Proof Polygonals are sums of $h_a(x)$

Let $a_0, \dots, a_n$ be partition defining points and
Let $\phi(x)$ be a polygonal function with slope
m_k on $[a_k, a_{k+1}]$ for $0 \le k \le n-1$

Define $b_0 = m_0$ and $b_k = m_k - m_{k-1}$ for $1 \le k \le n-1$

Claim: $\phi(x) = \phi(a_0) + \sum_{k=0}^{n-1} b_k h_{a_k}(x)$

$\phi(a_0) = \phi(a_0) + \sum_{k=0}^{n-1} b_k h_{a_k}(x) = \phi(a_0)$ because $a_0 \le a_k \Rightarrow h_{a_k}(a_0) = 0$

For $x \in [a_0, a_1]$, we find

$\phi(x) = \phi(a_0) + b_0 h_{a_0}(x) = \phi(a_0) + m_0(x - a_0)$

This is the unique Linear function w/ value $\phi(a_0)$ @ $x = a_0$ and slope m_0 so the statement holds on this subinterval.

Assume that $\phi(a_0) + \sum_{k=0}^{j} b_k h_{a_k}(x)$ agrees with $\phi(x)$

on $[a_0, a_{j+1}]$.

This means that for $x \ge a_j$, $\phi(a_0) + \sum_{k=0}^{j-1} b_k h_{a_k}(x) = m_j(x - a_j) + \phi(a_j)$
since this is the unique linear function with the desired slope and passing through proper points.

For $x \ge a_{j+1}$, we compute $\phi(a_0) + \sum_{k=0}^{j+1} b_k h_{a_k}(x) = m_j(x - a_j) + \phi(a_j) + (m_{j+1} - m_j)(x - a_{j+1})$
$= m_{j+1} x + C$ for some C. This has the correct slope and value @ $x = a_{j+1}$ so C must be "right". By induction we are done.

~~Nett~~ Proof of WAT

- Let $\epsilon > 0$. By our polygonal approximation theorem, we can choose a polygonal, ϕ, such that for all $x \in [a,b]$

$$\left| f(x) - \phi(x) \right| < \frac{\epsilon}{2}$$

Let $n+1$ be the number of points defining the partition which makes ϕ piece-wise linear.

$$\phi(x) = \phi(a_0 = a) + \sum_{k=0}^{n-1} b_k h_{a_k}(x) \quad \text{as previously proven}$$

For each k, Let $P_k(x)$ be a polynomial such that for all $x \in [a,b]$

$$\left| |x - a_k| - P_k(x) \right| < \frac{\epsilon}{n \cdot \max\{b_k\}}$$

We see that $\left| |x-a_k| + (x-a_k) - P_k(x) - (x-a_k) \right| < \frac{\epsilon}{n \max\{b_k\}}$

$$\left| \frac{1}{2}\left(|x-a_k| + x - a_k\right) - \frac{1}{2}\left(P_k(x) + x - a_k\right) \right| < \frac{\epsilon}{2n \cdot \max\{b_k\}}$$

$$\left| h_{a_k}(x) - q_k(x) \right| < \frac{\epsilon}{2n \cdot \max\{b_k\}}$$

$$\left| b_k h_{a_k}(x) - b_k q_k(x) \right| < \frac{\epsilon}{2n}$$

Proof of WAT (Cont.)

Let $Q(x) = \phi(a_0) + \sum_{k=0}^{n-1} b_k q_k(x)$

For all $x \in [a,b]$, we see

$$|f(x) - Q(x)| = |f(x) - \phi(x) + \phi(x) - Q(x)|$$
$$\leq |f(x) - \phi(x)| + |\phi(x) - Q(x)|$$
$$< \frac{\varepsilon}{2} + \left|\phi(a_0) + \sum_{k=0}^{n-1} b_k h_{a_k}(x) - \phi(a_0) - \sum_{k=0}^{n-1} b_k q_k(x)\right|$$
$$< \frac{\varepsilon}{2} + \sum_{k=0}^{n-1} |b_k h_{a_k}(x) - b_k q_k(x)|$$
$$< \frac{\varepsilon}{2} + \sum_{k=0}^{n-1} \frac{\varepsilon}{2n} = \frac{\varepsilon}{2} + n \cdot \frac{\varepsilon}{2n} = \varepsilon$$

Q is a linear combination of polynomials so Q is also a polynomial so we are done.

$\square$