# Random Sum of RV

Let $\xi_1, \xi_2, \xi_3 \cdots$ be iid, Let $N$ be RV non-negative and integer-value

$$X = \sum_{k=1}^{N} \xi_k$$

e.g. ① population growth

$N$: # of species

$\xi_1$: # of descendent each species

$$\mu = \mathbb{E}(\xi_k)$$

$$\sigma^2 = Var(\xi_k)$$

$$\nu = \mathbb{E}(N)$$

$$\tau^2 = Var(N)$$

$N$ and $\xi$ are independent.

$$\mathbb{E}(X) = \mu \cdot \nu = \mathbb{E}\xi_k \mathbb{E}N$$

① $$\mathbb{E}(X) = \sum_{n=0}^{\infty} \mathbb{E}(X \mid N=n)\,\mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} \mathbb{E}\Big(\sum_{k=1}^{N} \xi_k \,\Big|\, N=n\Big)\mathbb{P}(N=n)$$

property of conditional expectation →

$$= \sum_{n=0}^{\infty} \mathbb{E}\Big(\sum_{k=1}^{n} \xi_k \,\Big|\, N=n\Big)\mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} \underbrace{\mathbb{E}\Big(\sum_{k=1}^{n} \xi_k\Big)}_{=n\mu}\mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty} n\mu\,\mathbb{P}(N=n)$$

$$= \mu \underbrace{\sum_{n=0}^{\infty} n\,\mathbb{P}(N=n)}$$

$$= \mu \cdot \mathbb{E}(N)$$

$$= \mu \cdot \nu \quad (\mathbb{E}X = \mathbb{E}N\,\mathbb{E}\xi_k)$$

② $$Var(X) = \mathbb{E}\big[X - \mathbb{E}(X)\big]^2 = \mathbb{E}\big[(X - N\mu + N\mu - \mathbb{E}(X))^2\big]$$

$$= \mathbb{E}\big[\underbrace{(X-N\mu)^2}_{①} + \underbrace{2(X-N\mu)(N\mu-\mathbb{E}(X))}_{②} + \underbrace{(N\mu-\mathbb{E}(X))^2}_{③}\big]$$

① $$\mathbb{E}[(X-N\mu)^2] = \mathbb{E}\Big[\Big(\sum_{k=1}^{N}\xi_k - N\mu\Big)^2\Big] = \underbrace{\sum_{n=0}^{\infty}\mathbb{E}\Big[\Big(\sum_{k=1}^{N}\xi_k - N\mu\Big)^2 \,\Big|\, N=n\Big]\mathbb{P}(N=n)}_{\text{total law of expectation}}$$

$$= \sum_{n=0}^{\infty}\mathbb{E}\Big[\underbrace{\Big(\sum_{k=1}^{n}\xi_k - n\mu\Big)^2}\Big]\mathbb{P}(N=n)$$

$$= \Big[\sum_{k=1}^{n}(\xi_k - \mu)\Big]^2 \quad \boxed{n\mu = \sum_{k=1}^{n}\mu}$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{n}(\xi_k-\mu)(\xi_j-\mu)$$

$$= \sum_{n=0}^{\infty}\mathbb{E}\Big[\sum_{k=1}^{n}\sum_{j=1}^{n}(\xi_k-\mu)(\xi_j-\mu)\Big]\,\mathbb{P}(N=n)$$

(linearity) $$= \sum_{n=0}^{\infty}\Big(\sum_{k=1}^{n}\sum_{j=1}^{n}\underbrace{\mathbb{E}(\xi_k-\mu)(\xi_j-\mu)}\Big)\,\mathbb{P}(N=n)$$

independence $\longrightarrow$ $$= \begin{cases} \cancel{\mathbb{E}(\xi_k-\mu)\,\mathbb{E}(\xi_j-\mu)} \rightarrow 0 \quad \text{因为 } \mu=\mathbb{E}(\xi_k) & \text{if } k\neq j \\ \underbrace{\mathbb{E}(\xi_k-\mu)^2}_{=\sigma^2} & \text{if } k=j \end{cases}$$

$$= \sum_{n=0}^{\infty}\Big(\sum_{k=1}^{n}\sigma^2\Big)\mathbb{P}(N=n)$$

只剩 $n$-pair, $j=k$

$$= \sum_{n=0}^{\infty} n\sigma^2\,\mathbb{P}(N=n) = \sigma^2\cdot\nu$$

$$\mathbb{E}\big[\underbrace{(X-N\mu)^2}_{①}+2\underbrace{(X-N\mu)(N\mu-\mathbb{E}(X))}_{②}+\underbrace{(N\mu-\mathbb{E}(X))^2}_{③}\big] = \sigma^2\cdot\nu + ② + ③$$

② $$\mathbb{E}\big[(X-N\mu)(N\mu-\mathbb{E}(X))\big]$$

$$= \Big(\sum_{n=0}^{\infty}\mathbb{E}\Big(\sum_{k=1}^{N}\xi_k-N\mu\Big)(N\mu-\mu\nu)\ \Big|\ N=n\Big)\,\mathbb{P}(N=n)$$

$$= \Big[\sum_{n=0}^{\infty}\underbrace{\mathbb{E}\Big(\sum_{k=1}^{n}\xi_k-n\mu\Big)(n\mu-\mu\nu)}\Big]\,\mathbb{P}(N=n)$$

$$= (n\mu-\mu\nu)\,\underbrace{\mathbb{E}\Big(\sum_{k=1}^{n}\xi_k-n\mu\Big)}$$

$$= \mathbb{E}\Big[\sum_{k=1}^{n}(\xi_k-\mu)\Big] = \sum_{k=1}^{n}\overbrace{\mathbb{E}(\xi_k-\mu)}^{=0} = 0$$

$$=0$$

$$\mathbb{E}[\underbrace{(X-N\mu)^2}_{①}+\underbrace{2(X-N\mu)(N\mu-\mathbb{E}(X))}_{②}+\underbrace{(N\mu-\mathbb{E}(X))^2}_{③}] = \sigma^2\cdot\nu + 0 + ③$$

③ $\mathbb{E}(N\mu-\mathbb{E}(X))^2 = \mathbb{E}(N\mu-\mu\nu)^2$ ← 这个地方没有ξ，所以用不着 total law of expectation 只有当有两个独立RV才用

$$= \mu^2\underbrace{\mathbb{E}(N-\nu)^2}_{\tau^2}$$

$$= \mu^2\tau^2$$

$$\mathbb{E}[\underbrace{(X-N\mu)^2}_{①}+\underbrace{2(X-N\mu)(N\mu-\mathbb{E}(X))}_{②}+\underbrace{(N\mu-\mathbb{E}(X))^2}_{③}] = \sigma^2\cdot\nu + 0 + \mu^2\tau^2$$

$$= \mathbb{E}(N)\mathrm{Var}(\xi_k) + \underbrace{(\mathbb{E}(\xi_k))^2\mathrm{Var}(N)}_{\text{extra randomness induced by random variable } N}$$

$\mathrm{Var}(\sum_{k=1}^{n}\xi_k) = n\mathrm{Var}(\xi_k)$ ← if each $\xi_k$ is independent

$$\mathrm{Var}(X) = \nu\sigma^2 + \mu^2\tau^2$$

CDF: for some $x$

$$F_X(x) = \mathbb{P}(X\le x) = \sum_{n=0}^{\infty}\mathbb{P}(X\le x\mid N=n)\mathbb{P}(N=n) = \sum_{n=0}^{\infty}\mathbb{P}(\sum_{k=1}^{N}\xi_k\le x\mid N=n)\mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty}\mathbb{P}(\sum_{k=1}^{n}\xi_k\le x)\mathbb{P}(N=n)$$

$$= \sum_{n=0}^{\infty}\mathbb{P}(\sum_{k=1}^{n}\xi_k\le x)\mathbb{P}(N=n)$$

(接下来的情况取决于 $\sum_{k=1}^{n}\xi_k$，什么是伯努利，那么 $\sum_{k=1}^{n}\xi_k$ 就是二项)

( $\xi_k \sim Ber(p)$, $Y=\sum_{k=1}^{n}\xi_k \sim Bin(n,p)$

$\mathbb{P}(Y\le x) = F(x) = $ cdf of Bernalli )

let $G_x(n) = \mathbb{P}(\sum_{k=1}^{n}\xi_k\le x)$

$$= \sum_{n=0}^{\infty}G_x(n)\mathbb{P}(N=n) = \mathbb{E}\,G_x(N)$$