Problem Set 3 Revision. Due on Dec.9.

Problem 1
Let f be a bounded, measurable function on a set E of finite measure. For a measure. For a measurable subset $A \subset E$, prove that $\int_A f = \int_E f \cdot \chi_A$.

Proof: **ORIGINAL PROOF**: Since f is a bounded, measurable function on a set E of finite measure. By theorem 4, f is integrable over E. Now, consider that $\{\varphi_n\}$ and $\{\psi_n\}$ are two sequences of simple function such that $\varphi_n \leq f \cdot \chi_A \leq \psi_n$ and satisfies

$$\lim_{n\to\infty} \int_E \varphi_n = \int_E f \cdot \chi_A = \lim_{n\to\infty} \int_E \psi_n$$

Then we have $\varphi_n \leq f \leq \psi_n$ on A, so there must be

$$\int_A \varphi_n \leq \int_A f \leq \int_A \psi_n$$

for all n. By taking the limits of above inequality, we obtain $\int_A f = \int_E f \cdot \chi_A$. ■

In the original proof, I made some statement unclear, such as the reason of $\varphi_n \leq f \leq \psi_n$, and the structure of proof is not reasonable. There is no clear logic from every step to step. IN this new version, I chose a little bit different way to prove it but still work with simple functions.

New proof: Since f is a bounded measurable function on a set E of finite measure and $A \subset E$, f is bounded and measurable on A. Consider that for each simple function ψ on E such that $\varphi \leq f$, then because a measurable subset $A \subset E$, we have $\varphi \cdot \chi_A \leq f \cdot \chi_A$. Since φ is simple, we get $\int_A \varphi = \int_E \varphi \cdot \chi_A$. So,

$$\int_A \varphi = \int_E f \cdot \chi_A \leq \int_E f \cdot \chi_A$$

By taking sup, we have $sup \int_A \varphi = sup \int_E \varphi \cdot \chi_A \leq \int_E f \cdot \chi_A$.
Also, for each simple function ψ on E such that such that $\psi \geq f$, then because a measurable subset $A \subset E$, we have $\psi \cdot \chi_A \geq f \cdot \chi_A$. Since ψ is a simple function, we have $\int_A \psi = \int_E \psi \cdot \chi_A$. Therefore,

$$\int_A \psi = \int_E \psi \cdot \chi_A \geq \int_E f \cdot \chi_A$$

By taking inf, we obtain $int \int_A \psi = inf \int_E \psi \cdot \chi_A \geq \int_E f \cdot \chi_A$. Thus, we can conclude that

$$sup \int_E \varphi \cdot \chi_A \leq \int_E f \cdot \chi_A \leq inf \int_E \psi \cdot \chi_A$$

Then, given that f is a measurable function,

$$sup \int_A \varphi = \int_A f = inf \int_A \psi$$

Hence, $\int_E f \cdot \chi_A = \int_A f$. ■