

# Lecture #5

MA 511, Introduction to Analysis

June 1, 2021

# Convergence of a series

## Definition

Let  $(b_n)$  be a sequence. An **infinite series** is a formal expression of the form:

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + \cdots$$

We define the corresponding **sequence of partial sums**  $(s_m)$  to be:

$$s_m = \sum_{n=1}^m b_n = b_1 + b_2 + \cdots + b_m$$

We say that the series  $\sum_{n=1}^{\infty} b_n$  **converges** to  $B$  if the sequence  $(s_m)$  converges to  $B$ . In this case, we write  $\sum_{n=1}^{\infty} b_n = B$ .

Example:  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges by MCT although we cannot yet find what it converges to.

# Cauchy condensation test

Example: The “harmonic series”  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges. The argument for why it diverges can be generalized to a large class of infinite series, as follows.

## Theorem (Cauchy condensation test)

Suppose  $(b_n)$  is decreasing and satisfies  $b_n \geq 0$  for all  $n \in \mathbb{N}$ . Then, the series  $\sum_{n=1}^{\infty} b_n$  converges if and only if the series

$$\sum_{n=1}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + \cdots$$

converges.

## Corollary

The series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if and only if  $p > 1$ .

# Subsequences and convergence

## Definition

Let  $(a_n)$  be a sequence of real numbers, and let  $n_1 < n_2 < n_3 < n_4 < \dots$  be an increasing sequence of natural numbers. Then the sequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \dots)$$

is called a **subsequence** of  $(a_n)$  and is denoted by  $(a_{n_k})$ , where  $k \in \mathbb{N}$  indexes the sequence.

## Theorem

*Subsequences of a convergent sequence converge to the same limit as the original sequence.*

## Corollary

*For a real number  $b \in \mathbb{R}$ ,  $(b^n) \rightarrow 0$  if and only if  $-1 < b < 1$*

# Bolzano-Weierstrass theorem

- The contrapositive of the previous theorem can serve as a useful criterion for the divergence of a sequence.

## Corollary (Divergence criterion)

*If two subsequences converge to two different limits, then the original sequence diverges.*

Example: For  $(1, -1, 1, -1, 1, -1, \dots)$ , the even indexed terms converge to  $-1$  and the odd indexed terms to  $1$ , so the sequence diverges.

- When can we find a convergent subsequence? The Bolzano-Weierstrass theorem offers a partial answer.

## Theorem (Bolzano-Weierstrass theorem)

*Every bounded sequence contains a convergent subsequence.*

# The Cauchy criterion

## Definition

A sequence  $(a_n)$  is called a **Cauchy sequence** if, for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that whenever  $m, n \geq N$  it follows that  $|a_n - a_m| < \varepsilon$ .

## Theorem

*Every convergent sequence is a Cauchy sequence.*

## Lemma

*Cauchy sequences are bounded.*

## Theorem (Cauchy criterion)

*A sequence converges if and only if it is a Cauchy sequence.*

# Completeness of $\mathbb{R}$ revisited

- So, far we have proved the following implications:

$$\text{AoC} \Rightarrow \begin{cases} \text{NIP} \Rightarrow \text{BW} \Rightarrow \text{CC} \\ \text{MCT} \end{cases}$$

- This is not the whole story! It turns out that:

$$\text{AoC} \Leftrightarrow \text{MCT} \Leftrightarrow \text{BW}$$

- Neither NIP nor CC are enough to prove the Archimedean property, but if we assume the Archimedean property, then we have:

$$\text{AoC} \Leftrightarrow \text{NIP} \Leftrightarrow \text{MCT} \Leftrightarrow \text{BW} \Leftrightarrow \text{CC}$$

- So any of these could serve as our defining axiom of the real numbers! Each asserts the completeness of  $\mathbb{R}$  in its own language.
- Since we have an example of an ordered field that is not complete, i.e.  $\mathbb{Q}$ , it is impossible to prove any of these using only the field and order properties.