3.5.7 Consider the random walk Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0.3 & 0 & 0.7 & 0 \\ 2 & 0 & 0.3 & 0 & 0.7 \\ 3 & 0 & 0 & 0 & 1 \end{array}.$$

Starting in state 1, determine the probability that the process is absorbed into state 0. Do this first using the basic first step approach of equations (3.21) and (3.22) and second using the particular results for a random walk given in equation (3.42).

① $T = \{n : X_n = 0 \text{ or } 3\}$

$$u_1 = \mathbb{P}(X_T = 0 \mid X_0 = 1) = P_{10} + P_{11}u_1 + P_{12}u_2$$

$$u_2 = \mathbb{P}(X_T = 0 \mid X_0 = 2) = P_{20} + P_{21}u_1 + P_{22}u_2$$

$$\begin{cases} u_1 = 0.3 + 0.7u_2 \\ u_2 = 0.3u_1 \end{cases}$$

$$u_1 = 0.3 + 0.21u_1$$

$$0.79u_1 = 0.3$$

$$\Rightarrow \quad u_1 = 0.3797$$

$$u_i = \Pr\{X_n \text{ reaches state } 0 \text{ before state } N \mid X_0 = i\}$$

$$= \begin{cases} \dfrac{N-i}{N} & \text{when } p = q = \dfrac{1}{2}, \\ \dfrac{(q/p)^i - (q/p)^N}{1-(q/p)^N} & \text{when } p \neq q. \end{cases}$$

② $$u_i = \mathbb{P}(X_n = 0 \mid X_0 = i) = \begin{cases} \dfrac{N-i}{N} & p = q = \dfrac{1}{2} \\ \dfrac{(q/p)^i - (q/p)^N}{1-(q/p)^N} & p \neq p \end{cases}$$

$$\frac{(0.3/0.7)^1 - (0.3/0.7)^3}{1-(0.3/0.7)^3} = 0.3797$$

3.6.2 Customer accounts receivable at Smith Company are classified each month according to

0: Current
1: 30–60 days past due
2: 60–90 days past due
3: Over 90 days past due

Consider a particular customer account and suppose that it evolves month to month as a Markov chain $\{X_n\}$ whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.9 & 0.1 & 0 & 0 \\ 1 & 0.5 & 0 & 0.5 & 0 \\ 2 & 0.3 & 0 & 0 & 0.7 \\ 3 & 0.2 & 0 & 0 & 0.8 \end{array}.$$

Suppose that a certain customer's account is now in state 1: 30–60 days past due. What is the probability that this account will be paid (and thereby enter state 0: Current) before it becomes over 90 days past due? That is, let $T = \min\{n \geq 0; X_n = 0 \text{ or } X_n = 3\}$. Determine $\Pr\{X_T = 0 | X_0 = 1\}$.

$$U_{ik} = P_{ik} + \sum_{j=0}^{r-1} P_{ij} U_{jk}$$

$$U_{0,1} = P_{10} + \sum_{j=0}^{3} P_{1j} U_{j0}$$

$$= P_{10} + P_{10} U_{00} + P_{11} U_{10} + P_{12} U_{20} + P_{13} U_{30}$$

① $\because$ $U_{10} = 0.5 + \cancel{0.5 U_{00}} + 0.5 U_{20} = 0.5 + 0.5 U_{20}$

$U_{20} = 0.3$

$U_{10} = 0.5 + 0.5 \times 0.3 = 0.65$

3.6.8 Consider the Markov chain $\{X_n\}$ whose transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} \alpha & 0 & \beta & 0 \\ \alpha & 0 & 0 & \beta \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array},$$

where $\alpha > 0, \beta > 0$, and $\alpha + \beta = 1$. Determine the mean time to reach state 3 starting from state 0. That is, find $E[T|X_0 = 0]$, where $T = \min\{n \geq 0; X_n = 3\}$.

$T = \min\{ n \geq 0 : X_n = 3 \}$

$V_i = 1 + \sum_{j=0}^{r-1} P_{ij} V_j = E[T | X_0 = i]$

Q: $V_0 = 1 + \sum_{j=0}^{r-1} P_{0j} V_j = 1 + P_{00}V_0 + P_{01}V_1 + P_{02}V_2 + P_{03}V_3$

① $\cdots V_0 = 1 + \alpha V_0 + \beta V_2$

$V_1 = 1 + \sum_{j=0}^{3} P_{1j} V_j = 1 + P_{10}V_0 + P_{11}V_1 + P_{12}V_2 + P_{13}V_3$

② $\cdots V_1 = 1 + \alpha V_0 + \beta V_3$

$V_2 = 1 + P_{20}V_0 + P_{21}V_1 + P_{22}V_2 + P_{23}V_3$

③ $V_2 = 1 + \alpha V_0 + \beta V_1$

$$\begin{cases} V_0 = 1 + \alpha V_0 + \beta V_2 \\ V_1 = 1 + \alpha V_0 \\ V_2 = 1 + \alpha V_0 + \beta V_2 \end{cases}$$

$\Rightarrow V_0 = \dfrac{1 + \beta + \beta^2}{\beta^3}$

3.8.1 A population begins with a single individual. In each generation, each individual in the population dies with probability $\frac{1}{2}$ or doubles with probability $\frac{1}{2}$. Let X_n denote the number of individuals in the population in the nth generation. Find the mean and variance of X_n.

$P(\xi_1 = 0) = \frac{1}{2}$

$P(\xi_1 = 2) = \frac{1}{2}$

$\mu = 1$

$\sigma^2 = 1$

$\mathbb{E}[X_n] = \mu^n = 1^n = 1$

$Var(X_n) = n\sigma^2 = n$

3.9.1 Suppose that the offspring distribution is Poisson with mean $\lambda = 1.1$. Compute the extinction probabilities $u_n = \Pr\{X_n = 0 | X_0 = 1\}$ for $n = 0, 1, \ldots, 5$. What is u_∞, the probability of ultimate extinction?

$\lambda = 1.1 \quad X_0 = 1$

Q: $u_n = \mathbb{P}(X_n = 0 | X_0 = 1)$ for $n \in \{0,1,2,3,4,5\}$

$$\phi_n(s) = \mathbb{E}[S^n] = \sum_{k=0}^{\infty} s^k \, \mathbb{P}(N=n) = \sum_{k=0}^{\infty} s^k e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!}$$

$$= e^{-\lambda} \cdot e^{s\lambda}$$

$$= e^{s\lambda - \lambda} = e^{\lambda(s-1)} = e^{1.1(s-1)}$$

$$u_1 = \phi(u_0) = e^{1.1(u_0-1)} = e^{1.1(0-1)} = e^{-1.1} = 0.3329$$

$$u_2 = \phi(u_1) = \phi(0.3329) = e^{1.1(0.3329-1)} = 0.4801$$

$$u_\infty = \phi(u_\infty)$$

$$u = e^{1.1(u-1)}$$

$$\ln u = 1.1u - 1.1$$

$$u \approx 0.8238$$

3.9.6 Let $\phi(s) = as^2 + bs + c$, where a, b, c are positive and $\phi(1) = 1$. Assume that the probability of extinction is u_∞, where $0 < u_\infty < 1$. Prove that $u_\infty = c/a$.

$$a+b+c=1 \Rightarrow b = 1-c-a$$

$$u = au^2 + bu + c$$

$$= au^2 + bu - u + c$$

$$\Rightarrow au^2 + (b-1)u + c = 0$$

$$au^2 + (1-c-a-1)u + c = 0$$

$$au^2 - (a+c)u + c = 0$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-(a+c)) \pm \sqrt{(a+c)^2 - 4ac}}{2a}$$

$$= \frac{a+c \pm \sqrt{a^2+c^2-2ac}}{2a}$$

$$= \frac{(a+c) \pm \sqrt{(a-c)^2}}{2a}$$

$$= \frac{(a+c) \pm (a-c)}{2a}$$

① $\frac{a+c+a-c}{2a} = 1$ ② $\frac{a+c-a+c}{2a} = \frac{2c}{2a} = \frac{c}{a}$

3.9.8 Consider a branching process whose offspring follow the geometric distribution $p_k = (1-c)c^k$ for $k = 0, 1, \ldots$, where $0 < c < 1$. Determine the probability of eventual extinction.

$$\phi_n(s) = \mathbb{E}[s^n] = \sum_{k=0}^{\infty} s^k (1-c) c^k$$

$$= \sum_{k=0}^{\infty} (sc)^k (1-c)$$

$$= (1-c) \sum_{k=0}^{\infty} (sc)^k$$

$$\therefore \quad u_\infty = \phi(u_\infty)$$

$$u = (1-c) \sum_{k=0}^{\infty} (uc)^k$$

$$u = (1-c) \frac{1}{1-uc}$$

$$u = \frac{1-c}{1-uc}$$

$$u - u^2 c = 1-c$$

$$u^2 c - u - c + 1 = 0$$

$$(u-1)(uc + c - 1) = 0$$

$$\Rightarrow \quad u_1 = 1 \qquad u_2 = \frac{1-c}{c}$$

$\therefore$ u_2 is the sol.