$N \sim Poi(\lambda) \quad X \sim Bin(N, p)$

pmf of X:

$$P(X=k) = \sum_{n=0}^{\infty} P(X=k \mid N=n)\, P(N=n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \cdot e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= e^{-\lambda} \sum_{n=k}^{\infty} \frac{1}{k!(n-k)!} p^k (1-p)^{n-k} \lambda^n$$

$$= e^{-\lambda} \sum_{n=k}^{\infty} \frac{(p\lambda)^k}{k!} \cdot \frac{(1-p)^{n-k} \lambda^{n-k}}{(n-k)!} \qquad \text{Let } \lambda^n = \lambda^k \cdot \lambda^{n-k}$$

$$= e^{-\lambda} \frac{(p\lambda)^k}{k!} \sum_{n=k}^{\infty} \frac{[\lambda(1-p)]^{n-k}}{(n-k)!} \qquad \sum \frac{x^n}{n!} = e^x$$

$$= e^{-\lambda} \frac{(p\lambda)^k}{k!} \cdot e^{\lambda - \lambda p}$$

$$= e^{-\lambda + \lambda - \lambda p} \frac{(p\lambda)^k}{k!}$$

$$= e^{-\lambda p} \frac{(\lambda p)^k}{k!}$$

Random sum of RV , $X = \sum_{k=1}^{N} \xi_k$

$\mu = E(\xi_k) \quad \sigma^2 = Var(\xi_k) \quad \nu = E(N) \quad \tau^2 = Var(N)$

$$E(X) = \sum_{n=0}^{\infty} E(X \mid N=n)\, \mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} E\Big(\sum_{k=1}^{N} \xi_k \mid N=n\Big)\, \mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} E\Big(\sum_{k=1}^{n} \xi_k\Big)\, \mathbb{P}(N=n) = \sum_{n=0}^{\infty} n\mu\, \mathbb{P}(N=n)$$

$$= \mu \sum_{n=0}^{\infty} n \mathbb{P}(N=n) = \mu E(N) = \mu v$$

$$Var(X) = E[(X - E(X))^2] = E[(X - N\mu + N\mu - E(X))^2]$$

$$= E[\underbrace{(X-N\mu)^2}_{①} + \underbrace{2(X-N\mu)(N\mu - E(X))}_{②} + \underbrace{(N\mu - E(X))^2}_{③}]$$

$$① = E[(X-N\mu)^2] = \sum_{n=0}^{\infty} E[(\sum_{k=1}^{N} \xi_k - N\mu)^2 | N=n] \mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} E[(\underbrace{\sum_{k=1}^{n} \xi_k - n\mu)^2}] \mathbb{P}(N=n)$$

$$= [\sum_{k=1}^{n} (\xi_k - \mu)]^2 = \sum_{k=1}^{n} \sum_{j=1}^{n} (\xi_k - \mu)(\xi_j - \mu)$$

$$= \sum_{n=0}^{\infty} E[\sum_{k=1}^{n} \sum_{j=1}^{n} (\xi_k - \mu)(\xi_j - \mu)] \mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} (\sum_{k=1}^{n} \sum_{j=1}^{n} \mathbb{E}(\xi_k - \mu)(\xi_j - \mu)) \mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} \sum_{k=1}^{n} \mathbb{E}(\xi_k - \mu)^2 \mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} n\sigma^2 \mathbb{P}(N=n) = \sigma^2 E(N) = \sigma^2 \cdot v$$

$$② \; E(X - N\mu)(N\mu - E(X)) = \sum_{n=0}^{\infty} E[(\sum_{k=1}^{N} \xi_k - N\mu)(N\mu - \mu v) | N=n] \mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} \underbrace{E[(\sum_{k=1}^{n} \xi_k - n\mu)(n\mu - \mu v)]} \mathbb{P}(N=n)$$

$$= (n\mu - \mu v) \underbrace{E[(\sum_{k=1}^{n} \xi_k - n\mu)]}_{=0} = 0$$

③ $E[(N\mu - E(X))^2] = E[(N\mu - \mu\nu)^2]$

$$= \mu^2 \underbrace{E[(N-\nu)^2]}_{\tau^2}$$ 随机变量减均值

$$= \mu^2\tau^2$$

Ex: $X|Y=y \sim Poi(y)$, $Y \sim gamma(\alpha, \theta)$

α - positive integre. Find dist of X. $NegBin(\alpha, \frac{1}{1+\theta})$

作业

$$P(X=k) = \int_y f(x,y)dy = \int_y \mathbb{P}(X|Y=y) f_Y(y) dy$$

$$= \int_y \frac{e^{-y} y^k}{k!} \frac{\theta^\alpha y^{\alpha-1} e^{-\theta y}}{\Gamma(\alpha)} dy$$

$$= \frac{\theta^\alpha}{k!\Gamma(\alpha)} \int_0^\infty e^{-y} y^k y^{\alpha-1} e^{-\theta y} dy$$

$$= \frac{\theta^\alpha}{k!\Gamma(\alpha)} \int_0^\infty y^{k+\alpha-1} e^{-y(1+\theta)} dy$$

$$= \frac{\theta^\alpha}{k!\Gamma(\alpha)} \frac{\Gamma(k+\alpha)}{(1+\theta)^{k+\alpha}}$$

$$= \frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)} \frac{\theta^\alpha}{(1+\theta)^{k+\alpha}}$$

$$= \frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)} \left(\frac{\theta}{1+\theta}\right)^\alpha \left(1 - \frac{\theta}{1+\theta}\right)^k$$

$$\int_{-\frac{\pi}{2}}^{\arctan z}\underbrace{\int_{0}^{+\infty}\frac{1}{2\pi}e^{-r^2}\,r\,dr}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{+\infty} e^{-r^2}\,r\,dr$$, Let $u = -r^2$ $du = -2r$, 用 $\frac{1}{u'}du$ 替换 dr

$$= \frac{1}{2\pi}\int e^{u}\,r\left(-\frac{1}{2r}\right)du$$

$$= \frac{1}{2\pi}\int -\frac{e^{u}}{2}\,du$$

$$= \frac{1}{2\pi}\cdot -\frac{1}{2}\int_0^{\infty} e^{u}\,du$$

$$= \frac{1}{2\pi}\cdot -\underbrace{\frac{1}{2}e^{-r^2}}\Big|_0^{\infty}$$

$$= \frac{1}{2\pi}\cdot +\frac{1}{2} = \frac{1}{4\pi}$$

$$\int_{-\frac{\pi}{2}}^{\arctan z}\frac{1}{4\pi}\,d\theta = \frac{1}{4\pi}\theta\Big|_{-\frac{\pi}{2}}^{\arctan(z)} = \frac{1}{4\pi}\arctan(z) + \frac{1}{8}$$

Ex: ① $X \sim$ Poisson (λ) , Find dist of $2X$

② $X \sim$ Normal $(0,1)$, find dist of X^2

$\downarrow$ chi-square (1)

① $X, Y, \lambda_1, \lambda_2, Z = X+Y$ $\lambda = \lambda_1 + \lambda_2$

$$P(Z=z) = \sum_{k=0}^{z} P(X=k \text{ and } Y=z-k)$$

$$= \sum_{k=0}^{z} P(X=k)\,P(Y=z-k)$$

$$= \sum_{k=0}^{z}\frac{e^{-\lambda_1}\lambda_1^{k}}{k!}\,\frac{e^{-\lambda_2}\lambda_2^{z-k}}{(z-k)!} = \sum_{k=0}^{z}\frac{z!}{k!(z-k)!}\,\frac{e^{-\lambda_1}e^{-\lambda_2}\lambda_1^{k}\lambda_2^{z-k}}{z!}$$

$$= \sum_{k=0}^{z} \binom{z}{k} \frac{e^{-(\lambda_1+\lambda_2)} \lambda_1^k \lambda_2^{z-k}}{z!} = \frac{e^{-\lambda}}{z!} \underbrace{\sum_{k=0}^{z} \binom{z}{k} \lambda_1^k \lambda_2^{z-k}}_{\text{二项式定理}}$$

$$= \frac{e^{-\lambda}}{z!} (\lambda_1+\lambda_2)^z$$

$$= \frac{e^{-\lambda}}{z!} \lambda^z$$

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x+y)^n$$

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

② $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ Let $y = g(x) = x^2$

when $x<0$: $g^{-1}(y) = x = -\sqrt{y}$

$x>0$: $g^{-1}(y) = x = +\sqrt{y}$

$$\frac{d g^{-1}(y)}{dy} = \pm \frac{1}{2\sqrt{y}}$$

$$p(y) = \left|\frac{-1}{2\sqrt{y}}\right| \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \left|\frac{1}{2\sqrt{y}}\right| \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} = \frac{1}{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}$$