$\mu = \mathbb{E}[\xi]$ $\quad \sigma^2 = Var(\xi)$ $\quad X_{n+1} = \sum_{k=1}^{X_n} \xi_k^{(n+1)}$

$\mathbb{E}[X_n] = \mu^n$ $\quad Var(X_n) = \begin{cases} \mu^{n-1} \sigma^2 \frac{1-\mu^n}{1-\mu} & \mu \neq 1 \\ n\sigma^2 & \mu \neq 1 \end{cases}$

$\mu < 1$ expectation decreasing exponentially

Variance dereasing exponentially

$\mu > 1$ ... increasing

... increasing

$\mu = 1$ expectation constant

Extinction probability

① If $\mathbb{P}(\xi = 0) > 0$, can go extinct.

② Once $X_n = 0 \Rightarrow X_{n+1} = 0 = X_{n+2} \dots$,

Q: $\mathbb{P}(X_n = 0) = u_n$

$X_0 = 1$

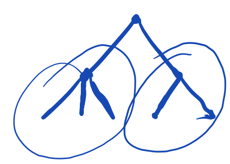
$X_1 = \sum_{k=0}^{X_0} \xi_k^{(1)} = \xi_1^{(1)}$

1st step analysis

$u_n = \mathbb{P}(X_n = 0) = \sum_{k=0}^{\infty} \underbrace{\mathbb{P}(X_n = 0 \mid X_1 = k)} \, \mathbb{P}(X_1 = k)$ $\quad \mathbb{P}(X_1 = k) = \mathbb{P}(\xi = k) = a_k$

$= \mathbb{P}($k independent branching process extinct in $(n-1)$ step$)$

$= (u_{n-1})^k$



$= \sum_{k=0}^{\infty} u_{n-1}^k a_k$

Bacteria die w.p. $\frac{1}{4}$

split w.p $\frac{3}{4}$

$a_0 = \frac{1}{4}$ $\quad a_2 = \frac{3}{4}$ $\quad$ no a_1 here

$u_1 = \mathbb{P}(X_1 = 0) = a_0 = \frac{1}{4}$

$u_2 = \frac{1}{4}(u_1)^0 + \frac{3}{4}(u_1)^2 = \frac{1}{4} + \frac{3}{4} \cdot (\frac{1}{4})^2$

$= (u_1)^0 \underbrace{\mathbb{P}(\xi = 0)}_{a_0} + (u_1)^2 \mathbb{P}(\xi = 2)$

$u_3 = \frac{1}{4} + \frac{3}{4} u_2^2$

$u_8 \approx 0.33 \dots$

$\lim_{n\to\infty} u_n = \frac{1}{3}$

Chapter 3.9 probability generating function

ξ to be some non-negative integer-valued R.V.

$$\phi_\xi(s) = \sum_{k=0}^{\infty} s^k \, \mathbb{P}(\xi=k)$$

u_n u_{n-1}

$$= \mathbb{E}[s^\xi]$$

Momentum generating function

$$M_\xi(t) = \mathbb{E}[e^{t\xi}]$$

$$s = e^t$$

property: ① convergence

$$\phi_\xi(1) = 1$$

$0 \le s \le 1$

$0 \le s^k \le 1$

$$\phi_\xi(k) = \sum \overset{\le 1}{s^k} \, \mathbb{P}(\xi=k) \le \sum \mathbb{P}(\xi=k)$$

$$\phi_\xi(k) \overset{=1}{\le} 1$$

$$\phi_\xi(1) = \sum_{k=0} 1 \cdot \mathbb{P}(\xi=k) = 1 \quad \text{[summing all the pmf]}$$

② Recover a_k from $\phi_\xi(s)$

$$\phi_\xi(0) = \sum_{k=1}^{\infty} 0^k \underbrace{\mathbb{P}(\xi=k)}_{a_k} + a_0 = a_0$$

$$s=0 \begin{cases} \phi_\xi'(0) = a_1 \\ \phi_\xi''(0) = 2a_2 \\ \phi_\xi^{(n)}(0) = n!\,a_n \end{cases}$$

$$\frac{d\phi_\xi(s)}{ds} = \phi_\xi'(s) = \sum_{k=1}^{\infty} k a_k s^{k-1}$$

③ $s=1$

$$\phi_\xi(1) = 1$$

$$\phi_\xi'(1) = \mathbb{E}[\xi]$$

$$\phi_\xi''(1) = \mathbb{E}[\xi(\xi-1)]$$

$$\phi_\xi^{(n)}(1) = \mathbb{E}[\xi\cdot(\xi-1)\cdot\cdots\cdot(\xi-n+1)]$$

$$Var(\xi) = \mathbb{E}[\xi^2] - (\mathbb{E}[\xi])^2$$

$$= \underbrace{\phi_\xi''(1)}_{\mathbb{E}[\xi^2-\xi]} + \underbrace{\phi_\xi'(1)}_{\mathbb{E}[\xi]} - [\phi_\xi'(1)]^2$$

e.g. ξ R.V

$$\phi_\xi(s) = \sum_{k=0}^{\infty} s^k a_k$$

$\xi \sim Bin(n,p)$

$$\mathbb{P}(\xi = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\phi_\xi(s) = \sum_{k=0}^{n} s^k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (ps)^k (1-p)^{n-k} = (ps+1-p)^n$$

④ Sum of independent variables.

If ξ, η independent

$$\phi_{\xi+\eta}(s) = \mathbb{E}[s^{\xi+\eta}] = \mathbb{E}[s^\xi \cdot s^\eta]$$

by independence $= \mathbb{E}[s^\xi]\,\mathbb{E}[s^\eta]$

$$= \phi_\xi(s)\,\phi_\eta(s)$$

If $\xi_1, \dots, \xi_n$ i.i.d

$$\phi_{\xi_1+\dots+\xi_n} = [\phi_{\xi_1}(s)]^n$$

☆ Random sum of R.V

N be R.V $\quad \xi_1$ be R.V and ind of N

$$g_N(s) = \mathbb{E}(s^N)$$

$$\phi_\xi(s) = \mathbb{E}[s^\xi]$$

e.g. $X = \sum_{k=1}^{N} \xi_k$

$$\mathbb{E}[s^X] = \mathbb{E}(s^{\sum_{k=1}^{N}\xi_k})$$

by total law of $\mathbb{E}$

$$= \sum_{n=0}^{\infty} \mathbb{E}[s^{\sum_{k=1}^{N}\xi_k} \mid N=n]\,\mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} \mathbb{E}[s^{\sum_{k=1}^{N}\xi_k} \mid N=n]\,\mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} \underbrace{\mathbb{E}[s^{\sum_{k=1}^{n}\xi_k}]}_{= s^{\xi_1}s^{\xi_2}\cdots s^{\xi_n}}\mathbb{P}(N=n)$$

since iid

$$= \sum_{n=0}^{\infty} [\mathbb{E}(s^{\xi_1})]^n\,\mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} [\phi_\xi(s)]^n\,\mathbb{P}(N=n)$$

$$= g_N(\phi_\xi(s))$$

$X_0 = 1 \qquad X_1 = \xi_1^{(1)}$

$$X_{n+1} = \sum_{k=1}^{X_n} \xi_k^{(k+1)}$$

$$\phi_1(s) = \mathbb{E}[s^{X_1}] = \mathbb{E}(s^{\xi_1^{(1)}}) = \phi_\xi(s)$$

$$\phi_2(s) = \mathbb{E}[s^{X_2}] = \mathbb{E}[s^{\sum_{k=1}^{X_1}\xi_k^{(2)}}] = \phi_1(\phi_1(s))$$

$\phi_{X_1}(\phi_\xi(s))$

$g_N \leftarrow$ generating func for X_1

$$\phi_3(s) = \phi_2(\phi_1(s)) = \phi_1(\phi_1(\phi_1(s)))$$

$$\phi_n(s) = \phi_{n-1}(\phi_1(s))$$

$$\phi_n(s) = \mathbb{E}[s^{X_n}] = \sum_{k=0}^{\infty} s^k\,\mathbb{P}(N=n)$$

$$u_n = \mathbb{P}(X_n = 0) = \phi_n(0)$$

$$= \phi_1(\phi_1(\cdots\phi_1(0)))$$ n times

☆ $u_n = \phi_1(u_{n-1})$

$$u_\infty = \lim_{n\to\infty} u_n$$

e.g. $a_0 = \frac{1}{4} \qquad a_2 = \frac{3}{4}$

$\phi(s) = \frac{1}{4} + \frac{3}{4}s^2 \ll$

$$\phi(s) = \sum_{k=0}^{\infty} s^k\,\mathbb{P}(N=n) = \sum_{k=0}^{2} s^k\,\mathbb{P}(N=n)$$

$$= s^0\,\mathbb{P}(N=n) + s^1\,\mathbb{P}(N=n) + s^2\,\mathbb{P}(N=n)$$

$u_0 = 0$

$u_1 = \phi(u_0) = \frac{1}{4}$

$u_2 = \phi(u_1) = \frac{1}{4} + \frac{3}{4}(\frac{1}{4})^2$

$= 1 \cdot \mathbb{P}(\xi=0) + s^2 \mathbb{P}(\xi=2)$

$= \frac{1}{4} + \frac{3}{4}s^2$

if $u_\infty = \lim_{n\to\infty} u_n$ exists, $u_\infty = \phi(u_\infty) = u$

$u = \frac{1}{4} + \frac{3}{4}u^2 \Rightarrow u_\infty = \frac{1}{3}$ or $\cancel{u=1}$

If $a_0 = \frac{3}{4}$ $a_2 = \frac{1}{4}$ $\phi(1) = 1$

$\phi(s) = \frac{3}{4} + \frac{1}{4}s^2 \Rightarrow \cancel{u_\infty = 3}$ or $\boxed{u_\infty = 1}$

find u_∞

$u = \phi(u)$

$1 = \phi(1) = \sum 1^k a_k = 1$ 1 will always be the solution

ϕ increasing in $[0,1]$

$\phi'(s) = a_1 + 2a_2 s + 3a_3 s^2 + \cdots > 0$

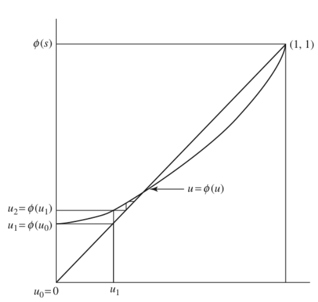
ϕ is convex since $\phi''(s) \geq 0$

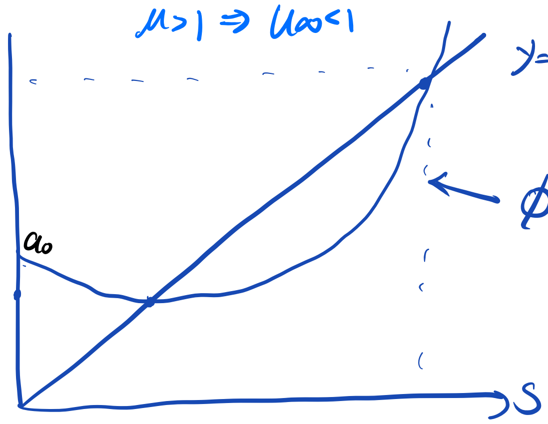
also, $\phi(0) = a_0$.

Therefore,

$\phi'(1) > 1$

$\mu > 1 \Rightarrow u_\infty < 1$



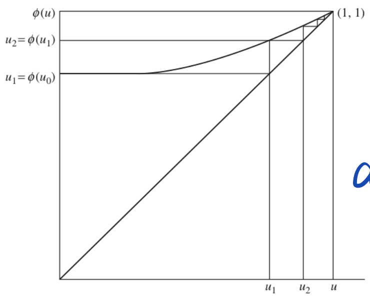


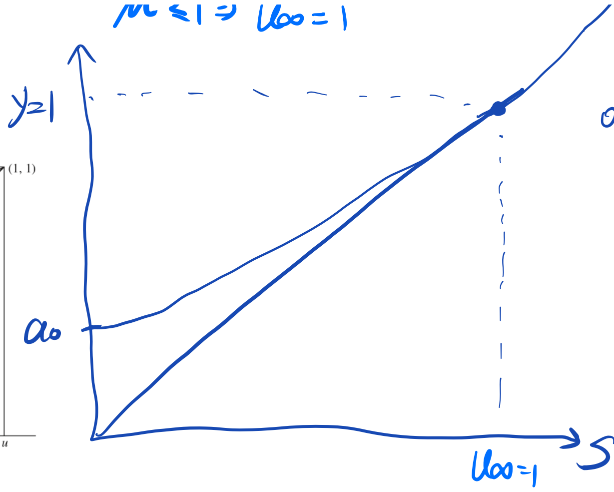
$\phi''(s) > 0$

$\phi'(1) \leq 1$ $\phi'(s) < 1$ $\forall s \in (0,1)$ /

first ↗

$\mu \le 1 \Rightarrow u_\infty = 1$





only 1 intersection

$u_\infty = 1$

Chose which ?

$\mu = \mathbb{E}[\xi] = \phi'(1)$