MF731

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Part 1.

1. (a) the model:

$$
S_0 \begin{cases} S_0\cdot(1+\sigma_0) \begin{cases} S_0\cdot(1+\sigma_0)(1+\sigma_1^u) \\ S_0\cdot(1-\sigma_0)\cdot(1-\sigma_1^u) \end{cases} \\ S_0\cdot(1-\sigma_0) \begin{cases} S_0\,(1-\sigma_0)(1+\sigma_1^d) \\ S_0\,(1-\sigma_0)(1-\sigma_1^d) \end{cases} \end{cases}
$$

$$C_1^m(1,S_0) = k_0 S_0$$

$$\therefore\ k_0 S_0 = E^Q[(S_1-S_0)^+] = \tfrac{1}{2}(S(1+\sigma_0)-S_0)^+ + \tfrac{1}{2}(S_0(1-\sigma_0)-S_0)^+$$

$$k_0 S_0 = \tfrac{1}{2} S_0\sigma_0$$

Similarly

$$\hat{C}^m(2) = k_1 S_0 = E^Q[(S_2-S_1)^+]$$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}(S_0(1+\sigma_0)(1+\sigma_1^u) - S_0(1+\sigma_0))^+ + \tfrac{1}{2}(S_0(1+\sigma_0)(1-\sigma_1^u) - S_0(1+\sigma_0))^+\right]$$

$$= \tfrac{1}{4}S_0(1+\sigma_0)\sigma_1^u + \tfrac{1}{4}S_0(1-\sigma_0)\sigma_1^d = \tfrac{1}{4}S_0[(1+2k_0)\sigma_1^u + (1-2k_0)\sigma_1^d)$$

$\therefore\ \sigma_1^u, \sigma_1^d$ lie on the line, $\quad (1+2k_0)\sigma_1^u + (1-2k_0)\sigma_1^d = 4k_1$

(b) Given $X = (S_2 - S_0)^+$,

$$P_a(x) = E^{Q_a}[e^{-rT}X] = E^{Q_a}[x]$$

$$k_1 S_0 = E^Q[(S_1 - S_0)^+] = \tfrac{1}{2}S_0\sigma_0 \quad \therefore\ \sigma_0 = 2k_0$$

$$C^m(2,S_0) = E^Q[(S_2 - S_1 + S_1 - S_0)^+] = (k_1+k_0)S_0$$

$$= \tfrac{1}{4}[S_0(\sigma_0+\sigma_1^u+\sigma_0\sigma_1^u) + S_0(\sigma_0-\sigma_1^u-\sigma_0\sigma_1^u)^+ + S_0(-\sigma_0+\sigma_1^d-\sigma_0\sigma_1^d)^+ + 0]$$

$$
= \begin{cases} \frac{S_0}{4}(2k_0 + (1-2k_0)\sigma_1^d, & \text{if } \sigma_1^u < \frac{2k_0}{1+2k_0},\ \sigma_1^d > \frac{2k}{1-2k_0} \quad ① \\ \frac{S_0}{4}\cdot 4k_0 & \text{if } \sigma_1^u > \frac{2k_0}{1+2k_0},\ \sigma_1^d > \frac{2k_0}{1-2k_0} \quad ② \\ \frac{S_0}{4}\cdot 4k_1 & \text{if } \sigma_1^u < \frac{2k_0}{1+2k_0},\ \sigma_1^d < \frac{2k_0}{1-2k_0} \quad ③ \\ \frac{S_0}{4}\cdot(2k_0 + (1+2k_0)\sigma_1^u) & \text{if } \sigma_1^u > \frac{2k_0}{1+2k_0},\ \sigma_1^d < \frac{2k_0}{1-2k_0} \quad ④ \end{cases}
$$

Since $0 < k_0 < \frac{1}{6}$, $\frac{k_0}{2} < k_1 < \min\{k_0, \frac{1}{4} - \frac{k_0}{2}\}$ $\therefore\ \frac{k_0}{2} < k_1 < k_0$

$\therefore$ ③ < ② < ① And ③ < ① < ④

$$
\mu_a(x) = \begin{cases} 2k_0 + (1-2k_0)\sigma_1^d - 4k_1 \\ 2k_0 + (1+2k_0)\sigma_1^u - 4k_1 \end{cases} = \begin{cases} 2k_0 - (1-2k_0)\sigma_1^u \\ 2k_0 - (1+2k_0)\sigma_1^d \end{cases}
$$

$\therefore\ (4k_0,\ 4k_1 + 2k_0)$

2.

a)

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the empirical VaR is : 24984.392523364484
theoretical VaR is:  22773.549452807296
onfidence interval for theoretical VaR:  21138.534984083803
```

b)

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24651.99266165352
```

Average Y_m: `22812.17217596515`

Confidence interval: `20687.144342383945 25094.040107888293`

Part 2:

1. the payoff of this defaultable ZCB under RT method is $\begin{cases} 1, & \tau > T \\ (1-\delta), & t<\tau\le T \end{cases}$

$$\overline{ZCB}^{RT}(t,T) = E\left[1_{\{\tau>T\}} e^{-\int_t^T r(u)du} + 1_{\{\tau\le T\}}(1-\delta) e^{-\int_t^T r(u)du} \mid \mathcal{F}_t\right]$$
$$= ZCB(t,T) + 1_{\{t<T\}}(1-\delta) e^{-\int_t^T r(u)du} \int_t^T r(s) e^{-\int_t^s r(u)du} ds$$

$$\overline{ZCB}^{RF}(t,T) = ZCB(t,T) + 1_{\{\tau>T\}}(1-\delta)\int_t^T r(s) e^{-\int_t^s (r(u)+r(u))du} ds$$

Assume that $r(t) \ge 0$, $t \le T$

$$e^{-\int_t^T r(u)du} \int_t^T r(s) e^{-\int_t^s r(u)du} ds = \int_t^T r(s) e^{-\int_t^s (r(u)+r(u))du} \cdot e^{-\int_s^T r(u)du} ds$$
$$\le \int_t^T r(s) e^{-\int_t^s (r(u)+r(u))du} ds$$

$\therefore$ RT gives a lower price, Since RF gives a default of $(1-\delta)$ which makes the default payment higher

2.

(a) Given $\{\tau > t\}$, $P_0(t,T) = e^{-\int_t^T r(u)du}$

$$P_1(t,T) = e^{-\int_t^T (r(u)+r(u))du}$$
$$P_1^{RF}(t,T) = e^{-\int_t^T (r(u)+r(u))du} + (1+\delta)\int_t^T r(s) e^{-\int_t^s (r(u)+r(u))du} ds$$

Then we have;

$$C(t,T) = -\frac{1}{T-t}\left(-\int_t^T (r(u)+r(u))du + \int_t^T r(u)du\right) = \frac{1}{T-t}\int_t^T r(u)du$$
$$C^{RF}(t,T) = -\frac{1}{T-t}\log\left(e^{-\int_t^T r(u)du} + (1-\delta)\int_t^T r(s) e^{-\int_t^s (r(u)+r(u))du} ds\right)$$

(b) By Hospital rule. $\lim_{T\downarrow t} C(t,T) = r(t)$

$\lim_{T\downarrow t} C^{RF}(t,T) = \delta r(t)$ where $\delta = 1$, 0 recovery

$C^{RF}(t,T) = C(t,T)$

3.

Take accured interest into account, we have:

premium side: $$V_t^{Prem}(t) = E^Q\left[\sum_{n=1}^{N\Delta} \frac{x}{\Delta}\cdot 1_{t_n<\tau} \mid \mathcal{F}_t\right] + E^Q\left[\sum_{n=1}^{N\Delta} 1_{t_{n-1}<\tau<t_n}\, x(T-t_{n-1}) \mid \mathcal{F}_t\right]$$
$$= E^Q\left[\sum_{n=1}^{N\Delta}\left(\frac{x}{\Delta}\cdot 1_{t_n<\tau} + x(\tau-t_{n-1})\cdot 1_{t_{n-1}<\tau<t_n}\right) \mid \mathcal{F}_n\right]$$
$$= E^Q\left[x\int_t^{t+N} 1_{t<\tau} \mid \mathcal{F}_t\right]$$
$$= x 1_{t<\tau}\int_t^{t+N} e^{-\int_t^s r(u)du} ds$$

And the default side: $\delta 1_{t<\tau}\int_t^{t+N} r(s) e^{-\int_t^s r(u)du} ds$

$$\therefore \quad x_t^Q = \delta \frac{\int_t^{t+N} r(s) e^{-\int_t^s r(u)du} ds}{\int_t^{t+N} e^{-\int_t^s r(u)du} ds}$$

4. (a) Since $F_\theta(x_1,x_2) = (1+e^{-x_1}+e^{-x_2}+(1-\theta)e^{-x_1-x_2})^{-1}$, $x_1, x_2 \in \mathbb{R}$

$F_{\theta,1}(x_1,\infty) = (1+e^{-x_1})^{-1}$ $\quad F_{\theta,2}(\infty,x_2) = (1+e^{-x_2})^{-1}$

which is the logistic distribution

(b) if $\theta = 0$, $F_\theta(x_1,x_2) = (1+e^{-x_1}+e^{-x_2}+e^{-x_1-x_2})^{-1} = F_{\theta,1}(x_1)F_{\theta,2}(x_2)$

independence of x_1, x_2

(c) $F_{\theta,1}^{-1}(y) = F_{\theta,2}^{-1}(y) = -\log\left(\frac{1}{y}-1\right)$

$$C_\theta(u_1,u_2) = F_\theta(F_{\theta,1}^{-1}(u_1), F_{\theta,2}^{-1}(u_2)) = \left(\frac{1}{u_1}+\frac{1}{u_2}-1+(1-\theta)\left(\frac{1}{u_1}-1\right)\left(\frac{1}{u_2}-1\right)\right)^{-1}$$
$$= \frac{u_1u_2}{u_1+u_2-u_1u_2+(1-\theta)(1-u_1)(1-u_2)} = \frac{u_1u_2}{1-\theta(1-u_1)(1-u_2)}$$