$$dS_t = \underset{\text{drift coeff.}}{\partial(S_t,t)}\,dt + \underset{\text{Vol}}{\sigma(S_t,t)}\,\underset{\text{BM}}{dZ_t}$$

$$V : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(S_t, t) \longmapsto V(S_t, t)$$ process index by time t.

$$\begin{aligned} \underset{\displaystyle V(S_{t+dt}, t+dt) - V(S_t, t)}{dV(S_t,t)} = {} & \partial_t V(S_t,t)\,dt + \partial_S V(S_t,t)\,dS_t \\ & + \frac{1}{2}\partial^2_{SS} V(S_t,t)\,\underset{\sigma^2(S_t,t)\,dt}{(dS_t)^2} \\ & + \partial^2_{St} V(S_t,t)\,\underset{0}{dS_t\,dt} \\ & + \frac{1}{2}\partial^2_{tt} V(S_t,t)\,\underset{0}{(dt)^2} \end{aligned}$$

	dS_t	dt
dS_t	$\sigma^2(S_t,t)dt$	0
dt	0	0

$$dV(S_t,t) = \partial_t V(S_t,t)dt + \partial_S V(S_t,t)\,\underline{\underline{dS_t}} + \frac{1}{2}\partial^2_{SS} V(S_t,t)\,\sigma^2(S_t,t)\,dt$$

$$\begin{aligned} dV(S_t,t) &= \partial_t V(S_t,t)\,dt + \partial_S V(S_t,t)\,(\partial(S_t,t)dt + \sigma(S_t,t)dZ_t) \\ &\quad + \frac{1}{2}\partial^2_{SS} V(S_t,t)\,\sigma^2(S_t,t)\,dt. \\ &= \left[\partial_t V(S_t,t) + \partial_S V(S_t,t)\,\partial(S_t,t) + \frac{1}{2}\partial^2_{SS} V(S_t,t)\sigma^2(S_t,t)\right] dt \\ &\quad + \partial_S V(S_t,t)\,\sigma(S_t,t)\,dZ_t \end{aligned}$$

Quadratic variation.

S_t. $[0, 1]$ $\quad 0 = t_0 \le t_1 \le \cdots \le t_N = 1$

$$t_n - t_{n-1} = \frac{1}{N}$$

$$\sum_{n=1}^{N} (S_{t_n} - S_{t_{n-1}})^2 \xrightarrow[N\to\infty]{} \text{Quadratic variation of } S \text{ on } [0, 1]$$

$$\langle S \rangle_1$$

<u>Example</u>: t.

$$\sum_{n=1}^{N} \underbrace{(t_n - t_{n-1})^2}_{\frac{1}{N}} = \sum_{n=1}^{N} \left(\frac{1}{N}\right)^2 = \frac{1}{N} \xrightarrow[N\to\infty]{} 0$$

$$\frac{1}{N}\sum_{n=1}^{N} N_{t_n}^2 \to \mathbb{E}[N_{t_n}^2]$$

Z_t.

$$\sum_{n=1}^{N} \underbrace{(Z_{t_n} - Z_{t_{n-1}})^2}_{\substack{\| \\ N_{t_n} \sim N(0, \sqrt{t_n - t_{n-1}}) = N(0, \sqrt{\frac{1}{N}})}} = \sum_{n=1}^{N} N_{t_n}^2 \xrightarrow[LLN]{N\to\infty} N\,\mathbb{E}[N_{t_n}^2] = 1$$

$$\mathbb{E}[N_{t_n}^2] = \mathbb{V}(N_{t_n}) = \frac{1}{N}$$

$$\sum_{n=1}^{N} (Z_{t_n} - Z_{t_{n-1}}) \underbrace{(t_n - t_{n-1})}_{\frac{1}{N}} \xrightarrow{N\to\infty} 0$$

GBM:

$$\frac{dS_t}{S_t} = \partial\, dt + \sigma\, dz_t. \qquad \partial,\ \sigma \text{ are constants.}$$

$$dS_t = \partial S_t\, dt + \sigma S_t\, dz_t$$

$$\underset{\substack{\| \\ V(S_t)}}{d \ln S_t} = \underset{\substack{\| \\ \frac{1}{S_t}}}{\partial_S V(S_t)}\, dS_t + \frac{1}{2} \underset{\substack{\| \\ -\frac{1}{S_t^2}}}{\partial^2_{SS} V(S_t)}\, (dS)^2$$

$$= \frac{1}{S_t} dS_t - \frac{1}{2} \frac{1}{S_t^2} (dS)^2$$

$$= \frac{1}{\cancel{S_t}} (\partial \cancel{S_t} dt + \sigma \cancel{S_t} dz_t) - \frac{1}{2} \frac{1}{\cancel{S_t^2}} \sigma^2 \cancel{S_t^2}\, dt$$

$$d \ln S_t = (\partial - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dz_t$$

Integrate with respect to time on $[0, t]$

$$\ln S_t - \ln S_0 = (\partial - \tfrac{1}{2}\sigma^2) t + \sigma (Z_t - Z_0) \qquad Z_0 = 0.$$

$$\ln S_t = \ln S_0 + (\partial - \tfrac{1}{2}\sigma^2) t + \cancel{\sigma(Z_t - Z_0)}\ \sigma Z_t$$

$$S_t = S_0\, e^{(\partial - \frac{1}{2}\sigma^2)t + \sigma Z_t} \qquad t \geq 0.$$

$$\ln S_t \sim N\left(\ln S_0 + (\partial - \tfrac{1}{2}\sigma^2)t,\ \sigma\sqrt{t}\right)$$

$\ln S$ has log-normal distribution.

$$S_t = S_0 \, e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma z_t}$$

$$\underbrace{dS_t}_{= V(t,\, z_t)} = \underbrace{\partial_t V(t, z_t)}_{= S_t(\alpha - \frac{1}{2}\sigma^2)} dt + \underbrace{\partial_z V(t, z_t)}_{= S_t \sigma} dz_t + \frac{1}{2} \underbrace{\partial^2_{zz} V(t, z_t)}_{= S_t \sigma^2} (dz_t)^2$$

$$= S_t(\alpha - \cancel{\tfrac{1}{2}\sigma^2})\, dt + S_t \sigma \, dz_t + \tfrac{1}{2} \cancel{S_t \sigma^2}\, dt$$

$$= S_t \alpha \, dt + S_t \sigma \, dz_t$$

$$\frac{dS_t}{S_t} = \alpha \, dt + \sigma \, dz_t .$$