MA583 Homework #3

Shi Bo

TOTAL POINTS

10 / 10

QUESTION 1

1 Exercise 4.1.2 **1 / 1**

✓ **- 0 pts** **Correct**

QUESTION 2

2 Problem 4.1.2 **2 / 2**

✓ **- 0 pts** **Correct**

- 2 pts Incorrect

- 0.5 pts Click here to replace this description.

QUESTION 3

3 Exercise 4.3.2 **1 / 1**

✓ **- 0 pts** **Correct**

QUESTION 4

4 Problem 4.3.1 **2 / 2**

✓ **- 0 pts** **Correct**

- 0.5 pts Click here to replace this description.

- 1.5 pts Click here to replace this description.

- 1 pts Click here to replace this description.

QUESTION 5

5 Exercise 4.4.2 **1 / 1**

✓ **- 0 pts** **Correct**

QUESTION 6

6 Problem 4.4.2 **2 / 2**

✓ **- 0 pts** **Correct**

- 0.5 pts Click here to replace this description.

- 2 pts Click here to replace this description.

QUESTION 7

7 Exercise 4.5.2 **1 / 1**

✓ **- 0 pts** **Correct**

4.1.2 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{c}
\begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\left\Vert \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array} \right\Vert
\end{array}.
$$

Determine the limiting distribution.

Any limiting distribution can solve $\mu = \mu P$, so

$\mu = \mu P$

$$
\left.\begin{array}{l}
0.6\mu_0 + 0.3\mu_1 + 0.4\mu_2 = \mu_0 \\
0.3\mu_0 + 0.3\mu_1 + 0.1\mu_2 = \mu_1 \\
0.1\mu_0 + 0.4\mu_1 + 0.5\mu_2 = \mu_2 \\
\mu_0 + \mu_1 + \mu_2 = 1
\end{array}\right\} \Rightarrow
\left\{\begin{array}{l}
\mu_0 = \frac{31}{66} \\
\mu_1 = \frac{16}{66} \\
\mu_2 = \frac{19}{66}
\end{array}\right.
$$

1 Exercise 4.1.2 **1 / 1**

✓ **- 0 pts** Correct

4.1.2 Five balls are distributed between two urns, labeled A and B. Each period, one of the five balls is selected at random, and whichever urn it's in, it is moved to the other urn. In the long run, what fraction of time is urn A empty?

Let X_n be # of balls in A.

Define P_{ij} is a probability where i is # of balls in A at time n, and j is # of balls in A at $n+1$.

So, we have $P_{i,j=i-1} = \frac{i}{5}$

$$P_{i,j=i+1} = \frac{5-i}{5}$$

$$P_{i,j=i} = 0$$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left\|\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right\| \end{array}$$

$$\begin{cases} \pi_0 = \frac{1}{5}\pi_1 \\ \pi_1 = \frac{2}{5}\pi_2 \\ \pi_2 = \frac{4}{5}\pi_1 + \frac{2}{5}\pi_3 \\ \pi_3 = \frac{3}{5}\pi_2 + \frac{4}{5}\pi_4 \\ \pi_4 = \frac{3}{5}\pi_3 + \pi_5 \\ \pi_5 = \frac{1}{5}\pi_4 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1 \end{cases}$$

by using Wolfram Alpha

$$\Rightarrow \pi_0 = \frac{1}{32}$$

2 Problem 4.1.2 2 / 2

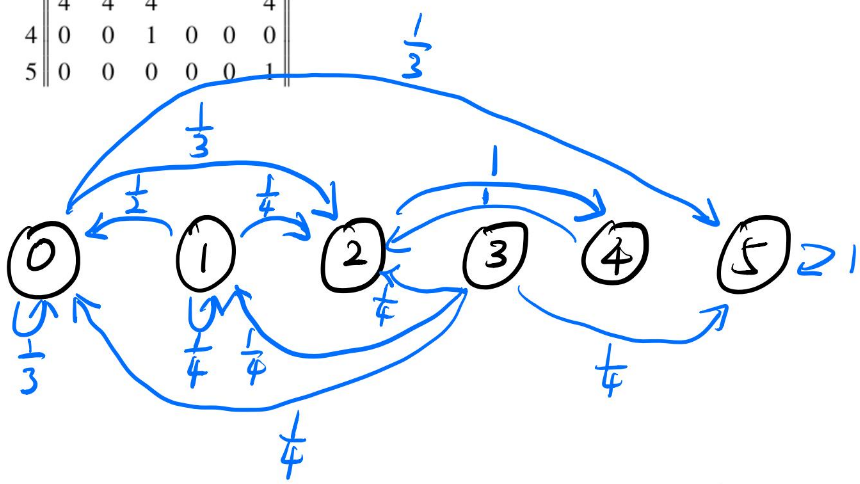
✓ **- 0 pts** Correct

- 2 pts Incorrect

- 0.5 pts Click here to replace this description.

4.3.2 Which states are transient and which are recurrent in the Markov chain whose transition probability matrix is

$$
\begin{array}{c|cccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\
1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\
4 & 0 & 0 & 1 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
?
$$



From chart above, we can see that state {0,1,3} there is a non-zero probability, then it will never return to state {0,1,3}. There are three recurrent state {2,4,5} For example, 2 can go to 4 and then return to 2, similar for {4} and {5}. So, they are recurrent.

3 Exercise 4.3.2 **1 / 1**

✓ **- 0 pts** Correct

4.3.1 A two-state Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \left\| \begin{array}{cc} 1-a & a \\ b & 1-b \end{array} \right\| \end{array}.$$

(a) Determine the first return distribution

$$f_{00}^{(n)} = \Pr\{X_1 \neq 0, \ldots, X_{n-1} \neq 0, X_n = 0 | X_0 = 0\}.$$

(b) Verify equation (4.16) when $i = 0$. (Refer to Chapter 3, (4.40).)

(a) $f_{00}^{(0)} = 0$ by definition. $\quad P_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)}, \quad n \geq 1,$

$f_{00}^{(n)} = \mathbb{P}\{X_n = 0 \text{ and } X_m \neq i \text{ for } m \in \{1, \cdots, n-1\} | X_0 = 0\}$

$f_{00}^{(1)} = \mathbb{P}(X_1 = 0 | X_0 = 0) = P_{00} = 1-a$

$f_{00}^{(2)} = \mathbb{P}(X_1 = 1 | X_0 = 0)\mathbb{P}(X_1 = 0 | X_0 = 1) = P_{01}P_{10} = a \cdot b$

$f_{00}^{(3)} = P_{01}P_{11}P_{10} = a \cdot (1-b) \cdot b$

$f_{00}^{(4)} = P_{01}P_{11}P_{11}P_{10} = P_{01}(P_{11})^{n-2}P_{10} = a \cdot (1-b)^{n-2} \cdot b$

$\vdots$

$f_{00}^{(n)} = P_{01}(P_{11})^{n-2}P_{10} = a \cdot (1-b)^{n-2} \cdot b$

(b) $P_{00}^{(n)} = \sum_{k=0}^{n} f_{00}^{(k)} P_{ii}^{(n-k)}$

$P_{00}^{(n)} = \frac{b}{a+b} + \frac{a}{a+b}(1-a-b)^n = \frac{b + a(1-a-b)^n}{a+b} \quad \forall n \geq 1$

$f_{00}^{(k)} = a \times b (1-b)^{k-2} \quad (k \neq 0)$

$\sum_{k=0}^{n} f_{00}^{(k)} P_{00}^{(n-k)} = f_{00}^{(1)} \cdot P_{00}^{(n-1)} + \sum_{k=2}^{n} f_{00}^{(k)} \cdot P_{00}^{(n-k)}$

$= (1-a)\frac{b + a(1-a-b)^{n-1}}{a+b} + \sum_{k=2}^{n} ab(1-b)^{k-2} \frac{b + a(1-a-b)^{n-k}}{a+b}$

①

Note that the geometric sum : $\sum_{k=0}^{N} r^k = \frac{1-r^{N+1}}{1-r}$, $r \neq 1$

$$① = \sum_{k=2}^{n} ab(1-b)^{k-2} \frac{b+a(1-a-b)^{n-k}}{a+b}$$

$$= \frac{ab}{a+b}\left(b\sum_{j=0}^{n-2}(1-b)^j + a(1-a-b)^{n-2}\sum_{j=0}^{n-2}\left(\frac{a-b}{1-a-b}\right)^j\right)$$

$$= \frac{ab}{a+b}\left(a-(a-b)^{n-1} - (1-a-b)^{n-1} + (1-a-b)^{n-1}\left(\frac{1-b}{1-a-b}\right)^{n-1}\right)$$

$$= \frac{ab}{a+b}\left(a-(a-b)^{n-1} - (1-a-b)^{n-1} + (1-b)^{n-1}\right)$$

$$= \frac{ab}{a+b}\left(1-(1-a-b)^{n-1}\right)$$

$$\therefore \quad (1-a)\frac{b+a(1-a-b)^{n-1}}{a+b} \sum_{k=2}^{n} ab(1-b)^{k-2} \frac{b+a(1-a-b)^{n-k}}{a+b}$$

$$= (1-a)\frac{b+a(1-a-b)^{n-1}}{a+b} + \frac{ab}{a+b}\left(1-(1-a-b)^{n-1}\right)$$

$$= \frac{(1-a)(b+a(1-a-b)^{n-1}(1-a-b))}{(1-a)(a+b)}$$

$$= \frac{b+a(1-a-b)^{n}}{a+b} = p_{00}^{(n)}$$

4 Problem 4.3.1 **2 / 2**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Click here to replace this description.
- **- 1.5 pts** Click here to replace this description.
- **- 1 pts** Click here to replace this description.

4.4.2 Determine the stationary distribution for the Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \end{array} \right\| \end{array}.$$

$$\left.\begin{aligned} \mu_0 &= \tfrac{1}{4}\mu_2 + \tfrac{1}{3}\mu_3 \\ \mu_1 &= \tfrac{3}{4}\mu_2 + \tfrac{2}{3}\mu_3 \\ \mu_2 &= \tfrac{1}{2}\mu_0 + \tfrac{1}{3}\mu_1 \\ \mu_3 &= \tfrac{1}{2}\mu_0 + \tfrac{2}{3}\mu_2 \\ \mu_0 + \mu_1 &+ \mu_2 + \mu_3 = 1 \end{aligned}\right\} \Rightarrow \begin{cases} \mu_0 = \frac{11}{73} \\ \mu_1 = \frac{51}{146} \\ \mu_2 = \frac{14}{73} \\ \mu_3 = \frac{45}{146} \end{cases}$$

5 Exercise 4.4.2 **1 / 1**

✓ **- 0 pts** Correct

4.4.2 Consider the Markov chain whose transition probability *matrix* is given by

$$\mathbf{P} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 0.1 & 0.4 & 0.2 & 0.3 \\ 2 & 0.2 & 0.2 & 0.5 & 0.1 \\ 3 & 0.3 & 0.3 & 0.4 & 0 \end{array}$$

(a) Determine the limiting probability π_0 that the process is in state 0.

(b) By pretending that state 0 is absorbing, use a first step analysis (Chapter 3, Section 3.4) and calculate the mean time m_{10} for the process to go from state 1 to state 0.

(c) Because the process always goes directly to state 1 from state 0, the mean *return* time to state 0 is $m_0 = 1 + m_{10}$. Verify equation (4.26), $\pi_0 = 1/m_0$.

(a)
$$0\pi_0 + 0.1\pi_1 + 0.2\pi_2 + 0.3\pi_3 = \pi_0$$
$$1\pi_0 + 0.4\pi_1 + 0.2\pi_2 + 0.3\pi_3 = \pi_1$$
$$0\pi_0 + 0.2\pi_1 + 0.5\pi_2 + 0.4\pi_3 = \pi_2 \quad \Rightarrow \quad \pi_0 = \frac{1449}{9999}$$
$$0\pi_0 + 0.3\pi_1 + 0.1\pi_2 + 0\pi_3 = \pi_3$$
$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

(b) $V_i = m_{10} = \mathbb{E}[T \mid X_0 = 1]$
$$V_1 = 1 + 0.4V_1 + 0.2V_2 + 0.3V_3$$
$$V_2 = 1 + 0.2V_1 + 0.5V_2 + 0.1V_3 \quad \Rightarrow \quad m_{10} = V_1 = \frac{950}{161} = \frac{8550}{1449}$$
$$V_3 = 1 + 0.3V_1 + 0.4V_2 + 0$$

(c)
$$m_0 = 1 + m_{10}$$
$$= 1 + \frac{950}{161}$$
$$= \frac{1111}{161}$$
$$= \frac{1}{\pi_0}$$

6 Problem 4.4.2 **2 / 2**

✓ **- 0 pts** **Correct**

- 0.5 pts Click here to replace this description.

- 2 pts Click here to replace this description.

4.5.2 Given the transition matrix

$$
\mathbf{P} = \begin{array}{c|ccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\
2 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\
4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
6 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{4} & \frac{1}{4} \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array},
$$

derive the following limits, where they exist:

(a) $\lim_{n\to\infty} P_{11}^{(n)}$ (e) $\lim_{n\to\infty} P_{21}^{(n)}$

(b) $\lim_{n\to\infty} P_{31}^{(n)}$ (f) $\lim_{n\to\infty} P_{33}^{(n)}$

(c) $\lim_{n\to\infty} P_{61}^{(n)}$ (g) $\lim_{n\to\infty} P_{67}^{(n)}$

(d) $\lim_{n\to\infty} P_{63}^{(n)}$ (h) $\lim_{n\to\infty} P_{64}^{(n)}$

4.5.2 **(a)** $\frac{3}{11}$, **(e)** $\frac{3}{11}$, **(b)** 0, **(f)** X, **(c)** $\frac{2}{33}$, **(g)** $\frac{1}{3}$, **(d)** $\frac{2}{9}$, **(h)** $\frac{4}{27}$.

First we divide MC into $C_1 = (1, 2)$
$C_2 = (3, 4, 5)$, $C_3 = (6)$, $C_4 = (7)$.

C_1 is aperiod, posive, recurrent, so $\lim_{n\to\infty} P_{11}^{(n)} = \pi_0$

$$
\left.\begin{aligned}
\tfrac{1}{3}\pi_0 + \tfrac{1}{4}\pi_1 &= \pi_0 \\
\tfrac{2}{3}\pi_0 + \tfrac{3}{4}\pi_1 &= \pi_1 \\
\pi_0 + \pi_1 &= 1
\end{aligned}\right\} \Rightarrow \begin{aligned} \pi_0 &= \tfrac{3}{11} \\ \pi_1 &= \tfrac{8}{11} \end{aligned}
$$

$$\lim_{n\to\infty} P_{11}^{(n)} = \pi_0 = \frac{3}{11}$$

$$
\begin{cases}
\pi_4 + \pi_5 = \pi_3 \\
\frac{2}{3}\pi_3 = \pi_4 \\
\frac{1}{3}\pi_3 = \pi_5 \\
\pi_3 + \pi_4 + \pi_5 = 1
\end{cases}
\Rightarrow
\begin{cases}
\pi_3 = \frac{1}{2} \\
\pi_4 = \frac{1}{3} \\
\pi_5 = \frac{1}{6}
\end{cases}
$$

For the transient class C_3, consider u as the probability that the ultimate absorption in C_1 starting from state 6:

$$u = \frac{1}{6} + \frac{1}{4}u \Rightarrow u = \frac{2}{9}$$

(a) $\lim_{n\to\infty} P_{11}^{(n)} = \pi_0 = \frac{3}{11}$

(b) $\lim_{n\to\infty} P_{3,1}^{(n)} = 0$

π_0

(c) $\lim_{n\to\infty} P_{6,1}^{(n)} = \mathbb{P}(\text{reach } 0\sim 5 \text{ before } 7)\,\mathbb{P}(\text{end up in } 1)$

$= u \cdot \pi_0 = \frac{2}{9} \cdot \frac{3}{11} = \frac{2}{33}$

(d) $\lim_{n\to\infty} P_{03}^{(n)} = \mathbb{P}(\text{end up in } 3) = \frac{2}{9}$

(e) $\lim_{n\to\infty} P_{21}^{(n)} = \pi_0 = \frac{3}{11}$

(f) $\lim_{n\to\infty} P_{33}^{(n)} =$ limit does not exist

(g) $\lim_{n\to\infty} P_{67}^{(n)} = \frac{1}{3}$

(h) $\lim_{n\to\infty} P_{04}^{(n)} = \frac{2}{3} \cdot u = \frac{4}{27}$

7 Exercise 4.5.2 **1 / 1**

✓ **- 0 pts** Correct