Sensitivity Analysis and Hedging V
- Principal Component Analysis.

<u>Idea</u>

When we computed key-rate durations to build hedging strategies we evaluated quantities like

1) $Px = f(y_{t_1}, \dots, y_{t_n})$

2) $\dfrac{\partial f}{\partial y_{t_i}}$

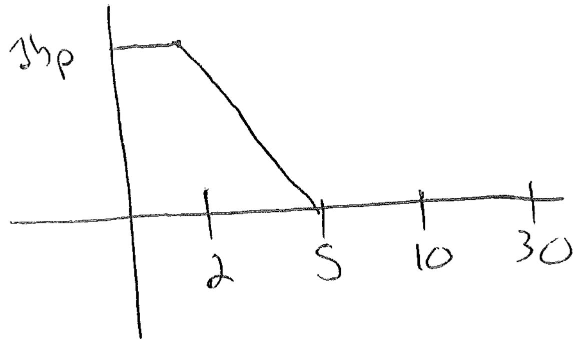
e.g. $y_{t_1}, \dots, y_{t_n} = 2, 5, 10, 30$ yr par rates

Px: price of an annuity

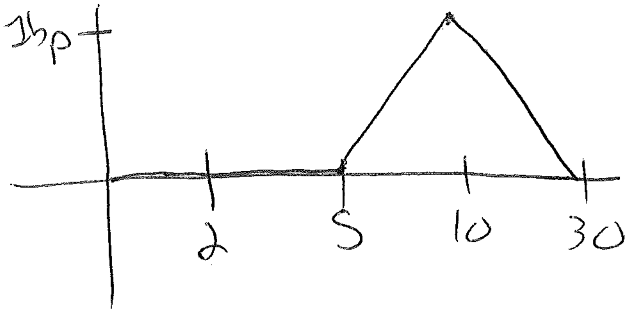
to get partials we had to shock key rates independently, but also do something with non-key rates.

Obtained yield curve shocks like

Y_2 :



Y_{10} :



etc.

This is good, but there are (at least) 2 drawbacks here

1) The shocks Y_{t_j} really are not economically meaningful
   - i.e. its not like this is a "typical" shock

2) In reality the key rates really are not independent so that we

③

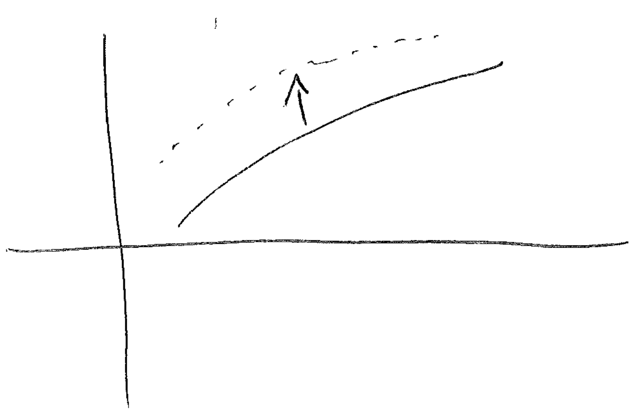
can't really "isolate" their effect by holding all other key rates constant.

Principal Component Analysis (PCA) tries to correct for this by identifying if there are yield curve movements which are typical and important.

i.e. if there is a way the <u>entire</u> yield curve moves which is important.

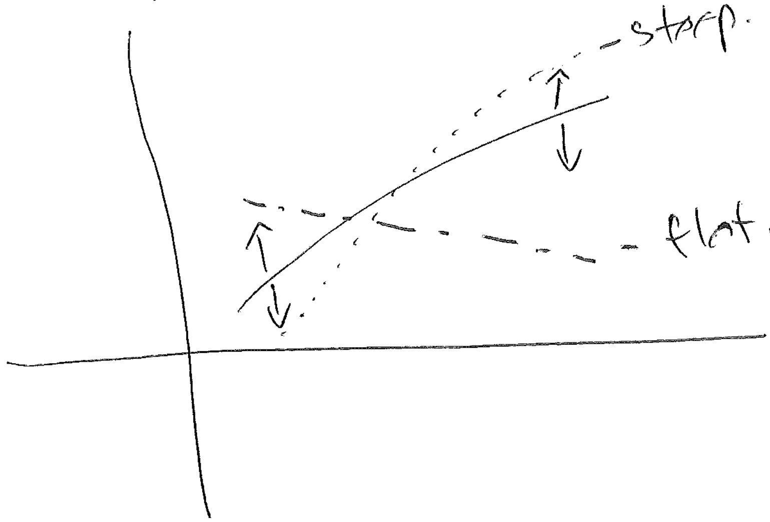
We have already seen a few yield-curve movements.

1) parallel

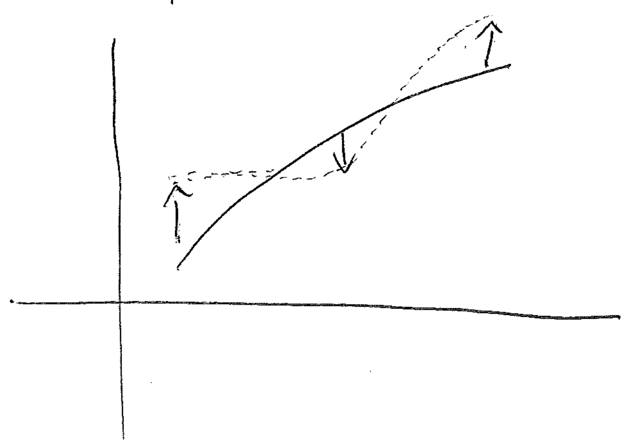


④

2) Steepening/ Flattening



3) Butterfly



PCA wants to know ~~to~~ which type(s) of movement is the most important.

Q. How does it do this?

Brief Review of Linear Algebra

Let V be an $N \times N$ symmetric, positive definite matrix

i.e. $V = V^T$ and $\forall\ x \in \mathbb{R}^N, x \neq 0$

we have $x^T V x > 0$

- equivalently, the (real) eigen-values of V are non-negative.

Next, let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$ be the eigen-values and let $e^{(n)}$ be the eigen-vector corresponding to λ_n for $n = 1, \dots, N$

We can choose $e^{(n)}$ so

1) $|e^{(n)}| = 1$ (unit length)

2) $e^{(n)\,T} e^{(m)} = 0 \quad n \neq m$

- the $e^{(n)}$ form an orthogonal basis for $\mathbb{R}^N$.

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Define C ($N \times N$ matrix) by

$$C = \left(\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(N)}\right)$$

$\uparrow$ n^{th} column is the vector $\beta^{(n)}$

Then

$$C^T C = \begin{pmatrix} \beta^{(1)} \\ \vdots \\ \beta^{(N)} \end{pmatrix} \left(\beta^{(1)}, \ldots, \beta^{(N)}\right)$$

$$= I_N \quad : \text{identity.}$$

$$\Rightarrow \quad C^T = C^{-1}$$

Next, set D ($N \times N$ matrix) by

$$D = C^T V C$$

Since

$$VC = V\left(\beta^{(1)}, \ldots, \beta^{(N)}\right)$$
$$= \left(\lambda_1 \beta^{(1)}, \ldots, \lambda_N \beta^{(N)}\right)$$

⑦

We have $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$ is diagonal.

Now, let $\Delta \in \mathbb{R}^N$ be a vector.
We can write

$$\Delta = \sum_{n=1}^{N} \alpha_n e^{(n)} \qquad \alpha_n = \Delta^T e^{(n)}$$

Now, in our case, we think of Δ as a yield curve shift. We can thus decompose Δ into the sum

$$\alpha_1 e^{(1)} + \alpha_2 e^{(2)} + \dots + \alpha_n e^{(n)}$$

of uncorrellated (i.e. orthogonal) shifts.

PCA then says that essentially only the first few shifts are important:

i.e. $\Delta \approx \alpha_1 e^{(1)} + \alpha_2 e^{(2)} + \alpha_3 e^{(3)}$

and in fact we call

$\alpha_1 e^{(1)}$: parallel shift/component.

$\alpha_2 \rho^{(2)}$: slope component

$\alpha_3 \rho^{(3)}$: curvature component.

Making this precise:

1) We first fix some reference maturities

T_n ; $n = 1, \ldots, N$

- similar to key rate durations
- typical set: $T = (.25, .5, 1, 2, 3, 5, 7, 10, 30)$
  - usually around 10

2) Now, if t is today, let

$y(t, T)$ = yield at t for a maturity of T

so $y(t, t + T_n)$ = yield at t for a maturity T_j years later.

think of $t \mapsto y(t, t + T_n)$ as the "rolling" yield of a newly issued T_j year bond.

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Typically, $y(t, t+T_n)$ is taken off the on-the-run par curve

Now, suppose we have observed yields at past dates $t_0, t_1, \ldots, t_M$ where

$$\Delta_t = t_m - t_{m-1} \qquad m = 1, \ldots, M$$

is approximately constant (and small)

– e.g. daily data.

We then compute

$$\Delta_{mn} = y(t_m, t_m + T_n) - y(t_{m-1}, t_{m-1} + T_n)$$

$$m = 1, \ldots, M$$

$$n = 1, \ldots, N$$

– M large, $N \approx 10$.

So if $\Delta^m \in \mathbb{R}^N$ is the vector

$$\Delta^m_n = \Delta_{mn}$$

we have a time series of data

$$\Delta^1, \Delta^2, \ldots, \Delta^M.$$

(10)

From this time series we compute the sample covariance matrix $V^M = V$ for Δ:

i.e. $\cancel{V_{n,k}}$

$$V = \frac{1}{M-1} \sum_{m=1}^{M} (\Delta^m - \bar{\Delta})(\Delta^m - \bar{\Delta})^T \text{ (matrix)}$$

$$\bar{\Delta} = \frac{1}{M} \sum_{m=1}^{M} \Delta^m \qquad \text{(vector)}$$

Since

$$V_{n,k} = \frac{1}{M-1} \sum_{m=1}^{M} (\Delta_{mn} - \bar{\Delta}_n)(\Delta_{mk} - \bar{\Delta}_k)$$

we see

$$V = V^T$$

and also one can show V is positive semi-definite.

$$\left(x^T A A^T x = |A^T x|^2 \right)$$

(12)

So, we denote by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$ the e'values and $e^{(n)}$ the e'vector for λ_n. We create C, D accordingly

We thus can write, for ~~n=1,...,N~~ $m=1,\dots,M$ that

$$\Delta^m = \sum_{n=1}^{N} \alpha_{nm} e^{(n)} \qquad \alpha_{nm} = \alpha_n^m$$
$$= e^{(n)T} \Delta^m$$

Typically, we retain only the first three terms in the sum to obtain

$$\Delta^m \approx \underbrace{\alpha_{1m} e^{(1)}}_{\text{parallel shift}} + \underbrace{\alpha_{2m} e^{(2)}}_{\text{slope}} + \underbrace{\alpha_{3m} e^{(3)}}_{\text{curvature}}$$

Another way to think about this:

assume that there are N factors $Z_1,\dots,Z_N$ and that our shifts

$\Delta^m = \{\Delta_{mn}\}$ can be thought as a function of these factors.

Also, we want the factors to be uncorrelated.

note $z_n \neq a^{(n)}$ b/c z_n is an actual number.

- we will show z_n is the weight of $a^{(n)}$ in our formula for Δ, so z_n tells us how we should scale the underlying (unit length) key shifts.

So, we set z via

$$\Delta^m = C z^m \qquad m = 1, \ldots, M.$$

then

$$\frac{1}{M-1} \sum_{m=1}^{M} (z^m - \bar{z})(z^m - \bar{z})^T$$

$$= \frac{1}{M-1} \sum_{m=1}^{M} (C^T(\Delta^m - \bar{\Delta}))(C^T(\Delta^m - \bar{\Delta}))^T$$

$$z^m = C^T \Delta^m.$$

(13)

$$= C^T V C \quad \text{(by construction of } V\text{)}$$

$$= D$$

$\Rightarrow$ factors Z uncorrellated. b/c empirical covariance matrix is diagonal.

Also, since $\Delta^m = \sum_{n=1}^{N} \alpha_{nm} e^{(n)}$ we have

$$Z^m = \sum_{n=1}^{N} \alpha_{nm} C^T e^{(n)} = \sum_{n=1}^{N} \alpha_{nm} \begin{pmatrix} e^{(1)} \\ \vdots \\ e^{(N)} \end{pmatrix} e^{(n)}$$

$$= \sum_{n=1}^{N} \alpha_{nm} \begin{pmatrix} 0 \\ \vdots \\ 1 \rightarrow n\text{th spot} \\ \vdots \\ 0 \end{pmatrix}$$

$$= \alpha^m$$

so $Z \doteq \alpha$ and hence

$$\Delta^m = \sum_{n=1}^{N} Z^m_n e^{(n)}$$

$$\approx \underset{\text{parallel}}{\underset{\uparrow}{Z^m_1 e^{(1)}}} + \underset{\text{slope}}{\underset{\uparrow}{Z^m_2 e^{(2)}}} + \underset{\text{curvature}}{\underset{\uparrow}{Z^m_3 e^{(3)}}}$$

so, the factors Z^m are the weights that we put on the component, normalized orthogonal shifts.

Z_1^m : weight of parallel shift

Z_2^m : weight of slope shift.

Z_3^m : weight of curvature shift.

so, how does this do?

- Pretty well
- first note: typically in practice the correlation matrix is used to mod-out general market volatility

$$\text{corr}_{ij} = \frac{\text{cov}_{ij}}{\sqrt{\text{cov}_{ii}\,\text{cov}_{jj}}} \qquad i,j = 1,\dots,N$$

- more stable over time.

- second note: the "parallel" shift, corresponding to the largest λ value typically explains 80% ~ 90% of observed fluctuations

e.g. 1993-1998 Treasury Bond Yields

1) maturity : 1yr 2yr 3yr 5yr 7yr 10yr 20yr 30yr
shift : .33 .35 .36 .36 .36 .36 .35 .35

- 91.7% of observed fluctuations.

for this data we also have

2) Slope : ~ 5.5 %

mat : 1 | 2 | 3 | 5 | 7 | 10 | 20 | 30
shift : -.59 | -.37 | -.23 | -.06 | .14 | .20 | .44 | .45

3) Curve : ~ 1.1 %

mat : 1 | 2 | 3 | 5 | 7 | 10 | 20 | 30.
Shift : .7 | -.3 | -.32 | -.3 | -.19 | -.12 | .26 | .32

- relative importance of these shifts stable over time
  - i.e. if we use data over different time periods we obtain approximately the same relative importance.

- shape of parallel / curve / slope shifts also relatively stable over time.

- see article by "Phoa" for more information:

- however we can certainly hedge against PCA shifts.

  e.g. find faces F_1, F_2, F_3 of par bonds at 2, 5, 10 yrs that DV01 match (e.g. normalized 1 bp shift) PCA shifts 1→3 against a given bond.

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we have ΔPx (Bond) ~~in~~ for shifts 1, 2, 3

we have ΔPx (2, 5, 10) for shifts 1, 2, 3

and unknown forces ~~1, 2,~~ f_1, f_2, f_3.

$\Rightarrow$ 3 equations, 3 unknowns, Linear

$\Rightarrow$ we should have a solution.