Sensitivity Analysis and Hedging II
- Duration and Convexity

Duration

Another measure of interest rate risk.
Assume the security has dependence on a factor y.
The Duration is

$$D = -\frac{1}{P_X}\frac{\Delta P_X}{\Delta y} = 10{,}000\,\frac{DV01}{P_X}$$

- measures sensitivity of the relative change in the price of the security to changes in y.

- if $P_X = f(y)$ then for small changes in y

$$D = -\frac{1}{P_X}\dot{f}(y) = -\frac{\dot{f}(y)}{f(y)}$$

Convexity

- measures the sensitivity of interest rate sensitivity to changes in rates
  i.e. a second derivative.

- measures it in relative price terms

- if $P_X = f(y)$

$$C = \frac{1}{P_X} f''(y) = \frac{1}{P_X} \frac{d^2 P_X}{dy^2}$$

- absent an explicit formula for f, we can approximate this numerically

- warning: you might sometimes see

$$C = \frac{1}{2} f''(y)/P_X$$

↑ built into definition.

- be careful to know if this is here!

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Approximations

1) 1^{st} order: $\frac{\Delta P}{P} = -D\Delta y$

2) 2^{nd} order: $\frac{\Delta P}{P} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$

rules of thumb

1) $D>0 \Rightarrow$ Px $\nearrow$ if rates $\searrow$
Px $\searrow$ if rates $\nearrow$

2) $D<0 \Rightarrow$ Px $\searrow$ if rates $\searrow$
Px $\nearrow$ if rates $\nearrow$

3) $C>0 \Rightarrow$ positive contribution to Px for both $\nearrow$ and $\searrow$ in rates

4) $C<0 \Rightarrow$ negative contribution to Px for both $\nearrow$ and $\searrow$ in rates.

$C>0$: Long Volatility or Long Convexity

- $C>0$ is typical, and generally a good thing.

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A note on scaling (or how D, C, DV01 change with F)

- write $Px(F)$, $DV01(F)$, $D(F)$, $C(F)$ to stress dependence on F.

- since

$$\frac{Px(F)}{F} = \frac{a}{2}\sum_{i=1}^{N} d(i/2) + d(N/2)$$

we have

$$\frac{Px(F)}{F} \text{ does not depend on } F.$$

$$\Rightarrow \quad Px(F) = \frac{F}{100}\, Px(100) \quad \text{e.g.}$$

Similarly

$$\Delta Px(F) = \frac{F}{100}\, \Delta Px(100) \qquad \text{so}$$

$$DV01(F) = -\frac{\Delta Px(F)}{10{,}000\,\Delta y}$$

$$= \frac{F}{100}\left(-\frac{\Delta Px(100)}{10{,}000\,\Delta y}\right)$$

$$= \frac{F}{100}\, DV01(100)$$

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- both price and DV01 scale linearly with f.

But

$$D(f) = -\frac{1}{Px(f)}\frac{\Delta Px(f)}{\Delta y} = -\frac{1}{\frac{f}{100}Px(100)}\frac{\frac{f}{100}\Delta Px(100)}{\Delta y}$$

$$= D(100)$$

- duration unchanged with f.
- Similarly, $C(f) = C(100)$ does not change with f.

Yield Based DV01, duration, convexity for Coupon Bonds

y: YTM.

Bond: coupon c, N remaining payments at times $1/2, 1, 3/2, \ldots, N/2$

- for now, we assume a coupon has just been paid.

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Then

$$\frac{P_X}{F} = \frac{a}{2}\sum_{i=1}^{N} \frac{1}{(1+y/2)^i} + \frac{1}{(1+y/2)^N}$$

$$\triangleq f(y)$$

So

$$DV01 = -\frac{F}{10{,}000} f'(y)$$

$$= \frac{F}{10{,}000}\left(\frac{a}{2}\sum_{i=1}^{N} \frac{i/2}{(1+y/2)^{i+1}} + \frac{N/2}{(1+y/2)^{N+1}}\right)$$

$$= \frac{F}{10{,}000}\frac{1}{2(1+y/2)}\left(\frac{a}{2}\sum_{i=1}^{N} \frac{i}{(1+y/2)^{i}} + \frac{N}{(1+y/2)^{N}}\right)$$

Alternatively, we can evaluate the sum above to get.

$$\frac{P_X}{F} = \frac{a}{y}\left(1-\frac{1}{(1+y/2)^N}\right) + \frac{1}{(1+y/2)^N}$$

So

$$DV01 = \frac{F}{10{,}000}\left(\frac{a}{y^2}\left(1-\frac{1}{(1+y/2)^N}\right) + \left(1-\frac{a}{y}\right)\frac{N/2}{(1+y/2)^{N+1}}\right)$$

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Note: if we think of T as the maturity so $N = 2T$ we have

$$DV01 = \frac{F}{10{,}000}\left(\frac{a}{y^2}\left(1 - \frac{1}{(1+y/2)^{2T}}\right) + \left(1 - a/y\right)\frac{T}{(1+y/2)^{2T+1}}\right)$$

As for Duration

Recall: $$DV01 = -\frac{F}{10000}\frac{\Delta P_x}{\Delta y}$$

$$= \frac{P_x}{10000}\left(-\frac{F\,\Delta P_x}{P_x\,\Delta y}\right)$$

$$= \frac{P_x \cdot D}{10000}$$

so

$$D = \frac{10{,}000\, DV01}{P_x}$$

$$= \frac{F}{P_x}\left(\frac{a}{y^2}\left(1 - \frac{1}{(1+y/2)^N}\right) + \left(1 - a/y\right)\frac{N/2}{(1+y/2)^{N+1}}\right)$$

$$= \frac{a/y^2\left(1 - \frac{1}{(1+y/2)^N}\right) + \left(1 - a/y\right)\frac{N/2}{(1+y/2)^{N+1}}}{\frac{a}{y}\left(1 - \frac{1}{(1+y/2)^N}\right) + \frac{1}{(1+y/2)^N}}$$

Zeros

Let $q = 0$ above, we have for $N = 2T$

$$D = \frac{N/2}{1 + y/2} = \frac{T}{(1+y/2)}$$

Now, as indicated by its name (i.e. "duration") we are inclined to think there is a connection between the duration of a bond and the average (yield weighted) time of the remaining cash flows.

This idea is made more precise through the quantity "Macauly Duration".

$$D_{mac} = (1 + y/2) D$$

- we immediately see that for zeros with T years to maturity

$$D_{mac} = T \quad \text{(exact time of flow).}$$

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More generally, if we think of a "bond" as having $2T$ remaining payments of size F_i $i=1,\dots,2T$ then its yield y satisfies

$$P_X = \sum_{i=1}^{2T} \frac{F_i}{(1+y/2)^i} = f(y)$$

so, its duration is

$$D = -\frac{\dot{f}(y)}{P_X} = \frac{1}{P_X(1+y/2)} \sum_{i=1}^{2T} \frac{i/2\, F_i}{(1+y/2)^i}$$

$$= \frac{1}{P_X(1+y/2)} \sum_{i=1}^{2T} \frac{T_i F_i}{(1+y/2)^i}$$

$T_i = i/2$ = time of i^{th} payment F_i.

Thus

$$D_{mac} = \frac{1}{P_X} \sum_{i=1}^{2T} \frac{T_i F_i}{(1+y/2)^i}$$

$$= \sum_{i=1}^{2T} T_i W_i \quad ; \quad W_i = \frac{F_i/(1+y/2)^i}{P_X}$$

$$= \frac{F_i/(1+y/2)^i}{\sum_{j=1}^{2T} F_j/(1+y/2)^j}$$

So, D_{mac} is a weighted average of the remaining payment times where the weights are the relative contribution of the ith payment to the current price, using the discounting factors implied by the yield.

- D_{mac} is an average time of the cash flows, taking into account ~~both size~~ NPV according to y.

Note

Using D_{mac} we have the first order approximation

$$\frac{\Delta P_x}{P_x} = -\frac{D_{mac}}{1+y/2}\Delta y.$$

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For Coupon Bonds we have

$$D_{mac} = (1+y/2) D$$

$$= \frac{F}{P_X}(1+y/2)\left(\frac{q}{y^2}\left(1 - \frac{1}{(1+y/2)^N}\right) + (1-q/y)\frac{N/2}{(1+y/2)^{N+1}}\right)$$

<u>example</u>

$q = 4\%$, $T = 20$ $(N = 2T = 40)$

$y = .04231$, $F = 100$

$\Rightarrow$ $D = 13.555$

$D_{mac} = 13.842$. ("average" time of cash flows is 13.8 yrs).

Par Coupon Bonds

Here: $F = P_X$, $q = y$, $N = 2T$

$\Rightarrow$

$$D_{mac} = \left(\frac{1}{y} + \frac{1}{2}\right)\left(1 - \frac{1}{(1+y/2)^{2T}}\right)$$

$$D = \frac{1}{y}\left(1 - \frac{1}{(1+y/2)^{2T}}\right)$$

Simpler formulas.

$$DV01 = \frac{F}{10000}\frac{1}{y}\left(1 - \frac{1}{(1+y/2)^{2T}}\right)$$

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Annuities

T years, $f_i = A$ $\quad i = 1, \dots, 2T$

$$\Rightarrow D = \frac{\cancel{A}}{P_X(1+y/2)} \sum_{i=1}^{2T} \frac{i/2}{(1+y/2)^i}$$

$$\left(\frac{P_X}{A} = \sum_{i=1}^{2T} \frac{1}{(1+y/2)^i} \right)$$

$$= \frac{1}{(1+y/2)} \frac{\sum_{i=1}^{2T} i/2/(1+y/2)^i}{\sum_{i=1}^{2T} \frac{1}{(1+y/2)^i}}$$

$$= [\text{algebra}] \quad \frac{1}{y} - \frac{1}{(1+y/2)} \frac{N/2}{((1+y/2)^N - 1)}$$

so

$$D_{mac} = \frac{1}{y} + \frac{1}{2} - \frac{N/2}{((1+y/2)^N - 1)}$$

Perpetuities $(N \rightarrow \infty)$

$$D = \frac{1}{y} \; ; \quad D_{mac} = \frac{1}{y} + \frac{1}{2}.$$

Yield-Based Convexity for Coupon Bonds

- again, assume a coupon has just been paid.

$$\frac{P_x}{F} = \frac{a}{2}\sum_{i=1}^{N} \frac{1}{(1+y/2)^i} + \frac{1}{(1+y/2)^N}$$

$$= \frac{a}{y}\left(1 - \frac{1}{(1+y/2)^N}\right) + \frac{1}{(1+y/2)^N}$$

$$= f(y)$$

$$\Rightarrow C = \frac{1}{P_x} F \cdot \ddot{f}(y) = \ddot{f}(y)/f(y)$$

(ugly formula)

$$C = \frac{F}{P_x}\left(\frac{2a}{y^3}\left(1 - \frac{1}{(1+y/2)^N}\right) - \frac{2a}{y^2}\frac{N/2}{(1+y/2)^{N+1}} + \left(1 - a/y\right)\frac{(N/2)^2 + N}{(1+y/2)^{N+2}} \right)$$

- easy to obtain numerically ...

Maturity Dependence of DV01, Duration, Convexity

- specify first for zeros with maturity T.

$$Px = F(1+y/2)^{-2T}$$

$$D = \frac{T}{1+y/2}$$

$$D_{mac} = T$$

$$C = \frac{T^2 + T/2}{(1+y/2)^2}$$

$$DV01 = \frac{Px \cdot D}{10{,}000} = \frac{1}{10{,}000} \cdot \frac{FT}{(1+y/2)^{2T+1}}$$

So, for $y>0$ fixed.

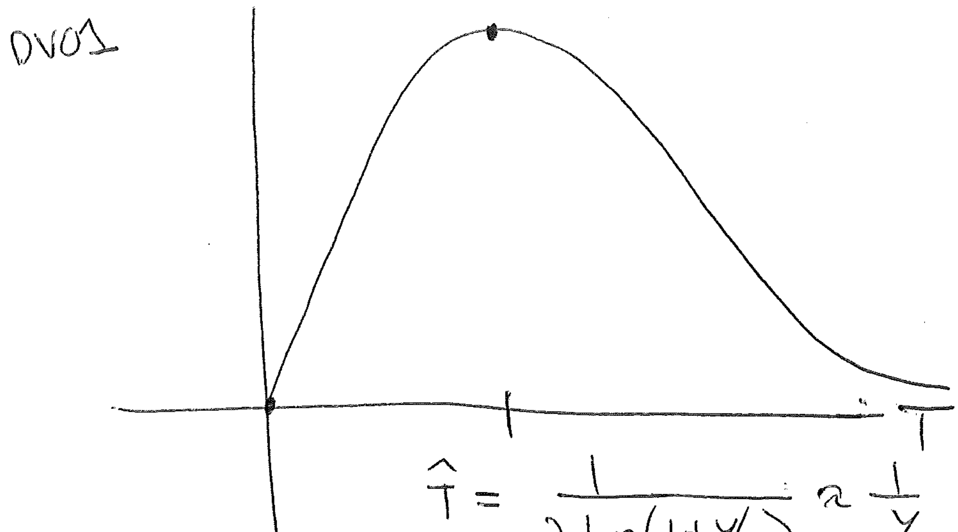
$Px \searrow$ as $T \nearrow$ (like $\frac{1}{K^{2T}}$)

$D \nearrow$ as $T \nearrow$ (linearly)

$D_{mac} \nearrow$ as $T \nearrow$ (exactly linearly)

$C \nearrow$ as $T \nearrow$ (quadratically)

As for DV01



$$\hat{T} = \frac{1}{2\ln(1+y/2)} \approx \frac{1}{y}$$

so DV01 can decrease with maturity if the maturity is large enough

- recall: $y = \hat{r}(T)$ so this happens if $T \cdot \hat{r}(T) > 1$.

e.g. $T = 20$, $\hat{r}(T) > .05$

Coupon Bonds

- complicated relationship wrt T (see formulas)

Rules of Thumb.

1) $D \nearrow$ as $T \nearrow$

- can actually be violated under extreme conditions

2) $D \searrow$ as $q \nearrow$

3) $D \searrow$ as $y \searrow$.

Precisely, if we think of

$$D_{mac} = D_{mac}(q, y, T) \quad q, y, T > 0$$

we have

1) for q, y fixed,

$$\lim_{T \to \infty} D_{mac}(q, y, T) = \frac{1}{q} + \frac{1}{y} \quad \text{(perpetuity)}$$

2) for q, y fixed, $q \geq y$

$$D_{mac}(q, y, T) \nearrow \quad \text{as } T \nearrow$$

3) for q, y fixed with $q < y$

$D_{mac}(q, y, T) \nearrow$ as $T \nearrow$ up to a level T^* and then $\searrow$ as $T \nearrow$ to $\frac{1}{y} + \frac{1}{q}$.

T^*: typically a mathematical curiosity

e.g. $y = 0.05$, $q = 0.01$

$\Rightarrow T^* \approx 53$ yrs.

4) for y, T fixed, $D_{mac}(q, y, T) \searrow$ as $q \nearrow$.

5) for q, T fixed, $D_{mac}(q, y, T) \searrow$ as $y \nearrow$.

Warning

Typically, for T fixed, changing q will also change y (and vice versa) so it is not too realistic to think of changing just q or just y.

DV01 rules of thumb

DV01 $\nearrow$ as $T \nearrow$, except for long maturity zeros.

DV01 $\nearrow$ as $q \nearrow$

DV01 $\searrow$ as $y \nearrow$

in fact

1) $\lim_{T \to \infty} DV01(F, q, y, T) = \dfrac{Fq}{10{,}000\, y^2}$

2) $q \geq y \Rightarrow DV01(F, q, y, T) \nearrow$ as $T \nearrow$.

3) $q < y \Rightarrow DV01(F, q, y, T) \nearrow$ as $T \nearrow$ up to T^{**} then decreases

4) $DV01(F, q, y, T) \nearrow$ as $q \nearrow$.

5) $DV01(F, q, y, T) \searrow$ as $y \nearrow$.

Convexity Rules of Thumb

1) $C \nearrow$ as $T \nearrow$ (except for deep discount bonds)

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2) $C \searrow$ as $a \nearrow$.

3) $C \searrow$ as $y \nearrow$.