Lecture #20

MA 511, Introduction to Analysis

June 24, 2021

Integrating Infinite Discontinuities

Lemma (Exercise 7.3.2)

Thomae's function is integrable on all intervals $[a, b]$ *and* $\int_a^b t = 0$

Lemma

The function $h(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{otherwise} \end{cases}$, *where* C *is the Cantor set, is integrable on* $[0, 1]$ *and* $\int_a^b h = 0$

- Thomae's function is discontinuous on a countable, dense subset of $\mathbb{R}$
- h discontinuous on an uncountable set of points

Measure Theory

- Measure Theory and Lebesgue integration are the main focus of MA 711/higher level Real Analysis courses
- In our definition of integration, we used the lengths of intervals to determine the areas of rectangles approximating the area under curves
- Measure theory is how we define this concept in general
- The Lebesgue measure on $\mathbb{R}$ gives intervals the lengths we expect and is the measure we implicitly use.

Definition

A set A is (Lebesgue) measure zero if, for all $\varepsilon > 0$, A can be covered by a countable collection of intervals, $\{(a_n, b_n)\}_{n=1}^{\infty}$, such that $\sum_{n=1}^{\infty} b_n - a_n \leq \varepsilon$

Measure Zero Sets

Theorem

Any set $A \subset \mathbb{R}$ which is finite or countable has measure zero

Lemma

Any subset of a measure zero set is measure zero

Theorem

The countable union of measure zero sets has measure zero.

- These sets will be crucial for our classification of Riemann integrable functions
- They are also important for the study of Lebesgue integration

α-Continuity Revisited

Definition (α-Continuity)

f is α-continuous at x if there exists $\delta > 0$ such that if, for all $y, z \in V_\delta(x)$, $|f(y) - f(z)| < \alpha$. The set of points where f is not α-continuous is D^α

Definition (Uniform α-Continuity)

f is uniformly α-continuous on $[a, b]$ if there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \alpha$

Theorem

If $\alpha < \alpha'$, then $D^{\alpha'} \subseteq D^\alpha$. The set of points where f is discontinuous is $D = \bigcup_{\alpha \in \mathbb{R}^+} D^\alpha = \bigcup_{n=1}^\infty D^{\frac{1}{n}}$

Theorem

For $\alpha > 0$, D^α is closed.

Lebesgue's Theorem

Theorem (Lebesgue's Theorem)

Let f be a bounded function on $[a, b]$. f is Riemann integrable if and only if the set of discontinuities of f has measure zero

- This limitation is not true for the Lebesgue integral (MA 711 material)

Non-Integrable Derivatives

- We will contruct a function which is differentiable but not continuous on the Cantor set
- We will modify this construction for a "fat" Cantor set so that f' is discontinuous on a set with positive measure

Lemma

The function g defined below is differentiable but g' is discontinuous at 0

$$g(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Non-Integrable Derivatives (Continued)

Definition

Define f_n to be a sequence of functions with the following properties

1. $f_n(x) = 0$ on C_n
2. For every boundary point of C_n, x_k, $f_n(x) = g(l_k(x))$ on a small interval in C_n^c touching x_k where $l_k(x)$ is a linear map taking 0 to x_k and $\delta > 0$ to C_n^c for δ small enough
3. f_n is differentiable and bounded by all functions $(x_k - x)^2$ on the domain it is not yet defined

- We have defined a sequence of functions which are differentiable everywhere but for which f_n' is not continuous at the boundary points of C_n

Non-Integrable Derivatives (Continued)

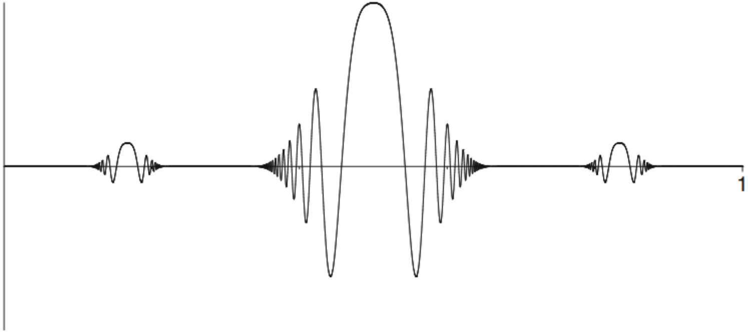

Figure 7.4: A GRAPH OF $f_2(x)$.

Non-Integrable Derivatives (Continued)

Theorem

The function $f = \lim_{n\to\infty} f_n$ is differentiable everywhere but f' is discontinuous on C

Corollary

f' is integrable on $[0,1]$ and $\int_0^x f' = f(x)$

Definition (Fat Cantor Set)

Let $\tilde{C}_0 = [0,1]$ and inductively define $\tilde{C}_n$ as follows:

1. Begin with a copy of $\tilde{C}_{n-1}$
2. For each of interval, remove the middle subinterval of length $\frac{1}{3^{n+1}}$

The set $\tilde{C} = \bigcap_{n=0}^{\infty} \tilde{C}_n$ is a fat Cantor set

Non-Integrable Derivatives (Continued)

Definition

Let $\tilde{f}_n$ be constructed in the same way as f_n but using the boundary points of $\tilde{C}_n$ for the points that $\tilde{f}'_n$ is discontinuous. Let $\tilde{f} = \lim_{n\to\infty} \tilde{f}_n$

Lemma

$\tilde{f}$ is differentiable everywhere and $\tilde{f}'$ is discontinuous on $\tilde{C}$. $\tilde{f}'$ is not integrable

Non-Integrable Derivatives (Continued)

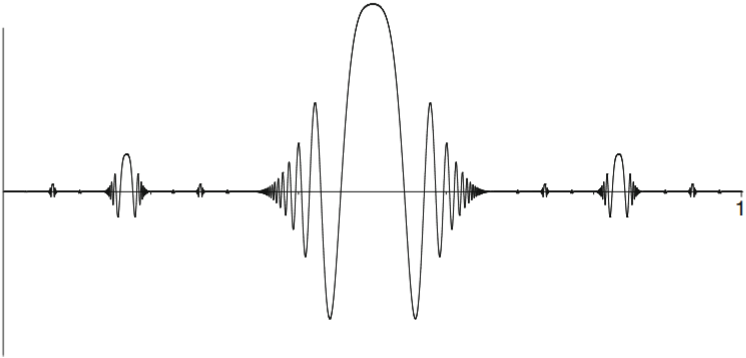

Figure 7.5: A DIFFERENTIABLE FUNCTION WITH A NON-INTEGRABLE DERIVATIVE.

Lesbesgue Integration

- The approach for defining the Lebesgue integral (assuming that we have defined the Lebesgue measure) is as follows:
  1. Define the integral of characteristic functions (1 on A and 0 otherwise) as the measure of A
  2. Define the integral of simple functions (positive linear combinations of characteristic functions) the as the obvious sum
  3. Define the integral of positive functions as the supremum over all simple functions less than or equal to f
  4. Extend to all functions by computing the positive and negative parts separately as positive integrals and then taking the difference
- All Riemann integrable functions are Lebesgue integrable but the reverse is not true
- There are functions which are not Lesbesgue integrable but which can be handled by improper Riemann integrals
- The Generalized Riemann Integral takes integration even further