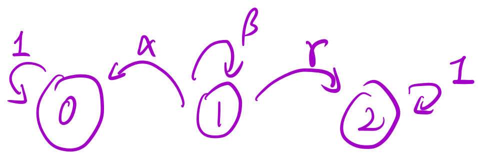
First step analysis

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$



$X_0 = 1$

Q1: prob I reach 0/2 ?

Q2: On average how many step ?

$T = \min\{n : X_n = 0 \text{ or } 2\}$ traslent: not absorbing

$Q_1 \Leftrightarrow \mathbb{P}(X_T = 0 \mid X_0 = 1)$

$Q_2 \Leftrightarrow \mathbb{E}[T \mid X_0 = 1]$ T is ending time. 从0到0的prob是1

Q_1: $u = \mathbb{P}(X_T = 0 \mid X_0 = 1)$ absorbing state

$$= \underbrace{\mathbb{P}(X_T = 0 \mid X_1 = 0)}_{1=} \mathbb{P}(X_1 = 0 \mid X_0 = 1) + \underbrace{\mathbb{P}(X_T = 0 \mid X_1 = 1)}_{=u} \mathbb{P}(X_1 = 1 \mid X_0 = 1)$$
$$+ \underbrace{\mathbb{P}(X_T = 0 \mid X_1 = 2)}_{0=} \mathbb{P}(X_1 = 2 \mid X_0 = 1)$$

$$= 1 \cdot \alpha + u \cdot \beta + 0 \cdot \gamma = \alpha + u\beta$$

$$\Rightarrow u = \frac{\alpha}{1-\beta} = \frac{\alpha}{\alpha+\gamma}$$

Q_2: $V = \mathbb{E}[T \mid X_0 = 1] = \mathbb{E}[T \mid X_1 = 0]\,\mathbb{P}(X_1 = 0 \mid X_0 = 1) +$
$\mathbb{E}[T \mid X_1 = 1]\,\mathbb{P}(X_1 = 1 \mid X_0 = 1) +$
$\mathbb{E}[T \mid X_1 = 2]\,\mathbb{P}(X_1 = 2 \mid X_0 = 1)$

$\mathbb{E}[T \mid X_1 = 2] = \mathbb{E}[T \mid X_1 = 0] = 1$ Take 1 unit of time to change state

$\mathbb{E}[T \mid X_1 = 1] = 1 + V$ We start where we want to reach

相当于已经从 X_0 走到 X_1 的这一步

$$= \alpha + \beta(1+V) + 1\cdot\gamma$$

$$= 1 + \beta V$$

row sum is 1

$$\Rightarrow V = \frac{1}{1-\beta}$$

$T \sim \text{geometric}(1-\beta)$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$T = \min\{n : X_n = 0 \text{ or } 3\}$

Q_1: $u_1 = \mathbb{P}(X_T = 0 \mid X_0 = 1)$

$u_2 = \mathbb{P}(X_T = 0 \mid X_0 = 2)$

Q_2: $V_1 = \mathbb{E}[T \mid X_0 = 1]$

$V_2 = \mathbb{E}[T \mid X_0 = 2]$

Q_1: $u_1 = \mathbb{P}(X_T = 0 \mid X_0 = 1) = \underline{\mathbb{P}(X_T = 0 \mid X_1 = 0) P_{10}} + \mathbb{P}(X_T = 0 \mid X_1 = 1) P_{11}$

$+ \mathbb{P}(X_T = 0 \mid X_1 = 2) P_{12} + \underline{\mathbb{P}(X_T = 0 \mid X_1 = 3)} P_{13}$

$$u_i = \sum_{k=0}^{N} P_{ik} \cdot u_k = u_1 = \sum_{k=0}^{3} P_{1k} \cdot u_k$$

$$= P_{10} + u_1 P_{11} + u_2 P_{12} + 0$$

$u_2 = \mathbb{P}(X_T = 0 \mid X_0 = 2) = \mathbb{P}(X_T = 0 \mid X_1 = 0) P_{20} + \mathbb{P}(X_T = 0 \mid X_1 = 1) P_{21}$

$+ \mathbb{P}(X_T = 0 \mid X_1 = 2) P_{22} + \mathbb{P}(X_T = 0 \mid X_1 = 3) P_{23}$

$$= P_{20} + u_1 P_{21} + u_2 P_{22} + 0$$

$$V_i = 1 + \sum_{k=0}^{N} P_{ik} \cdot V_k =$$

Q_2: $V_1 = \mathbb{E}[T \mid X_0 = 1] = 1 \cdot P_{10} + (1 + V_1) P_{11} + (1 + V_2) P_{12} + 1 \cdot P_{13}$

$$= 1 + V_1 P_{11} + V_2 P_{12}$$

$V_2 = \mathbb{E}[T \mid X_0 = 2] = 1 + V_1 P_{21} + V_2 P_{22}$

Q_3: $\mathbb{P}(X_T = 0)$? Initial distribution, P $\Rightarrow$ Markov Chain

$$M = \left(\mathbb{P}(X_0 = 0) \quad \cdots \quad \mathbb{P}(X_0 = 3) \right)$$

$$= (\,p_0 \quad p_1 \quad p_2 \quad p_3\,)$$

$$P(X_1=1) = P(X_1=1 \mid X_0=0)\,P(X_0=0) + \cdots + P(X_1=1 \mid X_0=3)\,P(X_0=3)$$

$$= \underset{\uparrow}{\mu}\, P$$

row vector times trasition matrix

$$P(X_k=0) = \underset{\uparrow}{\mu}\, P^k$$

Starting state

$$P(X_T=0) = u_1 \cdot \mathbb{P}(X_0=1) + u_2 \cdot \mathbb{P}(X_0=2)$$

stopping time

More than 2 absorbing states

$$P = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \left(\begin{array}{ccc|ccc} p_{00} & p_{01} & p_{02} & p_{03} & p_{04} & p_{05} \\ p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ .2 & .1 & .1 & .1 & .4 & .1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} \qquad \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

transient: 0, 1, 2

absorbing (diagnol): 3, 4, 5

$$T = \min\{n : X_n = 3 \text{ or } 4 \text{ or } 5\}$$

$$= \min\{n : X_n \geq 3\}$$

Q matrix: 写V的时候用到

R matrix: absorbing part

multiple starting state or multiple ending state

e.g. Q:

$$u_{03} = \mathbb{P}(X_T=3 \mid X_0=0) = p_{00}u_{03} + p_{01}u_{13} + p_{02}u_{23} + p_{03}\cdot 1 + 0$$

$$u_{13} = \mathbb{P}(X_T=3 \mid X_0=1) = p_{10}u_{03} + p_{11}u_{13} + p_{12}u_{23} + p_{13}\cdot 1 + 0$$

$$u_{23} = \mathbb{P}(X_T=3 \mid X_0=2) = p_{20}u_{03} + p_{21}u_{13} + p_{22}u_{23} + p_{23}\cdot 1 + 0$$

Q:

$$v_0 = \mathbb{E}[T \mid X_0=0] = 1 + p_{00}v_0 + p_{01}v_1 + p_{02}v_2$$

$$v_1 = \mathbb{E}[T \mid X_0=1] = 1 + p_{10}v_0 + p_{11}v_1 + p_{12}v_2$$

$$v_2 = \mathbb{E}[T \mid X_0=2] = 1 + p_{20}v_0 + p_{21}v_1 + p_{22}v_2$$

$$u_{04} = \mathbb{P}(X_T=4 \mid X_0=0) = p_{00}u_{04} + p_{01}u_{14} + p_{02}u_{24} + p_{03}u_{34} + p_{04}$$

$$v_3 = \mathbb{E}[T \mid X_0=3] = 1 + p_{30}v_0 + p_{31}v_1 + p_{32}v_2$$

General Form for stopping Prob

$X_n \in \{0, 1, 2, \dots, N\}$

$0, 1, 2, 3, \dots, r-1$ transient

$r, r+1, \dots, N$ absorbing

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

$(N-r+1) \times r$ $\begin{pmatrix} 0 & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

$T = \min\{n : X_n \geq r\}$

reach absorbing state and stop

$U_{ik} = \mathbb{P}(X_T = k \mid X_0 = i)$

start from i for $i = 0, \dots, r-1$

end with k $k > r$

$$U_{ik} = \sum_{j=0}^{r-1} P_{ij} U_{jk} + P_{ik}$$

solve some k, need $\begin{cases} U_{0k} \\ U_{1k} \\ \vdots \\ U_{k-1,k} \end{cases}$

$V_i = \mathbb{E}[T \mid X_0 = i]$

$V_i = 1 + \sum_{j=0}^{r-1} P_{ij} V_j \quad i \in \{0, 1, \dots, r-1\}$