Lecture #15

MA 511, Introduction to Analysis

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Continuity of Limiting Functions Revisited

Theorem (Cauchy Criterion for Sequences of Functions)

For each $n \in \mathbb{N}$, let $f_n : A \to \mathbb{R}$ be a function. (f_n) converges uniformly to f if and only if for every $\varepsilon > 0$ and $x \in A$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m \geq N$

Theorem (Continuous Limit Theorem)

Let (f_n) be a sequence of functions which converge uniformly to f on A. If all f_n are continuous, then f is continuous.

- Since we basically defined uniform convergence of functions as having the property that fixed the hole we discovered earlier, this result is unsurprising
- Uniform convergence of functions will be a necessary tool for many theorems about the behavior of function sequences in the coming chapter

More Convergence Results

Theorem (Dini's Theorem from 6.2.11)

Let $f_n \to f$ pointwise on a compact set K and $f_n(x)$ be increasing for all $x \in K$. If f_n and f are continuous, then $f_n \to f$ uniformly

Theorem (from 6.2.13)

Let A be a countable set $A = \{x_1, x_2, ...\}$, and (f_n) be a bounded sequence of functions on A. There is a subsequence (f_{n_k}) which converges pointwise to some function f on A.

Even More Convergence Results

Definition (Equicontinuity from 6.2.14)

A sequence of functions (f_n) is equicontinuous on A if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$ and $n \in \mathbb{N}$, $|x - y| < \delta$ implies that $|f_n(x) - f_n(y)| < \varepsilon$

Theorem (Arzela-Ascoli Theorem from 6.2.15)

Let f_n be bounded, and equicontinuous on $[0, 1]$. There exists a subsequence of functions f_{n_k} which converges uniformly to some function f

Differentiability of Limits

Theorem (Differentiable Limit Theorem)

Let $f_n \to f$ pointwise on $[a, b]$ and let f_n be differentiable. If (f_n') converges uniformly on $[a, b]$ to a function g, then f is differentiable and $f' = g$

Theorem

Let (f_n) be a sequence of differentiable functions on $[a, b]$ and (f_n') converge uniformly. If $f_n(x_0)$ is convergent for some $x_0 \in [a, b]$, then (f_n) converges uniformly on $[a, b]$

- Combining these, we only need to have that (f_n') converges uniformly and $f(x_0)$ converges for some $x_0 \in [a, b]$ to get both results

Definition (Convergence of Series of Functions)

For each $n \in \mathbb{N}$, let f_n and f be functions $A \to \mathbb{R}$. We say that the series $\sum_{n=1}^{\infty} f_n$ converges pointwise to f if the sequence of partial sum functions

$$s_m = \sum_{n=1}^{m} f_n$$

converges pointwise to f. We define uniform convergence similarly.

- Just as we did with sequences of functions, we will try to recover similar results as those about series of real numbers
- We will also need to restrict ourselves to uniform convergence to make many of these properties work

Term-by-Term Theorems

Theorem (Term-by-Term Continuity Theorem)

Let f_n be continuous on A and $\sum_{n=1}^{\infty} f_n$ converge uniformly to f on A. If this is true, then f is continuous

Theorem (Term-by-Term Differentiability Theorem)

Let f_n be differentiable on A and $\sum_{n=1}^{\infty} f_n'$ converge uniformly to g on A. If $\sum_{n=1}^{\infty} f_n$ converges at some $x_0 \in A$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly to some function f with $f' = g$

Cauchy Criterion

Theorem (Cauchy Criterion for Uniform Convergence of Series)

A series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A if and only if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{n=s+1}^{t} f_n \right| < \varepsilon$$

for all $t > s \geq N$ and $x \in A$

Corollary (Weierstrass M-Test)

Let f_n be a sequence of functions and M_n be a sequence of real numbers such that $|f_n| \leq M_n$ for all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A

Definition (Power Series)

Power series are expressions of the form

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

defined on whatever domain the series converges.

- It is possible for a function $f(x)$ to be defined in a different way but be representable as a power series
- a_n can be 0 so polynomials are also power series
- All power series converge at $x = 0$

Theorem

If $\sum_{n=1}^{\infty} a_n x^n$ converges at $x_0 \in \mathbb{R}$, then it converges absolutely for all $|x| < |x_0|$

Radius of Convergence

- The previous theorem shows there are only a few types of domains for power series: $\{0\}$, $\mathbb{R}$, and symmetric intervals around 0

Definition (Radius of Convergence)

Let the power series $\sum_{n=1}^{\infty} a_n x^n$ converge on some domain A. If $A = (-R, R), [-R, R), (-R, R]$, or $[-R, R]$, we define the radius of convergence to be R. If $A = \mathbb{R}$, we define the radius of convergence to be ∞

Theorem

If $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely at $x_0 \in \mathbb{R}$, then it converges uniformly on $[-|x_0|, |x_0|]$

Abel's Theorem

Lemma (Abel's Lemma)

Let (b_n) be a decreasing sequence of positive values and $\sum_{n=1}^{\infty} a_n$ be a series such that there exists $A > 0$ satisfying

$$|\sum_{n=1}^{k} a_n| \leq A$$

for all $k \in \mathbb{N}$. It then follows that

$$|\sum_{n=1}^{k} a_n b_n| \leq Ab_1$$

Theorem (Abel's Theorem)

If a series $\sum_{n=1}^{\infty} a_n x^n$ converges at $R > 0$, then it converges uniformly on $[0, R]$, A similar result holds for $-R$ and $[-R, 0]$

Properties of Power Series

Theorem

If $\sum_{n=1}^{\infty} a_n x^n$ converges pointwise on A, it converges uniformly on compact subsets of A

Theorem

If $\sum_{n=1}^{\infty} a_n x^n$ converges on $(-R, R)$, then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges on $(-R, R)$ as well. The convergence is uniform on compact subsets of $(-R, R)$

Theorem

Assume that $f(x) = \sum_{n=1}^{\infty} a_n x^n$ converges on an interval $A \subseteq \mathbb{R}$. f is continuous on A and infinitely differentiable on all intervals $(-R, R) \subseteq A$. The derivative of f is given by

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$