

# Lecture #24

MA 511, Introduction to Analysis

July 1, 2021

# Euler's Sum

- We have seen but never proven that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

- The prof in the book relies on expressing  $\sin(x)$  as an infinite sum AND an infinite product

$$\begin{aligned}\sin(x) &= - \sum_{n=1}^{\infty} \frac{(-x)^n}{(2n-1)!} \\ &= x \prod_{n=1}^{\infty} \left(1 - \frac{x}{n\pi}\right) \left(1 + \frac{x}{n\pi}\right)\end{aligned}$$

- There are dozens of proofs using all sorts of analytic techniques. Some are purely based on Taylor series and convergence properties while others use multivariable calculus
- 3blue1brown has an interesting informal explanation based around a geometric construction and the inverse square law for light intensity

# Wallis's Product

- Wallis's product comes from the special case of the  $\sin(x)$  product formula at  $\frac{\pi}{2}$

$$\prod_{n=1}^{\infty} \left( \frac{4n^2}{4n^2 - 1} \right) = \frac{\pi}{2}$$

- We define  $b_n = \int_0^{\frac{\pi}{2}} \sin^n(x) dx$  and use integration by parts and induction to prove that

$$b_0 = \frac{\pi}{2}$$

$$b_1 = 1$$

$$b_n = \frac{n-1}{n} b_{n-2}$$

# Wallis's Product (Continued)

- Some algebra/induction shows that

$$b_n = \begin{cases} b_0 \prod_{k=1}^{\frac{n}{2}} \left( \frac{k-1}{k} \right) & \text{if } n \text{ even} \\ b_1 \prod_{k=1}^{\frac{n-1}{2}} \left( \frac{k-1}{k} \right) & \text{if } n \text{ odd} \end{cases}$$

- If we look at  $\lim_{n \rightarrow \infty} \frac{b_{2n}}{b_{2n+1}}$ , some algebra will reveal that the limit is 1
- Algebra will also reveal that  $\frac{b_0}{b_1} \lim_{n \rightarrow \infty} \frac{b_{2n}}{b_{2n+1}}$  is the Wallis product which must therefore have value  $\frac{\pi}{2}$

# The $\arcsin(x)$ Formula

- We will also need the Taylor series for  $\arcsin(x)$  for this proof.
- $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$  so if we can find the Taylor series for  $\frac{1}{\sqrt{1-x^2}}$  we can apply term by term integration to find the series for  $\arcsin(x)$
- We can inductively show that the coefficients are  $a_0 = 1$  and  $a_n = \frac{(2n)!}{2^{2n}(n!)^2}$
- Some algebra can relate these terms to the Wallis product again, giving the following limit

$$\lim_{n \rightarrow \infty} \frac{1}{a_n \sqrt{n}} = \sqrt{\pi}$$

- We need to show that this converges on  $(-1, 1)$ , but our usual versions of the remainder theorem do not work. We will need a different version

# The arcsin(x) Formula (Continued)

## Theorem (Integral Form of the Remainder Theorem)

Let  $f$  be  $N + 1$  times continuously differentiable on  $(-R, R)$  and  $E_N(x)$  be the usual Taylor series error function. Then

$$E_N(x) = \frac{1}{N!} \int_0^x f^{(N+1)}(t)(x-t)^N dt$$

- The book has a description of how to prove this that you can look at if you are interested
- We claim that this form will let us show convergence of the formula for  $\frac{1}{\sqrt{1-x^2}}$  on  $(-1, 1)$
- With that being the case, we can conclude that for  $-1 < x < 1$

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} x^{2n+1}$$

# Evaluating the Sum

- $\theta = \arcsin(\sin(\theta))$  for  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- Using out series for  $\arcsin(x)$  we get

$$\theta = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} \sin^{2n+1}(\theta)$$

- We can related this back to our work on the Wallis product by integrating over  $[0, \frac{\pi}{2}]$

$$\int_0^{\frac{\pi}{2}} \theta d\theta = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} b_{2n+1}$$

## Evaluating the Sum (Continued)

- After some cancellation and some integration, we arrive at the formula

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

- From this and some clever algebra, we get our result

$$\begin{aligned}\frac{1}{(2k)^2} &= \frac{1}{4} \frac{1}{k^2} \\ \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\ \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{4}{3} \frac{\pi^2}{8} = \frac{\pi^2}{6}\end{aligned}$$



# The Riemann Zeta Function

- We know that  $\sum_{n=1}^{\infty} \frac{1}{n^s}$  converges for  $s \geq 2$  by the comparison test and that it diverges for  $s = 1$
- It seems natural to ask what  $\sum_{n=1}^{\infty} \frac{1}{n^s}$  looks like as a function of  $s$
- This is the beginning of the definition of the Riemann Zeta function
- Some complex analysis results show that  $\sum_{n=1}^{\infty} \frac{1}{n^s}$  converges for  $\Re(s) > 1$  so we can define  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$  on this domain
- We can use complex analytic continuation (expressing this function as a power series) we expand the domain to be  $\mathbb{C} \setminus \{1\}$
- $\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ is prime}} \left( \frac{1}{1-p^{-s}} \right)$  so the behavior of this function somehow encodes information about the behavior of prime numbers
- The Riemann Hypothesis states that  $\zeta(s) = 0$  only if  $s = -2n$  or if  $\Re s = \frac{1}{2}$ . If you can prove this, you get a million dollars and your name in every analysis and number theory textbook written for the rest of time