MF702 Midterm Exam Solution

October 28, 2020

Please write your answers on separate paper. Write down clearly the question number for each solution. Mark your submission with a page number in each page. Scan or take pictures of your solutions and update them to the assignment session on Questromtools.

1. (35 points)

(a) What are three major differences between forward and futures?

(b) What are the main assumptions for Black-Scholes-Merton pricing formula?

(c) Construct a long position of a forward with the forward price E with two options, each with strike price E.

(d) In early March 2020 when COVID-19 started to spread globally, volatility skew for equity indices became more steep (i.e., the slop of implied volatility with respect to moneyness because more negative). Give an explanation of this phenomenon.

(e) Consider a geometric Brownian motion

$$dX_t = \alpha X_t dt + \sigma X_t dZ_t,$$

where Z is a 1-dimensional Wiener process. Use Itô's formula to calculate the dynamics of X^p for some constant $p > 0$.

Solution:
(a) Futures need to be mark-to-market in its margin account, forward does not. Futures typically are traded in exchanges, forward contracts are typically OTC. Forwards are subject to more credit risk.
(b) Underlying stock dynamics is assumed to follow geometric Brownian motion. Interest rate is assumed to be constant. No friction in trading bonds and underlying stocks.
(c) A long position of a forward with the forward price E can be constructed by a long position of a European call and a short position of a European put, both with strike price E and maturity the same as the forward.
(d) Investors bought more out-of-money put options to protect themselves. This pushed up the price of put options with small moneyness K/S_0, hence increases

the implied volatility for small K/S_0 and the implied volatility skew becomes more steep.
(e)

$$\begin{aligned}
dX_t^p =& pX_t^{p-1}dX_t + \frac{1}{2}p(p-1)X^{p-2}d\langle X\rangle_t \\
=& p\alpha X^p dt + \frac{1}{2}p(p-1)\sigma^2 X^p dt + p\sigma X_t^p dZ_t \\
=& \Big(p\alpha + \frac{1}{2}p(p-1)\sigma^2\Big)X_t^p dt + p\sigma X_t^p dZ_t.
\end{aligned}$$

2. (25 points) Consider a 3-month *up-and-in* European call barrier option. Consider a two-period binomial model for this option. The annual log return for the up-side stock price is $u = 0.1$ and the annual log return for the down-side stock price is $d = -0.05$. This means that the stock price either becomes $S_i e^{u\times 1.5/12}$ or $S_i e^{d\times 1.5/12}$ after one period, for $i = 0$ or 1. The annual continuously compounding interest rate is $r = 0.02$.
At the end of the second period, the payoff of this barrier option is

$$V_2 = \begin{cases} (S_2 - K)_+ & \text{if } M_2 \geq B \\ 0 & \text{otherwise} \end{cases},$$

where K is the strike price, B is the *barrier* level, and $M_2 = \max_{0\leq i\leq 2} S_i$. In other words, if M_2 is larger than the barrier level B, the option is knocked in and the payoff at maturity is the standard European call; if M_2 does not reach B, the option is worthless.
Assume $B = 101$ and $K = 100$ for this problem.

(a) Draw a binomial tree model and write down stock prices on each node of the tree. Calculate the risk-neutral probabilities for this binomial tree model.

(b) Use the risk-neutral pricing method, find the risk-neutral price of this European call barrier option at time 0.

(c) If the seller of this option construct a replication strategy for this option, how many shares of stock the seller needs to buy/sell at time 0? Verify the replication strategy indeed replicates the barrier option payoff at the end of second period.

(d) If the barrier level B increases, will the option price for the European call barrier option increase or decrease? Please explain your reason.

Solution: (a) $S_1^u = S_0 e^{\frac{1.5}{12}\times u} = 101.2578$, $S_1^d = S_0 e^{\frac{1.5}{12}\times d} = 99.3769$, $S_1^{uu} = S_0 e^{\frac{3}{12}\times u} = 102.5315$, $S_1^{ud} = S_1^{du} = S_0 e^{\frac{1.5}{12}\times u + \frac{1.5}{12}d} = 100.6270$, $S_2^{dd} = S_0 e^{\frac{3}{12}\times d} = 98.7578$. The risk neutral probability for stock price to go up in each period is

$$p = \frac{e^{0.02\frac{1.5}{12}} - e^{-0.05\frac{1.5}{12}}}{e^{0.1\frac{1.5}{12}} - e^{-0.05\frac{1.5}{12}}} = 0.4643.$$

(b) Let V_i be the option value at the end of period i. Observe from the binomial tree that $M_2 \geq B$ if and only if $S_1 = S_1^u$. Then

$$V_2^{uu} = (S_2^{uu} - K)_+ = 2.5315,\ V_2^{ud} = (S_2^{ud} - K)_+ = 0.6270,\ V_2^{du} = V_2^{dd} = 0.$$

Therefore,

$$\begin{aligned} V_1^u =& e^{-0.02\frac{1.5}{12}}\left[pV_2^{uu} + (1-p)V_2^{ud}\right] = 1.5075, \\ V_1^d =& e^{-0.02\frac{1.5}{12}}\left[pV_2^{du} + (1-p)V_2^{dd}\right] = 0, \\ V_0 =& e^{-0.02\frac{1.5}{12}}\left[pV_1^{u} + (1-p)V_1^{d}\right] = 0.6982. \end{aligned}$$

(c) Let Δ_i be the number of shares of stocks in the replication portfolio at time i. Then the value of replication portfolio at the end of the first period is $\Delta_0 S_1 + e^{r\frac{1.5}{12}}(V_0 - \Delta S_0)$. To match option price at the end of the first period, we need

$$\begin{aligned} 101.2578\Delta_0 + e^{0.02\times\frac{1.5}{12}}(V_0 - 100\Delta_0) =& V_1^u, \\ 99.3769\Delta_0 + e^{0.02\times\frac{1.5}{12}}(V_0 - 100\Delta_0) =& V_1^d. \end{aligned}$$

Solving the previous equation, we obtain $\Delta_0 = 0.8015$.
Similarly, Δ_1^u needs to satisfy

$$\begin{aligned} S_2^{uu}\Delta_1^u + e^{0.02\times\frac{1.5}{12}}(V_1^u - S_1^u\Delta_1^u) =& V_2^{uu}, \\ S_2^{ud}\Delta_1^u + e^{0.02\times\frac{1.5}{12}}(V_1^u - S_1^u\Delta_1^u) =& V_2^{ud}. \end{aligned}$$

The previous system admits a unique solution Δ_1^u. From the previous equations, we can see that the replication portfolio replicates the option payoff at time 2 if $S_1 = S_1^u$. If $S_1 = S_1^d$, then the replication portfolio value at time 1 is zero. Therefore, its value at time 2 is also zero matching the option payoff at time 2 as well. Therefore, our replication strategy replicates the option payoff.
(d). The option price will decrease, because it is less likely to knock in the payoff, whose expected discounted value decreases under the risk-neutral measure.

3. (20 points) Let $T_c < T$ and $K > 0$ be given, a *chooser option* is a contract sold at time zero that confers on its owner the right to receive either a European call or a put time time T_c. The owner of the chooser option may wait until time T_c before choosing. The European call or put chosen expires at time T with strike price K.

(a) Let c_t and p_t be the European call and put prices at time t. Write down the value of the chooser option at time T_c.

(b) Let r be the annual continuously compounding interest rate. Show that the time-zero price of a chooser option is the sum of the time-zero price of a put, expiring at time T and having strike price K, and a call, expiring at time T_c and having strike price $Ke^{-r(T-T_c)}$. (Hint: use put-call parity.)

Solution: (a) The owner of the chooser option will choose the option with larger value at time T_c. Therefore the value of the chooser option at time T_c is

$$V_{T_c} = \max\{c_{T_c}, p_{T_c}\} = \max\{c_{T_c} - p_{T_c}, 0\} + p_{T_c}.$$

(b) By the put-call parity at time T_c,

$$c_{T_c} - p_{T_c} = S_{T_c} - Ke^{-r(T-T_c)}.$$

Therefore,

$$V_{T_c} = \max\{S_{T_c} - Ke^{-r(T-T_c)}, 0\} + p_{T_c},$$

where the first term on the right-hand side can be viewed as the payoff of a European call option with maturity at time T_c and strike price $Ke^{-r(T-T_c)}$, the second term is the price at time T_c of a European call with maturity T and strike price K.
Go back to time zero, the value of the chooser option should be the sum of a European call with payoff $\max\{S_{T_c} - Ke^{-r(T-T_c)}, 0\}$ and the price of a European put with maturity T and strike K.

4. (20 points) Consider a financial market in which investors can trade a risky stock with price S_0 at time 0 and borrow or lend at the risk-free rate $r > 0$ (one-period simple interest rate). Consider an one-period market model with three states: $\Omega = \{u, m, d\}$ with $S_1^u > S_1^m > S_1^d > 0$ and probabilities $p_u = \mathbb{P}[S_1 = S_1^u] > 0$, $p_m = \mathbb{P}[S_1 = S_1^m] > 0$, and $p_d = \mathbb{P}[S_1 = S_1^d] = 1 - p_u - p_m > 0$.

(a) Is the market complete?

(b) Show that the risk-neutral probability measure is not uniquely determined. What is the maximum possible range of the risk-neutral probability $\tilde{p}_u = \widetilde{\mathbb{P}}[S_1 = S_1(u)]$?

(c) Consider a European call option with maturity at time 1 and strike price $K = S_1^m$. Use your answers in part (b) to find the range of possible prices for this European call option at time zero.

(d) Suppose that the European call option considered in (c) is traded liquidly in the market with the price c, which is in the range of the risk-neutral prices you found in (c). Is the risk-neutral measure uniquely determined?

Solution: (a) A market is complete if any payoff of European options can be replicated. Consider a replication portfolio with cash position C_0 and stock position Δ_0 at time 0. The value of this replication portfolio is $C_0(1+r) + \Delta_0 S_1$ at time 1. Consider a European option whose payoff at time 1 is

$$c_1 = \begin{cases} c^u & S_1 = S_1^u \\ c^m & S_1 = S_1^m \\ c^d & S_1 = S_1^d \end{cases}$$

In order to replicate this European option, we need

$$\begin{aligned} C_0(1+r) + \Delta_0 S_1^u &= c^u, \\ C_0(1+r) + \Delta_0 S_1^m &= c^m, \\ C_0(1+r) + \Delta_0 S_1^d &= c^d. \end{aligned}$$

There are only two unknown C_0 and Δ_0 on the left-hand side. Therefore the system is over determined and may not always has a solution. Therefore, we may not always find a replication strategy for any European option, hence the market is not complete.

(b) Let $\tilde{p}^u = \widetilde{P}[S_1 = S_1^u]$ and $\tilde{p}^m = \widetilde{P}[S_1 = S_1^m]$ and $\tilde{p}^d = \widetilde{P}(S_1 = S_1^d)$. Then

$$S_0 = \frac{1}{1+r}\Big[\tilde{p}^u S_1^u + \tilde{p}^m S_1^m + \tilde{p}^d S_1^d\Big]. \tag{1}$$

The largest value of $\tilde{p}^u$ is obtained by solving

$$S_0 = \frac{1}{1+r}\Big[\tilde{p}^u S_1^u + (1-\tilde{p}^u) S_1^d\Big].$$

We obtain

$$\tilde{p}^u_{max} = \frac{(1+r)S_0 - S_1^d}{S_1^u - S_1^d}.$$

The smallest value of $\tilde{p}^d$ is obtained by solving

$$S_0 = \frac{1}{1+r}\Big[\tilde{p}^u S_1^u + (1-\tilde{p}^u) S_1^m\Big].$$

We obtain

$$\tilde{p}^u_{min} = \frac{(1+r)S_0 - S_1^m}{S_1^u - S_1^m}.$$

Therefore, $\tilde{p}^u \in [\tilde{p}^u_{min}, \tilde{p}^u_{max}]$.

(c) The payoff of the European option is nonzero only when $S_1 = S_1^u$. Therefore, the risk-neutral price is

$$c_0 = \frac{1}{1+r}\tilde{p}^u (S_1^u - S_1^m). \tag{2}$$

The minimal value of c_0 is when $\tilde{p}^u = \tilde{p}^u_{max}$ and $\tilde{p}^u = \tilde{p}^u_{min}$.

(d) When c_0 is observed from the market, we can then solve $\tilde{p}^u$ from (2). Then we can use (1) and $\tilde{p}^u + \tilde{p}^m + \tilde{p}^d = 1$ to solve $\tilde{p}^m$ and $\tilde{p}^d$. Then the risk neutral measure is uniquely determined.