MA583 Homework #4

Shi Bo

TOTAL POINTS

18 / 20

QUESTION 1

1 Exercise 5.1.3 **1 / 1**

✓ **- 0 pts** Correct

QUESTION 2

2 Exercise 5.1.9 **1 / 1**

✓ **- 0 pts** Correct

QUESTION 3

3 Problem 5.1.5 **2 / 2**

✓ **- 0 pts** Correct

- 2 pts Incorrect

QUESTION 4

4 Exercise 5.3.2 **1 / 1**

✓ **- 0 pts** Correct

QUESTION 5

5 Problem 5.3.1 **2 / 2**

✓ **- 0 pts** Correct

- 2 pts Incorrect

- 0.5 pts Click here to replace this description.

- 1.5 pts Click here to replace this description.

QUESTION 6

6 Exercise 5.4.2 **1 / 1**

✓ **- 0 pts** Correct

QUESTION 7

7 Problem 5.4.7 **0 / 2**

- 0 pts Correct

✓ **- 2 pts** Incorrect

- 1 pts Click here to replace this description.

QUESTION 8

8 Problem 5.4.9 **2 / 2**

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

- 2 pts Click here to replace this description.

- 1.5 pts Click here to replace this description.

- 0.5 pts Click here to replace this description.

QUESTION 9

9 Exercise 5.5.3 **1 / 1**

✓ **- 0 pts** Correct

QUESTION 10

10 Problem 5.5.5 **2 / 2**

✓ **- 0 pts** Correct

- 0.5 pts Click here to replace this description.

- 1.5 pts Click here to replace this description.

- 2 pts Click here to replace this description.

- 1 pts Click here to replace this description.

QUESTION 11

11 Exercise 5.6.2 **1 / 1**

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

QUESTION 12

12 Exercise 5.6.3 **1 / 1**

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

QUESTION 13

13 **3 / 3**

✓ **- 0 pts** Correct

5.1.3 Let X and Y be independent Poisson distributed random variables with parameters α and β, respectively. Determine the conditional distribution of X, given that $N = X + Y = n$.

$X \sim poi(\alpha)$, $Y \sim poi(\beta)$, $N \sim poi(\alpha+\beta)$

$$\mathbb{P}(X=x \mid N=n) = \mathbb{P}(X=x \mid X+Y=n) = \frac{P(X=x,\ X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=x,\ Y=n-x)}{P(X+Y=n)} = \frac{\mathbb{P}(X=x)\mathbb{P}(Y=n-x)}{P(X+Y=n)}$$

$$= \frac{e^{-\alpha}\frac{\alpha^x}{x!} \cdot e^{-\beta}\frac{\beta^{n-x}}{(n-x)!}}{e^{-(\alpha+\beta)}\frac{(\alpha+\beta)^n}{n!}}$$

$$= e^{-\alpha-\beta+\alpha+\beta}\frac{n!}{x!(n-x)!}\frac{\alpha^x\beta^{n-x}}{(\alpha+\beta)^n}$$

$$= C_n^x \frac{\alpha^x\beta^{n-x}}{(\alpha+\beta)^n} = C_n^x \frac{\alpha^x\ \beta^{n-x}}{(\alpha+\beta)^{n-x}(\alpha+\beta)^x}$$

$$= C_n^x \left(\frac{\alpha}{\alpha+\beta}\right)^x\left(\frac{\beta}{\alpha+\beta}\right)^{n-x}$$

$$= C_n^x \left(\frac{\alpha}{\alpha+\beta}\right)^x\left(1-\frac{\alpha}{\alpha+\beta}\right)^{n-x}$$

$Bin(n, \frac{\alpha}{\alpha+\beta})$

1 Exercise 5.1.3 **1 / 1**

✓ **- 0 pts** Correct

5.1.9 Let $\{X(t);\ t \geq 0\}$ be a Poisson process having rate parameter $\lambda = 2$. Determine the following expectations:

(a) $E[X(2)]$.
(b) $E\left[\{X(1)\}^2\right]$.
(c) $E[X(1)X(2)]$.

(a) $X(t) \sim poi(X(t) = x) = e^{-\lambda t} \frac{(\lambda t)^x}{x!}$

$$E[X(t)] = \sum_{x=0}^{\infty} x P(X=x) = \sum_{x=0}^{\infty} x e^{-\lambda t} \frac{(\lambda t)^x}{x!}$$

plug In $\rightarrow$ $= e^{-4} \sum_{x=0}^{\infty} \frac{4^x x}{x!}$

$t=2,\ \lambda=2$

$$= e^{-4}\left(\frac{4^0 \cdot 0}{0!} + \frac{4^1 \cdot 1}{1!} + \frac{4^2 \cdot 2}{2!} + \frac{4^3 \cdot 3}{3!} + \cdots\right)$$

$$= e^{-4}\left(4 + \frac{4^2}{1!} + \frac{4^3}{2!} + \frac{4^4}{3!} + \cdots\right)$$

$$= e^{-4} \cdot 4e^4$$

$$= 4$$

(b) $E[(X(1))^2] = Var(X(1)) + (E[X(1)])^2$

$$= \lambda + \lambda^2 = 6$$

(c) $E[X(1)X(2)] = E[(X(2) - X(1) + X(1))]X(1)$

$$= E[(X(2) - X(1))X(1)] + E[X(1)^2]$$

$$= E[X(2) - X(1)]\, E[X(1)] + E[(X(1))^2]$$

$$= \lambda \cdot \lambda + 6 = 4 + 6 = 10$$

2 Exercise 5.1.9 **1 / 1**

✓ **- 0 pts** Correct

5.1.5 For each value of $h > 0$, let $X(h)$ have a Poisson distribution with parameter λh. Let $p_k(h) = \Pr\{X(h) = k\}$ for $k = 0, 1, \ldots$. Verify that

$$\lim_{h\to 0} \frac{1 - p_0(h)}{h} = \lambda, \quad \text{or } p_0(h) = 1 - \lambda h + o(h);$$

$$\lim_{h\to 0} \frac{p_1(h)}{h} = \lambda, \quad \text{or } p_1(h) = \lambda h + o(h);$$

$$\lim_{h\to 0} \frac{p_2(h)}{h} = 0, \quad \text{or } p_2(h) = o(h).$$

Here $o(h)$ stands for any remainder term of order less than h as $h \to 0$.

$$P_0(h) = \mathbb{P}(X(h) = 0) = e^{-\lambda h} \frac{(\lambda h)^0}{0!} = e^{-\lambda h}$$

$$\lim_{h\to 0} \frac{1-e^{-\lambda h}}{h} = \lim_{h\to 0} \lambda e^{-\lambda h} = \lambda$$

(洛必达)

$$P_1(h) = \mathbb{P}(X(h) = 1) = e^{-\lambda h} \frac{\lambda h^1}{1!} = \lambda h e^{-\lambda h}$$

$$\lim_{h\to 0} \frac{\lambda h e^{-\lambda h}}{h} = \lambda e^{-\lambda h} = \lambda$$

$$P_2(h) = \mathbb{P}(X(h) = 2) = e^{-\lambda h} \frac{\lambda^2 h^2}{2}$$

$$\lim_{h\to 0} \frac{\frac{1}{2}\lambda^2 h^2 e^{-\lambda h}}{h} = \lim_{h\to 0} \frac{1}{2}\lambda^2 h e^{-\lambda h} = 0$$

3 Problem 5.1.5 **2 / 2**

✓ **- 0 pts** **Correct**

- 2 pts Incorrect

5.3.2 A radioactive source emits particles according to a Poisson process of rate $\lambda = 2$ particles per minute.

(a) What is the probability that the first particle appears some time after 3 min but before 5 min?

(b) What is the probability that exactly one particle is emitted in the interval from 3 to 5 min?

(a) According to the description of the question, exponential dist can be applied:

$$\mathbb{P}(3 < T_1 < 5) = (1 - e^{-\lambda t}) - (1 - e^{-\lambda t})$$

$$= (1 - e^{-2(5)}) - (1 - e^{-2(3)})$$

$$= 1 - e^{-10} - 1 + e^{-6}$$

$$= e^{-6} - e^{-10} \approx 0.0024$$

(b) $\mathbb{P}(X(5) - X(3) = 1) = \mathbb{P}(X(2) = 1)$

$$= e^{-\lambda t} \frac{(\lambda t)^1}{1!}$$

$$= e^{-4} \cdot 4 = 4e^{-4}$$

$$\approx 0.073$$

4 Exercise 5.3.2 **1 / 1**

✓ **- 0 pts** **Correct**

5.3.1 Let $X(t)$ be a Poisson process of rate λ. Validate the identity

$$\{W_1 > w_1, W_2 > w_2\}$$

if and only if

$$\{X(w_1) = 0, X(w_2) - X(w_1) = 0 \text{ or } 1\}.$$

Use this to determine the joint upper tail probability

$$\begin{aligned}\Pr\{W_1 > w_1, W_2 > w_2\} &= \Pr\{X(w_1) = 0, X(w_2) - X(w_1) = 0 \text{ or } 1\} \\ &= e^{-\lambda w_1}[1 + \lambda(w_2 - w_1)]e^{-\lambda(w_2 - w_1)}.\end{aligned}$$

Finally, differentiate twice to obtain the joint density function

$$f(w_1, w_2) = \lambda^2 \exp\{-\lambda w_2\} \quad \text{for } 0 < w_1 < w_2.$$

$\mathbb{P}(W_1 > w_1, W_2 > w_2)$

$= \mathbb{P}(X(w_1) = 0, X(w_2) - X(w_1) = 0) \cup \mathbb{P}(X(w_1) = 0, X(w_2) - X(w_1) = 1)$

they are independent, so,

$= \mathbb{P}(X(w_1) = 0, X(w_2) - X(w_1) = 0) + \mathbb{P}(X(w_1) = 0, X(w_2) - X(w_1) = 1)$

$= \mathbb{P}(X(w_1) = 0)\mathbb{P}(X(w_2) - X(w_1) = 0) + \mathbb{P}(X(w_1) = 0)\mathbb{P}(X(w_2) - X(w_1) = 1)$

$= e^{-\lambda w_1} e^{-\lambda(w_2 - w_1)} + e^{-\lambda w_1} \lambda(w_2 - w_1) e^{-\lambda(w_2 - w_1)}$

$= e^{-\lambda w_1} [1 + \lambda(w_2 - w_1)] e^{-\lambda(w_2 - w_1)}$

$$\frac{\partial \cdot}{\partial w_1 \partial w_2} = \frac{\partial}{\partial w_2}\left[-\lambda e^{-\lambda w_2}\right] = \lambda^2 e^{-\lambda w_2}$$

$\therefore f(w_1, w_2) = \lambda^2 e^{-\lambda w_2}$

5 Problem 5.3.1 **2 / 2**

- ✓ **- 0 pts** **Correct**
- **- 2 pts** Incorrect
- **- 0.5 pts** Click here to replace this description.
- **- 1.5 pts** Click here to replace this description.

5.4.2 Let $\{X(t); t \geq 0\}$ be a Poisson process of rate λ. Suppose it is known that $X(1) = 2$. Determine the mean of $W_1 W_2$, the product of the first two arrival times.

We know that $W_1, \ldots, W_n \mid X(t) = n$ follows the ordered uniform distribution $[0, t]$ from the lecture.

So,

$$\mathbb{E}[W_1 W_2 \mid X(1) = 2]$$

by symmetry,

$$= \mathbb{E}[U_1 U_2]$$

$$= \mathbb{E}[U_1]^2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

6 Exercise 5.4.2 **1 / 1**

✓ **- 0 pts** Correct

5.4.7 Let $W_1, W_2, \ldots$ be the event times in a Poisson process $\{X(t); t \geq 0\}$ of rate λ, and let $f(w)$ be an arbitrary function. Verify that

$$E\left[\sum_{i=1}^{X(t)} f(W_i)\right] = \lambda \int_0^t f(w)\mathrm{d}w.$$

From 5.4.1 Shot noise, we notice that (5.24):

$$I(t) = \sum_{k=1}^{X(t)} h(t-W_k)$$

So,

$$\mathbb{E}\left[\sum_{i=1}^{X(t)} f(W_i)\right] = \mathbb{E}\left[\sum_{k=1}^{X(t)} h(t-W_k)\right]$$

$$= \lambda \int_0^t h(t-W_k)\,d(t-W_k)$$

$$= \lambda \int_0^t f(w)\,dw$$

7 Problem 5.4.7 **0 / 2**

- **- 0 pts** Correct
- ✓ **- 2 pts Incorrect**
- **- 1 pts** Click here to replace this description.

5.4.9 Customers arrive at a service facility according to a Poisson process of rate λ customers per hour. Let $N(t)$ be the number of customers that have arrived up to time t, and let $W_1, W_2, \ldots$ be the successive arrival times of the customers. Determine the expected value of the product of the waiting times up to time t. (Assume that $W_1 W_2 \cdots W_{N(t)} = 1$ when $N(t) = 0$.)

$$\mathbb{E}[W_1 W_2 \cdots W_{N(t)}] = \sum_{n=0}^{\infty} \mathbb{E}[W_1 W_2 \cdots W_{N(t)} \mid N(t) = n]\, \mathbb{P}(N(t) = n)$$

by symmetry

$$= \sum_{n=0}^{\infty} \mathbb{E}[U_1 U_2 \cdots U_n]\, \mathbb{P}(N(t) = n)$$

$$= \sum_{n=0}^{\infty} [\mathbb{E}(U_1)]^n e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\frac{t}{2}\right)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$= e^{-\lambda t(1 - \frac{1}{2}t)}$$

8 Problem 5.4.9 2 / 2

✓ **- 0 pts** **Correct**

- 1 pts Click here to replace this description.

- 2 pts Click here to replace this description.

- 1.5 pts Click here to replace this description.

- 0.5 pts Click here to replace this description.

5.5.3 Defects (air bubbles, contaminants, chips) occur over the surface of a varnished tabletop according to a Poisson process at a mean rate of one defect per top. If two inspectors each check separate halves of a given table, what is the probability that both inspectors find defects?

According to the question, we know that the expected # of defect over half of the top is $\frac{1}{2}$.

So, suppose X_1 = # of defects found by first inspector

X_2 = # of defects found by second inspector

Then, $X_1 \sim poi(\frac{1}{2})$ and $X_2 \sim poi(\frac{1}{2})$

$$\mathbb{P}(\text{both find defects}) = \mathbb{P}(X_1 \neq 0 \text{ and } X_2 \neq 0)$$

$$= (1-\mathbb{P}(X_1=0))(1-\mathbb{P}(X_2=0))$$

$$= \left(1 - \frac{e^{-\frac{1}{2}}(\frac{1}{2})^0}{0!}\right)^2$$

$$= (1-e^{-\frac{1}{2}})^2 \approx 0.1548$$

9 Exercise 5.5.3 **1 / 1**

✓ **- 0 pts** Correct

5.5.5 Consider a two-dimensional Poisson process of particles in the plane with intensity parameter ν. Determine the distribution $F_D(x)$ of the distance between a particle and its nearest neighbor. Compute the mean distance.

Suppose the radius is d, so according to the question, we have

$$F_D(d) = P(D \le x) = 1 - P(D > x)$$

$= 1 - P($no other particles in disk with area πd^2 centered at the particle$)$

$$= 1 - e^{-\nu\pi x^2} \qquad x^2 > 0$$

$$f_D(x) = \frac{d F_D(x)}{dx} = 2x\nu\pi e^{-\nu\pi x^2}$$

$$E[D] = \int_0^\infty x \cdot 2x\nu\pi e^{-\nu\pi x^2} dx$$

$$= 2\pi\nu \int_0^\infty x^2 e^{-\nu\pi x^2} dx$$

let $u = \pi\nu x^2$ $\quad du = 2\pi\nu x\, dx$ $\quad x = \sqrt{\frac{u}{\pi\nu}}$

$$= \int_0^\infty \sqrt{\frac{u}{\pi\nu}}\, e^{-u} du = \frac{1}{\sqrt{\pi\nu}} \int_0^\infty u^{\frac{1}{2}} e^{-u} du$$

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \Rightarrow n = \frac{3}{2}$

$$= \frac{1}{\sqrt{\pi\nu}} \cdot \Gamma\left(\frac{3}{2}\right) = \frac{1}{\sqrt{\pi\nu}} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2\sqrt{\pi\nu}} \cdot \sqrt{\pi} = \frac{1}{2\sqrt{\nu}}$$

10 Problem 5.5.5 **2 / 2**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Click here to replace this description.
- **- 1.5 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.
- **- 1 pts** Click here to replace this description.

5.6.2 Shocks occur to a system according to a Poisson process of intensity λ. Each shock causes some damage to the system, and these damages accumulate. Let $N(t)$ be the number of shocks up to time t, and let Y_i be the damage caused by the ith shock. Then

$$X(t) = Y_1 + \cdots + Y_{N(t)}$$

is the total damage up to time t. Determine the mean and variance of the total damage at time t when the individual shock damages are exponentially distributed with parameter θ.

$N(t) \sim \text{poi}(\lambda)$, $Y_i \sim \exp(\theta)$, $X(t) = Y_1 + Y_2 + \cdots + Y_{N(t)}$

$$= \sum_{k=1}^{N(t)} Y_k$$

$$\mathbb{E}[X(t)] = \mathbb{E}[N(t)]\,\mathbb{E}[Y_1]$$

Random sum of R.V.

$$= \lambda t \cdot \frac{1}{\theta} = \frac{\lambda t}{\theta}$$

$$Var[X(t)] = \mathbb{E}[N(t)]\,Var(Y_1) + (\mathbb{E}[Y_1])^2 \cdot Var(N(t))$$

$$= \lambda t \cdot \frac{1}{\theta^2} + \frac{1}{\theta^2} \cdot \lambda t$$

$$= \frac{2\lambda t}{\theta^2}$$

11 Exercise 5.6.2 **1 / 1**

✓ **- 0 pts** **Correct**

- 1 pts Click here to replace this description.

5.6.3 Let $\{N(t); t \geq 0\}$ be a Poisson process of intensity λ, and let $Y_1, Y_2, \ldots$ be independent and identically distributed nonnegative random variables with cumulative distribution function $G(y) = \Pr\{Y \leq y\}$. Determine $\Pr\{Z(t) > z | N(t) > 0\}$, where

$$Z(t) = \min\{Y_1, Y_2, \ldots, Y_{N(t)}\}.$$

$$\mathbb{P}(Z(t) > z \mid N(t) > 0)$$

$$= \frac{\mathbb{P}(Z(t) > z, N(t) > 0)}{\mathbb{P}(N(t) > 0)}$$

$$= \frac{\sum_{n=1}^{\infty} \mathbb{P}(Z(t) > z \mid N(t) = n)\mathbb{P}(N(t) = n)}{\mathbb{P}(N(t) > 0)}$$

$$\mathbb{P}(Z(t) > z \mid N(t) = n) = \mathbb{P}(\min(y_1, y_2, \ldots, y_n) > z)$$

$$= \mathbb{P}(Y_1 > z)^n = (1 - G(z))^n$$

$\mathbb{P}(N(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$ and $\mathbb{P}(N(t) > 0) = 1 - e^{-\lambda t}$

So, $\mathbb{P}(Z(t) > z \mid N(t) > 0) = \dfrac{\sum_{n=1}^{\infty} (1 - G(z))^n e^{-\lambda t} \frac{(\lambda t)^n}{n!}}{1 - e^{-\lambda t}}$

$$= \frac{e^{-\lambda t}}{1 - e^{-\lambda t}} \sum_{n=1}^{\infty} (1 - G(z))^n (\lambda t)^n$$

$$= \frac{e^{-\lambda t}}{1 - e^{-\lambda t}} \left(e^{\lambda t(1 - G(z))} - 1\right) = \frac{e^{-\lambda G(z) t} - e^{-\lambda t}}{1 - e^{-\lambda t}}$$

12 Exercise 5.6.3 1 / 1

✓ **- 0 pts** Correct

- 1 pts Click here to replace this description.

13 **3 / 3**

✓ **- 0 pts** Correct