# HW Problems for Assignment 4 - Part 1
# Due 6:30 PM Tuesday, November 9, 2021

**1. (25 Points) Scenario Analysis for a hedged "squared distance" option in the Black-Scholes Model.** Assume, in addition to the money market account which yields a constant return $r > 0$, there are two stocks with dynamics

$$\begin{aligned} \frac{dS_t^{(1)}}{S_t^{(1)}} &= \mu_1 dt + \sigma_2 dW_t^{(1)}; \\ \frac{dS_t^{(2)}}{S_t^{(2)}} &= \mu_2 dt + \sigma_2 \left(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}\right) \end{aligned} \tag{0.1}$$

where $\mu_1, \mu_2 \in \mathbb{R}, \sigma_1, \sigma_2 > 0$, $-1 < \rho < 1$, and $W^{(1)}, W^{(2)}$ are two independent standard Brownian motions under the physical probability measure $\mathbb{P}$. As such, $\rho$ is correlation in the instantaneous asset returns. In our portfolio we are long a European option with time $T$ terminal payoff

$$V(T, S_T^{(1)}, S_T^{(2)}) = \left(S_T^{(1)} - S_T^{(2)}\right)^2,$$

the squared distance between the two stocks, and we are delta-hedging according to the (two dimensional) Black-Scholes delta hedging strategy (described below).

Recall that the arbitrage free price of this option at $t \leq T$ is

$$V(t, S_t^{(1)}, S_t^{(2)}; \sigma_t, \rho_t) = E^{\mathbb{Q}}\left[e^{-r(T-t)}\left(S_T^{(1)} - S_T^{(2)}\right)^2 \mid S_t^{(1)}, S_t^{(2)}\right],$$

Here $\mathbb{Q}$ is the risk neutral measure under which each risk asset has drift rate $r$. Above, by "$\sigma_t$" and "$\rho_t$" we mean that we have estimated the volatility and correlation at time $t$ to be $\sigma_t$, $\rho_t$ respectively and then to price the option, we assume the volatility and correlation are kept at $\sigma_t$, $\rho_t$ over $[t, T]$. While this is not entirely consistent with (0.1), it is common industry practice.

We are using the standard delta-hedging strategy. As such, our position at time $t$ is 1 unit notional of the option, and $-h_t^{(1)}$, $-h_t^{(2)}$ shares of the stocks, where

$$h_t^{(1)} = \partial_{s_1} V(t, s_1, s_2; \sigma_t, \rho_t); \qquad h_t^{(2)} = \partial_{s_2} V(t, s_1, s_2; \sigma_t, \rho_t).$$

The portfolio value at time $t$ is thus

$$V_t = V(t, S_t^{(1)}, S_t^{(2)}; \sigma_t, \rho_t) - h_t^{(1)} S_t^{(1)} - h_t^{(2)} S_t^{(2)}.$$

The horizon is 5 days, where one day is $\Delta = 1/252$. We hold a constant share position over $[t, t+5\Delta]$ so that at $t+5\Delta$ the value of our portfolio is

$$V_{t+5\Delta} = V(t, S^{(1)}_{t+5\Delta}, S^{(2)}_{t+5\Delta}; \sigma_{t+5\Delta}, \rho_{t+5\Delta}) - h^{(1)}_t S^{(1)}_{t+5\Delta} - h^{(2)}_t S^{(2)}_{t+5\Delta}.$$

Again, by $\sigma_{t+5\Delta}, \rho_{t+5\Delta}$ we mean the constant volatility and correlation, now estimated at $t+5\Delta$ which we use to price the option over $[t+5\Delta, T]$. We are interested in running a scenario analysis to see what our losses look like if at the end of 5 days the log returns $X^{(i)}_{t+5\Delta} = \log(S^{(i)}_{t+5\Delta}/S^{(i)}_t), i = 1,2$; volatility $\sigma_{t+5\Delta}$; and correlation $\rho_{t+5\Delta}$ takes certain values.

The scenarios under consideration are any combination of

$$\begin{aligned} X^{(i)}_{t+5\Delta} &= \pm 0.20, \pm 0.10, \pm 0.05; \qquad i = 1,2; \\ \frac{\sigma^{(i)}_{t+5\Delta}}{\sigma^{(i)}_t} &= .5, .75, 1.25, 1.5, 1.75, 2; \qquad i = 1,2; \end{aligned}$$

$$\rho_{t+5\Delta} - \rho_t = 0, \pm .5.$$

Thus, there are $N = 3888$ possible scenarios $x_1, ..., x_N$. For each scenario $x_n$, we will compute the full portfolio loss $\ell_n := L_{t+5\Delta}(x_n) = -(V_{t+5\Delta}(x_n) - V_t)$.

(a) Show that

$$V(t, s_1, s_2; \sigma, \rho) = e^{r(T-t)} \left( s_1^2 e^{\sigma_1^2 (T-t)} - 2 s_1 s_2 e^{\sigma_1 \sigma_2 \rho (T-t)} + s_2^2 e^{\sigma_2^2 (T-t)} \right).$$

(b) Output the worst case scenario risk measure

$$\varrho(L_{t+5\Delta}) = \max \left\{ \ell_n \;\middle|\; n = 1, ..., N \right\},$$

along with the log return/volatility combination which achieves this measure.

(c) Now, assume the following weights ($i = 1,2$)

$$X^{(i)}_{t+5\Delta} = \begin{cases} \pm 0.20 & \text{weight} = .5 \\ \pm 0.10 & \text{weight} = .75 \\ \pm 0.05 & \text{weight} = 1 \end{cases};$$

$$\frac{\sigma^{(i)}_{t+5\Delta}}{\sigma^{(i)}_t} = \begin{cases} .5, 2 & \text{weight} = .5 \\ .75, 1.75 & \text{weight} = 1.25 \\ 1.25, 1.5 & \text{weight} = .75 \end{cases};$$

$$\rho_{t+5\Delta} - \rho_t = \begin{cases} 0 & \text{weight} = .5 \\ \pm .5 & \text{weight} = 1.25 \end{cases};$$

so that, for example, the weight of $X^{(1)}_{t+\Delta} = -.1$, $X^{(2)}_{t+\Delta} = .2$, $\sigma^{(1)}_{t+5\Delta} = 2\sigma^{(1)}_t$, $\sigma^{(2)}_{t+5\Delta} = .5\sigma^{(2)}_t$ and $\rho_{t+5\Delta} - \rho_t = -.5$ is $w = .75 \times .5 \times .5 \times .5 \times 1.25 =$

15/128. Compute the scenario risk measure

$$\varrho(L_{t+5\Delta}) = \max\left\{w_n\ell_n \;\middle|\; n = 1, \dots, N\right\}.$$

along with the log return/volatility/correlation combination which achieves this measure. For this combination, what is the portfolio loss?

For parameter values use

$$\begin{aligned}
&r = 1.32\%; \qquad \mu_1 = 15.475\%; \qquad \mu_2 = 18.312\%;\\
&t = 0; \qquad \sigma_{1,t} = 22.14\%; \qquad \sigma_{2,t} = 30.36\% \qquad \rho_t = 0.237;\\
&T = 0.25; \qquad S_0^{(1)} = 158.12; \qquad S_0^{(2)} = 170.33; \qquad \Delta = 1/252.
\end{aligned}$$

**2. (25 Points) Stress Test for a Market-Cap Weighted Portfolio of Microsoft and Apple Stocks.** In this exercise you will perform the stress test shown in class, except that we will assume a negative shock to both stocks, not just one, with one negative shock dependent upon the other.

We have a market cap weighted portfolio of Microsoft and Apple stocks, where the caps are computed as of 9/1/16. We have daily log return data from 9/1/11 − 9/1/16. We shock the portfolio by assuming a large negative daily log return for Apple, which leads to a large negative log return for Microsoft. We want to see how this shock affects our losses over a $K$ day horizon. In particular, we wish to estimate the shocked $K$ day Value at Risk, and the frequency at which losses exceed the regulatory capital requirement estimated prior to the shock.

To perform the stress test:

(1) With the historical log return data, estimate the mean vector $\mu_{t+\Delta}$ and covariance matrix $\Sigma_{t+\Delta}$ as of time $t = 9/1/16$ using EWMA. You can start off with initial values of 0 for both and then update through time via the recursive equations

$$\begin{aligned}
\mu_{s+\Delta} &= \lambda\mu_s + (1-\lambda)X_s;\\
\Sigma_{s+\Delta} &= \theta\Sigma_s + (1-\theta)(X_s - \mu_s)(X_s - \mu_s)^{\mathrm{T}}.
\end{aligned}$$

(2) At $t$ assume $X_{t+\Delta} \overset{\mathcal{F}_t}{\sim} N\left(\mu_{t+\Delta}, \Sigma_{t+\Delta}\right)$. Estimate $\mathrm{VaR}_\alpha\left(L^{lin}_{t+\Delta}\right)$, and using the square root of time rule, estimate $\mathrm{VaR}_\alpha\left(L^{lin}_{t+K\Delta}\right)$ as well as the regulatory capital change $3 \times \mathrm{VaR}_\alpha\left(L^{lin}_{t+K\Delta}\right)$.

(3) Shock the system by assuming a large negative return for Apple, where below (1) corresponds to Microsoft, and (2) corresponds to Apple. Specifically:

(i) Set $X_{t+\Delta,(2)} = x_{(2)} = \mu_{t+\Delta,(2)} - 5 \times \sigma_{t+\Delta,(2)}$.

(ii) To obtain $X_{t+\Delta,(1)}$, first use the conditioning result which showed that given $X_{t+\Delta,(2)} = x_{(2)}$, $X_{t+\Delta,(1)}$ is normally distributed with

$$\widetilde{\mu}_{t+\Delta,(1)} = \mu_{t+\Delta,(1)} + \frac{\rho_{t+\Delta,(1)}\sigma_{t+\Delta,(1)}}{\sigma_{t+\Delta,(2)}}\left(x_{(2)} - \mu_{t+\Delta,(2)}\right);$$
$$\widetilde{\sigma}^2_{t+\Delta,(1)} = \sigma^2_{t+\Delta,(1)}\left(1 - \rho^2_{t+\Delta}\right).$$

Then, assume a $-5$ sigma (conditional) shock for Microsoft by setting $X_{t+\Delta,(1)} = x_{(1)} = \widetilde{\mu}_{t+\Delta,(1)} - 5\widetilde{\sigma}_{t+\Delta,(1)}$. This part varies from the lecture slides, and makes the stress test (with high probability) more stressful.

(iii) Use $x_{(1)}, x_{(2)}$ to update $\mu_{t+2\Delta}, \Sigma_{t+2\Delta}$ via EWMA.

(4) For days $k = 2, \dots, K$ sample $X_{t+k\Delta} \sim N(\mu_{t+k\Delta}, \Sigma_{t+k\Delta})$ and update $\mu_{t+(k+1)\Delta}, \Sigma_{t+(k+1)\Delta}$ via EWMA (the mean and covariance need not be updated for $k = K$).

(5) Estimate the linearized $K$ day portfolio loss by assuming the shares the stocks were held constant: i.e. by setting $L^{lin}_{t+K\Delta} = -\theta_t^{\mathrm{T}}\left(\sum_{k=1}^K X_{t+k\Delta}\right)$ where $\theta_t = S_t\lambda_t$ is the dollar position in the stocks at time $t$.

You will then repeat steps $(3)-(5)$ above $\hat{M}$ times to get the shocked losses

$$\ell_m = L^{Lin,m}_{t+K\Delta} = -\theta_t^{\mathrm{T}}\left(\sum_{k=1}^K X^m_{t+k\Delta}\right).$$

Using the losses $\{\ell_m\}_{m=1}^{\hat{M}}$ compute

(a) The average $K$-day portfolio loss.

(b) An estimate of the $K$-day $\mathrm{VaR}_\alpha$.

(c) The frequency with which the losses exceeded the initial $K$ day Value at Risk found using the square root of time rule.

(d) The frequency with which the losses exceed the regulatory capital found using the square root of time rule.

To obtain the frequencies, take the number of exceedances and multiply by $100/\hat{M}$ to get the percentage of days the losses exceeded the given values.

The log return data is in the file

"MSFT_AAPL_Log_Returns.csv".

This file has three columns: the date in numeric format sorted oldest to newest; the daily log return for MSFT; the daily log return for APPL. In order to help out the grader, you MUST write your program to use this data file, in this specific format. The market caps as of 9/1/16 are

MSFT: $448.77(b)$;   AAPL: $575.11(b)$.

For the confidence use $\alpha = .95$. For the number of days use $K = 10$. For the portfolio value use $\$1,000,000$. For the initial days of data use $M = 100$.

For the EWMA procedure use $\lambda = \theta = .97$. For the number of paths to run in your simulation use $\hat{M} = 50,000$.