Due on 2021/12/14 2p.m

Problem 1
Let $(X, \mathcal{M}, \mu)$ be a measure space and let g be a nonnegative function that is integrable over X. Define

$$\nu(E) = \int_E g d\mu$$

for all $E \in \mathcal{M}$.
(i) Prove that ν is a measure on $X, \mathcal{M}$.
(ii) Let f be a nonnegative function on X that is measurable with respect to $\mathcal{M}$. Show that

$$\int_X f d\nu = \int_X fg d\mu$$

(Hint: First establish this for the case when f is simple and then apply the Simple Approximation Lemma and the Monotone Convergence theorem.)

Proof: **(i)**
We first prove that $0 \leq \nu(E) \leq \infty$. Since g is integrable, we have that

$$\nu(E) = \int_E g d\mu \leq \int_X g d\mu < \infty$$

and because g is nonnegative, by monotonicity,

$$\nu(E) = \int_E g d\mu \geq 0$$

Also, since integration over $\emptyset$ is 0, it is obvious that $\nu(\emptyset) = \int_\emptyset g d\mu = 0$. Finally, let $\{E_k\}_{k=1}^\infty$ be any sequence of countable and disjoint set in $\mathcal{M}$. Define $E = \cup_{k=1}^\infty E_k$. Then, we can consider a step function χ_E such that $\chi_E \cdot g = \sum_{k=1}^\infty \chi_{E_k} \cdot g$ on X. Therefore, given that g is nonnegative, we can obtain

$$\int_X \chi_E \cdot g d\mu = \int_X (\sum_{k=1}^\infty \chi_{E_k} \cdot g) d\mu$$

$$\Rightarrow \int_E g d\mu = \sum_{k=1}^\infty \int_X \chi_{E_k} \cdot g d\mu = \sum_{k=1}^\infty \int_{E_k} g d\mu$$

$$\Rightarrow \nu(E) = \sum_{k=1}^\infty \nu(E_k)$$

Thus, ν is countably additive.
(ii)
We now show that for nonnegative measurable function f on X, the $\int_X f d\mu = \int_X fg d\mu$ holds. Consider a nonnegative simple function $\psi = \sum_{k=1}^n a_k \chi_{E_k}$, $E_k \in \mathcal{M}$. Then by linearity of integration and (i), we have

$$\int_X \psi d\nu = \sum_{k=1}^n a_k \nu(E_k) = \sum_{k=1}^n a_k \int_{E_k} g d\mu = \sum_{k=1}^n a_k \int_X g \chi_{E_k} d\mu = \int_X \sum_{k=1}^n a_k \chi_{E_k} g d\mu = \int_X \psi \cdot g d\mu$$

Thus, the result is true for nonnegative simple function. Now, since f is nonnegative measurable function on X, by simple approximation lemma, there exists a increasing sequence of nonnegative simple function $\{\psi_n\}_{n=1}^\infty$ on X such that $f = \lim_{n\to\infty} \psi_n$ on X. Also, we have $\lim_{n\to\infty} \psi_n g = fg$, and since $\{\psi_n\}$ and g are nonnegative, $\{\psi_n g\}$ is nonnegative. By monotone convergence theorem, we have

$$\int_X f d\nu = \lim_{n\to\infty} \int_X \psi_n d\nu = \lim_{n\to\infty} \int_X \psi_n g d\mu = \int_X fg d\mu$$

Therefore, proved. ■

Problem 2
Let μ be a measure on $(X, \mathcal{M})$ and let μ_1 and μ_2 be mutually singular measures on $X, \mathcal{M}$ such that $\mu = \mu_1 - \mu_2$. Show that $\mu_2 = 0$. Use this to establish the uniqueness assertion in the Jordan Decomposition Theorem.

We first prove that $\mu_2 = 0$.
Let μ be a measure and μ_1, μ_2 be mutually singular measures on $(X, \mathcal{M})$. Since μ_1 and μ_2 are mutually singular, we consider that X is a disjoint union of measurable sets A and B, $X = A \cup B$, such that $\mu_2(B) = \mu_1(A) = 0$. Assume $\mu = \mu_1 - \mu_2$ and since μ, μ_1 and μ_2 are all measures, they are all nonnegative. Then, for any measurable set $E \in \mathcal{M}$, we have $\mu_2(E) = \mu_2(A \cap E) + \mu_2(B \cap E)$. Since $A \cap E \subset A$, by monotonicity, $\mu_1(A \cap E) = 0$. Let $C \in \mathcal{M}$ be the set such that $\mu_1(C) = 0$, then we have $\mu(C) = \mu_2(C)$ and thus $\mu(C) + \mu_2(C) = 0$. Given that μ and μ_2 are measures, they are nonnegative, which implies $\mu_2(C) = 0$. Hence, $\mu_2 << \mu_1$. Since $\mu_2 << \mu_1$, we have $\mu_2(A \cap E) = 0$. Now, since $E \cap B \subset B$, by monotonicity, we have $\mu_2(E \cap B) = 0$ and so $\mu_2(E) = 0$ for any measurable set E, and thus $\mu_2 = 0$.
We then prove the uniqueness of Jordan Decomposition.
Consider that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. Then, there exists disjoint measurable sets A and B such that $A \cup B = X$, $\nu^+(B) = \nu^-(A) = 0$. Assume that there exists another pair μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$. Then, there exists disjoint measurable sets A' and B' such that $A' \cup B' = X$, and $\mu^+(B\,') = \mu^-(A\,') = 0$. Now, for any measurable set $E \in \mathcal{M}$, we define $\mu\,' = \mu(E \cap A)$, $\mu^{+'} = \mu^+(E \cap A)$ and $\mu^{-'} = \mu^-(E \cap A)$ where $\mu\,'$,$\mu^{+'}$and $\mu^{-'}$are measures, because $\mu^+$and μ^- are measures. Then, since $\mu^+(B\,')$ and $\mu^-(A) = 0$, we have $\mu^{+'} = \mu^+(B\,' \cap A) = 0$ and $\mu^{-'}(A\,') = \mu^-(A\,' \cap A) = 0$. We know that $A\,' \cap B\,' = X$, so $\mu^{+'}$ and $\mu^{-'}$are mutually singular. Then, for any $E \in \mathcal{M}$, we have

$$\mu\,'(E) = \mu(E \cap A) = \mu^+(A \cap E) - \mu^-(A \cap E) = \mu^{+'}(E) - \mu^{-'}(E)$$

Hence, given what we have proven above, we have $\mu^{-'}(E) = \mu^-(E \cap A) = 0$. Similarly, we can get $\mu^-(E \cap B) = 0$,$\nu^-(E \cap A\,') = 0$ and $\nu^-(E \cap B\,') = 0$. Thus, we have

$$\nu^-(E) = \nu^-(E \cap A\,') + \nu^-(E \cap B\,') = 0$$

and

$$\mu^-(E) = \mu^-(E \cap A) + \mu^-(E \cap B) = 0$$

So, we have $\nu^-(E) = \mu^-(E)$. Since $\nu^+(E) - \nu^-(E) = \mu^+(E) - \mu^-(E)$, we will get $\nu^+(E) = \mu^+(E)$. Hence, the proof is completed. ■

Problem 3
Show that the Radon-Nikodym Theorem for finite measures μ and ν implies the Theorem for $\sigma-$ finite measures μ and ν.

Proof: Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν be a measure on $\mathcal{M}$, which is absolutely continuous with respect to μ. Consider that there exists pairwise disjoint collections $\{X_i\}_{i=1}^{\infty}$ of sets in $\mathcal{M}$ such that $X = \cup_{i=1}^{\infty} X_i$, and $\mu(X_i) < \infty$ for every $i \in \mathbb{N}$. Let $\{E_i\}_{i=1}^{\infty}$ be a pairwise disjoint collection of measurable sets whose union is X. We then define a set function $\mu_i : \mathcal{M} \to [0, \infty]$ by $\mu_i(A) = \mu(A \cap E_i)$, for every $A \in \mathcal{M}$. Thus, each μ_i is a measure. Also, we have $\mu_i(X) = \mu(X \cap E_i) \leq \mu(X_i) < \infty$, so each μ_i is a finite measure. Similarly, let the set function $\nu_i : \mathcal{M} \to [0, \infty]$ by $\nu_i(A) = \nu(A \cap E_i)$, for every measurable set A. Next, let A be a measurable set such that $\mu_i(A) = 0$, and $\mu(A \cap E_i) = 0$. Since $\nu << \mu$, we obtain $\nu_i(A) = 0$ and $\nu(A \cap E_i) = 0$. Thus, we have $\nu_i << \mu_i$. By Radon-Nikodym theorem, for each $i \in \mathbb{N}$, there is a nonnegative measurable function f_i such that $\nu_i(E) = \int_E f_i d\mu_i$ for all measurable set E. Define f be a function $f(x) = f_i(x)$ if $x \in E_i$. Then, we consider $f = \sum_{i=1}^{\infty} f_i = \lim_{n \to \infty} \sum_{i=1}^{n} f_i$, therefore f is a nonnegative measurable function since it is pointwise limit of nonnegative measurable functions. Thus, we can conclude that, for any measurable set $G \in \mathcal{M}$ and ν is a measure,

$$\nu(G) = \sum_{i=1}^{\infty} \nu(G \cap E_i) = \sum_{i=1}^{\infty} \nu_i(G \cap E_i) = \sum_{i=1}^{\infty} \int_{G \cap E_i} f_i d\mu_i = \sum_{i=1}^{\infty} \int_{G \cap E_i} f d\mu = \int_G f d\mu$$

Hence, proved. ■

Problem 4
Prove the uniqueness assertion in the Lebesgue Decomposition Theorem. (Note: although this was a suggested problem, we did not discuss the Lebesgue Decomposition Theorem in class. Therefore, this problem requires that you first read the statement and proof of that Theorem in the book. You may use the results from parts (i) - (iii) of this problem in your solution to part (iv).)

Proof: **(iv)**
Let $(X, \mathcal{M}, \mu)$ be a σ- finite measure space and ν be a σ-finite measure on $\mathcal{M}$. Let $\nu = \nu_1 + \nu_2$ be Lebesgue decomposition of finite measure ν with respect to finite measure μ, so we have $\nu_1 \perp \mu$ and $\nu_2 << \mu$. In order to prove the uniqueness consider another pair $v_1\,'$ and $v_2\,'$ such that

$$\nu = \nu_1 + \nu_2 = \nu_1\,' + \nu_2\,'$$

so, $\nu_1\,' - \nu_1 = \nu_2 - \nu_2\,'$. Suppose that $\nu_1 \perp \mu, \nu_1\,' \perp \mu$, then there exists decomposition

$$\overline{A} \cup \overline{B} = X \quad \text{and} \quad A\,' \cup B\,' = X$$

such that $\nu_1(\overline{B}) = 0$, $\mu(\overline{A}) = 0$, $\nu_1\,'(B\,') = 0$ and $\mu(A\,') = 0$. By the decomposition, we can obtain that $(\overline{A} \cup A\,') \cup (\overline{B} \cup B\,') = X$ and $(\overline{A} \cup A\,') \cap (\overline{B} \cup B\,') = \emptyset$. Let $A = \overline{A} \cup A\,'$ and $B = \overline{B} \cup B\,'$, we have $(\nu_1 + \nu_1\,')(B) = \nu_1(B) + \nu_1\,'(B) = 0$ and $\mu(A) \leq \mu(\overline{A}) + \mu(A\,') = 0$, so $\mu(A) = 0$. Hence, $\nu_1 + \nu_1\,' \perp \mu$.
Next, we prove that $\nu_2\,' << \mu$ and $\nu_2 << \mu$, so $\nu_2\,' + \nu_2 << \mu$. For any measurable set $E \in \mathcal{M}$ such that $\mu(E) = 0$, we have $\nu_2\,'(E) = 0$ and $\nu_2(E) = 0$. Therefore, $(\nu_2\,' + \nu_2)(E) = \nu_2\,'(E) + \nu_2(E) = 0$ and so $\nu_2\,' + \nu_2 << \mu$, which implies that $\nu_2\,'(B) = \nu_2(B) = 0$ due to $\mu(B) = 0$. Next, for any measurable set $E \subset B$, we have

$$\nu_1\,'(E) - \nu_1(E) = \nu_2(E) - \nu_2\,'(E)$$

and also $\nu_2(E) \leq \nu_2(B) = 0$ and $\nu_2\,'(E) \leq \nu_2\,'(B) = 0$. Thus, we have $\nu_1(E) = \nu_1\,'(E)$. Then, for any measurable set G, we can conclude that

$$\nu_1(G) = \nu_1(G \cap B) + \nu_1(G \cap A) = \nu_1\,'(G \cap B) + 0 = \nu_1\,'(G \cap B) + \nu_1\,'(G \cap A) = \nu_1\,'(G)$$

Therefore, we proved $\nu_1 = \nu_1\,'$ on X. By the similar way, we can have $\nu_2 = \nu_2\,'$ on X. ■