MF 731 Corporate Risk Management
Final Exam, December 16, 2020

This is the final exam. There are 4 questions for a total 100 points. Each question may contain multiple parts. You have between 9:00 and 11:00 AM to complete the exam

The exam is closed book, notes, cheat sheets, calculator, smart phone and smart watch.

You must upload your answers to Questrom Tools by 11:15 AM. If you type your answers or write them on a note-taking software program, upload your file to Questrom tools. If you write your answers on paper, take a picture of each page you would like to submit and upload the picture file to Questrom tools.

Write your name on every page of your exam (i.e. on every sheet of paper that you turn in)!

If you are stuck on a problem, MOVE ON to other parts of the exam and come back later. Also if you unsure of the answer, write as much as you can so that you can receive partial credit. Blank answers will receive 0 points. Also, please explain your reasoning/provide a derivation for your answers. Answers with no explanation will also receive no credit. Good luck!

Probability Distributions.

(1) Bernoulli: $X \sim B(p)$.

$$\mathbb{P}[X=1]=p; \quad \mathbb{P}[X=0]=1-p; \qquad 0<p<1.$$

Mean: p. Variance: $p(1-p)$. Skewness: $\frac{1-2p}{\sqrt{p(1-p)}}$.

(2) Normal: $X \sim N(\mu,\sigma^2)$.

$$F(x)=\mathbb{P}[X\le x]=\int_{-\infty}^{x}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy; \qquad x\in\mathbb{R},\mu\in\mathbb{R},\sigma>0.$$

Mean: μ. Variance: σ^2. Skewness: 0.

(3) Exponential: $X \sim \operatorname{Exp}(\lambda)$.

$$F(x)=1-e^{-\lambda x}; \qquad x\ge 0,\lambda>0.$$

Mean: $\frac{1}{\lambda}$. Variance: $\frac{1}{\lambda^2}$. Skewness: 2.

(4) Poisson: $X \sim \operatorname{Poi}(\lambda)$.

$$p(x)=\mathbb{P}[X=x]=e^{-\lambda}\frac{\lambda^x}{x!}; \qquad x=0,1,2,...,\lambda>0.$$

Mean: λ. Variance: λ. Skewness $\lambda^{-1/2}$.

(5) Gamma: $X \sim \operatorname{Gamma}(\alpha,\beta)$.

$$F(x)=K(\alpha,\beta)\int_0^x y^{\alpha-1}e^{-\beta y}dy; \qquad x\ge 0,\alpha,\beta>0.$$

Mean: $\frac{\alpha}{\beta}$. Variance: $\frac{\alpha}{\beta^2}$. Skewness: $\frac{2}{\sqrt{\alpha}}$.

(6) Pareto: $X \sim \operatorname{Pareto}(x_m,\alpha)$.

$$F(x)=1-\left(\frac{x_m}{x}\right)^{\alpha}; \qquad x>x_m>0,\alpha>0.$$

(7) Generalized Extreme Value: $X \sim \mathrm{GEV}(\xi,\mu,\sigma^2)$.

$$F(x)=H_{\xi,\mu,\sigma}(x)=\begin{cases} e^{-e^{-\frac{x-\mu}{\sigma}}} & \xi=0,x\in\mathbb{R} \\ e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} & \xi\ne 0,1+\xi\frac{x-\mu}{\sigma}>0 \end{cases}; \quad \mu\in\mathbb{R},\sigma>0.$$

(8) Generalized Pareto: $X \sim \mathrm{GP}(\xi,\beta)$.

$$F(x)=G_{\xi,\beta}(x)=\begin{cases} 1-e^{-\frac{x}{\beta}} & \xi=0,x\ge 0 \\ 1-\left(1+\frac{\xi x}{\beta}\right)^{-\frac{1}{\xi}} & \xi\ne 0,1+\frac{\xi x}{\beta}\ge 0,x\ge 0 \end{cases}; \quad \beta>0.$$

Abbreviations.

(1) pdf: probability density function.
(2) pmf: probability mass function.
(3) cdf: cumulative distribution function.
(4) iid: independent, identically distributed.
(5) rv: random variable.

1. Mini Questions on Operational, Model, and Liquidity Risk.

(a) (**10 Points**) Outline the "Central Limit Theorem" approximation to the cdf of the compound sum $S_N = \sum_{n=1}^{N} X_n$ where $\{X_n\}_{n=1}^{\infty}$ are iid with common cdf G, and where N is a rv independent of $\{X_n\}_{n=1}^{\infty}$ taking values in $0, 1, 2, ...$ (here we define $S_N = 0$ if $N = 0$). Using the approximation, estimate $\mathrm{ES}_\alpha (S_N)$, for a given confidence α, when $N \sim \mathrm{Poi}(\lambda)$ and $G \sim \mathrm{Exp}(1/\beta)$.

(b) (**10 Points**) Describe the model risk associated to pricing exotic derivatives in the constant interest rate, stochastic volatility model

$$\frac{\Delta S_n}{S_n} = (e^r - 1) + \sigma_n Z^{\mathbb{Q}}; \qquad Z^{\mathbb{Q}} = \pm 1 \text{ with Prob. } .5, \tag{0.1}$$

calibrated to a given set of market call options prices. Here $\sigma = \{\sigma_n\}$ is the stochastic volatility process. What is the risk and how may we measure it?

(c) (**10 Points**) Assume we own $\lambda_t > 0$ shares of a stock with (theoretical mid-price) S_t at time t. Furthermore, assume a constant proportional bid-ask spread of s for the stock. If the theoretical mid-price S has a log return over $[t, t + \Delta]$ which is normally distributed with mean $\mu_{t+\Delta}$ and variance $\sigma_{t+\Delta}^2$, compute the liquidity adjusted value at risk LVaR_α assuming linearized portfolio losses.

2. Logistic Regression (LogR) and Linear Discriminant Analysis (LDA). Let $\{(x_n, g_n)\}_{n=1}^{N}$ be our classification training data, where $x_n \in \mathbb{R}^M$ and $g_n \in \{0, 1\}$.

(a) (**10 Points**)) Given the training data, derive (do not just write down) the first order optimality conditions for $\widehat{\beta} = (\widehat{\beta}_0, \widehat{\beta}_1)^{\mathrm{T}}$ in the LogR model. You do NOT have to solve for $\widehat{\beta}_0, \widehat{\beta}_1$: just set up the equations.

(b) (**10 Points**)) Derive (do not just write down) the expression for the logit transform of the posterior probabilities using LDA. You do NOT have to give the expressions for the optimal parameters in terms of the training data, but otherwise provide as much detail as you can.

(c) (**Extra Credit: 10 Points**) Assume the training data can be separated in that there exists $\check{\beta}_0, \check{\beta}_1$ such that for $n = 1, ..., N$

$$g_n = 1 \Leftrightarrow \check{\beta}_0 + x_n^{\mathrm{T}} \check{\beta}_1 > 0; \qquad g_n = 0 \Leftrightarrow \check{\beta}_0 + x_n^{\mathrm{T}} \check{\beta}_1 < 0.$$

Show that there is NO optimal $\widehat{\beta}$ for the LogR model. Thus, we see that if Logistic Regression works, it cannot be that *any* decision boundary strictly separates the training data.

3. EVT and Copulas.

(a) (**10 Points**) Let $\{X_i\}_{i=1}^{\infty}$ be iid rvs with common cdf F, and set $M_n = \max\{X_1, ..., X_n\}$. Furthermore, assume for n large we know $M_n \sim H_{\xi,\mu,\sigma}$ is GEV distributed. Write the pseudo-code for how you would implement the Block-Maxima method to obtain ξ, μ, σ. If it turns out that $\xi = 0$, how do we approximate $\mathrm{VaR}_\alpha(F)$?

(b) The assumption that $\{X_i\}_{i=1}^\infty$ is iid is questionable. To introduce dependence we recall that $U_i = F(X_i) \sim U(0,1)$ is uniformly distributed. Thus, assume for each $n = 1, 2, ...$ that $(U_1, ..., U_n)$ is sampled off the Archimedian copula

$$c(u_1, ..., u_n) = \psi\left(\sum_{i=1}^n \psi^{-1}(u_i)\right),$$

where $\psi : [0, \infty) \to [0, 1)$ is strictly decreasing with $\psi(0) = 1, \psi(\infty) = 0$ (c is multi-variate copula which you do NOT have to prove). Then, set $X_i = F^{-1}(U_i)$ for $i = 1, ..., n$.

(i) (**5 Points**) For $\psi(t) = 1/(1+t)$ show that

$$\mathbb{P}\left[M_n \le x\right] = \frac{1}{1 + n\left(\frac{1}{F(x)} - 1\right)}.$$

and note that like the iid case, $F(x) < 1$ implies $\mathbb{P}\left[M_n \le x\right] \to 0$.

(ii) (**5 Points**) Assume $F \sim \mathrm{Exp}\left(1/\beta\right)$. Using the above copula, identify $\{c_n, d_n\}$ so that $\mathbb{P}\left[M_n \le c_n x + d_n\right]$ as a non-degenerate limit $\widehat{H}(x)$. How does it compare to the iid case?

4. CVA for a "one premium payment" CDS. Assume at time $t = 0$ the buyer ("B") enters into an OTC CDS with a counter-party ("S"). The CDS maturity is T. Unlike in class, in this CDS, in order to receive the default protection over the period $[0, T]$, B need only make a *single* premium payment of x at time $T/2$, provided the underlying reference entity R has not defaulted. As in class, if R defaults at $\tau^R \le T$, S pays B the loss given default δ^R at this time. Below, we always assume the interest rate $r > 0$ is constant.

(a) (**10 Points**) Assume a constant loss given default of δ^S for the seller. Leaving the CDS market price process in the general form of $\{V(t)\}_{t \le T}$ derive (do not just write down) the general formula for the CVA at time 0. Justify your answers.

(b) (**10 Points**) To get a more concrete answer we must identify the price process $\{V(t)\}_{t \le T}$. To do this, assume a constant default intensity τ^R for R, and that for pricing purposes, the only information in the economy at t is if R has defaulted.

In this setting, show the value process takes the form $V(t) = 1_{\tau^R > t}\widetilde{V}(t)$ for a deterministic (non-random) function $\widetilde{V}$. Be as explicit as possible when describing $\widetilde{V}$ and identify the spread $\widehat{x}$ when makes $\widetilde{V}(0) = 0$ (fair swap spread).

(c) (**10 Points**) Now, assume the buyer and seller default times τ^B, τ^S are independent of each other (as well as τ^R) have constant intensities γ^B, γ^S. For a general spread x, identify the CVA at time 0. What does this specify to when $x = \widehat{x}$ from part (b)?