Recommended HW Problems for Assignment 6 - Part II
SOLUTIONS

Note : you do not have to turn in these problems, but you should definitely try them, as they will help you study for the final!

1. CVA, DVA for a Forward Contract in the Black-Scholes Model. In this exercise, we will again compute the CVA and DVA for an option in the Black-Scholes model, but to make the example a bit more realistic, we will consider a forward contract.

Recall that in the Black-Scholes model the asset X evolves under risk neutral measure $\mathbb{Q}$ according to

$$\frac{dX_t}{X_t} = rdt + \sigma dW_t^{\mathbb{Q}}; \quad X_0 = X_0$$

Above, the interest rate $r > 0$ and volatility $\sigma > 0$ are constant, and $W^{\mathbb{Q}}$ is a $\mathbb{Q}$ Brownian motion. Now, assume we have entered into a forward contract with maturity T on X. Recall that this is an agreement made at time 0 to purchase X at time T for a certain forward price K, which we determine at time 0. Since the contract costs nothing to enter into, the risk neutral theory of pricing tells us that the forward price K must satisfy

$$0 = E^{\mathbb{Q}}\left[e^{-rT}(X_T - K)\right] \Longrightarrow K = e^{rT}X_0,$$

where we have use that the discounted asset price is a Martingale under risk neutral measure. Thus, the time T cash flow of the option is $V(T) = X_T - X_0e^{rT}$.

The contract buyer B has default time τ^B, and contract seller S has default time τ^S. Assume under $\mathbb{Q}$ that

(i) τ^B, τ^S are independent of each other, as well as $W^{\mathbb{Q}}$.
(ii) τ^B, τ^S have constant intensities γ^B, γ^S.
(iii) B and S have deterministic losses given default δ^B, δ^S.

Lastly, set $\tau = \min\{\tau^B, \tau^S\}$ and $\xi \in \{B, S\}$ as the name of the first to default.

Our goal is to compute the CVA and DVA for the forward contract, from the perspective of the buyer B of the contract. To do this:

(a) Derive an explicit formula for the time $t \leq T$ price of the forward contract assuming no default:

$$V(t) = E^{\mathbb{Q}}\left[e^{-r(T-t)}(X_T - X_0e^{rT}) \mid \mathcal{F}_t\right].$$

(b) Using the independence of $W^{\mathbb{Q}}, \tau^S, \tau^B$, show that with $\gamma = \gamma^B + \gamma^S$ we have

$$\begin{aligned} \mathrm{CVA}(0) &= \delta^S E^{\mathbb{Q}}\left[\int_0^T \gamma^S V(t)^+ e^{-(r+\gamma)t} dt\right]. \\ \mathrm{DVA}(0) &= \delta^B E^{\mathbb{Q}}\left[\int_0^T \gamma^B V(t)^- e^{-(r+\gamma)t} dt\right]. \end{aligned}$$

(c) Denote by

$$C^{BS}(t, x; r, \sigma, K, T), P^{BS}(t, x; r, \sigma, K, T),$$

the Black-Scholes call and put prices for a given time t, current stock value x, risk free rate r, volatility σ, strike K and maturity T. Show that

$$\begin{aligned} \mathrm{CVA}(0) &= \delta^S \gamma^S \int_0^T C^{BS}(0, X_0; 0, \sigma, t, X_0) e^{-\gamma t} dt; \\ &= \frac{\delta^S \gamma^S}{\gamma} E^{\mathbb{Q}}\left[1_{\tau \leq T} C^{BS}(0, X_0; 0, \sigma, \tau, X_0)\right]. \\ \mathrm{DVA}(0) &= \delta^B \gamma^B \int_0^T P^{BS}(0, X_0; 0, \sigma, t, X_0) e^{-\gamma t} dt; \\ &= \frac{\delta^B \gamma^B}{\gamma} E^{\mathbb{Q}}\left[1_{\tau \leq T} P^{BS}(0, X_0; 0, \sigma, \tau, X_0)\right]. \end{aligned}$$

Thus, we can think of the CVA and DVA as the expected value of an at the money call/put option when the spot rate is 0 and when the maturity is the default time τ (if $\tau \leq T$ and up to a multiplicative constant). Note also the above formula does not depend up on r.

(d) Prove the explicit formula:

$$\mathrm{CVA}(0) = \frac{\gamma^S \delta^S X_0}{\gamma}\left(2\int_0^T N\left(\frac{1}{2}\sigma\sqrt{t}\right)\gamma e^{-\gamma t} dt - \left(1 - e^{-\gamma T}\right)\right),$$

where N is the standard normal c.d.f..

Solution:

(a) Since the discounted stock is a Martingale under $\mathbb{Q}$ we obtain

$$V(t) = E^{\mathbb{Q}}\left[e^{-r(T-t)}(X_T - X_0 e^{rT}) \mid \mathcal{G}_t\right] = X_t - X_0 e^{rt}.$$

(b) Since τ is independent of the filtration $\mathbb{G}$, and since by definition the value process V is $\mathbb{G}$ adapted:

$$\begin{aligned}
\text{CVA}(0) &= E^{\mathbb{Q}}\left[1_{\tau\le T}1_{\xi=S}\delta^S V(\tau)^+e^{-r\tau}\right];\\
&= \delta^S E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left[1_{\tau\le T}1_{\xi=S}V(\tau)^+e^{-r\tau} \mid W^{\mathbb{Q}}\right]\right];\\
&= \delta^S E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left[\int_0^T \gamma\frac{\gamma^S}{\gamma}V(t)^+e^{-(r+\gamma)t}dt \mid W^{\mathbb{Q}}\right]\right];\\
&= \delta^S E^{\mathbb{Q}}\left[\int_0^T \gamma^S V(t)^+e^{-(r+\gamma)t}dt\right].
\end{aligned}$$

Above, we have used the fact that since τ is independent of $W^{\mathbb{Q}}$, and as such the conditional and regular densities coincide at $\gamma(t)e^{-\int_0^t\gamma(u)du}$. We have also used that the probability of $\xi = S$ is γ^S/γ. A similar calculation gives

$$\text{DVA}(0)_t = \delta^B E^{\mathbb{Q}}\left[\int_0^T \gamma^B V(t)^-e^{-(r+\gamma)t}dt\right].$$

(c) Using that $V(t) = X_t - X_0e^{rt}$:

$$\begin{aligned}
\text{CVA}(0) &= \delta^S\gamma^S E^{\mathbb{Q}}\left[\int_0^T V(t)^+e^{-rt}e^{-\gamma t}dt\right];\\
&= \delta^S\gamma^S\int_0^T E^{\mathbb{Q}}\left[\left(e^{-rt}X_t - X_0\right)^+\right]e^{-\gamma t}dt.
\end{aligned}$$

Now, since $dX_t/X_t = rdt + \sigma dW_t^{\mathbb{Q}}$ we see that

$$e^{-rt}X_t = X_0e^{\sigma W_t^{\mathbb{Q}}-(1/2)\sigma^2 t}; \qquad t\ge 0.$$

But, this is just the stock price in a Black-Scholes model where the interest rate is 0. From here, we see that

$$E^{\mathbb{Q}}\left[\left(e^{-rt}X_t - X_0\right)^+\right] = C^{BS}(0, X_0; 0, \sigma, t, X_0),$$

i.e., the call price for an at-the-money call with maturity t, in a model where the interest rate is 0. Plugging this in, we see that

$$\begin{aligned}
\text{CVA}(0) &= \delta^S\gamma^S\int_0^T C^{BS}(0, X_0; 0, \sigma, t, X_0)e^{-\gamma t}dt;\\
&= \frac{\delta^S\gamma^S}{\gamma}\int_0^T C^{BS}(0, X_0; 0, \sigma, t, X_0)\gamma e^{-\gamma t}dt;\\
&= \frac{\delta^S\gamma^S}{\gamma}E^{\mathbb{Q}}\left[1_{\tau\le T}C^{BS}(0, X_0; 0, \sigma, \tau, X_0)\right],
\end{aligned}$$

where the last equality follows since τ has density $\gamma e^{-\gamma t}$. Similarly, for the DVA, using that $x^- = (-x)^+$:

$$\begin{aligned}
\mathrm{DVA}(0) &= \delta^B \gamma^B E^{\mathbb{Q}}\left[\int_0^T V(t)^- e^{-rt} e^{-\gamma t} dt\right]; \\
&= \delta^B \gamma^B \int_0^T E^{\mathbb{Q}}\left[\left(e^{-rt} X_t - X_0\right)^-\right] e^{-\gamma t} dt; \\
&= \delta^B \gamma^B \int_0^T E^{\mathbb{Q}}\left[\left(X_0 - e^{-rt} X_t\right)^+\right] e^{-\gamma t} dt; \\
&= \delta^B \gamma^B \int_0^T P^{BS}(0, X_0; 0, \sigma, t, X_0) e^{-\gamma t} dt; \\
&= \frac{\delta^B \gamma^B}{\gamma} \int_0^T P^{BS}(0, X_0; 0, \sigma, t, X_0) \gamma e^{-\gamma t} dt; \\
&= \frac{\delta^B \gamma^B}{\gamma} E^{\mathbb{Q}}\left[1_{\tau \le T} P^{BS}(0, X_0; 0, \sigma, \tau, X_0)\right].
\end{aligned}$$

(d) A straight-forward calculation shows that

$$\begin{aligned}
C^{BS}(0, X_0; 0, \sigma, t, X_0) &= X_0 \left(N\left(\frac{1}{2}\sigma\sqrt{t}\right) - N\left(-\frac{1}{2}\sigma\sqrt{t}\right)\right); \\
&= X_0 \left(2N\left(\frac{1}{2}\sigma\sqrt{t}\right) - 1\right),
\end{aligned}$$

where we have used that $N(-a) = 1 - N(a)$ for $a > 0$. We thus have

$$\begin{aligned}
\mathrm{CVA}(0) &= \delta^S \gamma^S \int_0^T C^{BS}(0, X_0; 0, \sigma, t, X_0) e^{-\gamma t} dt; \\
&= \frac{\delta^S \gamma^S X_0}{\gamma} \int_0^T \left(2N\left(\frac{1}{2}\sigma\sqrt{t}\right) - 1\right) \gamma e^{-\gamma t} dt; \\
&= \frac{\delta^S \gamma^S X_0}{\gamma} \left(2 \int_0^T N\left(\frac{1}{2}\sigma\sqrt{t}\right) \gamma e^{-\gamma t} dt - \left(1 - e^{-\gamma T}\right)\right).
\end{aligned}$$