HW Problems for Assignment 6 - Part 2
Due 6:30 PM Tuesday, December 7, 2021

1. (30 Points) CVA, DVA for a Forward Contract in the Black-Scholes Model. As in class, in this exercise we will compute the CVA and DVA for an option in the Black-Scholes model. However, to produce non-zero DVA we will consider a forward contract, whose terminal payoff may be negative.

Recall that in the Black-Scholes model the asset X evolves under risk neutral measure $\mathbb{Q}$ according to

$$\frac{dX_t}{X_t} = rdt + \sigma dW_t^{\mathbb{Q}}; \quad X_0 = X_0$$

Above, the interest rate $r > 0$ and volatility $\sigma > 0$ are constant, and $W^{\mathbb{Q}}$ is a $\mathbb{Q}$ Brownian motion. Now, assume at time 0 we have entered into a forward contract with maturity T on X. This is an agreement, made at time 0, to purchase X at time T for a certain forward price K, which must be determined at time 0. Since the contract costs nothing to enter into, the risk neutral theory of pricing tells us that the forward price K must satisfy

$$0 = E^{\mathbb{Q}}\left[e^{-rT}(X_T - K)\right] \Longrightarrow K = e^{rT}X_0,$$

where we have use that the discounted asset price is a Martingale under risk neutral measure. As such, the time T cash flow of the option is $V(T) = X_T - X_0 e^{rT}$.

The contract buyer B has default time τ^B, and contract seller S has default time τ^S. Set $\tau = \min\left\{\tau^B, \tau^S\right\}$ and $\xi \in \{B, S\}$ as the name of the first to default. Assume under $\mathbb{Q}$ that

(i) τ^B, τ^S are independent of each other, and independent of $W^{\mathbb{Q}}$.
(ii) τ^B, τ^S have constant intensities γ^B, γ^S.
(iii) B and S have deterministic losses given default δ^B, δ^S.

Our goal is to compute the CVA and DVA for the forward contract, from the perspective of the buyer B of the contract. To do this:

(a) (**5 Points**) Derive an explicit formula for the time $t \leq T$ price of the forward contract assuming no default:

$$V(t) = E^{\mathbb{Q}}\left[e^{-r(T-t)}(X_T - X_0 e^{rT}) \mid \mathcal{F}_t\right].$$

where $\mathcal{F}_t$ is the information in the Black-Scholes model at t ignoring any defaults.

(b) (**10 Points**) Using the independence of $W^{\mathbb{Q}}, \tau^S, \tau^B$, show that with $\gamma = \gamma^B + \gamma^S$ we have

$$\mathrm{CVA}(0) = \delta^S E^{\mathbb{Q}}\left[\int_0^T \gamma^S V(t)^+ e^{-(r+\gamma)t} dt\right].$$

$$\mathrm{DVA}(0) = \delta^B E^{\mathbb{Q}}\left[\int_0^T \gamma^B V(t)^- e^{-(r+\gamma)t} dt\right].$$

(c) (**10 Points**) Denote by

$$C^{BS}(t,x;r,K,T), \quad P^{BS}(t,x;r,K,T),$$

the Black-Scholes call and put prices for a given time t and current stock value x. Here, we keep r, K, T to highlight the dependence on the interest rate, strike and maturity. Show that

$$\begin{aligned}
\mathrm{CVA}(0) &= \delta^S\gamma^S \int_0^T C^{BS}(0, X_0; 0, X_0, t) e^{-\gamma t} dt; \\
&= \frac{\delta^S\gamma^S}{\gamma} E^{\mathbb{Q}}\left[1_{\tau\le T} C^{BS}(0, X_0; 0, X_0, \tau)\right]. \\
\mathrm{DVA}(0) &= \delta^B\gamma^B \int_0^T P^{BS}(0, X_0; 0, X_0, t) e^{-\gamma t} dt; \\
&= \frac{\delta^B\gamma^B}{\gamma} E^{\mathbb{Q}}\left[1_{\tau\le T} P^{BS}(0, X_0; 0, X_0, \tau)\right].
\end{aligned}$$

Thus, we may view the CVA and DVA in terms of the expected value of an at the money call/put option when the interest rate is 0, and the maturity is the default time τ (if $\tau \le T$).

(d) (**5 Points**) Prove the explicit formula:

$$\mathrm{CVA}(0) = \frac{\gamma^S\delta^S X_0}{\gamma}\left(2\int_0^T N\left(\frac{1}{2}\sigma\sqrt{t}\right)\gamma e^{-\gamma t} dt - \left(1 - e^{-\gamma T}\right)\right),$$

where N is the standard normal c.d.f..

2. (20 Points) CVA for a Call Option in the Black-Scholes Model with correlated default times for B, S. We retain the setting of problem one. However, now we wish to compute the CVA for a call option, but DO NOT want to assume the default times τ_B, τ_S are independent. Rather, we will assume the buyer and seller default times take the form

$$\tau_S = -\frac{1}{\gamma_S}\log(U_S); \qquad \tau_B = -\frac{1}{\gamma_B}\log(U_B),$$

where $\gamma_B, \gamma_S > 0$ are constant, and (U_S, U_B) are $U(0,1)$ random variables generated off of the Gumbel copula

$$c_\theta^{\mathrm{Gu}}(u_S, u_B) = e^{-\left((-\log(u_S))^\theta + (-\log(u_B))^\theta\right)^{1/\theta}}; \qquad \theta \ge 1.$$

We again write $\tau = \min\left\{\tau^B, \tau^S\right\}$ and $\xi \in \{B, S\}$ as the name of the first to default and assume

(i) τ^B, τ^S are independent of $W^{\mathbb{Q}}$.

(ii) B and S have deterministic losses given default δ^B, δ^S.

Our goal is to compute the CVA for the call option from the perspective of the buyer B. We will do this in the following steps.

(a) (**5 Points**) Using the independence of (τ, ξ) and $W^{\mathbb{Q}}$, show that

$$\mathrm{CVA}(0) = \delta^S C^{BS}(0, X_0) E^{\mathbb{Q}}\left[1_{\tau \le T} 1_{\xi = S}\right].$$

where $C^{BS}(0, X_0)$ is the time 0 price of the call option.

(b) (**10 Points**) For any (smooth) copula c show that

$$\mathbb{Q}\left[\tau \le T, \xi = S\right] = \int_{e^{-\gamma_S T}}^{1} \partial_{u_S} c\left(u_S, u_S^{\gamma_B/\gamma_S}\right) du_S.$$

(c) (**5 Points**) Specifying to the Gumbel case, show that

$$\mathrm{CVA}(0) = \delta^S C^{BS}(0, X_0) \frac{\gamma_S^\theta}{\gamma_S^\theta + \gamma_B^\theta}\left(1 - e^{-\left(\gamma_S^\theta + \gamma_B^\theta\right)^{1/\theta} T}\right)$$