Problem Set 1, due Thursday, September 23, at 1230pm: section 1.4 #35; section 1.5 #38; section 2.2 #7; and section 2.2 #8.

Problem 1
Show that the collection of Borel sets is the smallest σ-algebra that contains the closed sets.

Proof: Denote $\mathscr{B}$ as the collection of Borel sets, and $\sigma(\mathscr{F})$ as the smallest σ-algebra containing $\mathscr{F}$ for any collection of sets F. Consider that $\mathscr{F}$ is the collection of closed sets of $\mathbb{R}$ and $F \in \mathscr{F}$. Then, $\mathbb{R}\backslash\mathcal{F}$ is an open set, by the definition of Borel set, therefore $\mathbb{R}\backslash\mathcal{F} \in \mathscr{B}$. So we can conclude that $\mathscr{F} \subset \mathscr{B}$, since $\mathcal{F} = \mathbb{R}\backslash(\mathbb{R}\backslash\mathcal{F}) \in \mathscr{B}$, which implies that $\sigma(\mathscr{F}) \in \mathscr{B}$. Next, consider the collection of open sets A of $\mathbb{R}$ for which $a \in A$. We can see that $\mathbb{R}\backslash a \in \mathscr{F}$, then $a = \mathbb{R}\backslash(\mathbb{R}\backslash a) \in \sigma(\mathscr{F})$, which implies that A is a subset of $\sigma(\mathscr{F})$. Hence, we conclude $\sigma(A) \subseteq \sigma(\mathscr{F}) = \mathscr{B}$. ■

Problem 2
We call an extended real number a cluster point of a sequence $\{a_n\}$ if a subsequences converges to this extended real number. Show that $\liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$ and $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$.

Proof: First, we denote the largest cluster point of $\{a_n\}$ by a_{large} and the smallest cluster point of $\{a_n\}$ by a_{small}. Then, if $\lim_{n\to\infty}\sup_{k\geq n}\{a_k\} = +\infty$, since $\sup_{k\geq n}\{a_k\}$ is monotone increasing, hence $\sup_{k\geq n}\{a_k\} = +\infty$ for any $n > N$. Therefore, there is $n_1 \in \mathbb{N}$ such that $a_{n_1} > 1$ and $n_2 \in \mathbb{N}$ such that $a_{n_2} > 2\dots$ i.e. $\{a_{n_k}\} \to +\infty$ as $k \to \infty$. So, $+\infty$ is a cluster point and $a_{large} = +\infty$, i.e. $a_{large} = \lim_{n\to\infty}\sup_{k\geq n}\{a_k\}$. If $\lim_{n\to\infty}\sup_{k\geq n}\{a_k\} < +\infty$, then $\lim_{n\to\infty}\sup\{a_k\} = -\infty$ or a bounded real number. Suppose a is any cluster of $\{a_n\}$, then there is a subsequence of $\{a_n\}$ such that $a_{n_i} \to a$ as $i \to \infty$. Hence, for any $n \in \mathbb{N}$, we have $n_i \geq i \geq n$ and $a_{n_i} \leq \sup_{k\geq n}\{a_k\}$, $a = \lim_{i\to\infty} a_{n_i} \leq \sup_{k\geq n}\{a_k\}$ and $a_{large} \leq \lim_{n\to\infty}\sup_{k\geq n}\{a_k\}$. On the other hand, for any $y > a$, there are finitely many terms of $\{a_n\}$ in $[y, +\infty]$ at most. So, there is $N \in \mathbb{N}$ such that when $k > N$, we have $a_k < y$, i.e. $\sup_{k\geq n}\{a_k\} \leq y$ for $n > N$, i.e. $\lim_{n\to\infty}\sup_{k\geq n}\{a_k\} \leq y$. Let $y \to a^+$, then $\lim_{n\to\infty}\sup_{k\geq n}\{a_k\} \leq a_{large}$. Consequently, $a_{large} = \lim_{n\to\infty}\sup_{k\geq n}\{a_k\}$. If $\lim_{n\to\infty}\sup_{k\geq n}\{a_k\} = -\infty$, since $\inf_{k\geq n}\{a_k\}$ is monotone decreasing, hence $\inf_{k\geq n}\{a_k\} = -\infty$. For any $n > N$, choose n_1 such that $a_{n_1} < -1$ and $n_2 > n_1$ such that $a_{n_2} < -2\dots$ So we get a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ and $a_{n_i} \to -\infty$, i.e. $-\infty$ is a cluster point and is the smallest. So $a_{small} = \lim_{n\to\infty}\inf_{k\geq n}\{a_k\}$. If $\lim_{n\to\infty}\inf_{k\geq n}\{a_k\} > -\infty$, then $\lim_{n\to\infty}\inf_{k\geq n}\{a_k\} = +\infty$ or a bounded real number. Let a be any cluster point of $\{a_n\}$, then there is a subsequence $\{a_{n_i}\}$ and $a_{n_i} \to a, (i \to \infty)$. For any $n \in \mathbb{N}$, $n_i \geq i \geq n$, hence $a_{n_i} \geq \inf_{k\geq n}\{a_k\}$, $a = \lim_{i\to\infty} a_{n_i} \geq \inf_{k\geq n}\{a_k\}$ and $a \geq \lim_{n\to\infty}\inf_{k\geq n}\{a_k\}$, so $a_{small} \geq \lim_{n\to\infty}\inf_{k\geq n}\{a_k\}$. On the other hand, for any $y \geq a_{small}$, there are at most finitely many terms of $\{a_n\}$ in $(-\infty, y]$. Therefore, there is N such that $a_k > y$ when $k > N$. So, $\inf_{k\geq n}\{a_k\} \geq y$ when $n > N$. So, we have $\inf_{k\geq n}\{a_k\} \geq y$, i.e. $\lim_{n\to\infty}\inf_{k\geq n}\{a_k\} \geq y$. Let $y \to a^-_{small}$, then $\lim_{n\to\infty}\inf_{k\geq n}\{a_k\} \geq a_{small}$. Consequently, $a_{small} = \lim_{n\to\infty}\inf_{k\geq n}\{a_k\}$. ■

Problem 3
A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which

$$E \subseteq G \text{ and } m^*(G) = m^*(E)$$

Proof: Given set E is bounded and then we know that the outer measure of E is finite, $m^*(E) < \infty$. Consider a countable collection of nonempty, open, bounded intervals $\{I_{n,j}\}_{j=1}^{\infty}$ for any n, which covers E. So, we have

$$\sum_{j=1}^{\infty} m^*(I_{n,j}) \leq m^*(E) + 1/n, \forall n \in \mathbb{N}$$

Now, define $G = \cap_{n\in\mathbb{N}}\cup_{j\geq 1} I_{n,j}$ and $\cup_{j\geq 1} I_{n,j}$ is open for all n, and contains E. By definition and monotonicity of m^*, we have

$$m^*(E) \leq m^*(G) \leq m^*(\cup_{j=1}^{\infty} I_{n,j}) \leq \sum_{j=1}^{\infty} m^*(I_{n,j}) \leq m^*(E) + 1/n$$

This inequality holds for all n, then we can conclude that for any bounded set E , there exists a G_δ set for which $E \subseteq G$ and $m^*(E) = m^*(G)$. ■

Problem 4
Let B be the set of rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers b.Prove that $\sum_{k=1}^n m^*(I_k) \geq 1$.

Proof: Given that $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers B, where $B = \mathbb{Q} \cap [0, 1]$. Then, we have $B \subset \cup_{k=1}^n I_k$, or say $\mathbb{Q} \cap [0, 1] \subseteq \cup_{k=1}^n I_k$. Next, we take closure on both sides for $\mathbb{Q} \cap [0, 1] \subseteq \cup_{k=1}^n I_k$, we will have

$$[0, 1] \subseteq \cup_{k=1}^n \overline{I_k}$$

Then, by the monotonicity and subadditivity of m^* , the following inequality holds,

$$1 = m^*([0, 1]) \leq m^*(\cup_{k=1}^n \overline{I_k}) \leq \sum_{k=1}^n m^*(\overline{I_k}) = \sum_{k=1}^n m^*(I_k)$$

Hence, we conclude that $\sum_{k=1}^n m^*(I_k) \geq 1$. ■