

## HW Problems for Assignment 3

### Due 6:30 PM Tuesday, October 19, 2021

**1. (10 Points) The “Exponential Premium Principal” Risk Measure.** In this exercise we introduce another risk measure, which shares certain properties with Value at Risk and Expected Shortfall, but which is not a coherent risk measure. Namely, fix a parameter  $\alpha > 0$  and, for a random variable  $L$  define the “exponential premium principal” risk measure with parameter  $\alpha$  by the formula

$$\varrho_\alpha(L) := \frac{1}{\alpha} \log(E[e^{\alpha L}]).$$

- (a) Show that  $\varrho_\alpha$  is a convex risk measure. To show convexity, use Holder’s inequality.
- (b) By considering  $L \sim N(0, 1)$  show that  $\varrho_\alpha$  is not positive homogenous, and hence not coherent.

**2. (25 Points) Component risk measures for an equity portfolio.** In this exercise, we will compute component risk measures for an equally weighted portfolio of five stocks: Walmart, Target, Costco, Citigroup and JP Morgan. Closing price data is in the file “Five\_Stock\_Prices.csv”. In the file, the first column is the date (in Excel numeric format) while columns 2-6 give the stock price data. Data is sorted oldest to newest.

The hypothetical portfolio is fixed at 15 million dollars, and allocates 20% in each stock. As such the respective dollar positions are kept constant throughout time at **\$3 million in each stock**.

We use **linearized losses** and **normally distributed log returns**, with **EWMA** updating. As before, we use the oldest  $M$  returns to obtain an estimate for  $\mu, \Sigma$ , and then update there-after according to the EWMA procedure.

In this setting, compute on a daily basis throughout time the **percent component  $\varrho$  for  $\varrho$**  equal to

- (1) Value at Risk.
- (2) A spectral risk measure with exponential weighting function  $\phi_\gamma(u) = \frac{\gamma}{e^\gamma - 1} e^{\gamma u}, 0 \leq u \leq 1$ .

Additionally, **compute the percent contribution** to the **loss variance** across time.

Begin your calculations after the first  $M$  periods so, with  $N$  log returns, you will have  $N + 1 - M$  component risk measure estimates.

Note that for each day, the above quantities are vectors with five components (one for each stock). **Output a time evolution plot** for each of the above values, over the range  $t - (N - M)\Delta, \dots, t$ . Each plot will have five

graphs.

In addition to the above portfolio value and weights, use  $M = 50$  trail days, a VaR confidence of  $\alpha = .99$ , a spectral risk aversion of  $\gamma = 25$ , and EWMA parameters of  $\lambda = .94$ ,  $\theta = .96$ .

As we discussed in class, the component risk measure plots should be very close to the variance contribution plot. Is this the case?

**3. Spherical and elliptical random variables.** This exercise shows that many of the conclusions on component risk measures, and risk-measure based optimal investment, extend beyond the Gaussian setting.

We say a random vector  $Z \in \mathbb{R}^d$  is *spherical* if for all  $a \in \mathbb{R}^d$  the random variables  $a^T Z$  and  $|a|Z^{(1)}$  have the same distribution, where  $Z^{(1)}$  is the first component of  $Z$ . In other words  $\mathbb{P}[a^T Z \leq \tau] = \mathbb{P}[|a|Z^{(1)} \leq \tau]$  for all  $\tau \in \mathbb{R}$ . For more information on spherical random variables, see Chapter 6.3 of the class textbook.

We say a random vector  $X \in \mathbb{R}^d$  is *elliptical* if  $X = \mu + AZ$  where  $\mu \in \mathbb{R}^d$ ,  $A \in \mathbb{R}^{d \times d}$  and  $Z$  is spherical. Lastly, denote by  $\Sigma = AA^T$ .

- (a) **(5 Points)** Assume  $Z = \sqrt{W} \times \tilde{Z}$  where  $\tilde{Z} \sim N(0, 1_d)$ , and  $W \geq 0$  is a random scalar independent of  $\tilde{Z}$ . Show that  $Z$  is spherical. In particular, taking  $W \equiv 1$  shows that  $\tilde{Z}$  is spherical.
- (b) Assume a one period model with  $d$  stocks, whose log returns  $X$  are elliptically distributed. Write  $\theta$  as the vector of dollar positions.
  - (i) **(5 Points)** Show that the linearized losses  $L^{lin}$  have the same distribution as

$$-\theta^T \mu + \sqrt{\theta^T \Sigma \theta} \times Z^{(1)},$$

and hence for any cash-additive, positively homogenous risk measure  $\varrho$

$$\varrho(L^{lin}) = -\theta^T \mu + \sqrt{\theta^T \Sigma \theta} \times \varrho(Z^{(1)}).$$

- (ii) **(5 Points)** Argue why part (i) means we can make the same conclusions connecting  $\mathcal{R}'_{C, \%$  to the percentage contribution to variance, as we did when  $X \sim N(\mu, \Sigma)$ .