Due Date: Nov 11

**Problem 4**
23. Express a measurable function as the difference of non-negative measurable functions and thereby prove the general Simple Approximation Theorem based in the special case of a non-negative measurable function.

*Proof:* Assume $f$ is an extended real-valued function on a measurable set $E$. Let $f^+ = max\{f, 0\}$ and $f^- = max\{-f, 0\}$ then both $f^+$ and $f^-$ are greater or equal to zero. We can notice that $f = f^+ - f^-$. Since $f^+$ and $f^-$ are non-negative and the proof for the non-negative case implies that there exists sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of simple functions on E which converge pointwise on $E$ to $f^+$ and $f^-$, and $0 \leq \varphi_n \leq f^+$, $0 \leq \psi_n \leq f^+$.
Then we define $\phi_n = \psi_n - \varphi_n$, the $\phi_n$ is a sequence of simple function on $E$ for all n, by the result of problem 19 in the textbook. To see that $\phi_n$ converges pointwise on $E$ to $f$.
If $f < 0$, we have

$$\phi_n = -\psi_n \to -f^- = f$$

If $f \geq 0$, then $f^- = 0$ and $\psi_n = 0$ for all n and thus

$$\phi_n = \varphi_n \to f^+ = f$$

Hence, we conclude that

$$|\phi_n| = |\varphi_n - \psi_n| \leq \varphi_n + \psi_n \leq f^+ + f^- = |f|$$

on E for all n. Therefore, $|\phi_n| < |f|$ on E.

**I tried to modify my previous proof a little bit in order to make it nicer. Then, since Simple Approximation Theorem is an if and only if statement, we need to prove it in another direction.**

Assume function $f$ is non-negative and measurable, we may choose $\{\varphi_n\}$ to be increasing. Let n be a natural number and $E_n = \{x \in E | f(x) \leq n\}$. By the simple approximation lemma, applied to the restriction of $f$ to $E_n$ and with choice of $\epsilon = \frac{1}{n}$. Then there exists $\varphi_n$ and $\psi_n$ simple functions on $E_n$ such that

$$0 \leq \varphi_n \leq f \leq \psi_n \text{ on } E_n \text{ and } 0 \leq \psi_n - \varphi_n \leq \frac{1}{n} \text{ on } E_n$$

We can notice that

$$0 \leq \varphi_n \leq f \text{ and } 0 \leq f - \varphi_n \leq \psi_n - \varphi_n < \frac{1}{n} \text{ on } E_n$$

Extend $\varphi_n$ to rest of $E$ by letting $\varphi_n(x) = n$ if $f(x) > n$. So the $\varphi_n$ is a simple function defined on $E$ and satisfies $0 \leq \varphi_n \leq f$ on $E$. Now, we will show that $\varphi_n$ converges pointwise to $f$ on $E$. Let $x$ belongs to $E$. If $x$ is such that $f(x) < \infty$ , then we choose a $N \in \mathbb{N}$ for which $f(x) < N$. Since $x \in E_n$, for all $n \geq N$ we have

$$0 \leq f(x) - \varphi(x) < \frac{1}{n}$$

and thus $\varphi_n \to f$ pointwise.
If $x$ is such that $f(x) = \infty$, then we take $\varphi(x) = n$ for all n. It follows that $\varphi_n \to f$ pointwise. By replacing each $\varphi_n$ with $\max\{\varphi_1, \ldots, \varphi_n\}$ we obtain $\{\varphi_n\}$ increasing.