Wednesday, 9 December 2020
08:30

HW6

1. $$\max_{\Delta} \frac{\Delta^T \vec{r}}{\sqrt{\Delta^T \Sigma \Delta}}, \qquad \Delta^T \vec{1} = 1$$

$$\| $$

$$\sum_{i=1}^{M} \Delta_i = 1$$

unconstrained problem

$$\max_{\Delta, \lambda} \frac{\Delta^T \vec{r}}{\sqrt{\Delta^T \Sigma \Delta}} + \lambda(\Delta^T \vec{1} - 1)$$

1

First order condition.

with respect to Δ^T : $$\frac{\vec{r}}{\sqrt{\Delta^T \Sigma \Delta}} - \frac{\Delta^T \vec{r}\ \cancel{\tfrac{1}{2}}\,\cancel{2}\, \Sigma \Delta}{(\Delta^T \Sigma \Delta)^{\frac{3}{2}}} + \lambda \vec{1} = 0$$

withrespet to λ : $\Delta^T \vec{1} = 1$

$$\frac{\vec{r}}{\sqrt{\Delta^T \Sigma \Delta}} - \frac{\Delta^T \vec{r}\ \Sigma \Delta}{(\Delta^T \Sigma \Delta)^{\frac{3}{2}}} = -\lambda \vec{1}$$

Left multiply Δ^T on both sides

$$\frac{\Delta^T \vec{r}}{(\Delta^T \Sigma \Delta)^{\frac{1}{2}}} - \frac{\Delta^T \vec{r}\ \cancel{\Delta^T \Sigma \Delta}}{(\Delta^T \Sigma \Delta)^{\cancel{\frac{3}{2}}\frac{1}{2}}} = -\lambda \Delta^T \vec{1}$$

$$\underbrace{\hspace{6em}}_{=0}$$

$$\Rightarrow \lambda = 0.$$

$$\Rightarrow \quad \vec{r} = \frac{\Delta^T \vec{r}}{\Delta^T \Sigma \Delta} \Sigma \Delta$$

$$\vec{r} = c \Sigma \Delta$$

$$\Rightarrow \quad \Delta = \frac{1}{c} \Sigma^{-1} \vec{r}$$

plug Δ back into constrain

$$\Delta^T \vec{1} = \frac{1}{c} \vec{r}^T \Sigma^{-1} \vec{1} = 1$$

$$\Rightarrow c = \vec{r}^T \Sigma^{-1} \vec{1} = \vec{1}^T \Sigma^{-1} \vec{r}$$

$$\Rightarrow \Delta^*_{MSR} = \frac{\Sigma^{-1} \vec{r}}{\vec{1}^T \Sigma^{-1} \vec{r}}$$

(b) $\min \Delta^T \Sigma \Delta, \qquad \Delta^T \vec{1} = 1$

un constrained problem

$$\min_{\Delta, \lambda} \ \Delta^T \Sigma \Delta + \lambda(\Delta^T \vec{1} - 1)$$

FOC

$$\Delta^T : \quad 2\Sigma\Delta + \lambda\vec{1} = 0$$

$$\lambda : \quad \Delta^T \vec{1} = 1$$

$$\Delta = -\frac{\lambda}{2}\Sigma^{-1}\vec{1}$$

plug into the constrain

$$\Delta^T \vec{1} = -\frac{\lambda}{2} \vec{1}^T \Sigma^{-1} \vec{1} = 1$$

$$\Rightarrow -\frac{\lambda}{2} = \frac{1}{\vec{1}^T \Sigma^{-1} \vec{1}}$$

$$\Rightarrow \Delta^*_{GMV} = \frac{\Sigma^{-1}\vec{1}}{\vec{1}^T \Sigma^{-1} \vec{1}}$$

2. $r_f = 0.02$ $\qquad \vec{r}_e = \vec{r} - r_f \vec{1}$

(a) $\Delta_{MSR} = \dfrac{\Sigma^{-1} \vec{r}_e}{\vec{1}^T \Sigma^{-1} \vec{r}_e} = \begin{pmatrix} 0.5603 \\ 0.6886 \\ -0.2489 \end{pmatrix}$

$\bar{r}_{MSR} = \Delta_{MSR}^T \vec{r}$

(b) $\bar{r} = 0.20 \qquad \partial^*$

$$\bar{r} = \bar{r}(\partial) = (1-\partial)\, r_f + \partial\, \bar{r}_{MSR}$$

$$= r_f + \partial\,(\underbrace{\bar{r}_{msr} - r_f}_{\bar{r}_{MSR,e}})$$

$$\bar{r}_e = \bar{r} - r_f = \partial\,(\bar{r}_{MSR,e})$$

$$\Rightarrow \partial = \frac{\bar{r}_e}{\bar{r}_{MSR}} = 3.09$$

3. M assets, $\vec{r} = (\bar{r}_1, \cdots, \bar{r}_M)$ $\quad \Sigma$

I index, $\quad r_I, \quad \sigma_I^2$

cov is given as γ

portfolio.

(a) $\max_{\Delta} \operatorname{Corr}(r(\Delta), r_I)$, $\Delta^T \vec{1} = 1$

$$\operatorname{Corr}(r(\Delta), r_I) = \frac{\operatorname{cov}(\Delta^T \vec{r}, r_I)}{\operatorname{Var}(\Delta^T \vec{r})^{\frac{1}{2}} \sigma_I}$$

$$= \frac{\Delta^T \operatorname{cov}(\vec{r}, r_I)}{\sigma_I (\Delta^T \Sigma \Delta)^{\frac{1}{2}}}$$

$$\operatorname{Var}(\Delta^T \vec{r}) = \Delta^T \Sigma \Delta$$

$$= \frac{\Delta^T \gamma}{\sigma_I (\Delta^T \Sigma \Delta)^{\frac{1}{2}}} \quad \gamma \in \mathbb{R}^{\vec{r}}$$

$$\operatorname{Var}(\Delta^T \vec{r}) = \mathbb{E}[\Delta^T (\vec{r} - \bar{r})(\vec{r} - \bar{r})^T \Delta]$$

$$= \Delta^T \underbrace{\mathbb{E}[(\vec{r} - \bar{r})(\vec{r} - \bar{r})^T]}_{\Sigma} \Delta$$

using (a) in problem 1.

$$\Delta_{corr, I} = \frac{\Sigma^{-1} \gamma}{\vec{1}^T \Sigma^{-1} \gamma}$$

(b) $\operatorname{Var}(r(\Delta) - r_I)$, $\bar{r}(\Delta) = \bar{r}$, $\Delta^T \vec{1} = 1$

$$= \underset{\substack{\| \\ \Delta^T \Sigma \Delta}}{Var(r(\Delta))} - \underbrace{2\, cov(r(\Delta), r_I)}_{2\Delta^T \gamma} + \underset{\substack{\| \\ \sigma_I^2}}{Var(r_I)}$$

unconstrained problem

$$\min_{\Delta, \lambda_1, \lambda_2} \Delta^T \Sigma \Delta - 2\Delta^T \gamma + \lambda_1 (\Delta^T \vec{r} - \bar{r}) + \lambda_2 (\Delta^T \vec{1} - 1)$$

FOC:

$$\Delta^T: \quad 2\Sigma\Delta - 2\gamma + \lambda_1 \vec{r} + \lambda_2 \vec{1} = 0$$

$$\Delta = \Sigma^{-1}\gamma - \frac{\lambda_1}{2}\Sigma^{-1}\vec{r} - \lambda_2 \Sigma^{-1}\vec{1}$$

$$= \underset{\substack{\| \\ \partial_{c,I}}}{\vec{1}^T\Sigma^{-1}\gamma} \underbrace{\frac{\Sigma^{-1}\gamma}{\vec{1}^T\Sigma^{-1}\gamma}}_{\Delta^*_{corr,I}} - \underbrace{\frac{\lambda_1}{2}\vec{1}^T\Sigma^{-1}\gamma}_{\partial_M} \underbrace{\frac{\Sigma^{-1}\vec{r}}{1^T\Sigma^{-1}\vec{r}}}_{\Delta^*_{MSR}} - \underbrace{\frac{\lambda_2}{2}\vec{1}^T\Sigma^{-1}\vec{1}}_{\partial_G} \underbrace{\frac{\Sigma^{-1}\vec{1}}{\vec{1}^T\Sigma^{-1}\vec{1}}}_{\Delta^*_{GMV}}$$

$$\Delta = \partial_{c,I}\Delta^*_{corr,I} + \partial_M \Delta^*_{MSR} + \partial_G \Delta^*_{GMV}$$

$$\Delta^T \vec{r} = \bar{r}$$

$$\bar{r} = \partial_{c,I}\bar{r}_{corr,I} + \underline{\partial_M}\,\bar{r}_{MSR} + \underline{\partial_G}\,\bar{r}_{GMV}$$

$$\Delta^T \vec{1} = 1$$

$$1 = \partial_{c,I}\, 1 + \partial_M\, 1 + \partial_G\, 1$$