Due Date: Oct 19
Completed: **2021/10/09**

Problem 1

1. (10 Points) The "Exponential Premium Principal" Risk Measure. In this exercise we introduce another risk measure, which shares certain properties with Value at Risk and Expected Shortfall, but which is not a coherent risk measure. Namely, fix a parameter $\alpha > 0$ and, for a random variable L define the "exponential premium principal" risk measure with parameter α by the formula

$$\varrho_\alpha(L) := \frac{1}{\alpha}\log\left(E\left[e^{\alpha L}\right]\right).$$

(a) Show that ϱ_α is a convex risk measure. To show convexity, use Holder's inequality.

(b) By considering $L \sim N(0,1)$ show that ϱ_α is not positive homogenous, and hence not coherent.

Solution:

Problem 1

$$\varrho_\alpha(L) \triangleq \frac{1}{\alpha}\log(\mathbb{E}[e^{\alpha L}]) \qquad \alpha>0$$

(a) In order to prove the convex risk measure, we need to prove

① Monotonicity

$$\varrho_\alpha(L_1) \le \varrho_\alpha(L_2) \quad \text{if } \mathbb{P}(L_1 \le L_2)=1$$

② Cash-additivity

$$\varrho_\alpha(L-\ell) = \varrho_\alpha(L) - \ell$$

③ Definition of Convexity

$$\varrho_\alpha(\lambda L_1+(1-\lambda)L_2) \le \lambda\varrho_\alpha(L_1)+(1-\lambda)\varrho_\alpha(L_2) \quad \text{for } 0\le\lambda\le 1$$

③ We know that Holders inequality:

$$\mathbb{E}[XY] \le \mathbb{E}[X^p]^{1/p}\,\mathbb{E}[Y^q]^{1/q}$$

$$\frac{1}{p}+\frac{1}{q}=1 \quad \text{and} \quad p>1$$

where X, Y are random variables.

Thus,

$$\mathbb{E}[e^{\alpha(\lambda L_1+(1-\lambda)L_2)}] = \mathbb{E}[e^{\alpha\lambda L_1}e^{\alpha L_2(1-\lambda)}]$$

$$\le [(e^{\alpha\lambda L_1})^{\frac{1}{\lambda}}]^{\lambda}\,[(e^{\alpha L_2(1-\lambda)})^{\frac{1}{1-\lambda}}]^{1-\lambda}$$

where $p = \frac{1}{\lambda}$, $q = \frac{1}{1-\lambda}$ $\Rightarrow$ $\frac{1}{p} + \frac{1}{q} = \lambda + (1-\lambda) = 1$

Then, we use the inequality and take logarithm, for the left hand side.

$$LHS = \rho_\alpha\left(\frac{1}{\alpha}[\lambda(\alpha L_1) + (1-\lambda)(\alpha L_2)]\right) = \rho_\alpha(\lambda L_1 + (1-\lambda)L_2)$$

$$RHS = \lambda \rho_\alpha(L_1) + (1-\lambda)\rho_\alpha(L_2)$$

$$\therefore \ LHS < RHS \quad ③ \text{ proved}$$

For ①, we know that $\alpha > 0$, so for $L_1 \leq L_2$, monotonicity holds automatically.

For ②, $\rho_\alpha(L - \ell) = \frac{1}{\alpha}\log(\mathbb{E}[e^{\alpha(L-\ell)}])$

$$= \frac{1}{\alpha}\log(\mathbb{E}[e^{\alpha L}e^{-\alpha \ell}])$$

$$= \frac{1}{\alpha}\log(\mathbb{E}[e^{\alpha L}]) - \ell$$

(b) Let $L \sim N(0,1)$ then

$$\rho(\lambda L) = \frac{1}{\alpha}\log(\mathbb{E}[e^{\alpha\lambda L}]) = \frac{1}{2}\alpha\lambda^2$$

which is not linear in λ. So positive homogeneous fails.

Problem 2

In this setting, compute on a daily basis throughout time the percent component ρ for ρ equal t

(1) Value at Risk.

(2) A spectral risk measure with exponential weighting function $\phi_\gamma(u) = \frac{\gamma}{e^\gamma - 1}e^{\gamma u}, 0 \leq u \leq 1$.

Additionally, compute the percent contribution to the loss variance across time.

Output a time evolution plot for each of the above values, over the range $t - (N - M)\Delta, \ldots, t$. Each plots will have five graphs.

As we discussed in class, the component risk measure plots should be very close to the variance contribution plot. Is this the case?

Solution: Daily basis throughout time the percent component for VaR and Exponential Spectral Risk Measure are as follows(for more detail see codes):

Here, what is special to the exponential spectral risk measure is that we know

$$L^{linear}_{t+\Delta} = -V_t(W_t^T \mu_{t+\Delta} + \sqrt{W_t^T \Sigma_{t+\Delta} W_t} Z), where Z \sim N(0, 1)$$

However, for spectral risk measure we should have

$$L^{linear}_{t+\Delta} = -V_t(W_t^T \mu_{t+\Delta} + \sqrt{W_t^T \Sigma_{t+\Delta} W_t} \rho_{\phi_\tau}(Z))$$

$$where, \rho_{\phi_\tau}(Z) = \int_0^1 VaR_u \phi_\tau(u) du$$

$$= \int_0^1 N^{-1}(u)\phi_\tau(u) du$$

and this integral can not be solved analytically, thus we apply numerical integration.

$$= \int_0^1 N^{-1}(u)\phi_\tau(u)du$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} N^{-1}(\frac{i}{N})\phi_\tau(\frac{i}{N})$$

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In [5]: rel_comp_var_data
Out[5]:
array([[ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ],
       [13.77428284, 19.39604664, 17.1132544 , 26.02836216, 23.68805396],
       [12.92949115, 20.13392566, 17.51147034, 25.37133934, 24.05377351],
       ...,
       [11.38071245, 34.18136991, 11.84016826, 24.05834539, 18.539404  ],
       [11.42473447, 33.68163481, 11.71307267, 24.37406311, 18.80649494],
       [11.34250577, 34.35747129, 12.15555942, 23.78051148, 18.36395204]])
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In [6]: rel_comp_sp_data
Out[6]:
array([[ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ],
       [14.24353894, 19.48766435, 17.38563258, 25.6935983 , 23.18956583],
       [13.22061685, 20.382353  , 17.86901846, 24.8963317 , 23.63167999],
       ...,
       [11.3878768 , 33.32927912, 11.47705904, 24.79822228, 19.00756275],
       [11.41179049, 32.95035471, 11.36027794, 25.06571524, 19.21186162],
       [11.31584261, 33.74074885, 11.87749767, 24.37144748, 18.69446338]])
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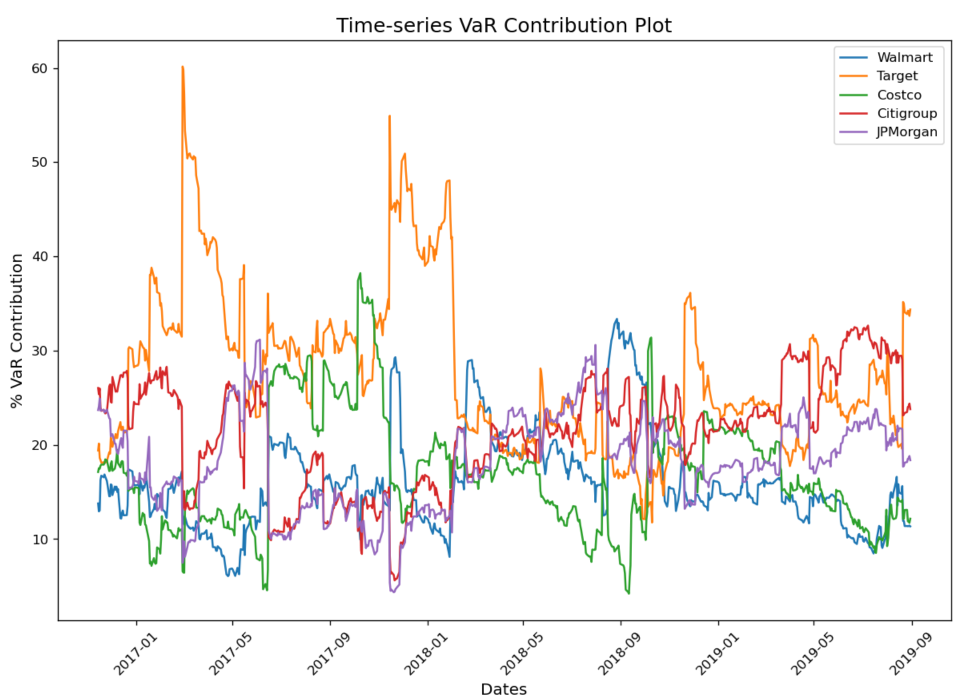
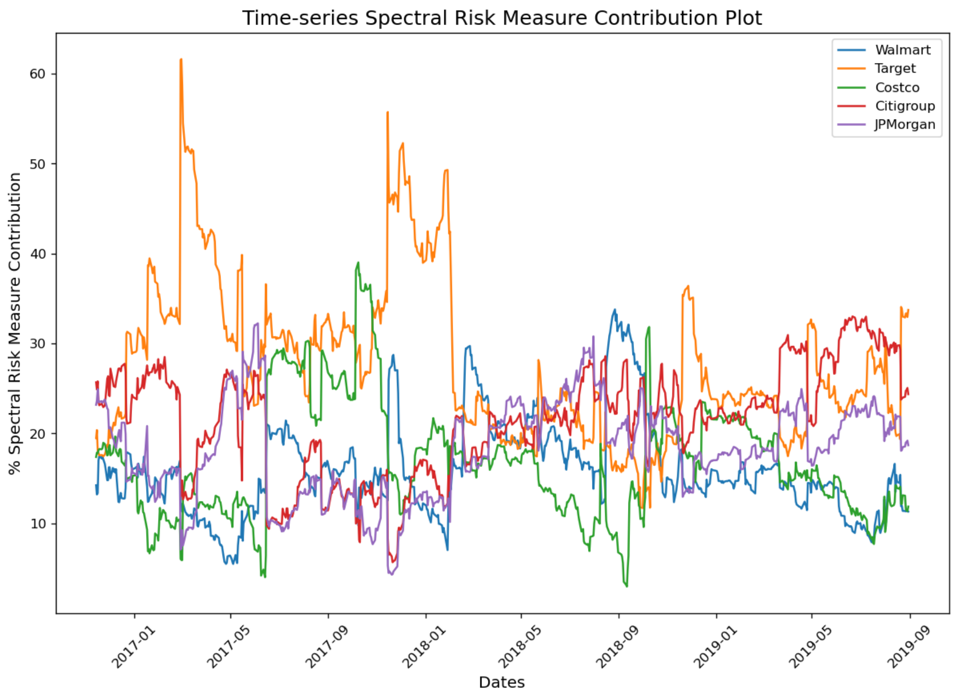
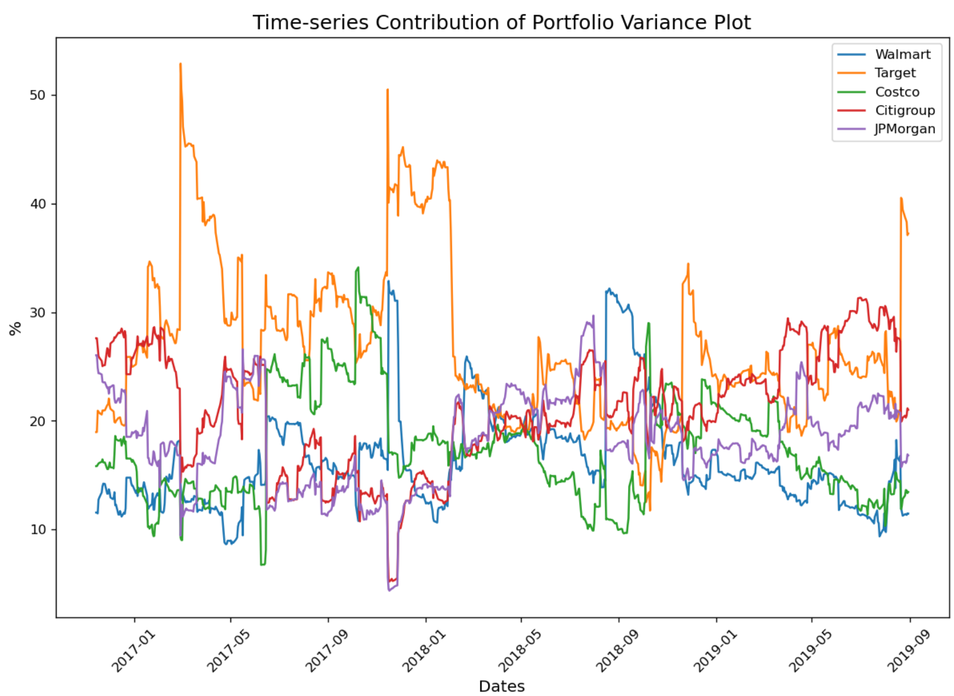
Furthermore, as the figures shown below, the component risk measure plots are very close to the variance contribution plot regardless of using VaR or Exponential Spectral Risk Measure. This is Reasonable, since as we learned in class, the final answer for the percentage component risk is irrelevant to the risk measure ρ and the log mean return μ is almost 0. Thus, this is the case. The equation is as following:

$$R_{C,\%}^{(i)}(\theta) = 100 \cdot \frac{(\Sigma\theta)^{(i)}\theta^{(i)}}{\theta^T\Sigma\theta}$$







Problem 3
Spherical and elliptical random variables. This exercise shows that many of the conclusions on component risk measures, and risk-measure based optimal investment, extend beyond the Gaussian setting.

Solution:

problem 3

(a) By definition, if Z is spherical, a^TZ and $|a|Z^T$ should have same distribution.

$$Z = \sqrt{W} \cdot \tilde{Z} \quad \text{where} \quad \tilde{Z} \sim N(0,1).$$

The charateristic function of a^TZ is,

$$\mathbb{E}[e^{i\gamma a^T Z}] = \mathbb{E}[\mathbb{E}[e^{i\gamma\sqrt{W}a^T\tilde{Z}}|W]] = \mathbb{E}[e^{-\frac{1}{2}\gamma^2 W|a|^2}]$$

Where the W and $\tilde{Z}$ are independent. Also, the characteristic function for $|a|Z^{(1)}$ is

$$\mathbb{E}[e^{i\gamma|a|Z^{(1)}}] = \mathbb{E}[\mathbb{E}[e^{i\gamma|a|\sqrt{W}\tilde{Z}^{(1)}}|W]] = \mathbb{E}[e^{-\frac{1}{2}\gamma^2 W|a|^2}]$$

Therefore, Z is spherical.

(b) (i) We know that $L^{lin} = -\theta^T X = -\theta^T(\mu + AZ) = -\theta^T\mu - \theta^T AZ$, where Z is spherical. Let $a = -A^T\theta$ and we proved that a^TZ and $|a|Z^{(1)}$ have same distribution, and

$$|a| = \sqrt{|a|^2} = \sqrt{a^Ta} = \sqrt{\theta^T AA^T\theta} = \sqrt{\theta^T\Sigma\theta}$$

Hence, L^{lin} has same distribution as $-\theta^T\mu + \sqrt{\theta^T\Sigma\theta}\, Z^{(1)}$. Because L is cash-additive and positive homogenuous, thus

$$\varrho(L^{lin}) = -\theta^T\mu + \sqrt{\theta^T\Sigma\theta} \cdot \varrho(Z^{(1)})$$

(b)(ii) Since $R_M^{lin,(i)}(\theta) = -\mu^{(i)} + \frac{(\Sigma\theta)^{(i)}}{\sqrt{\theta^T\Sigma\theta}} \cdot \varrho(Z^{(1)})$, so

$$R_{C\%}^{lin,(i)} = 100 \cdot \frac{-\theta^{(i)}\mu^{(i)} + \frac{\theta^{(i)}(\Sigma\theta)^{(i)}}{\sqrt{\theta^T\Sigma\theta}} \cdot \varrho(Z^{(1)})}{-\theta^T\mu + \sqrt{\theta^T\Sigma\theta}\,\varrho(Z^{(1)})}.$$

If $\mu = 0$, $R_{C\%}^{lin,(i)}(\theta) = 100 \cdot \frac{\theta^{(i)}(\Sigma\theta)^{(i)}}{\theta^T\Sigma\theta}$ which does not depend on ϱ.