Portfolio optimization:

wealth at time 1:

$$W_1 = \frac{W_0}{B_{0,1}} + \Delta\left(S_1 - \frac{S_0}{B_{0,1}}\right)$$

consider $B_{0,1} = (1+r)^{-1}$, where r is one-period simple interest rate. Then

$$W_1 = \underbrace{(1+r)W_0}_{\text{value from riskless investment}} + \underbrace{\Delta(S_1 - (1+r)S_0)}_{\text{excess value from risky investment}}$$

Agent preference: utility function $U : \mathbb{R} \rightarrow \mathbb{R}$.

Agent's portfolio choice problem:

$$\max_{\Delta} \mathbb{E}[U(W_1)] = \max_{\Delta} \mathbb{E}[U((1+r)W_0 + \Delta(S_1 - (1+r)S_0))]$$

consider $\mathbb{E}[U((1+r)W_0 + \Delta(S_1 - (1+r)S_0))]$ as a function wrt Δ. It is a concave ~~convex~~ function. Optimal Δ is specified by the first order condition (FOC)

$$0 = \mathbb{E}[U'((1+r)W_0 + \Delta(S_1 - (1+r)S_0))(S_1 - (1+r)S_0)]$$

$$0 = \mathbb{E}[u'(W_1)(S_1 - (1+r)S_0)] \qquad (1)$$

Define another measure $\mathbb{Q}$ via

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{u'(W_1)}{\mathbb{E}[u'(W_1)]} \qquad \mathbb{E}^{\mathbb{P}}[Z] = 1.$$

Then from (1), we have

$$0 = \mathbb{E}\left[\frac{u'(W_1)}{\mathbb{E}[u'(W_1)]}(S_1 - (1+r)S_0)\right]$$

$$= \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\left(S_1 - (1+r)S_0\right)\right]$$

$$= \mathbb{E}^{\mathbb{Q}}[S_1 - (1+r)S_0]$$

$$\Leftrightarrow \quad S_0 = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1]$$

i.e. Discounted stock price is a martingale under $\mathbb{Q}$

$\Rightarrow$ $\mathbb{Q}$ is a risk neutral measure.

Come back to FOC (1). Apply Taylor expansion to approximate U', consider $r=0$.

$$U'(W_1) \approx U'(W_0) + U''(W_0)(W_1 - W_0)$$

$$= U'(W_0) + U''(W_0)\,\Delta\,(S_1 - S_0)$$

Then

$$0 = \mathbb{E}[U'(W_1)(S_1 - S_0)] \approx U'(W_0)\,\mathbb{E}[S_1 - S_0]$$

$$+ U''(W_0)\,\mathbb{E}[\Delta(S_1 - S_0)^2]$$

$$\Rightarrow \Delta \approx -\frac{U'(W_0)}{U''(W_0)} \frac{\mathbb{E}[S_1 - S_0]}{\mathbb{E}[(S_1 - S_0)^2]}$$

- $-\dfrac{U'(W_0)}{U''(W_0)}$: $(\text{absolute risk aversion})^{-1}$
- $\mathbb{E}[S_1 - S_0]$: excess expected P&L.
- $\mathbb{E}[(S_1 - S_0)^2]$: variance.

$$\min_{\Delta,\lambda_1,\lambda_2} \left(\Delta^T \Sigma \Delta - \lambda_1 (\Delta^T \vec{r} - \bar{r}) - \lambda_2 (\Delta^T \vec{1} - 1) \right)$$

If $\Delta^T \vec{r} - \bar{r} \neq 0$, say $\Delta^T \vec{r} - \bar{r} > 0$, we can have $\lambda_1 \to \infty$ to get min $-\infty$. So we must have $\Delta^T \vec{r} - \bar{r} = 0$.

Then $\lambda_1 (\Delta^T \vec{r} - \bar{r}) = 0$.

$\lambda_2 (\Delta^T \vec{1} - 1) = 0$.

$$\Delta_f r_f + \Delta^T \vec{r} = \bar{r} \qquad \Delta_f + \Delta^T \vec{1} = 1$$

$$\Rightarrow (1 - \Delta^T \vec{1}) r_f + \Delta^T \vec{r} = \bar{r} \qquad \Rightarrow \Delta_f = 1 - \Delta^T \vec{1}$$

$$\Rightarrow \quad \Delta^T (\vec{r} - r_f) = \bar{r} - r_f$$