MF702 Final Exam

December 16, 2020

Please write your answers on separate paper. Write down clearly the question number for each solution. Mark your submission with a page number in each page. Scan or take pictures of your solutions and update them to the assignment session on Questromtools.

*There are **3 pages** in total for this exam.*

You may need the following formula throughout the exam:

1) $\sum_{i=1}^{n} \lambda^i = \frac{\lambda}{1-\lambda}(1-\lambda^n)$ for $0 < \lambda < 1$.

2) If $\lambda = (1+y/2)^{-1}$ then $\lambda/(1-\lambda) = 2/y$.

1. (20 Points) Consider a long-short market-neutral equity hedge fund. The hedge fund aims to choose a portfolio weight Δ to solve the following constrained optimization problem:

$$\begin{aligned} &\text{Maximize } \Delta^\top \overrightarrow{r} - \frac{\gamma}{2}\left(\Delta^\top \Sigma \Delta\right), \\ &\text{subject to } \Delta^\top \overrightarrow{1} = 0. \end{aligned}$$

Here $\overrightarrow{r}$ is the expected return of different stocks and Σ is the covariance matrix of stock returns. In the objective function, $\Delta^\top \overrightarrow{r}$ is the portfolio expected return, $\Delta^\top \Sigma \Delta$ is the variance of portfolio return, $\gamma/2$ can be thought as the risk aversion of the hedge fund. The constrain means that all weights adding up to zero, i.e., the sum of long weight is exactly the sum of short weight, therefore, the short position exactly finances the long position.

Show that the optimal portfolio is

$$\Delta^* = \frac{1}{\gamma}\Sigma^{-1}\left(\overrightarrow{r} - \ell\,\overrightarrow{1}\right), \quad \text{where } \ell = \frac{\overrightarrow{1}^\top \Sigma^{-1} \overrightarrow{r}}{\overrightarrow{1}^\top \Sigma^{-1} \overrightarrow{1}}.$$

2. (20 Points) Recall that for a given set of discount factors $d(t), t = .5, 1, 1.5, ..., T$ the par rates $C(t)$ are given by

$$1 = \frac{C(t)}{2}\sum_{j=1}^{2t} d\left(\frac{j}{2}\right) + d(t).$$

Assume the spot curve is upward sloping: i.e. $\hat{r}(t-.5) \leq \hat{r}(t)$ for $t = 1, 1.5, ..., T$. Show that

$$C(t) \leq \hat{r}(t); \qquad t = .5, 1, 1.5, ..., T.$$

3. (30 Points) Assume the current spot rate curve is flat at 4%. You have \$1,000,000 to invest and are considering buying ONE of the following three securities:

(1) A 15 year zero coupon bond.

(2) A 20 year par bond.

(3) A perpetuity with payment A. Payment is made twice per year and each payment is A.

In this problem, do the following:

(a) (8 Points) Find A so that the perpetuity has market value \$1,000,000.

(b) (8 points) Prove that the yield based duration and convexity for an perpetuity are

$$\begin{aligned} D_a &= \frac{1}{y}, \\ C_a &= \frac{2}{y^2}. \end{aligned}$$

(c) (14 Points) Now, the quants on the treasury desk have forecasted a $-10bp$ parallel shift in the spot curve. Given this, which of the following portfolios do you expect to perform the best:

(i) \$1,000,000 in the 15 year zero coupon bond.

(ii) \$1,000,000 in the 20 year par coupon bond.

(iii) \$1,000,000 in the perpetuity with payment A computed from part (a)

Justify your answer using a first order approximation.

In this problem, you can use the fact that the yield based duration formula for a T year par bond is

$$D_{T,\text{par}} = \frac{1}{y}\left(1 - \frac{1}{(1+y/2)^{2T}}\right).$$

4. (30 Points) You own a perpetuity which pays A semi-annually and has a market value of \$10,000,000. You are considering hedging this perpetuity with a barbell of zeros with maturities $T_1 < T_2$ which is market value and duration matched to the annuity. Assume the spot curve is flat at $y = \mathbf{5\%}$.

(a) (15 Points) Amongst zeros of maturities $2, 5, 10, 30$ years respectively, pick maturities T_1, T_2 so that $T_1 < D_a < T_2$ and T_1, T_2 are closest to D_a, where D_a is the duration for the perprtuity. For example, if $D_a = 7$ then take $T_1 = 5, T_2 = 10$. For these maturities find faces $F(T_1), F(T_2)$ so that you are duration and market value matched with the perpetuity.

Note: You can use the formula for D_a in Problem 2 (b). Here, write down your final answer as the system of equations satisfied by $F(T_1)$ and $F(T_2)$. Do not attempt to calculate the actual numbers for the faces.

(b) (15 Points) Assuming a upwards parallel shift of $10bp$ in the spot curve, how do you expect the hedging portfolio's performance to be relative to the perpetuity?

Note: You can use the formula for C_a in Problem 2 (b). You may assume in the barbell that 50% of the initial market value is in the T_1 year and 50% in the T_2 year. This is NOT the exact value but is not too far off.