Lecture #9

MA 511, Introduction to Analysis

June 7, 2021

Perfect sets

- One of the goals of topology is to strip away all of the extraneous information and isolate the properties responsible for a particular phenomenon we are studying.
- For example, we saw that the set $[0, 1]$ was uncountable, but what is it about this set that makes it uncountable? Is it a particular example of a more general phenomenon?

Definition

A set $P \subseteq \mathbb{R}$ is **perfect** if it is closed and contains no isolated points.

Example: Show that the Cantor set is perfect.

Theorem

A nonempty perfect set is uncountable.

Connected sets

Definition

Two nonempty sets $A, B \subseteq \mathbb{R}$ are **separated** if $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$. A set $E \subseteq \mathbb{R}$ is **disconnected** if it can be written as $E = A \cup B$, where A and B are nonempty separated sets. A set that is not disconnected is called **connected**.

Theorem

A set $E \subseteq \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B and x an element of the other.

Theorem

A set $E \subseteq \mathbb{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, it follows that $c \in E$ as well.

F_σ sets and G_δ sets

- The closer we look, the more intricate and enigmatic $\mathbb{R}$ looks.
  - Open sets are either a finite or countable union of open intervals.
  - Closed sets on the other hand do have a neat characterization. For example, the Cantor set is closed.

Definition

A set $A \subseteq \mathbb{R}$ is called an F_σ **set** if it can be written as the countable union of closed sets. A set $B \subseteq \mathbb{R}$ is called a G_δ **set** if it can be written as the countable intersection of open sets.

Proposition

A set A is a G_δ set if and only if its complement is an F_σ set.

- $(a, b]$ is both a G_δ set and an F_σ set.
- $\mathbb{Q}$ is an F_σ set and $\mathbb{I}$ is a G_δ set.

F_σ sets and G_δ sets (cont.)

- Recall that a set $G \subseteq \mathbb{R}$ is **dense** in $\mathbb{R}$ if, given any two real numbers $a < b$, it is possible to find a point $x \in G$ with $a < x < b$.

Theorem

If $\{G_1, G_2, G_2, \ldots\}$ is a countable collection of dense open sets, then the intersection $\bigcap_{n=1}^{\infty} G_n$ is not empty.

Corollary

The irrationals $\mathbb{I}$ is not an F_σ set and consequently $\mathbb{Q}$ is not a G_δ set.

- Can you find a set which is neither an F_σ set nor a G_δ set?

Nowhere-dense sets

Proposition

A set $G \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if and only if $\overline{G} = \mathbb{R}$.

Definition

A set E is **nowhere-dense** if $\overline{E}$ contains no nonempty open intervals.

Example: $\mathbb{Q} \subseteq \mathbb{R}$ is dense, while $\mathbb{Z} \subseteq \mathbb{R}$ is nowhere-dense.

Proposition

A set E is nowhere-dense in $\mathbb{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbb{R}$.

Baire's theorem

Theorem (Baire's theorem)

The set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

- Baire's theorem (also called the Baire Category theorem) offers another perspective on the size of $\mathbb{R}$.
- Sets that are countable unions of nowhere-dense are called "meager" or of **first category**, while sets that are not of first category are of **second category**. Thus the Baire Category theorem says that $\mathbb{R}$ is of second category.
- The Baire Category theorem generalizes to say that any complete metric space is of second category.
- Consider the complete metric space of continuous functions on $[0, 1]$ with metric $\sup |f(x) - g(x)|$. The set of functions that are differentiable at even one point is of first category. Thus, **most continuous functions do not have derivatives at any point**.