# Portfolio Optimization & Risk Management

**Goals:**

- Review some common techniques in portfolio optimization.
- Discuss methods for estimating expected returns and volatility.
- Describe a few different approaches to analyzing portfolio risk.
- Review some of the basic concepts in time series analysis.

# Unconstrained Mean Variance Optimization

- We know from economic theory that investors should prefer a higher return to less.
- We also know that investors should be risk averse, that is, all else equal, they should prefer a portfolio with less volatility to more volatility.
- **Question**: how do you reconcile this with the assumption of risk neutrality in option pricing models?
- Using this information, and a few additional assumptions we can write an investor utility of a portfolio as:

$$w^T R - a w^T C w \tag{1}$$

where a is an investor's risk aversion coefficient.

# Unconstrained Mean Variance Optimization

- As you saw in the last lecture, the solution to this optimization problem is:

$$w = \frac{1}{2a} C^{-1} R \tag{2}$$

- Because each investor's risk aversion coefficient will be different, under this model we can't identify a single portfolio or set of weights that all investors should hold (unless we add the assumptions of CAPM to this model).
- We can however say that some portfolios dominate others.
- For each risk aversion coefficient, there will be a single set of weights that are optimal for the investor. Aggregating this for all risk aversion coefficients provides us with an **efficient frontier** of portfolios.

# Physical vs. Risk Neutral Measure

- So far, we have focused on options pricing and yield curve construction and have focused on the **risk neutral measure**.
- When using the risk neutral measure, we rely on *hedging* and *arbitrage* arguments to construct market implied distributions under the assumption that investors are risk neutral.
- When working with portfolio optimization, we are leaving the risk-neutral measure and focusing on the **physical measure**.
- In the physical measure, we know that investors are not risk neutral but are, in fact, risk averse.
- Unlike in the risk neutral measure, when we work with the physical measure, we need to estimate expected returns.

# Mean Variance Optimization: Alternate Formulation

- We can rewrite the unconstrained maximization problem in (1) as a minimization problem with a single equality constraint.
- In particular, we can find the portfolio that has the minimum variance for a given expected return:

$$\max_{w} \left(-w^T C w\right) \tag{3}$$

$$\text{s.t.} w^T R = \hat{R} \tag{4}$$

# Mean Variance Optimization: Alternate Formulation

- Notice that a risk aversion parameter no longer shows up in our optimization.
- However, it is now an optimization with a constraint, so we need to know how to solve those. In particular, we need to know how to work with **Lagrange multipliers**.
- Also notice that we have switched the minimum for the negative of the maximum. Remember that we can always do this to switch between maximization and minimization problems.
- Repeating this process for a wide array of expected returns provides us with a similar efficient frontier.

# Mean Variance Efficient Frontier

- If we consider the solutions to (3) for many different expected returns, we obtain a set of optimal portfolios that are referred to as an efficient frontier.
- These efficient frontier portfolios dominate all other portfolios that are below the efficient frontier.
- It is common to plot the efficient frontier with volatility on the x-axis and expected return on the y-axis.

# Portfolio Optimization with Constraints

- Generally speaking, solving realistic portfolio optimization problems will require many additional constraints.
- These constraints will not however change the approach or formulation of the overall optimization problem. Instead they are tweaks.
- The most obvious constraint is that we want our portfolio to be fully invested. Incorporating this constraint leads to the following optimization problem:

$$\max_{w} \quad -w^T C w \tag{5}$$

$$\text{s.t.} \quad w^T R = \hat{R}, \quad w^T 1 = 1 \tag{6}$$

# Portfolio Optimization with Constraints

- Because this is an equality constraint, we can use Lagrange multipliers to compute the solution to this optimization problem analytically.
- Once we incorporate inequality constraints, we will lose analytically tractability and instead will have to rely on pre-built optimization packages and iterative optimization procedures.
- Python's scipy module has an optimize function to handle these types of problems.
- **IpOpt** is an optimization software application that I would highly recommend if you need to go beyond the scope of built-in packages.

# Global Minimum Variance Portfolio

- Another portfolio of interest is the fully-invested portfolio that has minimum variance.
- This is known as the **minimum variance portfolio**, and is the solution to the following optimization problem:

$$\max_{w} \quad -w^T C w \tag{7}$$

$$\text{s.t.} \quad w^T 1 = 1 \tag{8}$$

- Notice that the minimum variance portfolio does not depend on expected returns for the underlying assets. We will see later that this is a useful feature.

# What other types of Constraints are most common?

- Long Only Constraint
- Maximum Leverage Constraints
- Market Exposure Constraint
- Factor Exposure Constraints
- Tracking Error Constraints

# Example: Market Exposure Constraint

- A common constraint to add would be to limit the $\beta$ or market exposure of our portfolio.
- Mathematically, this would lead to the following optimization to solve:

$$\begin{aligned}
\max_{w} \quad & -w^T C w && (9)\\
\text{s.t.} \quad & w^T R = \hat{R} && (10)\\
& w^T 1 = 1 && (11)\\
& \beta_l \leq w^T \hat{\beta} \leq \beta_u && (12)
\end{aligned}$$

where $\beta_l$ is the minimum allowable beta, $\beta_u$ is the maximum allowable beta and $\hat{\beta}$ is a vector of market betas for each instrument in the optimization.

# Example: Tracking Error Constraint

- Active managers are commonly asked to generate superior returns compared to some benchmark.
- To do this, they may want to minimize their tracking error for a given level of expected outperformance.
- They can then make an efficient frontier analogously to what we saw before, except now tracking error would be on the x-axis and expected outperformance would be on the y-axis.

# Example: Tracking Error Constraint

- Mathematically, we can write this tracking error optimization problem as:

$$\max_{w} \quad -\sqrt{\text{Var}(r_p - r_b)} \tag{13}$$

$$\text{s.t.} \qquad w^T R = r_p \tag{14}$$

$$w^T 1 = 1 \tag{15}$$

  where $r_p$ is the return of the portfolio and $r_b$ is the return of the benchmark.

- Let's look at the definition of $\text{Var}(r_p - r_b)$:

$$\text{Var}(r_p - r_b) = \text{Var}(r_p) - 2\text{Cov}(r_p, r_b) + \text{Var}(r_b) \tag{16}$$

- The last term is the variance of the benchmark and is exogenous.

Therefore we can focus on the first two terms here.

# Example: Tracking Error Constraint

- That means we can rewrite our optimization as:

$$\max_{w} \quad -w^T C w + 2w^T \lambda \tag{17}$$

$$\text{s.t.} \quad w^T R = r_p \tag{18}$$

$$w^T 1 = 1 \tag{19}$$

where $\lambda$ is a vector of covariances between our instruments and the market.

# Estimation of Portfolio Optimization Inputs

- In order to set up our portfolio optimization, we needed to create estimates of:
  - Expected Returns
  - Asset Volatilities
  - Asset Covariances

# Estimation of Portfolio Optimization Inputs

- Remember that when we are doing this we are attempting to make a **forecast**.
- We are also trying to do so by looking at limited amounts of historical data.
- As a result, our estimates will necessarily be noisy with a significant amount of estimation error.
- Because of this, it is best practice to make sure that our portfolios are robust to these potential errors.

# Estimating Expected Returns

- In order to estimate expected returns we generally perform some sort of time series or econometric analysis.
- The simplest approach is clearly to use the historical return as the expected return. How well do you think this will work?
- In practice, we will often instead use an expected return model where we assume the next period return is a function of some combination of:
  - Lagged returns of the same asset
  - Lagged returns of other assets
  - Lagged factor returns

# Estimating Volatilities & Covariances

- In order to estimate volatilities and covariances we may choose to assume that the underlying volatilities are stationary and simply use the historical observed volatility.
- We may also choose to apply time-series techniques to our volatility estimates, such as GARCH or calibrating a stochastic volatility model.
- If we simply use historical observed volatility, we still must choose the period length of the lookback for our volatility estimate.

# Estimating Volatilities & Covariances

- We must also choose a weighting scheme on the historical data. Equal weighting is the simplest and leads to a traditional standard deviation type estimate.
- An Exponentially Weighted Moving Average Model (EWMA) is another approach that applies more weight to current observations and decreased weight to older observations.
- Also note that the covariance matrix is a constant in our portfolio optimization setup. Do you think this is a reasonable assumption?

# Problems with Mean-Variance Optimization

- In practice, many issues arise with portfolio optimization:
  - As you saw last class, portfolio optimization requires inversion of a covariance matrix that is often not full rank. If this is not handled properly it can lead to unstable and undesirable results.
  - Our portfolio optimization algorithms assume that the returns (and covariances) that we pass in are known with certainty, but in reality there is a great deal of estimation error. As a result, for highly correlated assets this can lead to implausible portfolios.

# Modern techniques for improving Portfolio Optimization

- A number of techniques have been created within the last few decades that attempt to obviate some of the weaknesses of mean-variance optimization:
  - Resampling an Efficient Frontier
  - Applying Shrinkage Techniques to a Covariance Matrix
  - Risk Parity Portfolios

# How do we manage portfolio risk?

- Once we have constructed a portfolio that we believe is robust and optimal to different market conditions, we need to know how to manage its risk.
- Proper risk management should be concerned with more than just estimating volatility.
- In particular, central to a good risk management process is understanding the behavior of your portfolio during **tail events**.
- We need to understand the portfolio's **skewness** and **kurtosis**.
- If we construct a portfolio with a high Sharpe ratio, it is of little use if its higher moments imply a high probability of blowing up.

# Most Common Risk Measures

- Volatility
- VaR
- CVaR
- Extreme Loss / Stress Test Scenarios

# VaR

- Value-at-Risk (VaR) measures the predicted maximum loss with a specific confidence level.
- In other words, it tells us with probability $1 - \epsilon$ that our portfolio won't lose more than $X$ in a given period.
- VaR is the most commonly used risk measure both for sell-side institutions and hedge funds.
- Despite its intuitive appeal, there are many problems with VaR:
  - It says nothing about the severity of the loss that we will incur should we exceed our threshold.
  - VaR on a portfolio of two assets is not always less than or equal to the individual VaR of the assets, which is not in line with basic diversification concepts.

# CVaR

- Conditional Value-at-Risk (CVaR) measures the expected loss conditional on exceeding a given threshold.

- Mathematically, for a given threshold $\epsilon$, CVaR is equal to:

$$\text{CVaR}_{\epsilon} = \int_{-\infty}^{\text{VaR}_{\epsilon}} P dP \tag{20}$$

  where $P$ is the portfolio profit/loss.

- CVaR does not suffer from the same issues as VaR in terms of coherence and consistency, and it also clearly measures the loss in the tail of the distribution. As a result, it has grown in popularity recently.

# VaR & CVaR in Portfolio Optimization

- If we want, we can reformulate our portfolio optimization problems in terms of VaR or CVaR instead of variance.
- If we choose to do this, we should keep in mind the advantages and disadvantages of each metric.
- As an example, we can build an efficient frontier of portfolios that are mean-CVaR optimal instead of mean-variance variance optimal.
- This would require solving an optimization problem of the following form:

$$\max_{w} \quad -\mathrm{CVaR}_{\epsilon}(w) \tag{21}$$

$$\text{s.t.} \quad w^T R = \hat{R}, w^T 1 = 1 \tag{22}$$

# Calculation of VaR / CVaR

- VaR and CVaR are crucial to firm-wide risk management procedures and calculation of these metrics for large complex portfolios is a common task for a quant.
- Remember that when we perform this task, we are interested in working with the physical measure and not the risk neutral measure.
- As a result, extracting the risk neutral distribution from options prices and integrating this density in order to calculate CVaR would not be a good approach.
- Instead, we want to work with historical data.

# Calculation of VaR / CVaR

- The two most common approaches to calculating VaR/CVaR for a complex portfolio are:
  - Historical Simulation
  - Monte Carlo / Gaussian Copula
- Note that both techniques invole simulation, and once the simulation has been properly defined the calculation of VaR or CVaR is the same, just with a slightly different formula.

# Historical Simulation

- Historical Simulation is a non-parametric approach that relies on resampling from our dataset repeatedly.
- Earlier in the course we talked about statistical bootstrapping, and historical simulation is just an application of this methodology.
- Note that because we are resampling from our historical dataset, we are assuming that our dataset is representative of the future, that is, we are assuming the data is stationary.
- To implement our historical simulation algorithm we begin by generating a series of simulated returns. Each simulated return will be a randomly selected day of historical returns (for all assets).
- We then apply these returns to our portfolio positions to compute the P&L of our portfolio.

# Gaussian Copula

- The second main approach to calculating VaR/CVaR is to use a Gaussian (or some other) Copula and create a simulation algorithm that generates simulated asset paths which can be used to construct risk metrics.
- Unlike historical simulation, the Gaussian Copula approach assumes that the joint distribution of returns is jointly normal with some mean and covariance matrix.
- We generally assume that the means are zero for risk purposes, and use the historical covariance matrix.

# Gaussian Copula

- We can then generate simulated returns that are jointly normal and use these to create our risk metric.
- In order to implement this algorithm, we need to know how to generate jointly normal random variables. Python and other advanced languages have built-in functions for doing this.

# Management of Non-Linear Risks

- When working with linear instruments, applying the results obtained from a historical simulation or Gaussian Copula based Monte Carlo is quite simple. We just take the simulated percent change and apply it the asset and multiply by our position size.
- This worked well because the change to the underlying asset was the only factor that affected our P&L for that position.
- When working with options and other non-linear instruments this will no longer be the case and other complexities will be introduced.
- For example, in addition to simulating asset prices, do we also need to simulate implied volatilities?

# Management of Non-Linear Risks

- We also know that using only an option's $\Delta$ would lead to a poor approximation of P&L especially in large moves, which are the most important for risk purposes.
- The most common approach to handling this is to use a Greeks based approach to approximate the P&L of the option / derivative.
- To do this robustly, we would need to include:
  - Changes to the options time to expiry
  - Changes to the underlying asset price
  - Changes to the options volatility
  - Changes to other factors such as interest rates
- We can then use changes to these quantities and the option's most dominant Greeks to construct a more accurate approximation.

# Working with Time Series Data

- A time series data set is a set of observations at discrete points that have a particular order, that is, the points in the time series are indexed by time.
- Points on the time series may or may not be correlated with each other, depending on the underlying process.
- This is inherently different from analyzing other types of data sets where there are simply a set of obvervations without an index.
- Because of this, we need different methods for working with time series data.

# What are our goals for working with Time Series?

- Calculate some sort of forecast (i.e. a forecasted future price of an asset)
- To better understand the statistical properties of the distribution of the underlying asset.
- To utilize information about the statistical properties of the distribution, or the conditional distribution, to inform trading decisions.

# White Noise

- White noise is the simplest building block of time series analysis.
- Mathematically, white noise can be written as:

$$\mathbb{E}\left[Z_{t_i}\right] = 0$$
$$\mathbb{E}\left[Z_{t_i}^2\right] = \sigma^2$$
$$\mathbb{E}\left[Z_{t_i} Z_{t_j}\right] = 0$$

- Most commonly, we will work with Gaussian Weight noise, in which case each observation is normally distributed.
- The sum of white noise terms is a random walk.
- White noise is stationary, but its sum, a random walk, is not.

# Autoregressive Models

- An autoregressive model is one where the current observation depends on the previous value, as well as a white noise component.
- Mathematically, we can write this as:

$$X_t = \phi X_{t-1} + Z_t$$

- This example includes dependency on only one lag, and is therefore an AR(1), however, we could have a situation with more terms on the right hand side.
- If $\phi = 1$, then we have a random walk and the data is non-stationary.
- If $\phi < 1$ then the process is stationary.

# Moving Average Models

- Moving average models are constructured based on moving averages, or weighted sums of the underlying components.
- Generally, the underlying components in the sum will be white noise terms.
- Mathematically, this can be written as:

$$X_t = Z_t + \phi_1 Z_{t-1} + \phi_2 Z_{t-2}$$

- The above equation is an MA(2) model, however, more or less terms can be included depending on the underlying data.

# ARMA Models

- AR and MA models can be combined into ARMA models.
- For example an ARMA(1,2) has AR length 1 and MA length 2 and can be written as:

$$X_t = \psi X_{t-1} + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + Z_t$$

- Note that this model has parameters $\psi_1$, $\phi_1$ and $\phi_2$.

# ARMA Models

- When working with ARMA models, model selection involves choosing the appropriate number of lag terms in each component.
- AIC and other information criteria exist that help us choose the optimal model.
- Once a model is selected, we need to estimate the models parameters (i.e. $\psi_1, \phi_1, \phi_2$ in the above model)
- Python and other advanced languages has built in functions for estimated ARMA models. In Python, the statsmodels module has an arima_model class which can do this.

# Autocorrelation Function

- An autocorrelation function defines the correlation between an observation and lagged observations with different offsets.
- Mathematically, this can be written as:

$$\rho_X(h) = \text{corr}(X_t, X_{t-h})$$

- Plots of the autocorrelation function are useful visual tools that show us the evolving dependency structure of our time series.
- It is also useful in model selection in particular choosing how many lags to include.

# Trading Autocorrelation: Momentum & Mean Reversion Strategies

- In practice, perhaps the simplest quant strategy is to examine a stock or other tradeable asset for momentum or mean-reversion.
- This is equivalent to asking whether there is autocorrelation in the data or whether the asset prices are stationary.
- If we do find this, then we know that the return in the subsequent period will be a function of the previous return and we can make a profitable investment strategy by adjusting our position based on the observed return.
- If we can't find mean-reversion or momentum in an individual asset, then we might look for it in a pair (this is known as Pairs trading) or larger set of assets.

# Cross Asset Autocorrelation

- In addition to looking for mean-reversion or momentum in an individual asset, we might also look to uncover lead-lag type relationships between assets.
- For example, we might speculate that large cap stocks move first, and then small cap stocks follow.
- Mathematically, we could write this as:

$$X_t = \psi X_{t-1} + \phi Y_{t-1} + Z_t$$
$$Y_t = \psi Y_{t-1} + \phi X_{t-1} + Z_t$$

- The $\phi$ coefficients represent the cross asset autocorrelation.
- As the number of assets grows, it may be intractable to include cross coefficients for all assets. So what can we do?