Due on 11/12 Tuesday

Problem 1
1.. (20 Points) An Optimal Liquidation Problem.

Solution:

1.(a) Assume $f_n \equiv f$ and f is strictly convex. So our goal is to solve a constrained optimization problem which is

$$\min_{\{\lambda_n\}} \sum_{n=1}^{N} \frac{1}{2}\lambda_n f\left(\frac{\lambda_n}{\lambda^M}\right) \quad \text{subject to} \quad \sum_{n=1}^{N} \lambda_n = \lambda$$

Applying Lagrange Multipliers, we have

$$\min_{\{\lambda_n\}} \left\{ \sum_{n=1}^{N} \frac{1}{2}\lambda_n f\left(\frac{\lambda_n}{\lambda^M}\right) + \theta\left(\lambda - \sum_{n=1}^{N}\lambda_n\right) \right\}$$

$$\Rightarrow \min \left\{ \sum_{n=1}^{N} \frac{1}{2}\lambda_n^2 f\left(\frac{1}{\lambda^M}\right) + \theta\left(\lambda - \sum_{n=1}^{N}\lambda_n\right) \right\}$$

By the first order condition, we have

$$\frac{2f}{\lambda^M}\lambda_n = \theta \Rightarrow \lambda_n = \frac{\theta\lambda^M}{2f}$$

In order to satisfy the constriant,

$$\lambda = \sum_{n=1}^{N}\lambda_n = N\left(\frac{\theta\lambda^M}{2f}\right) = \theta\frac{\lambda^M N}{2f} \Rightarrow \theta = \frac{2f\lambda}{\lambda^M N}$$

Thus, $\lambda_n = \frac{2f\lambda}{\lambda^M N} \cdot \cancel{\lambda^M} \cdot \cancel{2f} = \frac{\lambda}{N}$

So, for specific f (non-decreasing and strictly convex) $\lambda_n = \frac{\lambda}{N}$ is really an optimal condition.

(b) By the question, we have another optimization problem,

$$\min\left\{ \sum_{n=1}^{N} \frac{1}{2}\eta_n \frac{\lambda_n^{1+\beta_n}}{(\lambda^M)^{1+\beta_n}} + \theta\left(\lambda - \sum_{n=1}^{N}\lambda_n\right) \right\}$$

From (a), we will have $\lambda_n = \lambda^M \times \frac{\theta}{2f_n}$ for $n = 1, \ldots, N$,

and $\sum_{n=1}^{N}\lambda_n = \lambda \Rightarrow \frac{\lambda}{\lambda^M} = \frac{\sum_{n=1}^{N}\lambda_n}{\lambda^M} = g(\theta)$.

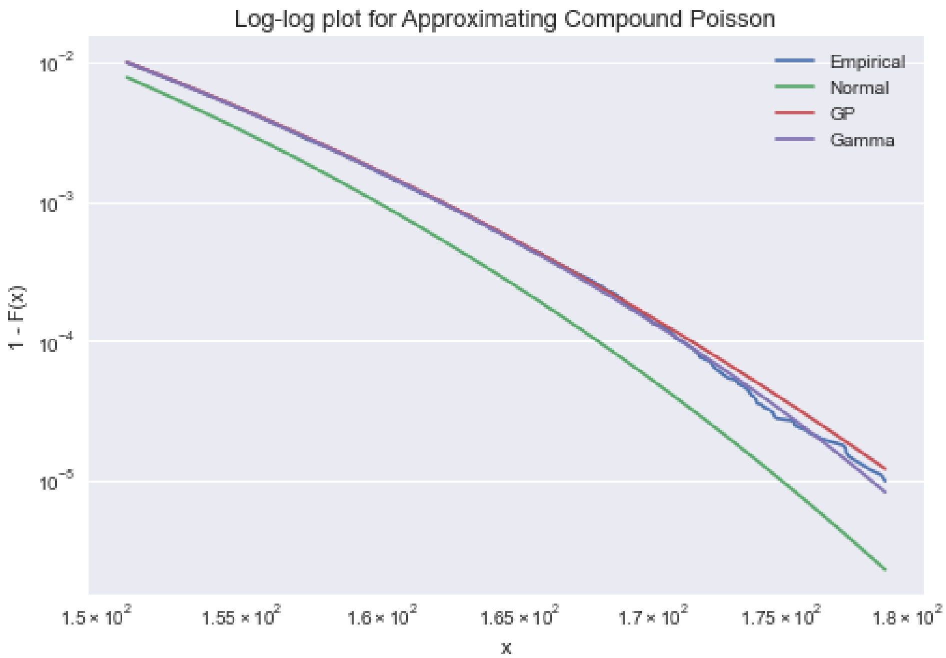
So, g is strictly increasing and $g(0) = 0$, $g(\infty) = \infty$.

There exists a unique $\tilde{g}$ such that $\lambda_n = \lambda^M \times \frac{\theta}{2f_n}$ for $n = 1, \ldots, N$,

since by the same procedure in (a), there has no explicit solution.

Problem 2
2. (30 Points) Approximating a Compound Poisson Random Variable and Risk Measure Estimation.

Solution: (a). GP and Gamma works well.



(b). Math problem result is as follows:

2.(b) $ES = \frac{a}{b} + \frac{F^{-1}(\alpha)}{b(1-\alpha)} f(F^{-1}(\alpha))$

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u\, du \qquad VaR_u = F^{-1}(u)$$

$$w = F^{-1}(u) \qquad u = F(w) \qquad du = f(w)$$

$$= \frac{1}{1-\alpha}\int_{F^{-1}(u)}^{\infty} w f(w)\, dw$$

$$f(w) = k(a,b)\, w^{a-1} e^{-bw}$$

$$= \frac{1}{1-\alpha}\int_{F^{-1}(u)}^{\infty} k(a,b)\, \underbrace{w^a}_{u}\, \underbrace{e^{-bw}}_{dv} \qquad \text{apply integration by parts}$$

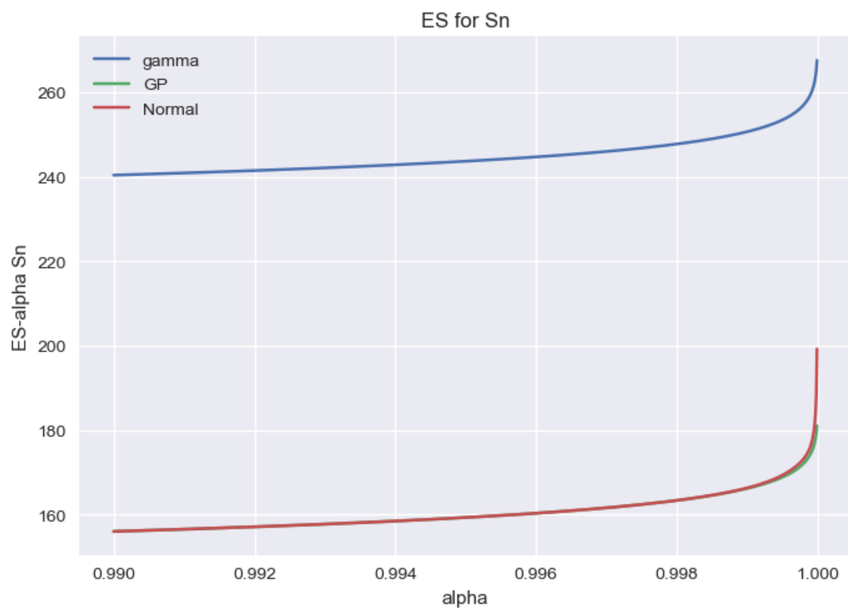
$$= \frac{k(a,b)}{1-\alpha}\left(-\frac{1}{b} e^{-bw} w^a \Big|_{F^{-1}(u)}^{\infty} + \frac{a}{b}\int_{F^{-1}(u)}^{\infty} w^{a-1} e^{-bw}\right)$$

$$= \frac{1}{b(1-\alpha)} F^{-1}(\alpha) \quad \frac{k(a,b) F^{-1}(\alpha)^{a-1} e^{-bF^{-1}(\alpha)}}{f(F^{-1}(\alpha))} + \frac{a}{b(1-\alpha)}\int_{F^{-1}(u)}^{\infty} k(a,b)\, w^{a-1} e^{-bw}\, dw$$

$$= \frac{1}{b(1-\alpha)} F^{-1}(\alpha) f(F^{-1}(\alpha)) + \frac{a}{b(1-\alpha)}\left(1 - F(F^{-1}(\alpha))\right)$$

$$= \frac{a}{b} + \frac{F^{-1}(\alpha)}{b(1-\alpha)} \times f(F^{-1}(\alpha))$$

For the ES, the normal and GP method are very closed to each other, but gamma reached about 250.



Problem 3
1. (20 Points) Calibrating to At the Money Options.

Solution:

part 2 ⟨1.⟩ (a)

$$S_0 = S_0 \quad \begin{cases} S_1^u = S_0(1+\sigma_0) & \begin{cases} S_2^{uu} = S_0(1+\sigma_0)(1+\sigma_1^u) \\ S_2^{ud} = S_0(1+\sigma_0)(1-\sigma_1^u) \end{cases} \\ S_1^d = S_0(1-\sigma_0) & \begin{cases} S_2^{du} = S_0(1-\sigma_0)(1+\sigma_1^d) \\ S_2^{dd} = S_0(1-\sigma_0)(1-\sigma_1^d) \end{cases} \end{cases}$$

$\because C^m(1, S_0) = \frac{\lambda_1 S_0}{4}$

$$\therefore \frac{\lambda_1 S_0}{4} = \mathbb{E}^{\mathbb{Q}}[(S_1 - S_0)^+] = \frac{1}{2}[S_0(1+\sigma_0) - S_0]^+ + \frac{1}{2}[S_0(1-\sigma_0) - S_0]^+$$
$$= \frac{1}{2}(S_0 + S_0\sigma_0 - S_0)^+ + \frac{1}{2}(S_0 - S_0\sigma_0 - S_0)^+$$
$$= \frac{1}{2}S_0\sigma_0$$

$$\Rightarrow \frac{S_0}{4}\lambda_1 = \frac{1}{2}S_0\sigma_0$$
$$\cancel{S_0}\lambda_1 = 2\cancel{S_0}\sigma_0$$
$$\sigma_0 = \frac{\lambda_1}{2}$$

Then, $\because C^m(2, S_0) = \frac{\lambda_2 S_0}{4}$

$$\therefore \frac{\lambda_2 S_0}{4} = \mathbb{E}^{\mathbb{Q}}[(S_2 - S_0)^+] = \frac{S_0}{4}\Big\{[(1+\sigma_0)(1+\sigma_1^u) - 1]^+ + [(1+\sigma_0)(1-\sigma_1^u) - 1]^+$$
$$+ [(1-\sigma_0)(1+\sigma_1^d) - 1]^+ + [\underbrace{(1-\sigma_0)(1-\sigma_1^d)}_{<1} - 1]^+\Big\}$$

$$\Rightarrow \frac{4 \times \lambda_2 S_0 / 4}{S_0} = \lambda_2 = \sigma_0 + (1+\sigma_0)\sigma_1^u + [\underbrace{\sigma_0 - (1+\sigma_0)\sigma_1^u}_{①}]^+ + [\underbrace{(1-\sigma_0)\sigma_1^d - \sigma_0}_{②}]^+$$

By considering the ① and ②, we have the final result,

$$\lambda_2 - \lambda_1 = \frac{1}{2}((2+\lambda_1)\sigma_1^u - \lambda_1)\mathbb{1}_{\{\frac{\lambda_1}{2+\lambda_1} < \sigma_1^u < 1\}} + \frac{1}{2}((2-\lambda_1)\sigma_1^d - \lambda_1)\mathbb{1}_{\{\frac{\lambda_1}{2-\lambda_1} < \sigma_1^d < 1\}}$$

(b) If $\sigma_1^u > \frac{2\lambda_2-\lambda_1}{2+\lambda_1} \Rightarrow \sigma_1^u > \frac{\lambda_1}{2+\lambda_1}$ $(\lambda_2 > \lambda_1)$

$$\lambda_2 - \lambda_1 = \underbrace{\frac{1}{2}\left[(2+\lambda_1)\sigma_1^u - \lambda_1\right]\mathbb{1}_{\left\{\frac{\lambda_1}{2+\lambda_1}<\sigma_1^u<1\right\}}}_{3} + \frac{1}{2}\left((2-\lambda_1)\sigma_1^d - \lambda_1\right)\mathbb{1}_{\left\{\frac{\lambda_1}{2-\lambda_1}<\sigma_1^d<1\right\}}$$

$$3 > \frac{1}{2}\left[(2+\lambda_1)\frac{2\lambda_2-\lambda_1}{2+\lambda_1} - \lambda_1\right] = \frac{1}{2}\left[2\lambda_2 - \lambda_1 - \lambda_1\right] = \lambda_2 - \lambda_1$$

Thus, there is no solution if $\sigma_1^u > \frac{2\lambda_2-\lambda_1}{2+\lambda_1}$.

However, for $\sigma_1^d = \frac{2\lambda_2-\lambda_1}{2-\lambda_1}$

$$\lambda_2 - \lambda_1 = \frac{1}{2}\left[(2+\lambda_1)\sigma_1^u - \lambda_1\right] + \lambda_2 - \lambda_1$$

In order to have a solution, $0 < \sigma_1^u \le \frac{\lambda_1}{2+\lambda_1}$.

For $\sigma_1^d = \frac{2\lambda_2-(2+\lambda_1)\sigma_1^u}{2-\lambda_1}$,

$$\lambda_2 - \lambda_1 = \{\cdots\} + \frac{1}{2}\left[(2-\lambda_1)\frac{2\lambda_2-(2+\lambda_1)\sigma_1^u}{2-\lambda_1} - \lambda_1\right]$$

$$= \{\cdots\} + \frac{1}{2}\left[2\lambda_2 - (2+\lambda_1)\sigma_1^u - \lambda_1\right]$$

In order to have a solution, $\frac{\lambda_1}{2+\lambda_1} < \sigma_1^u < \frac{2\lambda_2-\lambda_1}{2+\lambda_1}$.

Finally, $\sigma_1^u = \frac{2\lambda_2-\lambda_1}{2+\lambda_1}$,

$$\lambda_2 - \lambda_1 = \lambda_2 - \lambda_1 + \frac{1}{2}\left((2-\lambda_1)\sigma_1^d - \lambda_1\right),$$

when $\sigma_1^d \in \left(0, \frac{\lambda_1}{2-\lambda_1}\right)$, we have the solution.

Problem 4
2. (30 Points) VaR with Unknown Parameters.

Solution: (a) and (b)
The average VaR lies well in the middle of the confidence interval. The point estimation of VaR in (a) is approximately the same as the one in (b), and the confidence interval in (a) is smaller than the one in (b). This means the uncertainty of mean return adds to the uncertainty of the estimation.

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The empirical VaR is: 24984.392523364484
The Standard Estimats VaR is: 22773.549452807296
The standard estimator VaR CI is around: [ 21138.534984083803 , 24651.99266165352 ].
The Ym (The Average Simulated VaR) is: 22809.25553449109
The confidence interval (A, B) for the simulated average VaR is
 ( 21139.992273886077 , 24653.77741894026 )
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