Lecture #11

MA 511, Introduction to Analysis

June 9, 2021

Preservation of properties under a function

- Given a function $f : A \to \mathbb{R}$ and a subset $B \subseteq A$, the notation $f(B)$ (called the **image of B under f**) refers to the range of f over the set B and is given by $f(B) = \{f(x) : x \in B\}$.
- If B is *open*/*closed*/*bounded*/*compact*/*perfect*/*connected*, then is $f(B)$ also *open*/*closed*/*bounded*/*compact*/*perfect*/*connected*?
- We will consider the case when f is continuous, and if $f(B)$ has the same given property as B, we will say that f **preserves** that property.

Theorem (Topological characterization of continuity)

Let g be defined on all of $\mathbb{R}$. If B is a subset of $\mathbb{R}$, define the set $g^{-1}(B)$ (called the preimage of B under g) to be $g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\}$. Then, g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is open.

Extreme value theorem

- Is open-ness preserved by continuous maps?
- Is closed-ness preserved by continuous maps?

Theorem (Preservation of compact sets)

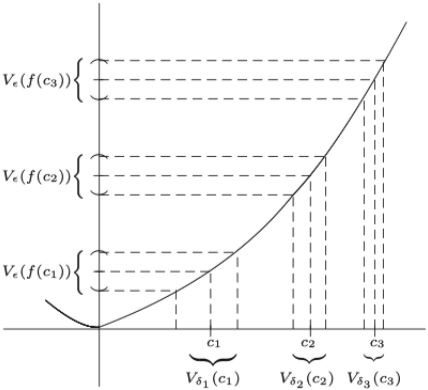
Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ be compact, then $f(K)$ is compact as well.

Theorem (Extreme value theorem)

If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains a maximum and minimum value. In other words, there exist $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.

Uniform continuity

- Sometimes when proving that a function f is continuous at c, the δ we respond with depends not just on the ε, but also on c



Definition

A function $f : A \to \mathbb{R}$ is **uniformly continuous on** A if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

Uniform continuity (cont.)

- If f is uniformly continuous on A, then f is also continuous on A, but the converse is not true.

Theorem (Sequential criterion for absence of uniform continuity)

A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\varepsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

- Uniform continuity is always in reference to a particular domain.

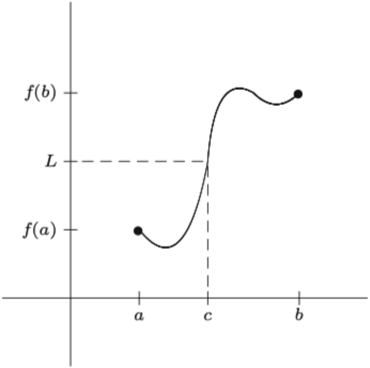
Theorem (Uniform continuity on compact sets)

A function that is continuous on a compact set K is uniformly continuous on K.

Intermediate value theorem

Theorem (Intermediate value theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$



Proofs of IVT

- There are several methods to prove the intermediate value theorem and each way isolates the interplay between continuity and completeness in a slightly different way.
- The first and potentially most useful (because it generalizes to higher dimensions) method uses the fact that continuous maps preserve connected-ness.

Theorem (Preservation of connected sets)

Let $f : G \to \mathbb{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected as well.

- A typical application of IVT is using it to prove the existence of roots, e.g. consider $f(x) = x^2 - 2$ on $[1, 2]$.
- So, there is some relationship between the continuity of f and the completeness of $\mathbb{R}$. We can also use AoC or NIP to prove IVT.