Fixed Income

Interest rate risk management

Goals:

- DV01, Duration, and Convexity
- Portfolio and yield curve strategies
- Multiple factor and PCA

Motivation

- We need a way to measure the sensitivity of a bond's price to interest rates
- Measuring price sensitivity to the whole curve can be difficult
- We will focus on several key interest rate factors (eg. short, medium, or long term rates)

One factor models

One single factor y, we can think of prices as a function of y

$$P_x = f(y)$$

Note: y may not be bond's YTM

First and second order approximations

Given $P_x = f(y)$

- First order approximation: $\Delta P_x = \dot{f}(y)\Delta y$
- Second order approximation: $\Delta P_x = \dot{f}(y)\Delta y + \frac{1}{2}\ddot{f}(y)(\Delta y)^2$

Example:

ZCB with maturity T, $y = \text{YTM}$

$$
\begin{aligned}
P_x =& f(y) = F(1+y/2)^{-2T} \\
\dot{f}(y) =& -\left(\tfrac{T}{1+y/2}\right)F(1+y/2)^{-2T} \\
=& -\left(\tfrac{T}{1+y/2}\right)f(y) \quad \text{Note: it is negative} \\
\ddot{f}(y) =& \tfrac{T^2+T/2}{(1+y/2)^2}f(y) \quad \text{Note: it is positive}
\end{aligned}
$$

So

$$
\begin{aligned}
\Delta P_x &\approx -\left(\tfrac{T}{1+y/2}\right)P_x\Delta y \quad \text{First order} \\
&\approx -\left(\tfrac{T}{1+y/2}\right)P_x\Delta y + \tfrac{T^2+T/2}{(1+y/2)^2}P_x(\Delta y)^2 \quad \text{Second order} \\
\frac{\Delta P_x}{P_x} &\approx -\left(\tfrac{T}{1+y/2}\right)\Delta y \\
&\approx -\left(\tfrac{T}{1+y/2}\right)\Delta y + \tfrac{T^2+T/2}{(1+y/2)^2}(\Delta y)^2
\end{aligned}
$$

DV01

Dollar value of a Basis Point

$$DV01 = -\frac{\Delta P_x}{10,000\Delta y}$$

DV01 tells us how much the price will change if y moves 1bp

Basis point (bp)

$100bp = 1\%$ or $1bp = 0.0001$

Changes in rates often quoted in bp

Why the minus sign?

Prices usually go down if rates go up. Use minus sign to think of DV01 as a positive number

If $P_x = f(y)$ where f is known and smooth

$$DV01 = -\frac{\dot{f}(y)}{10,000} \quad \text{if } \Delta y \text{ is small.}$$

Hedging with DV01

Matching DV01 among positions

See the example in note “Sensitivity Analysis 1 DV01” Pages 8 - 12.

Duration and Convexity

Duration

$$D = -\frac{1}{P_x}\frac{\Delta P_x}{\Delta y} = 10,000\frac{DV01}{P_x}$$

- Measures sensitivity of the relative change in the price of the security to changes in y
- if $P_x = f(y)$, then for small changes in y

$$D = -\frac{\dot{f}(y)}{P_x} = -\frac{\dot{f}(y)}{f(y)}$$

Duration and Convexity

Duration

$$D = -\frac{1}{P_x}\frac{\Delta P_x}{\Delta y} = 10,000\frac{DV01}{P_x}$$

- Measures sensitivity of the relative change in the price of the security to changes in y
- if $P_x = f(y)$, then for small changes in y

$$D = -\frac{\dot{f}(y)}{P_x} = -\frac{\dot{f}(y)}{f(y)}$$

Convexity

If $P_x = f(y)$,

$$C = \frac{\ddot{f}(y)}{P_x} = \frac{1}{P_x}\frac{d^2 P_x}{dy^2}$$

- measures the sensitivity of interest rate sensitivity to changes in rates (it is a second derivative)
- some textbook defines C as

$$C = \frac{1}{2}\frac{\ddot{f}(y)}{P_x}$$

Approximation:

- 1^{st} order:

$$\frac{\Delta P}{P} = -D\Delta y$$

- 2^{nd} order:

$$\frac{\Delta P}{P} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

The previous formulas imply

- When $D > 0$, rates $\uparrow \implies P \downarrow$; rates $\downarrow \implies P \uparrow$
- When $D < 0$, the effect is opposite
- When $C > 0$, positive contribution to P when rates vary (either $\uparrow$ or $\downarrow$)
- When $C < 0$, negative contribution to P
- $C > 0$: long volatility or long convexity

Approximation:

- 1^{st} order:

$$\frac{\Delta P}{P} = -D\Delta y$$

- 2^{nd} order:

$$\frac{\Delta P}{P} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

The previous formulas imply

- When $D > 0$, rates $\uparrow \implies P \downarrow$; rates $\downarrow \implies P \uparrow$
- When $D < 0$, the effect is opposite
- When $C > 0$, positive contribution to P when rates vary (either $\uparrow$ or $\downarrow$)
- When $C < 0$, negative contribution to P
- $C > 0$: long volatility or long convexity

Note on scaling: if $P(F) = F \times P(1)$ (i.e. price is linear in face value)

- DV01 is linear in F
- C does not depend on F

Yield based DV01, duration, convexity

Bond: annual coupon rate q,
N remaining payments at time $1/2, 1, 3/2, \dots, N/2$

Price satisfies

$$\begin{aligned}\frac{P_X}{F} =& \frac{q}{2}\sum_{i=1}^{N}\frac{1}{(1+y/2)^i} + \frac{1}{(1+y/2)^N}\\ =& \frac{q}{y}\left(1 - \frac{1}{(1+y/2)^N}\right) + \frac{1}{(1+y/2)^N}\\ =:& f(y)\end{aligned}$$

Then

$$\begin{aligned}DV01 =& -\frac{F}{10,000}\dot{f}(y)\\ =& \frac{F}{10,000}\left(\frac{q}{y^2}\left(1 - \frac{1}{(1+y/2)^N}\right) - \left(1 - \frac{q}{y}\right)\frac{N/2}{(1+y/2)^{N+1}}\right)\end{aligned}$$

If we think T as maturity so that $T = 2N$, we have

$$DV01 = \frac{F}{10,000}\left(\frac{q}{y^2}\left(1 - \frac{1}{(1+y/2)^{2T}}\right) - \left(1 - \frac{q}{y}\right)\frac{T}{(1+y/2)^{2T+1}}\right)$$

Duration:

$$
\begin{aligned}
D =& \frac{10,000 DV01}{P_X} \\
=& \frac{F}{P_X}\left(\frac{q}{y^2}\left(1 - \frac{1}{(1+y/2)^N}\right) - \left(1 - \frac{q}{y}\right)\frac{N/2}{(1+y/2)^{N+1}}\right) \\
=& \frac{\frac{q}{y^2}\left(1 - \frac{1}{(1+y/2)^N}\right) - \left(1 - \frac{q}{y}\right)\frac{N/2}{(1+y/2)^{N+1}}}{\frac{q}{y}\left(1 - \frac{1}{(1+y/2)^N}\right) + \frac{1}{(1+y/2)^N}}
\end{aligned}
$$

Specialize to Zero Coupon Bond:
Let $q = 0$ above, for $N = 2T$ we have

$$
D = \frac{N/2}{1 + y/2} = \frac{T}{1 + y/2}.
$$

Macaulay duration

As indicated by the name duration, we think there is a connection between the duration of a bond and the (weighted) average time of the remaining cash flow

This idea is made precise through Macaulay duration

$$D_{mac} = (1 + y/2)D$$

ZCB:

$$D_{mac} = T \quad \text{Exact maturity of cash flow}$$

Macaulay duration for coupon bond

Coupon bond with $2T$ remaining payments of size F_i at $i = 1, \ldots, 2T$

YTM ssatisfies

$$P_x = \sum_{i=1}^{2T} \frac{F_i}{(1+y/2)^i} = f(y)$$

Duration is

$$D = -\frac{\dot{f}(y)}{P_x} = \frac{1}{P_x(1+y/2)} \sum_{i=1}^{2T} \frac{(i/2)F_i}{(1+y/2)^i} = \frac{1}{P_x(1+y/2)} \sum_{i=1}^{2T} \frac{T_i F_i}{(1+y/2)^i},$$

where $T_i = i/2 =$ time of the i^{th} payment of F_i

$$D_{mac} = \sum_{i=1}^{2T} T_i w_i, \quad w_i = \frac{F_i/(1+y/2)^i}{\sum_{j=1}^{2T} F_j/(1+y/2)^j}$$

D_{mac} is a yield weighted average of the remaining payment times where the weights are the relative contribution of the i^{th} payment to the current price, using the discounting factors implied by the yield

Maturity dependence of DV01, Duration and Convexity

ZCB with maturity T:

$$
\begin{aligned}
P_X =& F(1+y/2)^{-2T} \\
D =& \frac{T}{1+y/2} \\
D_{mac} =& T \\
C =& \frac{T^2+T/2}{(1+y/2)^2} \\
DV01 =& \frac{1}{10,000}\frac{FT}{(1+y/2)^{2T+1}}
\end{aligned}
$$

For $y > 0$ fixed

- $T \uparrow \implies P_X \downarrow$
- $T \uparrow \implies D \uparrow$
- $T \uparrow \implies D_{mac} \uparrow$
- $T \uparrow \implies C \uparrow$
- $T \uparrow \implies DV01$ first increases then decreases when T is sufficiently large

Coupon bond

App for duration

- $T \uparrow \implies D \uparrow$, except extreme large y
- $q \uparrow \implies D \downarrow$, because higher weight for coupon payments, most of which are received before T
- $y \downarrow \implies D \uparrow$, because there is less discounting, the weight associated to the face value is higher
- $T \uparrow \implies C \uparrow$, except extreme large y
- $q \uparrow \implies C \downarrow$
- $y \uparrow \implies C \downarrow$

Warning: When we change q and T, typically YTM y changes as well. Therefore, we need to be careful about changing other parameters while keep y constant.

Portfolio DV01

Consider a portfolio of fixed income securities $S^1, \ldots, S^M$ with prices $P^1, \ldots$ and face values $F^1, \ldots$

Total value of portfolio

$$P_x(port) = \sum_{i=1}^{M} F^i P^i.$$

Let y be an interest rate factor, the portfolio DV01 is

$$DV01(port) = -\frac{\Delta P_x(port)}{10,000\Delta y}$$

Note $\Delta P_x(port) = \sum_{i=1}^{M} F^i \Delta P^i$, then

$$DV01(port) = \sum_{i=1}^{M} F^i \left(\frac{-\Delta P^i}{10,000\Delta y} \right) = \sum_{i=1}^{M} F^i DV01^i,$$

where $DV01^i$ is the DV01 for S^i

Portfolio duration and convexity

$$
\begin{aligned}
D(port) &= -\frac{\Delta P_x(port)}{P_x(port)\Delta y} = \sum_{i=1}^{M} \frac{-\Delta P_x^i}{P_x(port)\Delta y} \\
&= \sum_{i=1}^{M} \frac{F^i P^i}{P_x(port)} \left(\frac{-\Delta P^i}{P^i \Delta y} \right) = \sum_{i=1}^{M} \frac{F^i P^i}{P_x(port)} D^i
\end{aligned}
$$

$D(port)$ is a weighted average of the component durations D^i where the weight is the percentage of the portfolio total price in security S^i

Convexity of the portfolio is

$$
C(port) = \sum_{i=1}^{M} \frac{F^i P^i}{P_x(port)} C^i,
$$

where C^i is the convexity of S^i

Hedging based on duration and convexity

Basic idea: duration matching

If portfolio A and B have the same total price $P^A = P^B$ and durations $D^A = D^B$, then for small changes in y, the price changes ΔP^A and ΔP^B are approximately the same

See an example in the note "Portfolio Construction and Yield Cruve Strategies" pages 6-10

Implication of convexity: see the note "Portfolio Construction and Yield Cruve Strategies" pages 6-10

Multiple interest rate factors: basic idea

So far we have considered interest rate sensitivities with respect to a single interest rate factor and we build a portfolio of ZCBs so that prices and duration are marched between two portfolios A and B:

$$\hat{r}(T) \quad \rightarrow \quad \hat{r}(T) + \Delta y, \qquad \text{for all } T, \tag{1}$$

then

$$\Delta P_x^A \approx \Delta P_x^B.$$

(1) means that all rate movements can be described by a single factor.

We wish to relax this assumption by studying price sensitivities to changes in multiple factors.

We pick a few key factors $y_1, \ldots, y_N$ and assume that the bond prices are functions of these rates

$$P_x = f(y_1, \ldots, y_N).$$

For example, we can use $2, 5, 10, 30$ par yield as key factors $y_1, \ldots, y_4$

DV01, duration, and convexity

1-st order approximation:

$$\Delta P_x = \sum_{i=1}^{N} \frac{\partial f}{\partial y_i} \Delta y_i, \quad \Delta y_i \text{ small}$$

2-nd order:

$$\Delta P_x = \sum_{i=1}^{N} \frac{\partial f}{\partial y_i} \Delta y_i + \frac{1}{2} \sum_{i,j=1}^{N} \frac{\partial^2 f}{\partial y_i \partial y_j} \Delta y_i \Delta y_j$$

DV01, duration, and convexity

1-st order approximation:

$$\Delta P_x = \sum_{i=1}^{N} \frac{\partial f}{\partial y_i} \Delta y_i, \quad \Delta y_i \text{ small}$$

2-nd order:

$$\Delta P_x = \sum_{i=1}^{N} \frac{\partial f}{\partial y_i} \Delta y_i + \frac{1}{2} \sum_{i,j=1}^{N} \frac{\partial^2 f}{\partial y_i \partial y_j} \Delta y_i \Delta y_j$$

DV01 for the i-th factor:

$$DV01^i = -\frac{\frac{\partial f}{\partial y_i}}{10,000}$$

Duration for the i-th factor:

$$-\frac{f}{\partial y_i} \frac{1}{f}$$

Convexity for factors i and j:

$$\frac{\partial^2 f}{\partial y_i \partial y_j} \frac{1}{f}$$

Here DV01, duration are vectors and convexity is a matrix

Example

$y_1, \ldots, y_4$ are par yield for 2 yr, 5 yr, 10yr and 30 yr

we want to move y_1, but not move y_2, y_3 and y_4.

This isolates the impact from change in y_1.

When y_1 is changed, y_2, y_3, and y_4 remains unchanged. All other rates are changed by linear interpolations.

Initial par curve $y^*(t)$, $t = 1/2, 1, 3/2, \cdots 30$ with key factors $y_1^* = y^*(2)$, $y_2^* = y^*(5)$, $y_3^* = y^*(10)$, and $y_4^* = y^*(30)$

Let $d^*(t)$ be the initial discount factor calculated from $y^*(t)$. Then

$$\frac{f^*}{F} = \frac{P_x^*}{F} = \frac{q}{2} \sum_{i=1}^{N} d^*(t_i) + d^*(t_N)$$

Now $y_1^* \to y_1^* + 1bp$, calculate changes to all par curve while keeping y_2^*, y_3^*, y_4^* fixed and the new bond price $f(y_1, y_2^*, y_3^*, y_4^*)$.

$$\frac{\Delta P_x}{\Delta y_1} = \frac{f(y_1, y_2^*, y_3^*, y_4^*) - f(y_1^*, y_2^*, y_3^*, y_4^*)}{0.0001}.$$

Example

See Note on “Multiple rate factors” page 15 - 18.

hedging using key rates

See note "Multiple interest rate factors" pages 18-24

Principal component analysis

In the previous section, we consider $y_1, \ldots, y_N$ as key rates, we shock key rate independently and do something with non-key rates.

Two drawbacks:

- Shocks to key rates are not economically meaningful usually
- Key rates usually do not move independently

Principal component analysis ties to identify typical movements

Typical movement:

- Parallel
- Steepening / flattening
- Butterfly

Review of linear algebra

Let V be a symmetric, positive definite $N \times N$-matrix
System: $V = V^\top$ and Positive definite: $x^\top V x > 0$ for any $x \in \mathbb{R}^N \setminus \{0\}$

V has the spectral decomposition

$$V = CDC^\top, \quad \text{where}$$

- D is a diagonal matrix with diagonal elements $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$;
- $C = (\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(N)})$, where $\beta^{(n)}$ the n-th column vector such that $|\beta^{(n)}| = 1$ (unit length) and $(\beta^{(n)})^\top \beta^{(m)} = 0$ (linear independent).

  $\beta^{(1)}$ is the first principal component, $\beta^{(2)}$ is the second principal component...

  The previous property means that

  $$C^\top C = 1_N \quad \text{and} \quad C^\top = C^{-1}.$$

- $\mathrm{Tr}(V) = \mathrm{Tr}(CDC^\top) = \mathrm{Tr}(DC^\top C) = \mathrm{Tr}(D) = \sum_{n=1}^N \lambda_n$.

When V is the covariance matrix, $\beta^{(1)}$ explains the most proportion of variation in the date, $\beta^{(2)}$ explains the second most proportion of variation...

For any vector $\Lambda \in \mathbb{R}^N$, we can always write

$$\Lambda = \sum_{n=1}^{N} \alpha_n \beta^{(n)}, \qquad \text{where } \alpha_n = \Lambda^\top \beta^{(n)},$$

Note we can apply transpose to both sides of the previous equation and left product with $\beta^{(m)}$

$$\Lambda^\top \beta^{(m)} = \sum_{n=1}^{m} \alpha_n (\beta^{(n)})^\top \beta^{(m)} = \alpha_m,$$

the second equation follows from $(\beta^{(n)})^\top \beta^{(m)} = 1$ when $n = m$, and 0 when $n \neq m$

Therefore Λ can be decomposed into a linear combination of principal components.

Choosing the first three principal components, we can approximate

$$\Lambda \approx \alpha_1 \beta^{(1)} + \alpha_2 \beta^{(2)} + \alpha_3 \beta^{(3)}$$

Principal component

Let X be a random vector, μ is the mean vector and Σ is the covariance matrix of X. Σ has a spectral decomposition $\Sigma = CDC^\top$

The principal components transform of X is

$$Y = C^\top (X - \mu). \tag{2}$$

This can be thought of as a rotation and a centering of X. The i^{th} component of Y is known as the n^{th} principal component of X:

$$Y_n = (\beta^{(n)})^\top (X - \mu).$$

Simple calculations show

$$\mathbb{E}[Y] = 0 \quad Cov(Y) = C^\top \Sigma C = C^\top CDC^\top C = D.$$

Hence principal components of Y are uncorrelated and have variance $Var(Y_n) = \lambda_n$.

To measure the ability of the principal components to explain the variability in X, let us observe that

$$\sum_{n=1}^{d} Var(Y_n) = \sum_{n=1}^{d} \lambda_n = \mathrm{Tr}(\Sigma) = \sum_{n=1}^{d} Var(X_n).$$

Then, for $k \leq d$, the ratio $\sum_{i=1}^{k} \lambda_i / \sum_{i=1}^{d} \lambda_i$ represents the amount of variability explained by the first k principal components.

Apply PCA to yield curve

- Set of maturities $T = \{1/12, 1/4, 1/2, 1, 2, 3, 5, 7, 10, 30\}$, $N = 10$
- $y(t, t + T_j)$ is the yield at t for a maturity T_j years later
- We have observations $y(t_m, t_m + T_j)$ for $m = 1, \ldots, M$
- Compute $\Delta_n^m = y(t_m, t_m + T_n) - y(t_{m-1}, t_{m-1} + T_n)$, $m = 1, \ldots, M$ and $N = 1, \ldots, 10$.
- Let Δ^m be a vector $(\Delta_1^m, \cdots, \Delta_N^m)^\top$. Δ^m measures change of yields on day m across all maturities
- Compute the sample covariance matrix

$$\Sigma = \frac{1}{M-1} \sum_{m=1}^{M} (\Delta^m - \overline{\Delta})(\Delta^m - \overline{\Delta})^\top,$$

where $\overline{\Delta} = \frac{1}{M} \sum_{m=1}^{M} \Delta^m$ is the sample mean.

- Perform PCA on Σ

Note

- Sometimes we perform the PCA on the correlation matrix rather than the covariance matrix to mod-out market volatility
- The first three principal components are usually stable in yield curve application. The first principal component "parallel" shift usually explains $80\% \sim 90\%$ of variation in data
- Hedging against PCA shifts. Find faces F_1, F_2, F_3 of par bonds at $2, 5, 10$ years so that the DV01 of their portfolio matches the DV01 of the bond, when three preincipal components shift.

$$\Delta P_x(bond) = \Delta P_x(2, 5, 10), \qquad \text{against three shifts}$$

Solve the three equations to get three unknown F_1, F_2, F_3.