Fixed Income

Valuation of securities with deterministic cash flows

Goals:

- Time value of money
- spot rate, YTM
- Treasury yield curve

Debt securities

We broadly say a fixed income security is one which is backed by debt

Features of a loan

- Principal: size of the loan
- Interest: rate at which the remaining balance owed grows with time
- Payment structure: how the loan is repaid

Types of loans

- Zero-Coupon Bond: paid with a single lump-sum (principal plus interest) at maturity
  e.g. borrow $1 today and pay back $1.05 a year from now
- Coupon Bond: periodic interest payments with the principal (plus last interest payment) repaid at maturity
  e.g. Borrow $1 today, pay 2.5 cents in six months and $1.025 a year from now
- Amortizing Loan: periodic level payments where each payment has a portion of the principal and a portion representing the interest
  e.g. mortgage

Fixed income: The first type of debt instruments were issued by the U.S. government. The cash flows are fixed

Primary and secondary markets

- Primary markets
  - Borrowers issue debt
  - Investors provide capital to borrowers
- Issuers in primary markets
  - Governments (US Treasury, HM Treasury)
  - Federal agencies
  - Corporations and banks
- Secondary markets
  - Trading after issue
  - Mostly over-the-counter (OTC)

Overview of players in debt markets

- Issuers
  - Governments (Treasury sector)
  - Agencies (agency debenture securities)
  - Corporations (credit sector)
  - Commercial banks
  - States and municipalities (tax-exempt sector)
  - Special-purpose vehicles (SPVs)
  - Foreign institutions
- Intermediaries ("Sell-side")
  - Investment banks
  - Commercial banks
  - Dealers
  - Primary dealers
  - Interdealer brokers
  - Credit-rating agencies
- Investors ("Buy-side")
  - Government and sovereign wealth funds
  - Pension funds
  - Insurance companies
  - Mutual funds
  - Commercial banks
  - Asset management firms
  - Households

Players and their objectives

- Issuers
  - Sell at best price
  - Liquidity in secondary market
  - Modify and reverse issuance decisions (flexibility)
  - Minimize funding cost
- Intermediaries
  - Primary market services (auctions, underwriting, distribution)
  - Market-making in secondary market
  - Proprietary trading
  - Services in risk management etc
- Investors
  - Buy at fair price
  - Diversification
  - Modify and reverse investment decisions (flexibility)
  - Get advisory services

Key players and their activities

- Governments
  - Issuers
  - Activity driven by deficit/surplus produced by economy
  - Surplus might be invested abroad (China and Japan invest in the US)
  - Set fiscal policies and regulate markets
- Central banks
  - Set monetary policies (BoE independent since 1992)
  - Conduct open market operations
  - Inject liquidity
  - Conduct auctions
  - Try to influence interest rates to promote growth and control inflation
  - Try to maintain stability of financial system
- Federal agencies and government sponsored entities
  - Federal National Mortgage Association (Fannie Mae)
  - Federal Home Loan and Mortgage Corporation (Freddie Mac)
  - Full, partial or implicit government guarantees
- Corporations and banks
  - Short-term debt: commercial paper
  - Long-term debt: corporate bonds
  - Banks lend and borrow in interbank market
  - Benchmark interest rate is LIBOR

Key players and their activities

- Financial institutions and dealers
  - Intermediation
  - Investment
  - Issuance
  - Arbitrage
  - Securitization
- "Buy-side" institutions
  - Asset management firms, pension funds, university endowments, insurance companies
  - Manage money and invest assets
  - Care about risk-return, costs
- Households
  - Mortgages
  - Auto loans
  - Credit cards
  - Student loans
  - Pensions
  - Savings

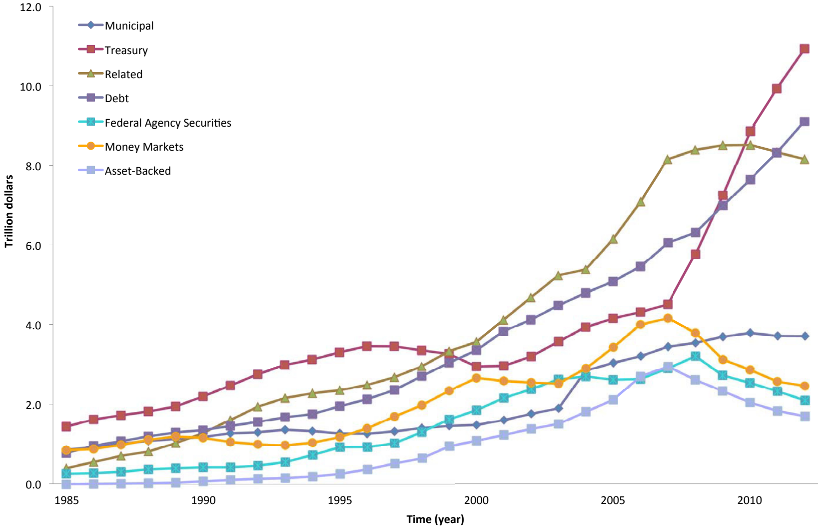
Fixed income markets 2012

Table: Fixed income markets 2012

Market	Market Values (USD bn)	Notional (USD bn)
U.S. Treasury Debt	10,920.9	
U.S. Municipal Debt	3,714.5	
U.S. Federal Agency Securities	2,095.8	
U.S. Money Market	2,460.8	
Mortgage-Backed Securities	2,572.2	
Asset-Backed Securities	8,168.1	
OTC Interest Rate Swaps	17,080.4	369,998.6
OTC Interest Rate Forwards	47.4	71,352.6
OTC Interest Rate Options	1,705.6	48,351.4
Exchange Traded Futures		22,641.3
Exchange Traded Options		25,909.7
U.S. Corporate Debt	9,100.7	
Credit Derivatives	847.6	25,068.7

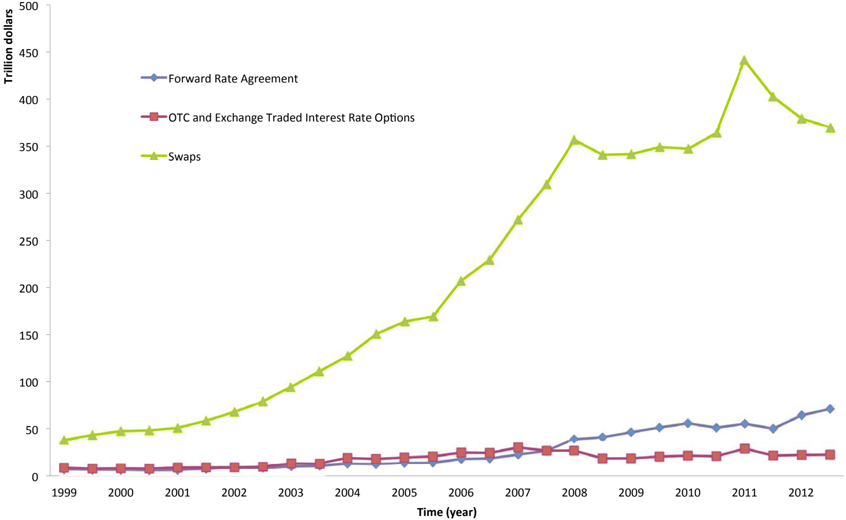
Source: Securities Industry and Financial Market Association (SIFMA) and Bank for International Settlements

Fixed income markets 1985–2012



Source: Securities Industry and Financial Market Association (SIFMA)

Fixed income derivatives (notional values) 1999–2012



The money market

The Money Market refers to the market for short term borrowing and lending, usually undertaken by banks.

- **Federal Funds Rate:** rate for borrowing/lending balances kept at the Federal Reserve
- **Eurodollar Rate:** rate of interest of dollar deposits at a European bank
- **LIBOR (London Interbank Offered Rate):** average interest rate the banks charge each other for short term uncollateralized borrowing/lending
- **Repo Rate:** interest rate charged for short term borrowing/lending with collateral

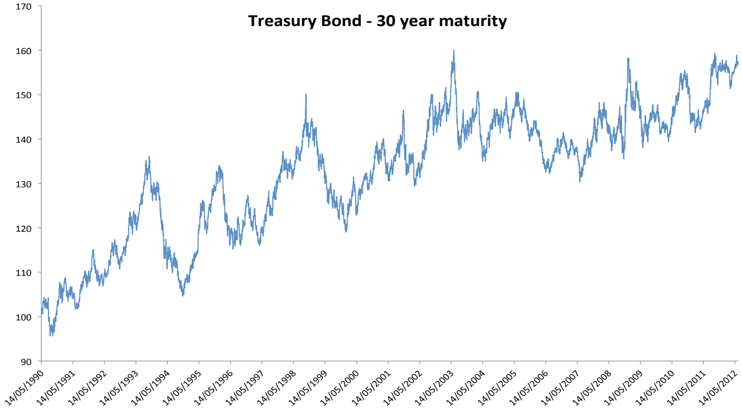
The MBS and ABS market

- The Mortgage-backed securities (MBS) market has grown significantly over the past years:
  - These securities allow local banks, which issue mortgages to individuals, to diversify their risk.
    - A bank issues mortgages to individuals living nearby.
    - These mortgages are susceptible to local events (e.g. a local company goes bankrupt leaving many mortgage holders without a job).
- Asset Backed Securities (ABS) are similar to MBS, but instead they are collateralized by other types of loans (auto loans, credit cards, etc.).

The derivatives market

- **Swaps**
  - Introduced in early 1980s to take advantage of apparent arbitrage opportunities in corporate bond markets.
  - Convenient for cash management and risk management. Ideal instrument to alter timing of payments and revenues and to change sensitivity of cash flows to fluctuations in interest rates.
- **Futures and forwards**
  - Two counterparties decide to exchange a security, or cash, or a commodity, at a prespecified time in the future for a price agreed upon today.
  - Futures are traded on regulated exchanges.
  - Forwards are traded in the over-the-counter (OTC) market.
- **Options**
  - Intuitively, an option is the financial equivalent of an insurance contract: It is a contract according to which the option buyer, who purchases the insurance, receives a payment from the options seller, who sold the insurance, only if some interest rate scenario occurs in the future.

Price of 30-year Treasury bond



- Source:Bloomberg

Debt securities vs. stocks: statistics

Table: Summary statistics

	US bonds	US large cap stocks	US small cap stocks
	Barclays US Aggregate	S&P 500 Composite	Russell 2000
	January 1989–June 2009		
Geometric mean (annualized)	7.08	8.04	7.51
Standard deviation (ann)	3.92	15.10	19.41
Max	3.80	10.83	15.28
Min	-3.42	-18.39	-23.32
Positive months	174	156	154
Negative months	72	90	92

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Negative months	72	90	92
	June 2007–June 2009		
Geometric mean (annualized)	5.98	-22.15	-23.00
Standard deviation (ann)	4.48	22.23	28.50
Max	3.66	9.14	14.37
Min	-2.39	-18.39	-23.32
Positive months	16	11	13
Negative months	9	14	12

Setting and conventions

Assumption: The dates and amounts of the cash flows are precisely known

You can think them as U.S. government bonds or corporate bonds with fixed coupon but no risk of default

Conventions

- Time is measured in years, the present time is $t = 0$
- no default risk
- trading has no transaction/funding costs, trading restrictions, or price impact

Basic products

Zero-coupon bond: face value: F, Maturity: T

Coupon bond:

- Face value F, Maturity T
- Annual coupon rate: q
- Number of coupons per year: m
- Bond pays Fq/m at time $t_i = i/m$, $i = 1, \ldots, mT$
- Bond pays F at time $t_{mT} = T$

Annuity

- Maturity T, payment A, and # of payments per year m
- Annuity pays A at time $t_i = i/m$, $i = 1, \ldots, m$
- Perpetuity: Annuity with $T = \infty$

Time value of money

Discounting factor:

$d(t)$ is the price today for receiving \$1 at time t.

Price of a (F, T) zero-coupon bond

Price of a (F, T, q, m) coupon bond

Price of an (A, T, m) annuity

Price of a perprtuity

Conventions

- Typically, prices will be quoted per \$100 of face values

  i.e., Face value is usually set as \$100

- Price is a theoretical guided price (to avoid easy arbitrage) here

  Market price could be different due to transaction costs, liquidity, default...

- The face value F of a bond is called Par Value
  - If price $> F$, then the bond is above par
  - If price $< F$, the bond is below par
  - If price $= F$, the bond is at par

Interest rates

\$1 today will be worth $\$\frac{1}{d(t)}$ at time t

Compounding frequency:

\$1 invested at a 5% rate of return yields, after one year:

- Annual compounding: $1 + 0.05 = 1.05$
- Semi-annual compounding: (typical for U.S. government bonds) $(1 + 0.05/2)(1 + 0.05/2) \approx 1.0506$
- Monthly compounding: $(1 + 0.05/12)^{12} \approx 1.0512$
- Continuous compounding: $\lim_{n\to\infty}(1 + 0.05/n)^n = e^{0.05} \approx 1.05127$

The amounts increase because interest earns interest

Spot rates

Let t be a multiple of $1/2$ year

Spot rate: the annual rate with semi-annual compounding which is consistent with the discounting factor $d(t)$

\$1 today earning (the annual rate) $\hat{r}(t)$ compounded semi-annually grows to \$$1/d(t)$ at time t

$$\left(1+\frac{\hat{r}(t)}{2}\right)^{2t}=\frac{1}{d(t)} \quad \Leftrightarrow \quad \hat{r}(t)=2\left(\left(\frac{1}{d(t)}\right)^{\frac{1}{2t}}-1\right)$$

Effective Spot Rates: defined for any $t>0$, not necessary multiple of $1/2$

$$(1+\hat{R}(t))^{t}=\frac{1}{d(t)} \quad \Leftrightarrow \quad \hat{R}(t)=\left(\frac{1}{d(t)}\right)^{\frac{1}{t}}-1$$

Note: $\hat{R}(t)\geq\hat{r}(t)$, because

$$(1+\hat{R}(t))^{t}=\left(1+\frac{\hat{r}(t)}{2}\right)^{2t}$$

Note: $\frac{1}{d(t)}$ is increasing in t, but it is not necessary that $\hat{r}(t)$ and $\hat{R}(t)$ increasing in t

Forward rates

Forward Loan: agreement to lend money at a future date

For example, agree to lend \$1 six months from now with the loan due six months after that at an annual interest rate of 6% compound semi-annually

- τ: time of agreement
- ζ: time of loan initiation
- T: time of loan settlement

Effective forward rate:
\$1 borrowed at ζ becomes $(1 + R^{for}_{\tau,\zeta,T})^{T-\zeta}$ at time T

Forward rate (semi-annual compounding, annual rate)
\$1 borrowed at ζ becomes $(1 + \frac{r^{for}}{2})^{2(T-\zeta)}$ at time T

Relation between different rates

consider $\tau = 0$

$$\left(1 + R^{for}_{0,\zeta,T}\right)^{T-\zeta} = \left(1 + \frac{r^{for}_{0,\zeta,T}}{2}\right)^{2(T-\zeta)} = \frac{d(\zeta)}{d(T)} = \frac{\left(1 + \hat{R}(T)\right)^{T}}{\left(1 + \hat{R}(\zeta)\right)^{\zeta}}.$$

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We define

$$f(t) = r^{for}_{0,t-1/2,t}, \qquad \text{for } t = k/2, k = 1, 2, \ldots$$

Then $f(1/2) = r^{for}_{0,0,1/2} = \hat{r}(1/2)$.

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We also have

$$\left(1 + \hat{r}(1)/2\right)^{2} = \left(1 + \frac{f(1/2)}{2}\right)\left(1 + \frac{f(1)}{2}\right).$$

More generally

$$\left(1 + \hat{r}(t)/2\right)^{2t} = \left(1 + \frac{f(1/2)}{2}\right)\left(1 + \frac{f(1)}{2}\right)\ldots\left(1 + \frac{f(t)}{2}\right).$$

Relation between different rates cont.

$$1 + f(t)/2 = \frac{(1+\hat{r}(t)/2)^{2t}}{(1+\hat{r}(t-1/2)/2)^{2t-1}} = \left(\frac{1+\hat{r}(t)/2}{1+\hat{r}(t-1/2)/2}\right)^{2t-1}(1+\hat{r}(t)/2)$$

Therefore

- When $\hat{r}(t) > \hat{r}(t-1/2)$ (upward sloping spot curve) $\Rightarrow f(t) > \hat{r}(t)$
- When $\hat{r}(t) < \hat{r}(t-1/2)$ (downward sloping spot curve) $\Rightarrow f(t) < \hat{r}(t)$

Par Rates

Consider a T year bond with coupon c and semi-annual payments

The (semi-annual) par rate $c(T)$ is defined as the value of the coupon which makes the bond par valued, i.e., $c(T)$ solves

$$1 = \frac{c(T)}{2} \sum_{t=1}^{2T} d(t/2) + d(T)$$

Note: when $T = 1/2$

$$1 = \frac{c(1/2)}{2} d(1/2) + d(1/2)$$

Then

$$c(1/2) = 2\left(\frac{1}{d(1/2)} - 1\right) = \hat{r}(1/2) = f(1/2).$$

Important note

- Discount factors are typically extracted from today's price data. So discount factors may be different tomorrow
- Spot curve could be different tomorrow
- We do no know what the spot rate is in six months. Therefore a six month spot loan made six months from now is not the same as the six month forward load with the six months maturity. The later is agreed today at a rate implied by today's discount factors.
- Similar for forward rate, we know $r^{for}_{0,\zeta,T}$, but we don't know $r^{for}_{\tau,\zeta,T}$ $(r > 0)$.

Arbitrage and discount factors

Spot curve (discount factor) is constructed from a particular set of benchmark securities

for example, the most recently issued, or "on-the-run" treasuries

We can then use these to obtain a theoretical price for other securities,

for example, "off-the-run" securities

If the traded price deviated from the theoretical price, there is arbitrage.

Example

Consider a zero-coupon bond with $F = \$100$ and maturity T

$$\frac{P_x}{F} = 100\frac{1}{(1+\hat{r}(T)/2)^{2T}}.$$

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$$\frac{P_x}{F} = 100 \frac{1}{(1+\hat{r}(T)/2)^{2T}}.$$

Consider a coupon bond with annual coupon q, face value F, N remaining payment dates and $1/2, 1, \ldots, N/2$

$$\begin{aligned}
\frac{P_x}{F} =& \frac{q}{2}\sum_{i=1}^{N} d(i/2) + d(N/2) \quad \text{(discount factors)} \\
=& \frac{q}{2}\sum_{i=1}^{N} \frac{1}{(1+\hat{r}(i/2)/2)^i} + \frac{1}{(1+\hat{r}(N/2)/2)^N} \quad \text{(spot rates)} \\
=& \frac{q}{2}\sum_{i=1}^{N} \left(\prod_{j=1}^{i}(1+f(j/2)/2)\right)^{-1} + \left(\prod_{j=1}^{N}(1+f(j/2)/2)\right)^{-1} \quad \text{(forward rate)}
\end{aligned}$$

Lots of different rates, we want a single rate to summarize the price

Yield to maturity

Coupon yield: q

Current yield: $q\frac{F}{p_x}$ Coupon divided by price

Yield to maturity (YTM)

The single interest rate that when used to discount all the bond's future cash flows produces the bond's current price

We seek y to solve

$$\frac{P_x}{F} = \frac{q}{2}\left(\frac{1}{1+y/2} + \frac{1}{(1+y/2)^2} + \cdots + \frac{1}{(1+y/2)^N}\right) + \frac{1}{(1+y/2)^N}.$$

Note: if the spot rate curve is flat, i.e., $\hat{r}(1/2) = \hat{r}(1) = \cdots = \hat{r}(N/2) = \tilde{y}$.
Then $y = \tilde{y}$.
But in general this does not happen

Note: In practice, bonds are quoted in terms of YTM y, not P_x/F

Relationship between y, q, P_x, F

1. $P_x > F \Leftrightarrow q > y$
2. $P_x = F \Leftrightarrow q = y$
3. $P_x < F \Leftrightarrow q < y$

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1. $P_x > F \Leftrightarrow q > y$
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Proof:
Let $\lambda = \frac{1}{1+y/2}$. Then

$$\frac{P_x}{F} = \frac{q}{2}\sum_{i=1}^{N}\lambda^i + \lambda^N = \lambda^N + \frac{q}{2}\frac{\lambda}{1-\lambda}(1-\lambda^N)$$

On the other hand

$$\lambda = \frac{1}{1+y/2} \Rightarrow \frac{\lambda}{1-\lambda} = \frac{2}{y}$$

Then

$$\frac{P_x}{F} = \lambda^N + \frac{q}{y}(1-\lambda^N)$$

The results follow from the identity above.

Effective yield to maturity

Consider a security which pays F_i at times T_i for $i = 1, \ldots, N$

The effective yield to maturity is the yield Y s.t.

$$P_x = \sum_{i=1}^{N} \frac{F_i}{(1+Y)^{T_i}}.$$

YTM for an annuity

$$P_x = \sum_{i=1}^{N} \frac{A}{(1+y/2)^i} = A\frac{\lambda}{1-\lambda}(1-\lambda^N), \quad \text{where } \lambda = \frac{1}{1+y/2}$$

YTM for a peretuity

$$P_x = A\frac{\lambda}{1-\lambda} = A\frac{2}{y}$$

$$y = \frac{2A}{P_x}$$

Warning

YTM does not reflect the yield obtained by investing in bond

For example, consider 2 payments

From the definition of YTM

$$P_x(1+y/2)^2 = \frac{q}{2}(1+y/2) + 1 + \frac{q}{2}$$

From the investment point of view

$$\text{Actual} = \frac{q}{2}(1+r/2) + 1 + \frac{q}{2},$$

where r is the interest rate 6 months from now

The two equations above are the same only when $r = y$

y is the yield of investment iff we reinvest the intermediate cash flow at y all the time

Example

Suppose that all bonds have a face value of \$100

- The price of a 1-year zero-coupon bond is \$95.2381
- The price of a 2-year coupon bond with an annual coupon of 6% is \$100

Questions:

1. Calculate the 1-year effective spot rate
2. What is the effectiveYTM on the 2-year bond?
3. Now suppose that in addition a 2-year zero coupon bond trades in the market at §88.9829
    - What is this bond's effective YTM?
    - Is the bond "cheap" or "expensive" compared to the other bonds?

Solution

1.
$$100\frac{1}{1+\hat{R}(1)} = 95.2381 \quad \Rightarrow \quad \hat{R}(1) = 5\%$$

2.
$$\frac{6}{1+y} + \frac{106}{(1+y)^2} = 100 \quad \Rightarrow \quad y = 6\%$$

3.
$$\frac{100}{(1+y)^2} = 88.9829 \quad \Rightarrow \quad YTM = 6.01\%$$

From the first two bonds, the 2-year spot rate is

$$\frac{6}{1.05} + \frac{106}{(1+\hat{R}(2))^2} = 100 \quad \Rightarrow \quad \hat{R}(2) = 6.03\%$$

The implies price of the third bond is

$$\frac{100}{(1+\hat{R}(2))^2} = 88.95 < 88.98.$$

Therefore the third bond is expensive even though it has a high YTM!

Yield curves

Zero-coupon yield curve: spot rate curve

$y = \hat{r}(T)$ for a T year zero coupon bond

Par-coupon yield curve:

YTM, as a function of T, for coupon bonds trading at par

Annuity yield curve

YTM, as a function of T, for a T-year annuity, paying every six months

Note: All these curves use some kind of interpolation to get the entire curve from a few key matruties

For example, par curve for treasures use data for 1 mo, 3 mo, 6 mo, 1 yr, 2 yr, 3 yr, 5 yr, 7 yr, 10 yr, 20 yr, 30yr

Treasury yield

U.S. government issues

- Treasury bills (T-bills): mature in 1 year or less, zero coupon
- Treasury notes (T-notes): maturities in 2, 5, 7, 10 years, coupon paid semi-annually
- Treasury bonds (T-bonds): maturity of 30 years, semi-annual coupon

Yields on treasuries are investment yields

Yields on T-bills: Treasury Department uses two methods

An investor that purchases a 91-day T-bill for $\$9,800$ per $\$10,000$ face value will have a yield of

$$\text{Discount yield} = \frac{10,000 - 9,800}{10,000} \times \frac{360}{91} = 7.91\%$$
$$\text{Investment yield} = \frac{10,000 - 9,800}{9,800} \times \frac{365}{91} = 8.19\%.$$

Note that the day counts are different

Yield on T-notes and bonds: both receive semi-annual coupon

$$\text{Treasury yield} = \left(C + \frac{FV - PP}{T}\right) \div \frac{FV + PP}{2},$$

where $C =$ coupon rate, $FV =$ fave value, $PP =$ purchase price, $T =$ time to maturity

For example, the yield on a 10-year note with 3% coupon purchased at a premium for $\$10,300$ and held to maturity is

$$\text{Treasury yield} = \left(300 + \frac{10,000 - 10,300}{10}\right) \div \frac{10,000 + 10,300}{2} = 2.66\%$$

Treasury yield curve

A function of treasury yield against the maturity

Interpolated from traded treasury bonds with several key maturities

Three forms of yield curve: (from investopedia.com)

Normal yield curve: longer maturity bonds have a higher yield compared to shorter-term bonds

A normal or up-sloped yield curve indicates yields on longer-term bonds may continue to rise, responding to periods of economic expansion. When investors expect longer-maturity bond yields to become even higher in the future, many would temporarily park their funds in shorter-term securities in hopes of purchasing longer-term bonds later for higher yields

In a rising interest rate environment, it is risky to have investments tied up in longer-term bonds when their value has yet to decline as a result of higher yields over time. The increasing temporary demand for shorter-term securities pushes their yields even lower, setting in motion a steeper up-sloped normal yield curve

Inverted yield curve

An inverted or down-sloped yield curve suggests yields on longer-term bonds may continue to fall, corresponding to periods of economic recession. When investors expect longer-maturity bond yields to become even lower in the future, many would purchase longer-maturity bonds to lock in yields before they decrease further.

The increasing onset of demand for longer-maturity bonds and the lack of demand for shorter-term securities lead to higher prices but lower yields on longer-maturity bonds, and lower prices but higher yields on shorter-term securities, further inverting a down-sloped yield curve.

Flat yield curve

typically happens in transition between the previous two patterns

Daily treasury yield curve can be found here