Example 4

$f(x)=x^2$ on $[0,4]$

$$\sum_{k=1}^{N} k^2 = \frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}$$

Let $P_N = \left\{ \frac{k}{N} \right\}_{k=0}^{4N}$

$M_k = \left\{ \frac{(k+1)^2}{N^2} \right\}$, $m_k = \left\{ \frac{k^2}{N^2} \right\}$, $\Delta x_k = \left\{ \frac{1}{N} \right\}$

$$U(f,P_N) = \sum_{k=0}^{4N} \frac{(k+1)^2}{N^2} \cdot \frac{1}{N} = \frac{1}{N^3} \cdot \sum_{\substack{j=1}}^{4N} j^2 = \frac{1}{N^3} \cdot \left(\frac{(4N)^3}{3} + \frac{(4N)^2}{2} + \frac{4N}{6} \right)$$

$j=k+1$ $\quad$ $j=1$

$$= \frac{64}{3} + \frac{2}{3N^2} + \frac{8}{N}$$

$$L(f,P_N) = \sum_{k=0}^{4N} \frac{(k)^2}{N^2} \cdot \frac{1}{N} = \frac{1}{N^3} \cdot \sum_{k=1}^{4N} k^2 = \frac{64}{3} + \frac{2}{3N^2} - \frac{8}{N}$$

$k=0 \Rightarrow k^2=0$

$$\frac{64}{3} + \frac{2}{3N^2} - \frac{8}{N} \le L(f) \le U(f) \le \frac{64}{3} + \frac{2}{3N^2} + \frac{8}{N}$$

$$\lim_{N\to\infty} \Rightarrow \frac{64}{3} \le L(f) \le U(f) \le \frac{64}{3}$$

$$\int_0^4 x^2\,dx = \frac{64}{3}$$

Proof Integrability Criterion

Suppose that for all $\epsilon>0$, we can choose $P_\epsilon \in \mathcal{P}$ such that $U(f,P_\epsilon)-L(f,P_\epsilon)<\epsilon$

$$U(f)-L(f) \le U(f,P_\epsilon)-L(f,P_\epsilon) < \epsilon$$

True for all $\epsilon>0$, so $U(f)-L(f)=0$

Suppose $U(f)=L(f)$. For all $\epsilon>0$

$\exists\ P_1, P_2 \in \mathcal{P}$ such that

$$U(f,P_1)-U(f)<\frac{\epsilon}{2}$$

$$L(f)-L(f,P_2)<\frac{\epsilon}{2}$$

Let $P_\epsilon = P_1 \cup P_2$. $U(f,P_\epsilon)-L(f,P_\epsilon) \le U(f,P_1)-L(f,P_2)$

$$< U(f)+\frac{\epsilon}{2}-\left(L(f)-\frac{\epsilon}{2}\right)=\epsilon$$

Proof Continuity $\Rightarrow$ Int. able

- f cont. on compact domain so f is uniformly cont. and bounded.

For $\varepsilon > 0$, choose $\delta > 0$ such that

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$

Let P be any partition such that $\Delta x_k < \delta$

The Extreme Value theorem says that

$M_k = f(z_k)$ and $m_k = f(w_k)$ for $z_k, w_k \in [x_k, x_{k+1}]$

$|w_k - z_k| \leq |x_{k+1} - x_k| < \delta$ so

$M_k - m_k = f(z_k) - f(w_k) < \frac{\varepsilon}{b-a}$

So, $U(f,P) - L(f,P) = \sum_{k=0}^{n-1} (M_k - m_k)\Delta x_k$

$< \frac{\varepsilon}{b-a}\left(\sum_{k=0}^{n-1} \Delta x_k\right) = \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon$

By Integrable Criterion we are done

$\square$

Let $\varepsilon > 0$ and M be a bound for f.

Let P be a partition of $[a, b]$ such that:

$x_1 < a + \frac{\varepsilon}{4M}$

Let P' be a partition of $[x_1, b]$ such that

$U(f, P') - L(f, P') < \frac{\varepsilon}{2}$

Let P'' be $P' \cup \{x_k\}_{k=2}^{n}$ from P

$U(f, P'' \cup P) - L(f, P'' \cup P) \leq (M_0 - m_0) \cdot \frac{\varepsilon}{4M} + \sum_{k=1}^{n-1} (M_k - m_k) \Delta x_k$

$< (M - (-M)) \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = 2M \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon$

$\square$

Example Ex. 7.3.3

$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ 0 & \text{else} \end{cases}$ on $[0,1]$

$P_{N,\delta} = \{0, \frac{1}{N}+\delta, \frac{1}{N-1}-\frac{\delta}{2}, \frac{1}{N-1}+\frac{\delta}{2}, \dots, \frac{1}{2}+\frac{\delta}{2}, 1-\delta, 1\}$

$M_k = \{1, 0, 1, 0, \dots, 0, 1\}, \quad m_k = \{0, \dots, 0\}$

$\Delta x_k = \{\frac{1}{N}+\delta, \frac{1}{N-1}-\frac{\delta}{2}-\frac{1}{N}-\delta, \delta, \dots, 1-\delta-\frac{1}{2}-\frac{\delta}{2}, \delta\}$

$L(f, P_{N,\delta}) = 0$

$U(f, P_{N,\delta}) = 1\cdot(\frac{1}{N}+\delta) + \delta + \delta + \dots \delta$

$= \frac{1}{N} + \delta \cdot N$ $\quad N+1?$
$\quad N-1?$

For δ small, N large, we can make $U(f, P_{N,\delta}) - L(f, P_{N,\delta})$ as small as desired.

So $\int_0^1 f(x)\,dx = 0$