Due on Oct 21 (Thursday)

Problem 1
24. Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

Proof: First, we can rewrite that

$$E_1 = (E_1 \cap E_2) \cup (E_1 \backslash E_2)$$
$$E_2 = (E_1 \cap E_2) \cup (E_2 \backslash E_1)$$
$$E_1 \cup E_2 = (E_1 \cap E_2) \cup (E_1 \backslash E_2) \cup (E_2 \backslash E_1)$$

Thus, by the finite additivity of m, we have

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1 \backslash E_2) + m(E_1 \cap E_2) + m(E_2 \backslash E_1) + m(E_1 \cap E_2)$$
$$= m(E_1) + m(E_2) \blacksquare$$

Problem 2
40.Show that there is an open set of real numbers that, contrary to intuition, has a boundary of positive measure. (Hint: Consider the complement of the generalized Cantor set of the preceding problem.)

Proof: Let F be as in problem 2.39, which is a generalized Cantor Set with measure $1 - \alpha$. Define its complement $\mathcal{O} = [0, 1]\backslash F$, then $\mathcal{O}$ is open. Since $\mathcal{O}$ is open, it does not contain any its boundary point. However, we know that $[0, 1]\backslash F$ is dense in $[0, 1]$ according to the problem 2.39, so every point in F will be a boundary point of $\mathcal{O}$. Therefore, $bd\mathcal{O} = F$, then $m(bd\mathcal{O}) = m(F) = 1 - \alpha$ for $0 < \alpha < 1$. $\blacksquare$

Problem 3
7. Let the function f be defined on a measurable set E. Show that f is measurable if and only if for each Borel set A, $f^{-1}(A)$ is measurable. (Hint: the collection of set A that have the property that $f^{-1}(A)$ is measurable is σ-algebra.)

Proof: Since every open set is a Borel set, by proposition 3.2, f is measurable if and only if for each open set O, $f^{-1}(O)$ is measurable. Given that $f^{-1}(A)$ is measurable for each Borel set A, then f is measurable because open sets are Borel sets. On the other hand, let

$$\mathscr{A} = \{A \subseteq \mathbf{R} : f^{-1}(A) \in m\}$$

By proposition 3.2, every open set in $\mathbf{R}$ belongs to $\mathscr{A}$. Then, we must show that $\mathscr{A}$ is a σ-algebra.

- Since $f^{-1}(\mathbf{R}) = E \in m$, so $\mathbf{R} \in \mathscr{A}$
- Suppose a countable collection of sets $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{A}$ for all n. Then, $f^{-1}(\cup_{n=1}^{\infty} A_n) = \cup_{n=1}^{\infty} f^{-1}(A_n)$ is measurable, because $f^{-1}(A_n) \in m$ for all n and the collection of measurable sets is closed under countable unions.
- Suppose $A \in \mathscr{A}$, we have $f^{-1}(A^c) = E\backslash f^{-1}(A) \in m$, because $f^{-1}(A)$ is measurable and the collection of measurable sets is closed under complement.

Hence, $\mathscr{A}$ is a σ-algebra on $\mathbf{R}$, which contains all open subsets of $\mathbf{R}$. It follows that $\mathbf{B} \in \mathscr{A}$ where $\mathbf{B}$ is a Borel-σ algebra. So, we have $f^{-1}(A)$ is measurable for each Borel set A. $\blacksquare$

Problem 4
23. Express a measurable function as the difference of non-negative measurable functions and thereby prove the general Simple Approximation Theorem based in the special case of a non-negative measurable function.

Proof: Assume f is an extended real-valued function on a measurable set E. Let $f^+ = max\{f, 0\}$ and $f^- = max\{-f, 0\}$ then both f^+ and f^- are greater or equal to zero. We can notice that $f = f^+ - f^-$. By simple Approximation Theorem, there exists sequences $\{\Phi_n\}$ and $\{\psi_n\}$ of simple functions on E which converge pointwise on E to f^+ and f^-, and $0 \leq \Phi_n \leq f^+$, $0 \leq \psi_n \leq f^+$. Then we define $\phi_n = \psi_n - \Phi_n$, the ϕ_n is a sequence of simple function on E for all n, by the result of problem 19 in the textbook. To see that ϕ_n converges pointwise on E to f. If $f < 0$, we have

$$\phi_n = -\psi_n \to -f^- = f$$

If $f \geq 0$, so $\psi_n = 0$ for all n and

$$\phi_n = \Phi_n \to f^+ = f$$

Hence, we conclude that

$$|\phi_n| = |\Phi_n - \psi_n| \leq \Phi_n + \psi_n \leq f^+ + f^- = |f|$$

on E for all n. ■