1. Max $\Delta^T\vec{\gamma} - \frac{\gamma}{2}(\Delta^T\Sigma\Delta)$ s.t $\Delta^T\vec{1}=0$

$$L = \Delta^T\vec{\gamma} - \frac{\gamma}{2}(\Delta^T\Sigma\Delta) - \lambda(\Delta^T\vec{1})$$

FOC: $\vec{\gamma} - \frac{\gamma}{2}(2\Sigma\Delta) - \lambda\vec{1} = 0$ ①

$-(\Delta^T\vec{1}) = 0$ ②

From ①: $\vec{\gamma} - \gamma\Sigma\Delta - \lambda\vec{1} = 0$

$$\gamma\Sigma\Delta = \vec{\gamma} - \lambda\vec{1}$$

$$\Sigma\Delta = \frac{1}{\gamma}(\vec{\gamma} - \lambda\vec{1})$$

$$\Delta = \frac{1}{\gamma}\Sigma^{-1}\vec{\gamma} - \frac{\lambda}{\gamma}\Sigma^{-1}\vec{1}$$

Multibly both side by $\vec{1}^T$,

$$\vec{1}\Delta = \frac{1}{\gamma}\vec{1}^T\Sigma^{-1}\vec{\gamma} - \frac{\lambda}{\gamma}\vec{1}^T\Sigma^{-1}\vec{1} = 0 \text{ by ②.}$$

$$\frac{1}{\gamma}\vec{1}\Sigma^{-1}\vec{\gamma} = \frac{\lambda}{\gamma}\vec{1}^T\Sigma^{-1}\vec{1}$$

$$\vec{1}\Sigma^{-1}\vec{\gamma} = \lambda\vec{1}^T\Sigma^{-1}\vec{1}$$

$$\lambda = \frac{\vec{1}\Sigma^{-1}\vec{\gamma}}{\vec{1}^T\Sigma^{-1}\vec{1}} \quad \text{(proved)}$$

2. $1 = \frac{C(t)}{2} \sum_{j=1}^{2t} d\left(\frac{j}{2}\right) + d(t)$

$d\left(\frac{j}{2}\right) = \frac{1}{\left(1+\hat{r}\left(\frac{j}{2}\right)/2\right)^{j}} \qquad d(t) = \frac{1}{\left(1+\hat{r}(t)/2\right)^{2t}}$

Let $\lambda = \frac{1}{1+\hat{r}(t)/2} < 1 \quad d(t) = \lambda^{2t} \qquad \frac{\lambda}{1-\lambda} = \frac{2}{\hat{r}(t)}$

$\sum_{j=1}^{2t} \lambda^{j} = \frac{\lambda}{1-\lambda}\left(1-\lambda^{2t}\right)$

$\therefore 1 \geq \frac{C(t)}{2}\left[\frac{\lambda}{1-\lambda}\left(1-\lambda^{2t}\right)\right] + \lambda^{2t}$

$= \frac{C(t)}{2}\left[\frac{2}{\hat{r}(t)}\left(1-\lambda^{2t}\right) + \frac{2}{C(t)}\lambda^{2t}\right] = \frac{C(t)}{\hat{r}(t)}\left(1-\lambda^{2t}\right) + \lambda^{2t}$

In order to meet the condition,

$\frac{C(t)}{\hat{r}(t)}$ must be $\leq 1 \Rightarrow C(t) \leq \hat{r}(t)$

3. (a) ∵ spot curve is flat ∴ YTM = spot rate

So, we have $y = \frac{2A}{P_x}$ $\quad 4\% = \frac{2A}{1000000}$

Solve for $A = 20000$

(b) $D = -\frac{1}{P_x}\frac{\Delta P_x}{\Delta y} = -y \cdot \frac{d(\frac{1}{y})}{d(y)} = -y \times \left(\frac{1}{-y^2}\right) = \frac{1}{y}$

$C = \frac{1}{P_x} \times \frac{d^2 P_x}{dy^2} = \frac{1}{[(\frac{1}{y}) \times \frac{d^2(\frac{1}{y})}{dy^2}]} = y \times \left[\frac{d(-\frac{1}{y^2})}{dy}\right] = y \times \frac{2}{y^3} = \frac{2}{y^2}$

(c) ~~(i) $\Delta P = -D \times P \times \Delta y = -\frac{15}{1+3.9\%} \times 1000000 \times -0.1\% = 14436.958$~~

~~(ii) $\Delta P = -D \times P \times \Delta y = -\frac{20}{1+3.5\%} \times 1000000 \times -0.1\% =$~~

~~(iii) $\frac{2}{(3.9\%)^2} =$~~

$\frac{\Delta P}{P} = \cancel{-D \times \Delta y}$

(i) $\Delta P = -D \times P \times \Delta y = -D \times 1000000 \times 0.1\% = 14710$

(ii) $= \frac{\Delta P}{P} = -D \times \Delta y = -13.68 \times -0.1\% = 0.01368$

$D_{1,par} = \frac{1}{4\%}\left(1 - \frac{1}{(1+2\%)^{60}}\right) = 13.68$

(iii) $D_{per} = \frac{1}{y} = \frac{1}{4\%} = 25$

$\frac{\Delta P}{P} = -D \times \Delta y = -25 \times -0.001 = 0.025$

the (iii) performs better.

4(a) X_A (perpuity) = 1M $\qquad D_{perp} = \frac{1}{5\%} = 20$

$X_B(ZCB, 10, 30) = F^{10}p^{10} + F^{30}p^{30}$

$p^{10} = \frac{1}{(1+\frac{5\%}{2})^{20}} = 0.6102 \qquad p^{30} = \frac{1}{(1+\frac{5\%}{2})^{60}} = 0.2272$

$D_{10} = \frac{10}{1+\frac{y}{2}} = 9.756 \qquad D_{30} = \frac{30}{1+\frac{y}{2}} = 29.268$

$D_A = D_B = 20 = \frac{F^{10}p^{10}}{10M\,p^{P}} \times 9.756 + \frac{F^{30}p^{30}}{10M\,p^{P}} \times 29.268$

$F^{10} = 7782891 \qquad F^{30} = 23097678$

~~(b) r = 5% + 0.1% = 5.1%~~

$\cancel{\Delta X_A = X_A^{new} - X_A^{old} = 10m(p^{P,new} - p^{P,old})}$

$\cancel{\Delta X_A = X_A D_A \Delta y}$

$\cancel{\Delta X_B = X_B^{new} - X_B^{old} = F^{10}(p^{10,new} - p^{10,old}) + F^{30}(p^{30,new} - p^{30,old})}$

$\cancel{\Delta X_B = X_B D_B \Delta y = 99969125 \times 0.1\%}$

(b) $r = 5\% + 0.1\% = 5.1\% \qquad C_a = \frac{2}{y^2} = \frac{2}{0.05^2} = 800$

we can get that the barbell's value decreases less than the perpulty due to the convexity effect, for initial market value is in T_1 and 50% in T_2.

$C_a = \frac{T^{10}\times(T^{10}+\frac{1}{m})}{(1+y/m)^2} \times w + \frac{T^{30}\times(T^{30}+\frac{1}{m})}{(1+y/m)^2} \times w$

$C^{10} = \frac{10^2+10/2}{(1+\frac{5\%}{2})^2} = 99.9 \qquad C^{30} = \frac{30^2+30/2}{(1+\frac{5\%}{2})^2} = 870.599 \qquad 508 < 800$, so

$99.9 \times 0.47 + 870.599 \times 0.53 = 508$