HW Problems for Assignment 5 - Part 2
Due 6:30 PM Wednesday, December 2, 2020

1. (10 Points) Recover at Treasury. Recall the recovery at face (RF) methodology for a defaultable zero coupon bond (ZCB): at the default time τ the bond holder receives $(1-\delta)$ in cash per unit notional. An alternative convention is "recovery at treasury" (RT). Here, at τ the bond holder receives (per unit notional) $(1-\delta)$ notional of a default free ZCB maturing at T. As such, the payment $(1-\delta)$ is at T, rather than at τ.

In the hazard rate model discussed in class, identify the time $t \leq T$ price of a defaultable ZCB using the RT methodology, given $\tau > t$. Assuming $r(t) \geq 0, t \leq T$, how do the RT and RF prices compare?

2. (10 Points) Credit Spreads for Defaultable Bonds. For the hazard rate model discussed in class, assume r, γ are continuous functions of time. For the ZCB maturity T, and current time $t \leq T$, denote by $p_0(t,T)$, $p_1(t,T)$, and $p_1^{RF}(t,T)$ the prices for a default-free ZCB; defaultable ZCB with 0 recovery; and defaultable ZCB with $(1-\delta)$ RF recovery respectively.

Define the credit spread for 0 recovery as

$$c(t,T) \triangleq -\frac{1}{T-t}\left(\log(p_1(t,T)) - \log(p_0(t,T))\right),$$

and the credit spread for RF recovery as

$$c^{RF}(t,T) \triangleq -\frac{1}{T-t}\left(\log(p_1^{RF}(t,T)) - \log(p_0(t,T))\right).$$

On the set $\{\tau > t\}$, compute:

(a) (**5 Points**) $c(t,T)$ and $c^{RF}(t,T)$.

(b) (**5 Points**) The limits $\lim_{T\downarrow t} c(t,T)$ and $\lim_{T\downarrow t} c^{RF}(t,T)$. These are the *instantaneous credit spreads*. Here, you should get very simple expressions. Can you explain *why* they are what they are?

3. (15 Points) Credit Default Swap Spreads and Accrued Interest. Here, for the hazard rate model discussed in class, we will assume 0 interest rates (i.e. $r(t) = 0$ for all $t \geq 0$). As shown in class, if we ignore the accrued interest paid by the borrower to the seller up default, the time t credit default swap spread for entering into a N year swap as

$$x_t = \delta \frac{\int_t^{t+N} \gamma(s) e^{-\int_t^s \gamma(u)du} ds}{\frac{1}{\Delta}\sum_{n=1}^{N\Delta} e^{-\int_t^{t+n/\Delta}\gamma(u)du}}, \tag{0.1}$$

In this exercise we will not ignore accrued interest, and see how things change. Below, we set $t_n = t + n/\Delta$ for $n = 1, ..., N\Delta$. Recall that if $t_{n-1} < \tau \leq t_n$ then upon default, the buyer must pay the seller accrued interest in the amount of $x(\tau - t_{n-1})$ where x is the swap spread. As the interest rate is 0, the fair value of these payments is

$$E^{\mathbb{Q}}\left[\sum_{n=1}^{N\Delta} 1_{t_{n-1}<\tau\leq t_n} x(\tau - t_{n-1}) \;\Big|\; \mathcal{F}_t\right].$$

Simplify this formula to show that, including accrued interest, the time t swap rate is

$$x_t^{ai} = \delta \frac{\int_t^{t+N} \gamma(s) e^{-\int_t^s \gamma(u)du} ds}{\int_t^{t+N} e^{-\int_t^s \gamma(u)du} ds}, \tag{0.2}$$

which admits a simple interpretation as a weighted average of the intensity function. Looking at the formulas including and non including accrued interest, do you expect much of a difference?

4. (15 Points) Parameterized Logistic Copula. Let X_1 and X_2 be random variables with joint cdf

$$F_\theta(x_1, x_2) = \left(1 + e^{-x_1} + e^{-x_2} + (1-\theta)e^{-x_1-x_2}\right)^{-1}, \qquad x_1, x_2 \in \mathbb{R},$$

where $\theta \in [-1, 1]$.

(a) (**5 Points**) Identify the marginal distributions $F_{\theta,1}$, $F_{\theta,2}$ for X_1 and X_2 respectively.

(b) (**5 Points**) Show that when $\theta = 0$, X_1 and X_2 are independent.

(c) (**5 Points**) Show that the copula of X_1 and X_2, defined abstractly by $c_\theta(u_1, u_2) = F_\theta(F_{\theta,1}^{-1}(u_1), F_{\theta,2}^{-1}(u_2))$, takes the form

$$C_\theta(u_1, u_2) = \frac{u_1 u_2}{1 - \theta(1-u_1)(1-u_2)}.$$