Sensitivity Analysis and Hedging III
- Portfolio Construction and Yield Curve Strategies

Portfolios of Fixed Income Securities

- suppose we want to build a portfolio of fixed income securities $S^1, \ldots, S^M$ with dollar prices $P^1, \ldots, P^M$
  - prices per unit face (not 100 face)

- if we have face (notional, shares) f^i in security S^i denote by

$$Port = \{(S^i, f^i)\}_{i=1,\ldots,M}$$

the portfolio

- the (total) value or price of our portfolio is

$$Px(Port) \triangleq X = \sum_{i=1}^{M} f^i \cdot p^i$$

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Portfolio DV01

Writing $Px^i = f^i p^i$ as the (total) price of our position in security S^i we have

$$Px(Port) = \sum_{i=1}^{M} Px^i$$

$$- Px^i = Px(f^i p^i)$$

$$= f^i \left(\frac{q^i}{2} \sum_{j=1}^{N^i} \frac{1}{(1+\hat{r}(j/2)/2)^j} + \frac{1}{(1+\hat{r}(N^i/2)/2)^{N^i}} \right)$$

if e.g. S^i is a q^i coupon bond paying semi-annually at times $j/2$, $j = 1, \ldots, N^i$.

Let y be an interest rate factor. The Portfolio DV01 by definition is

$$DV01(Port) = -\frac{\Delta Px(Port)}{10{,}000\, \Delta y}.$$

③

$$= \sum_{i=1}^{m} \frac{-\Delta Px(f^i s^i)}{10{,}000\, \Delta y} = \sum_{i=1}^{m} f^i \left(\frac{-\Delta Px(s^i)}{10{,}000\, \Delta y} \right)$$

$$= \sum_{i=1}^{m} f^i DV01^i$$

$DV01^i$: dollar value of a bp for 1 unit face in s^i.

Intuitively, this makes sense b/c the dollar value of a bp for 1 unit s^i is $DV01^i$

$\Rightarrow$ dollar value of a bp for f^i face of s^i is $f^i DV01^i$

$\Rightarrow$ dollar value of a bp for the portfolio is sum of DV01's for components of the portfolio

- DV01 is an absolute (e.g. \$) changes so it scales with size, composition.

Small Warning

- we are sometimes a bit loose about f^i.

  e.g. we own \$1M face of 2 yr zero, \$2M face of 5 yr zero.

- might just write S^1 as 1M face of 2yr zero, $f^1 = 1$, S^2 as 2M face of 5yr zero, $f^2 = 1$, as opposed to S^1 as 1 face of 2yr zero, $f^1 = 1M$, S^2: 1 face of 5yr zero, $f^2 = 2M$.

Back to general case (Px/unit face)

Duration.

By definition: $D(\text{Port}) = -\dfrac{\Delta Px(\text{Port})}{Px(\text{Port})\,\Delta y}$

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$$= \sum_{i=1}^{M} \frac{-\Delta P_X(f^i s^i)}{X \Delta y} = \sum_{i=1}^{M} \frac{f^i p^i}{X}\left(\frac{-\Delta P(s^i)}{p^i \Delta y}\right)$$

- recall: $P_X(s^i) = p^i$ and
$P_X(f^i s^i) = f^i P_X(s^i)$

$$= \sum_{i=1}^{M} \frac{f^i p^i}{X} D^i$$

D^i: duration of s^i, which is independant of f^i

$\Rightarrow$

$D(\text{Port})$ is a <u>weighted</u> <u>average</u> of the component durations D^i where the weights

$$w^i = \frac{f^i p^i}{\sum_{j=1}^{M} f^j p^j} \qquad i=1,\dots,M$$

is percentage of the portfolio total price in security ~~s~~ s^i.

(6)

Similarly

$$C(\text{Port}) = \sum_{i=1}^{M} \frac{F^i p^i}{X} C^i$$

C^i : convexity of S^i

Hedging based on Duration / Convexity

Basic Idea (Duration Matching)

If Portfolios A, B have the same total price $X_A = X_B$ and durations $D_A = D_B$ then for small changes in y (Δy small) the price changes $\Delta X_A, \Delta X_B$ will approximately be the same.

<u>example</u>

3 securities: 2, 5, 10 yr 0.

labelled S^1, S^2, S^3

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Portfolio A : 1M face in S^2

Portfolio B : F^1 face in S^1, F^3 face in S^3

Q.

If $\hat{r}(2) = 5.78\%$, $\hat{r}(5) = 6.02\%$, $\hat{r}(10) = 6.26\%$ and we use yield-based duration what should F^1, F^3 be so that we are price and duration matched?

A.

$$X_A = 1M\,P^2$$

$$X_B = F^1P^1 + F^3P^3$$

$$\Rightarrow \quad F^1P^1 + F^3P^3 = 1M\,P^2 \quad \text{(price match)}$$

$$D_A = D^2$$

if we are px-matched.

$$D_B = \frac{F^1P^1D^1 + F^3P^3D^3}{F^1P^1 + F^3P^3} = \frac{1}{1M\,P^2}\left(F^1P^1D^1 + F^3P^3D^3\right)$$

$$\Rightarrow \quad F^1P^1D^1 + F^3P^3D^3 = 1M\,P^2 \cdot D^2$$

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using i) $D(T\text{ yr } 0, \text{yield}=y) = \frac{T}{1+y/2}$

$y = \hat{r}(T)$

ii) $Px(1 \text{ face}, T \text{ yr } 0) = \frac{1}{(1+\hat{r}(T)/2)^{2T}}$

and plugging in we obtain.

$F^1 = 520{,}386$, $F^2 = 516{,}843$

with a common duration of

$D = 4.8539$.

<u>Now</u>, how does this trade work if there is a "parallel" shift in the spot curve by 35 bp?

i.e. the entire spot curve goes up by 35 bp.

$\hat{r}(2) = 6.13\%$ $\hat{r}(5) = 6.37\%$

$\hat{r}(10) = 6.61\%$.

Using the faces we obtain (exactly)

$\Delta X_A = -12{,}511.7$

-not too bad.

$\Delta X_B = -12{,}448.8$

We can also ~~the~~ look at the first order approximations

$$\Delta X_A \approx -X_A D_A \Delta y = -12{,}628.9$$

$$\Delta X_B \approx -X_B D_B \Delta y = -12{,}628.9$$

- match by construction!

- first order approximations capture virtually all the change.

Now, how does this trade work if we have a downward parallel shift?

$$\Delta \hat{r}(T) = -35 \text{ bp} \quad \forall\, T.$$

$$\Rightarrow \Delta X_A = 12{,}747.7 \quad \text{(actual)}$$

$$\Delta X_B = 12{,}813.2$$

vs.

$$\Delta X_A = \Delta X_B = 12{,}628.9 \quad \text{(1st order approximation)}$$

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Note/Warning: if you are duration matching, do NOT forget you have to price match as well in order to have your DOLLAR sensitivity to rate changes be equal.

i.e. $\Delta X \approx -X \cdot D \, \Delta y$

- both X, D have to match.

- alternatively, you could hedge DV01 directly

e.g. B: 1M face of S^1, S^3

A: F face of S^2

find F so that $DV01_A = DV01_B$

$$\Rightarrow F = \frac{1M \, DV01^1 + 1M \, DV01^3}{DV01^2}$$

- here $\Delta X_A \approx \Delta X_B$ but not necessarily $X_A = X_B$.

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Come back to the trades and assume we are long the 2,10 yr and short the 5yr:

- our value is $X_B - X_A$.

$D = 0$ with $D_B = D_A$ initially.

- in the +35 bp shift our gain/loss was (exact)

$$\Delta X_B - \Delta X_A = -12{,}448.8 + 12{,}511.7$$
$$= 62.9$$

- in the -35 bp shift, our gain/loss was (exact)

$$\Delta X_B - \Delta X_A = 12{,}813.2 - 12{,}747.7$$
$$= 65.5$$

- out performance in <u>both</u> up and down shifts!

- Why?

B/c we are long convexity!

$\Rightarrow$

$C_A = 25.916$

$C_B = 40.007$

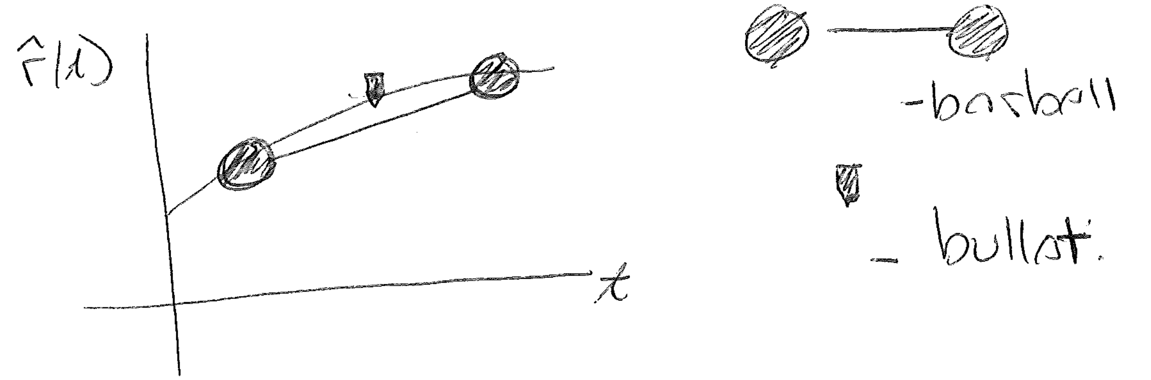
$\Rightarrow$ going long B and short A benefits us in both up and down movements.

In the above example, portfolio B is an example of a "Barbell"

- long a short and long maturity

Portfolio A is an example of a "Bullet"

- long a medium maturity



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In general, a barbell is long convexity versus a bullet and hence if the spot rate curve is relatively flat, and if you expect a roughly parallel shift in the yield curve, a ~~barb~~ barbell is preferable to a bullet.

why?

- assume spot rate curve is flat at a level (zero-yield) $y > 0$.

- portfolio A $\rightarrow$ bullet: own 1 face of a T_2 year zero

1) $\text{price}^{\text{bullet}} = \dfrac{1}{(1+y/2)^{2T_2}} = X = P(T_2)$

2) $D^{\text{bullet}} = \dfrac{T_2}{1+y/2} = D = D(T_2)$

3) $D_{mac}^{\text{bullet}} = T_2$

4) $C^{\text{bullet}} = \dfrac{T_2^2 + T_2/2}{(1+y/2)^2}$

Portfolio B (barbell)

- allocate $\alpha X = \alpha P(T_2)$ in T_1 year zero, $(1-\alpha)X = (1-\alpha)P(T_2)$ in T_3 year zero

$$T_1 < T_2 < T_3.$$

- to match durations/ price we must have (yield is same for each qty)

$$\alpha = \frac{T_3 - T_2}{T_3 - T_1}$$

$$f_1 = \frac{T_3 - T_2}{T_3 - T_1} \cdot \frac{P(T_2)}{P(T_1)}$$

$$f_3 = \frac{T_2 - T_1}{T_3 - T_1} \cdot \frac{P(T_2)}{P(T_3)}$$

so

$$P^{barbell} = P^{bullet} = P(T_2)$$

$$D^{barbell} = D^{bullet} = D = \frac{T_2}{1+y/2}.$$

But

$$C^{barbell} = \alpha C(T_1) + (1-\alpha) C(T_3)$$

$$= \alpha \frac{T_1^2 + T_1/2}{(1+y/2)^2} + (1-\alpha) \frac{T_3^2 + T_3/2}{(1+y/2)^2}$$

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set $g(t) = \frac{t^2 + t/2}{(1+y/2)^2}$ so

$c^{bullet} = g(T_2)$

$c^{barbell} = \alpha g(T_1) + (1-\alpha) g(T_3)$

<u>Note</u>

$g''(t) = \frac{2}{(1+y/2)^2} > 0$ so g is

convex.

Property of convex functions:

$$\alpha g(T_1) + (1-\alpha) g(T_3) \geq g(\alpha T_1 + (1-\alpha) T_3) \quad \forall_{\substack{T_1 < T_3 \\ \alpha \in (0,1)}}$$

But

$$\alpha T_1 + (1-\alpha) T_3 = \frac{T_3 - T_2}{T_3 - T_1} T_1 + \frac{T_2 - T_1}{T_3 - T_1} T_3$$

$$= T_2$$

so $c^{bullet} \leq c^{barbell}$

- Long convexity.

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so, for a roughly parallel shift we expect the barbell to out perform

- this trade (e.g. Long barbell versus short bullet) is an example of a "yield curve strategy".

  i.e. given an initial curve structure and a view on future movements of the yield curve, you design a duration matched, price matched pair of portfolios A, B and go long B, short A with the hope of gaining money on the yield curve shift.

- note: Long barbell, short bullet trades (or vice-versa) are called "butterfly trades"

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Q.

For a flat spot curve, will we always make money going long the barbell and short the bullet?

A.

NO! It depends on the type of yield curve shift!

Example

flat spot curve at 6%, you have \$ 1 M initial capital and can invest in 2, 5, 10 year zeros. You price and duration match portfolio B (Long 2, 10) and A (Long 5)

$\Rightarrow$

A: face $F_5 = 1.3439$ M.

~~duration D_5 = [illegible]~~

B: face $F_2 = 703,443$

$F_{10} = 677,292$

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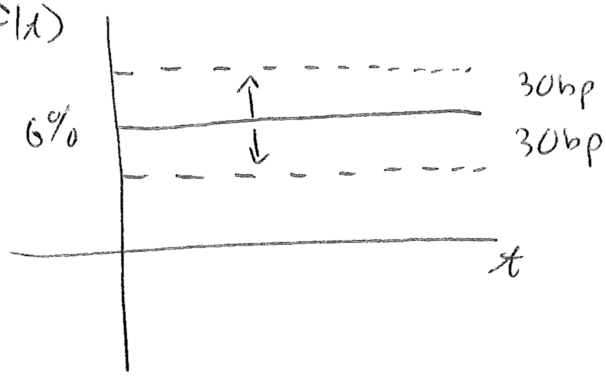
1) For a parallel shift (up, down) of 30 bp



If we are Long B and Short A

gain/loss (+30bp) $= \Delta X_B - \Delta X_A$

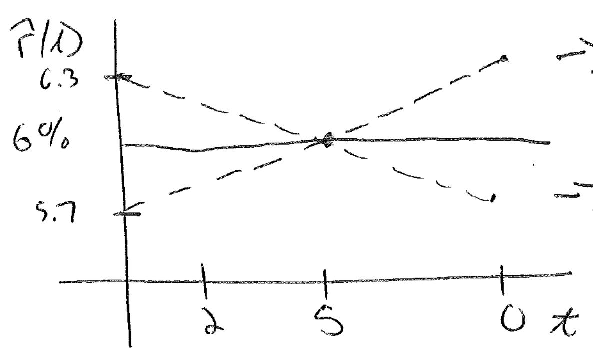
$= 62.61$ (62.4945)

gain/loss (−30 bp) $= 64.97$ (64.7790)

– out perform in both scenarios.

2) For a steepening ~~shift~~ ("flatting" or "inverse steepening")

shift of 30 bp



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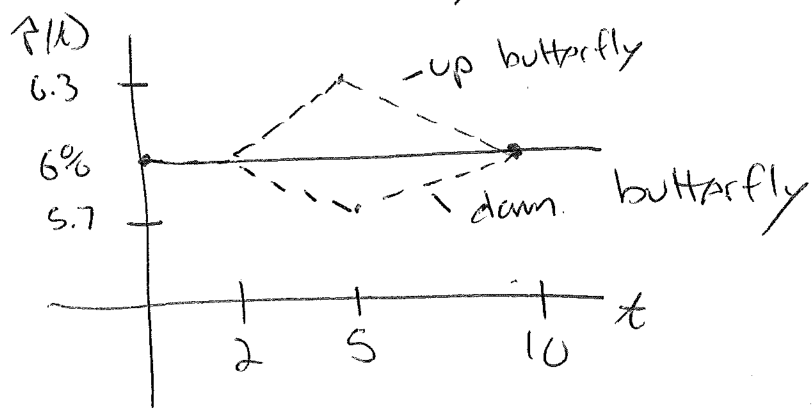
if we are long B and short A

gain/loss (steepening) = 7463.74 (-7103.01)

gain/loss (flattening) = -7102.86 (7463.59)

- lose money if short end rises, long end falls.

3) For a <u>butterfly</u> shift of 30 bp



gain/loss (up butterfly) = 14,447.1

gain/loss (down butterfly) = - 14,680.4

- lose money if intermediate rates fall.

so, depending on the type of shift, many things can happen.

Now, this example is a bit contrived b/c the initial curve was flat and the types of shifts very simple, but it should make you realize that long convexity, while generally a good thing, may lose money at times.

e.g. for downward butterfly we expect short/long zero prices to not move too much, but intermediate zero prices to increase (even if initial curve not flat). So, if this is our belief we want to be long the middle of the curve and short the ~~ext~~ ends.

~~Now, even for parallel shifts the convexity gain of a barbell is mitigated by the fact that the~~

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for a more detailed discussion on this, as well as the performance of bullets vs. barbells in many different types of shifts, see the "Yield Curve Strategies" article.

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If we are long B and short A

gain/loss