Derivatives Pricing

Goals:

- Review the basics of options pricing
- Review the Black-Scholes model, implied volatility skews and the process for pricing Exotics.
- Discuss the most common types of derivatives
- Describe the most common Greeks and how they are used in Risk Management and Trading

Derivatives Market

- The derivatives market consists of a wide array of products from vannilla options (calls & puts) to more exotic structures.
- More exotic structures include:
  - Lookback Options
  - American and European Digital Options
  - Basket Options (i.e. best-of, worst-of)
  - Variance & Volatility Swaps
  - Bermudan Swaptions
  - Barrier Options
- Options on VIX are an example of exotic options on S&P.

Mechanics of Derivatives Markets

- Many derivatives still trade over-the-counter (OTC), although some are exchange traded.
- Most exotics trade OTC.
- Derivatives play a critical role in allowing investors to hedge various market risks.
- Banks often issue **structured products** that create a supply & demand imbalance in the market that might make certain exotic options appealing.

Major Players in Derivatives Market

- Hedgers use derivatives to hedge market risk. For example, a long-only equity manager may buy VIX calls to protect against a broad market selloff.
- Banks make markets in derivatives markets, meaning that they attempt to match buyers and sellers.
- Banks may choose to warehouse certain risks on some derivative trades (i.e. they may sell you an option and delta-hedge it).
- Hedge funds also use derivatives to express views more precisely.
- Because hedgers are also price insensitive, hedge funds or other traders may choose to take the other side of these trades.

Derivatives Market By Asset Class

- **Equity**: Consists of listed exchange trade vanilla options as well as OTC exotics. Exotics include basket type structures, volatility derivatives and other bespoke structures (i.e. lookback options)
- **Rates**: Large market for vanilla swaptions and mid-curves, which are traded OTC. Exotics include CMS options (i.e. an option on the difference between the 30yr and the 2yr). Bermudan swaptions are a major feature of the rates derivatives market and present modeling challenges for banks.
- **FX**: Contains perhaps the deepest and most liquid OTC market for exotic options. Vanilla options are extremely liquid as well. Common exotics include knock-in and knock-out options, baskets, American style barrier options and volatility derivatives.

- **Credit**: Primarily an OTC market for derivatives with credit index swaptions and index tranches. Credit-default-swaps also make up a large portion of the credit derivatives market.

Volatility as an Asset Class

- Recently, many market participants have suggested that volatility should be treated as its own asset class with its own properties and risk/reward profile.
- Volatility surfaces are a fertile place for quants to explore, given the significant features in volatility skews, volatility term structures and the cheapness or richness in the level of volatility.
- As the importance of volatility trading and risk management grows, so to does the need for quants with a deep unerstanding of these markets.

Volatility as an Asset Class

- In order to understand how these markets work, we need to first understand how to value these instruments, starting with simple vanillas and including the most common multi-asset derivatives.
- Further, we need to understand how these products will react to different market moves. The behavior of derivatives is non-linear with respect to its underlying parameters, and we need to understand this non-linearity well in order to model, trade or risk manage these products.

Vanilla Options

- Recall that call & put options provide the buyer the right, but not the obligation to buy (or sell) an underlying asset for an agreed upon price.
- This leads to the following payoff for a vanilla call option:

$$c_T = (S_T - K)^+ \tag{1}$$

- In order to compute the fair price of this payoff today, we need to employ an options pricing model.

Why trade Vanilla Options?

- Investors who buy vanilla options are given access to **leverage** and to a leveraged payout should the option expire in-the-money.
- They are also given access to the **intersting non-linear properties** in the option's price as a function of its parameters.
- Generally speaking, we do not buy options to get long or short access to the **delta** of an underlying asset. We could do that by buying or selling the underlying.

Modeling Vanilla Options

- The **first fundamental theorem of asset pricing** states that there is no arbitrage if and only if there exists a risk-neutral measure, and if a risk-neutral measure exists then the price of a call option can be written as:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT}\left(S_T - K\right)^+\right] \tag{2}$$

  Here, we are taking expectation against some risk-neutral probability model.

- Notice that we are only concerned with the terminal value of the asset. Therefore, in order to price European options, we only need the distribution of the terminal value of the asset.

- In other cases, we will also need to know the distribution at each increment, conditional on each prevoius increment.

Modeling Vanilla Options

- In order to calculate the expectation in (5), we generally start by specifying some SDE that defines the underlying asset.
- So far in this course, we have seen the **Black-Scholes SDE** and the **Bachelier SDE**.
- The SDE for the Black-Scholes equation is:

$$dS_t = rS_t\,dt + \sigma S_t\,dW \tag{3}$$

- The SDE for Bachelier is:

$$dS_t = rS_t\,dt + \sigma\,dW \tag{4}$$

Black-Scholes & Bachelier Models

- The Black-Scholes model leads to log-normally distributed asset prices, and normally distributed log returns of the underlying asset.
- The Bachelier model, on the other hand, corresponds to a normal distribution of asset prices.
- In the Black-Scholes model, σ is interpreted as a percentage, whereas in the Bachelier model, σ is interpreted as an absolute level of vol.
- The Bachelier model allows for negative values in the asset with positive probability. This may be seen as a feature in some asset classes (i.e. rates), but a weaknesses in others (i.e. equities)

Black-Scholes Model

- As we have seen, the Black-Scholes SDE leads to a closed-form solution for vanilla option prices, commonly known as the Black-Scholes formula:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT}\left(S_T - K\right)_+\right] = \Phi(d_1)S_0 - \Phi(d_2)Ke^{-rT} \tag{5}$$

- Here, Φ is the CDF of the standard normal distribution and:

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)T\right), \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned} \tag{6}$$

- This formula can be proven by either solving the Black-Scholes PDE or integrating the payoff function for a call or put option against the log-normal distribution.

Black-Scholes Model: Implied Volatility

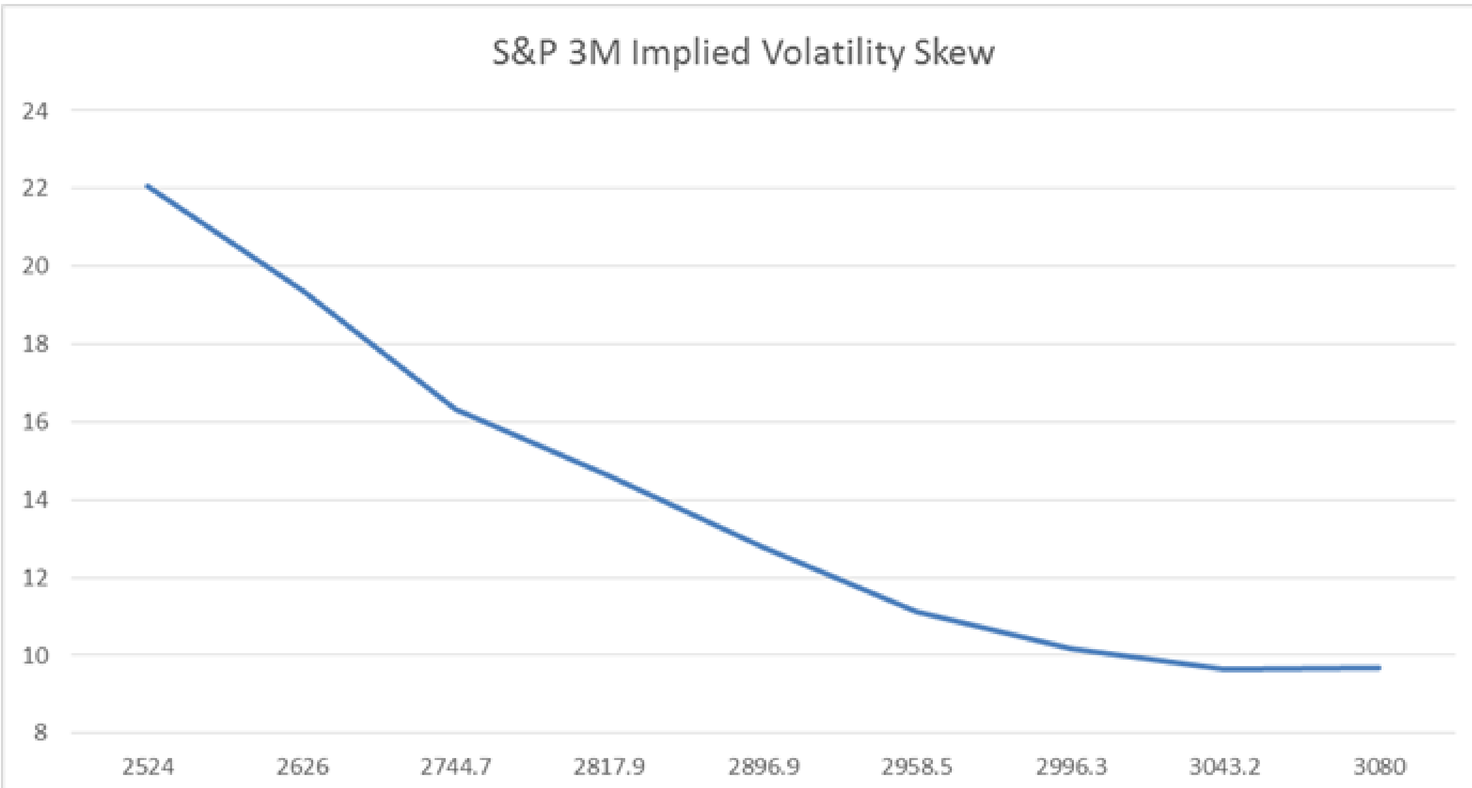
- The Black-Scholes formula tells us how to compute an option price given inputs r, T and σ.
- However in practice we observe the bid and offer prices in the market, and need to extract the value of σ_{implied} that matches the market price.
- This procedure of fitting an implied volatility to a market price requires a one-dimensional root-finding algorithm.
- In your coursework you will undoubtedly cover root finding using Newton's method, Bisection, etc. in greater detail.
- Many modern programming languages such as Python have built-in functions for these types of calculations. In Python we can use the **optimize.root** function in the **scipy** module.

Black-Scholes Model in Practice

- The Black-Scholes model tells us the volatility of the underlying asset. Because each asset can only have one volatility, we would expect that, if our model is correct, that the implied volatility that we extract to be the same for all strikes.
- Unfortunately this is not the case in almost all cases that involve market data. Volatility surfaces exhibit skews that are unexplained by the Black-Scholes model.
- **Question**: Is this related to the negative correlation you found between the S&P and its volatility?
- This means that for different strikes we extract different distributions of the underlying and we have no coherent distribution that explains market prices.

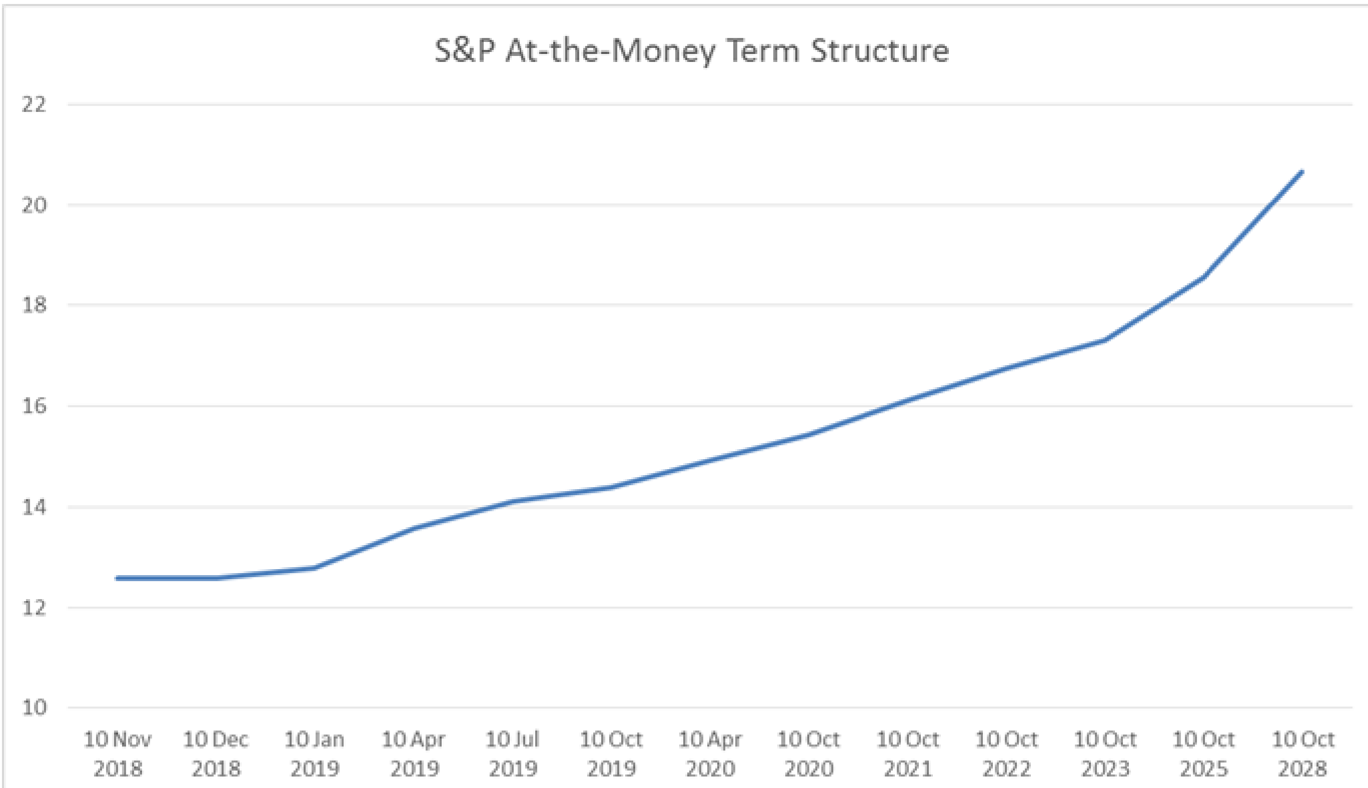
Example: Volatility Surface of S&P

The following example of short-dated S&P volatility skew shows how the extracted implied volatilities differ across strikes:



Example: Volatility Term Structure of S&P

The following shows the at-the-money S&P volatility term structure by expiry:



Black-Scholes Model: Comments

- Implied volatilities are inconsistent across strike and expiry.
- In order to fix this problem (a different implied distribution per strike), we need to choose a better model.
- In general, there are three techniques for trying to incorporate volatility skew into our options pricing model:
  - Local Volatility Models (see Dupire's Model)
  - Stochastic Volatility Models (i.e. Heston, SABR)
  - Jump Processes (i.e. Variance Gamma)

Black-Scholes Model: Comments

- In spite of its weaknesses Black-Scholes is still deeply rooted in the finance industry and is commonly used by traders.
- It is part of the nomenclature of traders, and does provide a more consistent way to compare options prices (as opposed to price)
- It can also still provide reasonable sensitivities, or Greeks, and help us to understand the evolution of option prices in time.

Working with more Complex SDE's

- Incorporation of any of the more complex SDE's mentioned above will help a great deal fitting the volatility surface.
- However, generally speaking these SDE's won't have closed-form solutions even for European call and put options.
- Therefore, use of these models requires a new set of tools.
- These models are beyond the scope of this course, but you should be aware of them and have a very solid understanding of the strengths and limitations of the Black-Scholes (& Bachelier) models.

What else can we do?

- Instead of incorporating a more complex SDE we could also try to fit an implied volatility function that would give us the implied vol. for any strike.
- For example we could fit the implied volatilies to a parabola (or a some higher dimension polynomial)..
- While tempting, and commonly done by traders in the industry, this method doesn't provide us with a coherent density for the underlying asset.
- It may help us to price non-traded European strikes and expiries, but it will be of little use when we turn our attention to exotics.

Black-Scholes Model Greeks

- The Black-Scholes model defines a set of Greeks that help us understand how an option will behave locally with respect to small changes in certain parameters:
  - **Delta**: $\frac{\partial C}{\partial S}$
  - **Gamma**: $\frac{\partial^2 C}{\partial S^2}$
  - **Vega**: $\frac{\partial C}{\partial \sigma}$
  - **Theta**: $\frac{\partial C}{\partial T}$
- These are the most dominant Greeks and the ones that you will work with most frequently.
- However, there are others, including Rho ($\frac{\partial C}{\partial r}$), Vanna ($\frac{\partial^2 C}{\partial S \partial \sigma}$) and Volga ($\frac{\partial^2 C}{\partial \sigma^2}$) that do come up.

Black-Scholes Model Greeks

- As the Black-Scholes formula is infinitely differentiable there are an infinite number of Greeks, but generally we don't look at anything beyond a second-order term.
- In the Black-Scholes model, each of these has an analytic solution. This will not be the case as you move to more realistic models.
- Note that these Greeks are model specific. If we choose another model that doesn't have a σ parameter, then vega will no longer be well defined.
- This is a problem you will face as traders generally think in terms of Black-Scholes Greeks but as we move to more realistic models there is no single parameter for volatlity.

How are Greeks used in Practice?

- Greeks are a critical component of hedging, both at the trade and portfolio level.
- Greeks can be used to set position limits and approximate portfolio risk.
- For example to compute VaR of an option's position, we might use our portfolio's delta and gamma and then use that to determine the value after a 2 or 3 standard deviation move in the underlying.
- Greeks also give us a good understanding of the general properties of our positions.

How are Greeks used in Practice?

- Traders also often communicate in the language of Greeks, and may even use Greeks in their trading decision making process.
- Greeks can be helpful for portfolio attribution. That is, if we have an option position we may want to do know what fraction of our P&L has come from delta / gamma / theta.
- Greeks can also be used to estimate P&L in real-time. While larger models may take more time to run, we can always multiply our respective portfolio Greeks by the market moves to get a good sense of P&L.

Black-Scholes Model PDE

- The Black-Scholes model is also defined by the following PDE:

$$\frac{\partial c_t}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c_t}{\partial S_t^2} + r S_t \frac{\partial c_t}{\partial S_t} - r c_t = 0, \tag{7}$$

  This can be seen by applying Ito's Lemma to the Black-Scholes SDE.

- This PDE can be solved with the appropriate boundary conditions in order to derive the Black-Scholes formula for vanilla options.

- This PDE can also be solved numerically to help us price certain exotic options. To do this, we need to know how to **discretize the PDE** and use **finite differences**.

- However, the PDE also tells us how option prices evolve over time, and as a result we can get a great deal of insight from it.

Tradeoff of Gamma vs. Theta

- For example, let's consider a Black-Scholes world and a delta-hedged portfolio.
- That is, a portfolio that is long one call option and short Δ shares of the underlying such that the portfolio delta is 0.
- What are the remaining risks for this portfolio?
- Using equation (7), we can see that a delta-hedged portfolio will only have exposure to the first two terms.
- These two terms are tied to the θ and γ of an option.
- Notice that the sign on both terms is positive, and also that we know that the sign of gamma is always positive and the sign of theta is always negative.

Tradeoff of Gamma vs. Theta

- So when we buy an option we have a tradeoff between the theta that we pay, and the gamma that we receive.
- You will often hear traders refer to this tradeoff.
- You will also hear them discuss whether there will be enough gamma (also known as realized vol.) in the underlying to justify the decay of the option price, or theta.

Pricing Exotics

- In your previous homeworks you saw examples of pricing exotic options, in particular you looked at pricing **Fixed Strike Lookback Options**.
- Unlike vanilla options, generally speaking, we don't observe market prices for these exotics. Therefore, the procedure that we used in the case of a European option does not work here as there is no input price to calibrate to the level of implied volatility.
- Additionally, we should no longer expect a closed-form solution for our exotic option price as the payoffs become more complex, even under the Black-Scholes model.

Pricing Exotics

- Because exotic options prices are unobservable, when pricing exotics we first look to the European Volatility Surface and find a set of parameters that match the set of European option prices.
- Note that this will require a more robust model than the Black-Scholes, which we know will fit poorly to the entire European options surface.
- Once we have this model **calibrated**, we can then use the calibrated set of parameters to price our exotic.
- We will also need a pricing method for pricing our exotic. Note that this will sometimes be different than the method you use to price Europeans.
- We will dive deeper into pricing exotics in the next lecture...