Lecture #21

MA 511, Introduction to Analysis

June 28, 2021

Riemann's Original Integral

Definition (Tagged Partition)

A tagged Partition is a partition $P \in \mathcal{P}$ along with a set of points $\{c_k\}$ such that $c_k \in [x_k, x_{k+1}]$

Definition (Riemann Sum)

Let $f : [a, b] \to \mathbb{R}$ be a function and $(P, \{c_k\}_{k=0}^{n-1})$ be a tagged partition of $[a, b]$. The Riemann sum of f associated to this tagged partition is

$$R(f.P) = \sum_{k=0}^{n-1} f(c_k)\Delta x_k$$

- The choice of $c_k = x_k$ and $\Delta x_k = \frac{b-a}{n}$ corresponds to the "left" sum of calculus
- Clearly, $L(f, P) \leq R(f, P) \leq U(f, P)$ for any choice of tags

An $\varepsilon - \delta$ Characterization of Integrability

Definition (δ-fine Partitions)

A partition is said to be δ-fine if $\Delta x_k < \delta$ for all $k = 0, ..., n-1$

Theorem (Limit Criterion for Riemann Integrability)

A bounded function f is Riemann integrable with $\int_a^b f = A$ if and only if for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$|R(f, P) - A| < \varepsilon$$

for any set of tags $\{c_k\}$ and any δ-fine partition P

- We have found another characterization (and Riemann's original definition) of the Riemann Integral
- This is the characterization which we will generalize

Gauges and Partitions

Definition (Gauge)

A function $\delta : [a, b] \to \mathbb{R}$ is called a gauge on $[a, b]$ if $\delta(x) > 0$ for all $x \in [a, b]$

Definition ($\delta(x)$-fine Tagged Partitions)

A tagged partition is said to be $\delta(x)$-fine if $\Delta x_k < \delta(c_k)$ for all $k = 0, ..., n-1$ and $\delta(x)$ a gauge on $[a, b]$

Theorem

For any gauge $\delta(x)$, there is a $\delta(x)$-fine tagged partition of $[a, b]$

- The existence of such partitions will be essential for generalizing integration

Gauge Integration

Definition (Gauge Integral)

A function $f : [a, b] \to \mathbb{R}$ is gauge integrable with $\int_a^b f = A$ if for all $\varepsilon > 0$, there exists a gauge on $[a, b]$, $\delta(x)$, such that

$$|R(f, P) - A| < \varepsilon$$

for all $\delta(x)$-fine tagged partitions P

Theorem

The gauge integral is well-defined

- All Riemann integrable functions are gauge integrable and the values of the integrals agree
- The gauge integral is more powerful than the Riemann integral (and Lebesgue integral)

Parametric Derivatives

Definition (Parametric Derivative)

A function $F : [a, b] \to \mathbb{R}$ is parametrically differentiable if there exists $\phi : [\alpha, \beta] \to [a, b]$ which is differentiable, strictly increasing, and such that $(F \circ \phi)$ is differentiable in the traditional sense. f is a parametric derivative of F if

$$(F \circ \phi)'(t) = f(\phi(t))\phi'(t)$$

- f is not unique. If $\phi'(t) = 0$, then $f(\phi(t))$ can have any value
- If $\phi'(t) \neq 0$, then $F'(\phi(t)) = f(\phi(t))$
- $\phi(t) = t$ give the normal derivative
- Many of our non-differentiable functions are parametrically differentiable

The Fundamental Theorem of Calculus

Theorem

If F is normally differentiable with $F'(x) = f(x)$, then $\int_a^b f = F(b) - F(a)$

Theorem

F is parametrically differentiable on $[a, b]$ with a parametric derivative f if and only if $\int_a^b f = F(b) - F(a)$

- All of the requirements from the Fundamental Theorem of Calculus can be dropped with these generalizations