MF702 PROBLEM SET 3: SOLUTIONS

1. Consider the following two-period model. The price of a stock is \$50, the simple compounding interest rate per period is 2%. After the first period, the price of the stock can go up to \$55 or drop to \$47. If the stock price jumps up in the first period, if the holder of an American call option exercises this option, the holder will receive a dividend of \$2 that is paid out by the stock. Assume that the option is exercised before dividends are paid out. After the dividend is paid out, in the second period the price can jump up to \$57 or down to \$48. If the stock price jumps down in the first period, in the second period it can jump up to \$48 or down to \$41. Consider an American call option with strike price $K = \$45$ that matures at the end of the second period.
   (a) Plot the binomial tree underlying this option. Calculate the risk neutral probability in each period. (Remember that after payout dividend, S^u becomes $55 - 2 = 53$)
   (b) Use the Snell envelop to price the option on the binomial tree.
   (c) Given the optimal exercise of the American call option.

   Solution: (2) If the stock price jumps up in the first period, the risk neutral probability in the second period $\tilde{p}^u$ satisfies

$$55 - 2 = \frac{1}{1 + 0.02}\Big(\tilde{p}^u 57 + (1 - \tilde{p}^u)48\Big).$$

   Therefore

$$\tilde{p}^u = \frac{(55 - 2) \times (1 + 0.02) - 48}{57 - 48} = 0.67.$$

   Similarly

$$\tilde{p}^d = \frac{47 \times (1 + 0.02) - 41}{48 - 41} = 0.99.$$

   The first period risk neutral probability is

$$\tilde{p} = \frac{50 \times (1 + 0.02) - 47}{55 - 47} = 0.5.$$

   (b) The payoff of the American call at maturity are

$$\begin{aligned} P_2^{uu} =& I_2^{uu} = (57 - 45)_+ = 12, \\ P_2^{ud} =& P_2^{du} = I_2^{ud} = I_2^{du} = (48 - 45)_+ = 3, \\ p_2^{dd} =& I_2^{dd} = (41 - 45)_+ = 0. \end{aligned}$$

Go one step backwards, In state u, if the holder were to exercise the option, he would receive the option payoff and the \$2 dividend. As a result, the exercise payoffs are

$$\begin{aligned} I^u =&(S^u - K)_+ + 2 = (55 - 45)_+ + 2 = 12, \\ I^d =&(S^d - K)_+ = (47 - 45)_+ = 2. \end{aligned}$$

The Snell-envelop at time 1 are

$$\begin{aligned} P_1^u =& \max\left\{I^u, \frac{1}{1+0.02}[\tilde{p}^u P_2^{uu} + (1-\tilde{p}^u)P_2^{ud}]\right\} = \max\{12, 8.85\} = 12, \\ P_1^d =& \max\left\{I^d, \frac{1}{1+0.02}[\tilde{p}^d P_2^{ud} + (1-\tilde{p}^d)P_2^{dd}]\right\} = \max\{2, 2.91\} = 2.91. \end{aligned}$$

Go back to the first period, if the holder exercise the option, the payoff is

$$I = (50 - 45)_+ = 5.$$

The Snell-envelop is

$$P = \max\{I, \frac{1}{1+0.02}\left[\tilde{p}P^u + (1-\tilde{p})P^d\right]\} = \max\{5, 6.86\} = 6.86.$$

Therefore the American call price at time 0 is 6.86.

(c) The optimal exercise rule is

$$\tau^*(\omega_1\omega_2) = \begin{cases} 1 & \omega_1 = u \\ 2 & (\omega_1, \omega_2) = du \\ \text{no exercise} & (\omega_1, \omega_2) = dd \end{cases}$$

2. Consider a two-period binomial model. Assume that the maturity is $T = 1$ and each period is $\Delta t = \frac{1}{2}$. The stock has an initial price os \$100 and can go up 15% (log return) or down 10% (log return) per annum with equal probabilities. Assume that the annual continuous compounding interest rate is $r = 0.02$. Consider an put-like option with intrinsic value $(K - S_t)_+$ if exercised at time $t \in \{0, 1, 2\}$. This option, however, can only be exercised at even times. That is, it can only be exercised in periods $t \in \{0, 2\}$, and it cannot be exercised in periods $t = 1$. Such as option is called a *Bermudan* option. Assume that the strike price is $K = 95$.
   (a) Draw a binomial tree for this model. Mark the nodes of the tree in which the option can be exercised.
   (b) Use backwards induction to show that the price of the Bermudan option is 1.29. Consider how the Snell envelop has to be adjusted for periods in which the option cannot be exercised.
   (c) What is the optimal exercise rule?

Solution: (b) The payoff of the option at time $t = 2\Delta$ are

$$I^{uu} = (K - S_2^{uu})_+ = 0, \quad I^{ud} = I^{du} = (K - S_2^{ud})_+ = 0, \quad I^{dd} = (K - S_2^{dd})_+ = 4.52.$$

The risk-neutral probability is

$$\tilde{p} = \frac{e^{r/2} - e^{d/2}}{e^{u/2} - e^{d/2}} = 0.46.$$

At time $t = \Delta$, the option cannot be exercised. Therefore, we need to use the Eropean risk-neutral pricing formula in this step. We have

$$\begin{aligned} P_1^u =& e^{-r\Delta}\Big(\tilde{p}I^{uu} + (1-\tilde{p})I^{ud}\Big) = 0, \\ P_1^d =& e^{-r\Delta}\Big(\tilde{p}I^{du} + (1-\tilde{p})I^{dd}\Big) = 2.42. \end{aligned}$$

Go back to time 0. If the option is exercised at time 0, its payoff is $I = (K - S_0)_+ = 0$. Therefore the Snell envelop at time 0 is

$$P_0 = \max\Big\{I, e^{-r\Delta}\Big(P_1^u\tilde{p} + P_1^d(1-\tilde{p})\Big)\Big\} = 1.29.$$

Therefore, the option price at time 0 is 1.29.

(c) The optimal exercise rule is

$$\tau^*(\omega_1\omega_2) = \begin{cases} 2, & (\omega_1\omega_2) = dd \\ \text{no exercise,} & \text{otherwise} \end{cases}$$

3. Consider an American call option with strike K and maturity $T = N$ on a non-dividend-paying stock with price process $(S_k : 0 \le k \le N)$. Show that the American call option is never optimal to be exercised before maturity. That is, show that for any $0 \le n < m \le N$,

$$(S_n - K)_+ \le \tilde{\mathbb{E}}\Big[e^{-r(m-n)}(S_m - K)_+\Big].$$

(Hint: Use Jensen's inequality and the face that the map $x \mapsto (x)_+$ is convex.)

Solution: It follows from the risk-neutral pricing formula, Jensen's inequality, and the convexity of $x \mapsto (x)_+$ that

$$\begin{aligned} (S_n - K)_+ =& \Big(\tilde{\mathbb{E}}_n[e^{-r(m-n)}S_m] - K\Big)_+ = \Big(\tilde{\mathbb{E}}_n\Big[e^{-r(m-n)}\Big(S_m - Ke^{r(m-n)}\Big)\Big]\Big)_+ \\ \le& \tilde{\mathbb{E}}_n\Big[\Big(e^{-r(m-n)}\Big(S_m - Ke^{r(m-n)}\Big)\Big)_+\Big] = \tilde{\mathbb{E}}_n\Big[e^{-r(m-n)}\Big(S_m - Ke^{r(m-n)}\Big)_+\Big] \\ \le& \tilde{\mathbb{E}}_n\Big[e^{-r(m-n)}(S_m - K)_+\Big]. \end{aligned}$$

In other words, $e^{-rn}(S_n - K)_+$ is a $\tilde{\mathbb{P}}$-submartingale.

4. Option pricing theory can also be used in corporate finance. Consider the following scenario. At time 0, a company needs cash to finance a large project. This company decides to take on some debt (that is, it borrows money from debt creditors) to finance this project. The total value of the company is C_0 at inception and C_T at maturity. This value includes all assets that this company holds. Any debt that the company takes on is a claim on the total value of the company. That is, if this company does not pay back its debt at time T, then it will declare bankruptcy and the debt creditors can confiscate all of ABS's asset C_T. We assume that the amount of debt that the company takes on has a face value of 100 at maturity. Therefore, if $C_T \geq 100$, the debt creditors will receive the full face value 100; if $C_T < 100$, the debt creditors will receive C_T (we assume that entering bankruptcy is costless).

   In this exercise, we price a convertible corporate debt in a two-period binomial model with periods $t \in \{0, 1, 2\}$. Assume $T = 2$ and $\Delta = 1$. A security is called *convertible debt* if the security is sold as corporate debt initially at time $t = 0$, but the holder of the security can decide to exchange the corporate debt for a share of equity at time $t = 1$. Suppose that the face value of debt at maturity is 100, and the company issue one equity share if the holder of the security decides to convert the debt. Suppose that the initial value of the company is $C_0 = 100$, and that in each period the value of the company can become e^u or e^d of the previous value. We assume that $u > 0 > d$ and $u + d > 0$.

   (a) Plot a binomial tree that corresponds to the pricing of the convertible debt in the two-period model.

   (b) Use the Snell envelop to determine the pricing of the convertible debt in this model. (Hint: At time 1, the holder will only exercise when the risk neutral value of conversion is larger than the value of keeping the debt.)

Solution: Proceed by backward induction. Suppose the price process of the convertible debt is CD. At maturity, the payoff of the secutiry if not converted is the same as the debt payoff π, that is

$$
\begin{aligned}
\pi_2^{uu} &= \min\{100, C_2^{uu}\} = 100 \min\{1, e^{2u\Delta t}\} = 100, \\
\pi_2^{ud} &= \min\{100, C_2^{ud}\} = 100 \min\{1, e^{(u+d)\Delta t}\} = 100, \\
\pi_2^{dd} &= \min\{100, C_2^{dd}\} = 100 \min\{1, e^{2d\Delta t}\} = 100 e^{2d\Delta t},
\end{aligned}
$$

where above equalities hold because $u > 0 > d$ and $u + d > 0$.

If the debt is converted to equity at time 1, then the payoff of the security is

$$\begin{aligned}\Pi_2^{uu} =& C_2^{uu} = 100e^{2u\Delta t},\\ \Pi_2^{ud} =& C_2^{ud} = 100e^{(u+d)\Delta t},\\ \Pi_2^{dd} =& C_2^{dd} = 100e^{2d\Delta t}.\end{aligned}$$

At time 1, if the holder of the security does not convert the debt, the value of security is equal to

$$\begin{aligned}\pi_1^u =& e^{-r\Delta t}\tilde{\mathbb{E}}_1[\pi_2|\omega_1 = u] = 100e^{-r\Delta t},\\ \pi_2^d =& e^{-r\Delta t}\tilde{\mathbb{E}}_1[\pi_2|\omega_2 = d] = 100e^{-r\Delta t}\Big(\tilde{p} + (1-\tilde{p})e^{2d\Delta t}\Big).\end{aligned}$$

If the holder decides to convert the debt, then the value of the security is equal to

$$\begin{aligned}\Pi_1^u =& e^{-r\Delta t}\tilde{\mathbb{E}}[\Pi_2|\omega_1 = u] = 100e^{-r\Delta t}\Big(\tilde{p}e^{2u\Delta t} + (1-\tilde{p})e^{(u+d)\Delta t}\Big)\\ \Pi_1^d =& e^{-r\Delta t}\tilde{\mathbb{E}}[\Pi_2|\omega_1 = d] = 100e^{-r\Delta t}\Big(\tilde{p}e^{(u+d)\Delta t} + (1-\tilde{p})e^{2d\Delta t}\Big).\end{aligned}$$

Therefore, the price of the convertible debt at time 1 is

$$\begin{aligned}CD_1^u =& \max\Big\{100e^{-r\Delta t}, 100e^{-r\Delta t}\Big(\tilde{p}e^{2u\Delta t} + (1-\tilde{p})e^{(u+d)\Delta t}\Big)\Big\},\\ CD_1^d =& \max\Big\{100e^{-r\Delta t}\Big(\tilde{p} + (1-\tilde{p})e^{2d\Delta t}\Big), 100e^{-r\Delta t}\Big(\tilde{p}e^{(u+d)\Delta t} + (1-\tilde{p})e^{2d\Delta t}\Big)\Big\}.\end{aligned}$$

We see that the holder will decide to convert in state u if and only if

$$\tilde{p}e^{2u\Delta t} + (1-\tilde{p})e^{(u+d)\Delta t} > 1 \Leftrightarrow e^{2u\Delta t}e^{r\Delta t} - e^{u\Delta t}\Big(1 + e^{(d+r)\Delta t}\Big) + 1e^{d\Delta t} > 0. \tag{1}$$

In state d, the holder will convert if

$$\tilde{p}e^{(u+d)\Delta t} + (1-\tilde{p})e^{2d\Delta t} > \tilde{p} + (1-\tilde{p})e^{2d\Delta t} \Leftrightarrow e^{(u+d)\Delta t} > 0. \tag{2}$$

The price of convertible debt in period 0 can be computed using the risk-neutral formula

$$CD_0 = \tilde{\mathbb{E}}\Big[e^{-r\Delta t}CD_1\Big] = e^{-r\Delta t}\Big(\tilde{p}CD_1^u + (1-\tilde{p})CD_1^d\Big).$$