Revision for Problem Set 1 (1.4#35) and (1.4#38)

Problem 1
Show that the collection of Borel sets is the smallest σ-algebra that contains the closed sets.

Proof: Denote B as the collection of Borel sets. By definition of Borel set, for every open set is a Borel set and since σ-algebra is closed with respect to formation of complements, we infer from proposition 11 that every closed set is a Borel set. To show B is the smallest, let F be any σ-algebra which contains all closed sets. By the definition (ii) of σ-algebra, F contains all open sets. Then we know that the Borel sets of real numbers is the smallest σ-algebra of sets of real numbers that contains all of the open sets of real numbers, so we can conclude $B \subset F$. Therefore, B is the smallest among all such F. ■

I tried to reprove this question in the way that uses definition of σ-algebra and Borel set such as "closed under complement". I think the previous proof that I wrote was a little bit confusing both in terms of the notations and overall structures, since I even feel this way when I read my own proof. Similar thing happens more severe in my proof of Problem 2. In this new vision, I changed my overall structure of the argument in order to make it clearer and typos of notations are gone. Finally, the proof is as simple and direct as possible than the previous one.

Problem 2
We call an extended real number a cluster point of a sequence $\{a_n\}$ if a subsequences converges to this extended real number. Show that $\liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$ and $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$.

Proof: Suppose that $\{a_n\}$ is a sequence of real numbers with $\limsup\{a_n\} = L$. We need to first show that L is a cluster point of the sequence. For any $\epsilon > 0$, there exists n_1 such that $a_k < L+\epsilon$ for $k \geq n_1$. Then, there exists $k_1 \geq n_1$ such that $a_{k_1} > L - \epsilon$, so $|a_{k_1} - L| < \epsilon$. Now, suppose that there are some $\{a_{k_1}, a_{k_2}, ..., a_{k_i}\}$ and $\{n_1, n_2, ..., n_i\}$ that are selected. Let $\max\{k_1, k_2, ..., k_i\} < n_{i+1}$, we have $|a_{i+1} - L| < \epsilon$. Hence, by the definition of cluster point, the subsequence $\{a_{k_i}\}$ converges to L and L is a cluster point of $\{a_n\}$. To see that L is the largest cluster point, consider a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = L$. If $L = \infty$, then $\{a_n\}$, by proposition 19 (ii), is not bounded above, so $\limsup a_n = \infty$. If $L < \infty$, then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $a_{n_k} > L - \epsilon$ for all $k \geq N$. Hence, by proposition 14, $\limsup\{a_n\} \geq L - \epsilon$ for all $\epsilon > 0$. Since ϵ was arbitrary, we can conclude $\limsup\{a_n\} \geq L$. To show that $\liminf\{a_n\} = L\,'$, where we suppose $L\,'$ is a cluster point, by proposition 19 (iii), we know that $\liminf\{a_n\} = -\limsup\{-a_n\}$. Next, consider a subsequence $\{-a_{n_k}\}$ and $\lim_{k\to\infty}\{-a_{n_k}\} = \limsup\{-a_n\}$. This implies, by the theorem 18, $\lim_{k\to\infty}\{a_{n_k}\} = \lim_{k\to\infty} -\{-a_{n_k}\} = -\limsup\{-a_n\} = \liminf\{a_n\}$. To see that $L\,'$ is the smallest cluster point, suppose that there exists a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = L\,'$. This implies that $\lim_{k\to\infty}\{-a_{n_k}\} = -L\,' \leq \limsup\{-a_n\}$, hence we have $L\,' \geq -\limsup\{-a_n\} = \liminf\{a_n\}$. ■

In this new vision, I tried to rearrange and changed my thoughts in a more reasonable and structural way in order to make the proof robust. Honestly speaking, my original one is very confusing for readers to understand what I wanted to convey and I have to revisit the topic. Now, the overall structure is clearer. This time, I first prove that L and $L\,'$ are the cluster points and then the largest and smallest respectively depending on the definition of cluster point, which more makes sense. Also, I tried to reduce the number of typos in my proof. Thank you.