Homework Solutions #4

CAS MA 511

Problem (5.2.2)**.** (a) is possible. Consider $f(x)$ to be the Dirichlet function and

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(b) is possible. Consider $g(x) = x$ and

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(c) is impossible. If g and $f + g$ are differentiable, then by the algebraic differentiability theorem, the difference $(f + g) - g = f$ must be differentiable.

(d) is possible. Let $d(x)$ be the Dirichlet function and consider $f(x) = x^2 d(x)$. If $f(x)$ were differentiable at any point other than 0, then it would be continuous at that point and hence $\frac{f(x)}{x^2} = d(x)$ would be continuous which is a contradiction. So, now we just need to show that $f(x)$ is indeed differentiable at $x = 0$. We have:

$$\lim_{x \to 0} \frac{x^2 d(x) - 0^2 d(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 d(x)}{x} = \lim_{x \to 0} x d(x) = 0$$

Thus, $f(x)$ is differentiable at $x = 0$ (and you can see why we needed to multiply by x^2.)

Problem (5.2.3)**.** For (a), we have:

$$\lim_{x \to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} = \lim_{x \to c} \frac{\frac{c-x}{xc}}{x - c} = -\lim_{x \to c} \frac{1}{xc} = -\frac{1}{c^2}$$

Thus, $h'(x) = -\frac{1}{x^2}$.

For (b), from part (a) and the chain rule, we have:

$$\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{[g(x)]^2}$$

Thus, by theorem 5.2.4 part (iii), we have:

$$\left(\frac{f}{g}\right)'(x) = f'(x)\frac{1}{g(x)} + f(x)\left(\frac{1}{g(x)}\right)' = f'(x)\frac{g(x)}{[g(x)]^2} - f(x)\frac{g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

For (c), we have:

$$\frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} = \frac{1}{x - c}\frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)} = \frac{1}{x - c}\frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{g(x)g(c)}$$

$$\frac{1}{x-c}\frac{f(x)g(c)-f(c)g(c)+f(c)g(c)-f(c)g(x)}{g(x)g(c)} = \frac{1}{g(x)g(c)}\left(g(c)\frac{f(x)-f(c)}{x-c} - f(c)\frac{g(x)-g(c)}{x-c}\right)$$

Now, when we take a limit as $x \to c$, we have:

$$\lim_{x\to c}\frac{1}{g(x)g(c)}\left(\frac{f(x)-f(c)}{x-c}g(c) - f(c)\frac{g(x)-g(c)}{x-c}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

Problem (5.2.12)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable on its domain, with $f'(x) \neq 0$. Then there exists a function f^{-1} defined on its range by $f^{-1}(y) = x$ where $f(x) = y$. Thus, by the chain rule:

$$\begin{aligned}
f^{-1}(f(x)) &= x \\
(f^{-1}(f(x)))' &= (x)' \\
(f^{-1})'(f(x))f'(x) &= 1 \\
(f^{-1})'(f(x)) &= \frac{1}{f'(x)} \\
(f^{-1})'(y) &= \frac{1}{f'(x)}
\end{aligned}$$

where $y = f(x)$. The second to last equality is where we use that $f'(x) \neq 0$.

Problem (5.3.6)**.** For (a), let $g(x) : [0,a] \to \mathbb{R}$ be differentiable, suppose $g(0) = 0$ and $|g'(x)| \leq M$ on $[0,a]$. Then, for $x \in [0,a]$ by the mean value theorem, there is some $0 \leq c \leq x \leq a$ such that:

$$g'(c) = \frac{g(x)-g(0)}{x-0} = \frac{g(x)}{x}$$

and hence $|g(x)| = |g'(c)x| \leq Mx$.

For (b), suppose that $h(x) : [0,a] \to \mathbb{R}$ is twice differentiable with $h'(0) = h(0) = 0$ and $|h''(x)| \leq M$ on $[0,a]$. If we try to apply the technique of (a), twice, we will get that $h(c) \leq Mx^2$, so we will try something different. Consider $f_1(x) = h(x) - M\frac{x^2}{2}$. Then, $f''(x) = h''(x) - M \leq 0$ on $[0,a]$ and hence by applying the mean value theorem twice, we see that $f_1(x) \leq 0$ on $[0,a]$ and hence $h(x) \leq M\frac{x^2}{2}$ on $[0,a]$. Similarly, if we let $f_2(x) = h(x) + M\frac{x^2}{2}$, then we will find that $f_2(x) \geq 0$ so that $h(x) \geq -M\frac{x^2}{2}$ on $[0,a]$ and hence $|h(x)| \leq M\frac{x^2}{2}$ on $[0,a]$

For (c), we can show that if $l(x) : [0,a] \to \mathbb{R}$ is differentiable n times with $l^{(n)}(0) = \cdots = l'(0) = 0$ and $|l^{(n)}(x)| \leq M$ on $[0,a]$, then $|l(x)| \leq M\frac{x^n}{n!}$. Then, by applying the mean value theorem n times to $f_1(x) = l(x) - M\frac{x^n}{n!}$ and $f_2(x) = l(x) + M^n\frac{x^n}{n!}$ we can show that $f_1(x) \leq 0$ and $f_2(x) \geq 0$ on $[0,a]$, since we have $f_1^{(n)}(x) = l^{(n)}(x) - M \leq 0$ and $f_2^{(n)}(x) = l^{(n)}(x) + M \geq 0$. Thus, we have $l(x) \leq M\frac{x^n}{n!}$ and $l(x) \geq -M\frac{x^n}{n!}$ and hence $|l(x)| \leq M\frac{x^n}{n!}$. (Note that we need the $\frac{1}{n!}$ to cancel all of the factors introduced by taking the n derivatives of x^n.)

Problem (5.3.10)**.** First, of all we have:

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} e^{-\frac{1}{x^2}} = 0$$

since $\frac{1}{x^2} \to \infty$ as $x \to 0$ and $e^{-x} \to 0$ as $x \to \infty$. Next, we have:

$$\lim_{x\to 0}|f(x)| = \lim_{x\to 0}|x|\left|\sin\left(\frac{1}{x^4}\right)\right|\left|e^{-\frac{1}{x^2}}\right| \leq \lim_{x\to 0}|xg(x)| = 0$$

since $g(x) \to 0$ as $x \to 0$. Since we have $|f(x)| \to 0$ as $x \to 0$, we must have that $f(x) \to 0$ as $x \to 0$. Thus both $f(x)$ and $g(x)$ tend to 0 as x tends to 0. Now, consider the quotient:

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} x \sin\left(\frac{1}{x^4}\right) = 0$$

since $-x \leq x \sin\left(\frac{1}{x^4}\right) \leq x$, which allows us to use the squeeze theorem. Finally, consider the quotient of derivatives:

$$\begin{aligned}
\lim_{x \to 0} \frac{f'(x)}{g'(x)} &= \lim_{x \to 0} \frac{\sin\left(\frac{1}{x^4}\right) e^{-\frac{1}{x^2}} - 4x^{-4} \cos\left(\frac{1}{x^4}\right) e^{-\frac{1}{x^2}} + 2x^{-2} \sin\left(\frac{1}{x^4}\right) e^{-\frac{1}{x^2}} +}{2x^{-3} e^{-\frac{1}{x^2}}} \\
&= \lim_{x \to 0} \frac{\sin\left(\frac{1}{x^4}\right) - 4x^{-4} \cos\left(\frac{1}{x^4}\right) + 2x^{-2} \sin\left(\frac{1}{x^4}\right)}{2x^{-3}} \\
&= \frac{1}{2} \lim_{x \to 0} x^3 \sin\left(\frac{1}{x^4}\right) - 4x^{-1} \cos\left(\frac{1}{x^4}\right) + 2x \sin\left(\frac{1}{x^4}\right) \\
&= -2 \lim_{x \to 0} x^{-1} \cos\left(\frac{1}{x^4}\right)
\end{aligned}$$

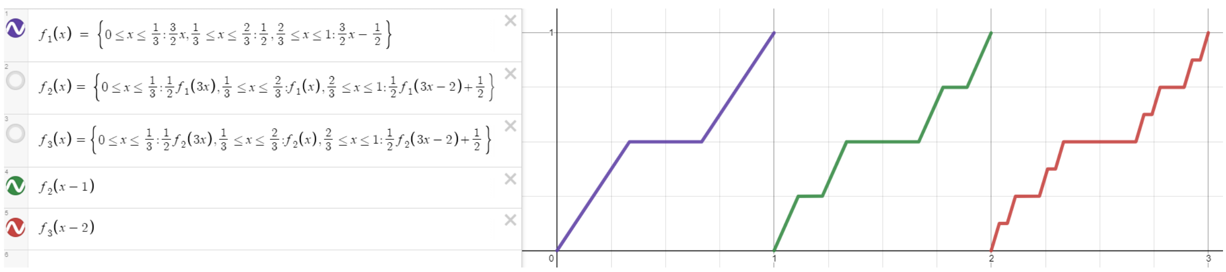
where the second to last equality follows by two applications of the squeeze theorem. Now this final limit diverges, since we can always find some small $x > 0$ such that $\frac{1}{x^4} = 2k\pi$ or $(2k+1)\pi$. This allows us to find sequences $(x_n) \to 0$ such that $\left(x_n^{-1} \cos\left(\frac{1}{x_n^4}\right)\right) \to \infty$ or $-\infty$. This does not contradict L'Hospital's rule, since the limit of the quotient of derivatives *does not* converge, so we cannot apply it.

Problem (5.4.2)**.** First of all, we have $0 \leq h(2^n x) \leq 1$, and thus we have $0 \leq \frac{1}{2^n} h(2^n x) \leq \frac{1}{2^n}$. Furthermore, we have:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2$$

Thus, by the comparison test, $\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$ also converges.

Problem (6.2.12)**.** For (a), we have the graphs of f_1, f_2, and f_3 below (spaced out so we can see all 3 without overlap):



Indeed, we observe that each is continuous, increasing and constant on $[0,1] \setminus C_n$.

For (b), we can continue this process to obtain a sequence (f_n). Now, we want to use the Cauchy criterion for uniform convergence. First of all, if $x \in \left[\frac{1}{3}, \frac{2}{3}\right]$:

$$|f_n(x) - f_m(x)| = 0$$

Now, if $x \in \left[0, \frac{1}{3}\right]$, then:

$$|f_n(x) - f_m(x)| = \frac{1}{2}|f_{n-1}(3x) - f_{m-1}(3x)| \leq \frac{1}{2}|f_{n-1}(x) - f_{m-1}(x)|$$

where the last inequality follows since each f_n is increasing. Thus, we have:

$$|f_n(x)-f_m(x)| \leq \frac{1}{2}|f_{n-1}(x)-f_{m-1}(x)| \leq \frac{1}{2^2}|f_{n-2}(x)-f_{m-2}(x)| \leq \cdots \leq \frac{1}{2^m}|f_{n-m}(x)-f_0(x)| \leq \frac{1}{2^m}$$

where this final inequality holds since $f_0(x) = x \leq 1$ and each $0 \leq f_n \leq 1$. Since, $\frac{1}{2^m} \to 0$, we have by the Cuachy criterion for uniform convergence that (f_n) converges uniformly.

For (c), let $f = \lim f_n$. Since each f_n is continuous, by theorem 6.2.6, f is also continuous. Now, consider $x, y \in [0,1]$ with $x \leq y$. Then, since f_n is increasing, we have $f_n(x) \leq f_n(y)$. Taking a limit as $n \to \infty$, we obtain that $f(x) \leq f(y)$, so that f is also increasing. Since $f_n(0)$ and $f_1(1) = 1$ for all n, we have $f(0) = 0$ and $f(1) = 1$.

Now, consider $x \in [0,1] \setminus C$. Then, since $C_n \subset C_{n+1}$, there must be some N such that $x \notin C_n$ for all $n \geq N$. Then, x is in some open interval of length $\frac{1}{3^N}$ on which f_n is constant for $n > N$. In particular, f is also constant on this interval and hence has $f'(x) = 0$. (Note that to say something about the derivative at x we need to know what is going on in a small neighborhood around x.) Since x was arbitrary, $f'(x) = 0$ on $[0,1] \setminus C$.

Problem (6.2.15)**.** Let (f_n) be a bounded, equicontinuous sequence on $[0,1]$. For (a), using exercise 6.2.13, since $Q \cap [0,1]$ is countable and (f_n) is bounded, we can produce a subsequence f_{n_k} that converges on $Q \cap [0,1]$. Denote this subsequence by g_k.

For (b), since (f_n) is equicontinuous, so is (g_k). Thus, there exists a $\delta > 0$ such that $|x - y| < \delta$ implies that $|g_k(x) - g_k(y)| < \frac{\varepsilon}{3}$.

Now, since $[0,1]$ is compact it can be covered in a finite number of δ-neighborhoods centered at rational points $r_1, \ldots, r_m$. Thus, since (g_k) converges to g, we can find a single N (by taking a maximum over the finite number δ-neighborhoods) such that for $s, t \geq N$ we have $|g_s(r_i) - g_t(r_i)| < \frac{\varepsilon}{3}$ for all i. If the number of points weren't finite the maximum might not exist.

For (c), for each $x \in [0,1]$, we must have that x is in some δ-neighborhood of one of the r_i and hence for $s, t \geq N$, we have:

$$\begin{aligned}
|g_s(x) - g_t(x)| &= |g_s(x) - g_s(r_i) + g_s(r_i) - g_t(r_i) + g_t(r_i) - g_t(x)| \\
&\leq |g_s(x) - g_t(r_i)| + |g_t(r_i) - s(r_i)| + |g_t(r_i) - g_t(x)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

Thus, (g_k) converges uniformly on $[0,1]$, by the Cauchy criterion.

Problem (6.3.4)**.** For any x, we have:

$$|h_n(x)| \leq \frac{1}{\sqrt{n}} \to 0$$

and hence $h_n(x) \to 0$ regardless of x, so that the convergence is uniform. On the other hand, consider the derivatives:

$$h_n'(x) = n\frac{\cos(nx)}{\sqrt{n}} = \sqrt{n}\cos(nx)$$

Suppose to reach a contradiction that $\cos(nx) = 0$ for all $n \geq N$, since otherwise the sequence will be unbounded because of the $\sqrt{n}$. In particular, $\cos(Nx) = 0$. However, this implies that $\cos(2Nx) \neq 0$, which is a contradiction. Thus, for any x, we can find arbitrarily large n for which $\cos(nx) \neq 0$ and hence $(h'(x)) = ((\sqrt{n}\cos(nx))$ is unbounded.

Problem (6.3.7)**.** Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a,b]$, and assume (f'_n) converges uniformly on $[a,b]$. Suppose there exists $x_0 \in [a,b]$ where $f_n(x_0)$ is convergent. Now, since $f_n(x) - f_m(x)$ is differentiable, we can apply the mean value theorem to find some c between x_0 and x such that:

$$f'_n(c) - f'_m(c) = \frac{f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))}{x - x_0}$$

Now, since the derivatives converge uniformly, we can choose N_1 such that for $n, m \geq N_1$ we have that $|f'_n(c) - f'_m(c)| < \min\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2(b-a)}\}$ and hence:

$$|f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| = |x - x_0||f'_n(c) - f'_m(c)| \leq |b-a||f'_n(c) - f'_m(c)| < \frac{\varepsilon}{2}$$

Now, since (f_n) converges at x_0, can choose N_2 such that for $n, m \geq N_2$ $|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$. Thus, for any $x \in [a,b]$ and $n, m \geq \max N_1, N_2$, we have:

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, (f_n) converges uniformly on $[a,b]$.

Problem (6.5.5)**.** For (a), we have:

$$\lim_{n\to\infty} \frac{(n+1)s^n}{ns^{n-1}} = \lim_{n\to\infty} s\frac{n+1}{n} = s < 1$$

Thus, by the ratio test, $\sum_{n=0}^{\infty} ns^{n-1}$ converges. Since this series converges, we must have $ns^{n-1} \to 0$ and hence this sequence must be bounded.

For (b), suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R,R)$ and pick t such that $|x| < t < R$. Then, we will show that the differentiated series converges absolutely at t and hence converges on $[-t,t]$ for any such t, which implies that it converges on $(-R,R)$. Since $\frac{|x|}{t} < 1$, (a) says that $n\left(\frac{|x|}{t}\right)^{n-1} < M$ for some M:

$$\sum_{n=1}^{\infty} |na_n x^{n-1}| = \sum_{n=1}^{\infty} (t^{n-1}|a_n|)\left(n\left(\frac{|x|}{t}\right)^{n-1}\right) \leq M\sum_{n=1}^{\infty} t^{n-1}|a_n| = \frac{M}{t}\sum_{n=1}^{\infty} t^n|a_n|$$

Now, since $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R,R)$ it converges absolutely on $[-t,t]$ by theorem 6.5.1 and theorem 6.5.2. Thus, $\sum_{n=1}^{\infty} t^n|a_n|$ converges so that by the comparison test, $\sum_{n=1}^{\infty} |na_n x^{n-1}|$ converges, so we are done.

Problem (6.5.8b)**.** Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R,R)$ and $f'(x) = f(x)$ for all $x \in (-R,R)$ and $f(0) = 1$. Then, since $f'(x) = f(x)$, differentiating term by term, we have:

$$\sum_{n=0}^{\infty} a_n x^n = f(x) = f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$$

So, by the uniqueness of power series representations (i.e. 6.5.8a), we have:

$$a_{n+1} = \frac{a_n}{n+1}$$

Now, since $f(0) = 1$, we have $a_0 = 1$, so that:

$$a_n = \frac{a_{n-1}}{n} = \frac{a_{n-2}}{n(n-1)} = \ldots = \frac{a_0}{n!} = \frac{1}{n!}$$

What function has this special property that $f'(x) = f(x)$?

Problem (6.6.5). For (a), since $f^{(n)}(x) = e^x$, we have that the Taylor coefficients are:

$$\frac{f^{(n)}(0)}{n!} = \frac{e^0}{n!} = \frac{1}{n!}$$

Now, let $R > 0$ and consider the series $\sum_{n=0}^{\infty} \frac{R^n}{n!}$. Then, we have:

$$\lim_{n\to\infty} \left| \frac{\frac{R^{n+1}}{(n+1)!}}{\frac{R^n}{n!}} \right| = \lim_{n\to\infty} \frac{R}{n+1} = 0 < 1$$

Thus, by the ratio test, this series converges. Moreover, the convergence is absolute (since the terms are positive) so that by theorem 6.5.2, the Taylor series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges on the closed interval $[-R, R]$.

For (b), we differentiate term by to term to compute $f'(x)$, as follows:

$$f'(x) = \sum_{n=1}^{\infty} n\frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

For (c), we have:

$$e^{-x} = f(-x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Then, we have:

$$\begin{aligned}
e^x e^{-x} &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right) \\
&= 1 + (x - x) + \left(\frac{x^2}{2} + x(-x) + \frac{(-x)^2}{2}\right) + \left(\frac{x^3}{6} + \frac{x^2}{2}(-x) + x\frac{(-x)^2}{2} + \frac{(-x)^3}{6}\right) + \cdots \\
&= 1 + (x - x) + (x^2 - x^2) + \left(\frac{x^3}{6} - \frac{x^3}{2} + \frac{x^3}{2} - \frac{x^3}{6}\right) + \cdots \\
&= 1
\end{aligned}$$