Ch3 Markov Chain

$\{X_t\}_{(t \in T)}$ Markov process iff given value X_t, the value of $X_{s\,(s>t)}$ is not influenced by the value of $X_{u\,(u<t)}$

不受它们影响

$$\mathbb{P}(X_{n+1} = j \mid X_0 = a_0, X_1 = a_1, \ldots, X_n = i) = \mathbb{P}(X_{n+1} = j \mid X_n = i)$$

Random Walk, start at 0 at t=0

$X_0 = 0$. flip coin $\begin{cases} \text{if head} & +1 \\ \text{if tail} & -1 \end{cases}$

$$X_{n+1} = \begin{cases} X_n + 1 & \text{w.p. } \frac{1}{2} \\ X_n - 1 & \text{w.p. } \frac{1}{2} \end{cases}$$

Notation

time stamp

$$P_{i,j}^{n,n+1} = \mathbb{P}(X_{n+1} = j \mid X_n = i)$$

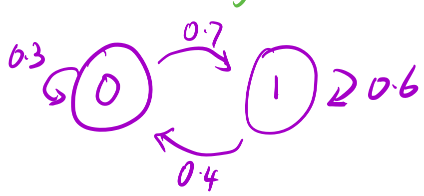
from to

e.g. 2 state Markov Chain

State space = $\{0, 1\}$

$P_{0,0} = 0.3$ $P_{0,1} = 0.7$ $P_{1,0} = 0.4$ $P_{1,1} = 0.6$

(0到0的概率在一个时间是0.3 ...)



Q: $\mathbb{P}(X_2 = 0 \mid X_0 = 1)$, 在两个时间内从1到0的概率是多少

$= P_{1,1} P_{1,0} + P_{1,0} P_{0,0}$

$= \mathbb{P}(X_2 = 0 \mid X_1 = 0, \cancel{X_0 = 1}) \cdot \mathbb{P}(X_1 = 0 \mid X_0 = 1) +$

$\mathbb{P}(X_2 = 0 \mid X_1 = 1, \cancel{X_0 = 1}) \cdot \mathbb{P}(X_1 = 1 \mid X_0 = 1)$

Total law of prob

Transition Martix

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} P_{0,0} & P_{0,1} \\ P_{1,0} & P_{1,1} \end{pmatrix} \end{array}$$

Matrix Multiplication

$$P^2 = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \checkmark & \checkmark \\ P_{1,1}P_{1,0}+P_{1,0}P_{0,0} & \checkmark \end{pmatrix} \end{array}$$

$$P_{1,0}^{n,n+2} = \sum_{k=0}^{1} P_{1,k}^{n,n+1} P_{k,0}^{n,n+1} = (P \cdot P)_{1,0} = (P^2)_{1,0}$$

这俩 0.1 意思不一样

element in matrix

Assumption: Stationary transition:

$$P_{i,j}^{n,n+1} = P_{i,j} = P(X_1 = j \mid X_0 = i)$$

与初始时间n无关

State $0, 1, 2, \ldots, N$

(finite, Countable)

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & , & \cdots & , N \end{array} \\ \begin{array}{c} 0 \\ \vdots \\ N \end{array} & \begin{pmatrix} P_{0,0}, & P_{0,1}, & \cdots, & P_{0,N} \\ P_{1,0} & P_{1,1} & \cdots & P_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ P_{N,0} & P_{N,1} & \cdots & P_{N,N} \end{pmatrix} \end{array}$$

1 step transition matrix

2-step:

$$P^2 = A = \begin{array}{c c} & \begin{array}{ccc} 0 & \cdots & N \end{array} \\ \begin{array}{c} 0 \\ \vdots \\ N \end{array} & \begin{pmatrix} a_{0,0} & & \\ & & \\ & & \end{pmatrix} \end{array}$$

$$a_{i,j} = \sum_{k=0}^{N} P_{i,k} P_{k,j} = (P \cdot P)_{i,j}$$

$$A = P^2$$

3-step: $a_{i,j} = \sum_{k=0}^{N} P_{i,k}^2 P_{k,j} = (P^2 \cdot P)_{i,j} = (P^3)_{i,j}$

How I from i to k in 2 step, then to j.

property: row sum is 1

col sum is not always 1.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.5 & 0 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{pmatrix} \end{array}$$

1 Q: prob going from 0 to 1, then to 2?

$\rightarrow 0.5 \times 0.5$

Q: prob going from 0 to 2 in 2 step?

$$\sum_{k=0}^{2} P_{0,k} P_{k,2}$$

"in" means by exactly 2 steps

"within" means in 1 or 2 steps

Q: prob going from 0 to 2 in 3 step?

$$\sum_{k=0}^{2} \sum_{j=0}^{2} P_{0,k} P_{k,j} P_{j,2}$$

Random Walk

State space $\{0, 1, \dots, N\}$

$$\begin{cases} +1 & w.p: \frac{1}{2} \\ -1 & w.p: \frac{1}{2} \end{cases}$$

$\{X_n\}_n \leftarrow$ transition matrix

$$\begin{array}{c} \\ 0 \\ \vdots \\ \\ N \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & N \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & & \vdots \\ 0 & & & & \vdots \\ \vdots & & & & \\ 0 & \cdots & & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

因为没法在一个time内移动两次

$P_{i,j} = 0$ if $|i-j| > 1$

$P_{i,i+1} = \frac{1}{2}$

$P_{i,i-1} = \frac{1}{2}$ $\quad i \in \{1, \dots, N-1\}$

$P_{0,0} = P_{0,1} = \frac{1}{2}$

$P_{N,N-1} = P_{N,N} = \frac{1}{2}$

Urn problem

2 baskets of balls $\quad$ $2a$ balls in total

A $\quad$ B

randomly pick 1 ball

put it into the other basket

X_n : # of balls in A

Y_n : $X_n - a$ X_n里多出a的

$\{-a, \ldots, a\}$ 因X_n里可以为0或2a个ball

what is transition matrix for Y_n?

$$\begin{array}{c|cccc} & -a & -a+1 & \cdots & a \\ \hline -a & \\ \vdots & \\ a & \end{array} \quad \begin{pmatrix} 0 & 1 & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

$$\frac{a-i}{2a} = \frac{a-(-a)}{2a} = 1$$

计算 X_n 的TM.

$P_{i,i+1} = \frac{2a-i}{2a}$

$P_{i,i-1} = \frac{2a+i}{2a}$

$P_{ij} = 0$ if $|i-j| > 1$ $P_{i,i+2} = 0$

开始时的球 in Y_n，所以 $i+a = X_n$ 就是X_n里有多少球也就是A里

$$P_{i,i+1} = \frac{2a-(i+a)}{2a} = \frac{a-i}{2a}$$ 从B移到A

Y_n 里的球多一个的时 X_n里也多一个，所以A里就得多一个，所以是从B里拿了一个球到A

$2a-(i+a)$ 就是在B里球的数量，所$P_{i,i+1}$的意思是你从B中拿到球的概率

$$P_{i,i-1} = \frac{i+a}{2a}$$ 从A到B

$i+a$ 就是A里的球的数量

Queueing problem

in each time ① if there exists passengers taxi pick up 1 passenger

② new passengers shows up

# of passengers show up will follow some distribution. ξ_t iid

pmf: $\mathbb{P}(\xi_t = k) = a_k$ $\sum_{k=0}^{\infty} a_k = 1$

the prob of number of k passengers will show up

X_n = # people in line Q: what is TM for X_n

State space $\{0, \ldots, \infty\}$

$$X_{n+1} = \begin{cases} \xi_{n+1} & \text{if } X_n = 0 \\ X_n - 1 + \xi_{n+1} & \text{if } X_n \geq 1 \end{cases}$$

ξ_{n+1} : # of people coming in time stamp n+1

$X_n - 1$: 接走一个人

if $i=0$ ($X_n=0$) $P_{0,j} = a_j$

当line里一个人也没有，从0个人到j个的概率表达为a_j or 来j个人在一个time stamp的概率

if $i \geq 1$ $P = \{ a_{j-(i-1)}$ if $j \geq i-1$

$l_{i,j} = \begin{cases} 0 & \text{if } j < i-1 \end{cases}$

$i \rightarrow j$ Start with i people, then we need to remove 1 people, in order to reach j person in line. how many people we need to reach j?

$\rightarrow j-(i-1)$

$i \quad -1 \qquad a_{j-(i-1)}$