MF702 PROBLEM SET 4: SOLUTIONS

1.

(i) Use Itô's formula to compute dW_t^4, where $\{W_t; t \geq 0\}$ is a Wiener process. Then write W_T^4 as the sum of a time integral and an integral with respect to dW_t.

(ii) Take expectations on both sides of the formula you obtained in (i), use the face that $\mathbb{E}W_t^2 = t$, derive the formula $\mathbb{E}W_T^4 = 3T^2$.

(iii) Use the method of (i) and (ii) to derive a formula for $\mathbb{E}W_T^6$.

Solution: (i)

$$\begin{aligned} dW_t^4 =& 4W_t^3 dW_t + \frac{1}{2} \times 12 W_t^2 d\langle W \rangle_t \\ =& 4W_t^3 dW_t + 6W_t^2 dt. \end{aligned}$$

Integrating both side on $[0, T]$, we obtain

$$W_T^4 - W_0^4 = \int_0^T 6W_t^2 dt + \int_0^T 4W_t^3 dW_t$$

Because $W_0 = 0$, therefore

$$W_T^4 = \int_0^T 6W_t^2 dt + \int_0^T 4W_t^3 dW_t.$$

(ii) Taking expectations on both sides and using the fact that $\int_0^\cdot 4W_t^3 dW_t$ is a martingale, we obtain

$$\mathbb{E}W_T^4 = \mathbb{E}\Big[\int_0^T 6W_t^2 dt\Big] = \int_0^T 6\mathbb{E}[W_t^2] dt = \int_0^T 6t dt = 3T^2,$$

where the second identity follows from the Fubini's theorem.

(iii) From Itô's formula,

$$dW_t^6 = 6W_t^5 dW_t + \frac{1}{2} \times 30 W_t^4 dt.$$

Integrating on $[0, T]$ and taking expectations, we obtain

$$\mathbb{E}\big[W_T^6\big] = \mathbb{E}\Big[\int_0^T 15W_t^4 dt\Big] = \int_0^T 15\mathbb{E}\big[W_t^4\big] dt = \int_0^T 45t^2 dt = 15T^3,$$

where we use (ii) in the third identity.

2. (Solving the Vasicek equation). The Vasicek interest rate stochastic differential equation is

$$dR_t = (\alpha - \beta R_t) dt + \sigma dW_t,$$

where α, β and σ are constants. We are going to solve this equation in this exercise.

(i) Use Itô's formula to compute $d(e^{\beta t}R_t)$. Simplify it so that you have a formula for $d(e^{\beta t}R_t)$ that does not involve R_t.

(ii) Integrate the equation you obtained in (i) and solve for R_t.

Solution: (i)

$$d\big(e^{\beta t}R_t\big) = e^{\beta t}\beta R_t dt + e^{\beta t}dR_t = e^{\beta t}\alpha dt + e^{\beta t}\sigma dW_t.$$

(ii) Integrating both sides on $[0, t]$, we obtain

$$e^{\beta t}R_t - R_0 = \int_0^t e^{\beta s}\alpha ds + \int_0^t e^{\beta s}\sigma dW_s.$$

Therefore

$$R_t = e^{-\beta t}R_0 + \int_0^t e^{\beta(s-t)}\alpha ds + \int_0^t e^{\beta(s-t)}\sigma dW_s.$$

3. For a European call expiring at time T with strike price K, the Black-Scholes-Merton price at time t, if the time-t stock price is x , is

$$c(t, x) = x\Phi(d_+(T-t, x)) - Ke^{-r(T-t)}\Phi(d_-(T-t, x)),$$

where

$$\begin{aligned} d_+(\tau, x) =& \frac{1}{\sigma\sqrt{\tau}}\Big[\log\frac{x}{K} + \Big(r + \frac{\sigma^2}{2}\Big)\tau\Big], \\ d_-(\tau, x) =& d_+(\tau, x) - \sigma\sqrt{\tau}, \end{aligned}$$

and $\Phi(y)$ is the cumulative standard normal distribution

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}dx = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}dz.$$

The purpose of this exercise is to show that the function c satisfies the Black-Scholes-Merton partial differential equation

$$c_t(t, x) + rxc_x(t, x) + \frac{1}{2}\sigma^2x^2c_{xx}(t, x) = rc(t, x), \quad 0 \le t < T, x > 0, \tag{1}$$

where c_t is the time derivative, c_x and c_{xx} are first and second partial derivatives with respect to x. c also satisfies the terminal condition

$$\lim_{t\uparrow T} c(t, x) = (x-K)_+, \quad x > 0, x \ne K. \tag{2}$$

For this exercise, we abbreviate $c(t, x)$ as c and $d_\pm(T-t, x)$ as $d_\pm$.

(i) Verify the equation

$$Ke^{-r(T-t)}\Phi'(d_-) = x\Phi'(d_+).$$

(ii) Show that $c_x = \Phi(d_+)$. This is the *delta* of the option. (Be careful! Remember that d_+ is a function of x.)

(iii) Show that
$$c_t = -rKe^{-r(T-t)}\Phi(d_-) - \frac{\sigma x}{2\sqrt{T-t}}\Phi'(d_+).$$
This is the *theta* of the option.

(iv) Use the formulas above to show that c satisfies (1).

(v) Show that for $x > K$, $\lim_{t\uparrow T} d_\pm = \infty$, but for $0 < x < K$, $\lim_{t\uparrow T} d_\pm = -\infty$. Use these equalities to derive the terminal condition (2).

Solution:

(i)

$$\Phi(d_+) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_+} e^{-\frac{x^2}{2}}dx \;\; implies \;\; \Phi'(d_+) = \frac{1}{\sqrt{2\pi}}e^{-\frac{{d_+}^2}{2}}$$
$$\Phi(d_-) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-} e^{-\frac{x^2}{2}}dx \;\; implies \;\; \Phi'(d_-) = \frac{1}{\sqrt{2\pi}}e^{-\frac{{d_-}^2}{2}}$$

So,
$$\frac{\Phi'(d_+)}{\Phi'(d_-)} = e^{\frac{{d_-}^2-{d_+}^2}{2}} = e^{-ln\frac{x}{K}-r(T-t)} = \frac{Ke^{-r(T-t)}}{x}.$$

Therefore,
$$Ke^{-r(T-t)}\Phi'(d_-) = x\Phi'(d_+).$$

(ii)

$$\begin{aligned}
c_x &= \Phi(d_+) + x\frac{\partial\Phi(d_+)}{\partial x} - Ke^{-r(T-t)}\frac{\partial\Phi(d_-)}{\partial x}\\
&= \Phi(d_+) + x\frac{\partial\Phi(d_+)}{\partial d_+}\frac{\partial d_+}{\partial x} - Ke^{-r(T-t)}\frac{\partial\Phi(d_-)}{\partial d_-}\frac{\partial d_-}{\partial x}\\
&= \Phi(d_+) + x\Phi'(d_+)\frac{1}{x\sigma\sqrt{T-t}} - Ke^{-r(T-t)}\Phi'(d_-)\frac{1}{x\sigma\sqrt{T-t}}\\
&= \Phi(d_+).
\end{aligned}$$

followed by $Ke^{-r(T-t)}\Phi'(d_-) = x\Phi'(d_+)$.

(iii)

$$\begin{aligned}
c_t &= x\frac{\partial\Phi(d_+)}{\partial t} - Ke^{-r(T-t)}\frac{\partial\Phi(d_-)}{\partial t} - Ke^{-r(T-t)}r\phi(d_-)\\
&= x\Phi'(d_+)\frac{\partial d_+}{\partial t} - Ke^{-r(T-t)}\Phi'(d_-) - rKe^{-r(T-t)}\Phi(d_-)\\
&= -\frac{\sigma}{2\sqrt{T-t}}x\Phi'(d_+) - rKe^{-r(T-t)}r\Phi(d_-).
\end{aligned}$$

(iv)

$$
\begin{aligned}
& c_t + rxc_x + \frac{1}{2}\sigma^2 x^2 c_{xx} \\
=& -\frac{\sigma}{2\sqrt{T-t}} x\Phi'(d_+) - rKe^{-r(T-t)} r\Phi(d_-) + rx\Phi(d_+) + \frac{1}{2}\sigma^2 x^2 \frac{\partial \Phi(d_+)}{\partial x} \\
=& - rKe^{-r(T-t)} r\Phi(d_-) + rx\Phi(d_+) \\
=& rc.
\end{aligned}
$$

(v)

If $x > K$, $\frac{x}{K} > 1$, $ln\frac{x}{K} > 0$,

$$
\begin{aligned}
\lim_{t\uparrow T} d_+ &= \lim_{t\uparrow T} \frac{1}{\sigma\sqrt{T-t}}[ln\frac{x}{K} + (r + \frac{\sigma^2}{2})(T-t)] \\
&= \lim_{t\uparrow T} \frac{1}{\sigma\sqrt{T-t}} ln\frac{x}{K} \\
&= \infty.
\end{aligned}
$$

Similarly, we can get $\lim_{t\uparrow T} d_- = \infty$. So, $\lim_{t\uparrow T} \Phi(d_+) = 1$ and $\lim_{t\uparrow T} \Phi(d_-) = 1$. Then, $\lim_{t\uparrow T} c(t,x) = x - K$.

If $0 < x < K$, $\frac{x}{K} < 1$, $ln\frac{x}{K} < 0$,

$$
\begin{aligned}
\lim_{t\uparrow T} d_+ &= \lim_{t\uparrow T} \frac{1}{\sigma\sqrt{T-t}}[ln\frac{x}{K} + (r + \frac{\sigma^2}{2})(T-t)] \\
&= \lim_{t\uparrow T} \frac{1}{\sigma\sqrt{T-t}} ln\frac{x}{K} \\
&= -\infty.
\end{aligned}
$$

Similarly, we can get $\lim_{t\uparrow T} d_- = -\infty$. So, $\lim_{t\uparrow T} \Phi(d_+) = 0$ and $\lim_{t\uparrow T} \Phi(d_-) = 0$. Then, $\lim_{t\uparrow T} c(t,x) = 0$.

Therefore, $\lim_{t\uparrow T} c(t,x) = (x - K)^+$.