Markowitz portfolio theory

Goals:

- Study the Markowitz portfolio allocation problem.
- Introduce the efficient frontier.

Relevant literature:

- Cvitanić & Zapatero Ch. 5, Luenberger Ch. 6

Introduction

- Last time, we discussed how a risk averse investor participates in a market with a risky and risk-free asset.
- We saw how the position in the risky asset balances a trade-off between risk, return, and risk aversion.
- Today, we will analyze the trade-off between risk and return more deeply when there are multiple risky assets.

Review: return, risk, and risk aversion

- Consider an investor with utility function U
- The investor allocates his initial wealth W_0 among a risky security with price S, and a ZCB with constant return r.
- We concluded that the optimal share position in the risky asset satisfies

$$\Delta^* \approx \frac{1}{A(W_0)} \frac{\mathbb{E}\left[(S_1 - S_0 e^r)\right]}{\mathbb{E}\left[(S_1 - S_0 e^r)^2\right]},$$

where

$$A(w) = -\frac{U''(w)}{U'(w)},$$

is the absolute risk-aversion.

Review: return, risk, and risk aversion

$$\Delta^* \approx \frac{1}{A(W_0)} \frac{\mathbb{E}\left[(S_1 - S_0 e^r)\right]}{\mathbb{E}\left[(S_1 - S_0 e^r)^2\right]}.$$

- There is a tradeoff between return, risk, and risk aversion.
- The investor allocates:
  - More to the risky asset, the higher the expected excess return.
  - Less to the risky asset, the higher the excess return variance.
  - Less to the risky asset, the higher his risk aversion.

But, what about correlations?

- This approximate relationship holds for one risky asset.
- What happens when there are multiple risky assets?
  - Here, risk and return do not tell the entire story.
  - Holding positively correlated assets may increase the risk of a portfolio, because assets have a tendency to underperform at the same time.
  - Alternatively, holding negatively correlated assets may reduce portfolio risk.
- We need to think about correlation.

Refresher: correlation

- The correlation $\rho_{X,Y}$ of two random variables X, Y is

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y},$$

  where $\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$ and $\sigma_X^2 = \mathrm{Var}(X)$, $\sigma_Y^2 = \mathrm{Var}(Y)$.

- From Cauchy-Schwarz: $\rho_{X,Y} \in [-1, 1]$.
- $\rho_{X,Y} > 0$: *positively correlated*. X, Y tend to move in the same direction.
- $\rho_{X,Y} < 0$: *negatively correlated*. X, Y tend to move in opposite directions.
- $\rho_{X,Y} = 0$: *uncorrelated*. No directional statement.
- Larger $|\rho_{X,Y}|$ indicates stronger dependency.

Correlation and portfolio allocation

- Suppose the investor participates in a market with two risky securities $S^{(1)}, S^{(2)}$, as well as the ZCB.
- Assume the investor is long in both $S^{(1)}, S^{(2)}$.
- What happens when $S^{(1)}, S^{(2)}$ are positively correlated?
  - If $S^{(1)}$ goes down, then not only does the investor lose money in $S^{(1)}$, but this also (on average) causes $S^{(2)}$ to drop, leading to additional losses.
  - In fact, the investor is still exposed to risks from $S^{(1)}$ even when she does not hold $S^{(1)}$.
- Thus, the investor may decide to put less money in $S^{(1)}$ than she would if the two risky assets were uncorrelated.

Correlation and portfolio allocation

- Conversely, what happens when $S^{(1)}, S^{(2)}$ are negatively correlated?
  - The portfolio volatility is decreased since movements in $S^{(1)}$ are offset by movements in $S^{(2)}$, and vice-versa.
  - E.g.: when $S^{(1)}$ underperforms, $S^{(2)}$ may perform well.
  - This is known as **diversification**.
- The investor may decide to put more money in $S^{(2)}$ than she would if the two risky assets were uncorrelated
- The decision to invest in risky securities depends, not only on individual security risk and return, but also on the correlation between risky securities.

Investment portfolio (2 risk securities)

- For today's analysis, rather than considering terminal wealth, we look at portfolio return.
- Suppose there are two risky securities with prices $S^{(1)}$, and $S^{(2)}$, and (for now) no ZCB.
- The investor starts off with W_0 and puts a fraction Δ_1 in security 1 and Δ_2 in security 2.
  - $\Delta_i, i = 1, 2$ are now fractions of wealth, not share positions.
  - Full investment (for now) implies $\Delta_1 + \Delta_2 = 1$.
- Lastly, for now, assume long positions so $0 < \Delta_1, \Delta_2 < 1$.

Asset return and volatility

- Denote the (random) return of $S^{(1)}, S^{(2)}$ over the period by

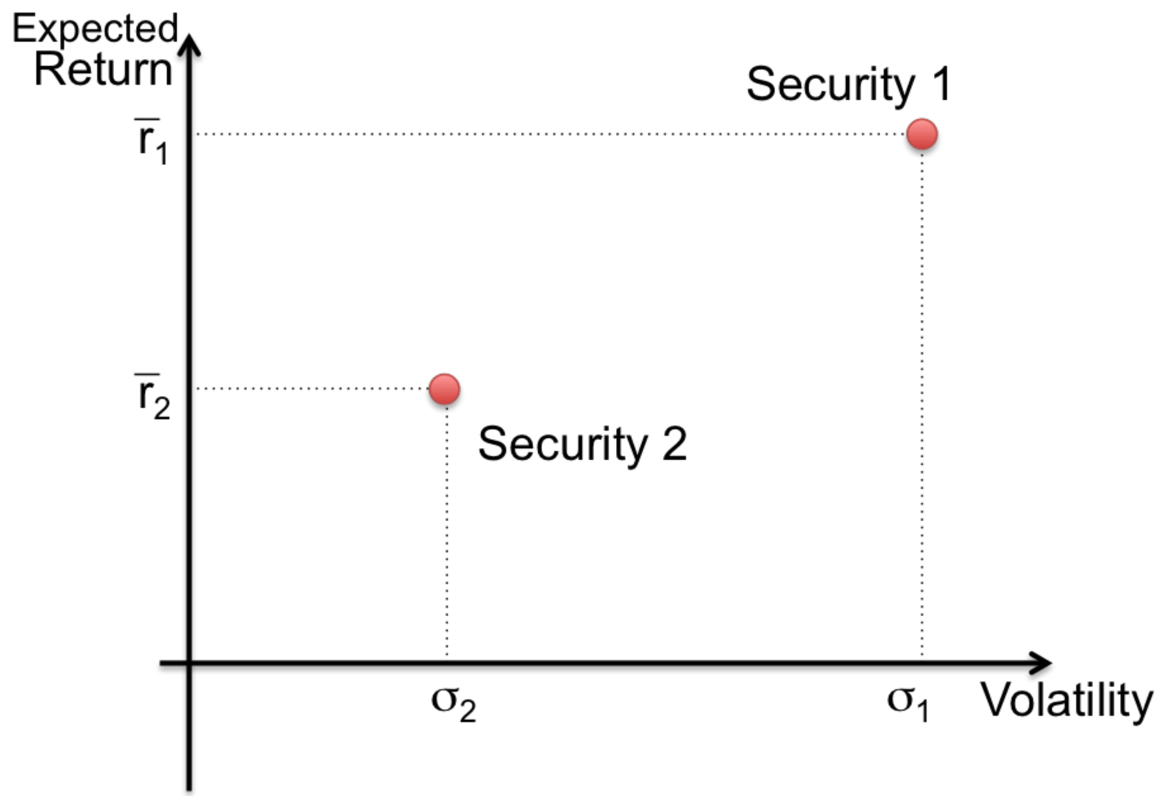
$$r_1 \triangleq \frac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}}; \qquad r_2 \triangleq \frac{S_1^{(2)} - S_0^{(2)}}{S_0^{(2)}}.$$

- Write $\bar{r}_1 = \mathbb{E}[r_1]$ and $\bar{r}_2 = \mathbb{E}[r_2]$ as the expected returns.
  - Without loss of generality we can assume $\bar{r}_1 > \bar{r}_2$.
- Next, define the variances

$$\sigma_1^2 = \mathrm{Var}(r_1); \qquad \sigma_2^2 = \mathrm{Var}(r_2),$$

  and assume $\sigma_1 > \sigma_2$.
- Lastly, let $\rho_{1,2}$ denote the correlation between r_1 and r_2.

Return vs volatility



Portfolio return

- For a portfolio $\Delta = (\Delta_1, \Delta_2)$ the (random) portfolio return is

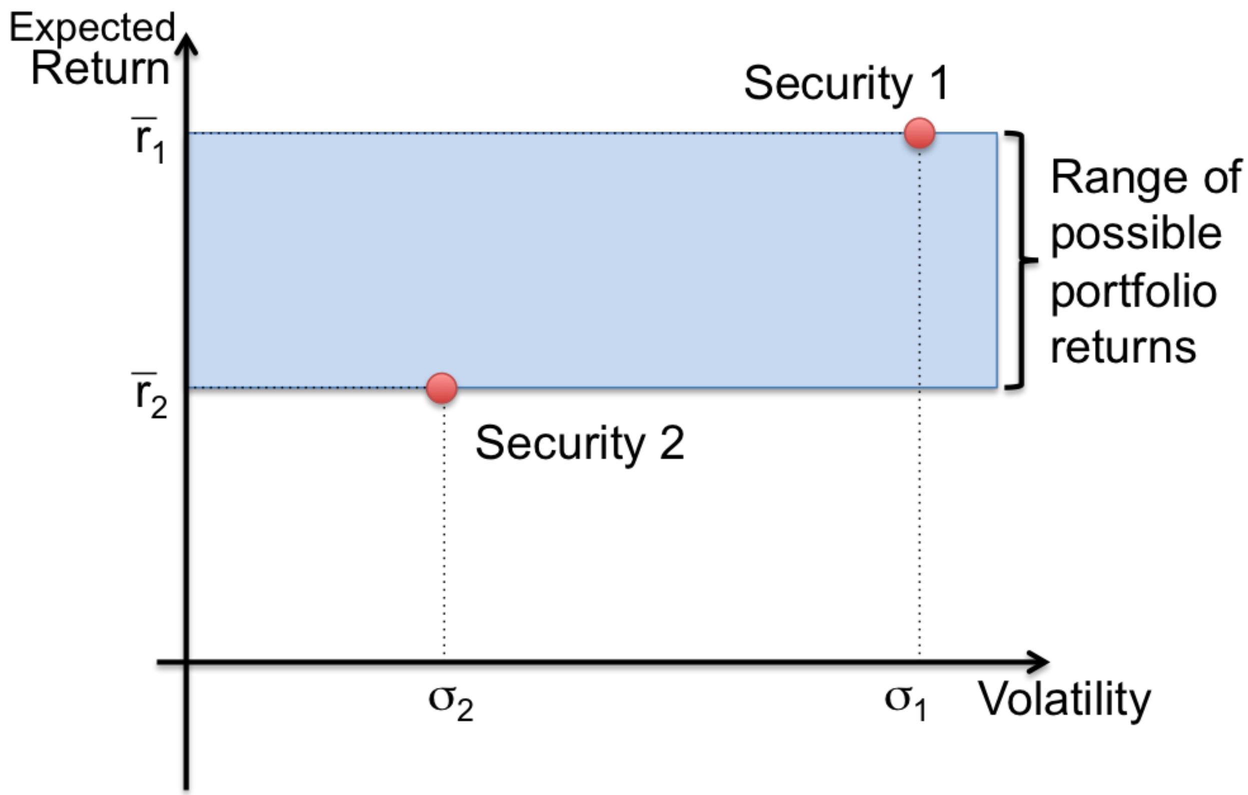
$$r(\Delta) \triangleq \frac{W_1 - W_0}{W_0} = \Delta_1 r_1 + \Delta_2 r_2 = \Delta_1 (r_1 - r_2) + r_2.$$

- The expected return of the portfolio is

$$\bar{r}(\Delta) = \mathbb{E}\left[\frac{W_1 - W_0}{W_0}\right] = \Delta_1 \left(\bar{r}_1 - \bar{r}_2\right) + \bar{r}_2.$$

- Since $\bar{r}_1 > \bar{r}_2$ by assumption:
  - The maximum expected return for any portfolio is $\bar{r}(\Delta) = \bar{r}_1$, obtained by setting $\Delta_1 = 1$ (all wealth in first asset).
  - The minimum expected return is $\bar{r}(\Delta) = \bar{r}_2$, obtained by setting $\Delta_1 = 0$ (all wealth in second asset).
  - Intermediate returns achievable by varying Δ_1 over $(0, 1)$.

Portfolio return



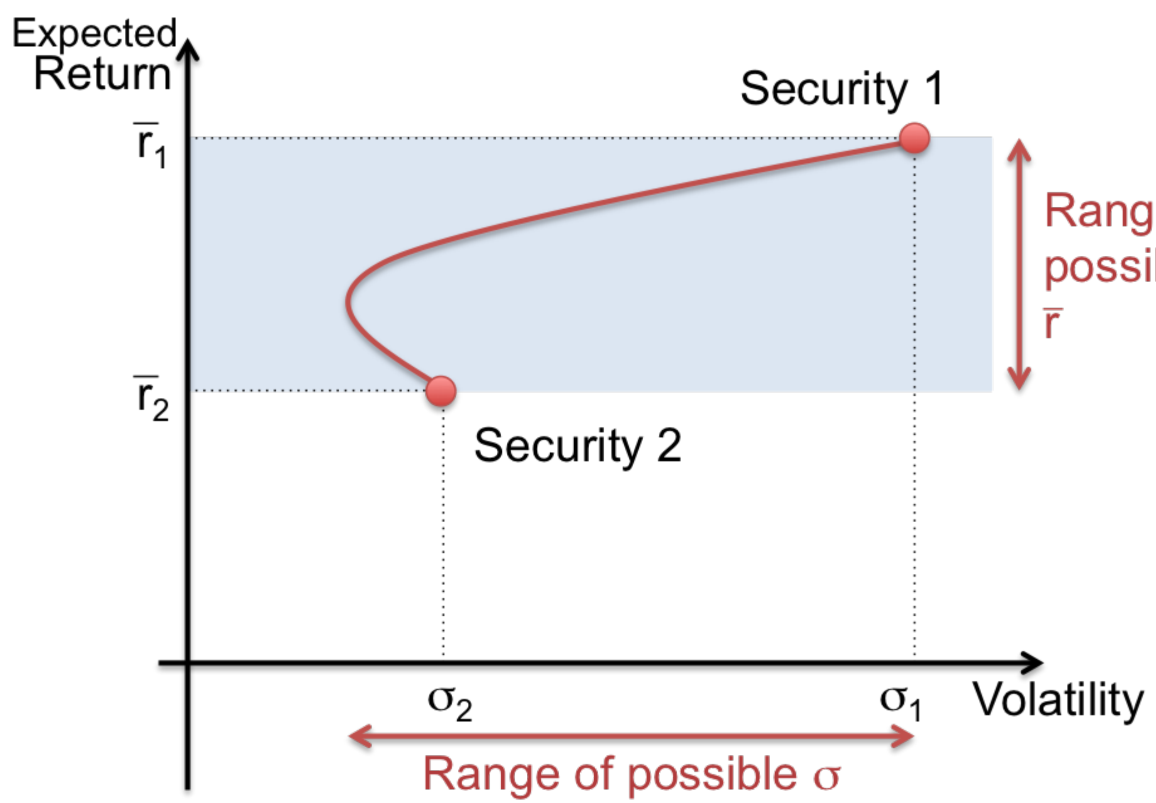
Portfolio volatility

- Expected return is not the only criterion for deciding a portfolio.
- The investor also is risk-averse and hence dislikes volatility.
- The portfolio return has variance

$$\begin{aligned}\sigma^2(\Delta) &= \mathrm{Var}(r(\Delta)) = \mathrm{Var}\left(\Delta_1 r_1 + \Delta_2 r_2\right);\\ &= \Delta_1^2\sigma_1^2 + \Delta_2^2\sigma_2^2 + 2\Delta_1\Delta_2\rho_{1,2}\sigma_1\sigma_2;\\ &= \left(\Delta_1(\sigma_1 - \sigma_2) + \sigma_2\right)^2 - 2\sigma_1\sigma_2(1-\rho_{1,2})\Delta_1(1-\Delta_1).\end{aligned}$$

- This defines a hyperbola in the return-volatility chart as Δ_1 ranges over $[0,1]$.
  - Identified via the map $\Delta_1 \to (\sigma(\Delta), \bar{r}(\Delta))$.

Portfolio volatility

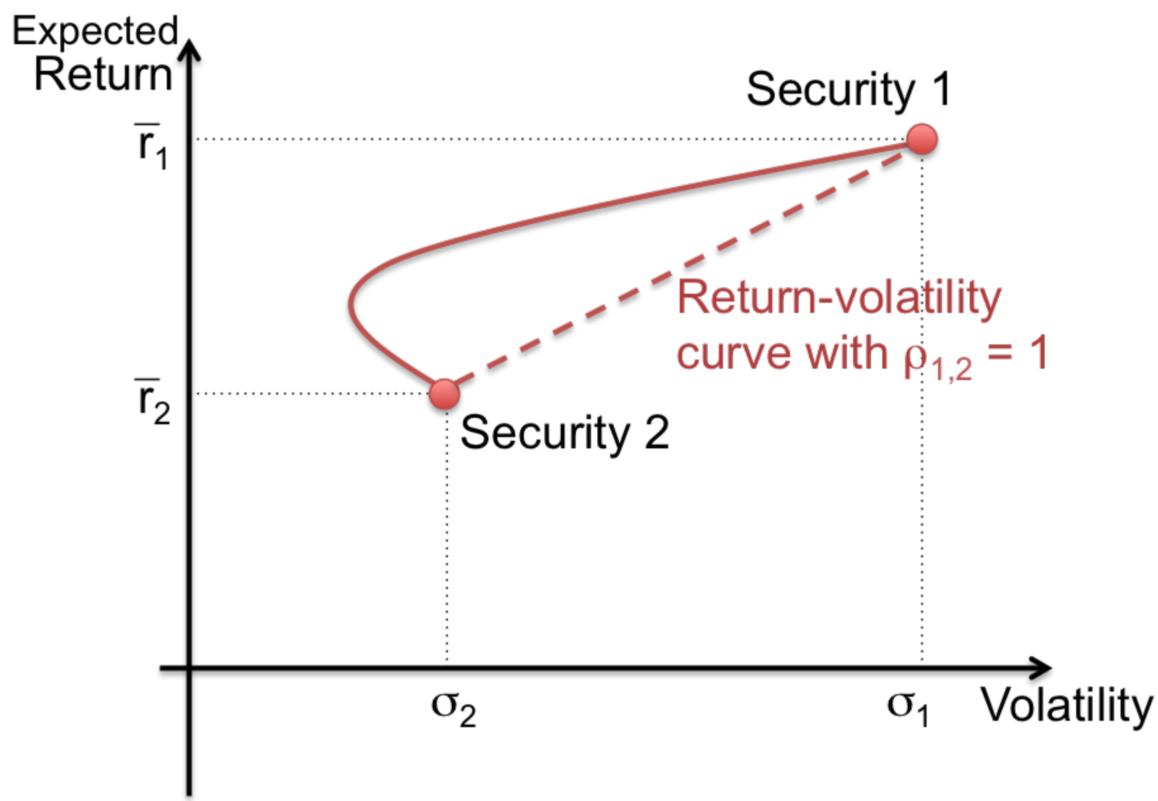


Portfolio volatility

$$\sigma^2(\Delta) = (\Delta_1(\sigma_1 - \sigma_2) + \sigma_2)^2 - 2\sigma_1\sigma_2(1 - \rho_{1,2})\Delta_1(1 - \Delta_1).$$

- For any $\Delta_1 \in [0, 1]$, the portfolio volatility is largest when $\rho_{1,2} = 1$.
  - Here, $\sigma(\Delta) = |\Delta_1(\sigma_1 - \sigma_2) + \sigma_2|$.
  - Since $\sigma_1 > \sigma_2$ by assumption, the maximum possible volatility is σ_1 when $\Delta_1 = 1$ (all wealth in first asset), and the minimum possible volatility is σ_2 when $\Delta_1 = 0$ (all wealth in second asset).

Portfolio volatility

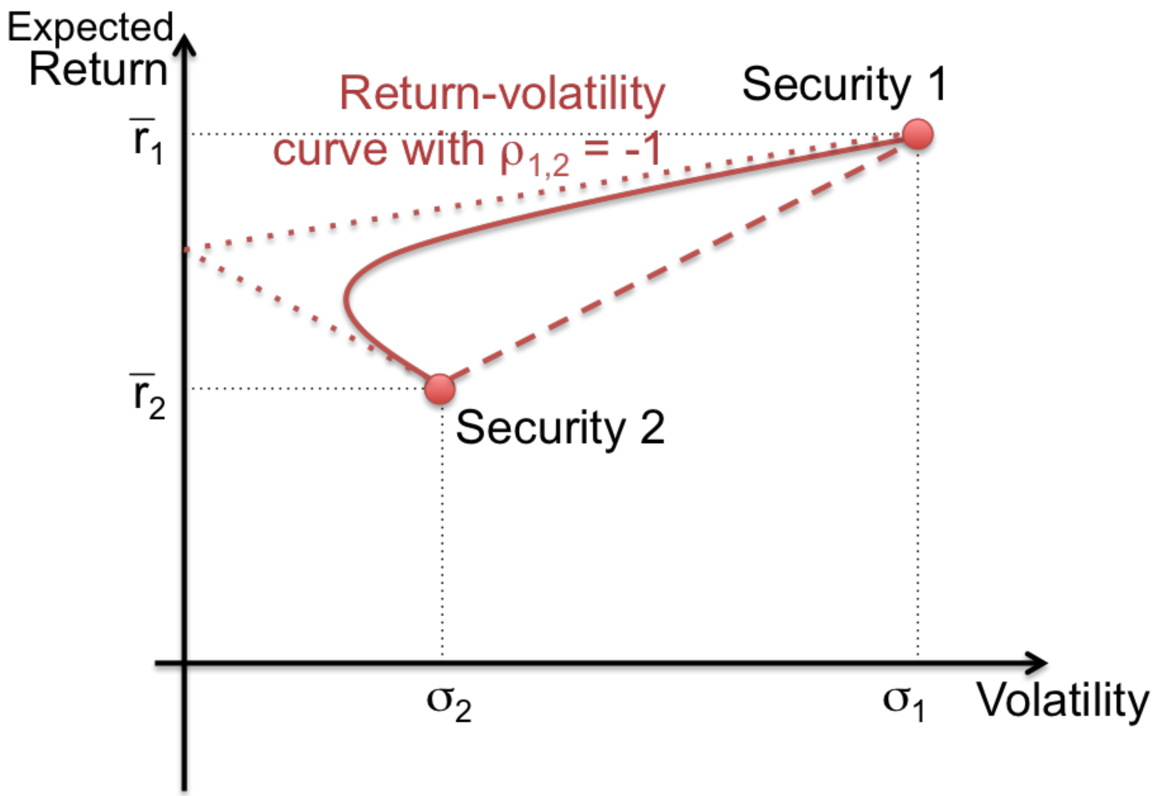


Portfolio volatility

$$\sigma^2(\Delta) = (\Delta_1(\sigma_1 - \sigma_2) + \sigma_2)^2 - 2\sigma_1\sigma_2(1 - \rho_{1,2})\Delta_1(1 - \Delta_1).$$

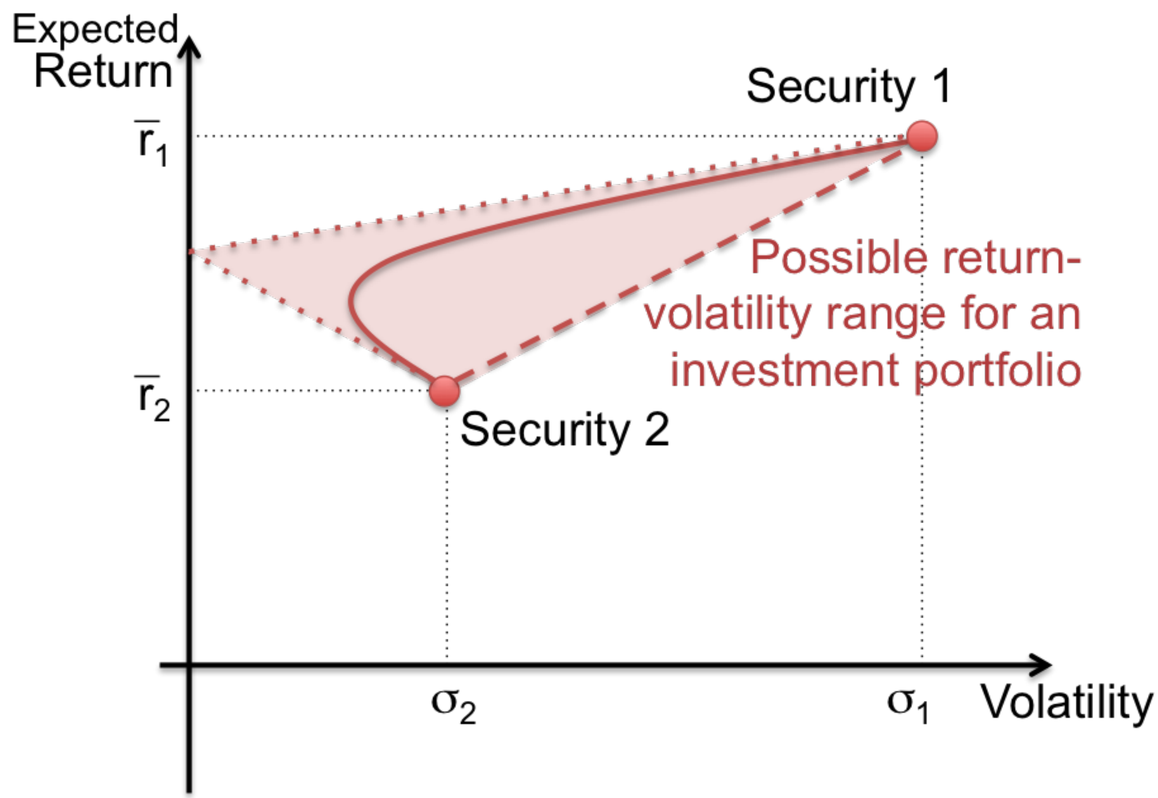
- For any given $\Delta_1 \in [0, 1]$, the portfolio volatility is smallest when $\rho_{1,2} = -1$.
  - Here, calculation shows $\sigma(\Delta) = |\Delta_1(\sigma_1 + \sigma_2) - \sigma_2|$.
  - Choosing $\Delta_1 = \sigma_2/(\sigma_1 + \sigma_2)$ yields zero portfolio volatility!
  - By holding the right amount of two negatively correlated assets, all portfolio volatility can be eliminated.

Portfolio volatility



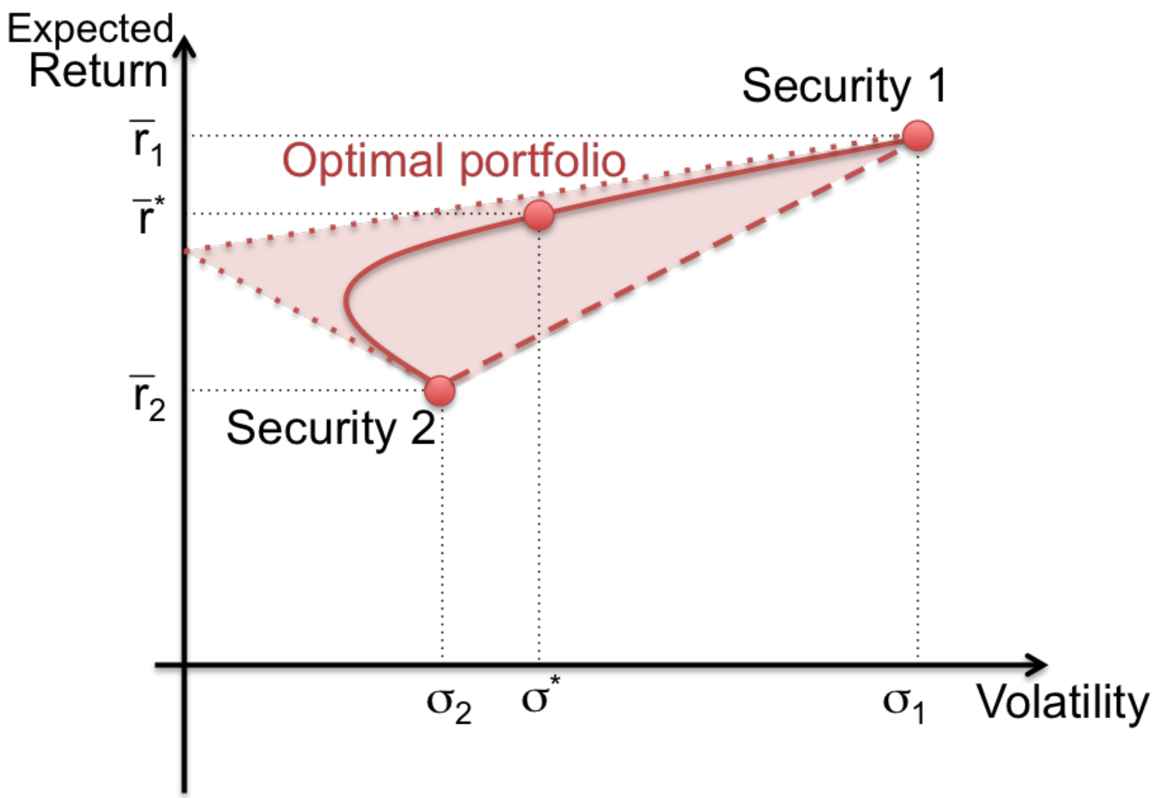
Return-volatility space

The map $\Delta_1 \to (\sigma(\Delta), \bar{r}(\Delta))$ traces out a curve in return-volatility space which lies in the triangle created as $\rho_{1,2}$ varies over $[-1, 1]$.



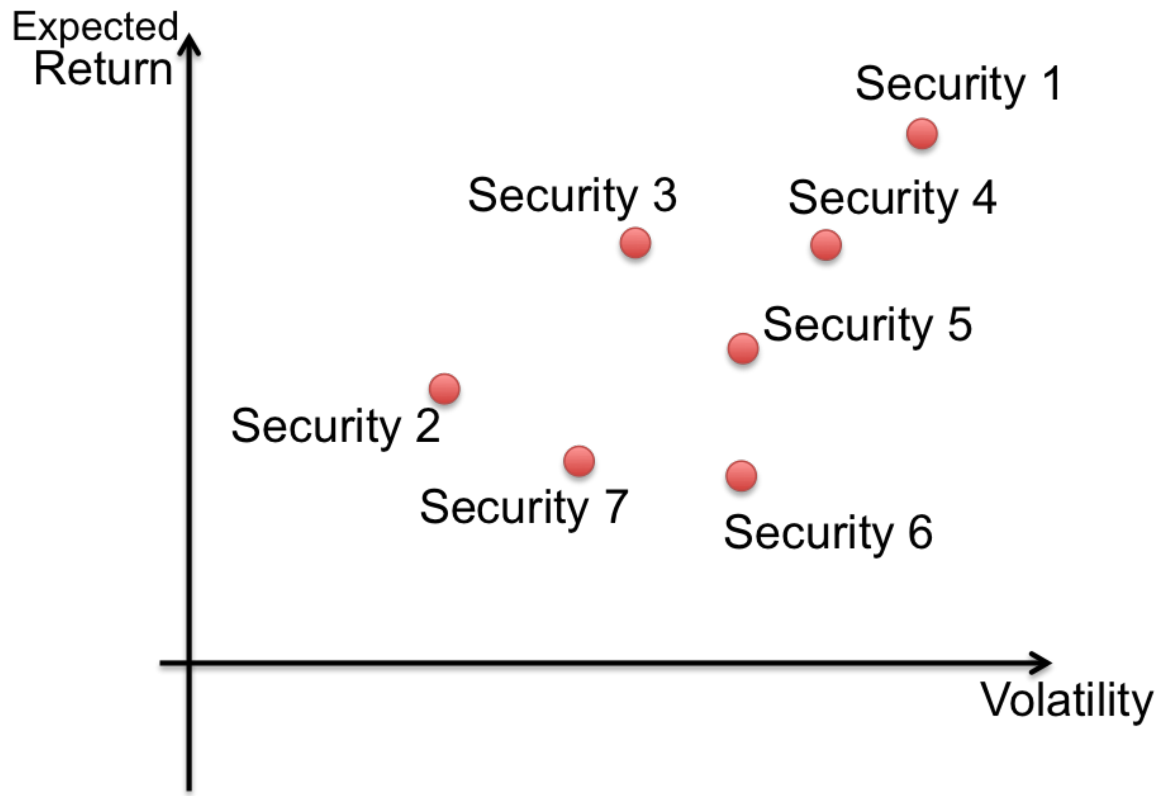
Optimal portfolio

The investor will choose the optimal portfolio $\Delta^* = (\Delta_1^*, 1 - \Delta_1^*)$ according to her preferences. This portfolio will lie on the return-volatility curve.



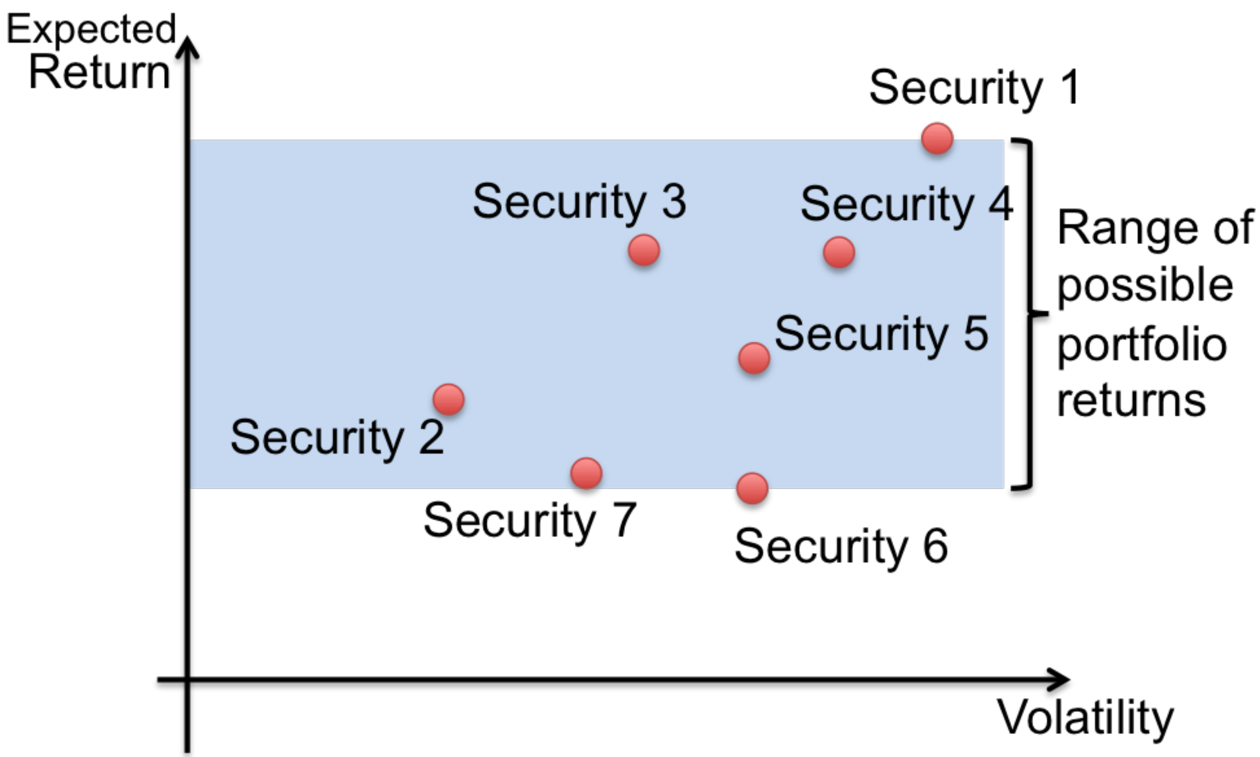
Many risky securities

What happens when there are more than two risky securities?



Many risky securities

As before, for long only portfolios, the expected return ranges from the lowest security expected return, to the highest security expected return.



Many risky securities

- Suppose there are M risky securities with expected returns $\bar{r}_m$, volatilities σ_m, and correlations $\rho_{m,k}$ for $m, k = 1, ..., M$.
- For a portfolio $\Delta = (\Delta_1, ..., \Delta_M)$ of fractions of wealth, the return, expected return and return variance are

$$
\begin{aligned}
r(\Delta) &= \sum_{m=1}^{M} \Delta_m r_m; \qquad \text{(random)} \\
\bar{r}(\Delta) &= \sum_{m=1}^{M} \Delta_m \bar{r}_m; \\
\sigma^2(\Delta) &= \sum_{m,k=1}^{M} \Delta_m \Delta_k \sigma_m \sigma_k \rho_{k,m}.
\end{aligned}
$$

- There are M control variables: the fractions $\Delta_1, \ldots, \Delta_M$ of the wealth invested in the risky securities.

Minimum variance for a target return

- Rather than assuming a specific utility function U for the investor, we assume the investor is "risk averse" in that she dislikes volatility.
- However, she is rational and wishes to achieve a certain target level of return.
- As such, she seeks the portfolio which minimizes variance, for a given level of return.

Minimum variance for a target return

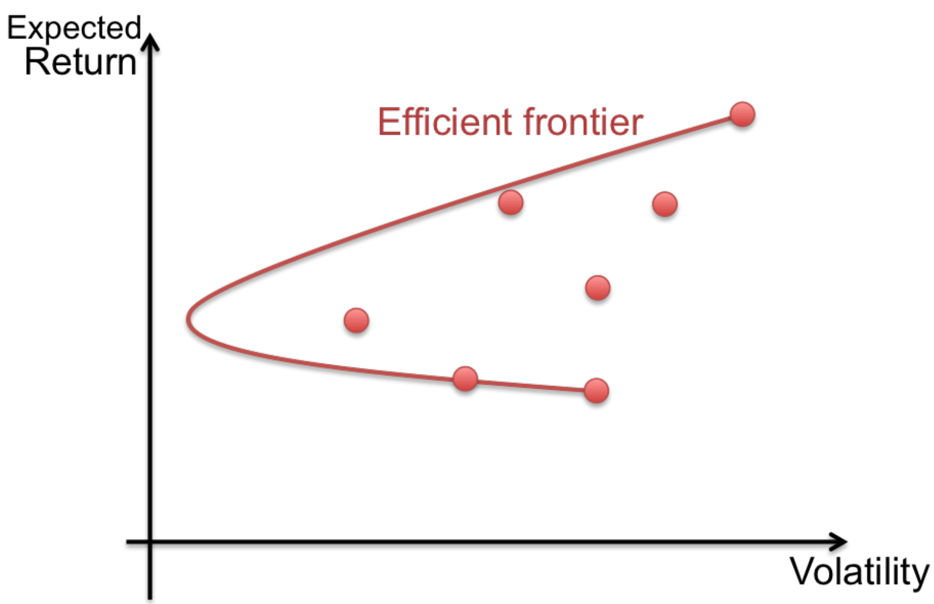
- Let $\bar{r}$ denote the target return.
- The investor wishes to choose a portfolio $\Delta^* = (\Delta_1^*, \ldots, \Delta_M^*)$ that solves:

$$\min_{\Delta} \sigma^2(\Delta) \text{ s.t. } \bar{r}(\Delta) = \bar{r}, \ \sum_{m=1}^{M} \Delta_m = 1$$

- Important notes:
  - We do not require long positions, so it may be that $\Delta_i < 0$ for some $i \in \{1, ..., M\}$.
  - There is (still for now) no ZCB, so the fractions in the risky assets must sum to 1.

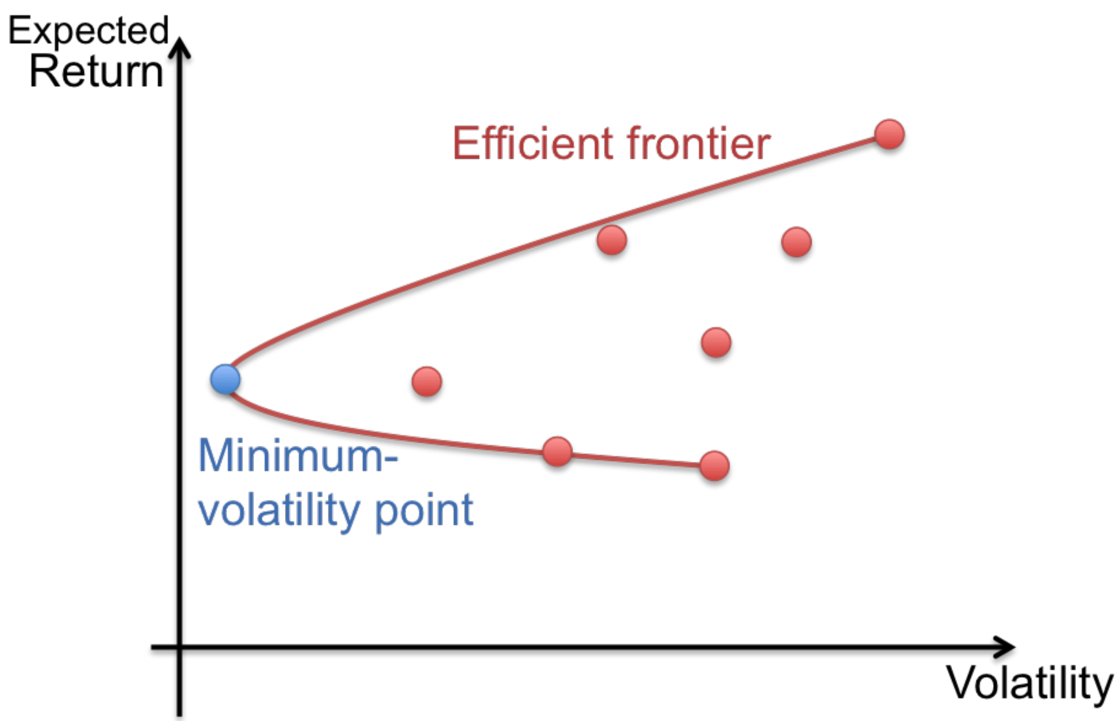
Efficient frontier

- Assume for a given $\bar{r}$ we can identify the variance optimal portfolio $\Delta^* = \Delta^*(\bar{r})$.
- The map $\bar{r} \to (\sigma(\Delta^*(\bar{r})), \bar{r})$ traces out the **efficient frontier**.
- Any risk-averse investor chooses a portfolio on the efficient frontier.



Global minimum variance (volatility)

The portfolio on the efficient frontier with the smallest volatility is called the **Global Minimum Variance** portfolio. Any rational investor would invest in portfolio that lies on the efficient frontier and above this point.



Markowitz

- We now compute the minimum variance portfolio for a given target expected return $\bar{r}$.

$$\min_{\Delta} \sigma^2(\Delta) \text{ s.t. } \bar{r}(\Delta) = \bar{r}, \ \sum_{m=1}^{M} \Delta_m = 1.$$

- Expanding this out, we need to solve

$$\min_{(\Delta_1,\ldots,\Delta_M)} \sum_{m,k=1}^{M} \Delta_m \Delta_k \sigma_m \sigma_k \rho_{k,m};$$
$$s.t. \ \sum_{m=1}^{M} \Delta_m \bar{r}_m = \bar{r}, \quad \sum_{m=1}^{M} \Delta_m = 1.$$

- The above problem is known as the **Markowitz portfolio allocation problem**.

Markowitz

- It is cleaner to use matrix notation. Thus, we set
  - $\Sigma_{k,m} \triangleq \sigma_m \sigma_k \rho_{k,m} = \mathrm{Cov}(r_k, r_m)$ for $k, m = 1, ..., M$.
    - Covariance matrix.
  - $\vec{r} = (\bar{r}_1, ..., \bar{r}_m)$: vector of expected security returns.
  - $\vec{1} = (1, ..., 1)$: vector of ones.
  - $\Delta = (\Delta_1, \ldots, \Delta_M)$: vector of portfolio weights.
- Then, our problem is to solve

$$\min_{\Delta} \Delta^{\mathrm{T}} \Sigma \Delta \quad \text{s.t.} \quad \Delta^{\mathrm{T}} \vec{r} = \bar{r}, \quad \Delta^{\mathrm{T}} \vec{1} = 1.$$

- The standard approach to solve this problem is to use Lagrange multipliers.

Lagrange multipliers

- Introduce two additional variables, λ_1 and λ_2, which are called the **Lagrange multipliers**.
- The solution to the constrained optimization problem can be obtained by first solving the unconstrained problem:

$$\min_{\Delta,\lambda_1,\lambda_2} \left(\Delta^{\mathrm{T}}\Sigma\Delta - \lambda_1\left(\Delta^{\mathrm{T}}\vec{r} - \bar{r}\right) - \lambda_2\left(\Delta^{\mathrm{T}}\vec{1} - 1\right)\right).$$

- The optimal solution $\Delta^*, \lambda_1^*, \lambda_2^*$ satisfies

$$\begin{aligned} 2\Sigma\Delta^* - \lambda_1^*\vec{r} - \lambda_2^*\vec{1} &= 0; \\ \lambda_1^*\left((\Delta^*)^{\mathrm{T}}\vec{r} - \bar{r}\right) &= 0; \\ \lambda_2^*\left((\Delta^*)^{\mathrm{T}}\vec{1} - 1\right) &= 0. \end{aligned}$$

- These equations are the **Karush-Kuhn-Tucker** (KKT) conditions.

Markowitz solution

- From the first equation we see

$$\Delta^* = \frac{1}{2}\lambda_1^* \Sigma^{-1}\vec{r} + \frac{1}{2}\lambda_2^* \Sigma^{-1}\vec{1}.$$

- A lengthy calculation gives explicit formulas for λ_1^*, λ_2^* so that equations $2, 3$ above are satisfied.
- Define the following two core portfolios
  - Maximum Sharpe Ratio: $\Delta_{\mathrm{MSR}} \triangleq \frac{\Sigma^{-1}\vec{r}}{\vec{1}^{\mathrm{T}}\Sigma^{-1}\vec{r}}$.
  - Global Minimum Variance: $\Delta_{\mathrm{GMV}} \triangleq \frac{\Sigma^{-1}\vec{1}}{\vec{1}^{\mathrm{T}}\Sigma^{-1}\vec{1}}$.
- By construction: $\Delta_{\mathrm{MSR}}^{\mathrm{T}}\vec{1} = 1$, $\Delta_{\mathrm{GMV}}^{\mathrm{T}}\vec{1} = 1$.
- In order for $(\Delta^*)^{\mathrm{T}}\vec{1} = 1$ it must be that for some α^*

$$\Delta^* = \alpha^* \Delta_{\mathrm{MSR}} + (1 - \alpha^*)\Delta_{\mathrm{GMV}}.$$

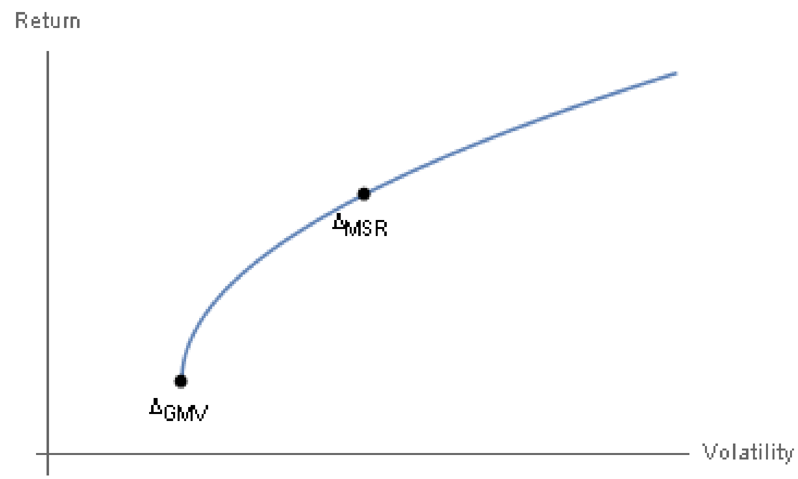
Two Fund Theorem

- Δ_{MSR} gives highest expected return per unit volatility. Solves

$$\max_{\Delta} \frac{\Delta^{\mathrm{T}}\vec{r}}{\sqrt{\Delta^{\mathrm{T}}\Sigma\Delta}} \quad \text{s.t.} \quad \Delta^{\mathrm{T}}\vec{1} = 1.$$

- Δ_{GMV} gives lowest possible variance. Solves

$$\min_{\Delta} \Delta^{\mathrm{T}}\Sigma\Delta \quad \text{s.t.} \quad \Delta^{\mathrm{T}}\vec{1} = 1.$$



Two Fund Theorem

- Both Δ_{MSR} and Δ_{GMV} lie on the efficient frontier.
  - However, they may not yield an expected return equal to our target $\bar{r}$.
- The optimal portfolio for a target return of $\bar{r}$ is a combination of the two core portfolios.

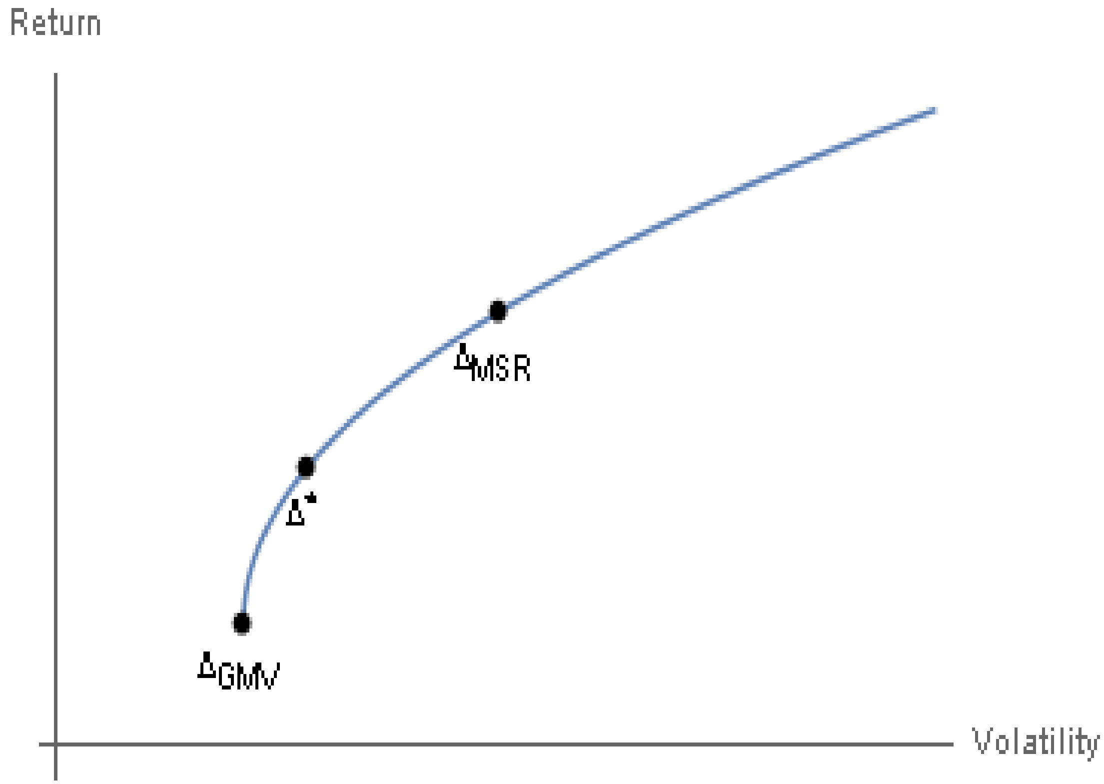
$$\Delta^* = \alpha^* \Delta_{\mathrm{MSR}} + (1 - \alpha^*)\Delta_{\mathrm{GMV}},$$

where α^* is chosen to achieve the target return:

$$\bar{r} \stackrel{?}{=} (\Delta^*)^{\mathrm{T}} \vec{r} = \alpha^* \bar{r}_{\mathrm{MSR}} + (1 - \alpha^*)\bar{r}_{\mathrm{GMV}};$$

$$\Longrightarrow \alpha^* = \alpha(\bar{r}) = \frac{\bar{r} - \bar{r}_{\mathrm{GMV}}}{\bar{r}_{\mathrm{MSR}} - \bar{r}_{\mathrm{GMV}}}.$$

Two Fund Theorem



Two-fund theorem

- In the Markowitz model, a volatility-averse investor only needs to decide how to allocate her wealth among two efficient portfolios.
- Instead of choosing the fractions of wealth $\Delta = (\Delta_1, \ldots, \Delta_M)$ invested in each security, she chooses her exposures $(\alpha^*, 1-\alpha^*)$ to the core portfolios Δ_{MSR}, Δ_{GMV}, in order to achieve the target expected return.
- This is the **two-fund theorem**.
- Any risk-averse investor needs to invest only in two *mutual funds* to achieve her investment goals.

Inclusion of a risk-free asset

- So far, we have ignored the risk-free asset (ZCB).
- How does the portfolio allocation problem change when the investor can put money in the ZCB?
- If the investor has a low target expected return, she may invest heavily in the ZCB.
- If the investor has a high target expected return, she may invest heavily in risky securities.
- The solution to the portfolio allocation problem changes when we introduce a risk-free security.

Markowitz with a risk-free asset

- The investor must decide what fraction Δ_f of her wealth to invest in the ZCB, and what fractions $\Delta = (\Delta_1, \ldots, \Delta_M)$ to invest in the M risky assets.
- Let r_f denote the (non-random) risk-free return. Since it is constant, we have $\mathrm{Var}(r_f) = 0$ and $\mathrm{Cov}(r_f, r_m) = 0$ for $m = 1, \ldots, M$.
  - Investing the ZCB changes the return, but does not add variance.
- The portfolio allocation problem is:

$$\min_{(\Delta_f, \Delta)} \Delta^{\mathrm{T}} \Sigma \Delta \quad \text{s.t. } \Delta_f r_f + \Delta^{\mathrm{T}} \vec{r} = \bar{r}, \Delta_f + \Delta^{\mathrm{T}} \vec{1} = 1.$$

Markowitz with a risk-free asset

- The investor can first choose the portfolio Δ of risky securities, and then buy the necessary amount of ZCB $\Delta_f = 1 - \Delta^{\mathrm{T}}\vec{1}$ to finance these investments.

- Therefore, we can rewrite the optimization problem as:

$$\min_{\Delta} \Delta^{\mathrm{T}}\Sigma\Delta \quad \text{s.t.} \quad \Delta^{\mathrm{T}}\vec{r}_e = \bar{r}_e,$$

  where

$$\vec{r}_e \triangleq \vec{r} - r_f\vec{1}; \qquad \bar{r}_e \triangleq \bar{r} - r_f.$$

- This is a similar problem. The differences are that now we are comparing expected **excess returns** to variances, and there is no restriction that $\Delta^{\mathrm{T}}\vec{1} = 1$.

Markowitz with a risk-free asset

- The first-order conditions for the Lagrange optimization problem are:

$$2\Sigma\Delta^* - \lambda_1^* \vec{r}_e = 0;$$
$$\lambda_1^* \left((\Delta^*)^{\mathrm{T}} \vec{r}_e - \bar{r}_e \right) = 0.$$

- In this case, a direct calculation shows

$$\Delta^* = \alpha(\bar{r})^* \times \frac{\Sigma^{-1}\vec{r}_e}{\vec{1}^{\mathrm{T}}\Sigma^{-1}\vec{r}_e}; \quad \alpha(\bar{r})^* = \bar{r}_e \times \frac{\vec{1}^{\mathrm{T}}\Sigma^{-1}\vec{r}_e}{\vec{r}_e^{\,\mathrm{T}}\Sigma^{-1}\vec{r}_e}.$$

- Invest in the Maximum Sharpe Ratio portfolio $\Delta_{\mathrm{MSR},e}$ which is found using the expected excess return $\vec{r}_e$ rather than the expected return $\vec{r}$.

Markowitz with a risk-free asset

- Thus

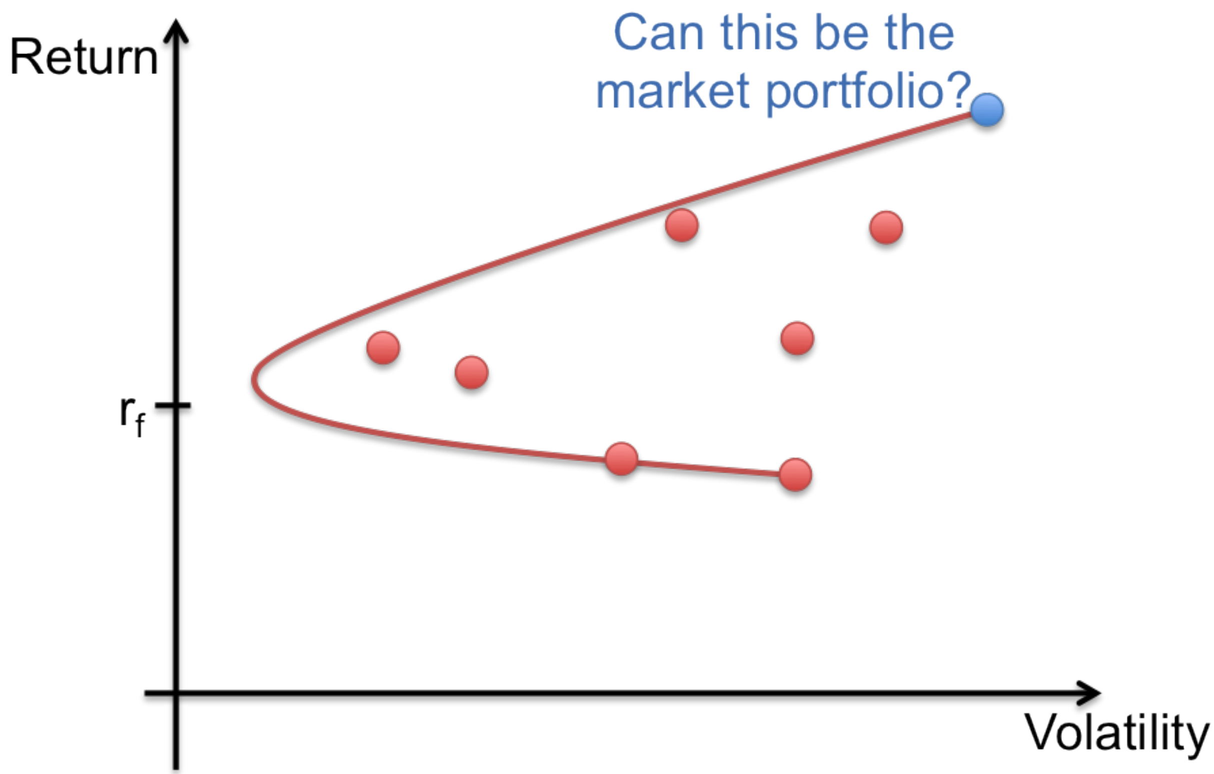
$$\begin{aligned} \Delta^* = \alpha(\bar{r})^* \Delta_{\mathrm{MSR,e}} : &\qquad \text{risky assets position.} \\ 1 - (\Delta^*)^{\mathrm{T}} \vec{1} = 1 - \alpha(\bar{r})^* : &\qquad \text{ZCB position.} \end{aligned}$$

- The constant $\alpha(\bar{r})^*$ is chosen so that the portfolio excess return is the target excess return.
- The risky portfolio $\Delta_{MSR,e}$ is the Maximum Sharpe Ratio portfolio when we use excess return instead of return. It is on the excess return - volatility efficient frontier.
- When there is a risk-free asset, the investor allocates a fraction Δ_f of her wealth in the risk-free asset, with the remaining fraction invested in the Maximum Sharpe Ratio portfolio.

One-fund theorem

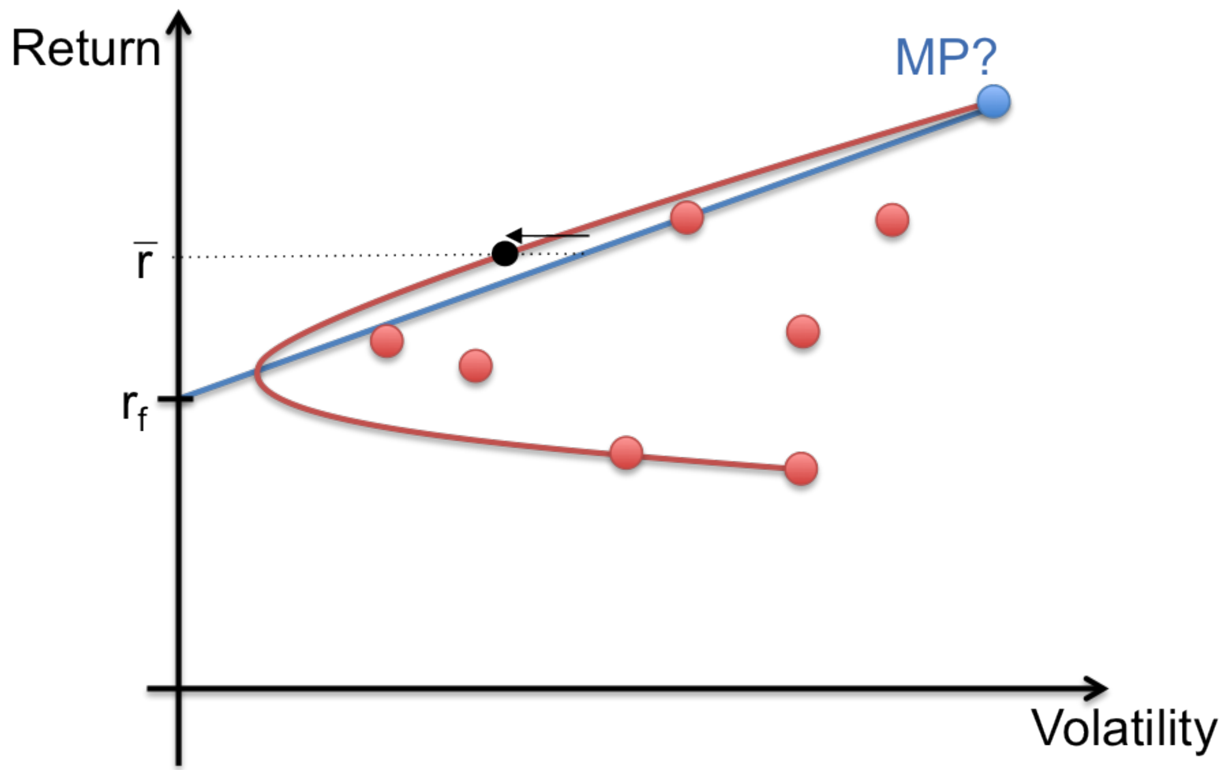
- Whenever there is a risk-free security, a volatility-averse investor invests in a single mutual fund, as well as the risk-free security.
- This is the **one-fund theorem**.
- Every investor holds the risk-free security and the “market portfolio“, but in different proportions.
- Thus, every investor holds the assets in the market portfolio.
- The market portfolio is the Maximum Sharpe Ratio portfolio in the excess return - volatility space.

Market portfolio: graphical representation



If the blue portfolio is the market portfolio, then the line between the risk-free rate and the blue portfolio contains all portfolios in which a volatility-averse investor would invest.

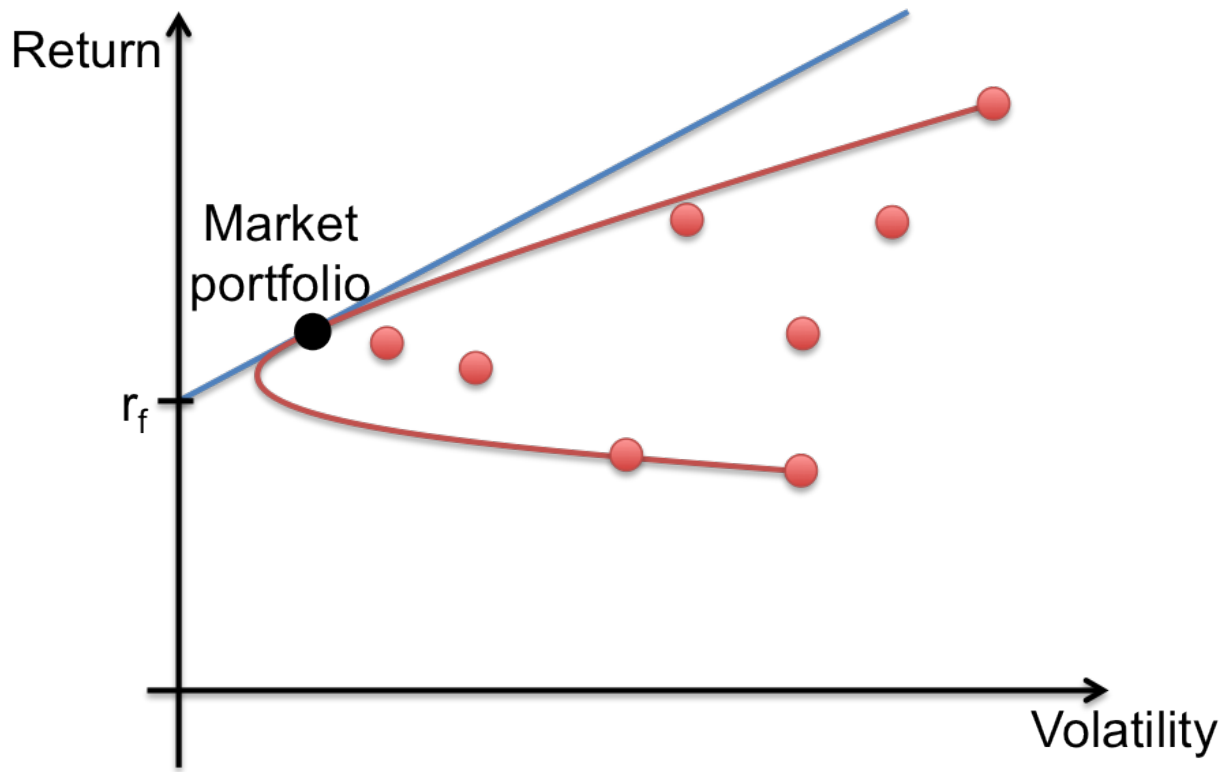
Market portfolio: graphical representation



The blue dot cannot be the market portfolio because an investor with target expected return $\bar{r}$ can achieve a smaller volatility by investing in the black portfolio.

Market portfolio: graphical representation

The market portfolio must lie on the tangential point of the line between the risk-free return and the efficient frontier.



Summary

- When choosing to invest in multiple risky securities, a risk-averse investor is not only worried about individual security risk and return, she is also worried about correlations.
- A rational, risk-averse investor will always try to obtain the greatest expected return with the smallest possible volatility.
- The efficient frontier contains all portfolios of risky securities that achieve a target expected return with minimal volatility.
- If there is no risk-free asset, a volatility-averse investor will choose a portfolio on the efficient frontier.

Summary

- Any portfolio on the efficient frontier can be constructed by investing in two efficient mutual funds: the Maximum Sharpe Ratio and Global Minimum Variance funds. The weight in each fund is determined by the target return.

- If there is a risk-free security, a volatility-averse investor will invest in the risk-free asset and a single mutual fund: the Maximum Sharpe Ratio fund (now computed using excess return). The weight is again determined by the target return.