# MF 731 Corporate Risk Management
# Practice Midterm Exam. Fall, 2021
# SOLUTIONS

This is the practice midterm exam. There are 5 questions for a total 100 points. Each question may contain multiple parts. You have 2 hours to complete the exam

**To get the most out of the exam** please take this exam as if it were the real exam! I.e. do not use notes, the class textbook, the internet, and especially do not look at the solutions ahead of time! Give yourself two hours to take the exam, and abide by this rule!

**1. Comparing VaR$_\alpha$ across Loss Operators (20 Points)** Consider a one stock portfolio where we hold $\lambda_t > 0$ shares of the stock constant over the period $[t, t+\Delta]$. Write $S_t > 0$ as the time $t$ price and $X_{t+\Delta}$ as the log return over $[t, t+\Delta]$, and assume we have a model for $X_{t+\Delta}$ given our information at time $t$.

(a) **(5 Points)** Write the full, linear and quadratic loss operators for the portfolio over $[t, t+\Delta]$.

(b) **(10 Points)** Can you order $\mathrm{VaR}_\alpha(L)$, $\mathrm{VaR}_\alpha\left(L^{lin}\right)$ and $\mathrm{VaR}_\alpha\left(L^{quad}\right)$? If so, provide the ordering. If not, explain why not.

(c) **(5 Points)** Discuss value at risk using quadratic losses. Can you think of any reasons for why we (at least in this situation) might not want to use quadratic losses to estimate the value at risk?

**Solution:**

(a) Write $\theta_t = \lambda_t S_t > 0$. From class we know

$$l_{[t]}(x) = -\theta_t(e^x - 1); \quad l^{lin}_{[t]}(x) = -\theta_t x; \quad l^{quad}_{[t]}(x) = -\theta_t\left(x + \frac{1}{2}x^2\right).$$

(b) By assumption $\theta_t > 0$. Since $e^x - 1 \geq x$ and $x + x^2/2 \geq x$ we deduce $L_{t+\Delta} \leq L^{lin}_{t+\Delta}$ and $L^{quad}_{t+\Delta} \leq L^{lin}$ and hence $\mathrm{VaR}_\alpha(L_{t+\Delta}) \leq \mathrm{VaR}_\alpha\left(L^{lin}_{t+\Delta}\right)$ and $\mathrm{VaR}_\alpha\left(L^{quad}_{t+\Delta}\right) \leq \mathrm{VaR}_\alpha\left(L^{lin}_{t+\Delta}\right)$. We can make no direct comparison between $\mathrm{VaR}_\alpha$ for $L^{quad}_{t+\Delta}$ and $L^{lin}_{t+\Delta}$ without knowing any more about the model for $X_{t+\Delta}$. This is because

$$e^x - 1 > x + \frac{1}{2}x^2 \qquad \Leftrightarrow \qquad x > 0,$$

yet for any reasonable model, we would not expect either $X_{t+\Delta} > 0$ (probability one) or $X_{t+\Delta} < 0$ (probability one).

(c) If we expect large (in magnitude) values of $X_{t+\Delta}$ we should be careful with the quadratic loss operator when estimating $\mathrm{VaR}_\alpha$. This is because $\mathrm{VaR}_\alpha$ focuses on large losses, which as our dollar position in the stock is positive, we associate with large negative dollar returns. This is consistent with the full and linear losses because for $\tau$

$$\begin{aligned} L_{t+\Delta} = \tau \quad &\Leftrightarrow \quad X_{t+\Delta} = \ln\left(1 - \frac{\tau}{\theta}\right), \\ L^{lin}_{t+\Delta} = \tau \quad &\Leftrightarrow \quad X_{t+\Delta} = -\frac{\tau}{\theta}. \end{aligned}$$

Now, the full losses cannot exceed $\theta$ so a large loss is $\tau \approx \theta$ which corresponds to $X_{t+\Delta} << 0$. Similarly, the linear losses can be arbitrarily high and this too corresponds to $X_{t+\Delta} << 0$. However, for the quadratic losses we have

$$L^{quad}_{t+\Delta} = \tau \quad \Leftrightarrow \quad (x+1)^2 = 1 - \frac{2\tau}{\theta},$$

and a large loss corresponds to $\tau \approx \theta/2$. But, this corresponds to log return near $-1$. The problem is the quadric term indicates gains for large magnitude $X_{t+\Delta}$ unlike for the full and linearized losses. For this reason, care should be used when making the quadratic approximation in this setting. We need to be very confident that the likelihood $X_{t+\Delta}$ takes very large values is very small.

**2. Back-Testing $\mathrm{VaR}_\alpha$ for a Discrete Loss Distribution (20 Points)** In class we developed the back-testing methodology assuming the conditional loss distribution of $L_{t+\Delta}$ given our time $t$ information $\mathcal{F}_t$ was continuous. In this problem we will see that a similar methodology can be developed for discrete distributions.

For each $t$ assume that conditional upon our information $\mathcal{F}_t$, the losses over $[t, t+\Delta]$ take the form

$$L_{t+\Delta} = \begin{cases} -1 & \text{Probability } = \frac{1-p}{2} \\ 1 & \text{Probability } = \frac{1-p}{2} \\ \mathcal{L}_{t+\Delta} & \text{Probability } = p \end{cases}.$$

Here, $\mathcal{L}_{t+\Delta} >> 1$ is not random (i.e. NOT an $\mathcal{F}_t$ measurable random variable), and $0 < p < 1/2$ is the same for all $t$.

(a) **(6 Points)** For $(1-p)/2 < \alpha < 1$, identify $\mathrm{VaR}_\alpha^t$ and $\mathrm{ES}_\alpha^t$.

(b) **(14 Points)** Assume we have historical loss data $\left\{L_{t-(m-1)\Delta}\right\}_{m=1}^M$ over the last $M$ days. Describe the back-testing methodology for how we would check if our methodology for estimating $\mathrm{VaR}_\alpha$ is sound. Here, assume $(1-p)/2 < \alpha < 1-p$.

**Solution:**

(a) From the description of $L_{t+\Delta}$ we immediately see for $(1-p)/2 < \alpha$ that

$$\mathrm{VaR}_\alpha^t = \begin{cases} 1 & \frac{1-p}{2} < \alpha \leq 1-p \\ \mathcal{L}_{t+\Delta} & 1-p < \alpha < 1 \end{cases}.$$

Using this we conclude

$$\mathrm{ES}_\alpha^t = \begin{cases} 1 + \frac{p}{1-\alpha}\left(\mathcal{L}_{t+\Delta} - 1\right) & \frac{1-p}{2} < \alpha \leq 1-p \\ \mathcal{L}_{t+\Delta} & 1-p < \alpha < 1 \end{cases}.$$

(b) It turns out the methodology is *almost* the same as in class. We just have to make one minor adjustment. From part (a) we know for $\alpha < 1-p$ that $\mathrm{VaR}_\alpha\left(L_{t+\Delta}\right) = 1$ and hence $\mathbb{P}\left[L_{t+\Delta} > \mathrm{VaR}_\alpha \middle| \mathcal{F}_t\right] = p$. This implies the exception random variables $\left\{I_{t+\Delta} := 1_{L_{t+\Delta} > \mathrm{VaR}_\alpha^t}\right\}_t$ are iid Bernoulli random variables with parameter $p$ and not $1-\alpha$ as in class. Thus the methodology goes through exactly as in the lecture notes but with $p$ replacing $1-\alpha$. Namely, we take our historical loss data $\left\{\widehat{L}_{t-(m-1)\Delta}\right\}$ as well as our historical $\mathrm{VaR}_\alpha$ estimates $\left\{\widehat{\mathrm{VaR}_\alpha}^{t-m\Delta}\right\}$ and count the number of exceedances

$$\widehat{N}^M = \sum_{m=1}^M 1_{\widehat{L}_{t-(m-1)\Delta} > \widehat{\mathrm{VaR}_\alpha}^{t-m\Delta}}.$$

If our model is sound, $\widehat{N}^M$ should be a sample off a Binomial distribution with parameters $M, p$. Therefore, using the Central Limit Theorem, for large $M$ we expect $(\widehat{N}^M - Mp)/\sqrt{Mp(1-p)}$ to be a sample off a $N(0,1)$ random variable. Thus, we can create a $100(1-\beta)\%$ confidence interval by looking at the range $(\tau_-^\beta, \tau_+^\beta)$ for $\widehat{N}^M$ where

$$\tau_\pm^\beta = Mp \pm z_{1-\frac{\beta}{2}} \sqrt{Mp(1-p)}, \quad z_\gamma = N^{-1}(\gamma).$$

**3. True/False (5 points each).** Identify if each of the statements below is true or false. If it is true provide a short explanation or proof for why it is true. If it is false, provide a short explanation, proof, or counter-example for why it is false. Answers with no explanation will not be given any credit.

(a) For loss random variables $L_1$ and $L_2$, if $L_1 \geq L_2$ with probability one then $\mathrm{ES}_\alpha(L_1) \geq \mathrm{VaR}_\alpha(L_2)$.

(b) Assume that given $\mathcal{F}_t$ we model $X_{t+\Delta} = \sigma_{t+\Delta} Z$ where $\sigma_{t+\Delta}$ is known given $\mathcal{F}_t$, and $Z$ is independent of $\mathcal{F}_t$ with mean 0 and variance 1. Then, EWMA produces a higher estimate of $\mathbb{E}\left[\sigma^2_{t+2\Delta} \middle| \mathcal{F}_t\right]$ than GARCH.

(c) BASEL introduced $\mathrm{VaR}_\alpha$ as an appropriate tool for measuring risk in the amendment to the first BASEL accord.

(d) All coherent risk measures are convex.

**Solution:**

(a) TRUE. $L_1 \geq L_2$ implies $\mathrm{ES}_\alpha(L_1) \geq \mathrm{ES}_\alpha(L_2)$. And, we know that the expected shortfall always exceeds the value at risk so that $\mathrm{ES}_\alpha(L_2) \geq \mathrm{VaR}_\alpha(L_2)$.

(b) FALSE. In the above setting, EWMA produces $\mathbb{E}\left[\sigma^2_{t+2\Delta} \middle| \mathcal{F}_t\right] = \sigma^2_{t+\Delta}$ while GARCH produces $\mathbb{E}\left[\sigma^2_{t+2\Delta} \middle| \mathcal{F}_t\right] = \alpha_0 + (\alpha_1 + \beta_1)\sigma^2_{t+\Delta}$. Since $\alpha_1 + \beta_1 < 1$ there is no (uniform) statement we can make comparing the two values.

(c) FALSE. The amendment to BASEL I came in 1996, three years after $\mathrm{VaR}_\alpha$ was first suggested (on a wide-spread scale) to be appropriate for measuring market risk. Rather, the 1996 amendment allowed firms to use their own internal models to estimate $\mathrm{VaR}_\alpha$.

(d) TRUE. Let $0 < \lambda < 1$. By sub-additivity and positive homogeneity

$$\varrho(\lambda L_1 + (1-\lambda)L_2) \leq \varrho(\lambda L_1) + \varrho((1-\lambda)L_2) = \lambda\varrho(L_1) + (1-\lambda)\varrho(L_2)$$

**4. (20 Points) Coherent measures and portfolio variance.** Consider $d$ stocks $S^{(1)}, ...., S^{(d)}$ and an investment window $[t, t+\Delta]$. Write $X = (X^{(1)}, ..., X^{(d)})$ as the log return over $[t, t+\Delta]$, and assume that given $t$, $X \sim N(\mu, \Sigma)$. Assume we have two portfolios $P^1, P^2$ with respective dollar positions (at $t$) of $\theta$ and $\psi$. We use linearized losses $L^{1,'}, L^{2,'}$ and further assume given $t$

(1) $L^{1,'}$ and $L^{2,'}$ have the same conditional mean.
(2) $L^{1,'}$ has a strictly lower conditional variance than $L^{2,'}$.

Let $\varrho$ be a risk measure. Is it possible for $\varrho$ to be coherent if $\varrho(L^{2,'}) < \varrho(L^{1,'})$?

**Solution**

The answer is YES or NO, depending on if $\varrho(Z) < 0$ or $\varrho(Z) > 0$ for $Z \sim N(0,1)$. As such, since all reasonable risk measures assign $\varrho(Z) > 0$ the typical answer is NO. In fact, in this situation, $\varrho$ cannot even be cash additive and positively homogeneous. To see this, we can write

$$L^{1,'} = \theta^{\mathrm{T}} X \sim N\left(\theta^{\mathrm{T}}\mu, \theta'\Sigma\theta\right) = \theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta} Z^1,$$

where $Z^1 \sim N(0,1)$ given $t$. As such, for any cash-additive, positively homogenous risk measure $\varrho$ we know

$$\varrho(L^{1,'}) = -\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta}\,\varrho(Z^1).$$

Similarly, we have

$$\varrho(L^{2,'}) = -\psi^{\mathrm{T}}\mu + \sqrt{\psi^{\mathrm{T}}\Sigma\psi}\,\varrho(Z^2),$$

where $Z^2$ is also $N(0,1)$ given $t$. As $L^{1,'}$ and $L^{2,'}$ have the same mean, we know $\theta^{\mathrm{T}}\mu = \psi^{\mathrm{T}}\mu$. Next, since $Z^1, Z^2$ have the same distribution, $\varrho(Z^1) = \varrho(Z^2)$. Therefore

$$\varrho(L^{1,'}) - \varrho(L^{2,'}) = \left(\sqrt{\theta^{\mathrm{T}}\Sigma\theta} - \sqrt{\psi^{\mathrm{T}}\Sigma\psi}\right) K; \qquad K = \varrho(Z^1) = \varrho(Z^2).$$

Since $L^{1,'}$ has a strictly lower variance than $L^{2,'}$ we know

$$\sqrt{\theta^{\mathrm{T}}\Sigma\theta} - \sqrt{\psi^{\mathrm{T}}\Sigma\psi} < 0,$$

Thus, $\varrho(L^{1,'}) - \varrho(L^{2,'}) < 0$ if and only if $K > 0$, giving the result.

**5. Time Aggregation for a Forward Contract using GARCH (20 Points).** Consider a stock model where the log return and variance follow (under the physical measure) a GARCH process but with drift $r > 0$, the constant interest rate. The physical and risk neutral measures are the same, and the GARCH model is

$$
\begin{aligned}
X_{t+\Delta} &= \left(r\Delta - \frac{1}{2}\sigma_{t+\Delta}^2\right) + \sigma_{t+\Delta} Z_{t+\Delta}, \\
\sigma_{t+\Delta}^2 &= \alpha_0 + \alpha_1\left(X_t - \left(r\Delta - \frac{1}{2}\sigma_t^2\right)\right)^2 + \beta_1\sigma_t^2.
\end{aligned} \tag{0.1}
$$

Here, $\alpha_0, \alpha_1, \beta_1 > 0$, $\alpha_1 + \beta_1 < 1$ and the $\{Z_{t+k\Delta}\}_{k=-\infty}^{\infty}$ are iid $N(0,1)$ random variables. The information at $t$ is generated by $\{Z_{t+k\Delta}\}_{k=-\infty}^{0}$. You may assume this model has a solution.

Fix a strike $K > 0$ and maturity $T = k_T\Delta$. A forward contract with maturity $T$ and strike $K$ is an option with payoff $S_T - K$ at $T$. To ease notation write $t_k = k\Delta$ for any integer $k$.

(a) **(6 Points)** As the physical and risk neutral measures are the same, for integers $k \le k_T$ the forward price at $t_k$ is

$$V_{t_k} = \mathbb{E}\left[e^{-r(T-t_k)}(S_T - K)\middle|\mathcal{F}_{t_k}\right].$$

Show that $\mathbb{E}\left[e^{-r(T-t_k)}S_T\middle|\mathcal{F}_{t_k}\right] = S_{t_k}$ and hence if $K = S_{t_0}e^{r(T-t_0)}$ the forward contract has 0 value at $t_0 = 0$. **Note:** you must prove your result. Just saying the discounted stock price is a martingale under the risk neutral measure is not sufficient!

(b) **(14 Points)** Fix $K$ from part (a). Assume our portfolio consists of $N$ forward contracts, and our horizon is $[t_m, t_M]$, where $0 < m < M < k_T$. Write the pseudo code for how you estimate/obtain the value at risk for the $m$ day full losses $L := -\left(V_{t_M} - V_{t_m}\right)$, $\mathrm{VaR}_\alpha(L)$ given our information at $t_m$. Here, you may assume that $\alpha_0, \alpha_1, \beta_1$ are given and $S_{t_m}, X_{t_m}, \sigma_{t_m+\Delta}$ are known at $t_m$.

**Solution:**

(a) By construction of the model

$$
\begin{aligned}
\mathbb{E}\left[S_T\middle|\mathcal{F}_{T-\Delta}\right] &= S_{T-\Delta}\mathbb{E}\left[e^{X_T}\middle|\mathcal{F}_{T_\Delta}\right], \\
&= S_{T-\Delta}e^{\left(r\Delta - \frac{1}{2}\sigma_T^2\right)}\mathbb{E}\left[e^{\sigma_T Z_T}\middle|\mathcal{F}_{T-\Delta}\right], \\
&= S_{T-\Delta}e^{r\Delta}.
\end{aligned}
$$

Above, the last line follows because $\sigma_T$ is known given $\mathcal{F}_{T-\Delta}$, $Z_T$ is independent of $\mathcal{F}_{T-\Delta}$ and $Z \sim N(0,1)$. Repeating this argument

$$
\begin{aligned}
\mathbb{E}\left[S_T\middle|\mathcal{F}_{T-2\Delta}\right] &= \mathbb{E}\left[\mathbb{E}\left[S_T\middle|\mathcal{F}_{T-\Delta}\right]\middle|\mathcal{F}_{T-2\Delta}\right] \\
&= e^{r\Delta}\mathbb{E}\left[S_{T-\Delta}\middle|F_{T-2\Delta}\right] = e^{2r\Delta}S_{T-2\Delta},
\end{aligned}
$$

and hence

$$\mathbb{E}\left[e^{-r(T-t_k)}S_T \middle| \mathcal{F}_{t_k}\right] = e^{-r(t_K-t_k)\Delta}S_{t_k}e^{r(t_K-t_k)\Delta} = S_{t_k}.$$

(b) From part $(a)$ we know for $t_k$ our portfolio value is $N(S_{t_k} - S_{t_0}e^{r(t_k-t_0)\Delta})$. Therefore

$$L = -N\left(S_{t_m}\left(e^{\sum_{k=m+1}^{M} X_{t+k\Delta}} - 1\right) - S_{t_0}e^{rm\Delta}\left(e^{r(M-m)\Delta} - 1\right)\right). \tag{0.2}$$

Therefore to obtain the value at risk, we run the following simulation: for $p = 1, ..., P$

(i) For $k = m+1, ..., M-1$ we sample $Z^p_{t_k} \sim N(0,1)$ and set both $X^p_{t_k}$, $\sigma^{2,p}_{t_k+\Delta}$ via the formulas in (0.1).

(ii) For $k = M$ we sample $Z^p_{t_M} \sim N(0,1)$ and set $X^p_{t_M}$ as in (0.1).

(iii) We then compute $\ell_p = L^p$ as in (0.2).

We then estimate the value at risk as $\ell_{(\lceil P\alpha \rceil)}$.