Due Date: Tuesday, Oct 5.
Please, if you want to check my answers or see more details, visit the Python and R file!

Problem Part 1-1(a)
VaR for a Portfolio of Microsoft, Apple and Google Stocks.
(a) Estimate the mean vector and covariance matrix for the daily log returns using EWMA. Start with some initial value μ_0, Σ_0 (e.g. the sample mean and variance, you can also set both to 0, it will not matter much) and then update via the EWMA formulas

$$\mu_{t+\Delta} = \lambda \mu_t + (1-\lambda) X_t$$

$$\Sigma_{t+\Delta} = \theta \Sigma_t + (1-\theta)(X_t - \mu_t)(X_t - \mu_t)^T$$

This will give you an estimate for μ, Σ as of 8/31/21, which is then used over the next period 8/31/21 9/1/21. Use $\lambda = \theta = 0.97$.

Solution: The estimate for μ is around $[0.0026, 0.0024, 0.0031]$ for Microsoft, Apple and Google. The estimate for Σ is shown in the figure 2.

Microsoft	Apple	Google
0.00163981	0.00220594	0.00181736
0.00193922	0.00264448	0.0020953
0.00240395	0.00314604	0.00278399
0.00212764	0.00304884	0.00254031
0.0017211	0.00308512	0.00245115
...	...	...
0.00284679	0.00187758	0.00282694
0.00270086	0.00156755	0.00285809
0.00232836	0.00135425	0.00259829
0.00232164	0.00152839	0.00302842
0.00263687	0.00238145	0.00312769

Figure 1: Time-varying EWMA estimates for μ

	Microsoft	Apple	Google
Microsoft	0.000100658	8.73025e-05	5.71717e-05
Apple	8.73025e-05	0.000168033	7.68443e-05
Google	5.71717e-05	7.68443e-05	0.000109096

Figure 2: Last day EWMA estimates for Σ

Problem Part 1-1(b)
Estimate the VaR, for a 95% confidence, of the market cap weighted portfolio in the following ways.
(i) Using the empirical distribution of the log returns and the loss operators.
(ii) Assuming the log returns are normally distributed with the estimated mean vector and covariance matrix, and using loss operators.
How do the VaR estimates compare? Is any one (or more than one) estimate different from the rest? If so, please explain why you think this is the case.

Solution: **(i)** Empirical VaR of log return is: 26105.33683230377
The Empirical VaR of full loss is: 27946.13666729291
The Empirical VaR of 1st approx is: 28346.96860329337
The Empirical VaR of 2nd approx is: 27942.2868718431
(ii) By using the simulation method for EWMA estimates,

The simulation full loss is 13107.32441227952.
The simulation quadratic loss is 13106.867403124746.

By using the simulation method for standard estimates,
The simulation full loss is 30406.840196808043.
The simulation quadratic loss is 30400.149173925623.
The linear loss is around 32000 by using the explicit formula (you can see my code).

It is obvious that VaR estimates by using the empirical distribution is around 28000, so is standard estimates. However, the VaR by using EWMA estimates was pretty low which is around 13000. The reason may be that EWMA (by focusing more on recent returns) is not catching the large losses (high covariance) which happened during the beginning of COVID. So the large "effect" (such as Covid or financial crisis) has been "filtered out". Since EWMA always assigns larger weights to most recent data.

Problem Part 1-2
Using the above methodology, for a lot of 100 puts (i.e. multiply your losses by 100):
(1) Estimate the one day VaR over $[t, t + \Delta]$.
(2) Estimate the K day VaR over $[t, t + K\Delta]$
(3) Compare the K day VaR with $\sqrt{K}$ times the one day VaR, to see how the square root of time rule holds up.

Solution: You can check my codes in the R file. The 1-day VaR is around 29 and 10-day aggregation VaR by sqrt is about 92.

(1)(2) 1-day VaR and sqrt(10)-VaR

```
Lss <- c()
for(i in 1:N){
  Lss <- c(Lss,100*Loss(S0,TT,k,r,mu,sigma,t,delta))
}

VaR_10_sqrt <- sqrt(K) * quantile(Lss,0.95)  # 96
quantile(Lss,0.95)
```

```
##      95%
## 29.40189
```

```
VaR_10_sqrt
```

```
##      95%
## 92.97694
```

The 10-days VaR over $[t, t + K\delta]$ is around 88.

```
    Loss_K <- c(Loss_K,tmpt3)
  }
  result <- c(result,sum(Loss_K))
}
100 * quantile(result,0.95) # 88
```

```
##      95%
## 88.65726
```

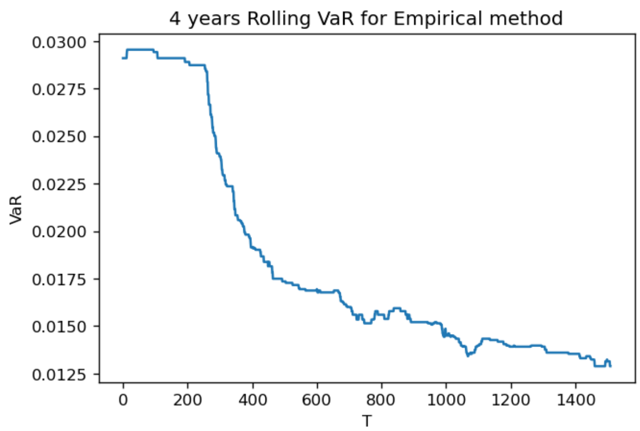
Problem Part 1-3
produce a series of one-day VaR estimates (using the full loss operator) and exceedances using

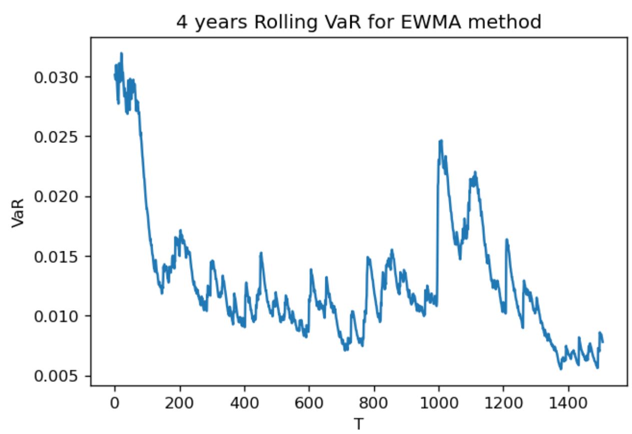
(a) The empirical distribution method.
(b) The normal log returns method with EWMA updating.

Solution: I used the initial μ and Σ as the mean and covariance for the first 1010 days.

```
The exceedences for empirical distribution is:  44
The exceedence for using EWMA method is  79
```

The exceedances by using empirical method is 44 and 79 for EWMA updating.





From the pic we can see that, the sample path for EWMA VaR is more spiky since it assigns more weights on the most recent data, so there may be more exceedances.

Problem Part2-1
Practice with ES.
(a)$L \sim Exp(1/\theta)$ is exponentially distributed with mean θ.
(b) $L \sim Pareto(1, 1/\xi)$ is Pareto distributed with unit scale and $1/\xi$ shape parameter for $\xi > 0$. Here, L has CDF

$$F(\tau) = \begin{cases} 0 & \tau \leq 1 \\ 1 - \tau^{-\frac{1}{\xi}} & \tau > 1 \end{cases}$$

How does your answer depend on ξ ?

Solution: **(a)** $VaR_\alpha = \inf[l | F(l) \geq \alpha]$
$= \inf[l | 1 - e^{-\frac{1}{\theta}l} \geq \alpha]$

$\Rightarrow 1 - e^{-\frac{1}{\theta}l} = \alpha$
$ln(1-\alpha) = -\frac{1}{\theta}l$
$l = -\theta ln(1-\alpha)$

$ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 F^{-1}(u)du$
$= \frac{1}{1-\alpha}\int_\alpha^1 \theta ln(1-u)du$
$= \frac{\theta}{1-\alpha}\int_\alpha^1 ln(1-u)du$
By integration by parts and change of variable,
$= -\theta[ln(-\alpha+1)-1]$ ■
(b)
$VaR_\alpha = inf[l|F(l) \geq \alpha] = inf[l|1 - l^{-\frac{1}{\xi}} \geq \alpha]$
$1 - l^{-\frac{1}{\xi}} = \alpha$
$l^{-\frac{1}{\xi}} = 1 - \alpha$
$l = VaR_\alpha = (1-\alpha)^{-\xi}$

If $\xi < 1$, we have
$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 \frac{1}{(1-u)^\xi}du$
$= \frac{1}{(1-\xi)(1-\alpha)^\xi}$

If $\xi \geq 1$, the $ES_\alpha = \infty$ for all $\alpha \in [0,1]$ ■

Problem Part2-2
Time aggregated risk measures for a constant weight portfolio of equities.
(a)(5 Points) (pen and paper problem)
(b)Write the simulation to obtain the time-aggregated risk measures, building into your simulation that (1) weights are held constant and (2) we use EWMA each day to obtain a new conditional mean and covariance estimate.

Solution: You can see Python codes for more details.
(a)
We know that the loss can be written as $V_{t+k\Delta} = V_{t+(k-1)\Delta} - V_{t+k\Delta}$, then
$\because V_{t+k\Delta} = -V_{t+(k-1)\Delta}(W^Te^{X_{t+k\Delta}} - 1)$
$L_{t+K\Delta} = -V_{t+(k-1)\Delta}(W^Te^{X_{t+k\Delta}} - 1)$
$\therefore V_{t+k\Delta} = V_{t+(k-1)\Delta} + V_{t+(k-1)\Delta}(W^Te^{X_{t+k\Delta}} - 1)$
$= V_{t+(k-1)\Delta}(1 + W^Te^{X_{t+k\Delta}} - 1)$
$= V_{T+(K-1)\Delta}W^Te^{X_{t+k\Delta}}$
$\therefore L_{t+K\Delta} = -V_{t+(k-1)\Delta}(W^Te^{X_{t+k\Delta}} - 1)$
$= -V_{t+(k-2)\Delta}W^Te^{X_{t+(k-1)\Delta}}(W^Te^{X_{t+k\Delta}} - 1)$
$= -V_{t+(k-3)\Delta}W^Te^{X_{t+(k-2)\Delta}} \cdot W^Te^{X_t+(k-1)\Delta}(W^Te^{X_{t+k\Delta}} - 1)$
$= \dots$
$= -V_t\prod_{k=1}^{k-1}W^Te^{X_{t+k\Delta}} \cdot (W^Te^{X_{t+k\Delta}} - 1)$
$\therefore L_{t+K\Delta} = -V_t(\prod_{k=1}^{K}w^Te^{X_{t+k\Delta}} - 1)$
(b)

```
In [28]: runcell('Historical method', 'D:/Courses/731/HWs/HW2/ShiBo731HW2Part2.py')
The 1-day VaR and ES are around:  25986.55749217227  and  30036.9719608208
The 10-day-sqrt VaR and ES are around:  82176.7102221776  and  94985.24541080762
```

```
K = 10 VaR and ES via simulation are:  91290.31711189671  and  108242.50869102548
```