Option pricing: continuous time models

Goals:

- Black-Scholes-Merton model of option pricing
- Greeks of options
- Implied volatility

Relevant literature:

- Hull Chap. 15, 16, 19, 20

Wiener process (standard Brownian motion)

- Markov (path independence):

$$z_{t_2} - z_{t_1} \text{ is independent of } z_{t_4} - z_{t_3} \text{ for } t_1 < t_2 < t_3 < t_4$$

- Martingale:

$$\mathbb{E}[z_{t_2} \mid z_{t_1}] = z_{t_1} \qquad t_1 < t_2$$

- Finite quadratic variation:

$$\sum_{i=1}^{n} (\Delta z)^2 = n(\sqrt{\Delta t})^2 = n\Delta t$$

- Limit:

$$\Delta t \to 0 : \quad \mathrm{d}z \sim \mathcal{N}(0, \sqrt{\mathrm{d}t})$$

ITÔ PROCESS

- General Itô process:

$$\mathrm{d}s_t = \alpha(s_t, t)\,\mathrm{d}t + \sigma(s_t, t)\,\mathrm{d}z_t \tag{1}$$

- The stochastic integral $\int_0^t \sigma(s_u, u) dz_u$, $t \geq 0$, is a martingale,

$$\mathbb{E}\Big[\int_0^t \sigma(s_u, u) dz_u \Big| \mathcal{F}_{\tilde{t}}\Big] = \int_0^{\tilde{t}} \sigma(s_u, u) dz_u, \qquad \text{for any } 0 \leq \tilde{t} \leq t.$$

- Consider a function of s_t, $V(s_t, t)$
- Itô's lemma:

$$\begin{aligned} \mathrm{d}V(s,t) &= \partial_s V\,\mathrm{d}s + \partial_t V\,\mathrm{d}t + \tfrac{1}{2}\,\partial^2_{ss} V\,\mathrm{d}s^2 \\ &= \Big(\tfrac{1}{2}\,\sigma(s,t)^2 \partial^2_{ss} V + \alpha(s,t)\partial_s V + \partial_t V\Big)\,\mathrm{d}t + \sigma(s,t)\partial_s V\,\mathrm{d}z_t, \end{aligned} \tag{2}$$

where $\partial_t V, \partial_s V, \partial^2_{ss} V$ are partial derivatives.

(...)dt is called the drift term and (...)dz_t is called the martingale term

Geometric Brownian motion

- Asset price ollows geometric brownian motion if:

$$\frac{\mathrm{d}s_t}{s_t} = \alpha\,\mathrm{d}t + \sigma\,\mathrm{d}z_t$$

where α and σ are constants.

- Itô's lemma:

$$\begin{aligned}\mathrm{d}\ln s_t &= \tfrac{1}{s}\,\mathrm{d}s - \tfrac{1}{2}\,\tfrac{1}{s^2}\,\mathrm{d}s^2 \\ &= \tfrac{1}{s}\,(\alpha s\,\mathrm{d}t + \sigma s\,\mathrm{d}z) - \tfrac{1}{2}\,\tfrac{1}{s^2}\,\sigma^2 s^2\,\mathrm{d}t \\ &= \left(\alpha - \tfrac{1}{2}\,\sigma^2\right)\,\mathrm{d}t + \sigma\,\mathrm{d}z\end{aligned}$$

- $\ln s_t$ is a normal random variable:

$$\ln s_t = \ln s_0 + \left(\alpha - \tfrac{1}{2}\,\sigma^2\right) t + \sigma\left[z_t - z_0\right], \quad z_0 = 0$$

$$\mathbb{E}\left[\ln s_t\right] = \ln s_0 + \left(\alpha - \tfrac{1}{2}\,\sigma^2\right) t, \quad \mathbb{V}\left[\ln s_t\right] = \sigma^2 t$$

- s_t is a lognormal random variable:

$$s_t = s_0 \exp\left[\left(\alpha - \tfrac{1}{2}\,\sigma^2\right) t + \sigma z_t\right]$$

Risk-neutral valuation

- Consider an asset price following geometric brownian motion

$$\frac{ds_t}{s_t} = \alpha dt + \sigma dz_t.$$

- Suppose that there exists a risk neutral measure $\tilde{\mathbb{P}}$ such that the discounted asset price $\{e^{-rt}s_t; t \geq 0\}$ is a martingale under $\tilde{\mathbb{P}}$,

$$\frac{ds_t}{s_t} = rdt + \sigma d\tilde{z}_t,$$

where

$$d\tilde{z}_t = dz_t + \frac{\alpha - r}{\sigma}dt,$$

$\tilde{z}$ is a Wiener process under $\tilde{\mathbb{P}}$ (see Girsanov theorem from MF795)

Risk-neutral valuation

- For an European option with maturity T and payoff $g(s_T)$, its arbitrage-free price at time t is

$$V_t = \tilde{\mathbb{E}}\left[e^{-r(T-t)} g(s_T) \middle| \mathcal{F}_t\right]$$

- The discounted price $e^{-rt} V_t$ is a martingale under $\tilde{P}$

  Proof: For any $s \leq t$,

$$\tilde{\mathbb{E}}\left[e^{-rt} V_t \middle| \mathcal{F}_s\right] = \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[e^{-rT} g(s_T) \middle| \mathcal{F}_t\right] \middle| \mathcal{F}_s\right] = \tilde{\mathbb{E}}\left[e^{-rT} g(s_T) \middle| \mathcal{F}_s\right] = e^{-rs} V_s,$$

- $d(e^{-rt} V_t)$ should have "0-drift". Think V as a function of time and the current asset price, i.e., $V_t = V(t, s_t)$ for a function V, then

$$\begin{aligned} d(e^{-rt} V_t) =& e^{-rt}\Big(- rV + \partial_t V + rs\partial_s V + \frac{1}{2}\sigma^2 s^2 \partial_{ss}^2 V\Big)(t, s_t) dt \\ &+ e^{-rt} \partial_s V \sigma s_t dz_t \end{aligned}$$

  "0-drift" $\Rightarrow$

$$\partial_t V + \frac{1}{2}\sigma^2 s^2 \partial_{ss}^2 V + rs\partial_s V - rV = 0.$$

Black-Scholes-Merton Equation

$$
\begin{aligned}
\partial_t V + \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} V + rs\partial_s V - rV =& 0, \quad s > 0, t \in [0, T) \\
V(T, s) =& g(s).
\end{aligned}
$$

The second equation is called terminal condition.

- European call: $g(s) = (s - K)_+$
- European put: $g(s) = (K - s)_+$

This is a partial differential equation. We will learn numeric solver for this type of equations in MF796.

BSM option pricing: Call option

European call option with maturity T and strike K.

Arbitrage-free price at time t with $s_t = s$ is

$$C(s, T-t) = s_t \Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{3}$$

$$d_1 = \frac{\ln[s_t/K] + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left[s_t e^{r(T-t)}/K\right]}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$= \frac{\ln\left[s_t e^{r(T-t)}/K\right]}{\sigma\sqrt{T-t}} - \frac{\sigma}{2}\sqrt{T-t}$$

(See class note for derivation)

BSM option pricing: Put option

From put-call parity, for a European put option with maturity T with strike K,

Arbitrage-free price at time t with $s_t = s$ is

$$P(s, T - t) = Ke^{-r(T-t)}\Phi(-d_2) - s_t\Phi(-d_1).$$

Self financing portfolio

Consider the asset price

$$\frac{ds_t}{s_t} = \alpha dt + \sigma dz_t.$$

Consider a self-financing portfolio with value P following

$$\begin{aligned} dP_t =& (P_t - \pi_t) r dt + \frac{\pi_t}{s_t} ds_t \\ =& (P_t - \pi_t) r dt + \pi \alpha dt + \pi_t \sigma dz_t. \end{aligned} \tag{4}$$

- π_t is how much money invested in the asset and $P_t - \pi_t$ is how much money in the bank account
- $(P_t - \pi_t) r dt$ is the interest collected over dt time
- $\frac{\pi_t}{s_t}$ is the number of shares invested at time t
- $\frac{\pi_t}{s_t} \approx \frac{\pi_t}{s_t}(s_{t+dt} - s_t)$ is the P& L from asset holding asset over dt time

Delta hedging

Let P_t be the value of the repetition portfolio for a derivative.

From the no arbitrage pricing theory,

$$P_t = V_t, \quad \text{for all } t \in [0, T],$$

where V_t is the arbitrage-free price of the derivative.

Therefore, we must have $dP_t = dV_t$. Comparing dz_t terms on both sides of the previous equation, we obtain

$$\frac{\pi_t}{s_t} = \partial_s V(t, s_t), \quad t \in [0, T].$$

$\#$ of shares in the repetition portfolio is always the partial derivative of the arbitrage-free price evaluated at the current time and the current asset price.

Market price of risk

Consider the asset price

$$\frac{ds_t}{s_t} = \alpha dt + \sigma dz_t.$$

Suppose that the price of a security is $V_t = V(t, s_t)$.

Itô's lemma:

$$dV_t = \left(\partial_t V + \alpha s \partial_s V + \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} V\right)(t, s_t)dt + \partial_s V(t, s_t)s_t \sigma dz_t.$$

Instantaneous return

$$\begin{aligned}
\frac{dV_t}{V_t} =& \mu_t dt + \nu_t dz_t, \\
\mu_t =& \frac{1}{V}\left(\partial_t V + \alpha s \partial_s V + \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} V\right)(t, s_t) \\
\nu(s,t) =& \frac{1}{V}\partial_s V(t, s_t)s_t\sigma
\end{aligned}$$

Market price of risk

Consider two securities with prices V_t^1 and V_t^2:

$$\frac{dV_t^1}{V_t^1} = \mu_t^1 dt + \nu_t^1 dz_t$$
$$\frac{dV_t^2}{V_t^2} = \mu_t^2 dt + \nu_t^2 dz_t$$

Riskless borrowing/leading at continuous compounding rate r

Consider a portfolio:

- 1 unit of V^1
- $-h$ units of V^2
- $-(V^1 - hV^2)$ borrowing

The portfolio value W follows

$$dW_t = dV_t^1 - hdV_t^2 - (V_t^1 - hV_t^2)rdt$$
$$= \Big(V^1(\mu^1 - r) - hV^2(\mu^2 - r)\Big)dt + (V^1\nu^1 - hV^2\nu^2)dz$$

Market price of risk

For the portfolio to be riskless, we must have

$$\begin{aligned} V^1(\mu^1 - r) =& hV^2(\mu^2 - r) \\ V^1\nu^1 =& hV^2\nu^2 \end{aligned}$$

The previous two equations imply

$$\frac{\mu^1 - r}{\nu^1} = \frac{\mu^2 - r}{\nu^2}$$

- $\frac{\mu - r}{\nu}$ is the Sharpe ratio: excess expected return over risk
- No arbitrage implies the same Sharpe ratio for every security
- $\lambda = \frac{\mu - r}{\nu}$ is the market price of risk

Dividend

Assume the underlying asset pays a proportional dividend continuously in time

Risk-neutral dynamics for s_t:

$$\frac{ds_t}{s_t} = (r-\delta)dt + \sigma dz_t.$$

Set $b = r - \delta$

The BSM equation is

$$\partial_t V + \frac{1}{2}\sigma^2 s^2 \partial_{xx}^2 V + bs\partial_s V - rV = 0.$$

European call option price at t

$$C(s, T-t) = s_t e^{(b-r)(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2).$$

European put option price at t:

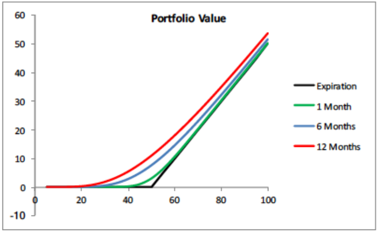
$$P(s, T-t) = Ke^{-r(T-t)}\Phi(-d_2) - s_t e^{(b-r)(T-t)}\Phi(-d_1).$$

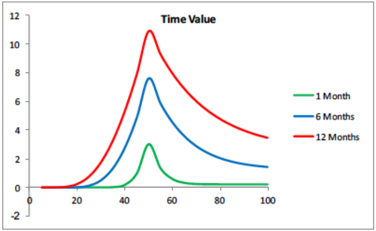
European call: time value

The Black-Scholes price of an European call with strike K, time-to-maturity $T - t$, and time t underlying price s is

$$c(s, T-t) = e^{-r(T-t)}\left[se^{b(T-t)}\Phi(d_1) - K\Phi(d_2)\right]$$

Time value: $c(s, T-t) - (s-K)_+$



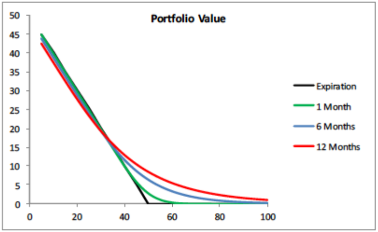


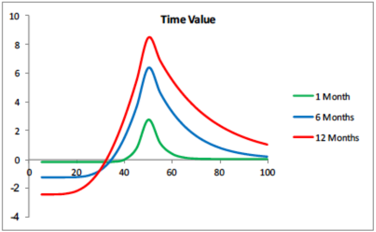
European put: time value

The Black-Scholes price of an European put with strike K, time-to-maturity $T-t$, and time t underlying price s is

$$p(s, T-t) = e^{-r(T-t)}\left[K\Phi(-d_2) - se^{b(T-t)}\Phi(-d_1)\right]$$

Time value: $p(s, T-t) - (K-s)_+$





American put option

Let $P(s,t)$ be the arbitrage-free price of an American put in the BSM model

t is the calendar time, rather than time to maturity $T-t$

Before exercise, $P(s,t)$ satisfies PDE

$$\partial_t P + \frac{1}{2}\sigma^2 s^2 \partial^2_{ss} P + bs\partial_s P - rP = 0.$$

with the terminal condition

$$P(s,T) = (K-s)_+$$

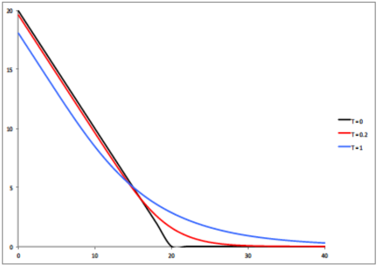
Snell envelop

$$P(s,t) = \max\left\{\tilde{P}(s,t), K-s\right\}$$
$$\tilde{P}(s,t) = \lim_{\Delta t \to 0} \tilde{\mathbb{E}}_t[P(s+\Delta s, t+\Delta t)]$$

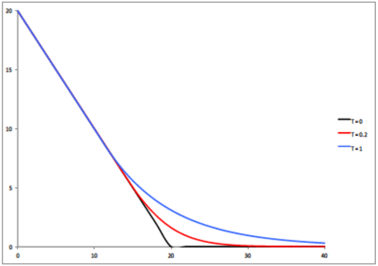
There is no closed-form solution for $P(s,t)$ for finite maturity T

American vs European Put

- $P(s,t) > p(s,t)$ when $t < T$
- American put option price is increasing in time to maturity



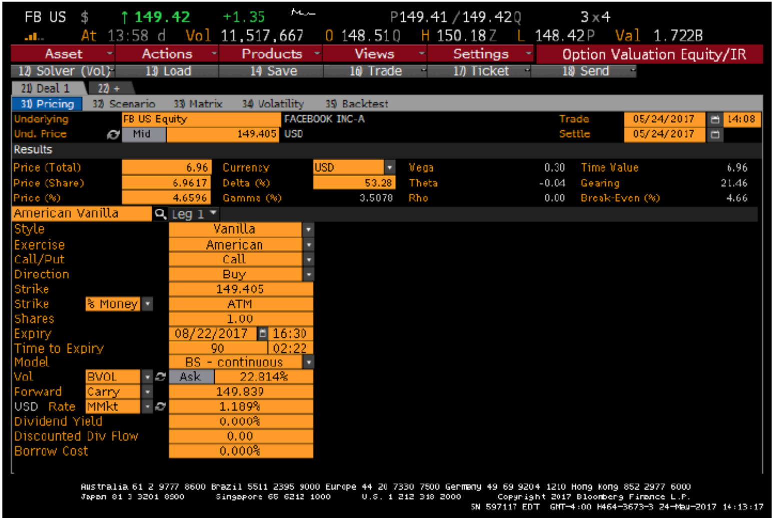
European Put



American Put

Bloomberg call pricer

FB US Equity OVME



BSM European call

BSM: current time t, maturity T, current underlying price K, strike K

$$c\left(s_t, K, T-t\right) = \left[s_t e^{b(T-t)}\,\Phi(d_1) - K\Phi(d_2)\right] e^{-r(T-t)}$$

Parameters: $b = r - \delta$

$$d_1 = \frac{\ln\left[s(t)e^{b(T-t)}/K\right]}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}$$

$$\begin{aligned} d_2 &= d_1 - \sigma\sqrt{T-t} \\ &= \frac{\ln\left[s(t)e^{b(T-t)}/K\right]}{\sigma\sqrt{T-t}} - \frac{\sigma}{2}\sqrt{T-t} \end{aligned}$$

European Call Greeks

- Delta: sensitivity of call option price with respect to underlying price, it is also the number of shares hold in the repetition portfolio at time t

$$\Delta_c \equiv \frac{\partial c(s, K, T-t)}{\partial s} = e^{(b-r)(T-t)}\Phi(d_1) > 0$$

$$0 < \Phi(d_1) < 1$$

  Long underlying in the repetition portfolio

- Gamma: sensitivity of Delta with respect to underlying price

$$\begin{aligned}\Gamma_c &= \frac{\partial^2 c}{\partial s^2} \\ &= \frac{\phi(d_1)e^{(b-r)(T-t)}}{s_t\sigma\sqrt{T-t}} > 0\end{aligned}$$

$$\phi(d_1) = \frac{\partial \Phi(d_1)}{\partial d_1}$$

  As underlying price increases, the repetition portfolio holds more shares

European Call Greeks

- Vega: sensitivity of call option price with respect to the underlying (instantaneous) volatility

$$
\begin{aligned}
Vega_c &= \frac{\partial c}{\partial \sigma} \\
&= s_t e^{(b-r)(T-t)} \phi(d_1) \sqrt{T-t} > 0
\end{aligned}
$$

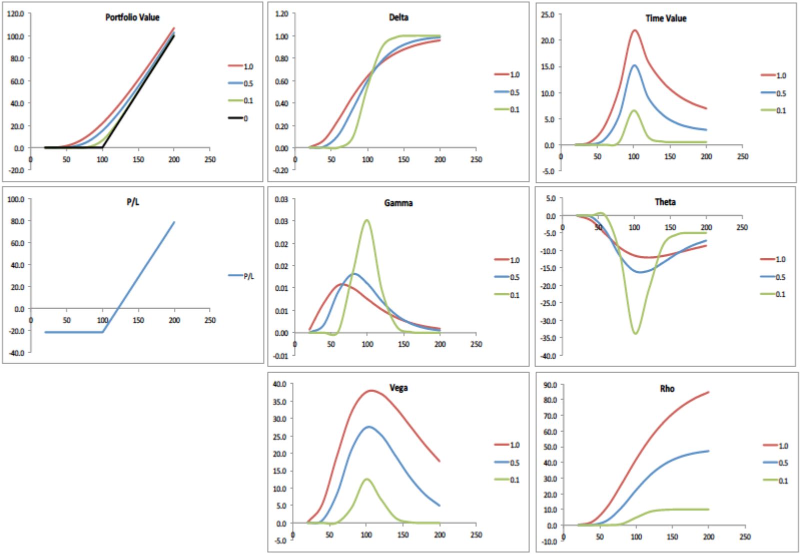
- Theta: sensitivity with respect to calender time

$$
\begin{aligned}
\Theta_c &= \frac{\partial c(s, K, T-t)}{\partial t} \\
&= -\frac{s_t e^{(b-r)(T-t)} \, \phi(d_1) \sigma}{2\sqrt{T-t}} \\
&\quad - (b-r) s_t \, e^{(b-r)(T-t)} \, \Phi(d_1) - rK \, e^{-r(T-t)} \, \Phi(d_2) \gtreqless 0
\end{aligned}
$$

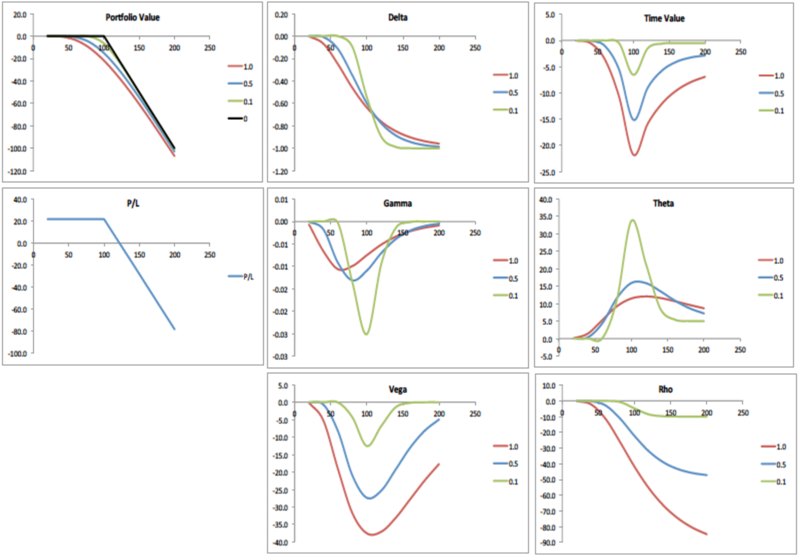
- Rho: sensitivity respect to the interest rate

$$
\rho_c = \frac{\partial c}{\partial r} = (T-t) K e^{-r(T-t)} \Phi(d_2) \geq 0
$$

Long Call Greeks



Short Call Greeks



BSM: European put

BSM:

$$p\left(s, K, T-t\right) = \left[K\Phi(-d_2) - s_t e^{b(T-t)}\,\Phi(-d_1)\right] e^{-r(T-t)}$$

Parameters: $b = r - \delta$

$$d_1 = \frac{\ln\left[s_t e^{b(T-t)}/K\right]}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}$$

$$\begin{aligned} d_2 &= d_1 - \sigma\sqrt{T-t} \\ &= \frac{\ln\left[s_t e^{b(T-t)}/K\right]}{\sigma\sqrt{T-t}} - \frac{\sigma}{2}\sqrt{T-t} \end{aligned}$$

European Put Greeks

- Delta

$$\Delta_p \equiv \frac{\partial p(s, K, T-t)}{\partial s} = -e^{(b-r)(T-t)}\Phi(-d_1) < 0$$

Short underlying in the repetition portfolio

- Gamma

$$\begin{aligned}\Gamma_p &= \frac{\partial^2 p}{\partial s^2} \\ &= \frac{\phi(d_1)e^{(b-r)(T-t)}}{s_t\sigma\sqrt{T-t}} > 0\end{aligned}$$

$$\phi(d_1) = \frac{\partial \Phi(d_1)}{\partial d_1}$$

Short less underlying in the repetition portfolio when the underlying price increases

European Put Greeks

- Vega

$$
\begin{aligned}
Vega_p &= \frac{\partial p}{\partial \sigma} \\
&= s_t e^{(b-r)(T-t)} \phi(d_1) \sqrt{T-t} > 0
\end{aligned}
$$

European put option price increases with respect to underlying (instantaneous) volatility

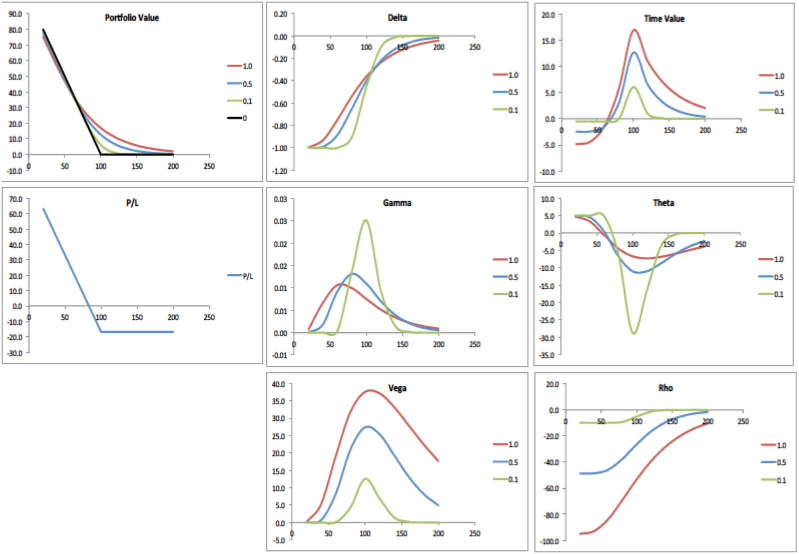
- Theta

$$
\begin{aligned}
\Theta_p &= \frac{\partial p(s, K, T-t)}{\partial t} \\
&= -\frac{s_t e^{(b-r)(T-t)}\, \phi(d_1)\sigma}{2\sqrt{T-t}} \\
&\quad + (b-r)s_t\, e^{(b-r)(T-t)}\, \Phi(-d_1) + rK\, e^{-r(T-t)}\, \Phi(-d_2) \gtreqless 0
\end{aligned}
$$

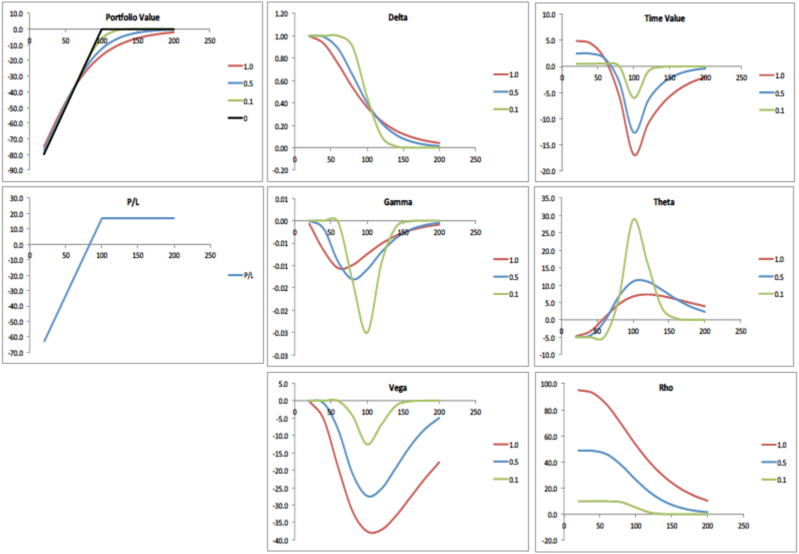
- Rho

$$
\rho_p = \frac{\partial p}{\partial r} = (T-t)Ke^{-r(T-t)}\Phi(-d_2) \leq 0
$$

Long Put Greeks



Short Put Greeks



Implies volatility

BSM model assumes that the instantaneous volatility is constant

But this may not be the case in reality.

For a European Call with maturity T and strike K

- Let $C_{BS}(S_0, \sigma, K, T)$ be the Black-Scholes-Merton price when the underlying stock price is S_0, volatility is σ, strike is K, and time-to-maturity is T
- Let $C_{market}(S_0, K, T)$ be the market observed price

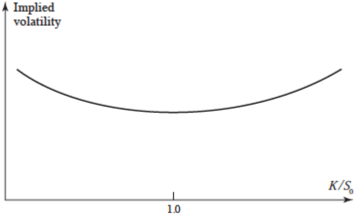
The Implied Volatility σ_{imp} is

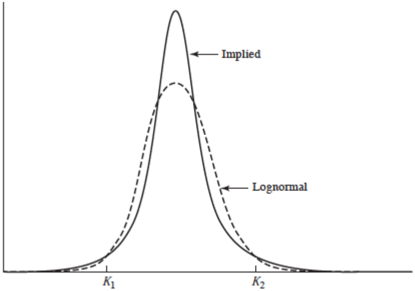
$$C_{BS}(S_0, \sigma_{imp}, K, T) = C_{market}(S_0, K, T).$$

- If the market stock return follows from GBM with vol σ, then $\sigma_{imp} = \sigma$
- But σ_{imp} is not a constant, specially after the market crash of 1987
- The implied volatility for the put option is the same as call option, why?

Volatility smile for foreign currency options

The left panel below is the implied volatility for foreign currency options in term of moneyness (i.e. K/S_0). The right panel is the implied distribution of the underlying price at T





Volatility smile for foreign currency options

- The implied distribution has heavier tail than log-normal both deep in-the-money and deep out-of-the-money

Volatility smile for foreign currency options

- The implied distribution has heavier tail than log-normal both deep in-the-money and deep out-of-the-money
- For a deep out-of-the-money call option with the strike K_2 (K_2/S_0 well above 1), there is larger probability to be in the money under the implied distribution than under the log-normal distribution
  $\implies$ higher market price for the call option than the BSM price
  $\implies$ higher implied volatility out of money.
  Similar for deep-in-the money call option with strike K_1

Volatility smile for foreign currency options

- The implied distribution has heavier tail than log-normal both deep in-the-money and deep out-of-the-money

- For a deep out-of-the-money call option with the strike K_2 (K_2/S_0 well above 1), there is larger probability to be in the money under the implied distribution than under the log-normal distribution
  $\Longrightarrow$ higher market price for the call option than the BSM price
  $\Longrightarrow$ higher implied volatility out of money.
  Similar for deep-in-the money call option with strike K_1

- One reason for implied volatility smile is stochastic volatility and and jumps of the underlying prices

Volatility smile for foreign currency options

- The implied distribution has heavier tail than log-normal both deep in-the-money and deep out-of-the-money
- For a deep out-of-the-money call option with the strike K_2 (K_2/S_0 well above 1), there is larger probability to be in the money under the implied distribution than under the log-normal distribution
  $\implies$ higher market price for the call option than the BSM price
  $\implies$ higher implied volatility out of money.
  Similar for deep-in-the money call option with strike K_1
- One reason for implied volatility smile is stochastic volatility and and jumps of the underlying prices
- When option has longer maturity, the volatility smile becomes flater
  $\Leftarrow$ distribution of long horizon prices is closer to the log-normal

Volatility smile for foreign currency options

- The implied distribution has heavier tail than log-normal both deep in-the-money and deep out-of-the-money

- For a deep out-of-the-money call option with the strike K_2 (K_2/S_0 well above 1), there is larger probability to be in the money under the implied distribution than under the log-normal distribution
  $\Longrightarrow$ higher market price for the call option than the BSM price
  $\Longrightarrow$ higher implied volatility out of money.
  Similar for deep-in-the money call option with strike K_1

- One reason for implied volatility smile is stochastic volatility and and jumps of the underlying prices

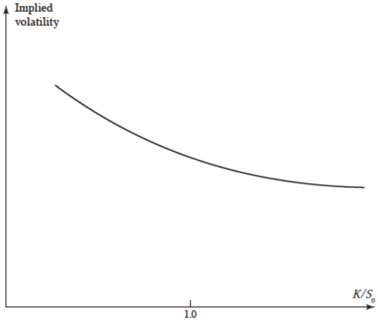
- When option has longer maturity, the volatility smile becomes flater
  $\Leftarrow$ distribution of long horizon prices is closer to the log-normal

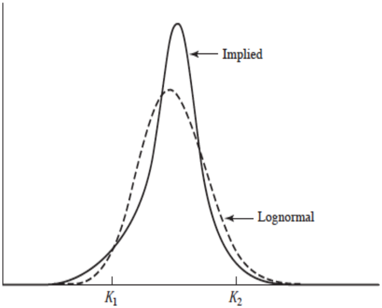
- In 1980, most people believe the stock price follows GBM, hence all options are priced using BSM. This means the deep-out/in-the-money options are less expensive that it should be. Few people made big money by buying deep-out/in-the-money options and wait. But this trading opportunity disappeared after 1987 crash

Volatility skew for equity options

Volatility skew: volatility decreases as the strike price increases

The volatility used to price low-strike-price option (a deep-out-of-the-money put or a deep-in-the-money call) is significantly higher than that used to price a high-strike-price option (i.e., a deep-in-the-money put or a deep-out-of-the money call)





Volatility skew for equity options

- For a deep-out-of-the-money call with strike K_2 (k_2/S_0 well above 1), the probability to be in-the-money is lower for the implied distribution than for the lognormal distribution
  $\Longrightarrow$ lower option price
  $\Longrightarrow$ lower implied vol

Volatility skew for equity options

- For a deep-out-of-the-money call with strike K_2 (k_2/S_0 well above 1), the probability to be in-the-money is lower for the implied distribution than for the lognormal distribution
  $\Longrightarrow$ lower option price
  $\Longrightarrow$ lower implied vol

- For a deep-out-of-the-money put option with a strike price of K_1. The probability of the stock price below k_1 is higher for the implied probability than for the lognormal distribution
  $\Longrightarrow$ higher option price
  $\Longrightarrow$ higher implied vol

Volatility skew for equity options

- For a deep-out-of-the-money call with strike K_2 (k_2/S_0 well above 1), the probability to be in-the-money is lower for the implied distribution than for the lognormal distribution
  $\Longrightarrow$ lower option price
  $\Longrightarrow$ lower implied vol

- For a deep-out-of-the-money put option with a strike price of K_1. The probability of the stock price below k_1 is higher for the implied probability than for the lognormal distribution
  $\Longrightarrow$ higher option price
  $\Longrightarrow$ higher implied vol

- Leverage effect: a negative correlation between equity prices and volatility. Stock price declines are accompanied by increases in volatility, making even greater declines possible. Stock price increases are accompanied by decreases in volatility, making further stock price increases less likely.

- Crshophobia: Volatility skew only appeared after the stock market crash of October 1987. One explanation is that traders are concerned about another crash coming and they demand deep-out-of-the-money put option for protection