MF702 PROBLEM SET 5: SOLUTIONS

1. Write an Excel spreadsheet which takes as input any one of the four quantities

(1) Discount Factors : $d(t_j)_{j=1,\dots,J}$
(2) Spot Rates : $\hat{r}(t_j)_{j=1,\dots,J}$
(3) Forward Rates : $f(t_j)_{j=1,\dots,J}$.
(4) Par Rates : $c(t_j)_{j=1,\dots,J}$.

and computes the other three. Above, we always assume that t_j is a multiple of $1/2$. Make sure your spreadsheet plots each of the four quantities.
For input data use the spreadsheet "Rates_Discount_Factors.xlsx". Using actual Treasury yield curve data, this file contains

(a) Spot rates from September 1, 1995.
(b) Discount factors from September 2, 2005.
(c) Par rates from September 3, 2015.

It will be interesting to compare the three sets of rates. Data is given in six month increments from .5 to 20 years. Note that Treasury data is for the 6 month, 1,2,3,5,7,10 and 20 year periods. To obtain the rest of the data, I have used a cubic-spline interpolation.
Note: Ideally, you should write a macro with a command button that automatically calculates the rates/discount factors. However, if you cannot write a macro, it is OK to hard code the formulas into your spreadsheet.

2. (semi-annual) Coupon bonds with a maturity $T = 10$ years, face value $F = \$10,000$, and annual coupon rate $q = 4\%$ are currently trading at $\$9,060$ per bond. Coupon bonds with maturity $T = 10$ years, face value $\$5,000$ and annual coupon rate $q = 8\%$ are currently trading at $\$6,146.50$ per bond. Determine the value of the following securities:

(1) A zero coupon bond with face value $\$20,000$ and maturity 10 years.
(2) An annuity that will make payments of $\$500$ twice per year for the next 10 years.
(3) A coupon bond with maturity 10 years, face value $\$7,500$ and annual coupon rate $q = 6\%$.

Solution

(a) To find the 10 year 0 price for 20,000 face note that

$$\frac{9060}{10000} = 0.02 \times \sum_{i=1}^{20} d(i/2) + d(10),$$
$$\frac{6146.50}{5000} = 0.04 \times \sum_{i=1}^{20} d(i/2) + d(10).$$

This implies

$$d(10) = 2\frac{9060}{10000} - \frac{6146.50}{5000} = 0.5827.$$

Thus, the price for 20,000 face is

$$20,000 \times .5827 = 11,654.$$

(b) For the annuity with amount $A = 500$ paid twice per year we have

$$\mathrm{Px} = 500\left(\sum_{i=1}^{20} d(i/2) = \frac{\frac{9060}{10000} - d(10)}{0.02}\right) = 8,082.5.$$

(c) For the 6% coupon 10 year bond with 7500 face we have

$$\begin{aligned}\mathrm{Px} &= 7500\left(0.03 \times \sum_{i=1}^{20} d(i/2) + d(10)\right),\\ &= 7500\left(\frac{.03}{.02}\left(\frac{9060}{10000} - d(10)\right) + d(10)\right)\\ &= 8007.375.\end{aligned}$$

3. Let $T > \eta > 0$ be given and let $\hat{R}(\eta), \hat{R}(T)$ denote the effective spot rates for dates η and T. Let $R^{\text{for}}_{0,\eta,T}$ denote the effective forward rate agreed upon at time 0 for borrowing between time η and T. Recall that for every dollar borrowed at time η the amount to be repaid at time T is $\left(1 + R^{\text{for}}_{0,\eta,T}\right)^{T-\eta}$. Assume that $\hat{R}(T) > \hat{R}(\eta)$. Show that $R^{\text{for}}_{0,\eta,T} > \hat{R}(T)$.

Solution: Recall how we obtained the effective forward rate $R^{\text{for}}_{0,\eta,T}$: i.e. that 1 dollar at time η is worth $d(\eta)$ dollars today and $(1+R^{\text{for}}_{0,\eta,T})^{T-\eta}$ dollars at time T is worth $(1+R^{\text{for}}_{0,\eta,T})^{T-\eta} d(T)$ dollars today. Thus, to make the agreement worth no money today we must have

$$\left(1 + R^{\text{for}}_{0,\eta,T}\right)^{T-\eta} = \frac{d(\eta)}{d(T)} = \frac{(1+\hat{R}(T))^T}{(1+\hat{R}(\eta))^\eta},$$

where the last equality comes from the definition of the effective spot rate. We thus have

$$\left(1 + R^{\text{for}}_{0,\eta,T}\right)^{T-\eta} = \left(\frac{1+\hat{R}(T)}{1+\hat{R}(\eta)}\right)^{\eta} \left(1 + \hat{R}(T)\right)^{T-\eta}.$$

The conclusion now readily follows.

4. Assume that the spot curve is flat at 6%. A portfolio consists of three zero-coupon bonds each having a face value of $\$1,000,000$. The maturities of the bonds are 2, 5, and 10 years.

(a) Find the DV01 and duration of the portfolio of bonds.
(b) Suppose that you are asked to purchase a single par-coupon bond having maturity of 7 years such that the DV01 of the par-coupon bond will match the DV01 of portfolio of zero-coupon bonds. What should the face of the par-coupon bond be?
(c) Suppose that the 2 year spot rate increases by 50 basis points (bp), the 5 year rate increases by 42 bp and the 10 year rate increases by 38 bp. Compute the exact price change of the portfolio of the zero coupon bonds.
(d) Now assume a *parallel* shift in the (flat) spot curve from parts $(a),(b)$: i.e. all the spot rates move by the same amount. What parallel shift would explain the price change in part (c)? Try to find a shift which exactly explains the price change. Is this possible? What if you use the first order approximation implied by the portfolio's duration? What shift does this yield? Compare the results.

Solution:

(a) For a face F in a T-year zero we know the price is

$$\mathrm{Px} = F\left(1+\frac{\hat{r}(T)}{2}\right)^{-2T} = f(y); \qquad y = \hat{r}_c(T).$$

Thus, for the $2,5$ and 10 year zeros (labeled $\mathbf{B}_1, \mathbf{B}_2$ and $\mathbf{B}_3$ respectively) we have (since $F = 1,000,000$ for each bond and the flat spot curve implies $y = \hat{r}(T) = 0.06$ for each T)

$$\begin{aligned}
\mathrm{Px}(\mathbf{B}_1) &= 1,000,000\,(1+.06/2)^{-4} = 888,487;\\
\mathrm{Px}(\mathbf{B}_2) &= 1,000,000\,(1+.06/2)^{-10} = 744,094;\\
\mathrm{Px}(\mathbf{B}_3) &= 1,000,000\,(1+.06/2)^{-20} = 553,676.
\end{aligned}$$

Using that $DV01 = -(1/10,000)f'(y)$ we see that

$$DV01 = \frac{F}{10,000}\left(\frac{T}{1+y/2}\right)(1+\frac{y}{2})^{-2T},$$

which gives

$$DV01(\mathbf{B}_1) = \frac{1,000,000}{10,000}\frac{2}{1+.06/2}\left(1+\frac{.06}{2}\right)^{-4} = 172.522;$$
$$DV01(\mathbf{B}_2) = \frac{1,000,000}{10,000}\frac{5}{1+.06/2}\left(1+\frac{.06}{2}\right)^{-10} = 361.211;$$
$$DV01(\mathbf{B}_3) = \frac{1,000,000}{10,000}\frac{10}{1+.06/2}\left(1+\frac{.06}{2}\right)^{-20} = 537.549.$$

Denote by X the portfolio. Since DV01 is additive:

$$\begin{aligned} DV01(X) &= DV01(\mathbf{B}_1) + DV01(\mathbf{B}_2) + DV01(\mathbf{B}_3) \\ &= 172.522 + 361.211 + 537.549 = 1071.28. \end{aligned}$$

As for the durations D, recall that $D = 10,000 \times DV01/\mathrm{Px}$. For the zeros this specifies to

$$D = \frac{T}{1+\hat{r}(T)/2}.$$

Using this formula and that $\hat{r}(T) = y = .06$ for each T we have

$$D(\mathbf{B}_1) = \frac{2}{1+.06/2} = 1.94175;$$
$$D(\mathbf{B}_2) = \frac{5}{1+.06/2} = 4.85437;$$
$$D(\mathbf{B}_3) = \frac{10}{1+.06/2} = 9.70874.$$

For the portfolio (yield) duration we have that

$$\mathrm{Px}(X) = P(\mathbf{B}_1) + P(\mathbf{B}_2) + P(\mathbf{B}_3) = 2,186,256,$$

so that (since the flat spot curve implies the yield of X is $y = \hat{r}(T) = 0.06$)

$$D(X) = -\frac{1}{X}\frac{\Delta X}{\Delta y} = -\sum_{i=1}^{3}\frac{1}{X}\frac{\Delta P(\mathbf{B}_i)}{\Delta y} = \sum_{i=1}^{d}\frac{\mathrm{Px}(\mathbf{B}_i)}{X}D(\mathbf{B}_i).$$

Using the above prices and durations we see that $D(X) = 4.90007$.

(b) For a par coupon bond with maturity T_0 we can relate the DV01 in terms of the yield y and face F by

$$DV01 = \frac{F}{10,000}\frac{1}{y}\left(1 - \frac{1}{(1+y/2)^{2T_0}}\right).$$

Since the spot curve is flat at $\hat{r}(T) = y = 0.06$ for all T we know the yield of the par bond is y and hence with $\mathbf{B}$ being the 7 year par bond that

$$DV01(\mathbf{B}) = 0.000564804 \times F.$$

We thus want

$$0.000564804 \times F = 1071.28 \Longrightarrow F = 1,896,729.9.$$

(c) Under the given rate changes we have that $\hat{r}(2) = .065$, $\hat{r}(5) = .0642$ and $\hat{r}(10) = 0.0638$. Using these values (which are still the respective yields since we have zeros) we obtain

$$\mathrm{Px}(\mathbf{B}_1) = 1,000,000 \left(1 + \frac{0.065}{2}\right)^{-4} = 879,913;$$
$$\mathrm{Px}(\mathbf{B}_2) = 1,000,000 \left(1 + \frac{0.0642}{2}\right)^{-10} = 729,092;$$
$$\mathrm{Px}(\mathbf{B}_3) = 1,000,000 \left(1 + \frac{0.0638}{2}\right)^{-20} = 533,639.$$

The portfolio value is thus

$$\mathrm{Px}(X) = \mathrm{Px}(\mathbf{B}_1) + \mathrm{Px}(\mathbf{B}_2) + \mathrm{Px}(\mathbf{B}_3) = 2,142,644.$$

The price change is thus

$$\Delta\mathrm{Px}(X) = 2,142,644 - 2,286,256 = -43,612.$$

(d) Now, assume a parallel shift in the spot curve so all the spot rates have moved from 0.06 to some level y. We are thus trying to find y so the portfolio price $\mathrm{Px}(X) = 2,142,644$. Using the formula for the zero price above we want y so that

$$\begin{aligned} 2,142,644 &= \mathrm{Px}(\mathbf{B}_1) + \mathrm{Px}(\mathbf{B}_2) + \mathrm{Px}(\mathbf{B}_3), \\ &= 1,000,000 \left((1 + y/2)^{-4} + (1 + y/2)^{-10} + (1 + y/2)^{-20}\right). \end{aligned}$$

Solving this for y we see that $y = 0.0641327$. Thus an upward parallel shift of 41.3 basis points will yield the same price. Using the first order approximation based on the portfolios duration, we are looking for Δy so that

$$\Delta\mathrm{Px}(X) = -D(X)\mathrm{Px}(X)\Delta y.$$

Thus, using the original $\mathrm{Px}(X) = 2,286,256$, duration $D(X) = 4.90007$ and $\Delta\mathrm{Px}(X) = -43,612$ we see that

$$\Delta y = .004071 \Longrightarrow y = 0.0641007.$$

Thus, the first order approximation does well in suggesting that an 40.7 bp upwards parallel shift explains the actual price change, when the exact parallel shift was an upwards shift of 41.3 bp.

5. Consider a zero-coupon bond with face value \$10,000 and maturity 10 years. Assume that $\hat{r}(10) = 4.8736\%$.

(a) Compute the convexity of the bond.

(b) Compute the exact price change in the bond corresponding to a 35 bp increase in $\hat{r}(10)$.

(c) Compute the first-order approximation to the price change in the bond for a 35 bp increase in $\hat{r}(10)$.

(d) Compute the second-order approximation to the price change in the bond for a 35 bp increase in $\hat{r}(10)$.

Solution: Recall for F face in a T year zero the price is $\mathrm{Px} = F(1+\hat{r}_c(T)/2)^{-2T} = Ff(y)$ for $y = \hat{r}_c(T)$. The duration is

$$D = -\frac{\dot{f}(y)}{f(y)} = \frac{T}{1+y/2},$$

and convexity is

$$C = \frac{\ddot{f}(y)}{f(y)} = \frac{T^2 + T/2}{(1+y/2)^2}.$$

(a) For $\hat{r}(10) = y = 4.8736\%$ and $T = 10$ we have

$$C = \frac{105}{(1+.048736/2)^2} = 100.064.$$

(b) If $\hat{r}(10)$ moves from 4.8736% to 5.2236% the price of $10,000$ face of the zero moves by

$$\mathrm{Px}(4.8736\%) = 10000\left(1+\frac{.048736}{2}\right)^{-20} = 6178.46;$$
$$\mathrm{Px}(5.2236\%) = 10000\left(1+\frac{.052236}{2}\right)^{-20} = 5971.09,$$

so the exact price change is $\Delta\mathrm{Px} = 5971.09 - 6178.46 = -207.362$.

(c) The first order approximation to the price change is $\Delta\mathrm{Px} = -D\mathrm{Px}\Delta y$ where use the duration and price as implied by the original spot rate $y = \hat{r}(10) = 0.048736$. Thus, we obtain

$$D = \frac{10}{1+.048736/2} = 9.76212,$$

and so, using $\mathrm{Px} = 6178.46$, $\Delta y = .0035$ we obtain $\Delta\mathrm{Px} = -211.102$, which is not too far from the exact price change of -207.362.

(d) The second order approximation to the price change is

$$\Delta\mathrm{Px} = \mathrm{Px}\left(-D\Delta y + (1/2)C(\Delta y)^2\right).$$

From the above we have that $\mathrm{Px} = 6178.46$, $\Delta y = 0.0035$, $D = 9.76212$ and $C = 100.064$ and thus we obtain $\Delta\mathrm{Px} = -207.315$. This value is very close to the exact price change of -207.362.

6. Suppose that the zero-coupon yield curve is upward sloping. In particular we have $\hat{r}(t+.5) \geq \hat{r}(t)$ for $t = .5, 1, \ldots, 10$. Consider the following securities

- **A**: a zero-coupon bond with maturity 10 years.
- **B**: an annuity with maturity 5 years.
- **C**: an annuity with maturity 10 years.
- **D**: a coupon bond with maturity 10 years.

If possible, order these securities by yield to maturity from lowest to highest. Give a BRIEF explanation of your ordering. If it is not possible to order the yields (or if more information is needed), give an explanation why not (including what additional assumptions you may need to provide an ordering).

Solution: We have

$$y_{\mathbf{B}} \leq y_{\mathbf{C}} \leq y_{\mathbf{D}} \leq y_{\mathbf{A}}.$$

First, $y_A = \hat{r}(10)$. We claim that $y_D < \hat{r}(10)$, hence $y_D < y_A$. To this end, we have

$$\frac{P_D}{F} = \frac{q}{2}\sum_{i=1}^{20} \frac{1}{\left(1+\frac{\hat{r}(i/2)}{2}\right)^i} + \frac{1}{\left(1+\frac{\hat{r}(10)}{2}\right)^i} = \frac{q}{2}\sum_{i=1}^{20} \frac{1}{(1+y_D/2)^i} + \frac{1}{(1+y_D/2)^{20}}.$$

If $y_D = \hat{r}(10)$, because $\hat{r}(10)$ is larger than all other short rates, we must have

$$\frac{q}{2}\sum_{i=1}^{20} \frac{1}{(1+y_D/2)^i} + \frac{1}{(1+y_D/2)^{20}} > \frac{q}{2}\sum_{i=1}^{20} \frac{1}{\left(1+\frac{\hat{r}(i/2)}{2}\right)^i} + \frac{1}{\left(1+\frac{\hat{r}(10)}{2}\right)^i} = \frac{P_D}{F}.$$

This means that $y_D = \hat{r}10$ introduces too much discounting. Therefore, $y_D < \hat{r}(10) = y_A$.

We next claim that for an annuity B with $T \leq 10$ years it holds that $y_B \leq \hat{r}(10) = y_{\mathbf{A}}$. Indeed, by definition of the annuity yield

$$\sum_{i=1}^{2T}(1+y_B/2)^{-i} = \sum_{i=1}^{2T}(1+\hat{r}(i/2)/2)^{-i} \geq \sum_{i=1}^{2T}(1+\hat{r}(10)/2)^{-i},$$

where the inequality follows because $\hat{r}(i/2) \leq \hat{r}(10)$ for $i = 1, ..., 2T \leq 20$. Since the map $y \mapsto \sum_{i=1}^{2T}(1+y/2)^{-i}$ is decreasing in y it follows that $y_B \leq \hat{r}(10) = y_{\mathbf{A}}$.

Next, we claim that for annuities B_1, B_2 with respective maturities T_1, T_2 that if $T_1 < T_2$

then $y_{B_1} \leq y_{B_2}$. Indeed, we have

$$\begin{aligned}
\sum_{i=1}^{2T_2}(1+y_{B_2}/2)^{-i} &= \sum_{i=1}^{2T_2}(1+\hat{r}(i/2)/2)^{-i};\\
&= \sum_{i=1}^{2T_1}(1+\hat{r}(i/2)/2)^{-i} + \sum_{i=2T_1+1}^{2T_2}(1+\hat{r}(i/2)/2)^{-i};\\
&= \sum_{i=1}^{2T_1}(1+y_{B_1}/2)^{-i} + \sum_{i=2T_1+1}^{2T_2}(1+\hat{r}(i/2)/2)^{-i},
\end{aligned}$$

where the last equality follows by definition of the annuity yield y_{B_1} for the T_1 year annuity. Now, we just showed above that $y_{B_1} \leq \hat{r}(T_1)$ so that

$$y_{B_1} \leq \hat{r}(T_1) \leq \hat{r}(i/2),\ i = 2T_1+1, ..., 2T_2.$$

Using this above we have

$$\begin{aligned}
\sum_{i=1}^{2T_2}(1+y_{B_2}/2)^{-i} &\leq \sum_{i=1}^{2T_1}(1+y_{B_1}/2)^{-i} + \sum_{i=2T_1+1}^{2T_2}(1+y_{B_1}/2)^{-i},\\
&= \sum_{i=1}^{2T_2}(1+y_{B_1}/2)^{-i}.
\end{aligned}$$

But, this implies $y_{B_1} \leq y_{B_2}$, which is what we wanted to show.

Using the three above results we can now say

$$y_{\mathbf{D}} \leq y_{\mathbf{A}}; \qquad y_{\mathbf{B}} \leq y_{\mathbf{C}} \leq y_{\mathbf{A}}.$$

It remains to show $y_{\mathbf{D}} \geq y_{\mathbf{C}}$. To see this, note that

$$\begin{aligned}
\frac{q}{2}\sum_{i=1}^{20}(1+\hat{r}(i/2)/2)^{-i} &= \frac{q}{2}\sum_{i=1}^{20}(1+y_{\mathbf{D}}/2)^{-i} + (1+y_{\mathbf{D}}/2)^{-20} - (1+\hat{r}(10)/2)^{-20};\\
&\geq \frac{q}{2}\sum_{i=1}^{20}(1+y_{\mathbf{D}}/2)^{-i},
\end{aligned}$$

where the last inequality followed since $y_{\mathbf{D}} \leq \hat{r}(10)$. Dividing the above by $q/2$ we see that

$$\sum_{i=1}^{20}(1+y_{\mathbf{D}}/2)^{-i} \leq \sum_{i=1}^{20}(1+\hat{r}(i/2)/2)^{-i} = \sum_{i=1}^{2T}(1+y_{\mathbf{C}}/2)^{-i}$$

where the last equality follows by definition of $y_{\mathbf{C}}$. Thus, we have $y_{\mathbf{D}} \geq y_{\mathbf{C}}$.

7. Assume that the spot curve is flat at some level $y > 0$. Let η denote the Macauley duration of a 10 year par-coupon bond (η is assumed to be a multiple of six months). Consider the following securities:

- **A**: a 10 year premium bond.
- **B**: a zero-coupon bond with maturity 10.
- **C**: a zero-coupon bond with maturity η.

If possible, order these securities by their Macauley duration from largest to smallest. Given a brief explanation of your reasoning. If it is not possible to order the securities by duration based upon the given information, please explain why not, including any additional assumptions you would need to provide the ordering.

Solution: We have

$$D_{\text{mac}}(\mathbf{A}) \leq D_{\text{mac}}(\mathbf{C}) \leq D_{\text{mac}}(\mathbf{B}).$$

To see this, recall that we showed that for a fixed yield y (which is the common value of the spot curve here) and maturity T that $D_{\text{mac}}(q, y, T)$ is decreasing in q. Now, for a premium bond we know the coupon q is above the yield y so there fore $D_{\text{mac}}(\mathbf{A}) \leq \eta$, which was the Macauley duration for a 10 year par bond. But, by definition of Macauley duration we have that $D_{\text{mac}}(\mathbf{C}) = \eta$ and $D_{\text{mac}}(\mathbf{B}) = 10$. It thus remains to show that $\eta \leq 10$. But this follows because

$$D_{\text{mac}}(10 \text{ year par bond}) = \eta = \sum_{i=1}^{20} T_i w_i,$$

where $T_i = i/2$ and, since for par bonds the coupon q is the yield y (which is also the common spot rate),

$$w_i = \frac{F_i/(1+y/2)^i}{\sum_{j=1}^{20} F_j/(1+y/2)^j}; \qquad F_i = Fy/2(i = 1, ..., 19), F_{20} = F(1+y/2).$$

The only way η could be 10 is if $w_i = 0, i = 1, ..., 19$ and $w_{20} = 1$, but this evidently cannot happen since $y > 0$. Thus, $\eta \leq 10$.