Recommended HW Problems for Assignment 6 - Part II
SOLUTIONS

Note : this is the second half of HW 6, covering material from Lecture 13 on Tuesday, December 4th. The first half of HW 6 was posted on November 27th, and covered material from Lecture 12 also on the 27th. You do not have to turn in these problems, but you should definitely try them, as they will help you study for the final!

1. CVA, DVA for a Forward Contract in the Black-Scholes Model. In class, we computed the CVA and DVA for a call option in the Black-Scholes model. This example was a bit contrived since call options may be purchased over an exchange, where there is no counter-party risk. In this example, we will again compute the CVA and DVA for an option in the Black-Scholes model, but to make the example a bit more realistic, we will consider a forward contract.

Recall that in the Black-Scholes model the asset X evolves under risk neutral measure $\mathbb{Q}$ according to

$$\frac{dX_t}{X_t} = rdt + \sigma dW_t^{\mathbb{Q}}; \quad X_0 = X_0$$

Above, the interest rate $r > 0$ and volatility $\sigma > 0$ are constant, and $W^{\mathbb{Q}}$ is a $\mathbb{Q}$ Brownian motion. Now, assume we have entered into a forward contract with maturity T on X. Recall that this is an agreement made at time 0 to purchase X at time T for a certain forward price K, which we determine at time 0. Since the contract costs nothing to enter into, the risk neutral theory of pricing tells us that the forward price K must satisfy

$$0 = E^{\mathbb{Q}}\left[e^{-rT}(X_T - K)\right] \Longrightarrow K = e^{rT}X_0,$$

where we have use that the discounted asset price is a Martingale under risk neutral measure. Thus, the time T cash flow of the option is $V(T) = X_T - X_0e^{rT}$.

The contract buyer B has default time τ^B, and contract seller S has default time τ^S. Assume under $\mathbb{Q}$ that

(i) τ^B, τ^S are independent of each other, as well as $W^{\mathbb{Q}}$.
(ii) τ^B, τ^S have constant intensities γ^B, γ^S.
(iii) B and S have deterministic losses given default δ^B, δ^S.
(iv) The filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$ is generated by $W^{\mathbb{Q}}$.

Lastly, set $\tau = \min\left\{\tau^B, \tau^S\right\}$ and $\xi \in \{B, S\}$ as the name of the first to default.

Our goal is to compute the CVA and DVA for the forward contract, from the perspective of the buyer B of the contract. To do this:

(a) Derive an explicit formula for the time $t \leq T$ price of the forward contract assuming no default:

$$V(t) = E^{\mathbb{Q}}\left[e^{-r(T-t)}(X_T - X_0e^{rT}) \mid \mathcal{G}_t\right].$$

(b) Using the independence of $W^{\mathbb{Q}}, \tau^S, \tau^B$, show that with $\gamma = \gamma^B + \gamma^S$ we have

$$\begin{aligned} \mathrm{CVA}(0) &= \delta^S E^{\mathbb{Q}}\left[\int_0^T \gamma^S V(t)^+ e^{-(r+\gamma)t}dt\right]. \\ \mathrm{DVA}(0) &= \delta^B E^{\mathbb{Q}}\left[\int_0^T \gamma^B V(t)^- e^{-(r+\gamma)t}dt\right]. \end{aligned}$$

(c) Denote by

$$C^{BS}(t,x;r,\sigma,K,T), P^{BS}(t,x;r,\sigma,K,T),$$

the Black-Scholes call and put prices for a given time t, current stock value x, risk free rate r, volatility σ, strike K and maturity T. Show that

$$\begin{aligned} \mathrm{CVA}(0) &= \delta^S\gamma^S \int_0^T C^{BS}(0,X_0;0,\sigma,t,X_0)e^{-\gamma t}dt; \\ &= \frac{\delta^S\gamma^S}{\gamma}E^{\mathbb{Q}}\left[1_{\tau\leq T}C^{BS}(0,X_0;0,\sigma,\tau,X_0)\right]. \\ \mathrm{DVA}(0) &= \delta^B\gamma^B \int_0^T P^{BS}(0,X_0;0,\sigma,t,X_0)e^{-\gamma t}dt; \\ &= \frac{\delta^B\gamma^B}{\gamma}E^{\mathbb{Q}}\left[1_{\tau\leq T}P^{BS}(0,X_0;0,\sigma,\tau,X_0)\right]. \end{aligned}$$

Thus, we can think of the CVA and DVA as the expected value of an at the money call/put option when the spot rate is 0 and when the maturity is the default time τ (if $\tau \leq T$ and up to a multiplicative constant). Note also the above formula does not depend up on r.

(d) Prove the explicit formula:

$$\mathrm{CVA}(0) = \frac{\gamma^S\delta^S X_0}{\gamma}\left(2\int_0^T N\left(\frac{1}{2}\sigma\sqrt{t}\right)\gamma e^{-\gamma t}dt - \left(1 - e^{-\gamma T}\right)\right),$$

where N is the standard normal c.d.f..

Solution:

(a) Since the discounted stock is a Martingale under $\mathbb{Q}$ we obtain

$$V(t) = E^{\mathbb{Q}}\left[e^{-r(T-t)}(X_T - X_0e^{rT}) \mid \mathcal{G}_t\right] = X_t - X_0e^{rt}.$$

(b) Since τ is independent of the filtration $\mathbb{G}$, and since by definition the value process V is $\mathbb{G}$ adapted:

$$\begin{aligned}
\mathrm{CVA}(0) &= E^{\mathbb{Q}}\left[1_{\tau\leq T}1_{\xi=S}\delta^S V(\tau)^+ e^{-r\tau}\right];\\
&= \delta^S E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left[1_{\tau\leq T}1_{\xi=S}V(\tau)^+ e^{-r\tau} \mid \mathcal{G}_\infty\right]\right];\\
&= \delta^S E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left[\int_0^T \gamma\frac{\gamma^S}{\gamma}V(t)^+ e^{-(r+\gamma)t}dt \mid \mathcal{G}_\infty\right]\right];\\
&= \delta^S E^{\mathbb{Q}}\left[\int_0^T \gamma^S V(t)^+ e^{-(r+\gamma)t}dt\right].
\end{aligned}$$

Above, we have used the fact that since τ is independent of $\mathcal{G}_\infty$, it has $\mathcal{G}_\infty$ conditional density equal to its regular density of $\gamma(t)e^{-\int_0^t\gamma(u)du}$. Also, we have used that the probability of $\xi = S$ is γ^S/γ. A similar calculation gives

$$\mathrm{DVA}(0)_t = \delta^B E^{\mathbb{Q}}\left[\int_0^T \gamma^B V(t)^- e^{-(r+\gamma)t}dt\right].$$

(c) Using that $V(t) = X_t - X_0e^{rt}$:

$$\begin{aligned}
\mathrm{CVA}(0) &= \delta^S\gamma^S E^{\mathbb{Q}}\left[\int_0^T V(t)^+ e^{-rt}e^{-\gamma t}dt\right];\\
&= \delta^S\gamma^S\int_0^T E^{\mathbb{Q}}\left[\left(e^{-rt}X_t - X_0\right)^+\right]e^{-\gamma t}dt.
\end{aligned}$$

Now, since $dX_t/X_t = rdt + \sigma dW_t^{\mathbb{Q}}$ we see that

$$e^{-rt}X_t = X_0e^{\sigma W_t^{\mathbb{Q}} - (1/2)\sigma^2 t}; \qquad t \geq 0.$$

But, this is just the stock price in a Black-Scholes model where the interest rate is 0. From here, we see that

$$E^{\mathbb{Q}}\left[\left(e^{-rt}X_t - X_0\right)^+\right] = C^{BS}(0, X_0; 0, \sigma, t, X_0),$$

i.e., the call price for an at-the-money call with maturity t, in a model where the interest rate is 0. Plugging this in, we see that

$$\begin{aligned}
\mathrm{CVA}(0) &= \delta^S\gamma^S\int_0^T C^{BS}(0, X_0; 0, \sigma, t, X_0)e^{-\gamma t}dt;\\
&= \frac{\delta^S\gamma^S}{\gamma}\int_0^T C^{BS}(0, X_0; 0, \sigma, t, X_0)\gamma e^{-\gamma t}dt;\\
&= \frac{\delta^S\gamma^S}{\gamma}E^{\mathbb{Q}}\left[1_{\tau\leq T}C^{BS}(0, X_0; 0, \sigma, \tau, X_0)\right],
\end{aligned}$$

where the last equality follows since τ has density $\gamma e^{-\gamma t}$. Similarly, for the DVA, using that $x^- = (-x)^+$:

$$
\begin{aligned}
\mathrm{DVA}(0) &= \delta^B \gamma^B E^{\mathbb{Q}}\left[\int_0^T V(t)^- e^{-rt} e^{-\gamma t} dt\right]; \\
&= \delta^B \gamma^B \int_0^T E^{\mathbb{Q}}\left[\left(e^{-rt} X_t - X_0\right)^-\right] e^{-\gamma t} dt; \\
&= \delta^B \gamma^B \int_0^T E^{\mathbb{Q}}\left[\left(X_0 - e^{-rt} X_t\right)^+\right] e^{-\gamma t} dt; \\
&= \delta^B \gamma^B \int_0^T P^{BS}(0, X_0; 0, \sigma, t, X_0) e^{-\gamma t} dt; \\
&= \frac{\delta^B \gamma^B}{\gamma} \int_0^T P^{BS}(0, X_0; 0, \sigma, t, X_0) \gamma e^{-\gamma t} dt; \\
&= \frac{\delta^B \gamma^B}{\gamma} E^{\mathbb{Q}}\left[1_{\tau \leq T} P^{BS}(0, X_0; 0, \sigma, \tau, X_0)\right].
\end{aligned}
$$

(d) A straight-forward calculation shows that

$$
\begin{aligned}
C^{BS}(0, X_0; 0, \sigma, t, X_0) &= X_0\left(N\left(\frac{1}{2}\sigma\sqrt{t}\right) - N\left(-\frac{1}{2}\sigma\sqrt{t}\right)\right); \\
&= X_0\left(2N\left(\frac{1}{2}\sigma\sqrt{t}\right) - 1\right),
\end{aligned}
$$

where we have used that $N(-a) = 1 - N(a)$ for $a > 0$. We thus have

$$
\begin{aligned}
\mathrm{CVA}(0) &= \delta^S \gamma^S \int_0^T C^{BS}(0, X_0; 0, \sigma, t, X_0) e^{-\gamma t} dt; \\
&= \frac{\delta^S \gamma^S X_0}{\gamma} \int_0^T \left(2N\left(\frac{1}{2}\sigma\sqrt{t}\right) - 1\right) \gamma e^{-\gamma t} dt; \\
&= \frac{\delta^S \gamma^S X_0}{\gamma} \left(2\int_0^T N\left(\frac{1}{2}\sigma\sqrt{t}\right) \gamma e^{-\gamma t} dt - \left(1 - e^{-\gamma T}\right)\right).
\end{aligned}
$$