Sensitivity Analysis and Hedging IV
- Multiple Interest Rate Factors.

Basic Idea.

So, far, we have considered interest rate sensitivities with respect to a single interest rate factor.

e.g. $y_0 = YTM = \hat{r}(T)$ for a T-year

We then built up a way to construct portfolios of zeros which are price and duration matched. Those portfolios then display roughly the same price sensitivity to parallel shifts in the spot curve.

I.e. if $\hat{r}(T) \mapsto \hat{r}(T) + \Delta y \quad \forall T$

then $\Delta Px(\text{Port } A) \approx \Delta Px(\text{Port } B)$, with small adjustments/differences arising due to e.g. convexity differences.

Inherently there is a problem with this though: it assumes that movements in the spot curve are perfectly (positively) correlated.

- more generally that all rate movements can be described by a single factor.

We now wish relax this assumption by studying price sensitivities to changes in multiple rates at the same time.

~ Quick Aside:

- when we yield based duration matched portfolios of zeros, and hence first order hedged against parallel shifts in the spot curve, we actually provided a way to do the same for coupon bonds as well.

Why? B/c we can think of a coupon bond as a portfolio of zeros.

E.g. 100M face of a 9-coupon bond paying $9/2$ at $t = 1/2, 1, 3/2, \ldots 2$ and also 1 at $t=2$.

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Cashflows (9 %)

$t = 1/2$	$t = 1$	$t = 3/2$	$t = 2$
$100 \cdot M \times \frac{9}{200}$	$100M \times 9/200$	$100M \times 9/200$	$100M \times (1 + 9/200)$

we can think of this as a portfolio of securites S^i with face F^i for

$$S^i : i/2 \text{ yr } 0 \qquad i = 1, \ldots, 4$$

$$F^i = 100M \cdot 9/200 \qquad i = 1, 2, 3$$

$$F^4 = 100M(1 + 9/200)$$

so, in theory if we yield-based duration match any portfolios A, B of zeros, we have done so for any portfolios A, B of coupon bonds

Problem: for reasonably large portfolios we have payments at all times $t = i/2$, $i = 1, \ldots, N$

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so, we are essentially duration matching the whole spot curve

- matching each cash flow using appropriate zero is an example of <u>immunization</u> and ~~it~~ is often ~~prac~~ impractical/ prohibitively expensive.

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Multiple factors

Since full immunization is not realistic, we instead focus on hedging against a few key-rate shifts

i.e. we pick a few (3 or 4) <u>key</u> interest rates $y_1, \ldots, y_N$ and think of bond prices as functions of those factors

$$Px(\text{Bond}) = f(y_1, \ldots, y_4)$$

<u>Q.</u>

What rates?

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It depends a bit on what our hedging instruments are:

- hedge with zeros → natural to use spot rates
- hedge with par bonds ↦ natural to use par rates.

To mix things up, let's use par-rates, or par yields.

E.g. we decide the key rates are the 2, 5, 10 and 30 year par yields.

- label y_1, y_2, y_3, y_4.

For an abstract fixed income security with deterministic cash flows, we write

$$Px = f(y_1, y_2, y_3, y_4)$$

1st order approximation:

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$$\Delta Px = \sum_{i=1}^{4} \frac{\partial f}{\partial y_i} \Delta y_i \qquad \Delta y_i \text{ small.}$$

2nd order

$$\Delta Px = \sum_{i=1}^{4} \frac{\partial f}{\partial y_i} \Delta y_i + \frac{1}{2} \sum_{i,j=1}^{4} \frac{\partial^2 f}{\partial y_i \partial y_j} \Delta y_i \Delta y_j$$

DV01's

$$DV01^i = \frac{-\partial f/\partial y_i}{10{,}000} \quad : \text{DV01 wrt. } i^{th} \text{ factor, including pace.}$$

Durations

$$\frac{-\partial f/\partial y_i}{Px} \quad : \text{Durations wrt } i^{th} \text{ factor}$$

Convexity : $\frac{\partial^2 f}{\partial y_i \partial y} \cdot \frac{1}{Px}$: Convexity wrt factors i, j

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Here: DV01, Duration is now a <u>vector</u> and Convexity is a <u>matrix</u>.

Now, think of multi-variable calculus.

$$f = f(y_1, \dots, y_N)$$

$$\frac{\partial f}{\partial y_i} = \lim_{\varepsilon \to 0} \frac{f(y_1, \dots, y_{i-1}, y_i + \varepsilon, y_{i+1}, \dots, y_n) - f(y_1, \dots, y_n)}{\varepsilon}$$

- keep y_j $j \neq i$ constant and perturb y_i.

We want to do something similar here but we need to be careful.

Reason 1) if we move y_1 (2yr par yield) we want to make sure that y_2, y_3, y_4 (5yr, 10yr, 30 yr par yields) do not move.

reason 2) we have to do something with par yields not at 2, 5, 10, 30 yrs.

typical coupon bond has flows at every six months. We are explicitly measuring sensitivities to 4 particular points. What do we do with the other points?

- Is it reasonable to assume no other points move if 2yr goes up 1 bp?
- If other points are to move then how?

Abstractly: consider a coupon bond with 30 yr maturity:

$$Px = f(y(1/2), y(1), y(3/2), \ldots, y(30))$$

$y(t) \rightarrow t$ yr par yield.

we estimate sensitivities by condensing

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down to

$$Px = f(y(2), y(5), y(10), y(30))$$

to get $\frac{\partial f}{\partial y(2)}$ what do we do with

$y(1), y(3/2), y(5/2), y(3)$ etc?

In fact, we do wish to move these rates!

<u>Key rates</u>: $y(2), y(5), y(10), y(30)$ most important.

- measure $\Delta y(j)$ keeping $\Delta y(k) = 0$ $j \neq k$ to isolate effect of key rate $y(j)$.
- let $\Delta y(t)$ move a bit if $\Delta y(j)$ moves for t near j to reflect fact that $y(j)$ is an important, or key rate.

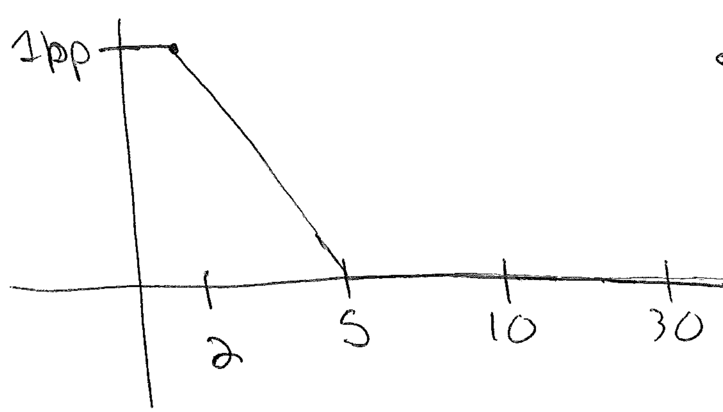
How do we do this?

- to measure $\frac{\partial f}{\partial y(j)}$ we shift the <u>whole</u> par yield curve so that
  1) other key rates do not move
  2) rates near j do move.

example: $y_1 = y(2)$, $y_2 = y(5)$, $y_3 = y(10)$, $y_4 = y(30)$

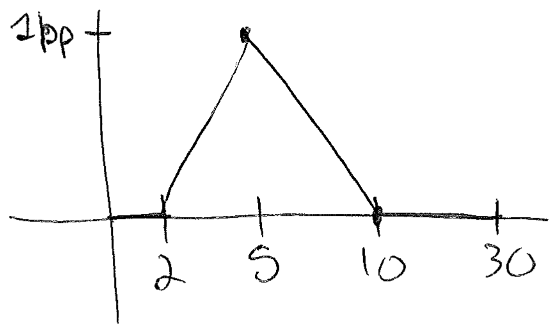
1) shift y_1 up 1bp

$\Rightarrow$ Yield curve shift of



shift $= Y_1(t)$ at t.

2) shift y_2 up 1 bp



$Y_2(t)$

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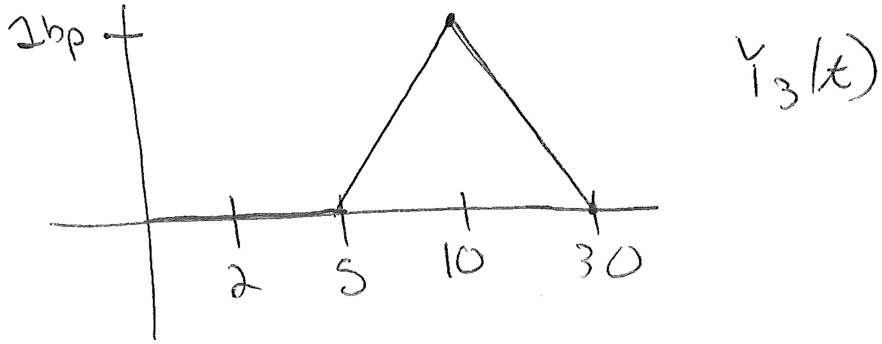
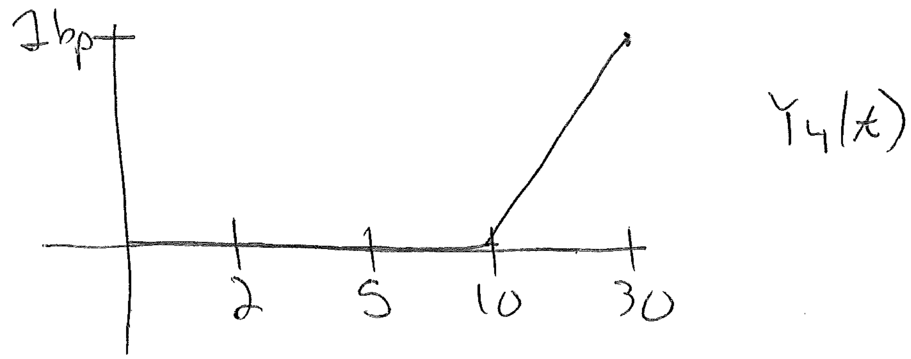
3) shift y_3 up 1 bp



4) shift y_4 up 1 bp



We can use these shifts to compute the "partials".

<u>Plan.</u>

a) Initial Par curve : $y^*(t)$ $\quad t = 1/2, 1, 3/2, \ldots, 30$

$$y_1^* = y^*_{(2)};\; y_2^* = y^*_{(5)};\; y_3^* = y^*_{(10)};\; y_4^* = y^*_{(30)}$$

b) Initial Bond Price

P_x^* = price w/ initial par curve.

Note: if $d^*(t)$ are the initial discount factors calculated from $y^*(t)$.

$$\frac{Px^*}{F} = \frac{a}{2}\sum_{i=1}^{N} d^*(t_i) + d^*(t_N)$$

N remaining payments at $t_1, \ldots, t_N$.

c) write initial bond price as

$$Px^* = f(y_1^*, y_2^*, y_3^*, y_4^*)$$

d) now, to get ~~sen~~ sensitivity wrt y_i employ the shift $Y_i(t)$ $t = 1/2, 1, \ldots, 30$ to the whole yield curve

- recall, shift was 1 bp
  - can be changed to X bp
- think of X bp for now.

this gives a new ~~yield~~ (par) yield curve $Y(t)$ $t = 1/2, 1, \ldots, 30$.

and hence the new discount factors

$d(t)$ $t = 1/2, 1, \ldots, 30$

e) compute the new bond price

$$\frac{P_X}{F} = \frac{q}{2} \sum_{i=1}^{N} d(t_i) + d(t_N)$$

write $P_X = f(y_1, y_2, y_3, y_4)$

?) then if $Y_i(t)$ (shift) is based of on X bp shift

$$\frac{\partial f}{\partial y_i} \approx \frac{\Delta P_X}{\Delta y_i} = \frac{f(y_1, \ldots, y_4) - f(y_1^*, \ldots, y_4^*)}{X \times .0001}$$

$$\Rightarrow DV01^i = -\frac{f(y_1, \ldots, y_4) - f(y_1^*, \ldots, y_4^*)}{X}$$

$$D^i = \frac{10{,}000 \cdot DV01^i}{f(y_1^*, y_2^*, y_3^*, y_4^*)}$$

Notes: 1) for key spot rate durations the calculations are simpler b/c the spot rate change at in-between points is explicity (e.g. linear interpolation) known and you can use this directly to price, without going through discount factors.

- in fact, for par key-rate durations you could compute spot rates to price if you wish.

2) (as in your spreadsheet) we can get $d(t)$ from $y(t)$ via

$$d(1/2) = \frac{1}{1+y(1/2)/2} \;;$$

$$d\left(\frac{n+1}{2}\right) = \frac{2 - y\left(\frac{n+1}{2}\right)\sum_{i=1}^{n} d(i/2)}{2 + y\left(\frac{n+1}{2}\right)}$$

3) Flow (both par, spot key rate shifts)

a) Reference Yield Curve → Reference Price

b) Ref. Y.C. + Perturbation → New Y.C.

c) New Y.C. ⟼ New Price

d) $\dfrac{\partial f}{\partial y_i} = \dfrac{\text{New Price} - \text{Ref. Price}}{.0001 \cdot \text{Bp size of } y_i \text{ perturbation}}$

Example

Ref YC: flat par curve at 5%
(⇒ flat spot curve at 5%)

Bond: Annuity (30 yr) with A = \$3250 every six months

$$\text{Ref Px} = A \sum_{i=1}^{30} \frac{1}{(1+y/2)^i} \qquad \begin{aligned} A &= \$3250 \\ y &= .05 \end{aligned}$$

$$= 100{,}453.13.$$

Perturbations: 1 bp shocks to 2, 5, 10, 30 yr par rate with associated shifts $Y_i(k)$ $i = 1,..,4$

$\Rightarrow$ new discount factors, spot rates

- compute using your spreadsheet.

$\Rightarrow$ $y_i^* = 5\%$ $i = 1,..,4$

$\Rightarrow$ $f(5.01, 5, 5, 5) = 100{,}452.15$

$f(5, 5.01, 5, 5) = 100{,}449.36$

$f(5, 5, 5.01, 5) = 100{,}410.77$

$f(5, 5, 5, 5.01) = 100{,}385.88$

↳ based off new d.f.

$\Rightarrow$

Rate	DV01	Duration
2yr	.98	.098
5yr	3.77	.375
10yr	42.36	4.217
30yr	67.25	6.695.

1st order approx: ($10000 \Delta y = 1$ here)

$$\Delta P_x = -(.98 + 3.77 + 42.36 + 67.25)$$
$$= -114.36$$

$$\Rightarrow P_x = 100{,}338.77$$

Now, we shifted each key rate up 1b (with associated Y_i shifts)

If we shifted the entire par curve up 1 bp

$$P_x = 3250 \sum_{i=1}^{30} \frac{1}{(1+.0501/2)^i} = 100{,}338.81$$

- very good agreement.

But, with this data we can now approximate non-parallel shifts effects on prices

e.g. $\Delta y(2) = 5$ bp ; $\Delta y(5) = 3$ bp

$\Delta y(10) = 2$bp ; $\Delta y(30) = -1$ bp.

$$\Delta P_x \approx -(.98 \times 5 + 3.77 \cdot 3 + 42.36 \cdot 2 + 67.25 \cdot (-1)) = -33.68$$

- for small Δy_i we do not have to compute brand new $\cancel{\text{x}}\, Y_i(t)$, we can scale linearly off the 1 bp-based $Y_i(t)$.

Hedging using Key Rates.

- suppose we have sold the annuity and now we want to hedge it with 2, 5, 10, 30 yr par coupon bonds.

- by "Hedge" we mean that we want to DV01 match at 2, 5, 10, 30 yrs using the above methodology.

  . first we have to compute DV01's, Durations for the

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par bonds using the above methodology

Doing this we get

Bond	DV01 (100F)	Duration
2yr	.018810	1.8810
5yr	.04376	4.376
10yr	.077946	7.7946
30yr	.154543	15.4543

- par bond price is 100 so

$D = 100 \cdot DV01$

For our annuity we have

$DV01^{(2yr)} = .98$

$DV01^{(5yr)} = 3.77$

$DV01^{(10yr)} = 42.36$

$DV01^{(30yr)} = 67.25$

So, matching DV01's gives

$$\frac{F_1}{100} \cdot .01881 = .98$$

$$\frac{F_2}{100} \cdot .04376 = 3.77$$

$$\frac{F_3}{100} \cdot .077946 = 42.36$$

$$\frac{F_4}{100} \cdot .154543 = 67.25$$

$\Rightarrow$

$F_1 = 5,209.99$ $\quad F_2 = 8,632.93$

$F_3 = 54,345.32$ $\quad F_4 = 43,515.40.$

A (slightly) simpler way to hedge:

- Volatility Weighted Hedging
- comes from a-priori knowledge of how yield changes interact accross maturities.

Example: Suppose we sell the 20 yr par bond and want to hedge by going long the 30 yr par bond

Suppose experience tells us that a 1 bp change in 30 yr yields leads to a 1.1 bp change in 20yr yields.

$$\Rightarrow \Delta Px_{20}(F_{20}) = -100 \cdot F_{20} \cdot \Delta y_{20} \cdot DV01_{20}(100)$$

$$= -100 \cdot F_{20} \cdot 1.1\ (\Delta y_{30} = 1bp) \cdot DV01_{20}(100)$$

$$\Delta Px_{30}(F_{30}) = -100 \cdot F_{30} \cdot (\Delta y_{30} = 1bp)\, DV01_{30}(100)$$

so, to match price changes

$$-100 \cdot F_{20} \cdot 1.1 \cdot 1bp \cdot DV01_{20}(100)$$

$$= -100 \cdot F_{3} \cdot 1bp \cdot DV01_{30}(100)$$

or

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$$f_{20} \cdot 1.1 \cdot DV01_{20}(100) = f_{30} \cdot DV01_{30}(100)$$

example

20 yr par yield is 5.8 %

30 yr par yield is 6.0 %

using, for par bonds

$$DV01_T(100) = \frac{1}{100} \frac{1}{y_T} \left(1 - \frac{1}{(1+y_T/2)^{2T}}\right)$$

T: maturity

we get

$$DV01_{20}(100) = .117465$$

$$DV01_{30}(100) = .138378$$

$\Rightarrow$ if $f_{20} = 10{,}000{,}000$ then

$$f_{30} = 9{,}337{,}575 = 10 \cdot M \times 1.1 \times \frac{.117465}{.138378}$$

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Example: sell 10M face of par 20 yr.
hedge with 10yrs, 30yr. (equal ~~terms~~ $ amounts)

Par Yields

$y_{10} = 5.2\%$ $y_{20} = 5.8\%$ $y_{30} = 6\%$

Changes

$\Delta y_{30} = 1bp \Rightarrow \Delta y_{20} = 1.1\ bp$
$\Delta y_{10} = 1.2\ bp$

DV01s(100)

$DV01_{10}(100) = .077215$

$DV01_{20}(100) = .117465$

$DV01_{30}(100) = .138378.$

How do we price match based on DV01 here? If $\Delta y_{30} = 1bp$

$\Delta Px_{20}(f_{20}) = -100 \cdot f_{20} \cdot 1.1 \cdot 1bp \cdot DV01_{20}(100)$

$\Delta Px_{10}(f_{10}) = -100 \cdot f_{10} \cdot 1.2 \cdot 1bp \cdot DV01_{10}(100)$

$\Delta Px_{30}(f_{30}) = -100 \cdot f_{30} \cdot 1bp \cdot DV01_{30}(100)$

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we want equal \$ amounts in 10yrs, 30yr

- since both are par bonds ($Px = 100$)

this means we want equal faces

as well.

$$F_{10} = F_{30} = F.$$

to match prices for $F_{30} = 10m$.:

$$10m \cdot 1.4 \cdot DV01_{30}(100)$$

$$= F(1.2 \cdot DV01_{10}(100) + DV01_{30}(100))$$

$$\Rightarrow F = \frac{10 \cdot m \cdot 1.4 \cdot .117465}{1.2 \cdot .077215 + .138378}$$

$$= 5{,}592{,}700.$$