Salins **MA 583 Midterm Exam** July 21, 2016

Instructions:

- Explain all of your steps.
- There is no need to simplify arithmetic (unless stated otherwise).
- You may not use notes, books, calculators, phones, or the internet.
- Do not cheat.

1. (20 points) Let X_n be a Markov chain with one-step transition probability matrix

$$P = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ .3 & .4 & .2 & .1 \\ .1 & .2 & .5 & .2 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

Let $T = \min\{n \geq 0 : X_n = 0 \text{ or } X_n = 3\}$.

(a) Find $\mathbb{P}(X_T = 0|X_0 = 1)$ and $\mathbb{P}(X_T = 0|X_0 = 2)$.

First-step analysis. Let $u_1 = \mathbb{P}(X_T = 0|X_0 = 1)$, $u_2 = \mathbb{P}(X_T = 0|X_0 = 2)$. From the matrix, we see that

$$\begin{aligned} u_1 &= .3 + .4u_1 + .2u_2 \\ u_2 &= .1 + .2u_1 + .5u_2. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} 0 &= .3 - .6u_1 + .2u_2 \\ 0 &= .1 + .2u_1 - .5u_2. \end{aligned}$$

If we multiply the second equation by 3 and add the equations together we see

$$0 = .6 - 1.3u_2, \qquad u_2 = \frac{6}{13}.$$

Plug this back into the equations

$$u_1 = -\frac{1}{2} + \frac{5}{2}u_2 = -\frac{1}{2} + \frac{30}{26} = \frac{17}{26}.$$

ANSWER: $u_1 = \frac{17}{26}$, $u_2 = \frac{6}{13}$.

(b) Assume that X_0 is equally likely to start in state 1 or 2. That is, X_0 is a random variable with $\mathbb{P}(X_0 = 1) = .5$ and $\mathbb{P}(X_0 = 2) = .5$. Find $\mathbb{P}(X_T = 0)$. Law of total probability.

$$\mathbb{P}(X_T = 0) = \mathbb{P}(X_T = 0|X_0 = 1)\mathbb{P}(X_0 = 1) + \mathbb{P}(X_T = 0|X_0 = 0)\mathbb{P}(X_0 = 0)$$

$$= \frac{17}{52} + \frac{3}{13} = \frac{29}{52}.$$

2. (20 points) Let X_n be a Markov chain with one-step transition probability matrix

$$P = \begin{Vmatrix} .9 & .1 & 0 & 0 \\ .9 & 0 & .1 & 0 \\ .9 & 0 & 0 & .1 \\ 1 & 0 & 0 & 0 \end{Vmatrix}.$$

(a) Show that P is regular. The easiest way to do this is to show that the Markov chain irreducible, aperiodic, and finite. It clearly has only 4 states. State 0 is clearly aperiodic because it can return to itself in one step. There is only class because any state can go directly to 0, and 0 can go to each of the other states.

(b) Find the limiting distribution $\pi = \lim_{n\to\infty} xP^n$ where x is any initial pmf. Because the Markov chain is regular, there exists a unique stationary distribution which is also the limiting distribution. Solve $\pi = \pi P$.

$$\begin{aligned} \pi_0 &= .9\pi_0 + .9\pi_1 + .9\pi_2 + 1\pi_3 \\ \pi_1 &= .1\pi_0 \\ \pi_2 &= .1\pi_1 \\ \pi_3 &= .1\pi_2 \\ 1 &= \pi_0 + \pi_1 + \pi_2 + \pi_3. \end{aligned}$$

This system of equations is redundant (always true for stochastic matrices) so we can ignore the top equation because it is the most complicated. The other equations tell us that $\pi_1 = .1\pi_0$, $\pi_2 =$

$.01\pi_0$, $\pi_3 = .001\pi_0$. If we combine them together and make sure they add up to 1 we can solve that $1 = 1.111\pi_0$, $\pi_0 = \frac{1000}{1111}$. It follows that $\pi_1 = \frac{100}{1111}$, $\pi_2 = \frac{10}{1111}$, $\pi_3 = \frac{1}{1111}$.

3. (10 points) Roll a fair six-sided die over and over again. Let X_n be the maximum value of the die in the first n rolls.

   Find the state space for X_n and write down the first-step transition probability matrix. Explain your work. If $X_n = 1$, then X_{n+1} will be whatever the next roll of the dice is. Each outcome has probability $\frac{1}{6}$.

   If $X_n = 2$, then X_{n+1} can 't be 1, $X_{n+1} = 2$ if the next die is 1 or 2 (probability $\frac{2}{6}$) or it takes the other values with probability $\frac{1}{6}$.

   The other rows are similar.

$$P = \begin{Vmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

4. (30 points) Let ξ be a Poisson(2) random variable. That is,

$$\mathbb{P}(\xi = k) = e^{-2}\frac{2^k}{k!}, \quad k = 0, 1, 2, 3, 4, ...$$

   Let X_n be a branching process with $X_0 = 1$ and new generations distributed like ξ.

   (a) Find the generating function $\phi_\xi(s) = \mathbb{E}(s^\xi)$. For full credit, simplify your answer so that it does not include an infinite sum. By definition,

$$\phi_\xi(s) = \mathbb{E}(s^\xi) = \sum_{k=0}^{\infty} s^k e^{-2}\frac{2^k}{k!}$$

   By the Taylor series for exponentiation,

$$= e^{-2}\sum_{k=0}^{\infty}\frac{(2s)^k}{k!} = e^{2s-2}$$

(b) Write down an equation for the extinction probability of this branching process. DO NOT TRY TO SOLVE THE EQUATION. Explain a procedure that you could use to approximate the solution. For any $n \geq 0$, let $u_n = \mathbb{P}(X_n = 0)$. Let $u_\infty = \lim_{n\to\infty} u_n$. A first-step analysis shows that

$$u_n = \sum_{k=1}^{\infty} (u_{n-1})^k \mathbb{P}(\xi = k) = \phi_\xi(u_{n-1}).$$

This equation shows that u_∞ must be a fixed point solving

$$u_\infty = e^{2u_\infty - 2}$$

We don't know how to solve for this algebraically, but we can approximate it recursively by letting $u_0 = 0$, $u_1 = e^{2u_0-2}$, $u_2 = e^{2u_1-2}$, and repeating this process over and over.

(c) Find the generating function of X_2. Generation functions of random sums of random variables have a nice formula. Recall that

$$X_2 = \sum_{k=1}^{X_1} \xi_k^{(1)},$$

where $\xi_k^{(1)}$ are independent Pois(2) random variables that are also independent of X_1. This means that $\phi_{X_2}(s) = \phi_{X_1}(\phi_\xi(s))$. Because we assumed that $X_0 = 1$, X_1 has the same distribution as ξ. Therefore,

$$\phi_{X_2}(s) = \phi_\xi(\phi_\xi(s)) = e^{2(e^{2s-2})-2}$$

(d) Find $\mathbb{P}(X_2 = 1)$. There are more than one way to do this problem. One way involves the generating function.

$$\mathbb{P}(X_2 = 1) = \phi'_{X_2}(0).$$

If you forgot this, you can see it by taking the derivative of

$$\phi_{X_2}(s) = \sum_{k=0}^{\infty} s^k \mathbb{P}(X_2 = k) = \mathbb{P}(X_2 = 0) + \mathbb{P}(X_2 = 1)s + \mathbb{P}(X_2 = 3)s^3 + ...$$

$$\phi'_{X_2}(s) = \mathbb{P}(X_2 = 1) + 2\mathbb{P}(X_2 = 2)s + 3\mathbb{P}(X_2 = 3)s^2 + ...$$

If you plug in $s = 0$, you see that

$$\phi'_{X_2}(0) = \mathbb{P}(X_2 = 1).$$

In this problem,

$$\phi'_{X_2}(s) = \frac{d}{ds}\left(e^{2e^{2s-2}-2}\right) = 4e^{2s-2}e^{2e^{2s-2}-2}.$$

Plugging in zero,

$$\mathbb{P}(X_2 = 1) = 4e^{2e^{-2}-4}.$$

5. (20 points) Let X_n be a Markov chain with state space $\{0, 1, 2, 3, 4, 5, 6\}$ and one-step transition probability matrix

$$P = \begin{Vmatrix} 0 & 0 & 0 & 0 & .5 & .5 & 0 \\ 0 & 0 & 0 & 0 & .7 & .3 & 0 \\ 0 & 0 & .4 & .6 & 0 & 0 & 0 \\ 0 & 0 & .3 & .7 & 0 & 0 & 0 \\ .1 & .9 & 0 & 0 & 0 & 0 & 0 \\ .9 & .1 & 0 & 0 & 0 & 0 & 0 \\ .1 & .1 & .1 & .1 & .1 & .1 & .4 \end{Vmatrix}.$$

(a) What are the communicating classes in this Markov chain? The communicating classes are $\{0, 1, 4, 5\}$, $\{2, 3\}$, and $\{6\}$

(b) What is the period of each class? $\{0, 1, 4, 5\}$ has period 2. The other classes have period 1.

(c) Which classes are recurrent and which classes are transient? The class $\{6\}$ is transient. The other classes are recurrent.

(d) Find $\lim_{n\to\infty} \mathbb{P}(X_n = 2|X_0 = 6)$.

We need to do this in 2 steps. First, let's find the probability that the Markov chain enters class $\{2, 3\}$ if it starts in state 6.

Let $T = \min\{n \geq 0 : X_n \in \{0, 1, 2, 3, 4, 5\}\}$. Let $u = \mathbb{P}(X_T \in \{2, 3\}|X_0 = 6)$. By a first-step analysis,

$$u = 0(.4) + 1(.2) + .4u.$$

The above expression is a consequence of the fact that there is a .4 chance of the first step going to $\{0, 1, 4, 5\}$, a .2 chance of the first step going to $\{2, 3\}$, and a .4 chance of the first step staying in 6 in which case the experiment restarts. Solving the first step analysis, we see that $u = \frac{1}{3}$.

Then, we need to find the long-time behavior of the chain in the case it reaches class $\{2, 3\}$. The submatrix associated with class $\{2, 3\}$ is regular, so it has a unique invariant distribution. We can solve for the unique invariant distribution.

$$\pi = \pi \begin{Vmatrix} .4 & .6 \\ .3. & .7 \end{Vmatrix}.$$

This means that $\pi_2 = .4\pi_2 + .3\pi_3$. $\pi_2 = \frac{1}{2}\pi_3$. Combining this with the requirement that $\pi_2 + \pi_3 = 1$, we conclude that $\pi_2 = \frac{1}{3}$, $\pi_3 = \frac{2}{3}$.

The answer is the product of $u\pi_2 = \frac{1}{9}$ because

$$\lim_{n\to\infty} \mathbb{P}(X_n = 2|X_0 = 6)$$

$$= \lim_{n\to\infty} \mathbb{P}(X_n = 2|X_T \in \{2, 3\})\mathbb{P}(X_T \in \{2, 3\}|X_0 = 6) = \pi_2 u.$$