## MF702 PROBLEM SET 6: SOLUTION

**1.** Consider $M$ assets whose return is a $M$-dimensional random vector $r$. The expected value of $r$ is a $M$-dimensional column vector $\overrightarrow{r}$ and the covariance matrix is $\Sigma$.

(a) The Maximum Sharpe Ratio portfolio $\Delta_{MSR}$ solves

$$\max_{\Delta} \frac{\Delta^\top \overrightarrow{r}}{\sqrt{\Delta^\top \Sigma \Delta}}, \quad \text{s.t. } \Delta^\top \overrightarrow{1} = 1.$$

Use the Lagrange multiplier method to show that $\Delta_{MSR} = \frac{\Sigma^{-1}\overrightarrow{r}}{\overrightarrow{1}\Sigma^{-1}\overrightarrow{r}}$.

(b) The Global Minimum Variance portfolio $\Delta_{GMV}$ solves

$$\min_{\Delta} \Delta^\top \Sigma \Delta, \quad \text{s.t. } \Delta^\top \overrightarrow{1} = 1.$$

Show that $\Delta_{GMV} = \frac{\Sigma^{-1}\overrightarrow{1}}{\overrightarrow{1}^\top\Sigma^{-1}\overrightarrow{1}}$.

**Solution:**

(a) Let us consider the unconstrained problem

$$\max_{\Delta,\lambda} \frac{\Delta^\top \overrightarrow{r}}{\sqrt{\Delta^\top \Sigma \Delta}} - \lambda\big(\Delta^\top \overrightarrow{1} - 1\big).$$

Take the first order condition, we obtain

$$\begin{aligned} &\frac{\overrightarrow{r}}{(\Delta^\top\Sigma\Delta)^{\frac{1}{2}}} - \frac{\Delta^\top\overrightarrow{r}\Sigma\Delta}{(\Delta^\top\Sigma\Delta)^{\frac{3}{2}}} = \lambda\overrightarrow{1}, \\ &\Delta^\top\overrightarrow{1} = 1. \end{aligned}$$

From the first equation above, left multiply by $\Delta^\top$, we observe the right-hand side is zero, which means $\lambda = 0$. Then the first equation yields

$$\overrightarrow{r} = \frac{\Delta^\top\overrightarrow{r}}{\Delta^\top\Sigma\Delta}\Sigma\Delta.$$

Note that $\frac{\Delta^\top\overrightarrow{r}}{\Delta^\top\Sigma\Delta}$ is a constant, rename is as $c$. Then $\overrightarrow{r} = c\Sigma\Delta$, which implies

$$\Delta = \frac{1}{c}\Sigma^{-1}\overrightarrow{r}.$$

To satisfy $\Delta^\top\overrightarrow{1} = 1$, we need $c = \overrightarrow{1}^\top\Sigma^{-1}\overrightarrow{r}$. Therefore

$$\Delta = \frac{\Sigma^{-1}\overrightarrow{r}}{\overrightarrow{1}^\top\Sigma^{-1}\overrightarrow{r}}.$$

(b) Consider the unconstrained problem

$$min_{\Delta,\lambda}\Delta^\top\Sigma\Delta + \lambda\big(\Delta^\top\overrightarrow{1} - 1\big).$$

Take the first order condition, we obtain

$$\begin{aligned} 2\Sigma\Delta &= -\lambda\vec{1}, \\ \Delta^\top\vec{1} &= 1. \end{aligned}$$

The first equation implies

$$\Delta = -\lambda\Sigma^{-1}\vec{1}.$$

In order to satisfy the second equation, we need $-\lambda = \left((\vec{1})^\top\Sigma^{-1}\vec{1}\right)^{-1}$. Therefore

$$\Delta = \frac{\Sigma^{-1}\vec{1}}{\vec{1}^\top\Sigma^{-1}\vec{1}}.$$

**2.** You can invest in three assets with expected returns and return covariance matrix given by

$$\vec{r} = \begin{pmatrix} 0.10 \\ 0.09 \\ 0.16 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.09 & -0.03 & 0.084 \\ -0.03 & 0.04 & 0.012 \\ 0.084 & 0.012 & 0.16 \end{pmatrix}.$$

You can also invest in the risk-free asset with return $r_f = 0.02$.

(a) What is the market portfolio ($\Delta_{MSR}$) in this setting? Compute the optimal weights invested in each risky security for this portfolio.

(b) Find the minimum-volatility portfolio which achieves an expected return of $\bar{r} = 0.20$.

**Solution:**

(a) The market portfolio is the Maximum Sharpe Ratio portfolio with respect to the excess return, and takes the form

$$\Delta_{MSR,e} = \frac{\Sigma^{-1}\vec{r}_e}{\vec{1}^T\Sigma^{-1}\vec{r}_e}.$$

Using the given values, a direct calculation gives

$$\Delta_{MSR,e} = \begin{pmatrix} 0.5603 \\ 0.6886 \\ -0.2489 \end{pmatrix}.$$

(b) Using the above values, the excess return of the market portfolio is

$$\Delta_{MSR,e}^\top\vec{r}_e = 0.058178,$$

while the target excess return is $\bar{r}_e = \bar{r} - r_f = 0.18$. Thus,

$$\alpha^*(\bar{r}) = \frac{\bar{r}_e}{\Delta_{MSR,e}^\top\vec{r}_e} = 3.09395.$$

Therefore the minimum volatility portfolio which achieves the target return is

$$\Delta^*(\bar{r}) = \alpha^*(\bar{r})\Delta_{MSR,e} = 3.0935 \begin{pmatrix} 0.5603 \\ 0.6886 \\ -0.2489 \end{pmatrix} = \begin{pmatrix} 1.7335 \\ 2.1306 \\ -0.7702 \end{pmatrix}.$$

The residual fraction $1 - \alpha^*(\bar{r}) = -0.209395$ in put into the ZCB. Thus we are highly leveraged in the market portfolio, with a significant short position in the ZCB.

**3.** (ETF construction) Often we wish to track an index $I$, but for practical reasons (such as wishing to avoid transactions costs) we cannot invest in all the securities which comprise $I$. In this setting, we wish to construct a portfolio that mimics the index most closely, in the sense of minimizing the tracking error, while still maintaining a target level of return.

More precisely, suppose we may invest in a subset of the index, comprising of $M$ stocks. The stocks have (random) returns $r_1, \dots, r_M$ with expected values $\overrightarrow{r} = (\bar{r}_1, \dots, \bar{r}_M)$ and covariance matrix $\Sigma = \{\Sigma_{mk}\}_{m,k=1}^M$ where $\Sigma_{mk} = Cov(r_m, r_k)$. On the other hand, the index $I$ has random return $r_I$ with expected value $\bar{r}_I$ and variance $\sigma_I^2$. We are able to estimate the covariance of the index and stock returns, obtaining the vector $\Upsilon = \{\Upsilon_m\}_{m=1}^M$, $\Upsilon_m = Cov(r_m, r_I)$. Lastly, we assume the investor may not allocate money to the ZCB.

We wish to find the portfolio weights $\Delta = (\Delta_1, \dots, \Delta_M)$ whose random return $r(\Delta)$ minimize

$$Var(r(\Delta) - r_I), \tag{1}$$

still yielding an expected return of $\bar{r}(\Delta) = \bar{r}$, where $\bar{r}$ is the target return.

(a) Consider the (related) problem of trying to find the portfolio which is most closely correlated with the index, but has no opinion about the expected return: i.e.

$$\max_{\Delta} Corr(r(\Delta), r_I), \quad \text{s.t. } \Delta^\top \overrightarrow{1} = 1.$$

Explicitly identify the optimal portfolio $\Delta_{Corr,I}$.

(b) Now, come back to the problem in (1):

$$Var(r(\Delta) - r_I), \quad \text{s.t.} \bar{r}(\Delta) = \bar{r}, \Delta^\top \overrightarrow{1} = 1.$$

Show that the optimal portfolio admits the decomposition

$$\Delta^* = \alpha_{C,I}\Delta_{Corr,I} + \alpha_M \Delta_{MSR} + \alpha_G \Delta_{GMV}.$$

Explicitly identify the constants $\alpha_{C,I}$, $\alpha_M$, and $\alpha_G$.

**Solution:**

(a) Note that

$$Corr(r(\Delta), r_I) = \frac{Cov(r(\Delta), r_I)}{\sigma_I \sqrt{Var(r(\Delta))}} = \frac{\Delta^\top \Upsilon}{\sigma_I \sqrt{\Delta^\top \Sigma \Delta}},$$

where the equalities follow since $Cov(r(\Delta), r_I) = \sum_{m=1}^{M} \Delta_m Cov(r_m, r_I)$, and because $Var(r(\Delta)) = \Delta^\top \Sigma \Delta$. Thus, as $\sigma_I$ plays no role, we see that the problem of finding the maximal correlated portfolio is, up to the notational difference of having $\Upsilon$ instead of $\overrightarrow{r}$, the same as Problem 1 (a). Therefore, we already know the answer is

$$\Delta_{corr,I} = \frac{\Sigma^{-1}\Upsilon}{\overrightarrow{1}^\top \Sigma^{-1} \Upsilon}.$$

(b) First, note that

$$Var(r(\Delta) - r_I) = Var(r(\Delta)) + \sigma_I^2 - 2Cov(r(\Delta), r_I) = \Delta^\top \Sigma \Delta + \sigma_I^2 - 2\Delta^\top \Upsilon.$$

Thus, using Lagrange multipliers, and noting that the term $\sigma_I^2$ does not depend on $\Delta$, the unconstrained problem is

$$\min_{\Delta, \lambda_1, \lambda_2} \Delta^\top \Sigma \Delta - 2\Delta^\top \Upsilon - \lambda_1(\Delta^\top \overrightarrow{r} - \bar{r}) - \lambda_2(\Delta^\top \overrightarrow{1} - 1).$$

The first order conditions with respect to $\Delta$ is

$$\begin{aligned} \Delta^* =& \Sigma^{-1}\Upsilon + \frac{\lambda_1}{2}\Sigma^{-1}\overrightarrow{r} + \frac{\lambda_2}{2}\Sigma^{-1}\overrightarrow{1} \\ =& \alpha_{C,I}\Delta_{Corr,I} + \alpha_M \Delta_{MSR} + \alpha_G \Delta_{GMV}, \end{aligned}$$

where

$$\alpha_{C,I} = \overrightarrow{1}^\top \Sigma^{-1} \Upsilon, \quad \alpha_M = \frac{\lambda_1}{2} \overrightarrow{1}^\top \Sigma^{-1} \overrightarrow{r}, \quad \alpha_G = \frac{\lambda_2}{2} \overrightarrow{1}^\top \Sigma^{-1} \overrightarrow{1}.$$

As $\alpha_{C,I}$ si explicitly known, it remains to find $\alpha_M, \alpha_G$, which enforce the target return and fully invested restrictions:

$$\begin{aligned} 1 =& \alpha_{C,I} + \alpha_M + \alpha_G, \\ \bar{r} =& \alpha_{C,I}\bar{r}_{corr,I} + \alpha_M \bar{r}_{MSR} + \alpha_G \bar{r}_{GMV}. \end{aligned}$$

From above equations, we can solve for

$$\begin{aligned} \alpha_M =& \frac{\bar{r} - \bar{r}_{GMV} - \alpha_{C,I}(\bar{r}_{corr,I} - \bar{r}_{GMV})}{\bar{r}_{MSR} - \bar{r}_{GMV}}, \\ \alpha_G =& \frac{\bar{r}_{MSR} - \bar{r} - \alpha_{C,I}(\bar{r}_{MSR} - \bar{r}_{corr,I})}{\bar{r}_{MSR} - \bar{r}_{GMV}}. \end{aligned}$$