

MF 731 Corporate Risk Management

Final Exam, December 14, 2021

This is the final exam. There are 4 questions for a total 100 points. Each question may contain multiple parts. You have between 9:00 and 11:00 AM to complete the exam

The exam is closed book, notes, cheat sheets, calculator, smart phone and smart watch.

You must upload your answers to Questrom Tools by 11:15 AM. If you type your answers or write them on a note-taking software program, upload your file to Questrom tools. If you write your answers on paper, take a picture of each page you would like to submit and upload the picture file to Questrom tools.

Write your name on every page of your exam (i.e. on every sheet of paper that you turn in)!

If you are stuck on a problem, MOVE ON to other parts of the exam and come back later. Also if you are unsure of the answer, write as much as you can so that you can receive partial credit. Blank answers will receive 0 points. Also, please explain your reasoning/provide a derivation for your answers. Answers with no explanation will also receive no credit. Good luck!

Probability Distributions.

- (1) Bernoulli: $X \sim B(p)$.

$$\mathbb{P}[X = 1] = p; \quad \mathbb{P}[X = 0] = 1 - p; \quad 0 < p < 1.$$

$$\text{Mean: } p. \text{ Variance: } p(1 - p). \text{ Skewness: } \frac{1-2p}{\sqrt{p(1-p)}}.$$

- (2) Normal: $X \sim N(\mu, \sigma^2)$.

$$F(x) = \mathbb{P}[X \leq x] = \int_{-\infty}^x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy; \quad x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0.$$

$$\text{Mean: } \mu. \text{ Variance: } \sigma^2. \text{ Skewness: } 0.$$

- (3) Exponential: $X \sim \text{Exp}(\lambda)$.

$$F(x) = 1 - e^{-\lambda x}; \quad x \geq 0, \lambda > 0.$$

$$\text{Mean: } \frac{1}{\lambda}. \text{ Variance: } \frac{1}{\lambda^2}. \text{ Skewness: } 2.$$

- (4) Poisson: $X \sim \text{Poi}(\lambda)$.

$$p(x) = \mathbb{P}[X = x] = e^{-\lambda} \frac{\lambda^x}{x!}; \quad x = 0, 1, 2, \dots, \lambda > 0.$$

$$\text{Mean: } \lambda. \text{ Variance: } \lambda. \text{ Skewness } \lambda^{-1/2}.$$

- (5) Gamma: $X \sim \text{Gamma}(\alpha, \beta)$.

$$F(x) = K(\alpha, \beta) \int_0^x y^{\alpha-1} e^{-\beta y} dy; \quad x \geq 0, \alpha, \beta > 0.$$

$$\text{Mean: } \frac{\alpha}{\beta}. \text{ Variance: } \frac{\alpha}{\beta^2}. \text{ Skewness: } \frac{2}{\sqrt{\alpha}}.$$

- (6) Pareto: $X \sim \text{Pareto}(x_m, \alpha)$.

$$F(x) = 1 - \left(\frac{x_m}{x}\right)^\alpha; \quad x > x_m > 0, \alpha > 0.$$

- (7) Generalized Extreme Value: $X \sim \text{GEV}(\xi, \mu, \sigma^2)$.

$$F(x) = H_{\xi, \mu, \sigma}(x) = \begin{cases} e^{-e^{-\frac{x-\mu}{\sigma}}} & \xi = 0, x \in \mathbb{R} \\ e^{-(1+\xi\frac{x-\mu}{\sigma})^{-\frac{1}{\xi}}} & \xi \neq 0, 1+\xi\frac{x-\mu}{\sigma} > 0 \end{cases}; \quad \mu \in \mathbb{R}, \sigma > 0.$$

- (8) Generalized Pareto: $X \sim \text{GP}(\xi, \beta)$.

$$F(x) = G_{\xi, \beta}(x) = \begin{cases} 1 - e^{-\frac{x}{\beta}} & \xi = 0, x \geq 0 \\ 1 - \left(1 + \frac{\xi x}{\beta}\right)^{-\frac{1}{\xi}} & \xi \neq 0, 1 + \frac{\xi x}{\beta} \geq 0, x \geq 0 \end{cases}; \quad \beta > 0.$$

Abbreviations.

- (1) pdf: probability density function.
- (2) pmf: probability mass function.
- (3) cdf: cumulative distribution function.
- (4) iid: independent, identically distributed.
- (5) rv: random variable.

1. (30 Points) Short Questions on Model Risk and Extreme Value Theory.

- (a) **(15 Points)** Outline the method by which we account for uncertainty in our VaR estimates due to uncertainty in our estimation of the log return volatility σ . Here, you may assume that at time t , we are using linearized losses (for a given dollar position $\theta_t > 0$ in the asset), and given our information at t , the next day's log return is normally distributed with mean $\mu_{t+\Delta}$ and variance $\sigma_{t+\Delta}^2$. You may also assume we have realizations $\{x_n\}_{n=1}^N$ (either through historical data or Monte Carlo simulation) of the log returns.
- (b) **(15 Points)** Outline the threshold exceedances method for estimating the VaR for a given (loss) random variable X . You may assume we have realizations $\{x_n\}_{n=1}^N$ (either through historical data or Monte Carlo simulation) of X which will yield $\xi > 0$ in the generalized Pareto fitting procedure.

2. (30 Points) CVA for a Defaultable Bond using Recovery at Treasury. Assume at time $t = 0$ the buyer ("B") enters into the following OTC contract with a counter-party ("S"). For a given reference entity R , if R defaults prior to T , then S will give B $\delta^R \in (0, 1)$ notional of a default free zero coupon bond (ZCB) maturing at T . In exchange, B will pay S a one time payment at time 0 to make the contract fairly valued.

Below, we assume the interest rate r and all default intensities $\gamma^R, \gamma^B, \gamma^S$ are constant.

- (a) **(15 Points)** For $t \leq T$ and assuming R has not defaulted by t , identify the theoretical value of the OTC contract $V(t)$, ignoring default of B or S . In particular, find the time 0 value of the contract $V(0)$ which B must pay S at initiation.
- (b) **(15 Points)** Assuming the default times of R, B and S are independent, identify the time 0 CVA of the contract. Be as explicit as possible.

3. (20 Points) A Better Way to Estimate Liquidity Adjusted Value at Risk. One problem with the way we computed the liquidity adjusted VaR over the horizon $[t, t + \Delta]$ was that we added the theoretical VaR (accounting for the log return over $[t, t + \Delta]$) to the cost of liquidating our position at t . This is inconsistent. We either liquidate at t and do not hold the position over $[t, t + \Delta]$, or we liquidate at $t + \Delta$ and should compute the liquidity cost then.

To remedy this, assume we hold $\lambda_t > 0$ shares of the stock over $[t, t + \Delta]$ and then liquidate at $t + \Delta$, facing a proportional bid-ask spread of $s_{t+\Delta}$. Assume further that for the theoretical mid-price S , the log return $X_{t+\Delta} \stackrel{\mathcal{F}_t}{\sim} N(\mu_X, \sigma_X^2)$ is normally distributed given our information at t .

- (a) **(10 Points)** Assume the spreads $s_t = s_{t+\Delta} = s$ are constant. Explicitly identify the VaR for linearized losses.

- (b) **(10 Points)** Assume instead that $s_{t+\Delta} = s_t + \zeta_{t+\Delta}$ where $\zeta_{t+\Delta} \stackrel{\mathcal{F}_t}{\sim} N(\mu_s, \sigma_s^2)$ is independent of $X_{t+\Delta}$. Write down the main steps required to estimate the VaR for full losses including the liquidation costs in a Monte Carlo simulation.

Hint: The “value” of our portfolio at both t and $t + \Delta$ is the value assuming we liquidate. In particular to obtain V_t , we (pretend we) sold our shares at S_t^b .

4. (20 Points) LDA with Exponential Distributions. Assume we have training data $\{(g_n, x_n)\}_{n=1}^N$ where for each n , $x_n \in \mathbb{R}_{++}^M$. In other words, each of the M data features has strictly positive numerical value. In this exercise, we ask what happens if use the LDA methodology, not with Gaussian conditional distributions, but rather with exponential conditional distribution. More precisely, let $\lambda_0, \lambda_1 \in \mathbb{R}_{++}^M$ and assume that given $G = k (k = 0, 1)$ the random variable X has pdf given by

$$f_k(x) = \prod_{m=1}^M \lambda_k^{(m)} e^{-\lambda_k^{(m)} x^{(m)}}, \quad x \in \mathbb{R}_{++}^M.$$

Lastly, set $\pi = \mathbb{P}[G = 1]$ so that $1 - \pi = \mathbb{P}[G = 0]$.

- (a) **(10 Points)** Set $\text{logit}(y) := \log(y/(1 - y))$. If our classification rule is

$$G(x) = 1_{\text{logit}(\mathbb{P}[G=1 \mid X=x]) \geq 0},$$

is the decision boundary linear? Justify your result.

- (b) **(10 Points)** The optimal $\pi, \lambda_0, \lambda_1$ are found by maximum likelihood, where we maximize $\ell(\pi, \lambda_0, \lambda_1)$ defined by

$$\ell(\pi, \lambda_0, \lambda_1) := \log \left(\prod_{n=1}^N \mathbb{P}[G = g_n, X = x_n] \right),$$

where $\mathbb{P}[G = g_n, X = x_n]$ is the joint pmf/pdf of (G, X) . Explicitly identify the optimal $\hat{\pi}, \hat{\lambda}_1, \hat{\lambda}_2$, and argue why they are “natural”.