Instructions:

- Explain all of your steps.
- There is no need to simplify arithmetic (unless stated otherwise).
- You may not use notes, books, calculators, phones, or the internet.
- Do not cheat.

1. (20 points) Let X_n be a Markov chain with one-step transition probability matrix
$$P = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ .3 & .4 & .2 & .1 \\ .1 & .2 & .5 & .2 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$
Let $T = \min\{n \geq 0 : X_n = 0 \text{ or } X_n = 3\}$.
   (a) Find $\mathbb{P}(X_T = 0|X_0 = 1)$ and $\mathbb{P}(X_T = 0|X_0 = 2)$.
   (b) Assume that X_0 is equally likely to start in state 1 or 2. That is, X_0 is a random variable with $\mathbb{P}(X_0 = 1) = .5$ and $\mathbb{P}(X_0 = 2) = .5$. Find $\mathbb{P}(X_T = 0)$.

2. (20 points) Let X_n be a Markov chain with one-step transition probability matrix
$$P = \begin{Vmatrix} .9 & .1 & 0 & 0 \\ .9 & 0 & .1 & 0 \\ .9 & 0 & 0 & .1 \\ 1 & 0 & 0 & 0 \end{Vmatrix}.$$
where $0 < p < 1$.
   (a) Show that P is regular.
   (b) Find the limiting distribution $\pi = \lim_{n\to\infty} xP^n$ where x is any initial pmf.

3. (10 points) Roll a fair six-sided die over and over again. Let X_n be the maximum value of the die in the first n rolls.

   Find the state space for X_n and write down the first-step transition probability matrix. Explain your work.

4. (30 points) Let ξ be a Poisson(2) random variable. That is,

$$\mathbb{P}(\xi = k) = e^{-2}\frac{2^k}{k!}, \quad k = 0, 1, 2, 3, 4, ...$$

   Let X_n be a branching process with $X_0 = 1$ and new generations distributed like ξ.

   (a) Find the generating function $\phi_\xi(s) = \mathbb{E}(s^\xi)$. For full credit, simplify your answer so that it does not include an infinite sum.

   (b) Write down an equation for the extinction probability of this branching process. DO NOT TRY TO SOLVE THE EQUATION. Explain a procedure that you could use to approximate the solution.

   (c) Find the generating function of X_2.

   (d) Find $\mathbb{P}(X_2 = 1)$.

5. (20 points) Let X_n be a Markov chain with state space $\{0, 1, 2, 3, 4, 5, 6\}$ and one-step transition probability matrix

$$P = \begin{Vmatrix} 0 & 0 & 0 & 0 & .5 & .5 & 0 \\ 0 & 0 & 0 & 0 & .7 & .3 & 0 \\ 0 & 0 & .4 & .6 & 0 & 0 & 0 \\ 0 & 0 & .3 & .7 & 0 & 0 & 0 \\ .1 & .9 & 0 & 0 & 0 & 0 & 0 \\ .9 & .1 & 0 & 0 & 0 & 0 & 0 \\ .1 & .1 & .1 & .1 & .1 & .1 & .4 \end{Vmatrix}.$$

   (a) What are the communicating classes in this Markov chain?

   (b) What is the period of each class?

   (c) Which classes are recurrent and which classes are transient?

   (d) Find $\lim_{n\to\infty} \mathbb{P}(X_n = 2 | X_0 = 6)$.