Due on Nov.18

Problem 1
Let f be a bounded, measurable function on a set E of finite measure. For a measure. For a measurable subset $A \subset E$, prove that $\int_A f = \int_E f \cdot \chi_A$.

Proof: Since f is a bounded, measurable function on a set E of finite measure. By theorem 4, f is integrable over E. Now, consider that $\{\varphi_n\}$ and $\{\psi_n\}$ are two sequences of simple function such that $\varphi_n \leq f \cdot \chi_A \leq \psi_n$ and satisfies

$$\lim_{n\to\infty} \int_E \varphi_n = \int_E f \cdot \chi_A = \lim_{n\to\infty} \int_E \psi_n$$

Then we have $\varphi_n \leq f \leq \psi_n$ on A, so there must be

$$\int_A \varphi_n \leq \int_A f \leq \int_A \psi_n$$

for all n. By taking the limits of above inequality, we obtain $\int_A f = \int_E f \cdot \chi_A$. ■

Problem 2
Let $\{f_n\}$ be a sequence of integrable function on E for which $f_n \to f$ almost everywhere on E, with f integrable over E. Show that $\int_E |f - f_n| \to 0$ if and only if $\int_E |f_n| \to \int_E |f|$.

Proof: We first suppose that $\int_E |f - f_n| \to 0$. Since $||f_n| - |f|| \leq |f - f_n|$ on E for all n and $|f_n| - |f| \to 0$ a.e. on E. Then by the previous result we have proven (General Lebesgue Dominated Convergence Theorem),

$$\lim_{n\to\infty} \int_E (|f_n| - |f|) = 0$$

Given that f is integrable, by linearity of integrability, we have

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$$

Conversely, we suppose $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. Then we have

$$\lim_{n\to\infty} \int_E (|f_n| + |f|) = 2 \int_E |f| < \infty$$

We know that $|f_n - f| \leq |f_n| + |f|$ on E for all n and by $|f - f_n| \to 0$ a.e. on E and general Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_E |f - f_n| = 0$$

Hence, proved. ■

Problem 3
Let f be integrable over $\mathbb{R}$. Show that for each $t \in \mathbb{R}$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x+t)dx$$

Proof: Given that f is integrable over $(-\infty, +\infty)$. Suppose $f = \chi_E$ on a measurable set E. For any $t \in \mathbb{R}$, we have

$$\chi_E(x+t) = \chi_{E-t}(x)$$

for all $x \in \mathbb{R}$. By the translation invariance of Lebesgue measure, we have

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \chi_E(x)dx = m(E) = m(E-t) = \int_{-\infty}^{\infty} \chi_{E-t}(x)dx = \int_{-\infty}^{\infty} \chi_E(x+t)dx = \int_{-\infty}^{\infty} f(x+t)dx$$

This result also applies to simple function via linearity of integration. Then, by Simple Approximation Theorem, if f is a measurable non-negative function, there exists an increasing sequence $\{\varphi_n\}$of non-negative simple functions such that $\{\varphi_n\}$converges pointwise to f. So for each x, $\{\varphi_n(x+t)\}$ is an increasing sequence which converges to $f(x+t)$. Now, since $\int \chi_E(x)dx = m(E) = m(E-t) = \int \chi_E(x+t)dx$ for any measurable set E, we have $\int \varphi_n(x)dx = \int \varphi_n(x+t)$. By Monotone Convergence Theorem, we have

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \varphi_n(x+t)dx = \int_{-\infty}^{\infty} f(x+t)dx$$

The result can also be extended to general integrable function f by using the result of non-negative function to positive and negative parts of f. ■

Problem 4
Let f be of bounded variation on [a,b]. Show that

$$\int_a^b |f\,'| \leq T_a^b(f)$$

Proof: Given that f is of bounded variation on $[a,b]$, we define $u(x) = T_a^x$ for all $x \in [a,b]$. Since $f \in BV[a,b]$, by the corollary we have proven in class (if $f \in BV[a,b]$, then $f\,'$ exists a.e. on $[a,b]$), f is differentiable a.e. on $[a,b]$.
Next, we define $g(x) = P_a^x(f)$ and $h(x) = N_a^x(f)$, then $g(x)$ and $h(x)$ are increasing, so $g\,'$and $h\,'$ exist on interval $[a,b]$. We know that

$$u(x) = T_a^x = P_a^x + N_a^x = g(x) + h(x)$$

and

$$f(x) = P_a^x - N_a^x + f(a) = g(x) - h(x) + f(a)$$

Then suppose that there exists a set $E \subseteq [a,b]$ with $m(E) = 0$ such that f and u are differentiable on $[a,b]\backslash E$. We have

$$u\,'(x) = g\,'(x) + h\,'(x) \quad \text{and} \quad f\,'(x) = g\,'(x) - h\,'(x)$$

Since $g\,'(x) \geq 0$ and $h\,'(x) \geq 0$,

$$|f\,'(x)| \leq |g\,'(x)| + |h\,'(x)| = g\,'(x) + h\,'(x) = u\,'(x)$$

Hence, we can conclude that $|f\,'(x)| \leq u\,'(x)$ a.e. on $[a,b]$. By monotonicity of integration and corollary 4 in Royden's book (Let f be an increasing function on the closed, bounded interval $[a,b]$. Then $f\,'$ is integrable over $[a,b]$ and $\int_a^b f \leq f(b) - f(a)$), we have

$$\int_a^b |f\,'(x)| \leq \int_a^b u\,'(x) \leq u(b) - u(a) = T_a^b(f)$$

Therefore, proved.

When I use some corollaries or theorems in the book and class, I restated them in the parenthesis in order to not make confusion here due to the different versions we use. Sorry for the messy structure of the proof.