Capital asset pricing model

Goals:

- Introduce the capital asset pricing model.
- Define the beta of both a security and a portfolio.
- Introduce the concepts of systematic and idiosyncratic risks.
- Discuss investment implications.

Relevant literature:

- Cvitanić & Zapatero Ch. 13, Luenberger Ch. 7

Introduction

- Last time we analyzed how a rational investor who dislikes volatility chooses an optimal portfolio. This is the Markowitz model.
- If there is a risk-free asset, the investor will invest in the risk-free security and the Maximum Sharpe Ratio (MSR), or market, portfolio.
- Only one fund of risky securities is needed to satisfy the demand of all investors.

Introduction

- The Markowitz model identifies demand for financial assets.
- The capital asset pricing model (CAPM), on the other hand, tells us the prices for these assets given the demand.
- CAPM also provides a more intuitive understanding of the market portfolio, and how the market prices securities according to their risk profiles.

CAPM

- Like Markowitz, CAPM assumes investors are rational and volatility averse.
- Thus, every investor allocates a fraction α of wealth to the MSR portfolio and a fraction $1 - \alpha$ to the risk free security.
- The investor chooses $\alpha = \alpha(\bar{r})$ to achieve a target return $\bar{r}$, corresponding to her preferences.
- The goal of CAPM is to explain how asset prices and returns are related to the portfolios held by investors.

Market portfolio

- Suppose there are
  - J investors.
  - M risky securities with total outstanding shares $K_1, ...K_M$.
- Investor j has initial wealth W_0^j, and allocates a fraction Δ_m^j to the m^{th} risky security, and Δ_f^j to the risk-free security, such that $\Delta_f^j + \sum_{m=1}^{M} \Delta_m^j = 1$.
- By definition, the price of the m^{th} risky security is

$$S_0^{(m)} = \frac{\$ \text{ in } S^{(m)}}{\text{Shares of } S^{(m)}} = \frac{\sum_{j=1}^{J} \Delta_m^j W_0^j}{K_m}.$$

- This enforces that supply $(S_0^{(m)} K_m)$ is equal to demand $(\sum_{j=1}^{J} \Delta_m^j W_0^j)$.

Market portfolio

- The one-fund theorem implies $\Delta_m^j = \alpha^j \Delta_m^*$ where we have set $\Delta^* \triangleq \Delta_{MSR,e}$ as Maximum Sharpe Ratio portfolio using excess return.
- Since $\Delta_f^j = 1 - \alpha^j$ we see that $\Delta_m^j = (1 - \Delta_f^j)\Delta_m^*$. As a result

$$S_0^{(m)} K_m = \Delta_m^* \sum_{j=1}^{J} (1-\Delta_f^j) W_0^j \quad \Leftrightarrow \quad \Delta_m^* = \frac{S_0^{(m)} K_m}{\sum_{j=1}^{J}(1-\Delta_f^j) W_0^j}.$$

- Furthermore, since $\sum_{m=1}^{M} \Delta_m^* = 1$:

$$\sum_{m=1}^{M} S_0^{(m)} K_m = \sum_{m=1}^{M} \sum_{j=1}^{J} (1-\Delta_f^j)\Delta_m^* W_0^j = \sum_{j=1}^{J} (1-\Delta_f^j) W_0^j.$$

Market portfolio

- Therefore

$$\Delta_m^* = \frac{S_0^{(m)} K_m}{\sum_{j=1}^{J} (1 - \Delta_f^j) W_0^j} = \frac{S_0^{(m)} K_m}{\sum_{l=1}^{M} S_0^{(l)} K_l}.$$

- The MSR portfolio is exactly the portfolio of relative market capitalizations!
  - This is why we call the MSR portfolio the market portfolio.
- You can think of the market portfolio as a market-cap weighted index, such as the S&P 500.

How does the market figure out its portfolio?

- On the one hand: Δ^* gives the best excess return per unit volatility.
- On the other hand: Δ^* is the portfolio of relative capitalizations.
- How does this work? How do we connect these two facts?
- This occurs when demand and supply interact; the market portfolio is an equilibrium.
- Heuristically:
  - If there is more demand, the price will rise and its expected return will fall. This will in turn decrease future demand.
  - If there is more supply, the price will fall and the expected return will rise. This will in turn increase future demand.

Is this equilibrium realistic?

- Assumptions underlying CAPM:
  - Investors are volatility adverse.
  - Everybody agrees on the expected returns and volatilities.
  - Supply and demand are matched immediately.
- These are not very realistic assumptions.
- However, the result is intuitive and realistic:
  - Large investors, such as banks and hedge funds, enter the market with large demand or supply.
  - They move prices to levels they believe are acceptable. That is, they construct the market portfolio.
  - Private investors invest in mutual funds or hedge funds that hold some proportion of the market portfolio.

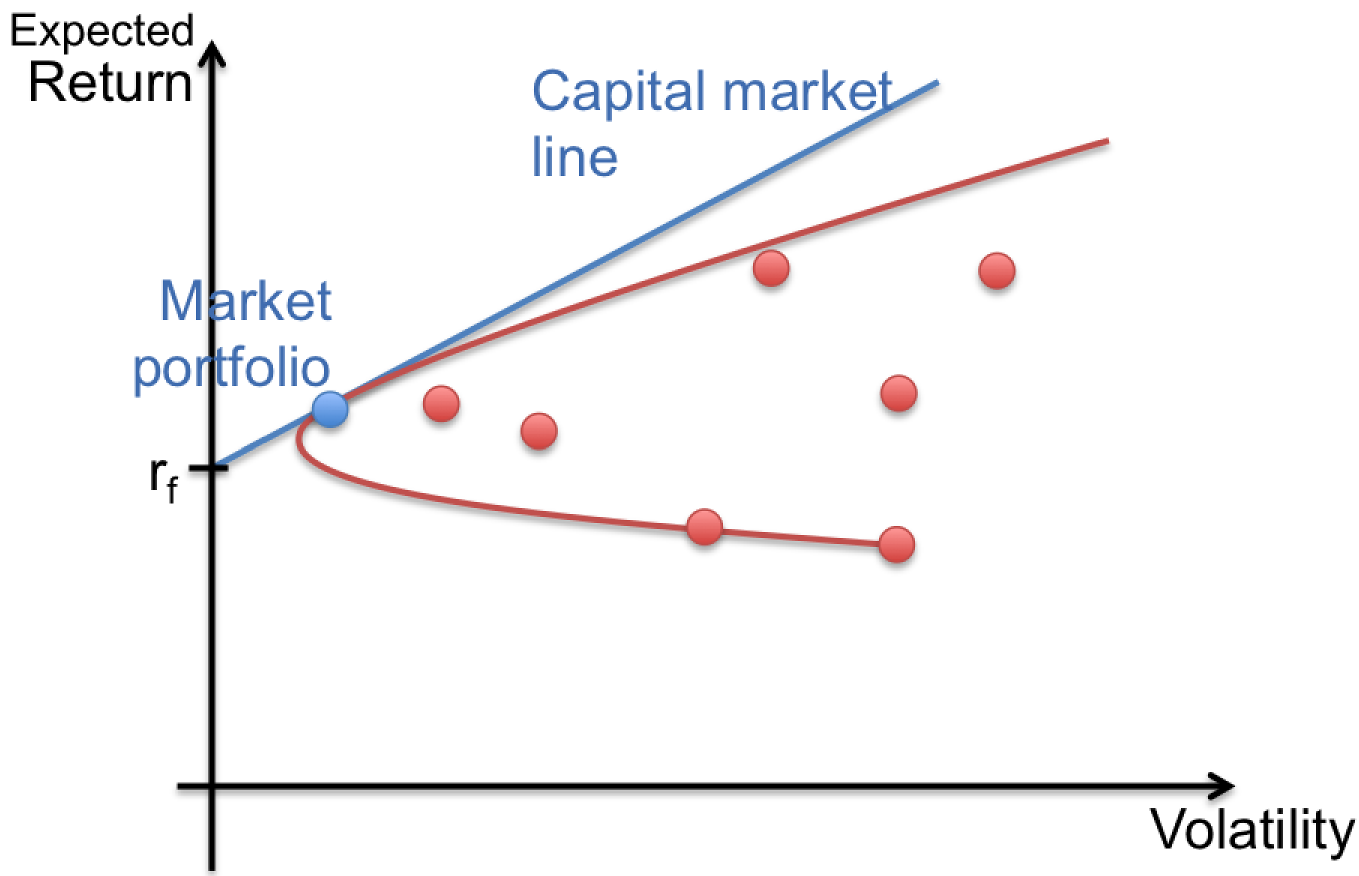
Capital market line

- Every investor holds the risk-free asset and the market portfolio in proportions that reflect risk his preferences.
- Consequently, he chooses a portfolio that lies on the line connecting the risk-free asset and the market portfolio.
- This line, in expected return vs volatility space, is called the **capital market line**.
- It is determined by the equation:

$$\bar{r} = r_f + \frac{\bar{r}^* - r_f}{\sigma^*}\sigma,$$

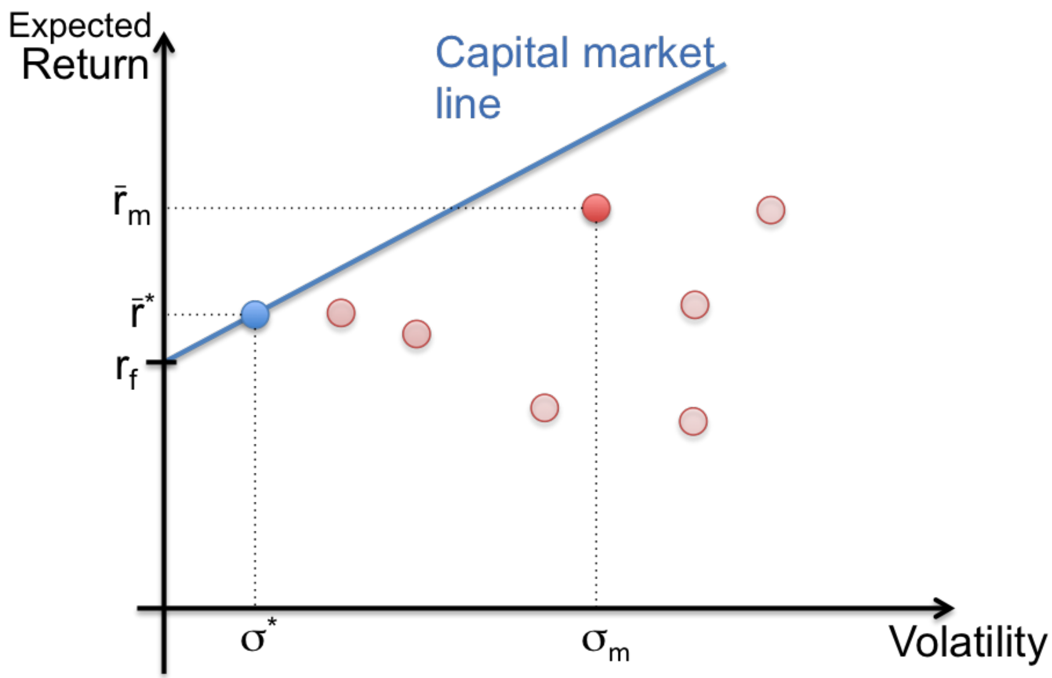
where $\bar{r}^* = \bar{r}_{MSR,e}$ is the return of the market portfolio, $\sigma^* = \sigma_{MSR,e}$ is its volatility, and σ is the volatility of the efficient portfolio for target return $\bar{r}$.

Capital market line



Derivation of CAPM pricing formula

- Consider the m^{th}asset with expected return $\bar{r}_m$ and volatility σ_m.
- The market portfolio has expected return $\bar{r}^*$ and volatility σ^*.
- How does the market portfolio relate to the m^{th} security?



Derivation of CAPM pricing formula

- Recall: the market portfolio solves

$$\max_{\Delta}\left\{\frac{\Delta^{\mathrm{T}}\left(\vec{r}-r_f\vec{1}\right)}{\sqrt{\Delta^{\mathrm{T}}\Sigma\Delta}}\quad s.t.\ \Delta^{\mathrm{T}}\vec{1}=1\right\}.$$

- Here
  - $\vec{r}=(\bar{r}_1,...,\bar{r}_M)$: vector of expected returns.
  - $\vec{1}=(1,...,1)$: vector of ones.
  - $\Sigma=\{\Sigma_{k,m}\}_{k,m=1}^{M}$: covariance matrix.
- The first order conditions for optimality imply

$$\begin{aligned}\vec{r}-r_f\vec{1}&=\frac{(\Delta^*)^{\mathrm{T}}(\vec{r}-r_f\vec{1})}{(\Delta^*)^{\mathrm{T}}\Sigma\Delta^*}\Sigma\Delta^*\\&=\frac{\bar{r}^*-r_f}{(\sigma^*)^2}\Sigma\Delta^*.\end{aligned}$$

Derivation of the CAPM pricing formula

- Next, let r_m be the (random) return of the m^{th} security and r^* be the (random) return of the market. portfolio.

- By definition $\mathrm{Var}(r^*) = (\sigma^*)^2$, and

$$\mathrm{Cov}(r_m, r^*) = \sum_{k=1}^{M} \Delta_k^* \mathrm{Cov}(r_m, r_k) = (\Sigma\Delta^*)_m .$$

- Thus

$$\begin{aligned} \bar{r}_m - r_f &= \frac{\bar{r}^* - r_f}{(\sigma^*)^2} (\Sigma\Delta^*)_m \\ &= \frac{\mathrm{Cov}(r_m, r^*)}{\mathrm{Var}(r^*)} (\bar{r}^* - r_f). \end{aligned}$$

Derivation of CAPM pricing formula

- Define

$$\beta_m \triangleq \frac{\mathrm{Cov}(r_m, r^*)}{\mathrm{Var}(r^*)}; \quad m = 1, ..., M.$$

- It follows that in equilibrium

$$\bar{r}_m = r_f + \beta_m(\bar{r}^* - r_f).$$

- The expected return of the m^{th} security is the risk-free return <u>plus</u> a multiple of the excess return of the market portfolio.
- The multiplication factor is called the **beta** of the m^{th} security.
- This is the CAPM pricing equation.

Interpretation of the beta

- The pricing equation

$$\bar{r}_m = r_f + \beta_m(\bar{r}^* - r_f),$$

  tells us that the expected return of the m^{th} security is determined by its beta.

- Assuming $\bar{r}^* > r_f$ (which is virtually always the case), we see that securities with higher betas will have higher expected returns.

- On the other hand, securities with lower betas will have lower expected returns.

- As a result, we can think of β_m as indicating the compensation for investing in the m^{th} security.

Interpretation of the beta

- But what exactly do investors get compensated for?
- We defined the beta as

$$\beta_m = \frac{\mathrm{Cov}(r_m, r^*)}{\mathrm{Var}(r^*)}.$$

- Higher $\beta_m > 0$ indicates the m^{th} return is more strongly correlated with the market portfolio return.
- Lower $\beta_m > 0$ indicates the m^{th} return is relatively independent of the market portfolio return.
- Negative β_m indicates the m^{th} return is negatively correlated with the market portfolio return.
- As a result, the beta measures the level of dependence between a security and the market portfolio.

Interpretation of the beta

- If $\beta_m > 0$, the security tends to move in the same direction as the market. CAPM tells us that the expected return of the security will be larger than the risk-free return:

$$\bar{r}_m = r_f + \beta_m(\bar{r}^* - r_f) > r_f.$$

- The investor gets compensated for being exposed to the risk of under-performing, when the market is under-performing.
- Given that investors are risk-averse, they require compensation to invest in securities that do badly when it matters the most: when the market is doing badly.

Interpretation of the beta

- If $\beta_m < 0$, the security tends to move opposite to the market. CAPM tells us that the expected return of the security will be smaller than the risk-free return:

$$\bar{r}_m = r_f + \beta_m(\bar{r}^* - r_f) < r_f.$$

- When $\beta_m < 0$, the security can be used to diversify against the market portfolio.
- The investor is willing to accept a lower expected return on a security with negative beta in exchange for the beneficial effect of diversification.

Portfolio market line

What does CAPM tell us about the beta of efficient portfolios?

- We know that any efficient portfolio is composed by a fraction α invested in the market portfolio and a fraction $1-\alpha$ invested in the risk-free security.
- Denote by r the (random) return of this portfolio. We have

$$r = \alpha r^* + (1-\alpha) r_f.$$

- As a result, the beta of an efficient portfolio is:

$$\beta = \frac{\mathrm{Cov}(r, r^*)}{\mathrm{Var}(r^*)} = \frac{\alpha \mathrm{Cov}(r^*, r^*)}{\mathrm{Var}(r^*)} = \alpha \frac{\mathrm{Var}(r^*)}{\mathrm{Var}(r^*)} = \alpha.$$

- Consequently, efficient portfolios have positive β when long the market portfolio, else they have negative β, as one would expect.

Portfolio market line

- Since

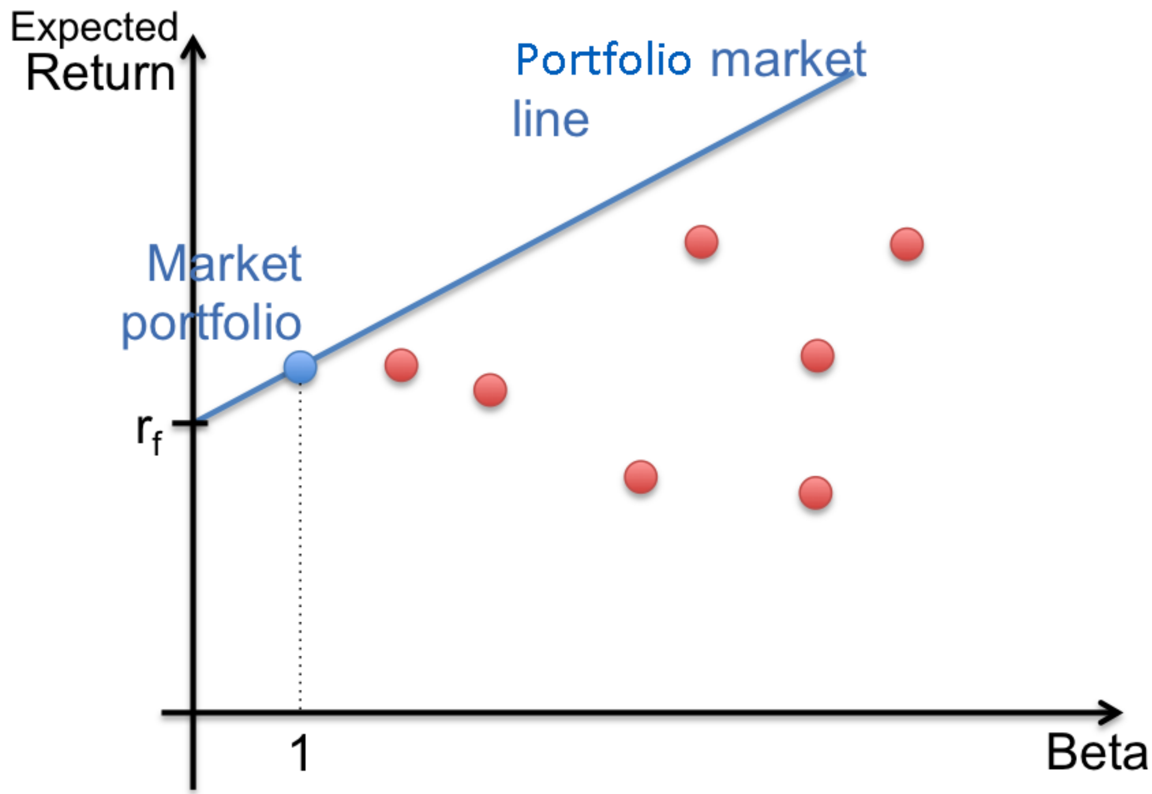
$$r = \alpha r^* + (1 - \alpha) r_f; \qquad \alpha = \beta,$$

  we see that

$$\bar{r} = r_f + \alpha(\bar{r}^* - r_f) = r_f + \beta(\bar{r}^* - r_f).$$

- In This is the **portfolio market line**. It links the expected return of an efficient portfolio to the excess expected return of the market through the beta.

Portfolio market line



Portfolio vs capital market line

- The capital market line and the portfolio market line contain the same portfolios, just expressed in different ways.
- The capital market line lies in expected return vs. volatility space, while the portfolio market line lies in expected return vs. beta space.
- The capital market line contains all efficient portfolios for a rational, volatility-averse investor.
- The portfolio market line, on the other hand, is valid under the assumptions of CAPM. It tells us that the portfolio β is the primary driver of expected returns.

Systematic and idiosyncratic risk

- CAPM tells us that the m^{th} security has expected return

$$\bar{r}_m = r_f + \beta_m(\bar{r}^* - r_f); \quad \beta_m = \frac{\mathrm{Cov}(r_m, r^*)}{\mathrm{Var}(r_m, r^*)}.$$

- The above equation relates averages, or expectations. What about the (random) return r_m?

- A standard regression analysis shows we may write

$$r_m = r_f + \beta_m(r^* - r_f) + \epsilon_m,$$

for a random variable ϵ_m with $\mathbb{E}[\epsilon_m] = 0$ and $\mathrm{Cov}(\epsilon_m, r^*) = 0$ (i.e. zero mean and uncorrelated with the market portfolio).

Systematic and idiosyncratic risk

- Assuming $r_m = r_f + \beta_m(\bar{r}^* - r_f) + \epsilon_m$, what is the volatility of this return?

- Write $\mathrm{Var}(\epsilon_m) = \sigma_{\epsilon_m}^2$. Then

$$\begin{aligned}\sigma_m^2 &= \mathrm{Var}(r_m) = \mathrm{Var}(r_f + \beta_m(r^* - r_f) + \epsilon_m);\\ &= \beta_m^2(\sigma^*)^2 + \sigma_{\epsilon_m}^2 + \beta_m \mathrm{Cov}(r^*, \epsilon_m);\\ &= \beta_m^2(\sigma^*)^2 + \sigma_{\epsilon_m}^2.\end{aligned}$$

- Thus, β_m captures the dependency of r_m on the market, and ϵ_m captures fluctuations of the security return that are orthogonal to the fluctuations of the market.

Systematic and idiosyncratic risk

The variance of the return r_m consists of two components:

1. The first component captures all variance arising from the fluctuations of the market return. This is called **systematic risk**.

2. The second component captures all variance that is orthogonal to the market. The fluctuations that cause this variance are specific to the security. This is known as **idiosyncratic risk**.

$$\sigma_m^2 = \underbrace{\beta_m^2(\sigma^*)^2}_{\text{Systematic risk}} + \underbrace{\sigma_{\epsilon_m}^2}_{\text{Idiosyncratic risk}}$$

Systematic and idiosyncratic risk

$$\sigma_m^2 = \underbrace{\beta_m^2(\sigma^*)^2}_{\text{Systematic risk}} + \underbrace{\sigma_{\epsilon_m}^2}_{\text{Idiosyncratic risk}}$$

- For a given variance, suppose a security has a low beta. Then most of its volatility is driven by idiosyncratic risk.
- By holding this security, the investor is exposing herself to security-specific risks. These risks can be diversified by investing in the market, given that the idiosyncratic risk factor ϵ_m and the market return r^* are uncorrelated.

Systemic and idiosyncratic risk

$$\sigma_m^2 = \underbrace{\beta_m^2(\sigma^*)^2}_{\text{Systematic risk}} + \underbrace{\sigma_{\epsilon_m}^2}_{\text{Idiosyncratic risk}}$$

- As such, the market does not believe it needs to pay a compensation for carrying idiosyncratic risk: $\mathbb{E}[\epsilon_m] = 0$.
- The only risk that cannot be diversified is systematic risk arising from the market itself. Because of this, an investor demands compensation for carrying systematic risk: this is the risk premium $r^* - r_f$.

Overview of CAPM

- CAPM tells us that, under certain assumptions, the return of any security is of the form:

$$r_m = r_f + \beta(r^* - r_f) + \epsilon_m,$$

where ϵ_m is a idiosyncratic risk factor, uncorrelated with the market.

- The investor only gets compensated for carrying systematic market risk:

$$\bar{r}_m = r_f + \beta_m(\bar{r}^* - r_f).$$

- Idiosyncratic risk can be diversified away.
- The beta of a security determines how the security is priced by the market.

Pricing in CAPM

- CAPM establishes a connection between the return of a security and the return of the market portfolio.
- How does CAPM relate to pricing?
- We know that the return of the m^{th} security is

$$r_m = \frac{S_1^{(m)} - S_0^{(m)}}{S_0^{(m)}} = r_f + \beta_m(r^* - r_f) + \epsilon_m.$$

- Consequently:

$$S_0^{(m)} = \frac{S_1^{(m)}}{1 + r_m} = \frac{S_1^{(m)}}{1 + r_f + \beta_m(r^* - r_f) + \epsilon_m}.$$

Pricing in CAPM

- Thus, taking expectations (physical measure) and noting that $S_0^{(m)}$ is not random:

$$S_0^{(m)} = \mathbb{E}\left[\frac{S_1^{(m)}}{1 + r_f + \beta_m(r^* - r_f) + \epsilon_m}\right].$$

- This is the **CAPM price** of the m^{th} security.
- It is the price implied by supply and demand (for the entire market) under the assumptions of CAPM.

CAPM price vs risk-neutral price

- If markets are complete and arbitrage free, the second FTAP implies there is a unique equivalent martingale, or risk-neutral, measure.
- In this case, the CAPM and the risk-neutral price coincide.
- The difference is that the CAPM and risk-neutral prices express the compensation that the investor gets for carrying risks in different ways.

CAPM price vs risk-neutral price

- The risk-neutral price compensates for bearing risks, by adjusting the probability distribution of outcomes. Unfavorable outcomes become more likely, making the security look more risky than it is, thus demanding higher compensation.

- The CAPM price compensates for risk by adjusting the interest rate with which future prices are discounted:

$$S_0^{(m)} = \mathbb{E}\left[\frac{S_1^{(m)}}{1 + r_m^{\mathrm{CAPM}}}\right],$$

  where the (random) interest rate r_m^{CAPM} satisfies

$$r_m^{\mathrm{CAPM}} = r_f + \beta_m(r^* - r_f) + \epsilon_m.$$

- Though conceptually different, both approaches lead to the same price in complete, arbitrage free markets.

Sharpe ratio

- For a given portfolio Δ with $\Delta_f = 1 - \Delta^{\mathrm{T}}\vec{1}$, the expected return and return variance are:

$$\bar{r}(\Delta) = \Delta_f r_f + \sum_{m=1}^{M} \Delta_m \bar{r}_m = r_f + \Delta^{\mathrm{T}}\left(\vec{r} - r_f \vec{1}\right);$$

$$\sigma^2(\Delta) = \sum_{m=1}^{M}\sum_{k=1}^{M} \Delta_m \Delta_k \sigma_m \sigma_k \rho_{m,k} = \Delta^{\mathrm{T}}\Sigma\Delta.$$

- The **Sharpe ratio** of the portfolio Δ is defined as

$$s(\Delta) = \frac{\bar{r}(\Delta) - r_f}{\sigma(\Delta)}.$$

- The Sharpe ratio measures excess return per unit of volatility. The higher the Sharpe ratio, the more excess return can be achieved with a given exposure to volatility.

Sharpe ratio in CAPM

- The Sharpe ratio is the slope of the line connecting the risk-free asset and the portfolio Δ in expected return-volatility space.
- In CAPM, the market portfolio maximizes this slope.
- As a result, the Sharpe ratio of any portfolio is bounded above by the Sharpe ratio of the market portfolio in the CAPM model:

$$s(\Delta) \leq s(\Delta^*).$$

Performance evaluation in CAPM

- A good way to measure the performance of a portfolio is to compare its Sharpe ratio to that of the market.
- The closer the portfolio Sharpe ratio lies to the market Sharpe ratio, the better the portfolio performance:

$$s(\Delta) \approx s(\Delta^*) \qquad \text{good performance.}$$
$$s(\Delta) \ll s(\Delta^*) \qquad \text{bad performance.}$$

- In reality, there are portfolios that experience Sharpe ratios larger than the market Sharpe ratio.
- If $s(\Delta) > s(\Delta^*)$, we say that Δ **beats** the market: it achieves a unit of excess return with less volatility than the market.

Investment implications of CAPM

- CAPM does not necessarily hold in reality.
- A deviation from CAPM is a return that has a constant component α_m:

$$r_m = \alpha_m + r_f + \beta_m(r^* - r_f) + \epsilon_m$$

- CAPM says $\alpha_m = 0$. However, this does need to be true in reality.
- In fact, most sophisticated investment managers look to construct portfolios that achieve "positive alpha".
- Positive alpha is excess return that the investment portfolio generates <u>independently</u> of how the market performs.
- Generating positive alpha is the ultimate goal of any investment manager, as it is compensation that the manager receives beyond the performance fees derived from the portfolio Sharpe ratio

Summary

- CAPM connects many of the topics we talked about in this course.
  - Investors hold assets in proportion to their Sharpe ratios.
  - Prices reflect the riskiness of the security payoff and its correlation with the market portfolio.
- CAPM holds under the assumptions that investors are rational, volatility averse, and agree on the expected returns, covariance. This last assumption is probably the most unrealistic.
- Despite this, CAPM yields intuitive and reasonable conclusions.