Credit Risk

MF 731 Corporate Risk Management

Outline

Credit risk basics.

Bond pricing with deterministic hazard rates.

Credit default swaps.

Sampling default times.

Treating multiple defaults: copulas.

(these notes complement MF772 (Credit Risk))

Credit Risk Basics

Recall the definition of credit risk:

"The risk of not receiving promised payments of outstanding debts due to the default of the borrower."

Two main types of default.

Issuer: of a security or bond. issurer cannot pay you back the principle or interest.

Counterparty: in an over the counter derivative contract.

This week: issuer default.

Next week: counterparty default.

Hazard Rate Models

A bond is issued by a company which might default.

We do not know exactly when default may occur, though.

Let τ denote the (random) default time.

$(\Omega, \mathcal{F}, \mathbb{Q})$: probability space. $\mathbb{Q}$: risk-neutral measure.

$\tau : \Omega \mapsto [0, \infty]$ a random variable.

$\tau = \infty$: no default.

Hazard Rate Models

We will assume:

- τ is a continuous r.v..
  - 1) Hence $\mathbb{Q}[\tau = 0] = 0$: no default at 0. no chance of imidiate defult
- 2) $\mathbb{Q}[\tau = \infty] = 0$.
  - Company will default before the end of time.
  - Assumption not necessary, but makes the formulas nicer.
- 3) $\mathbb{Q}[\tau > t] > 0$ for all $t \geq 0$.
  - Some chance of not defaulting over any horizon $[0, t]$.

As such $F(t) = \mathbb{Q}[\tau \leq t] = 1 - e^{-\int_0^t \gamma(u)du}$. exp dist

- γ: the *hazard* function of τ.

$\gamma(t) \equiv \gamma > 0 \quad \forall t > 0$

$F(t) = 1 - e^{-\gamma t} \Rightarrow \tau \overset{\mathbb{Q}}{\sim} \exp(\gamma)$

$$F(t) = 1 - e^{-\int_0^t \gamma(u)du}$$

$\frac{Q(t < \tau \le t+h)}{Q(\tau > t)}$

Since $\gamma(t) = \frac{d}{dt}(-\log(1 - F(t)))$:

$= \frac{1}{h} \frac{1 - e^{-\int_0^{t+h} \gamma(u)du} - (1 - e^{-\int_0^t \gamma(u)du})}{1 - e^{-\int_0^t \gamma(u)du}}$

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{1}{h} \mathbb{Q}\left[\tau \le t + h \,\middle|\, \tau > t\right] &= \lim_{h \downarrow 0} \frac{F(t+h) - F(t)}{h(1 - F(t))}; \\
&= \frac{1}{1 - F(t)} \frac{d}{dt} F(t); \\
&= \frac{d}{dt}\left(-\log(1 - F(t))\right); \\
&= \gamma(t).
\end{aligned}$$

$t \quad t+h$

instantenous

likelihood rate of default

(slope)

$-\log(1 - F(t)) = \int_0^t \gamma(u)du$

$Q[\tau \le t+h \mid \tau > h] = \gamma(t)h$

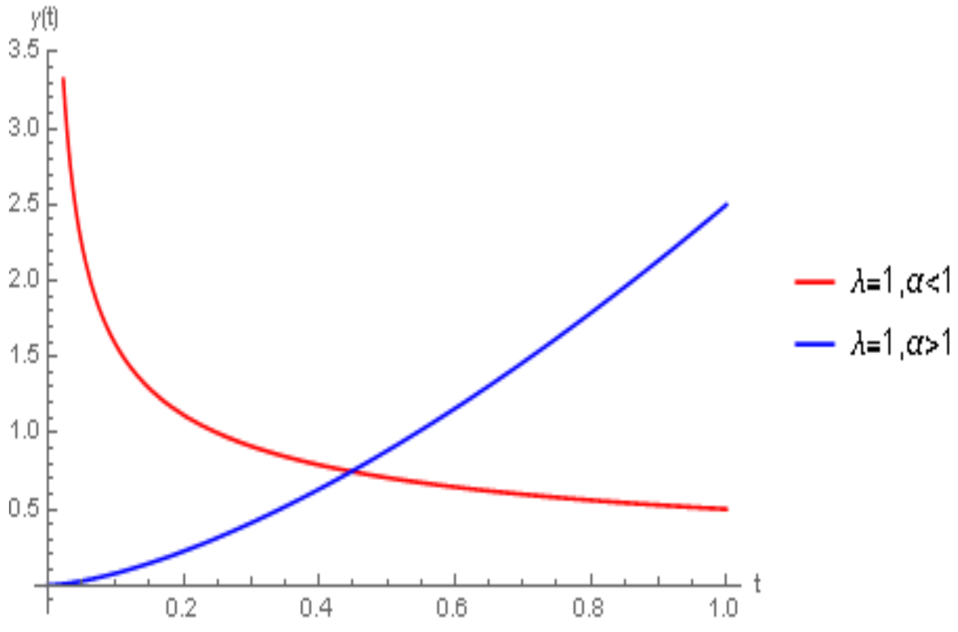
$\gamma(t)$: rate of default at t, given no default up to t.

Typically, we start with γ and then produce τ.

Example: Weibull Distribution

$F(t) = 1 - e^{-\lambda t^{\alpha}}$ for $\lambda, \alpha > 0$. $\alpha=1$, exponential dist.

$\gamma(t) = \lambda \alpha t^{\alpha - 1}$.



Information Flow

At t, our information $\mathcal{F}_t$ consists only of knowing the answer to

"Has the company defaulted by time t?"

No other randomness to add to our information set.

$\{\omega \mid \tau(\omega) \leq t\} \in \mathcal{F}_t \quad - \forall t \geq 0$

The event $\{\tau \leq t\}$ is $\mathcal{F}_t$ measurable for all $t \geq 0$.

τ is a $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ stopping time.

Conditional Expectations w.r.t. $\mathbb{F}$

unconditional

$$F(t) = 1 - e^{-\int_0^t \gamma(u)du} \Rightarrow \tau \text{ has p.d.f. } \gamma(t)e^{-\int_0^t \gamma(u)du} . = f(t)$$

$$\mathbb{Q}\left[t < \tau \le \ell \mid \tau > t\right] = \int_t^\ell \gamma(u)e^{-\int_t^u \gamma(v)dv}du.$$

$$\mathbb{Q}\left[t < \tau \le \ell\right] = \int_t^\ell \gamma(u)e^{-\int_0^u \gamma(v)dv}du.$$

$$\mathbb{Q}\left[\tau > t\right] = e^{-\int_0^t \gamma(v)dv}.$$

$$\mathbb{Q}(t < \tau \le \ell \mid \tau > t) = \frac{\mathbb{Q}(t < \tau \le \ell)}{\mathbb{Q}(\tau > t)} = \frac{\int_t^\ell \gamma(u) e^{-\int_0^u \gamma(u)du} du}{1-(1-e^{-\int_0^t \gamma(u)du})} = \underbrace{\int_t^\ell \gamma(u) e^{-\int_t^u \gamma(u)du} du}_{\text{cond. cdf}}$$

Given $\tau > t$, τ has p.d.f. $\gamma(u)e^{-\int_t^u \gamma(v)dv}$. ← cond. pdf

Using this, one can show for all g that

$$E^{\mathbb{Q}}\left[\mathbb{1}_{\tau>t}g(\tau) \mid \mathcal{F}_t\right] = \mathbb{1}_{\tau>t}\underbrace{\int_t^\infty g(u)\gamma(u)e^{-\int_t^u \gamma(v)dv}du}_{= E[g(\tau) \mid \tau > t] \text{ by def}}.$$

Given $\tau > t$.

Risk Neutral Bond Pricing

As default is the only source of uncertainty, the money market rate $r = \{r(t)\}_{t \geq 0}$ is deterministic.

We consider two securities:

- Default-free zero coupon bond (ZCB): pays \$1 at maturity T with certainty.
- Defaultable ZCB: pays \$1 at T if the firm has not defaulted.
  - For now, no recovery of principal.

Risk Neutral Bond Pricing

$p_0(t, T)$: default-free ZCB price at t.

$$p_0(t,T) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} \;\middle|\; \mathcal{F}_t\right] = e^{-\int_t^T r(u)du}.$$

r is not random.

$p_1(t, T)$: defaultable ZCB price at t.

$$p_1(t,T) = E^{\mathbb{Q}}\left[\underbrace{1_{\tau>T}}_{\text{if no default}} e^{-\int_t^T r(u)du} \;\middle|\; \mathcal{F}_t\right] = 1_{\tau>t} e^{-\int_t^T (r(u)+\gamma(u))du}.$$

Conditional expectation result for $g(y) = 1_{y>T}$.

$g(u) = 1_{u>T}$

$$p_1(t,T) = e^{-\int_t^T r(u)du} 1_{\tau>t} \times \int_t^\infty g(u)\gamma(u) e^{-\int_t^u \gamma(u)du} du$$

$$= e^{-\int_t^T r(u)du} 1_{\tau>t} \int_T^\infty \gamma(u) e^{-\int_t^u \gamma(u)du} du$$

$$= e^{-\int_t^T r(u)du} 1_{\tau>t} \left(-e^{-\int_t^u \gamma(u)du}\Big|_{u=t}^{\cancel{u=\infty}\to 0}\right)$$

$$= 1_{\tau>t} e^{-\int_t^T (r(u)+\gamma(u))du}$$

Effect of default on pricing:

We discount at the higher rate $r + \gamma$.

$\mathbb{Q}(\tau=\infty)=0 = e^{-\int_0^\infty \gamma(u)du}$

Recovery

In practice, when a bond defaults, there is some recovery of principal.

Result of a complicated legal process.

We consider the easiest recovery model.

Upon default the bond holder receives $(1 - \delta)$ in cash.

δ: loss given default.

"Recovery at Face" (RF) methodology.

Bond Pricing with Recovery

$$p_1^{RF} = E^{Q}\left[1_{\tau>t}\left(1_{\tau>T} e^{-\int_t^T r(v)dv} + (1-\delta)1_{t<\tau\le T} e^{-\int_t^\tau r(u)du}\right) \mid \mathcal{F}_t\right]$$

$$= p_1(t,T) + 1_{\tau>t} E^{Q}\left[(1-\delta)1_{t<\tau\le T} e^{-\int_t^\tau r(u)du} \mid \mathcal{F}_t\right]$$

$$= p_1(t,T) + 1_{\tau>t}\int_t^T (1-\delta) e^{-\int_t^u r(u)du} \; r(u) e^{-\int_t^u r(u)du} \, du$$

RF payoffs

$1_{\tau>T}$ at T (no default).

$(1-\delta)1_{\tau\le T}$ at τ (default at τ).

Price at t given $\tau > t$:

$$p_1^{RF}(t,T) = E^{\mathbb{Q}}\left[1_{\tau>T} e^{-\int_t^T r(u)du} + (1-\delta)1_{t<\tau\le T} e^{-\int_t^\tau r(u)du} \;\middle|\; \mathcal{F}_t\right];$$

$$= p_1(t,T) + (1-\delta)1_{\tau>t}\int_t^T \gamma(s) e^{-\int_t^s (r(u)+\gamma(u))du} ds.$$

(*cond. expect. result for* $g(y) = 1_{y\le T} e^{-\int_t^y r(u)du}$).

Summary of Bond Prices

Default free ZCB:

$$p_0(t,T) = e^{-\int_t^T r(u)du}.$$

Defaultable ZCB with 0 recovery:

$$p_1(t,T) = 1_{\tau>t} e^{-\int_t^T (r(u)+\gamma(u))du}.$$

Defaultable ZCB with RF recovery:

$$p_1^{RF}(t,T) = p_1(t,T) + (1-\delta)1_{\tau>t}\int_t^T \gamma(s) e^{-\int_t^s (r(u)+\gamma(u))du} ds.$$

Credit Default Swaps

A credit default swap (CDS) is a contract between two parties regarding the default of an underlying reference entity (R).

The protection buyer (B) pays the protection seller (S) a fixed premium to ensure against default of R.

There are many technical details regarding the CDS contract. Basic mechanics:

Today $= t$. Maturity $= t + N$. N year swap.

CDS Mechanics

Premium payments occur at $t_n = t + n/\Delta$, $n = 1, ..., N\Delta$.

Δ: payment frequency. Typically: $\Delta = 4$ (quarterly).

At t_n, if default has not occurred $(\tau > t_n)$, B pays x/Δ to S.

x: annualized CDS spread.

E.g.: $x = 200bp$, $\Delta = 4$. Quarterly payment of $50bp$.

After default, no further payments are made.

CDS Mechanics

If R defaults at $\tau \in (t_{n-1}, t_n]$:

Warning: many notions of default. E.g. bankruptcy, failure to pay a coupon, corporate restructuring.

S pays the loss given default δ to B. Payment is either in cash (cash settlement) or bond buy-back (physical delivery).

Idea: if B owns the bond, B will get $(1 - \delta)$ from R upon default. Thus, B needs a residual protection of δ.

B pays accrued interest $x(\tau - t_{n-1})$ to S

For simplicity we will ignore this.

Pricing CDS: Premium Side

Value of the premium side (ignoring accrued interest).

B→S

$$V_t^{prem}(x) = E^{\mathbb{Q}}\left[\sum_{n=1}^{N\Delta} \frac{x}{\Delta} 1_{t_n<\tau} e^{-\int_t^{t_n} r(u)du} \;\middle|\; \mathcal{F}_t\right];$$

$$= \frac{x}{\Delta}\sum_{n=1}^{N\Delta} p_1(t, t_n);$$

$$= \frac{x}{\Delta} 1_{\tau>t} \sum_{n=1}^{N\Delta} e^{-\int_t^{t_n}(r(u)+\gamma(u))du}.$$

Pricing CDS: Default Side

Value of the default side.

$S \to B$ $$V_t^{def} = E^{\mathbb{Q}}\left[\delta 1_{t<\tau\le t+N} e^{-\int_t^\tau r(u)du} \mid \mathcal{F}_t\right];$$
$$= \delta 1_{\tau>t} \int_t^{t+N} \gamma(s) e^{-\int_t^s (r(u)+\gamma(u))du} ds.$$

The CDS spread equates the two sides.

$x = x_t$ so that $V_t^{prem}(x_t) = V_t^{def}$.

No payment at contract initiation.

$$\boxed{x_t = \delta \frac{\int_t^{t+N} \gamma(s) e^{-\int_t^s (r(u)+\gamma(u))du} ds}{\frac{1}{\Delta}\sum_{n=1}^{N\Delta} e^{-\int_t^{t_n}(r(u)+\gamma(u))du}}.}$$

A Note on Calibration

$$x_t = \delta \frac{\int_t^{t+N} \gamma(s) e^{-\int_t^s (r(u)+\gamma(u))du} ds}{\frac{1}{\Delta} \sum_{n=1}^{N\Delta} e^{-\int_t^{t_n} (r(u)+\gamma(u))du}}.$$

We can use observed CDS spreads to approximate the default intensity γ under $\mathbb{Q}$.

Assume r, γ are small and constant.

$$e^{-\int_t^{t_n} (r(u)+\gamma(u))du} \approx 1.$$

This yields the approximation

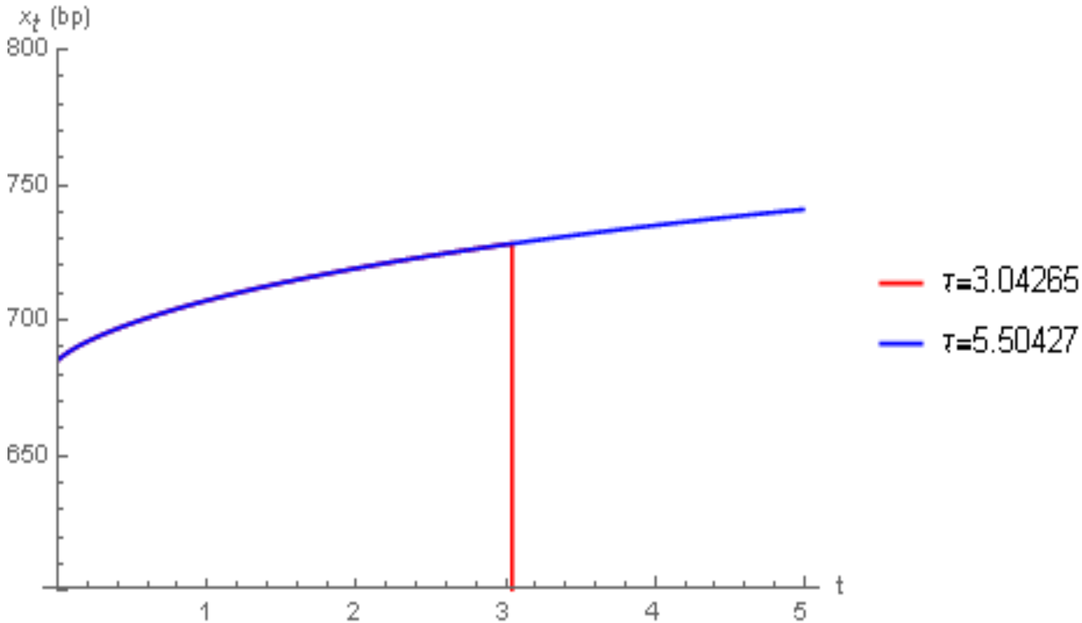
$$x_t \approx \delta\gamma \implies \gamma \approx x_t/\delta.$$ — useful if simple approx to the hazard rate

$= \frac{\delta N \gamma}{N} = \delta\gamma$

Example: Weibull intensity $\gamma(t) = \lambda\alpha t^{\alpha-1}$

$$x_t = \delta \frac{\int_t^{t+N} \gamma(s) e^{-\int_t^s (r(u)+\gamma(u))du} ds}{\frac{1}{\Delta}\sum_{n=1}^{N\Delta} e^{-\int_t^{t_n}(r(u)+\gamma(u))du}}.$$

$\lambda = 0.125$, $\alpha = 1.05$, $r \equiv 5\%$, $\delta = 0.5$, $N = 5$, $\Delta = 4$.



Sampling τ

How did we obtain τ in the last picture?

Recall: if $X \sim F$ then $F(X) \sim U(0,1)$.

Proof (for continuous F).

$\mathbb{P}[F(X) \leq t] = \mathbb{P}[X \leq F^{-1}(t)] = F(F^{-1}(t)) = t$.

Thus, $X \sim F^{-1}(U)$ where $U \sim U(0,1)$.

To sample X, we sample $U \sim U(0,1)$ and set $X = F^{-1}(U)$.

Provided we can invert F.

Sampling τ with $F(t) = 1 - e^{-\int_0^t \gamma(v)dv}$

$\mathbb{Q}(\tau = \infty) = \lim_{t\to\infty} \mathbb{Q}(\tau > t) = \lim_{t\to\infty} 1 - F(t)$

$= \lim_{t\to\infty} 1 - (1 - e^{\int_0^t \gamma(u)du})$

$= e^{-\int_0^\infty \gamma(u)du} = 0$

$$F(t) = u \iff \int_0^t \gamma(v)dv = -\log(1-u).$$

Recall: τ continuous with $F(\infty) = \mathbb{Q}[\tau < \infty] = 1$

This implies $\gamma(v) > 0$ and $\int_0^\infty \gamma(v)dv = \infty$.

For any $u \in (0,1)$ there is a unique t such that $\int_0^t \gamma(v)dv = -\log(1-u)$.

Thus, to simulate τ, we let $U \sim (0,1)$ and set

$$\tau = \min\left\{t \geq 0 \;\middle|\; \int_0^t \gamma(v)dv = -\log(U)\right\}.$$

$U \sim U(0,1)$ implies $1 - U \sim U(0,1)$.

Sampling τ : examples

Constant $\gamma(v) = \gamma > 0$.

$F(t) = 1 - e^{-\gamma t}$ so $\tau \sim \mathrm{Exp}(\gamma)$.

$\tau = \min\left\{t \mid \gamma t = -\log(U)\right\} = -\frac{1}{\gamma}\log(U)$. Sample exp dist

Weibull $\gamma(v) = \lambda\alpha t^{\alpha-1}$ so $\int_0^t \gamma(v)dv = \lambda t^\alpha$.

$\tau = \min\left\{t \mid \lambda t^\alpha = -\log(U)\right\} = \left(-\frac{1}{\lambda}\log(U)\right)^{1/\alpha}$.

Multiple Defaults

What if we own a security whose payoff depends on the default of two reference entities?

τ_1: default time of first company.

τ_2: default time of second company.

Information $\mathcal{F}_t$: we know if defaults have occurred or not by t, but nothing else.

E.g.:ZCB which pays if neither company defaults.

Time 0 price: $E^{\mathbb{Q}}\left[1_{\min\{\tau_1,\tau_2\}>T}e^{-\int_0^T r(v)dv}\right]$. 都不违约

Multiple Defaults

How can we price in this situation?

Easy way: assume τ_1, τ_2 independent! E.g. price $= e^{-\int_0^T r+\gamma_1+\gamma_2 \, du}$

$$\mathbb{Q}\left[\min\{\tau_1, \tau_2\} > T\right] = \mathbb{Q}\left[\tau_1 > T\right]\mathbb{Q}\left[\tau_2 > T\right] = e^{-\int_0^T (\gamma_1(v)+\gamma_2(v))dv}.$$

Independence implies $\min\{\tau_1, \tau_2\}$ has intensity $\gamma_1 + \gamma_2$.

But, assuming independence ignores <u>contagion</u>: not good

Default of one company increases the default likelihood for the other.

Akin to ignoring correlations of equity log returns.

Multiple Defaults and Dependence

How can we model dependence?

Recall how we sample τ_1, τ_2.

$$\tau_1 = \min \left\{ t \;\middle|\; \int_0^t \gamma_1(v)dv = -\log(U_1) \right\}.$$

$u_1, u_2 \overset{Q}{\sim} U(0,1)$

$$\tau_2 = \min \left\{ t \;\middle|\; \int_0^t \gamma_2(v)dv = -\log(U_2) \right\}.$$

Idea: introduce dependence for τ_1, τ_2 by assuming U_1, U_2 are DEPENDENT $U(0,1)$ r.v..

Joint Distributions and Copulas

How can we sample dependent (U_1, U_2)? Copula

A copula c is any map on $[0,1]^2$ such that

c is a joint cdf.

$$c(u_1, u_2) = \widetilde{\mathbb{P}}[U_1 \leq u_1, U_2 \leq u_2] \text{ for some } \widetilde{\mathbb{P}} \text{ and } U_1, U_2.$$

c gives rise to $U(0,1)$ marginals.

$$c(u_1, 1) = \widetilde{\mathbb{P}}[U_1 \leq u_1] = u_1,\ c(1, u_2) = \widetilde{\mathbb{P}}[U_2 \leq u_2] = u_2.$$

Definition extends to d r.v. $U_1, \ldots, U_d$.

Copula Examples

Independence copula: $c(u_1, u_2) = u_1 u_2$.

Corresponds to $U_1 \perp\!\!\!\perp U_2$.

$= \mathbb{P}(U_1 < u_1, U_1 < u_2)$

Minimum copula: $c(u_1, u_2) = \min\{u_1, u_2\}$.

Corresponds to $U_1 = U_2$.

Perfectly positive dependence structure.

$\mathbb{P}(1 - U_2 \leq u_1, U_2 \leq u_2) = \mathbb{P}(1 - u_1 \leq U_2 \leq u_2)$

Maximum copula: $c(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\}$. $= (u_2 + u_1 - 1)^+$

Corresponds to $U_1 = 1 - U_2$.

Perfectly negative dependence structure.

Sklar's Theorem ($\Rightarrow$)

Let F be a joint cdfwith marginals F_1, F_2. There exists a copula c such that $F(x_1, x_2) = c(F_1(x_1), F_2(x_2))$.

(proof for cont. r.v.)

Set $c(u_1, u_2) = F\left(F_1^{-1}(u_1), F_2^{-1}(u_2)\right)$ so identity holds.

Then $c(u_1, u_2) = \widetilde{\mathbb{P}}\left[X_1 \leq F_1^{-1}(u_1), X_2 \leq F_2^{-1}(u_2)\right] = \widetilde{\mathbb{P}}[\underbrace{F_1(X_1)}_{\sim U(0,1)} \leq u_1, \underbrace{F_2(X_2)}_{\sim U(0,1)} \leq u_2] = \widetilde{\mathbb{P}}[U_1 \leq u_1, U_2 \leq u_2]$.

This allows us to easily simulate (U_1, U_2).

(1) Sample $(X_1, X_2) \sim F$.

(2) Set $U_1 = F_1(X_1), U_2 = F_2(X_2)$.

$$c(u_1, 1) = F(F^{-1}(u_1), \infty) = F(F^{-1}(u_1)) = u_1$$

Sklar's Theorem ($\Leftarrow$)

For any copula c and cdf (F_1, F_2), the function $c(F_1(x_1), F_2(x_2))$ is a joint cdf with marginals F_1, F_2.

$(x_1, x_2) \mapsto c(F_1(x_1), F_2(x_2))$

(proof for cont. r.v.)

$$c(F_1(x_1), F_2(x_2)) = \widetilde{\mathbb{P}}[U_1 \leq F_1(x_1), U_2 \leq F_2(x_2)] = \widetilde{\mathbb{P}}\left[F_1^{-1}(U_1) \leq x_1, F_2^{-1}(U_2) \leq x_2\right] = \widetilde{\mathbb{P}}[X_1 \leq x_1, X_2 \leq x_2]$$

$F(x_1, 1) = \widetilde{\mathbb{P}}[U_1 \leq F_1(x_1)] = F_1(x_1)$ (same for $F(1, x_2)$).

This allows us to build joint distributions using

- Pre-specified marginal distributions.
- Pre-specified copula.

A very flexible framework.

Copulas: General Bounds

Theorem: $\max\{u_1 + u_2 - 1, 0\} \leq c(u_1, u_2) \leq \min\{u_1, u_2\}$.

$u_1 = 1 - u_2$ $\qquad\qquad$ $u_1 = u_2$

Holds for any copula.

Proof: $c(u_1, u_2) = \widetilde{\mathbb{P}}[U_1 \leq u_1, U_2 \leq u_2]$ and

$\widetilde{\mathbb{P}}[U_1 \leq u_1, U_2 \leq u_2] \leq \min\left\{\widetilde{\mathbb{P}}[U_1 \leq u_1], \widetilde{\mathbb{P}}[U_2 \leq u_2]\right\} =$
$\min\{u_1, u_2\}$.

$\widetilde{\mathbb{P}}[U_1 \leq u_1, U_2 \leq u_2] = 1 - \widetilde{\mathbb{P}}[U_1 > u_1 \bigcup U_2 > u_2] \geq$
$1 - \widetilde{\mathbb{P}}[U_1 > u_1] - \widetilde{\mathbb{P}}[U_2 > u_2] = u_1 + u_2 - 1$.

All copulas lie in between those associated with $(U, 1 - U)$ and (U, U).

Gauss Copula

Let $X \sim N(0, \Sigma)$ for $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.

X has standard normal marginals (with cdf N).

$F_1 = F_2 \sim N(0,1)$

Gauss copula:

$$\begin{aligned} c_\rho^{Ga}(u_1, u_2) &= \widetilde{\mathbb{P}}\left[X_1 \leq N^{-1}(u_1), X_2 \leq N^{-1}(u_2)\right]; \\ &= \int_{-\infty}^{N^{-1}(u_1)} \int_{-\infty}^{N^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}} \, dxdy \end{aligned}$$

Archimedean Copulas

We can generate a large class of copulas as follows:

Let $\psi : [0, \infty) \to [0, 1]$ be continuous, decreasing, with $\psi(0) = 1$, $\lim_{t \uparrow \infty} \psi(t) = 0$.

Archimedean copula: $c(u_1, u_2) = \psi\left(\psi^{-1}(u_1) + \psi^{-1}(u_2)\right)$.

E.g.: Gumbel. $\psi(t) = e^{-t^{1/\theta}}$ for $\theta \geq 1$.

$$c_\theta^{Gu}(u_1, u_2) = e^{-\left((-\log(u_1))^\theta + (-\log(u_2))^\theta\right)^{1/\theta}}.$$

$\lim_{\theta \to \infty} e^{-(-\log(u_1))\left(1 + \left(\frac{-\log(u_2)}{-\log(u_1)}\right)^\theta\right)^{1/\theta}} = \min\{u_1, u_2\}$

At $\theta = 1$ we obtain $u_1 u_2$. As $\theta \uparrow \infty$ we obtain $\min\{u_1, u_2\}$. "1)"

θ allows us to span from independence to minimum copulas.

Archimedean Copulas

E.g.: Clayton. $\psi(t) = (1 + \theta t)^{-1/\theta}$ for $\theta > 0$.

$$c_\theta^{CL}(u_1, u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$$

As $\theta \downarrow 0$ we obtain $u_1 u_2$. As $\theta \uparrow \infty$ we obtain $\min\{u_1, u_2\}$.

As before, θ allows us to span from independence to minimum copulas.

Pricing with Copulas

Let's come back to our application:

Pricing securities with dependence on multiple defaults.

Examples

No default ZCB: payoff of 1 only if $\tau_1 > T, \tau_2 > T$.

At most one default ZCB: payoff of 1 if $\tau_1 > T$ or $\tau_2 > T$.

How can we use copulas to price these bonds?

Pricing with Copulas

price at 0 $= \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(u)du} 1_{\tau_1>T, \tau_2>T}]$

Abstractly, we have the formulas

$$p_{0d}(0,T) = e^{-\int_0^T r(v)dv} E^{\mathbb{Q}}\left[1_{\min\{\tau_1,\tau_2\}>T}\right].$$

$$p_{1d}(0,T) = e^{-\int_0^T r(v)dv} E^{\mathbb{Q}}\left[1_{\max\{\tau_1,\tau_2\}>T}\right].$$

To simplify things, assume constant γ_1, γ_2:

$$\tau_1 = -\frac{1}{\gamma_1}\log(U_1), \quad \tau_2 = -\frac{1}{\gamma_2}\log(U_2), \quad (U_1, U_2) \sim c.$$

price at 0 $= e^{-\int^T r(u)du} \mathbb{Q}(\tau_1>T, \tau_2>T)$

This implies

$= e^{-\int^T r(u)du} \mathbb{Q}(U_1 < e^{-\gamma_1 T}, U_2 < e^{-\gamma_2 T})$

$$\{\min\{\tau_1,\tau_2\} > T\} = \left\{U_1 < e^{-\gamma_1 T}, U_2 < e^{-\gamma_2 T}\right\}.$$

$= e^{-\int^T r(u)du} c(e^{-\gamma_1 T}, e^{-\gamma_2 T})$

$$\{\max\{\tau_1,\tau_2\} > T\} = \left\{U_1 < e^{-\gamma_1 T}, U_2 < e^{-\gamma_2 T}\right\} \cup \left\{U_1 < e^{-\gamma_1 T}, U_2 \geq e^{-\gamma_2 T}\right\} \cup \left\{U_1 \geq e^{-\gamma_1 T}, U_2 < e^{-\gamma_2 T}\right\}.$$

Pricing with Copulas

For $p_{0d}(0, T)$ we have the formula

$$p_{0d}(0, T) = e^{-\int_0^T r(v)dv} c\left(e^{-\gamma_1 T}, e^{-\gamma_2 T}\right).$$

The formula for $p_{1d}(0, T)$ is more involved, but

We can always price $p_{1d}(0, T)$ using simulation.

For $m = 1, \ldots, M$ sample $(U_1^m, U_2^m) \sim c$, and set $\tau_1^m = -(1/\gamma_1)\log(U_1^m)$, $\tau_2^m = -(1/\gamma_2)\log(U_2^m)$.

Output $p_{1d}(0, T) = e^{-\int_0^T r(v)dv} \frac{1}{M} \sum_{m=1}^M 1_{\max\{\tau_1^m, \tau_2^m\} > T}$.

Note: we can price options with generic time T payoffs $g(\tau_1, \tau_2)$ this way.

Pricing with Copulas

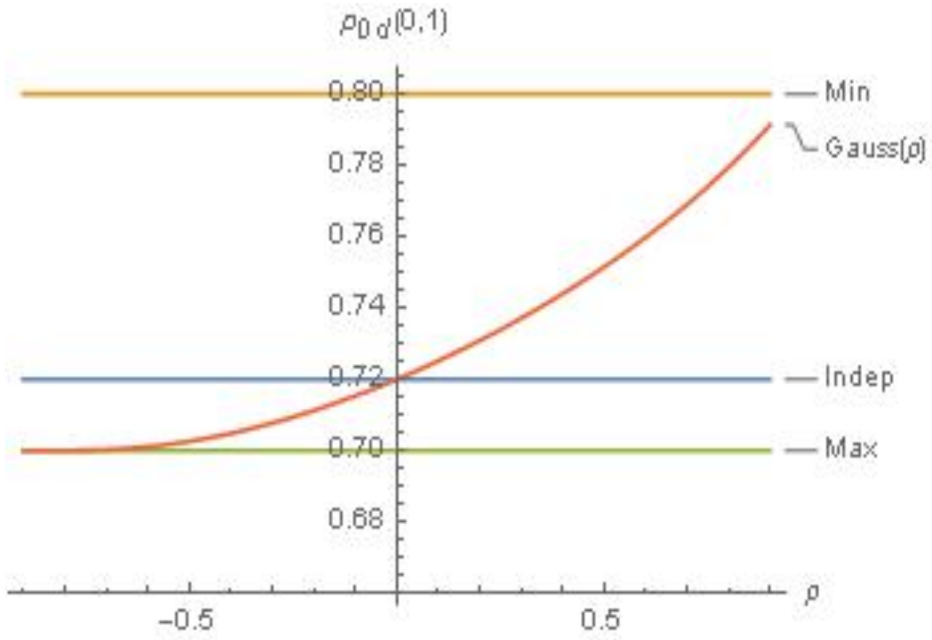
We have many copulas from which to choose.

- Independent, Minimum, Maximum, Gauss (ρ), Clayton (θ), Gumbel (θ).
- General archimedean (ψ).
- Generic $c(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2))$.

How do we choose? What are these copulas doing? What are the parameters doing?

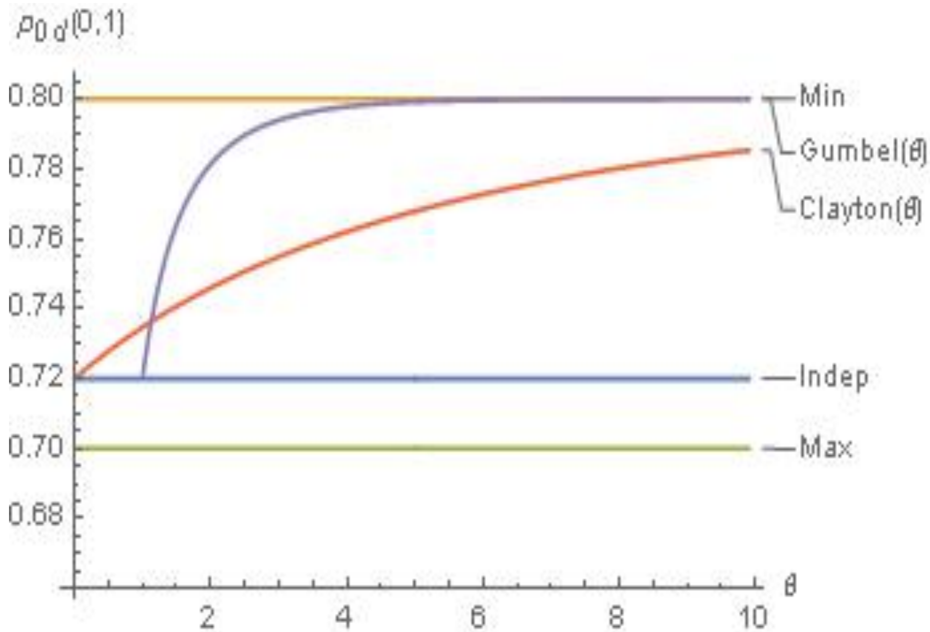
- We will focus on $p_{0d}(0, T) = c(e^{-\gamma_1 T}, e^{-\gamma_2 T})$ (set $r \equiv 0$).

$p_{0d}(0, T)$ for the Gauss Copula



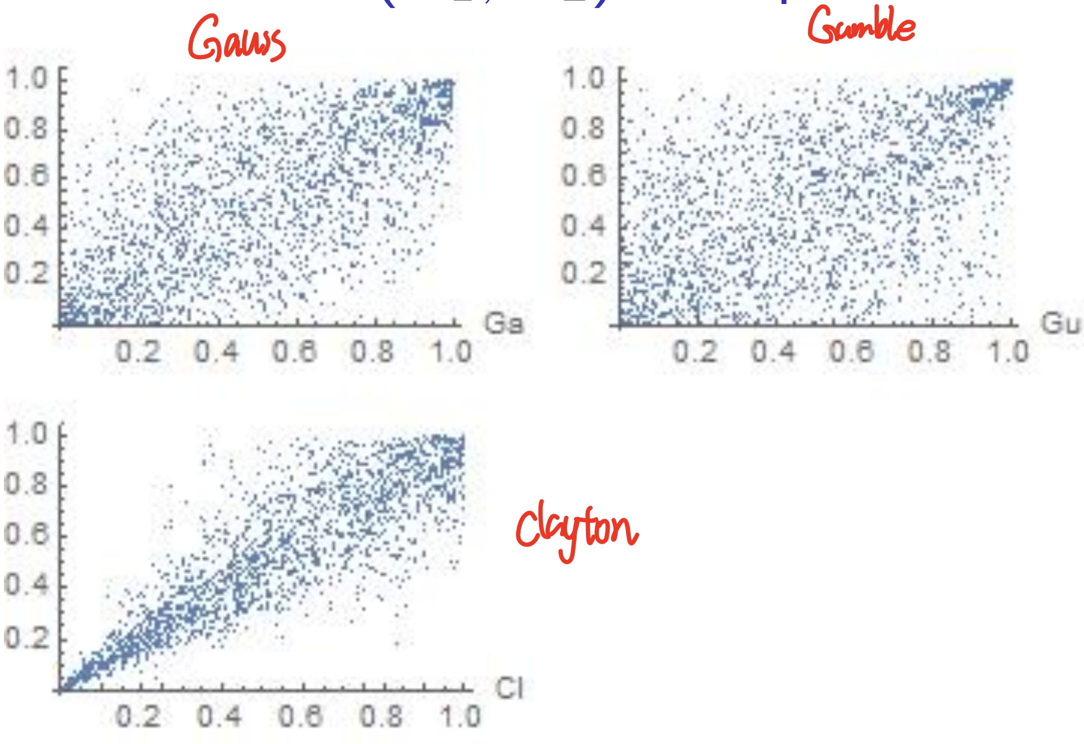
$T = 1$, γ_1: 10% one-year default prob. γ_2: 20% one-year default prob.

$p_{0d}(0, T)$ for the Gumbel,Clayton Copulas



$T = 1$, γ_1: 10% one-year default prob. γ_2: 20% one-year default prob.

5000 (U_1, U_2) Samples



Gauss (0.7); Gumbel (1.61); Clayton (5.2).

Gumbel: upper tail dep., Clayton: lower tail dep..

Tail dependence: alignment of extreme values.