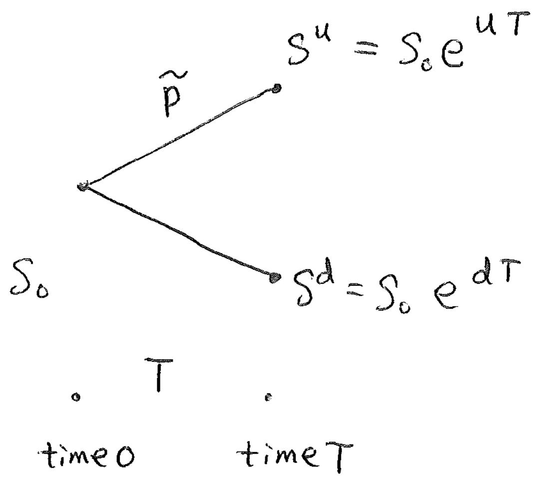


log return. between 0 to T

$$\log \frac{S_T}{S_0} = \begin{cases} \log \frac{S^u}{S_0} = uT \\ \log \frac{S^d}{S_0} = dT \end{cases}$$

$0, \quad \frac{T}{2}, \quad T$

$$\log \frac{S_T}{S_0} = \log \frac{S_{\frac{T}{2}}}{S_0} + \log \frac{S_T}{S_{\frac{T}{2}}}$$

$$\begin{array}{ccc} \| & \| & \| \\ (u+d)\frac{1.5}{12} & d\,\frac{1.5}{12} & u\,\frac{1.5}{12} \end{array}$$

Discounted stock price is a martingale.

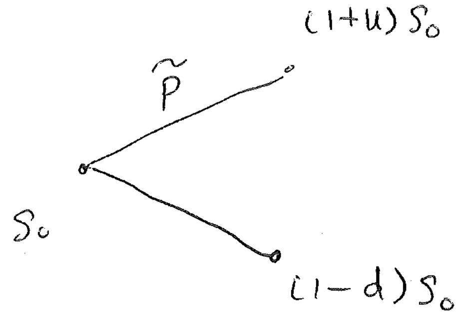
$$\overset{1}{\cancel{S_0}} = \tilde{\mathbb{E}}[e^{-rT} S_T]$$

$$= \tilde{p}\, e^{-rT} \cancel{S_0} e^{uT} + (1-\tilde{p})\, e^{-rT} \cancel{S_0} e^{dT}$$

$\Rightarrow \tilde{p}$

$$1 = e^{-rT} e^{dT} + \tilde{p}\,(e^{-rT} e^{uT} - e^{-rT} e^{dT})$$

$$\tilde{p} = \frac{e^{rT} - e^{dT}}{e^{uT} - e^{dT}}$$



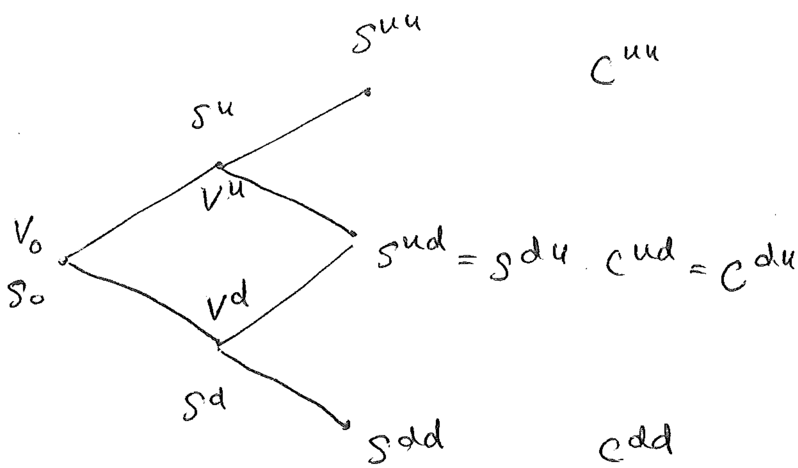
$$\cancel{S_0} = \tilde{\mathbb{E}}\left[e^{-rT} S_T \right]$$

$$= \tilde{p} e^{-rT} (1+u) \cancel{S_0} + (1-\tilde{p}) e^{-rT} (1-d) \cancel{S_0}$$

$\tilde{p}$:

$$1 = e^{-rT}(1-d) + \tilde{p}\, e^{-rT}\big((1+u) - (1-d)\big)$$

$$\tilde{p} = \frac{e^{rT} - (1-d)}{u+d}$$



- S^u $\quad V^u$ $\quad \Delta^u$ shares in stocks.

  $V^u - \Delta^u S^u$ in bank account.

$$\underline{\underline{\Delta^u}}\, S^{uu} + e^{r\frac{1.5}{12}}\left(\underline{\underline{V^u}} - \Delta^u S^u\right) = C^{uu}$$

$$\Delta^u S^{ud} + e^{r\frac{1.5}{12}}\left(V^u - \Delta^u S^u\right) = C^{ud}$$

- S_0. $\quad \Delta^0$ shares in stocks.

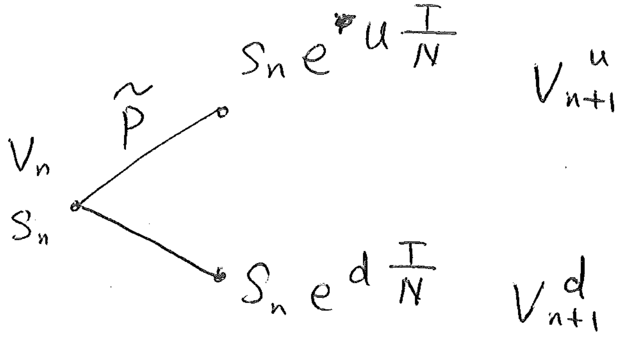
  $V_0 - \Delta^0 S_0$ in bank account

$$\underline{\underline{\Delta^0}}\, S^u + e^{r\frac{1.5}{12}}\left(\underline{\underline{V_0}} - \Delta^0 S_0\right) = V^u$$

$$\Delta^0 S^d + e^{r\frac{1.5}{12}}\left(V_0 - \Delta^0 S_0\right) = V^d$$

V_0 = time 0 value of call option.

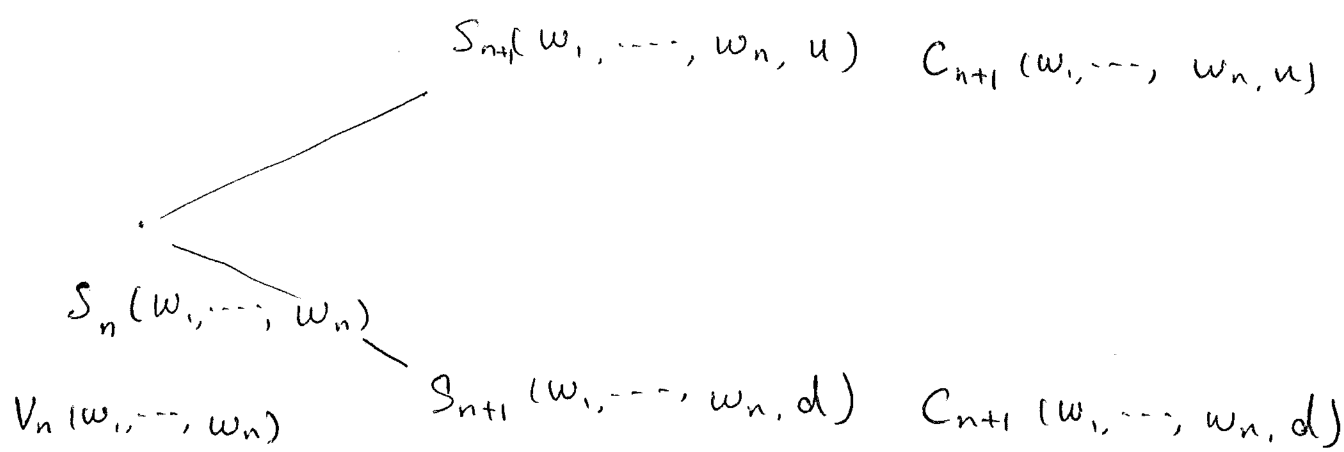
N period, maturity T. length of each period $\frac{T}{N}$



$$V_n = \tilde{\mathbb{E}}\left[e^{-r\frac{T}{N}} V_{n+1}\right]$$

$$V_n = e^{-r\frac{T}{N}}\left[\tilde{p} V_{n+1}^u + (1-\tilde{p}) V_{n+1}^d\right]$$

for all $n = 0, \cdots, N-1$.



$\Delta_n(\omega_1, \cdots, \omega_n)$ in stock,

$V_n(\omega_1, \cdots, \omega_n) - \Delta_n(\omega_1, \cdots, \omega_n) S_n(\omega_1, \cdots, \omega_n)$

$$\to \Delta_n(\omega_1, \cdots, \omega_n) S_{n+1}(\omega_1, \cdots, \omega_n, u) + e^{r\Delta t}\big[V_n(\omega_1, \cdots, \omega_n) - \Delta_n(\quad) S_n(\quad)\big]$$

$$= C_{n+1}(\omega_1, \cdots, \omega_n, u)$$

$$\to \Delta_n(\quad) S_{n+1}(\quad d) + e^{r\Delta t}[\quad]$$

$$= C_{n+1}(\omega_1, \cdots, \omega_n, d)$$

$S_{\Delta t}$

$$S_0 \;\begin{cases} S_0 e^{u} & p = \frac{1}{2} \\ S_0 e^{d} & 1-p = \frac{1}{2} \end{cases}$$

$$\ln\left(\frac{S_{\Delta t}}{S_0}\right) = \begin{cases} u = \sigma & p = \frac{1}{2} \\ d = -\sigma & 1-p = \frac{1}{2} \end{cases}$$

$$\begin{aligned} \operatorname{Var}\left(\ln\left(\frac{S_{\Delta t}}{S_0}\right)\right) &= p\,u^2 + (1-p)\,d^2 \\ &= \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 \\ &= \sigma^2 \end{aligned}$$

Stock with dividend

binomial tree price after dividend payment

$$S_0 \;\begin{cases} S_0 e^{u\Delta t} & S_0 e^{u\Delta t} e^{\delta \Delta t} \\ S_0 e^{d\Delta t} & S_0 e^{d\Delta t} e^{\delta \Delta t} \end{cases}$$

after dividend. before dividend.

$$\begin{aligned} S_0 &= e^{-r\Delta t}\,\tilde{\mathbb{E}}[\text{ pre-dividend stock price }] \\ &= \underline{\underline{e^{-r\Delta t}}}\left[\tilde{p}\, S_0 e^{u\Delta t} \underline{e^{\delta\Delta t}} + (1-\tilde{p})\, S_0 e^{d\Delta t} \underline{e^{\delta\Delta t}}\right] \end{aligned}$$

$$S_0 = e^{-(\underline{\underline{r}}-\delta)\Delta t}\left[\tilde{p}\, S_0 e^{u\Delta t} + (1-\tilde{p})\, S_0 e^{d\Delta t}\right]$$

$$\tilde{p} = \frac{e^{(r-\delta)\Delta t} - e^{d\Delta t}}{e^{u\Delta t} - e^{d\Delta t}}$$