Recommended HW Problems for Assignment 6 - Part I
SOLUTIONS

Note : you do not have to turn in these problems, but you should definitely try them, as they will help you study for the final!

1. Recover at Treasury. Recall the recovery at face (RF) methodology for a defaultable zero coupon bond (ZCB): at the default time τ, for every unit notional the bond holder owns, she receives $(1-\delta)$ in cash. An alternative way to model recovery is "recovery at treasury" (RT). Here, at the default time τ, rather than receiving cash, the bond holder receives $(1-\delta)$ notional of a default free ZCB maturing at T.

In the hazard rate model discussed in class, identify the time t price ($t \le T$) of a defaultable ZCB using the RT methodology, given $\tau > t$. Assuming the money market rate is non-negative, how do the two prices compare?

Solution The payoff for the defaultable ZCB under the RT method is

(1) 1 at T if no default.
(2) $(1-\delta)$ at T if default at $\tau \le T$.

This has time t price

$$\begin{aligned} p_1^{RT}(t,T) &= E^{\mathbb{Q}}\left[\left(1_{\tau>T} + (1-\delta)1_{t<\tau\le T}\right)e^{-\int_t^T r(u)du} \mid \mathcal{F}_t\right]; \\ &= p_1(t,T) + (1-\delta)1_{\tau>t}e^{-\int_t^T r(u)du}\int_t^T \gamma(s)e^{-\int_t^s \gamma(u)du}ds. \end{aligned}$$

Recall the the (RF) price

$$p_1^{RF}(t,T) = p_1(t,T) + (1-\delta)1_{\tau>t}\int_t^T \gamma(s)e^{-\int_t^s (r(u)+\gamma(u))du}ds.$$

To compare the two, note that if the money market rate is non-negative then

$$\begin{aligned} e^{-\int_t^T r(u)du}\int_t^T \gamma(s)e^{-\int_t^s \gamma(u)du}ds &= \int_t^T \gamma(s)e^{-\int_t^s (r(u)+\gamma(u))du} \times e^{-\int_s^T r(u)du}ds; \\ &\le \int_t^T \gamma(s)e^{-\int_t^s (r(u)+\gamma(u))du}ds. \end{aligned}$$

Thus, the RT method gives a lower price. Intuitively this is clear, since the RF method involves a cash flow of $(1-\delta)$ at $\tau \le T$, while the RT method involves a cash flow of $(1-\delta)$ at T. Since rates are non-negative, it is more valuable to receive money earlier.

2. Credit Spreads for Defaultable Bonds. For the hazard rate model discussed in class, assume the money market and intensity functions are continuous in time. For the maturity T and current time $t \leq T$, let $p_0(t,T)$, $p_1(t,T)$, and $p_1^{RF}(t,T)$ be the time-t prices for a default-free ZCB, defaultable ZCB with 0 recovery, and defaultable ZCB with $(1-\delta)$ RF recovery respectively.

Define the credit spread for 0 recovery as

$$c(t,T) \triangleq -\frac{1}{T-t}\left(\log(p_1(t,T)) - \log(p_0(t,T))\right),$$

and the credit spread for RF recovery as

$$c^{RF}(t,T) \triangleq -\frac{1}{T-t}\left(\log(p_1^{RF}(t,T)) - \log(p_0(t,T))\right).$$

On the set $\{\tau > t\}$, compute:

(a) $c(t,T)$ and $c^{RF}(t,T)$.

(b) The limits $\lim_{T\downarrow t} c(t,T)$ and $\lim_{T\downarrow t} c^{RF}(t,T)$. These are the *instantaneous credit spreads*. Here, you should get very simple expressions. Can you explain *why* they are what they are?

Solution:

(a) On $\{\tau > t\}$ we have

$$\begin{aligned} p_0(t,T) &= e^{-\int_t^T r(u)du}; \\ p_1(t,T) &= e^{-\int_t^T (r(u)+\gamma(u))du}; \\ p_1^{RF}(t,T) &= e^{-\int_t^T (r(u)+\gamma(u))du} + (1-\delta)\int_t^T \gamma(s)e^{-\int_t^s (r(u)+\gamma(u))du}ds. \end{aligned}$$

From here it follows that

$$\begin{aligned} c(t,T) &= \frac{1}{T-t}\int_t^T \gamma(u)du; \\ c^{RF}(t,T) &= -\frac{1}{T-t}\log\left(e^{-\int_t^T \gamma(u)du} + (1-\delta)e^{\int_t^T r(u)du}\int_t^T \gamma(s)e^{-\int_t^s ((r(u)+\gamma(u))du)}ds\right). \end{aligned}$$

(b) Since γ is continuous, and using l'Hospital's rule, we obtain

$$\begin{aligned} \lim_{T\downarrow t} c(t,T) &= \gamma(t); \\ \lim_{T\downarrow t} c^{RF}(t,T) &= \delta\gamma(t). \end{aligned}$$

So, the RF recovery method yields an instantaneous credit spread of $\delta\gamma(t)$ where as the 0 recovery $(\delta = 1)$ gives $\gamma(t)$.

3. Parameterized Logistic Copula. Let X_1 and X_2 be random variables with joint cdf

$$F_\theta(x_1, x_2) = \left(1 + e^{-x_1} + e^{-x_2} + (1-\theta)e^{-x_1-x_2}\right)^{-1}, \qquad x_1, x_2 \in \mathbb{R},$$

where $\theta \in [-1, 1]$.

(a) Identify the marginal distributions $F_{\theta,1}$, $F_{\theta,2}$ for X_1 and X_2 respectively.
(b) Show that when $\theta = 0$, X_1 and X_2 are independent.
(b) Show that the copula of X_1 and X_2, defined abstractly by $c_\theta(u_1, u_2) = F_\theta(F_{\theta,1}^{-1}(u_1), F_{\theta,2}^{-1}(u_2))$, takes the form

$$C_\theta(u_1, u_2) = \frac{u_1 u_2}{1 - \theta(1-u_1)(1-u_2)}.$$

Solution:

(a) From the formula we immediately see that

$$F_{\theta,1}(x_1) = F_\theta(x_1, \infty) = \frac{1}{1+e^{-x_1}}; \quad F_{\theta,2}(x_2) = F_\theta(\infty, x_2) = \frac{1}{1+e^{-x_2}}.$$

Thus, the marginals are the same, and have the (univariate) logistic distribution.

(b) Note that

$$\begin{aligned} F_0(x_1, x_2) &= \frac{1}{1+e^{-x_1}+e^{-x_2}+e^{-x_1-x_2}} = \frac{1}{1+e^{-x_1}} \times \frac{1}{1+e^{-x_2}}; \\ &= F_{0,1}(x_1)F_{0,2}(x_2), \end{aligned}$$

proving independence.

(c) We see that

$$F_{\theta,1}^{-1}(y) = F_{\theta,2}^{-1}(y) = -\log\left(\frac{1}{y} - 1\right),$$

for $0 < y < 1$. Therefore

$$\begin{aligned} c(u_1, u_2) &= \frac{1}{\frac{1}{u_1} + \frac{1}{u_2} - 1 + (1-\theta)\left(\frac{1}{u_1} - 1\right)\left(\frac{1}{u_2} - 1\right)}; \\ &= \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2 + (1-\theta)(1-u_1)(1-u_2)}; \\ &= \frac{u_1 u_2}{1 - \theta(1-u_1)(1-u_2)}. \end{aligned}$$

4. On Archimedean Copulas. In this exercise we will prove a result which gives a method to sample off of the archimedean copula

$$c(u_1, u_2) = \psi\left(\psi^{-1}(u_1) + \psi^{-1}(u_2)\right),$$

in the special case when $\psi(t) = E\left[e^{-tX}\right]$ is the Laplace transform for a random variable X, which we assume is continuous with strictly positive pdf on $(0, \infty)$. Note (you do not have to prove this) that such a ψ satisfies the requirements of being strictly decreasing with $\psi(0) = 1$ and $\psi(\infty) = 0$.

Prove the following statement: let Y_1, Y_2 be i.i.d. exponential random variables with rate 1 which are also independent of X, and set $(U_1, U_2) = (\psi(Y_1/X), \psi(Y_2/X))$. Then U had joint cdf given by c.

Based upon the above, we see that to sample off of c, we need only sample X, Y_1, Y_2 and then set $U_1 = \psi(Y_1/X)$ and $U_2 = \psi(Y_2/X)$.

Solution Since ψ is decreasing

$$\widetilde{\mathbb{P}}\left[U_1 \leq u_1, U_2 \leq u_2\right] = \widetilde{\mathbb{P}}\left[Y_1 \geq X\psi^{-1}(u_1), Y_2 \geq X\psi^{-1}(u_2)\right].$$

By first conditioning on X and using the i.i.d. property of Y_1, Y_2, as well as their independence from X we see that

$$\widetilde{\mathbb{P}}\left[U_1 \leq u_1, U_2 \leq u_2\right] = E^{\widetilde{\mathbb{P}}}\left[e^{-(\psi^{-1}(u_1)+\psi^{-1}(u_2))X}\right] = \psi(\psi^{-1}(u_1)+\psi^{-1}(u_2)) = c(u_1, u_2),$$

giving the result.