HW Problems for Assignment 5 - Part 1
Due 6:30 PM Tuesday, November 23, 2021

1. (20 Points) An Optimal Liquidation Problem. Consider a (stylized version of a) typical optimal liquidation problem. At $t = 0$ (today), the investor owns λ shares of a stock. The investor wishes to liquidate his position by a terminal time N. However, he faces price impact: if on day n, he sells λ_n shares of the stock, the proportional bid-ask spread s_n takes the form

$$\frac{1}{2}s_n = f_n\left(\frac{\lambda_n}{\lambda^M}\right),$$

where the $\{f_n\}_{n=1}^N$ are arbitrary (smooth) non-decreasing functions. The idea is that each day the investor may face a different impact function.

The investor wants to identify the liquidation policy $\{\lambda_n\}_{n=1}^N$ which minimizes the total liquidation cost $\sum_{n=1}^N (1/2)\lambda_n s_n$ subject to the liquidation constraint $\sum_{n=1}^N \lambda_n = \lambda$.

(a) First, assume $f_n \equiv f$ does not depend on n and f is strictly convex. Identify the optimal liquidation policy and show that it is independent of f.

(b) Now, assume $f_n(y) = \eta_n y^{p_n}$ so that $\eta_n > 0$ is the price elasticity of demand on day n and $p_n > 0$ is a scaling factor. Show that there exists an optimal liquidation policy, and identify the policy up a "Lagrange Multiplier". When $p_n \equiv 1$ for each n provide an explicit formula for λ_n.

2. (30 Points) Approximating a Compound Poisson Random Variable and Risk Measure Estimation. Let $S_N = \sum_{k=1}^N X_k$ where $N \sim \mathrm{Poi}(\lambda)$ and $\{X_k\}$ are i.i.d. (and independent of N) log-normal $LN(\mu, \sigma^2)$ random variables. You will estimate the tail of the c.d.f. three ways: using the normal, translated gamma, and generalized Pareto approximations.

(1) For the normal approximation assume

$$S_N \sim E[S_N] + \sqrt{\mathrm{Var}[S_N]}Z; \qquad Z \sim N(0,1).$$

To compute the mean and variance, use the formulas in class.

(2) For the translated gamma approximation assume

$$S_N \sim k + Y; \qquad Y \sim \mathrm{Gamma}(a, b).$$

Here, k, a, b are chosen to match the mean, variance, skewness of S_N.

(3) For the generalized Pareto approximation, used the methodology described in the lecture notes. Take $u = \mathrm{VaR}_{\alpha_0}$ associated to the sampled data empirical cdf so $F(u) = \alpha_0$. As in the slides, set M as the number of simulation runs.

For parameter values use

$$\lambda = 100; \qquad \mu = .10; \qquad \sigma = .4; \qquad M = 1,000,000; \qquad \alpha_0 = .99.$$

Produce the following two plots:

(a) A log-log plot of $1 - F_{S_N}(x)$ versus x with the three above approximations. For comparison purposes, sample M NEW (i.e. not from part (c) which was used to fit the GP distribution) copies of S_N and plot the tail of the empirical c.d.f. as well. For the range of x, choose the low value to be VaR_{α_0} associated to the empirical distribution. For the high value, take $\mathrm{VaR}_{.99999}$ of the empirical distribution. Which approximation works best?

(b) A plot of $\alpha \to \mathrm{ES}_\alpha(S_N)$ for each of three approximation schemes. How do the expected shortfall functions compare? **Math Problem:** to compute the expected shortfall as a function of α using the Gamma approximation, you must first prove that if F and f are the CDF and PDF of a Gamma (a, b) random variable (so that $f(y)$ is proportional to $y^{a-1}e^{-by}$) then

$$\mathrm{ES}_\alpha = \frac{a}{b} + \frac{F^{-1}(\alpha)}{b(1-\alpha)} \times f\left(F^{-1}(\alpha)\right).$$

To show this, use that $\int_\alpha^1 F^{-1}(u)du = \int_{F^{-1}(\alpha)}^\infty wf(w)dw$ and integration by parts.