Final Exam

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Instructions

• Submit your solutions as ONE single PDF file. And clearly indicate the question you are answering.

• Solutions are due before 11:15 a.m. on Thursday, August 12th, 2021. If you cannot upload on gradescope, then you must send us email BEFORE the upload time.

• Start each Big Question in new page

• Explain all of your steps. YOU MUST WRITE AT LEAST ONE SENTENCE OF EXPLANATION IN EACH SUB PROBLEM.

• Unless stated otherwise, you need to solve any linear algebra problems by hand and you must show all of your work.

• You do not need to use a calculator. I will grade the arithmetic formula as the final answer, not necessarily decimal approximation.

• You may use your notes, the textbook, and anything on the course webpage at learn.bu.edu.

• You may not refer to any other sources including other books or anything on the internet.

• You may not collaborate with anybody.

• Do not cheat.

Question 1

(20pts) A single fair dice is rolled repeatedly. The game stops the first time that the sum of two successive rolls is either 4 or 7.
(a) What is the probability that the game stops at a sum of 4?
(b) Let N denote the number of tosses in a game. Find the expected value of N.

Question 2

(20pts) Use the transition matrix below to find the limits of the following probabilities.(Do not just write down the numbers. You have to show the steps of finding stationary distribution and etc.)

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \end{array}$$

(a) $\lim_{n\to\infty} \mathrm{P}_{5,0}^{(n)}$

(b) $\lim_{n\to\infty} \mathrm{P}_{5,1}^{(n)}$

(c) $\lim_{n\to\infty} \mathrm{P}_{5,2}^{(n)}$

(d) $\lim_{n\to\infty} \mathrm{P}_{5,3}^{(n)}$

(e) $\lim_{n\to\infty} \mathrm{P}_{5,4}^{(n)}$

(f) $\lim_{n\to\infty} \mathrm{P}_{5,5}^{(n)}$

Question 3

(20pts) Let $\{X(t) : t \geq 0\}$ be a Poisson process of rate λ

(a) Let $Y_k \overset{\text{iid}}{\sim}$ Bernoulli(p), and $Z(t) = \sum_{k=1}^{X(t)} Y_k$, show that $X(t) - Z(t)$ is a Poisson process with rate $\lambda(1-p)$.

(b) Let $W_1, W_2, ...$ be the event times in $X(t)$, and let $f(w)$ be an arbitrary function. Verify that

$$\mathbb{E}[\prod_{i=1}^{X(t)} f(W_i)] = e^{\lambda(\int_0^t f(u)du - t)}$$

Question 4

(30 pts) Let $X(t)$ be a continuous time Markov chain with state space $\{0, 1, 2, 3, 4\}$. So the amount of time process stayed at each state is exponentially distributed. Let $P_{ij}(t) = \mathbb{P}(X(t) = j | X(0) = i)$. Suppose the infinitesimal generator of P(t) is the matrix,

$$A = \lim_{h \to 0} \frac{P(t) - I}{h} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 2 & -5 & 2 & 1 & 0 \\ 0 & 0 & -4 & 4 & 0 \\ 3 & 0 & 2.5 & -6 & 0.5 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

(a) Let $\tau = \min\{t \geq 0 : X(t) \neq X(0)\}$ be the time of the first jump. Find $\mathbb{E}[\tau | X(0) = 1]$. Show all of the work and explain crucial steps.

(b) Let $\tau = \min\{t \geq 0 : X(t) \neq X(0)\}$ the time of the first jump. Find $\mathbb{P}(X(\tau) = 2 | X(0) = 3)$. Show all of the work and explain crucial steps.

(c) Let $T = \min\{t \geq 0 : X(t) = 0 \text{ or } X(t) = 4\}$ be the first time continuous time Markov chain enters 0 or 4. And define

$$u_i = \mathbb{P}(X(T) = 0 | X(0) = i)$$

Use first step analysis, set up a system of equations for u_1, u_2, u_3. Explain all terms you used in your equations but NO NEED TO SOLVE THE EQUATION.

(d) Find the stationary distribution of the process.

Question 5

(10pts) Let $X(t) = 5 - 2t + 2B(t)$ where B(t) is a standard Brownian motion with B(0) = 0.
(a) Calculate E(X(2)X(1)). Show all of your work and explain each step.
(b) Calculate P(X(2)+X(1)≥5). Express your answer in terms of $\Phi(\cdot)$, the CDF of standard normal distribution. Show all of your work and explain each step.