No-arbitrage pricing of American options

Goals:

- Derive the risk-neutral price for American options
- Introduce stopping times
- Understand the optimal exercise problem
- Introduce the concept of super-replication

Relevant literature:

- Shreve Ch. 4, Hull Ch. 12

Notation

- We consider a multiperiod binomial model with N periods with Δt time between periods; i.e., $N = \frac{T}{\Delta t}$
- For $1 \leq n \leq N$, let $\omega_n \in \Omega = \{u, d\}$ denote if the stock price goes up (u) or down (d) in the n-th period
- Let $S_n(\omega_1 \dots \omega_n)$ denote the stock price in the n-th period if the stock price goes up and down according to $\omega_1 \dots \omega_n$
- Let $C_n(\omega_1 \dots \omega_n)$ denote the value of a Americal call option in the n-th period if the stock price moved as in $\omega_1 \dots \omega_n$
- Let $\Delta_n(\omega_1 \dots \omega_n)$ denote the number of shares of a replicating portfolio in the n-th period if the stock price moved as in $\omega_1 \dots \omega_n$
- Let $V_n(\omega_1 \dots \omega_n)$ denote the value of a portfolio in the n-th period if the stock price moved as in $\omega_1 \dots \omega_n$

For simplicity, sometimes we will drop $(\omega_1 \dots \omega_n)$ from the notation

Review

- In order to price a European call option, we construct a portfolio consisting of a loan and the stock that replicates the option payoff
- The value of the replicating portfolio at time zero has to agree with the price of the call option because of no-arbitrage
- This price is equal to the expected discounted payoff of the option, computed with respect to the risk-neutral probability distribution of the stock:

$$c = \tilde{\mathbb{E}}\left[e^{-rT}(S_T - K)_+\right]$$

  for a call
- Under the risk-neutral distribution, the stock price moves up during the time interval Δt with probability

$$\tilde{p} = \frac{e^{r\Delta t} - e^{d\Delta t}}{e^{u\Delta t} - e^{d\Delta t}}$$

Review

- Under the risk-neutral probability, the stock grows with the risk-free interest rate
- When pricing an option, the investor changes her views on the distribution on the stock price to become indifferent towards carrying the risks involved with the option
- Markets are complete in the multiperiod binomial model: any European derivative can be replicated by a portfolio consisting of a loan and the stock

American options

- Unlike European options, you can exercise an American option at **any** time on or before maturity
- Because of this, an American option is always more valuable than a European option - it gives you more rights:

$$C \geq c \quad \text{and} \quad P \geq p$$

- Two main questions arise:
  1. When is the best time to exercise an American option?
  2. Given the best time to exercise an option, what is the price of an American option?

We will address both questions in this lecture

American option pricing

Let's start by answering the second question

- Consider an N-period binomial model
- Suppose that you know with certainty that the option will be exercised in period $n^* \in \{1, \ldots, N\}$
- If you are 100% certain, then you can just view the American option as a European option with maturity n^*
- In this case, the price of an American call option is

$$\tilde{\mathbb{E}}\left[e^{-rn^*\Delta t}\left(S_{n^*} - K\right)_+\right]$$

American option pricing

- An American option also gives you the right to exercise at other times besides n^*
- Because of this, the American call option has to be priced higher than a European option that only pays off at time n^*:

$$C \geq \tilde{\mathbb{E}}\left[e^{-rn^*\Delta t}\left(S_{n^*} - K\right)_+\right]$$

for any $n^* \in \{0, \ldots, N\}$

Stopping time

- In general, you do not know in advance when you will want to exercise the option: the exercise time is random
- Let τ be a random time at which an American option is exercised
- τ is a random variable valued in $\{0, 1, \ldots, N, \infty\}$
  - $\tau = 0$ means that the option is exercised immediately
  - $\tau = \infty$ means that the option is never exercised
- The decision to exercise or not in period n is based only on the stock prices that have been observed until period n $\{\tau \leq n\} \in \mathcal{F}_n$
  - There exist no signals about future stock prices that may influence your decision to exercise
- Such an exercise time τ is also called a **stopping time**

American call price

- As argued before, the American option must be valued higher than an option that only pays off in period τ, even for random τ:

$$C \geq \tilde{\mathbb{E}}\left[e^{-r\tau\Delta t}(S_\tau - K)_+\right]$$

- The above inequality has to hold for any stopping time τ
- Consequently, we can conclude that:

$$C \geq \sup_{\text{stopping time } \tau} \tilde{\mathbb{E}}\left[e^{-r\tau\Delta t}(S_\tau - K)_+\right]$$

American call price

From the option seller's point of view, if the option holder exercises at time τ, then the initial capital required to form the replication portfolio is

$$\tilde{\mathbb{E}}\left[e^{-r\tau\Delta t}(S_\tau - K)_+\right].$$

Because the option holder has freedom to choose any exercise time, then it make sense to require the initial capital

$$C = \sup_{\text{stpping time } \tau} \tilde{\mathbb{E}}\left[e^{-r\tau\Delta t}(S_\tau - K)_+\right]$$

as the American call price at time 0.

Optimal exercise and no-arbitrage

- Thus, there has to exist an optimal exercise time τ^* such that

$$\tilde{\mathbb{E}}\left[e^{-r\tau^*\Delta t}(S_{\tau^*}-K)_+\right] = \sup_{\text{stopping time } \tau} \tilde{\mathbb{E}}\left[e^{-r\tau\Delta t}(S_\tau - K)_+\right]$$

- We will now proceed to compute the optimal stopping time τ^*

Optimal exercise

Suppose you hold an American call option and need to decide whether to exercise in the n-th period

- If you exercise, you will get a payoff of $S_n - K$
- If you don't exercise, the option will be valued C_{n+1} in the next period and you could sell it for that price. Thus, you can view the American option in this period as a European option which expires in the next period and has payoff of C_{n+1}. Your valuation of such an option is

$$\tilde{\mathbb{E}}_n \left[e^{-r\Delta t} C_{n+1} \right]$$

  according to the risk-neutral pricing theorem. The subscript n indicates that this is the n-period conditional expectation

Thus, you will only exercise when the payoff is large enough:

$$S_n - K \geq \tilde{\mathbb{E}}_n \left[e^{-r\Delta t} C_{n+1} \right]$$

Optimal exercise

Theorem. *Let C_n denote the price of an American call in period $n \in \{0, \ldots, N\}$. Define $C_{N+1} = 0$, $C_N = (S_N - K)_+$, and*

$$\tau^* = \min\left\{n \in \{0, 1, \ldots, N\} : S_n - K \geq \tilde{\mathbb{E}}_n \left[e^{-r\Delta t} C_{n+1}\right]\right\}$$

Then τ^* is the optimal exercise time for the American call option

Proof. We need to prove that *(i)* τ^* is a stopping time and *(ii)* that τ^* is optimal. For *(i)*, note that τ^* only takes on values in $\{0, 1, \ldots, N, \infty\}$. Furthermore,

$$\tau^* \leq n \Leftrightarrow (S_j - K)_+ \geq \tilde{\mathbb{E}}_j \left[e^{-r\Delta t} C_{j+1}\right] \text{ for some } j \leq n$$

Thus, the event $\{\tau^* \leq n\}$ is determined only by past or current stock prices. It follows that τ^* is a stopping time. For *(ii)*, note that a risk-averse investor will exercise as soon as she first believes that it is worth to exercise to not face further risks (Thm 4.4.5 of Shreve). $\square$

Risk-neutral price of an American call

- The Americal call is exercised optimally at the stopping time

$$\tau^* = \inf \left\{ n \in \{0, 1, \ldots, N\} : S_n - K \geq \tilde{\mathbb{E}}_n \left[e^{-r\Delta t} C_{n+1} \right] \right\},$$

- In order to be able to evaluate this stopping time, though, we need to know the value process $(C_n : 0 \leq n \leq N)$ of the American call
- How much would an investor that exercises the option optimally be willing to pay for an American call in each period of time?

Risk-neutral price of an American call

- As for the European option, we can compute the American call option prices using backward induction
- At maturity, the option is only exercised if it is in the money, otherwise it expires worthless. Thus:

$$C_N = (S_N - K)_+$$

- In period n, the option is either exercised or it is held until the next period
- If it is exercised, the option is worth $(S_n - K)_+$
- If it is held, then we can view the option as a European option to sell an American option in period $n+1$ for the price C_{n+1}. In this case, its price is $\tilde{\mathbb{E}}_n[e^{-r\Delta t} C_{n+1}]$

Risk-neutral price of an American call

- If the option is exercised according to the optimal time τ^*, then the investor will exercise in period n if

$$S_n - K \geq \tilde{\mathbb{E}}_n[e^{-r\Delta t} C_{n+1}] \geq 0,$$

in which case the price is $S_n - K$, and will hold if

$$S_n - K < \tilde{\mathbb{E}}_n[e^{-r\Delta t} C_{n+1}],$$

in which case the price is $\tilde{\mathbb{E}}_n[e^{-r\Delta t} C_{n+1}]$

- Thus, the price of the option in period n is

$$C_n = \max\{(S_n - K)_+, \tilde{\mathbb{E}}_n[e^{-r\Delta t} C_{n+1}]\}$$

Risk-neutral price of an American call

Overall, we have that

$$C = \tilde{\mathbb{E}}\left[e^{-r\tau^*\Delta t}(S_{\tau^*} - K)_+\right]$$

for the optimal exercise time

$$\tau^* = \inf\left\{n \in \{0, 1, \ldots, N\} : S_n - K \geq \tilde{\mathbb{E}}_n\left[e^{-r\Delta t}C_{n+1}\right]\right\}.$$

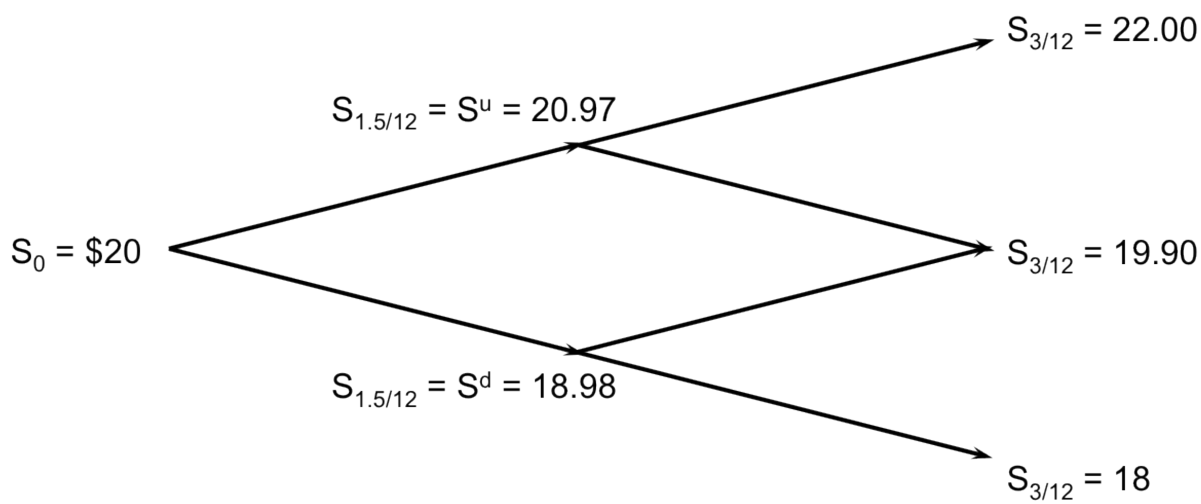
Here, we have $C_N = (S_N - K)_+$ and

$$C_n = \max\left\{(S_n - K)_+, \tilde{\mathbb{E}}_n\left[e^{-r\Delta t}C_{n+1}\right]\right\}$$

for $0 \leq n < N$.

Example, optimal exercise

Suppose you were to buy an American call with maturity $T = 3$ months and strike $K = \$21$ on a stock with $S_0 = \$20$. Let I denote the intrinsic value of the American option. Take $\Delta t = \frac{1.5}{12}$



Optimal exercise and no-arbitrage

- We established that the price of an American call is:

$$C = \sup_{\text{stopping time } \tau} \tilde{\mathbb{E}}\left[e^{-r\tau\Delta t}(S_\tau - K)_+\right]$$

- Is the supremum achieved?
- Suppose it is not achieved so that for every stopping time $\bar{\tau}$:

$$\tilde{\mathbb{E}}\left[e^{-r\bar{\tau}\Delta t}(S_{\bar{\tau}} - K)_+\right] < \sup_{\text{stopping time } \tau} \tilde{\mathbb{E}}\left[e^{-r\tau\Delta t}(S_\tau - K)_+\right] = C$$

- Then there exists no demand for the option!
  - No matter when the option holder exercises, her expected payoff will be smaller than what she paid for the option
- In this case, the price of the option would be $C = 0$, which is a contradiction

Example, optimal exercise

We proceed by backward induction. At maturity $T = 3$ months:

- If maturity, we set $C_{3/12} = (S_2 - K)_+$
- If $S_{3/12} = S^{uu}$

  $$C_{3/12} = C^{uu} = (S_{3/12} - K)_+ = 22 - 21 = 1.$$

  Thus, the option is exercised
- If $S_{3/12} = S^{ud}$, then

  $$C_{3/12} = C^{ud} = (S_{3/12} - K)_+ = (19.90 - 21)_+ = (-1.10)_+ = 0.$$

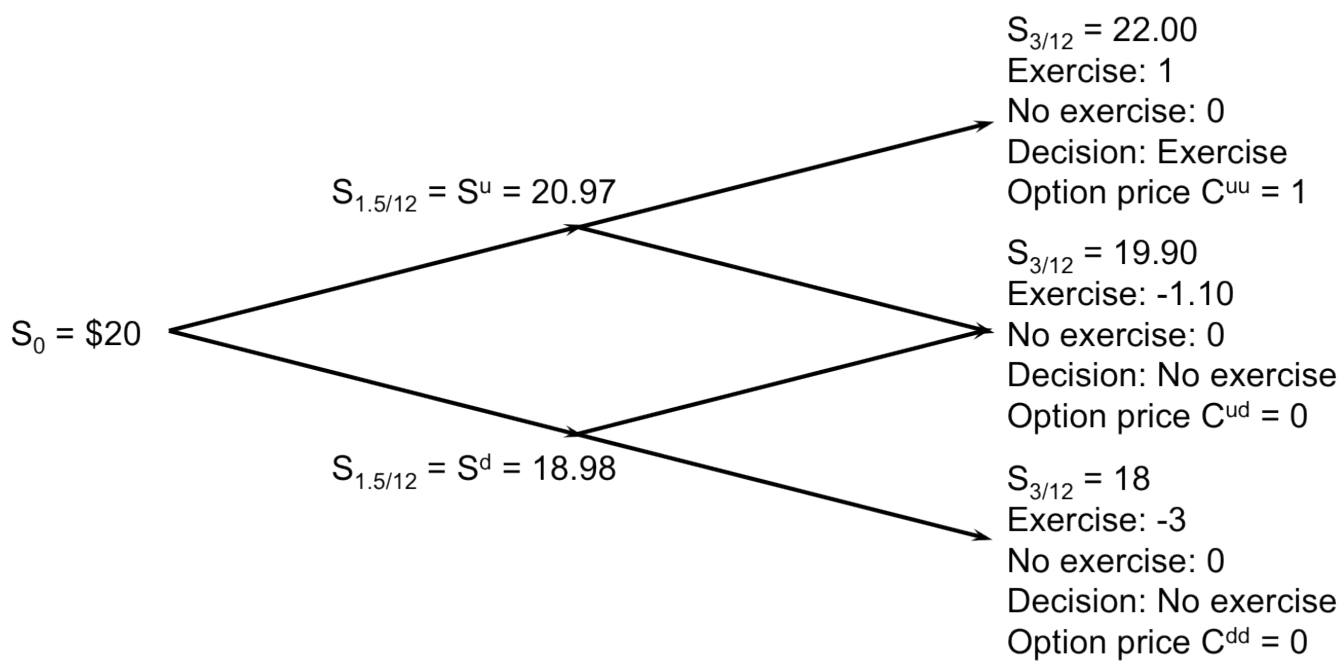
  Thus, the option is not exercised

Example, optimal exercise

- If $S_{3/12} = S^{dd}$, then

$$C_{3/12} = C^{dd} = (S_{3/12} - K)_+ = (18 - 21)_+ = 0.$$

Thus, the option is not exercised



Example, optimal exercise

At time $t = 1.5$ months, we have:

- If $S_{1.5/12} = S^u$, then

$$S_{1.5/12} - K = 20.97 - 21 = -0.03,$$

$$\tilde{\mathbb{E}}_1 \left[e^{-r\frac{1.5}{12}} C_{3/12} \right] = e^{-r\frac{1.5}{12}} \left[\tilde{p} C^{uu} + (1-\tilde{p}) C^{ud} \right] = 0.51.$$

  Thus, the option is not exercised. The option price is $C^u = 0.51$.

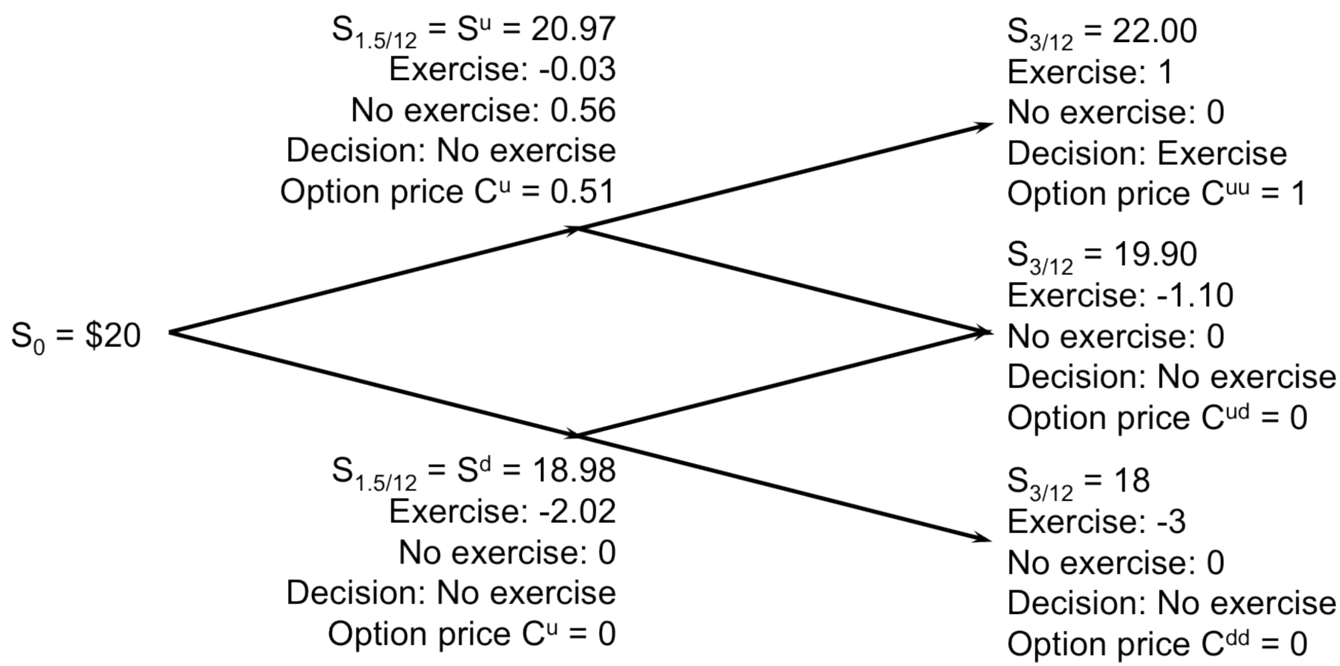
- If $S_{1.5/12} = S^d$, then

$$S_{1.5/12} - K = 18.98 - 21 = -2.02,$$

$$\tilde{\mathbb{E}}_1 \left[e^{-r\frac{1.5}{12}} C_{3/12} \right] = e^{-r\frac{1.5}{12}} \left[\tilde{p} C^{ud} + (1-\tilde{p}) C^{dd} \right] = 0.$$

  Thus, the option is not exercised. The option price is $C^d = 0$.

Example, optimal exercise



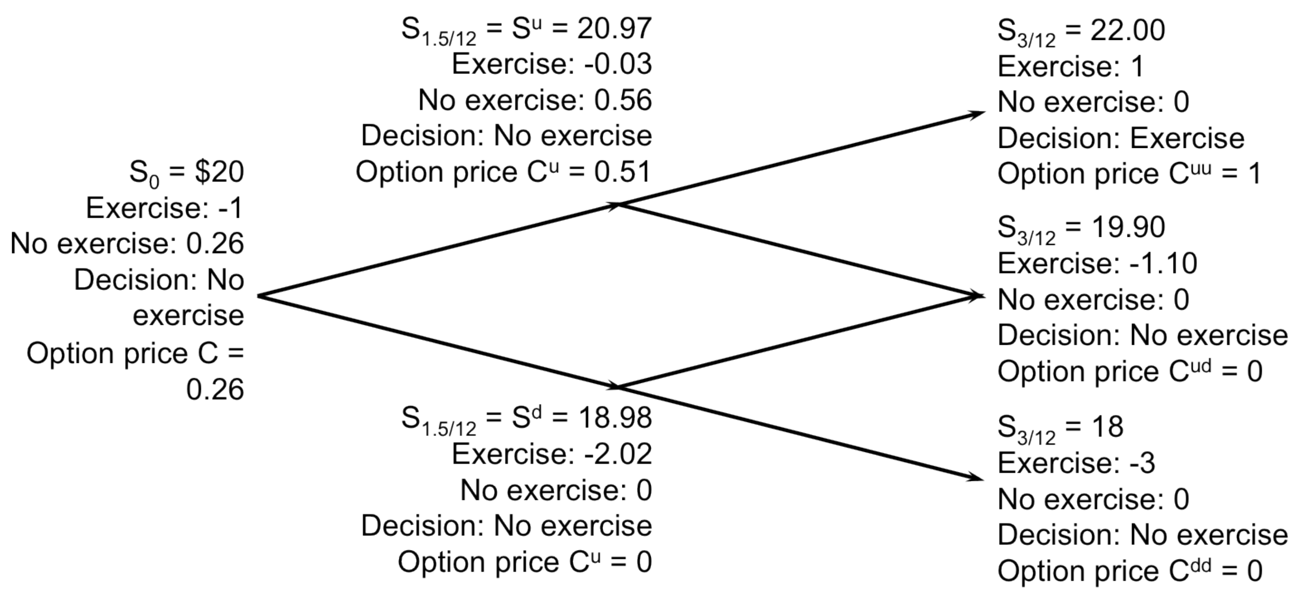
Example, optimal exercise

Finally, at time $t = 0$ months, we have:

$$S_0 - K = 20 - 21 = -1,$$

$$\tilde{\mathbb{E}}\left[e^{-r\frac{1.5}{12}} C_{1.5/12}\right] = e^{-r\frac{1.5}{12}}\left[\tilde{p}C^u + (1-\tilde{p})C^d\right] = 0.26.$$

Thus, the option is not exercised. The option price is $C = 0.26$.

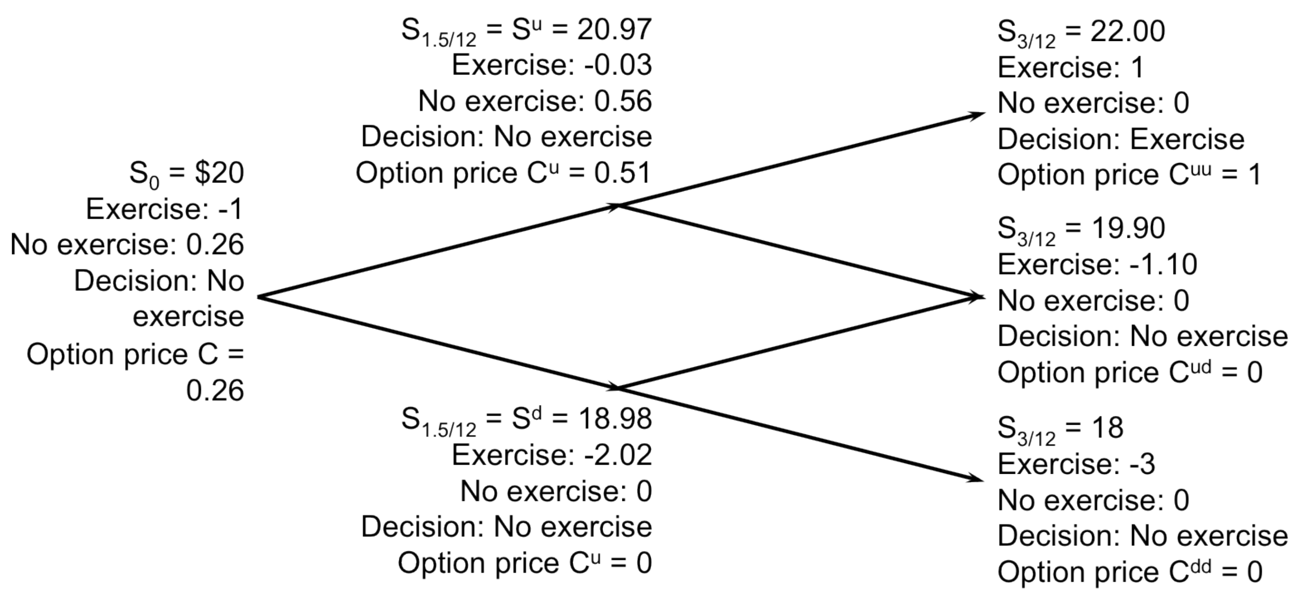


Example, optimal exercise

The optimal exercise time is (for $\Delta t = 1.5/12$)

$$\tau^* = \begin{cases} 2 & \text{if } \omega_1 = \omega_2 = u, \\ \infty & \text{otherwise} \end{cases}$$

It is not optimal to exercise this option early! (We already knew this as this is an American call on a non-dividend paying stock)



Risk-neutral price of an American derivative

The arguments generalize for any American derivative:

- Suppose Π is the price of a derivative that can be exercised at any time on or before period N and promises a payoff $g(S_n) \geq 0$ if exercised in period n

- Then:

$$\Pi = \tilde{\mathbb{E}}\left[e^{-r\tau^*\Delta t} g(S_{\tau^*})\right]$$

Here, τ^* is the optimal exercise time defined as

$$\tau^* = \inf\left\{n \in \{0, 1, \ldots, N\} : g(S_n) \geq \tilde{\mathbb{E}}_n\left[e^{-r\Delta t}\Pi_{n+1}\right]\right\},$$

where $\Pi_{N+1} = 0$, $\Pi_N = g(S_N)$, and

$$\Pi_n = \max\left\{g(S_n), \tilde{\mathbb{E}}_n\left[e^{-r\Delta t}\Pi_{n+1}\right]\right\}$$

for $0 \leq n < N$

Snell envelope

- In order to construct the optimal exercise time, we needed to figure out the first time τ^* when the intrinsic value of the option is higher than the expected discounted price from the next period
- Computing τ^* may be burdensome
- We can compute the American derivative price much faster by only using the recursive equations $\Pi_N = g(S_N)$ and

$$\Pi_n = \max\left\{g(S_n), \tilde{\mathbb{E}}_n\left[e^{-r\Delta t}\Pi_{n+1}\right]\right\}$$

  for $0 \leq n < N$
- The recursive system above is called **snell envelope**
- The snell envelope provides an efficient way to compute the price of an American derivative without computing τ^*

American option pricing

We have demonstrated that the risk-neutral price of an American call option in period n is defined recursively through the equation:

$$C_n = \max\left\{g(S_n), \tilde{\mathbb{E}}_n\left[e^{-r\Delta t}C_{n+1}\right]\right\}$$

Here, $C_{N+1} = 0$ and $C_N = (S_N - K)_+$.

- Can this price process be replicated?
- Replication is important as it guarantees that there exists a hedging portfolio
- It also guarantees that the price process $(C_n : 0 \leq n \leq N)$ does not allow any arbitrages

Can we replicate as for European options?

- For European options, we established that a replicating portfolio only needs to match the payoff at maturity: $v_N = (S_N - K)_+$ almost surely for a European call guarantees $v_n = c_n$ for all $n < N$
- This will not suffice for American options
  - The American option can be exercised at any time during its lifetime
- In order to hedge an American option, we have to ensure that the value of the replicating portfolio at any time is at least as large as the intrinsic value of the option
- Otherwise, the option seller may sit on large losses when the option is exercised unexpectedly by the option buyer

Super-replicating portfolio

- Let V denote the value of a portfolio satisfying

$$V_n \geq (S_n - K)_+$$

  for all $0 \leq n \leq N$

- A portfolio that is always valued at least as much as the intrinsic value of the underlying option is called **super-replicating portfolio**

Super-replicating portfolio

- If V is the value process of a super-replicating portfolio, then

$$V_{\tau^*} \geq (S_{\tau^*} - K)_+ = C_{\tau^*}$$

- Thus, the super-replicating portfolio pays off at least as much as the American call when the option is exercised optimally
- The no-arbitrage assumption implies that the initial value of a super-replicating portfolio will be at least as large as the price of the American call option:

$$V_0 \geq C$$

Super-replicating portfolio as a hedge

- A super-replicating portfolio V provides a hedge for the American call option
  - The super-replicating portfolio is initially valued as least as much as the option
  - Even when the option is not exercised optimally, the super-replicating portfolio provides enough liquidity to the option seller to pay off the option
  - A portfolio that holds a long position in a super-replicating portfolio and a short position in the corresponding American option has non-negative cash flows when the option is exercised
- The additional cash flows provide protection against the sub-optimal exercise of the option

Delta-hedge price of an American call

- Suppose V is a super-replicating portfolio with

$$V_{\tau^*} = C_{\tau^*}$$

- Can we construct such a super-replicating portfolio that pays off exactly as much as the American call when it is exercised optimally?
- If yes, the no-arbitrage assumption implies that $V_0 = C$
- Then, V replicates the option when exercised optimally and the American call can be hedged with minimal costs

As we will show, one can always construct a super-replicating portfolio that replicates the payoff of the option at the optimal exercise time τ^*

Replication for American derivatives

Theorem. *Consider an N-period binomial model with $d < r < u$ and*

$$\tilde{p} = \frac{e^{r\Delta t} - e^{d\Delta t}}{e^{u\Delta t} - e^{r\Delta t}}.$$

Let Π be the price of an American derivative with maturity $T = N\Delta$ that pays off $g(S_n)$ if exercised in the n-th period. Let Π_n denote the snell envelope n-th period price of the derivative. Define a portfolio consisting of savings and shares of the stock with initial value $V_0 = \Pi$ and n-th period value $V_n = \Delta_{n-1} S_n + e^{r\Delta t}\left(V_{n-1} - \Delta_{n-1} S_{n-1}\right)$ for

$$\Delta_n = \frac{\Pi_{n+1}(u) - \Pi_{n+1}(d)}{S_{n+1}(u) - S_{n+1}(d)}.$$

Then, $V_n = \Pi_n$ for all $0 \leq n \leq \tau^*$. Thus, V is a portfolio that replicates the risk-neutral price process of the American derivative.

Replication for American derivatives

Proof. Fix N. Proceed by induction. The claim for $n = 0$ holds by definition. Suppose now the claim holds for $n < \tau^*$. We will show that it also holds for $n+1$. Suppose that $\omega_{n+1} = u$ for now, so that $S_{n+1} = S_{n+1}(u)$. The induction assumption and the snell envelope property imply that $V_n = \Pi_n \geq e^{-r\Delta t}\tilde{\mathbb{E}}[\Pi_{n+1}]$.

Thus:

$$\begin{aligned}
V_{n+1}(u) = \frac{\Pi_{n+1}(u) - \Pi_{n+1}(d)}{S_{n+1}(u) - S_{n+1}(d)} S_{n+1}(u) + e^{r\Delta t} \left(V_n - \frac{\Pi_{n+1}(u) - \Pi_{n+1}(d)}{S_{n+1}(u) - S_{n+1}(d)} S_n \right) \\
= (\Pi_{n+1}(u) - \Pi_{n+1}(d)) \frac{S_{n+1}(u) - e^{r\Delta t} S_n}{S_{n+1}(u) - S_{n+1}(d)} + e^{r\Delta t} V_n \\
= (\Pi_{n+1}(u) - \Pi_{n+1}(d))(1 - \tilde{p}) + e^{r\Delta t} V_n \\
= (\Pi_{n+1}(u) - \Pi_{n+1}(d))(1 - \tilde{p}) + e^{r\Delta t} \Pi_n \\
\geq (\Pi_{n+1}(u) - \Pi_{n+1}(d))(1 - \tilde{p}) + e^{r\Delta t} e^{-r\Delta t} \left[\tilde{p}\Pi_{n+1}(u) - (1 - \tilde{p})\Pi_{n+1}(d)\right] \\
= \Pi_{n+1}(u)
\end{aligned}$$

The inequalities are equalities if $n < \tau^*$ since then $V_n = \Pi_n$ by the induction assumption and $\Pi_n = e^{-r\Delta t}\tilde{\mathbb{E}}[\Pi_{n+1}]$ because it is not optimal to exercise the option before time τ^*. Using the analogous argument for $\omega_n = d$, we conclude that $V_{n+1} \geq g(S_{n+1})$ and $V_{n+1} = \Pi_{n+1}$ if $n+1 \leq \tau^*$. $\square$

What do we learn?

- A super-replicating portfolio with $V_{\tau^*} = \Pi_{\tau^*}$ exists
- As a result, any American derivative can be hedged
- Furthermore, the super-replicating portfolio replicates the price of the option in each period
- Thus, the option price is also a no-arbitrage price

Example, super-replication

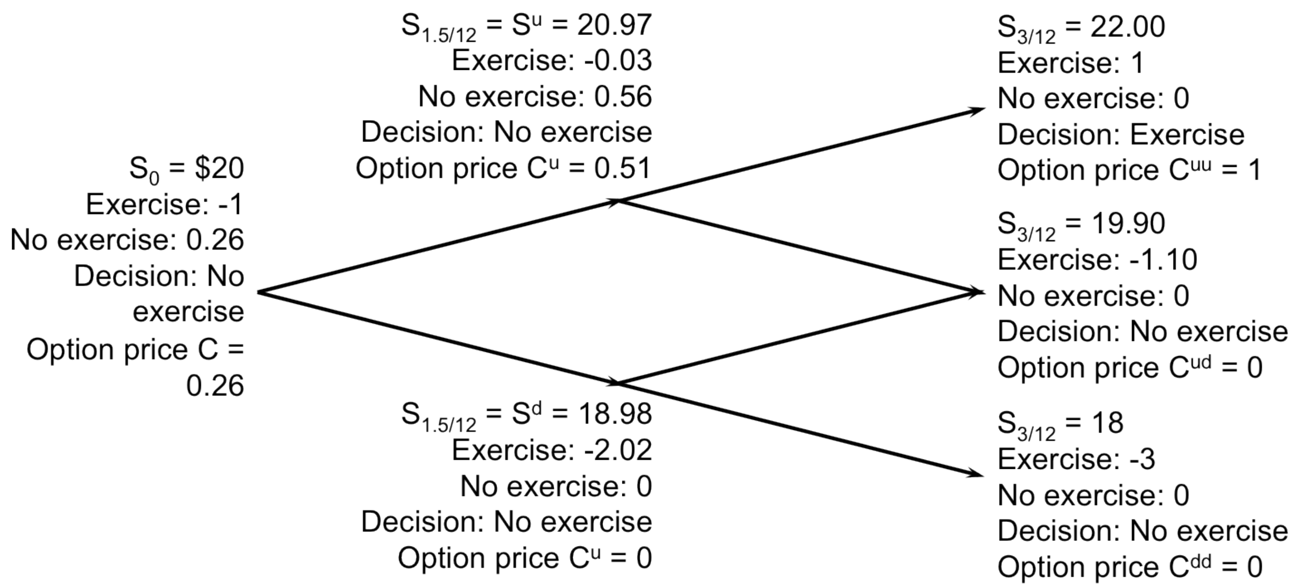
Suppose we were to price an American call with maturity $T = 3$ months and strike $K = \$21$ on a stock with $S_0 = \$20$. Take $r = 0.005$. We construct a super-replicating portfolio consisting of a loan and the stock.

- The super-replicating portfolio borrows V_0 initial cash and holds Δ_n shares of the stock in the n-th period
- The value of the super-replicating portfolio in period n is

$$V_n = \Delta_{n-1} S_n + e^{r\Delta t} \left(V_{n-1} - \Delta_{n-1} S_{n-1} \right)$$

Let I denote the intrinsic value process of the American option

Example, super-replication



Example, super-replication

- In period $n = 0$, we have

$$\Delta_0 = \frac{\Pi_1(u) - \Pi_1(d)}{S_1(u) - S_1(d)} = \frac{0.51 - 0}{20.97 - 18.98} = 0.2563$$

- In period $n = 1$, if $\omega_1 = u$, we have

$$\Delta_1(u) = \frac{\Pi_2(uu) - \Pi_2(ud)}{S_2(uu) - S_2(ud)} = \frac{1 - 0}{22 - 19.90} = 0.4762$$

- If $\omega_1 = d$, we have

$$\Delta_1(d) = \frac{\Pi_2(du) - \Pi_2(dd)}{S_2(du) - S_2(dd)} = \frac{0 - 0}{19.90 - 18} = 0$$

Example, super-replication

- The portfolio value in period $n = 0$ is

$$V_0 = \Pi_0 = 0.26$$

- The portfolio values in period $n = 1$ are

$$\begin{aligned}
V_1(u) &= \Delta_0 S_1(u) + e^{r\Delta t}\,(V_0 - \Delta_0 S_0) \\
&= 0.2563 \times 20.97 + 0.9994(0.26 - 0.2563 \times 20) \\
&= 0.51 = C_1(u) > I_1(u) = 0, \\
V_1(d) &= \Delta_0 S_1(d) + e^{r\Delta t}\,(V_0 - \Delta_0 S_0) \\
&= 0.2563 \times 18.98 + 0.9994(0.26 - 0.2563 \times 20) \\
&= 0 = C_1(d) = I_1(d) = 0
\end{aligned}$$

Example, super-replication

- The portfolio values in period $n = 2$ are

$$
\begin{aligned}
V_2(uu) &= \Delta_1(u)S_2(uu) + e^{r\Delta t}\left(V_1(u) - \Delta_1(u)S_1(u)\right) \\
&= 0.4762 \times 22 + 0.9994(0.51 - 0.4762 \times 20.97) \\
&= 1 = C_2(uu) = I_2(uu), \\
V_2(ud) &= \Delta_1(u)S_2(ud) + e^{r\Delta t}\left(V_1(u) - \Delta_1(u)S_1(u)\right) \\
&= 0.4762 \times 19.90 + 0.9994(0.51 - 0.4762 \times 20.97) \\
&= 0 = C_2(ud) = I_2(ud), \\
V_2(dd) &= \Delta_2(d)S_2(dd) + e^{r\Delta t}\left(V_1(d) - \Delta_1(d)S_1(d)\right) \\
&= 0 \times 18 + 0.9994(0 - 0 \times 18.98) \\
&= 0 = C_2(ud) = I_2(ud)
\end{aligned}
$$

- Thus, the portfolio is super-replicating

Summary

- The risk-neutral price of an American derivative with payoff function g is

$$\Pi = \sup_{\text{stopping time } \tau} \tilde{\mathbb{E}}\left[e^{-r\tau\Delta t} g(S_\tau)\right]$$

- The supremum above is achieved:

$$\Pi = \tilde{\mathbb{E}}\left[e^{-r\tau^*\Delta t} g(S_{\tau^*})\right]$$

for the optimal exercise time

$$\tau^* = \min\left\{n \in \{0, 1, \ldots, N\} : g(S_n) \geq \tilde{\mathbb{E}}_n\left[e^{-r\Delta t}\Pi_{n+1}\right]\right\},$$

where $\Pi_N = g(S_N)$ and

$$\Pi_n = \max\left\{g(S_n), \tilde{\mathbb{E}}_n\left[e^{-r\Delta t}\Pi_{n+1}\right]\right\}$$

for $0 \leq n < N$

Summary

- The snell envelope computes the price of an American derivative without the need to compute the optimal exercise time τ^* by solving the recursive equations:

$$\Pi_N = g(S_N),$$

$$\Pi_n = \max\left\{g(S_n), \tilde{\mathbb{E}}_n\left[e^{-r\Delta t}\Pi_{n+1}\right]\right\} \quad \text{for} \quad 0 \leq n < N.$$

- There exists a portfolio that replicates the price process of the American derivative

- Such a portfolio is given by an initial investment of $V_0 = \Pi$ and

$$\Delta_n = \frac{\Pi_{n+1}(u) - \Pi_{n+1}(d)}{S_{n+1}(u) - S_{n+1}(d)}.$$

  shares of the stock in the n-th period

- The price Π is a delta-hedge and no-arbitrage price

American option price duality

- Let $\tilde{V}$ be the value process of a super-replicating portfolio, and let V be the value process of the replicating portfolio
- Then, $\tilde{V}_n \geq \Pi_n = V_n$ for all $0 \leq n \leq \tau^*$
- As a result, the price of the American derivative is equal to the initial value of the super-replicating portfolio with the smallest initial value:

$$\Pi_0 = \inf_{\text{SRP } \tilde{V}} \tilde{V}_0,$$

where SRP = super-replicating portfolio

American option price duality

- On the other hand, we know that the price of an American call is also equal to the largest risk-neutral discounted payoff exercised at a stopping time:

$$\Pi = \sup_{\text{ST } \tau} \tilde{\mathbb{E}} \left[e^{-r\tau\Delta t} g(S_\tau) \right],$$

where ST = stopping time

- As a result, the American derivative price satisfies the following duality property:

$$\sup_{\text{ST } \tau} \tilde{\mathbb{E}} \left[e^{-r\tau\Delta t} g(S_\tau) \right] = \Pi = \inf_{\text{SRP } \tilde{V}} \tilde{V}_0$$

- This means that the price of the derivative is equal to the maximum profit a risk-neutral investor expects from the option, and equal to the smallest cost of entering a hedging portfolio

Dividend paying stocks

- If a stock guarantees a dividend payment, then this payment is risk-free in the eye of the investor
- Thus, the investor will remove any dividend payments from her calculations of the risk-neutral probabilities
- If the stock pays a dividend yield of $q\%$, then the correct risk neutral probabilities are:

$$\tilde{p} = \frac{e^{(r-q)\Delta t} - e^{d\Delta t}}{e^{u\Delta t} - e^{d\Delta t}}$$

$$\tilde{q} = 1 - \tilde{p} = \frac{e^{u\Delta t} - e^{(r-q)\Delta t}}{e^{u\Delta t} - e^{d\Delta t}}$$

- For no-arbitrage, we have to assume that $d < r - q < u$

American vs European options

Suppose you have two derivatives with the same payoff function g and the same maturity T. The only difference is that one is of American type (price Π) and one is of European type (price π)

- The European derivative can only be exercised at time $\tau = N$. The American derivative is optimally exercised at time

$$\tau^* = \inf \left\{ \in \{0, 1, \ldots, N\} : g(S_n) \geq \tilde{\mathbb{E}}_n \left[e^{-r\Delta t} \Pi_{n+1} \right] \right\}$$

- Consequently, the price of the American derivative is always higher than the price of the European derivative:

$$\Pi \geq \pi$$

American vs European options

- In any period, the price of the American derivative is higher than its intrinsic value:

$$\Pi_n = \max\left\{g(S_n), \tilde{\mathbb{E}}_n\left[e^{-r\Delta t}\Pi_{n+1}\right]\right\} \geq g(S_n)$$

  This is because the holder of the American derivative could exercise anytime and claim the payoff $g(S_n)$

- The holder of the European derivative can only exercise at maturity. Thus, the n-th period price of the European derivative may be smaller than the intrinsic value of the option
  - You hold a European put that is in the money, but you know that the stock price will rise above the strike by maturity. Then, the option will be worthless as you will not exercise it at maturity, even though the intrinsic value of the option today is positive given that the option is in the money

American vs European options

- An American option holds time value:
  - Intrinsic value: $g(S_n)$
  - Continuation value: $\mathbb{E}_n[e^{-r\Delta t}\Pi_{n+1}]$
  - Option price: $\Pi_n = \max\{g(S_n), \mathbb{E}_n[e^{-r\Delta t}\Pi_{n+1}]\}$
  - Time value: $\Pi_n - g(S_n) \geq 0$
- The time value on an option measures how valuable it is to continue to hold the option. It is always non-negative for American options
- The time value of a European option may become negative
  - In what scenarios can this occur?

American vs European options

- The European derivative can be replicated in the multiperiod binomial model given that markets are complete
- The American derivative can only be replicated if it is exercised according to the optimal exercise rule τ^*. In general, an American derivative can only be super-replicated
- The additional value of a super-replicating portfolio provides a hedge against the risk of a sub-optimal exercise