Change of RVs for Bivariate case:

Suppose that X_1 and X_2 have joint density $f_{X_1, X_2}(x_1, x_2)$,

and $Y_1 = u_1(X_1, X_2)$, $Y_2 = u_2(X_1, X_2)$, where u_1 and u_2

are 2 functions of X_1 and X_2.

We want to find the joint distribution of Y_1 and Y_2, $f_{Y_1, Y_2}(y_1, y_2)$.

need to find the inverse functions of u_1 and u_2,

namely, $\boxed{\text{to represent } X_1 \text{ and } X_2 \text{ in terms of } Y_1 \text{ and } Y_2:}$

$$X_1 = V_1(Y_1, Y_2), \quad X_2 = V_2(Y_1, Y_2).$$

Then, $$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(V_1(y_1, y_2), V_2(y_1, y_2)) \cdot |J|$$

"Jacobian"

$$|J| = \det \begin{bmatrix} \frac{\partial V_1(y_1, y_2)}{\partial y_1} & \frac{\partial V_1(y_1, y_2)}{\partial y_2} \\ \frac{\partial V_2(y_1, y_2)}{\partial y_1} & \frac{\partial V_2(y_1, y_2)}{\partial y_2} \end{bmatrix}$$

$$= \frac{\partial V_1(y_1, y_2)}{\partial y_1} \cdot \frac{\partial V_2(y_1, y_2)}{\partial y_2} - \frac{\partial V_2(y_1, y_2)}{\partial y_1} \cdot \frac{\partial V_1(y_1, y_2)}{\partial y_2}$$

Example: X and Y have joint PDF $f_{X,Y}(x,y)$ and $U = \frac{X}{Y}$, $V = Y$. Question: what is the joint PDF of U and V ?

Since $U = \frac{X}{Y}$ and $V = Y \Longrightarrow \boxed{X = U \cdot V,\ Y = V}$

Thus, $$J = \det \begin{bmatrix} \frac{\partial uv}{\partial u} & \frac{\partial uv}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \end{bmatrix} = \det \begin{bmatrix} v & u \\ 0 & 1 \end{bmatrix} = v$$

$$\Longrightarrow f_{U,V}(u,v) = f_{X,Y}(uv, v) \cdot |v|$$

Application: if $X, Y \overset{iid}{\sim} N(0,1)$, want to find the distribution of $\frac{X}{Y}$.

Since $X, Y \overset{iid}{\sim} N(0,1)$, $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$

Let $U = \frac{X}{Y}$, $V = Y$. then the joint PDF of U and V is as following:

$$f_{U,V}(u,v) = f_{X,Y}(uv, v) \cdot |v|$$

$$= \frac{1}{2\pi} e^{-\frac{u^2v^2+v^2}{2}} \cdot |v|$$

Then, $f_U(u) = \int_v f_{U,V}(u,v)\, dv$

$w = -(u^2v^2+v^2)/2$

$dw = -(2u^2v + 2v/2) = u^2v + v = -v(u^2+1)$

$$\int_0^\infty e^{-\frac{u^2v^2+v^2}{2}} v\,dv,$$

$$= \int_0^\infty e^{w}\, v \cdot \frac{1}{-v(u^2+1)}\,dw$$

$$= \int_0^\infty e^{w} \frac{1}{-(u^2+1)}\,dw$$

$$= \int_0^\infty -\frac{e^w}{u^2+1}\,dw$$

$$= -\frac{1}{u^2+1} e^w = -\frac{1}{u^2+1} e^{-\frac{u^2v^2+v^2}{2}}\Bigg|_0^\infty$$

$$= 0 - \left(-\frac{1}{u^2+1}\right)$$

$$= \frac{1}{u^2+1}$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-\frac{u^2v^2+v^2}{2}} |v|\,dv$$

$$= \frac{1}{\pi}\int_0^{+\infty} e^{-\frac{u^2+1}{2}v^2}\, v\,dv$$

$$= \frac{1}{\pi}\left[-\frac{1}{u^2+1} e^{-\frac{u^2+1}{2}v^2}\right]\Bigg|_{v=0}^{+\infty}$$

$$= \frac{1}{\pi(1+u^2)} \sim \text{Cauchy}$$

$f_U(u)$ is just equivalent to the PDF of $\frac{X}{Y}$ since $U = \frac{X}{Y}$

This is a much simpler way to obtain the distribution of the ratio of random variables compared to the method of polar coordinate we used during discussion.