$$\frac{dS_t}{S_t} = r\,dt + \sigma\, d\tilde{Z}_t \qquad \tilde{\mathbb{P}}$$

$\tilde{Z}$ is a BM under $\tilde{\mathbb{P}}$.

$$d(\underbrace{e^{-rt}}_{X_t}\,\underbrace{S_t}_{Y_t})$$

$$d(X_t Y_t) = dX_t\, Y_t + X_t\, dY_t + dX_t\, dY_t$$

$$d(e^{-rt} S_t) = e^{-rt}(-r)dt\; S_t + e^{-rt}\, \underbrace{dS_t}_{=\, S_t r\, dt + S_t \sigma\, d\tilde{Z}_t}$$

$$+\; \cancel{e^{-rt}(-r)dt\; \underbrace{dS_t}_{=\,(S_t r\,dt + S_t\sigma\, d\tilde{Z}_t)}} \to 0$$

	dt	$d\tilde{Z}_t$
dt	0	0
$d\tilde{Z}_t$	0	dt

$$= e^{-rt}\left(\cancel{-r S_t\, dt} \cancel{+ r S_t\, dt} + \sigma S_t\, d\tilde{Z}_t\right)$$

$$= e^{-rt}\, \sigma\, S_t\, d\tilde{Z}_t$$

$$V_t = V(S_t, t)$$

$$dS_t = r S_t\, dt + \sigma S_t\, d\tilde{z}_t$$

$$\underbrace{e^{-rt} V_t}_{\text{martingale}} = \underbrace{e^{-rt} V(S_t, t)}_{\text{martingale.}}$$

$$d\left(e^{-rt} V(S_t, t)\right) = e^{-rt}(-r)\,dt\, V(S_t,t) + e^{-rt}\, dV(S_t, t)$$

$$\left(\partial_t V(S_t,t) + \partial_s V(S_t,t)\, dS_t + \tfrac{1}{2}\partial^2_{ss} V(S,t)(dS_t)^2\right)$$

$$+ e^{-rt} \cancel{(-r)\,dt\, dV(S_t, t)} \to 0$$

$$= e^{-rt}\Big\{ \underbrace{-rV + \partial_t V + \partial_s V\, r S_t + \tfrac{1}{2}\partial^2_{ss} V\, \sigma^2 S_t^2}_{=0} \Big\}_{(S_t,t)} dt$$

$$+ e^{-rt}\, \partial_s V(S_t, t)\, \sigma S_t\, d\tilde{z}_t$$

$$-rV + \partial_t V + \partial_s V\, r S + \tfrac{1}{2}\partial^2_{ss} V\, \sigma^2 S^2 = 0.$$

for all (S, t)

$$V_t = \tilde{\mathbb{E}}\left[e^{-r(T-t)} g(S_T) \mid \mathcal{F}_t \right]$$

$$e^{-rt} V_t = \tilde{\mathbb{E}}\left[\underbrace{\cancel{e^{-rt}} e^{-r(T-\cancel{t})}} g(S_T) \mid \mathcal{F}_t \right]$$

$$= \tilde{\mathbb{E}}\left[e^{-rT} g(S_T) \mid \mathcal{F}_t \right] \qquad (*)$$

$$\tilde{\mathbb{E}}\left[e^{-rt} V_t \mid \mathcal{F}_s \right] = \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[e^{-rT} g(S_T) \mid \mathcal{F}_t \right] \mid \mathcal{F}_s \right] \qquad s \leq t$$

$$\overset{\text{Tower property}}{=} \tilde{\mathbb{E}}\left[e^{-rT} g(S_T) \mid \mathcal{F}_s \right]$$

$$\overset{(*)\ t \Leftrightarrow s}{=} e^{-rs} V_s$$

European call $\qquad (S_T - K)_+$

S_T : log-normal distribution.

$$S_T = S_t \exp\left((r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma(\tilde{Z}_T - \tilde{Z}_t) \right)$$

$$\tilde{Z}_T - \tilde{Z}_t \sim N(0, \sqrt{T-t}) \qquad \tilde{Z}_T - \tilde{Z}_t = \sqrt{T-t}\, N \qquad N \sim N(0,1)$$

$$C(S_t, T-t) = \tilde{\mathbb{E}}_t[e^{-r(T-t)} g(S_T)]$$

$$g(S_T) = (S_T - K)_+ = (S_T - K)\, 1_{\{S_T \geq K\}}$$

$$= \tilde{\mathbb{E}}_t[e^{-r(T-t)} S_T 1_{\{S_T \geq K\}}] - \tilde{\mathbb{E}}_t[e^{-r(T-t)} K 1_{\{S_T \geq K\}}]$$

$$=: \mathrm{I} - \mathrm{II}$$

$$\mathrm{II} = K e^{-r(T-t)} \tilde{\mathbb{E}}_t[1_{\{S_T \geq K\}}] = K e^{-r(T-t)} \tilde{\mathbb{P}}_t(S_T \geq K)$$

$$= K e^{-r(T-t)} \tilde{\mathbb{P}}_t\Big(S_t \exp\big((r - \tfrac{1}{2}\sigma^2)(T-t) + \sigma \underbrace{(\tilde{Z}_T - \tilde{Z}_t)}_{=\sqrt{T-t}\,N}\big) \geq K \Big)$$

$$\tilde{\mathbb{P}}\left(N \geq \frac{\log(K/S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

$$= 1 - \Phi\left(\frac{\log(K/S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

$$= K e^{-r(T-t)} \Phi\left(- \frac{\log(K/S_t) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

$$I = \tilde{\mathbb{E}}_t\Big[\cancel{e^{-r(T-t)}}\; \underset{\substack{\| \\ S_t \exp\left((r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\,N\right)}}{S_T}\; 1_{\{S_T \geq K\}}\Big]$$

$$= S_t e^{-\frac{1}{2}\sigma^2(T-t)}\, \tilde{\mathbb{E}}_t\Big[e^{\sigma\sqrt{T-t}\,\underline{N}}\, 1_{\left\{\underline{N} \geq \frac{\log(K/S_t)-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right\}}\Big]$$

$$= S_t e^{-\frac{1}{2}\sigma^2(T-t)} \int_{\frac{\log(K/S_t)-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}^{\infty} e^{\sigma\sqrt{T-t}\,n}\, \frac{1}{\sqrt{2\pi}} e^{-\frac{n^2}{2}}\, dn$$

$$e^{-\frac{n^2}{2}+\sigma\sqrt{T-t}\,n} = e^{\underbrace{-\frac{n^2}{2}+\sigma\sqrt{T-t}\,n-\frac{1}{2}\sigma^2(T-t)}_{-\frac{1}{2}(n-\sigma\sqrt{T-t})^2}}\, e^{\frac{1}{2}\sigma^2(T-t)}$$

$$= S_t \int_{\frac{\log(K/S_t)-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\underbrace{n-\sigma\sqrt{T-t}}_{\tilde{n}})^2}\, dn.$$

$$= S_t \int_{\frac{\log(K/S_t)-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}-\sigma\sqrt{T-t}}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\tilde{n}^2}}\, d\tilde{n}$$

$$= S_t \left\{ 1 - \Phi\left(\frac{\log(K/S_t)-(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \right\}$$

$$= S_t\, \Phi\left(-\frac{\log(K/S_t)-(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$