{N}$ such that $n_F > a$. and we say that F is **complete** if every nonempty subset of F that is bounded above has a supremum in F. For example, both $\mathbb{Q}$ and $\mathcal{R}$ are archimedean ordered fields, and $\mathcal{R}$ is complete (thanks to Proposition A.11), but $\mathbb{Q}$ is not. In general, by arguing as in the proof of Proposition 1.3, we readily see that a complete ordered field is archimedean.

Let F be an archimedean ordered field and let $a \in F$. Then there exist $m, n \in \mathbb{N}$ such that $m_F > -a$ and $n_F > a$, that is, $-m_F < a < n_F$. Thus if k is the largest among the finitely many integers satisfying $-m \leq k \leq n$ and $k_F \leq a$, then k_F is called the **integer part** of a in F, and is denoted by $\lfloor a \rfloor$. Note that $\lfloor a \rfloor \leq a < \lfloor a \rfloor + 1_F$. The following result is similar to Proposition A.9. The proof is similar to that of Proposition 1.6, but this time we include it.

Proposition A.12. *Let F be an archimedean ordered field and let $a_1, a_2 \in F$ satisfy $a_1 < a_2$. Then there is $r \in \mathbb{Q}$ such that $a_1 < r_F < a_2$.*

Proof. Since F is archimedean, there exists $n \in \mathbb{N}$ such that $n_F > (a_2 - a_1)^{-1}$, that is, $(n_F)^{-1} < (a_2 - a_1)$. Let $m \in \mathbb{N}$ be such that $m_F = \lfloor n_F a_1 \rfloor + 1_F$. Then $m_F - 1_F \leq n_F a_1 < m_F$. Hence

$$a_1 < m_F (n_F)^{-1} \leq (n_F a_1 + 1_F)(n_F)^{-1} = a_1 + (n_F)^{-1} < a_1 + (a_2 - a_1) = a_2.$$

Thus $r = m/n \in \mathbb{Q}$ satisfies $a_1 < r_F < a_2$. □

Corollary A.13. *Let F be an archimedean ordered field. Suppose $a \in F$ satisfies $|a| < \epsilon_F$, that is, $-\epsilon_F < a < \epsilon_F$, for all $\epsilon \in \mathbb{Q}^+$. Then $a = 0_F$.*

Proof. In case $a > 0_F$, by Proposition A.12, there exists $r \in \mathbb{Q}$ such that $0_F < r_F < a$. Thus the hypothesis is contradicted if we take $\epsilon = r$. Likewise, we arrive at a contradiction if $a < 0_F$. Hence $a = 0_F$. □

Let K and F be ordered fields. A map $f : K \to F$ is called an **order isomorphism** if f is both one-one and onto, and f preserves algebraic and order operations, that is, for all $x, y \in K$,

$$f(x + y) = f(x) + f(y), \quad f(xy) = f(x)f(y), \quad \text{and} \quad x < y \Longrightarrow f(x) < f(y).$$

If such a map exists, then we say that F is **order isomorphic** to K.

Proposition A.14. *Every complete ordered field is order isomorphic to $\mathcal{R}$.*

Proof. Let F be a complete ordered field. Define $f : \mathcal{R} \to F$ by

$$f(x) := \sup F_x, \quad \text{where} \quad F_x := \{r_F : r \in \mathbb{Q} \text{ and } r \le x\} \quad \text{for } x \in \mathcal{R}.$$

Note that f is well-defined. Indeed, given any $x \in \mathcal{R}$, by Proposition A.9, there exist $s, t \in \mathbb{Q}$ such that $x - 1 < s < x < t < x + 1$. It follows that $s_F \in F_x$ and t_F is an upper bound of F_x. Thus $\sup F_x$ exists, since F is complete. Note also that $f(r) = r_F$ for all $r \in \mathbb{Q}$. Indeed, $f(r) < r_F$ as well as $f(r) > r_F$ will both lead to a contradiction using Proposition A.12.

Let $x, y \in \mathcal{R}$ be such that $x < y$. Then $x < (x+y)/2 < y$, and using Corollary A.9, we can find $u, v \in \mathbb{Q}$ such that $x < u < (x+y)/2 < v < y$. Now u_F is an upper bound of F_x and v_F is an element of F_y. Hence we obtain $f(x) \le u_F < v_F \le f(y)$. Thus f is order-preserving, and therefore one-one.

To show that f is onto, suppose $a \in F$. In case $a = r_F \in \mathbb{Q}_F$ for some $r \in \mathbb{Q}$, then $a = f(r)$. Now suppose $a \notin \mathbb{Q}_F$. Let $\mathbb{Q}_a := \{r \in \mathbb{Q} : r_F \le a\}$. Since F is complete, it is archimedean, and therefore by Proposition A.12, there exist $r, s \in \mathbb{Q}$ such that $a - 1 < r_F < a$ and $a < s_F < a + 1$. This implies that the set $\mathbb{Q}_a$ is nonempty and bounded above. Hence $x := \sup \mathbb{Q}_a$ is a well-defined element of $\mathcal{R}$. We shall now show that $f(x) = a$.

First, suppose $x \in \mathbb{Q}$. Then $f(x) = x_F$. Now if $x_F < a$, then by Proposition A.12, there exists $r \in \mathbb{Q}$ such that $x_F < r_F < a$, and this leads to a contradiction, because on the one hand $x < r$, since $x, r \in \mathbb{Q}$ and $x_F < r_F$, but on the other hand, $r \le x$, since $r_F < a$ implies $r \in \mathbb{Q}_a$ and $x = \sup \mathbb{Q}_a$. Likewise, if $x_F > a$, then by Proposition A.12, there exists $s \in \mathbb{Q}$ such that $a < s_F < x_F$, but then s is an upper bound of $\mathbb{Q}_a$ (because $r \in \mathbb{Q}$ and $r_F \le a$ implies $r_F < s_F$ and hence $r < s$), and therefore $x = \sup \mathbb{Q}_a \le s$, which implies $x_F \le s_F$, and this contradicts $s_F < x_F$. Thus $f(x) = a$ when $x \in \mathbb{Q}$.

Next, suppose $x \notin \mathbb{Q}$. Let $r \in \mathbb{Q}$ with $r \le x$. Since $x \notin \mathbb{Q}$, we obtain $r < x$, and since $x = \sup \mathbb{Q}_a$, there exists $s \in \mathbb{Q}_a$ such that $r < s \le x$. Consequently, $r_F < s_F \le a$. It follows that a is an upper bound of F_x. Hence $f(x) \le a$. Furthermore, if $f(x) < a$, then by Proposition A.12, there exists $t \in \mathbb{Q}$ such that $f(x) < t_F < a$. But then $t \in \mathbb{Q}_a$, and so $t \le x$, which implies $t_F \le f(x)$, and so we obtain a contradiction. It follows that $f(x) = a$. Thus f is onto.

It remains to show that f preserves the algebraic operations. Let $x, y \in \mathcal{R}$ and let $\epsilon \in \mathbb{Q}^+$ be given. By Proposition A.9, there exist $r, s, u, v \in \mathbb{Q}$ such that

$$x - \frac{\epsilon}{4} < r < x < s < x + \frac{\epsilon}{4} \quad \text{and} \quad y - \frac{\epsilon}{4} < u < y < v < y + \frac{\epsilon}{4}.$$

Then $0 < s - r < \epsilon/2$ and $0 < v - u < \epsilon/2$. Since f is order-preserving, $r_F < f(x) < s_F$ and $u_F < f(y) < v_F$. Hence $r_F + u_F < f(x) + f(y) < s_F + v_F$. Moreover, $r + u < x + y < s + v$, and again since f is order-preserving, $r_F + u_F < f(x+y) < s_F + v_F$. Consequently,

$$f(x+y) - f(x) - f(y) < (s_F - r_F) + (v_F - u_F) < \frac{\epsilon_F}{2_F} + \frac{\epsilon_F}{2_F} = \epsilon_F.$$

By a similar argument, $f(x+y) - f(x) - f(y) > -\epsilon_F$. Now by Corollary A.13, we obtain $f(x+y) = f(x) + f(y)$. Thus, f preserves addition.

To show that f preserves multiplication, first consider $x, y \in \mathcal{R}$ with $x > 0$ and $y > 0$. Let $\epsilon \in \mathbb{Q}^+$ be given. By Proposition A.8, there exists $n \in \mathbb{N}$ such that $n > x$, $n > y$, and $n > \epsilon/6$. Now using Proposition A.9 and the assumption that $x > 0$ and $y > 0$, we obtain $r, s, u, v \in \mathbb{Q}^+$ such that

$$x - \frac{\epsilon}{6n} < r < x < s < x + \frac{\epsilon}{6n} \quad \text{and} \quad y - \frac{\epsilon}{6n} < u < y < v < y + \frac{\epsilon}{6n}.$$

Next, we use the usual trick of adding and subtracting suitable terms to write

$$xy - ru = (x-r)y + r(y-u) \quad \text{and} \quad sv - xy = (s-x)(v-y) + x(v-y) + y(s-x).$$

Consequently, $xy - ru < (\epsilon/6n)y + r(\epsilon/6n) < (\epsilon/6) + (\epsilon/6) < \epsilon/2$ and

$$sv - xy < \left(\frac{\epsilon}{6n}\right)^2 + x\frac{\epsilon}{6n} + y\frac{\epsilon}{6n} < \frac{\epsilon}{6n} + \frac{\epsilon}{6} + \frac{\epsilon}{6} \leq \frac{\epsilon}{6} + \frac{\epsilon}{3} = \frac{\epsilon}{2}.$$

Thus $sv - (\epsilon/2) < xy < ru + (\epsilon/2)$, and since f is order-preserving, we obtain

$$s_F v_F - \frac{\epsilon_F}{2_F} < f(xy) < r_F u_F + \frac{\epsilon_F}{2_F}.$$

Again, since f is order-preserving, by arguing as before, we obtain

$$s_F v_F - \frac{\epsilon_F}{2_F} < f(x)f(y) < r_F u_F + \frac{\epsilon_F}{2_F}.$$

It follows that $-\epsilon_F < f(xy) - f(x)f(y) < \epsilon_F$. By Corollary A.13, we obtain $f(xy) = f(x)f(y)$. Finally, since f preserves addition, it is easily seen that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in \mathcal{R}$. Hence the result just proved, namely $f(xy) = f(x)f(y)$ for all positive $x, y \in \mathcal{R}$, implies that $f(xy) = f(x)f(y)$ for all $x, y \in \mathcal{R}$. So f preserves multiplication as well. □

Remark A.15. In view of the results of the previous section and the uniqueness result in Proposition A.14, it makes sense to refer to any set satisfying the algebraic, order, and completeness properties as **the** set of all real numbers. The construction of $\mathcal{R}$ given in the previous section is one of the ways of constructing $\mathbb{R}$. Several other constructions are possible. The most prominent among these is a construction due to Dedekind, where the basic idea is to determine a real number x by means of the pair (L_x, R_x) of subsets of $\mathbb{Q}$, where $L_x := \{r \in \mathbb{Q} : r < x\}$ and $R_x := \{r \in \mathbb{Q} : r \geq x\}$. Such a pair is called a **Dedekind cut**, or simply a **cut**. More formally, a **cut** is a pair (L, R) of nonempty disjoint subsets of $\mathbb{Q}$ such that (i) $L \cup R = \mathbb{Q}$, (ii) L is downwards closed, that is, $s \in L$ whenever $s < t$ for some $t \in L$, (iii) R is upwards closed, that is, $s \in R$ whenever $s > t$ for some $t \in R$, and (iv) L has no maximum element. One defines addition, multiplication, and an order on the set of all cuts, and shows that this set is a complete ordered field. For more details about this approach, one can refer to the essays of Dedekind [24] or the appendix to Chapter 1 of Rudin [71]. For a host of other constructions for $\mathbb{R}$, we refer to the article of Weiss [87]. At any rate, by Proposition A.14, all these constructions yield essentially the same ordered field. ◇

Remark A.16. In this book, we have used the word completeness (of $\mathbb{R}$, or more generally, of any ordered field F) to mean that every nonempty subset that is bounded above has a supremum. This is sometimes referred to as **order completeness**, especially when contrasted with other notions such as **monotone completeness** and **Cauchy completeness** that are defined as follows. An ordered field F is said to be **monotone complete** (resp. **Cauchy complete**) if every monotonic bounded sequence (resp. Cauchy sequence) in F is convergent. Note that in an ordered field, the notions of absolute value and of a sequence being monotonic, bounded, convergent, or Cauchy are defined in the same way as in the case of $\mathbb{R}$. Arguing as in the proofs of Propositions 2.8 and 2.22, we readily see that for every ordered field F,

$$F \text{ is complete} \Longrightarrow F \text{ is monotone complete} \Longrightarrow F \text{ is Cauchy complete.}$$

It can be shown that Cauchy completeness implies (order) completeness, provided that the ordered field is archimedean. (Compare Exercise 2.42 of Chapter 2.) Thus for an archimedean ordered field, the three notions of completeness are equivalent. Moreover, an ordered field that is monotone complete is necessarily archimedean (because otherwise, the sequence (n_F) would be convergent, and if n_F converges to a, then there exists $n_0 \in \mathbb{N}$ such that $a - 1_F < n_F < a + 1_F$ for $n \geq n_0$, but then $(n+2)_F > a + 1_F$, which would be a contradiction!) and consequently, order complete. However, there do exist ordered fields that are Cauchy complete, but not archimedean and therefore neither order complete nor monotone complete; see, for instance, Ex. 4 and 7 in Chapter 1 of Gelbaum and Olmsted [32]. For more on various notions of completeness and related matters, see the article of Hall and Todorov [36]. Finally, we remark that the notion of Cauchy completeness can be readily defined for any field F that (is not necessarily ordered, but) has an "absolute value function", that is, a map from F to $\mathbb{R}^+$ given by $a \longmapsto |a|$ satisfying for all $a, b \in F$, the following: (i) $|a| = 0 \iff a = 0_F$, (ii) $|ab| = |a||b|$, and (iii) $|a + b| \leq |a| + |b|$. In the next section, we will formally introduce the field $\mathbb{C}$ of complex numbers and show that $\mathbb{C}$ is not an ordered field, but $\mathbb{C}$ has an absolute value function. Moreover, it is easy to show (using Proposition 2.22) that $\mathbb{C}$ is Cauchy complete. $\diamond$