Exotic Options

Goals:

- Discuss some of the most common exotic options structures
- Describe the most common techniques and algorithms for pricing exotics.
- Review some common approaches to modeling realized volatility.

Single-Asset Exotics

- Single Asset Exotics have complex payoffs but those payoffs are only tied to the behavior of a single underlying.
- These exotics may be a function of the entire path of the underlying asset.
- Recall that European options are only a function of the terminal value of the underlying asset and do not depend on any intermediate steps of the assets path.
- This property simplifies the valuation process for Europeans.
- Most exotics will depend on not only the terminal value of the underlying, but also the underlying asset's path.
- Path-dependent exotics have a more complex valuation process, and will generally require new sets of tools to value consistently.

European Digital Options

- A European digital options with strike K pays a predefined amount if an asset is above (or below) K at expiry.
- Euroepan digi calls pay out if the asset is above K at expiry.
- Euroepan digi puts pay out if the asset is below K at expiry.
- Like vanilla options, these only depend on the terminal distribution of the underlying asset, making their valuation less challenging.
- Mathematically, the value today of a European digital call can be written as:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT} 1_{S_T > K}\right] = \int_{-\infty}^{\infty} e^{-rT} 1_{S_T > K} \; \Phi(S_T) \tag{1}$$

American Digital Options

- American digital options with strike K pay a predefined amount if an asset is above (or below) K at any point prior to expiry.
- In FX markets the family of these products are commonly referred to as **one-touch** or **no-touch** options.
- **One-touches** pay if a barrier above or below is reached.
- **No-touches** pay if a barrier is not reached.
- Mathematically, the an upside one-touch can be priced by calculating the following expectation:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT} 1_{M_T > K}\right] \tag{2}$$

Where M is the maximum attained by the asset over the period.

American vs. European Digital Options

- As we can see from (3), European digitals are equivalent to CDF of the risk-neutral density. Therefore, knowledge of the CDF of the underlying distribution is sufficient for pricing a European digi.
- From (4) we can see that knowledge of the distribution of the maximum value is sufficient for pricing an American Digital.
- The distribution of the maximum is known analytically in some simple cases, such as Geometric Brownian Motion, but in general must be calculated numerically.

Barrier Options

- Barrier options are standard vanilla options that also possess a barrier which causes the option to either **knock-in** or **knock-out**.
- Common barrier options include:
  - Up-and-Out Calls / Puts
  - Up-and-In Calls / Puts
  - Down-and-Out Calls / Puts
  - Down-and-In Calls / Puts
  - Double Knock In Calls / Puts
  - Double Knock Out Calls / Puts
- Just as we saw with digital options, barriers can be observed continuously or only at a specific time.

Barrier Options

- Pricing continous barriers is significantly more challenging as it requires knowledge of the asset's entire path.
- Mathematically, the price of an up-and-out call with a continously monitored barrier can be written as:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT}(S_T - K)^{+} 1_{M_T < B}\right] \tag{3}$$

- If the barrier were European then the price could be written as:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT}(S_T - K)^{+} 1_{S_\tau < B}\right] \tag{4}$$

- Where in both cases B is the barrier level.
- Note that in the European barrier case the barrier date, τ, need not match the expiry of the European, T.

Lookback Options

- As you saw in your homework, lookback options give the buyer the right to choose the optimal exercise point at expiry.
- This means that we get to exercise at the maximum for a lookback call and at the minimum for a lookback put.
- Lookbacks can be either **Fixed Strike** or **Floating Strike**.
- Fixed Strike lookbacks set the strike at trade initiation and determine the payoff of the option based on the optimal point.
- Mathematically, the price of a fixed strike lookback call option can be written as:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT}(M_T - K)^+\right] \tag{5}$$

  Where M_T is the maximum attained over the period.

Lookback Options

- Floating Strike lookbacks set the strike based on the optimal point in the asset and calculate the payoff from the terminal asset value.
- Mathematically, the price of a floating strike lookback option can be written as:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT}(S_T - KM_T)^+\right] \tag{6}$$

  Where M_T is now the minimum attained over the period and K is the percent of the strike that you observe.
- Lookback options are path-dependent and depend on the joint distribution of the terminal asset and the asset's max / min.
- Lookbacks define how frequently the price is sampled (daily, weekly)

Asian Options

- Asian options define their payoff based on the average value attained by the asset over a given period.
- Similar to lookback options, Asian options can be either fixed or floating strike (which are defined in the same manner).
- Asian options can also be defined to be based on an arithmetic average over a given period or a geometric average.
- They also must define the frequency of observations in the average calculation.
- Because they depend on the average of the asset, they are also path-dependent.

Volatility / Variance Swaps

- **Vol & variance swaps** have payoffs linked to realized volatility.
- They are swaps, meaning the require the buyer to exchange realized volatility/variance for some set strike.
- However, because they are functions of realized volatility, they are path-dependent and hard-to-value derivatives.
- Volatility / Variance Swaps provide investors a direct way to express view on the cheapness or richness of **realized volatility**.

- Mathematically, the price of a volatility swap can be written as:

$$\sigma_R = \sqrt{\frac{1}{T}\sum_{i=1}^{N}\left(\frac{S_t - S_{t-1}}{S_{t-1}}\right)^2} \tag{7}$$

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT}(\sigma_R - \sigma_K)\right] \tag{8}$$

- Note that the price of a variance swap would be same, without the outer square root.
- Also note that exercise is not conditional on a positive payoff at expiry.

Pricing Techniques for Single Asset Exotics: Black-Scholes

- In a Black-Scholes setting, closed form solutions can be derived for many exotic options, such as Lookbacks and Digitals.
- The closed-form solutions will clearly be payoff dependent. So we will need to derive and implement a new formula for every payoff.
- If we use a technique like simulation, the overhead for implementing a new payoff will be significantly smaller if it is done correctly. (How?)

Pricing Techniques for Single Asset Exotics: Black-Scholes

- Additionally, Black-Scholes exotic option prices won't be consistent with the volatility surface. This could lead to large pricing error in cases where the skew or slope of the term structure are significant.
- So we have a tradeoff. In this context we will have (relatively) simple formulas for pricing exotics, but will have larger error.
- If we move to a more complex model, we will reduce the potential pricing error, but will need a more complex calibration procedure and will need to compute a numerical approximation to the exotic price.

Pricing Techniques for Single Asset Exotics: Incorporating the Volatility Smile

- If we want to incorporate the volatility smile, the most common approach is to specify a more realistic SDE with more parameters.
- As an example, many people use the Heston Stochastic Volatility Model:

$$\begin{aligned} dS_t &= rS_t\,dt + \sigma_t S_t\,dW_t^1 \\ d\sigma_t^2 &= \kappa(\theta - \sigma_t^2)\,dt + \xi\sigma_t\,dW_t^2 \\ Cov(dW_t^1, dW_t^2) &= \rho\,dt \end{aligned} \tag{9}$$

- Using this, or some other SDE, we can match volatility surfaces fairly accurately, however, implementation of this model requires a more intensive research proces.

Pricing via Risk Neutral Densities

- One method to price a European derivative with an arbitrary payout at expiry is to use the market implied risk-neutral density.
- European options, such as vanilla calls and puts, or digitals, are only functions of the terminal distribution of the asset.
- As a result, if we could extract that risk-neutral distribution, then we can price any European derivative.
- It turns out this is indeed possible. The simplest method for doing so is called Breden-Litzenberger.
- In the Breden-Litzenberger approach we differentiate the price of a call (or put) with respect to strike twice. This gives us:

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT}\phi(K) \tag{10}$$

Pricing via Simulation

- Perhaps the most straight-forward and intuitive way to price these exotics, especially those that are path-dependent.
- If we know the SDE that we want to simulate from, then we can **discretize** the SDE and simulate from the discretized SDE.
- For example, in the case of Black-Scholes, the SDE is:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{11}$$

- Which leads to the following discretized SDE:

$$\hat{S}_{t_{j+1}} = \hat{S}_{t_j} + \mu \hat{S}_{t_j} \Delta t + \sigma \hat{S}_{t_j} \sqrt{\Delta t} Z_j \tag{12}$$

- This discretization scheme is known as **Euler's scheme**. Z_j is a standard normal and $\sqrt{\Delta t} Z_j$ is an increment of a Brownian Motion.

Pricing via PDE's

- Another option would be to price these options by numerically solving the PDE corresponding to the model's dynamics.
- For example, recall that the Black-Scholes PDE is:

$$\frac{\partial c_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c_t}{\partial S_t^2} + rS_t \frac{\partial c_t}{\partial S_t} - rc_t = 0, \tag{13}$$

- If we were to replace each term in the PDE with a numerical, **finite difference approximation**, at each step, and apply the **appropriate boundary conditions**
- As we increase the complexity of the SDE that we are working with (i.e. by using stochastic volatility), we will also increase the complexity of the PDE that we are solving, in some cases adding an extra dimension.

How do we find Input Parameters?

- The risk-neutral density approach that we discussed was non-parametric.
- Other methods that begin with specification of an SDE rely on a set of model parameters.
- The process of finding the set of parameters that best match market data is known as a calibration. In the case of Black-Scholes, this means extracting a single implied volatility.
- In the case of Heston, we have more parameters to extract and we must run a multi-dimensional optimization to solve for them.
- This optimization will require many evaluations of the model with different sets of parameters. So we need to make sure that each evaluation is fast to make sure the calibration is practical.

Multi-Asset Options

- Multi-Asset options have payoffs that depend on multiple assets.
- The payoffs may be linked to only the assets values at expiries, or it may be a function of the entire path of each asset.
- Even in the case of multi-asset options with European payoffs on each asset, a multi-dimensional density will be required.
- Modeling these types of payoffs requires knowledge of the correlation structure of the underlying assets.

Uses of Multi-Asset Options

- In some cases, such as interest rate markets, multi-asset options present themselves naturally.
- In other cases, they may be used to hedge specific market or firm-wide risks.
- Investors may also use these types of options to isolate cheapness or richness in the implied correlations of different assets.
- In practice, hedge funds also use multi-asset options as a means of cheaply instrumenting their macro views.

Dual Digitals

- A Dual Digital Option pays a predefined amount if two assets are above / below a certain level at expiry.
- Note that the barrier on one asset may be above while the other may be below.
- Mathematically, the price of a call dual digital can be written as:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT} 1_{S_T > K_1} 1_{X_T > K_2}\right] \tag{14}$$

- Dual digitals are similar to European single asset digis in that they only depend on the distribution of the terminal value of the asset.
- Unlike single asset digitals they are concerned with the **joint distribution of the two assets**, and the **correlation structure between them**.

Basket Options

- Basket options have their payoff based on a linear combination
- Mathematically, the price of a basket call option can be written as:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT}\left(\sum_{i=1}^{N} w_i S_{T,i} - K\right)^+\right] \tag{15}$$

  Where the w_i's represent the weights of each underlying asset, the $S_{T,i}$'s represent the terminal asset prices.
- Similarly, best-of and worst-of option are defined on the max or min payoff of a set of underlyings.
- These products depend on the **correlation** between the assets in the basket.

Multi-Asset Knockout Options

- The barrier in a knock-in or knock-out option may be on the same asset, or on a different asset.
- The barrier may still be European or American, that is the barrier may be monitored continuously or only a single date.
- Mathematically, the price for a multi-asset up-and-out-call can be written as:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT}(S_{1,T} - K)^{+} 1_{M_{2,T} < B}\right] \tag{16}$$

Here the subscripts 1 & 2 refer to different assets.

Pricing Techniques for Multi-Asset Options

- Pricing multi-asset options is generally speaking more complex than pricing single-asset exotics.
- In the case of a multi-asset option, we need to specify a joint model for the underlying assets.
- An important component of this model will be a specification of the correlation structure between the assets, which can have a significant impact on the price.
- Calibration of the model to market data will also be more complex as we may potentially need to match multiple volatility surfaces.
- Once we have the model specified and calibrated, then **simulation** will be an important tool for valuation.

Designing Code for Exotic Options Pricing

- You may notice that there is a great deal of similarity in the payoffs of these exotics.
- They all require as input the distribution of the path of one or more assets.
- To value these, we need to specify the following:
  - A model (i.e. Black-Scholes)
  - A set of model parameters
  - A pricing technique (i.e. simulation)
  - A definition of a payoff

Designing Code for Exotic Options Pricing

- Remember that we want to write our code generically.
- To do this, we should separate calculation of the payoff from the model and pricing technique.
- Further, we should use polymorphism and an instrument or payoff class hierarchy to ensure that we can dynamically switch between payoff types at run-time
- For example, consider an exotic options pricer application that specifies a model and a payoff at run-time.
- At this point in the course, you should understand how to architect this and be able to write the program such that the proper model and instrument payoff functions will be applied.

Exotic Option Greeks

- In some rare cases, such as when employing the Black-Scholes model, we may find some exotic options Greeks that we are able to derive a closed-form-solution for.
- Generally speaking though, we will need to use a numerical approximation for these derivatives.
- Notice that their will be a common set of Greeks that apply to all our exotics.
- We also can expect that we will almost always be computing them the same way.
- This means that we can write code such that we only have to implement the definition of the Greeks once at the top of the class hierarchy (recall the **Template Method** from our last lecture).

Calculating Greeks via Finite Differences

- The most common method for calculating Greeks numerically is via finite differences, as you have seen on your homework.
- The motivation for this comes from Taylor Series expansion.
- In particular, consider the Taylor Series expansion of a function f, centered on x:

$$f(x+h) = f(x) + hdf + \frac{h^2}{2}df^2 + O(df^3) \tag{17}$$

Eliminating terms of second order or greater and solving for the derivative, df gives us:

$$df = \frac{f(x+h) - f(x)}{h} \tag{18}$$

This leads to a first order forward difference approximation.

Calculating Greeks via Finite Differences

- We could also create a similar first order backwards difference approximation:

$$f(x-h) = f(x) + hdf + \frac{h^2}{2}df^2 + O(df^3) \tag{19}$$

$$df = \frac{f(x)-f(x-h)}{h} \tag{20}$$

- If we look at the second order terms in both the forward and backward difference approximations, we can see that we eliminate another term by using a **central difference**

Calculating Greeks via Finite Differences

- Specifically, if we subtract 19 from 17 we get:

$$f(x+h) - f(x-h) = \quad f(x) + 2hdfO(df^3) \tag{21}$$

Notice that the df^2 term went away, giving us the following more accurate estimate:

$$df = \frac{f(x+h) - f(x-h)}{2h} \tag{22}$$

- Applying similar methods leads to the following finite difference estimate for a second derivative:

$$df^2 = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} \tag{23}$$

Hedging Greeks

- A natural use for Greeks is constructing portfolios that are immunized from a source of uncertainty.
- **Delta**, is most common, but not the only Greek to hedge.
- Theoretical hedging arguments assume continuous trading with no transactions costs or market frictions. In reality, we will choose to update our hedging portfolio at discrete times.
- This presents us with a tradeoff when hedging. Hedging too frequently will result in high transactions costs. Conversely, hedging too sparsely will result in larger portfolio volatility.
- Any practical hedging strategy requires a **re-balance scheme**:
  - Calendar Based Re-Balance
  - Move Based Re-Balance

Delta Hedging

- Delta-hedging is done by buying / selling shares of the underlying.
- If we are long a call option, then the option's delta is positive, and therefore we will want to sell shares to neutralize out delta.
- If we are long a put option, then our delta is negative, and we will need to buy shares to neutralize the delta.
- In the context of a Black-Scholes model, the delta of a European call can be calculated as:

$$\Delta = e^{-rt} N(d1) \tag{24}$$

- Note that the delta in more complex models will differ from its B-S counterpart, and that the delta for more complex instruments will not generally have an analytical solution.

Vega Hedging

- In some cases, we may want to hedge an options exposure not to the underlying asset but to a movement to the implied volatility of the option. This is known as vega hedging.
- To do this, we need to start with a hedging portfolio that has isolated exposure to vega with no other exposures.
- A common portfolio to use is an at-the-money straddle. Note that this straddle will be delta-neutral and will have exposure to vega.
- In order to choose the correct number of units, we need to calculate the vega of the portfolio to hedge, as well as of the straddle, which will give us the hedge ratio.

Delta-Hedged Straddle Portfolios

- Consider a portfolio that is long one unit of a call and one unit of a put with the same strike.
- This portfolio is known as a **straddle**.
- This portfolio will have no exposure to the underlying asset initially, that is it will be delta-neutral (Why?)
- However, this portfolio will still have other Greeks, most notably **gamma**, **theta** and **vega**
- Additionally, as the asset moves, the trade will accumulate delta.
- To eliminate this, traders often delta-hedge the straddle to eliminate this exposure.

Implied vs. Realized Volatility

- Consider a portfolio that buys a straddle and delta-hedges continuously until expiry. What are the factors that drive the profitability for this strategy?
- The cost of that portfolio is a function of the implied volatility when the trade is put on.
- The payoff of that portfolio is a function of the realized volatility that we experience during the life of the trade.
- So when we buy a straddle and continuously delta-hedge, we are selling implied volatility and buying realized volatility.
- Therefore, the profits of this strategy will be highly correlated with the spread between implied and realized vol.

Modeling Realized Volatility

- Modeling realized volatility is of interested to many market participants.
- As we saw, traders buying or selling straddles and delta-hedging them are implicitly making a judgment on expected future realized volatility.
- Additionally, many firms try to maintain a constant portfolio volatility and will as a result scale up or down their portfolio risk based on its expected future realized volatility.
- Additionally, risk managers make assumptions about expected future realized volatility when computing firm-wide risk-metrics such as VaR, CVaR or other measures.

Mean-Reversion of Volatility

- There is much research that shows that volatility exhibits mean-reverting behavior.
- In particular, we often see that market volatility spikes are followed by quick reversions (with the financial crisis being a fairly notable exception)
- Theoretically understanding of these phenomenon are useful not only to options traders but also to portfolio managers and risk managers more generally.
- As a result, it is a common place to apply time series techniques in order to try generate
- The underlying goal of this process will be to create an estimate of realized volatility that is better than today's known implied vol.

Applying Time Series Techniques to Realized Volatility

- Earlier in the course, you saw examples of using ARMA models in order to test for momentum or mean-reversion in stock prices.
- One approach you used was a regression based approach, and you also saw how we could use Python to perform stationarity tests (i.e. Dickey-Fuller) and obtain the autocorrelation coefficients (i.e. by plotting the Autocorrelation function)
- ARCH & GARCH models are the equivalent mean-reversion models for volatility.
- Time permitting, we will touch on applications of the models later in the course.