Utility and preferences

Goals:

- Introduce preference relations and utility functions.
- Review some common utility functions.
- Define the notion of risk aversion.

Relevant literature:

- Cvitanić & Zapatero Ch. 4, Luenberger Ch. 9

Introduction

- We have learned a lot about pricing and hedging.
  - We know that no-arbitrage is equivalent to the existence of a risk-neutral measure.
  - We know that market completeness is equivalent to the existence of a unique risk-neutral measure.
  - Under no arbitrage, we can price derivatives with the risk-neutral pricing formula.
  - Under market completeness, we can construct replicating portfolios.
- However, we have never asked <u>why</u> we need all of this.
  - Why would an investor want to buy a derivative?

Introduction

- As we have informally mentioned, an investor will participate in the markets according to her preferences.
  - If she is a speculator seeking high returns, she may want to invest aggressively in risky securities.
  - If she is a hedger with low tolerance for risks, she may want to invest in safer securities.
- In this lecture, we will formalize the concept of investor preferences, and their relation to wealth.

Wealth

- Consider a one period model with $M+1$ assets.
  - M risky assets: stocks, options, interest rate derivatives,etc.
  - A ZCB maturing at the end of period 1.
- The price process of the m^{th} risky security is $S^{(m)}$.
- Suppose we start out with initial wealth W_0.
- Using W_0, we buy $\Delta^{(m)}$ units of the m^{th} security, and ρ units of the bond:

$$W_0 = \rho B_{0,1} + \sum_{m=1}^{M} \Delta^{(m)} S_0^{(m)};$$

$$\Longrightarrow \rho = \frac{1}{B_{0,1}} \left(W_0 - \sum_{m=1}^{M} \Delta^{(m)} S_0^{(m)} \right).$$

Wealth

- At 1, our wealth in scenario ω_1 is

$$W_1(\omega_1) = \rho + \sum_{m=1}^{M} \Delta^{(m)} S_1^{(m)}(\omega_1);$$
$$= \frac{W_0}{B_{0,1}} + \sum_{m=1}^{M} \Delta^{(m)} \left(S_1^{(m)}(\omega_1) - \frac{S_0^{(m)}}{B_{0,1}} \right).$$

- We then use this wealth to finance our consumption at 1.
  - E.g.: we buy a car!
- Our goal is to choose the portfolio $\Delta = (\Delta^{(1)}, \ldots, \Delta^{(M)})$ which yields the best possibility for consumption, taking into account both randomness (i.e. scenario dependence) and our preferences, or attitudes towards wealth.

Preference relation

- The final wealth W_1 is random: it depends on the realization of the $S^{(m)}$.
- It is doubtful a portfolio exists which outperforms every other portfolio in all states of nature.
  - Typically, one portfolio outperforms in one state of nature, while another outperforms in another state of nature.
- Since we cannot chose an "ω-by-ω" best portfolio, we must use a **preference relation** to rank portfolios.

Preference relation

- Let us use the symbol "$\succeq$" to rank portfolios.
  - I.e. $\Delta_1 \succeq \Delta_2$ means we like portfolio Δ_1 more than Δ_2.
- What reasonable/good properties should $\succeq$ have?
  1. **Completeness:** Either $\Delta_1 \succeq \Delta_2$ or $\Delta_2 \succeq \Delta_1$.
  2. **Transitivity:** If $\Delta_1 \succeq \Delta_2$ and $\Delta_2 \succeq \Delta_3$, then also $\Delta_1 \succeq \Delta_3$.
  3. **Monotonicity:** If $W_1^{\Delta_1}(\omega) \geq W_1^{\Delta_2}(\omega)$ for all ω, then $\Delta_1 \succeq \Delta_2$.
  4. **Convexity (diversification):** Let $\Delta_3 = \alpha\Delta_1 + (1-\alpha)\Delta_2$ for $\alpha \in (0,1)$. If $\Delta_1 \succeq \Delta$ and $\Delta_2 \succeq \Delta$, then also $\Delta_3 \succeq \Delta$.
- An ordering $\succeq$ satisfying $1-4$ above is a **preference relation**. We assume every investor has a preference relation reflecting attitudes towards reward (wealth) and risk.

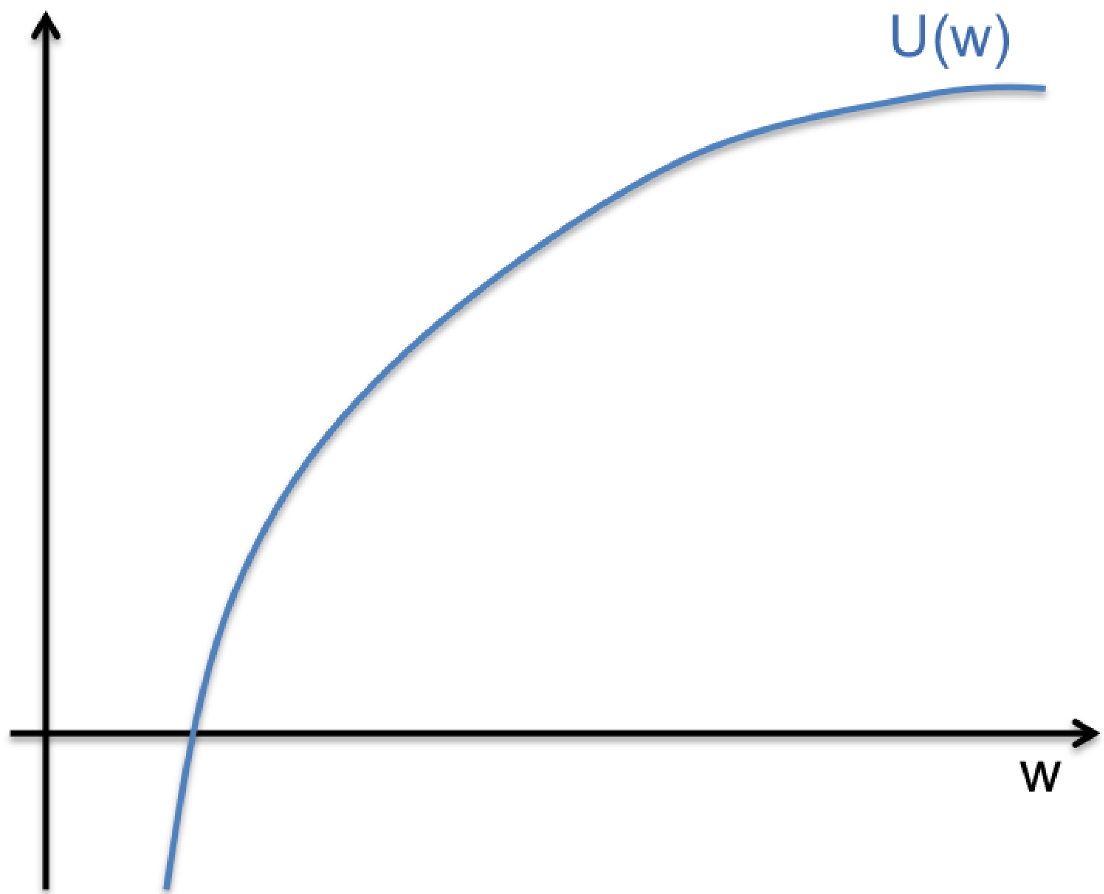
Preference relations and utility functions

- Preference relations are very abstract.
- We would like a concrete way to “score” portfolios.
  - High score = good portfolio.
  - Low score = bad portfolio.
- A **utility function** U enables such a scoring.
  - $U : \mathbb{R} \to \mathbb{R}$ is a function.
  - If our wealth is w, then $U(w)$ marks the “utility” (i.e. enjoyment) we get from w.
  - We can then say a portfolio is good if, on average, we get high utility from its terminal wealth.

What properties should U have?

- **More wealth is better:** $U'(w) > 0$.
  - We call U' the **marginal** utility.
  - We want marginal utility to be positive.
- **Diminishing value of $\$1$:** $U''(w) < 0$.
  - If we are poor, one more dollar is very good for us.
  - If we are rich, one more dollar is ok, but not as good.
  - Thus, our marginal utility decreases with wealth, or $U'' < 0$.
- Utility functions are concave and increasing.

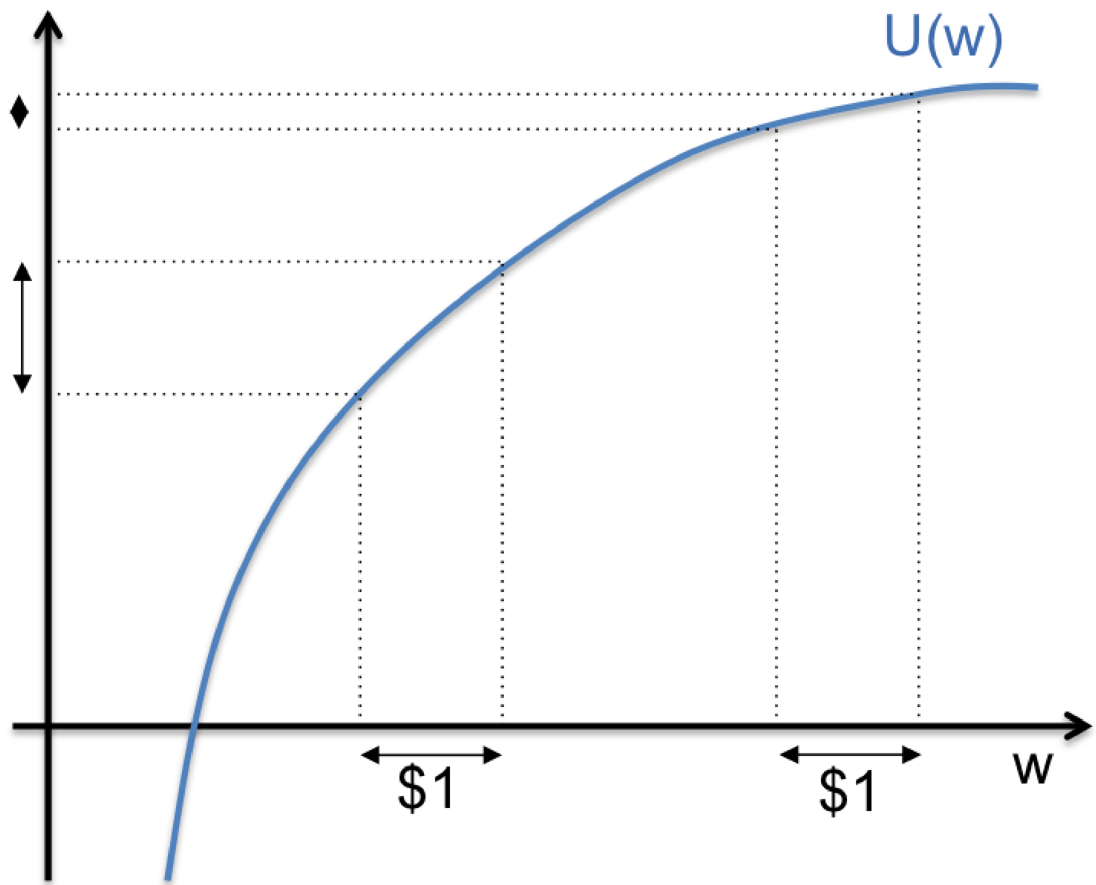
The graph of a utility function



Law of diminishing returns

- U being concave and increasing has important implications for how the investor values wealth.
- As we saw before, $U' > 0$ tells us the investor prefers more wealth to less wealth.
- However, $U'' < 0$ tells us the additional utility derived from an additional dollar decreases as investor wealth increases.
  - If we get $\$100$, we are very happy.
  - If Warren Buffet gets $\$100$, he probably wouldn't care.
- This is the **law of diminishing return**: More wealth is good, but not as good the wealthier you become.

Law of diminishing returns



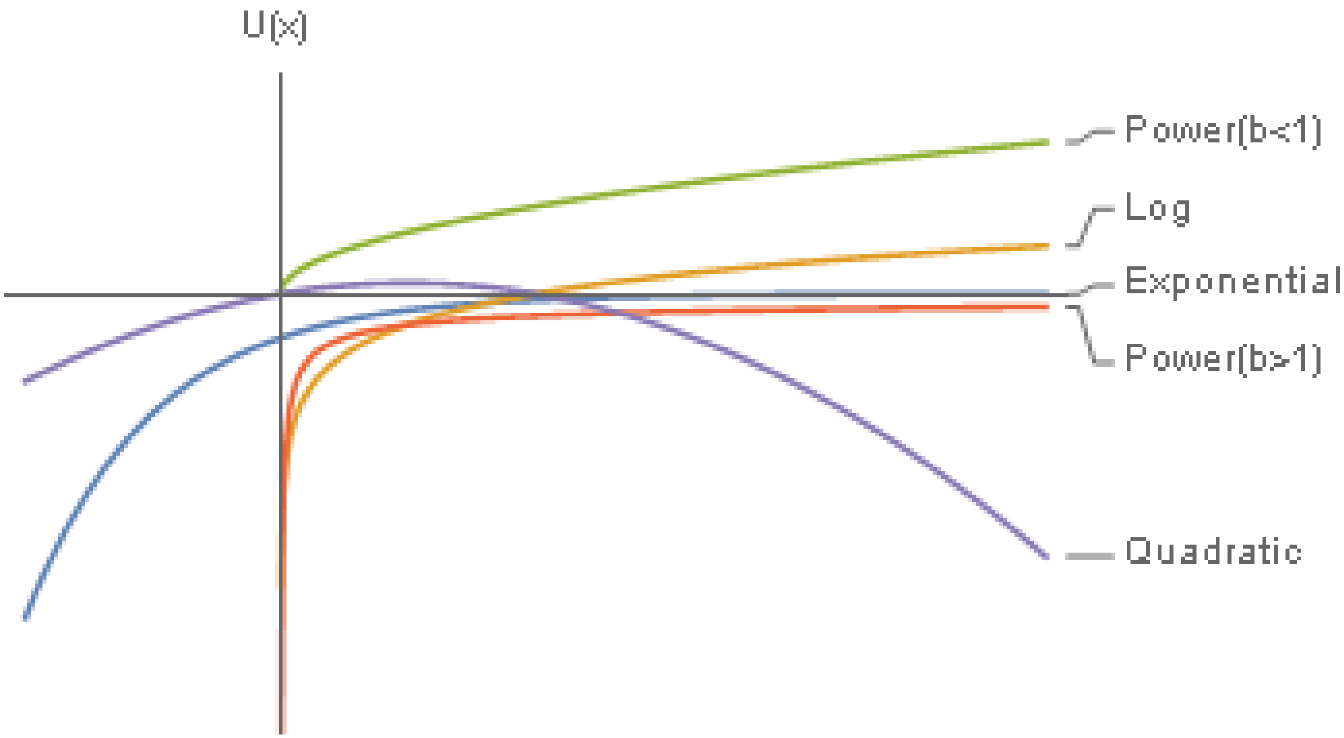
Law of diminishing returns & risk preferences

- The law of diminishing returns illuminates investor risk preferences.
- An investor with low wealth sees a much larger drop in utility losing $\$1$, then rise in utility by gaining $\$1$.
  - She prefers safe securities which have small risk of losing value.
- An investor with high wealth is more or less utility-indifferent between gaining and losing $\$1$.
  - She will be more accepting of risk and tend to invest in securities with both high reward and high risk.

Common utility functions

- Utility functions commonly used in the industry:
  - **Exponential:** $U(w) = -e^{-aw}$ for $a > 0$ and $w \in \mathbb{R}$.
  - **Logarithmic:** $U(w) = \ln(w)$ for $w > 0$.
  - **Power:** $U(w) = \frac{w^{1-b}}{1-b}$ for $b > 0$, $b \neq 1$, and $w > 0$.
  - **Quadratic:** $U(w) = w - cw^2$ for $c > 0$.
    - This “utility function” is only increasing for $w < \frac{1}{2c}$.
    - It is not a true utility function, but people use it none the less.
- As we will see, each of these makes different assumptions on the risk preferences of the investor.

Common utility functions



Utility functions and preference relations

- Given a utility function U we can create a preference relation $\succeq$.
- Recall the wealth at 1:
  - $W_1 = \frac{W_0}{B_{0,1}} + \sum_{m=1}^{M} \Delta^{(m)} \left(S_1^{(m)} - \frac{S_0^{(m)}}{B_{0,1}} \right)$.
- Inputs: initial wealth W_0, portfolio $\Delta = (\Delta^{(1)}, \ldots, \Delta^{(M)})$.
  - Write W_1^{Δ} to stress dependence.
  - Important note: W_1^{Δ} is linear in Δ:

$$W_1^{\alpha\Delta_1 + \beta\Delta_2} = \alpha W_1^{\Delta_1} + \beta W_1^{\Delta_2}.$$

- We then define a preference relation $\succeq$ via
  - $\Delta_1 \succeq \Delta_2 \Longleftrightarrow \mathbb{E}\left[U(W_1^{\Delta_1})\right] \geq \mathbb{E}\left[U(W_1^{\Delta_2})\right]$.
  - $\succeq$: “Expected utility from terminal wealth”.

$$\Delta_1 \succeq \Delta_2 \Longleftrightarrow \mathbb{E}\left[U(W_1^{\Delta_1})\right] \geq \mathbb{E}\left[U(W_1^{\Delta_2})\right]$$

Is $\succeq$ really a preference relation? YES.

- Completeness, Transitivity: ✓.
- Monotonicity: ✓ since $U' > 0$.
- Convexity: ✓ since $U'' < 0$. Indeed:

$$\begin{aligned}
&\mathbb{E}\left[U\left(W_1^{\alpha\Delta_1+(1-\alpha)\Delta_2}\right)\right] \\
&\quad = \mathbb{E}\left[U\left(\alpha W_1^{\Delta_1} + (1-\alpha)W_1^{\Delta_2}\right)\right] \text{ (linearity in } \Delta\text{);} \\
&\quad \geq \alpha\mathbb{E}\left[U(W_1^{\Delta_1})\right] + (1-\alpha)\mathbb{E}\left[U(W_1^{\Delta_2})\right] \text{ (concavity of } U\text{).}
\end{aligned}$$

Portfolio choice problem

- U allows us to "optimally invest" by choosing the portfolio which maximizes the expected utility from terminal wealth (i.e. $\succeq$).
- Indeed, we want Δ^* that solves

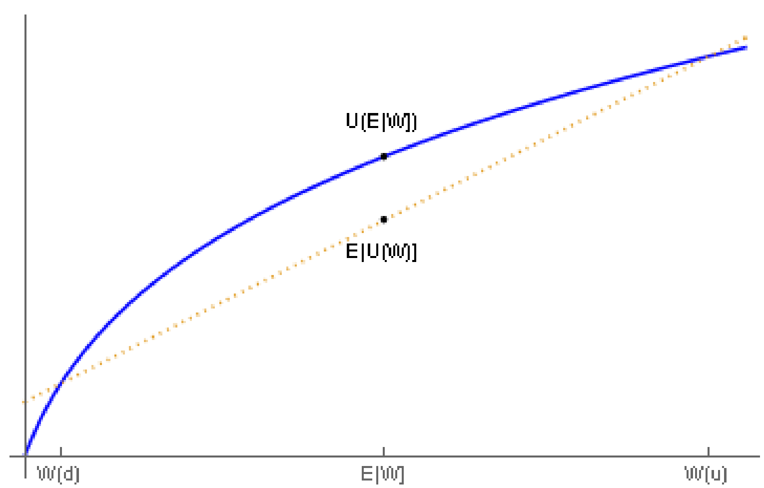
$$\max_{\Delta} \mathbb{E}\left[U(W_1^{\Delta})\right]; \quad W_1^{\Delta} = \frac{W_0}{B_{0,1}} + \sum_{m=1}^{M} \Delta^{(m)} \left(S_1^{(m)} - \frac{S_0^{(m)}}{B_{0,1}}\right).$$

- In the next class we will characterize the optimal portfolio Δ^*
- Important note: it is clear from the formula that the utility functions $U(x)$ and $aU(x) + b$ ($a > 0, b \in \mathbb{R}$) both yield the same portfolios.
  - Ordinal preferences, not cardinal preferences.

Risk averse investors

A concave utility function implies the investor is risk averse.

- Given a random payoff W, the investor obtains expected utility $\mathbb{E}\left[U(W)\right]$.
- Given the non-random average payoff $\mathbb{E}\left[W\right]$, the investor obtains expected utility $U\left(\mathbb{E}\left[W\right]\right)$.
- By concavity (Jensen's inequality) $U\left(\mathbb{E}\left[W\right]\right) \geq \mathbb{E}\left[U(W)\right]$ so the investor prefers the non-random average payoff.



Absolute risk aversion coefficient

- We can measure how risk averse the investor is.
- Qualitatively: more concave U (i.e. more negative U'') implies more risk averse.
- However, U'' by itself is not a good measure of risk aversion as it depends on the scale of the utility (recall: U and $aU, a > 0$ yield the same optimal portfolios).
- The **absolute risk aversion coefficient** is:

$$A(w) = -\frac{U''(w)}{U'(w)}.$$

- The higher $A(w)$, the more risk averse.

Absolute risk aversion coefficient

Each utility function makes different assumptions about investor risk aversion:

- **Exponential utility:** $A(w) = -\frac{-a^2 e^{wa}}{ae^{wa}} = a$. The absolute risk aversion coefficient is constant. Investors with different wealths are equally risk averse.
- **Logarithmic utility:** $A(w) = -\frac{-1/w^2}{1/w} = \frac{1}{w}$. Investors with high wealth are less risk averse than investors with low wealth.
- **Power utility:** $A(w) = -\frac{-bw^{-b-1}}{w^{-b}} = b\frac{1}{w}$. Again, investors with high wealth are less risk averse than investors with low wealth.
- **Quadratic utility:** $A(w) = \frac{1}{\frac{1}{2c} - w}$. Investors with wealth close to $\frac{1}{2c}$ are infinitely risk averse.

Relative risk aversion coefficient

- The relative risk aversion coefficient indicates the level of risk aversion for investments that are proportional to the wealth of the investor.
- The **relative risk aversion coefficient** is defined as:

$$R(w) = wA(w),$$

where $A(w)$ is the absolute risk aversion coefficient.

- An investor with higher R will be more risk averse towards investments that take on a large fraction of her wealth.

Relative risk aversion coefficient

These are the relative risk aversion coefficients for different utilities:

- **Exponential utility:** $R(w) = wA(w) = aw$. Rich investors are more risk averse towards large bets than poor investors are towards small bets.
- **Logarithmic utility:** $R(w) = wA(w) = w\frac{1}{w} = 1$. All investors are equally risk averse on a relative scale.
- **Power utility:** $R(w) = wA(w) = wb\frac{1}{w} = b$. Again, all investors are equally risk averse on a relative scale.
- **Quadratic utility:** $R(w) = wA(w) = \frac{w}{\frac{1}{2c} - w}$. Investors with wealth close to $\frac{1}{2c}$ are infinitely risk averse.

Utility and asset pricing

- By specifying the absolute and relative risk aversion coefficients, we can capture the risk preferences of an investor.
- Using these preferences, we can then proceed to both optimally invest and price assets.
- Regarding asset pricing, we will see:
  - Highly risk averse investors will not demand risky assets, and hence will assign low prices to those assets.
  - Risk friendly investors will demand for risky assets, and hence will assign high prices to those assets.
- In the next class, we will see how to derive asset prices from a given utility function, through the lens of optimal investment.