Due on Nov 9

Problem 1
(a) Show that $V(t, s_1, s_2; \sigma, \rho) = e^{r(T-t)}\left(s_1^2 e^{\sigma_1^2(T-t)} - 2 s_1 s_2 e^{\sigma_1\sigma_2\rho(T-t)} + s_2^2 e^{\sigma_2^2(T-t)}\right)$
(b) Output the worst case scenario risk measure $\varrho(L_{t+5\Delta}) = \max\{\ell_n \mid n = 1, \ldots, N\}$ along with the log return/volatility combination which achieves this measure.
(c) Compute the scenario risk measure $\varrho(L_{t+5\Delta}) = \max\{w_n \ell_n \mid n = 1, \ldots, N\}$ along with the log return/volatility/correlation combination which achieves this measure. For this combination, what is the portfolio loss?

Solution: **(a)** Setting:
$\mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(S_T^{(1)} - S_T^{(2)})^2 | S_t^{(1)}, S_t^{(2)}]$
where $\mathbb{Q}$-measure: $\frac{dS_t^{(1)}}{S_t} = rdt + \sigma_1 dW_t^{\mathbb{Q},1}$
$\frac{dSt^2}{St} = rdt + \sigma_2(\rho(W_T^{\mathbb{Q},1} - W_t^{\mathbb{Q},2}) + \sqrt{1-\rho}(W_T^{\mathbb{Q},2} - W_t^{\mathbb{Q},2}))$
So, we have
$S_T^{(1)} = S_t^{(1)} e^{r(T-t)+\sigma_1(W_T^{\mathbb{Q},1} - W_t^{\mathbb{Q},1}) - \frac{1}{2}\sigma_1^2(T-t)}$
$S_T^{(2)} = S_t^{(2)} e^{r(T-t)-\sigma_2(\rho(W_T^{\mathbb{Q},1} - W_T^{\mathbb{Q},2}) + \sqrt{1-\rho^2}(W_T^{\mathbb{Q},2} - W_T^{\mathbb{Q},2})) - \frac{1}{2}\sigma_2^2(T-t)}$
$\therefore \mathbb{E}[e^{r(T-t)}(S_t^{(1)} e^{(r-\frac{1}{2}\sigma_1^2)(T-t)+\sigma_1\sqrt{T-t}Z_1} - S_t^{(2)} e^{(r-\frac{1}{2}\sigma_2^2)(T-t)+\rho\sigma_2\sqrt{T-t}Z_1+\sqrt{1-\rho^2}\sigma_2\sqrt{T-t}Z_2})^2]$
$\mapsto S_t^{(1)}, S_t^{(2)}$ are constant
$\mapsto Z_1$ is dependent of Z_2 and both are $\sim N(0,1)$ r.v.
$= e^{-r(T-t)}(S_1^2 e^{\sigma_1^2(T-t)} - 2S_1S_2 e^{\sigma_1\sigma_2\rho(T-t)} + S_2^2 e^{\sigma_2^2(T-t)})$
(b) The worst case scenario risk measure along with the log return/ Volatility Correlation combination is about **1000**.($X_1 = X_2 = 0.2, \sigma_1 = \sigma_2 = 0.5, \rho = 0.5$)
(c) The worst case scenario risk measure with **weights** along with the log return/ Volatility Correlation combination is about **1300**($X_1 = X_2 = 0.05, \sigma_1 = \sigma_2 = 0.75, \rho = 0.5$).

Problem 2
(a) The average K-day portfolio loss.
(b) An estimate of the K-day VaR_α.
(c) The frequency with which the losses exceeded the initial K day Value at Risk found using the square root of time rule.
(d) The frequency with which the losses exceed the regulatory capital found using the square root of time rule.

Solution: If you want to check the correctness of result, please run my code.

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The Initial one day VaR: [[14383.84618561]]
The Initial 10 day VaR: [[45485.71546004]]
3x Initial 10 day VaR (capital charge): [[136457.14638012]]
Calculating (Shocked) Case...
The average K-day portfolio loss: 70508.71938205088
The Estimate of the K-day VaRα: 157287.0948226779
The frequency with which the losses exceeded the initial K day VaR is 70.024
 The frequency with which the losses exceed the regulatory capital  9.152
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Problem 3
(10 Points) GEV Distributions for Finite Discrete Distributions.

Solution:

1. Suppose there exists $C_n > 0$ and $d_n \in \mathbb{R}$ such that $F(C_n x + d_n)^n$ approaches to $H(X)$ and $H(X)$ degenerated, thus

$$F(C_n x + d_n)^n = \left(\sum_{k=1}^{\lfloor C_n x + d_n \rfloor} P(k)\right)^n$$

as $n \to \infty$, we have

$$\sum_{k=1}^{\lfloor C_n x + d_n \rfloor} P(k) = \begin{cases} = 1 & \text{if } C_n x + d_n \geq k \\ < 1 & \text{if } C_n x + d_n < k \end{cases}$$

Then $\lim_{n \to \infty} F(C_n x + d_n)^n = 1_{C_n x + d_n \geq k}$

The above shows that $H(X)$ is degenerated. Thus, there is no such $C_n > 0$ and $d_n \in \mathbb{R}$ such that $H(X)$ a non-degenerated cdf.

Problem 4
(20 Points, 10 points each part) GP Distributions for U(0, 1) and N(0, 1) random variables.

Solution: **(a)**

2. (a) We observe that for $0 < u < 1$ and $0 < x < 1 - u$:

$$F_u(x) = \mathbb{P}[X \leq u + x \mid X > u] = \frac{\mathbb{P}(u \leq X \leq x + u)}{\mathbb{P}(u > X)} = \frac{X}{1 - u}$$

Since $G_{-1,\beta}(x) = \frac{x}{\beta}$. So we can take $\beta(u) = \frac{1}{1-u}$.

(b)

(b) From the hint, we can see that for $x > 0$:

$$\frac{1}{\sqrt{2\pi}} \frac{x}{1+x^2} e^{-\frac{x^2}{2}} \leq 1 - F(x) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}} \quad \circledast$$

Next, $F_u(x) = \mathbb{P}[X \leq x + u \mid X > u] = \frac{\mathbb{P}[u < X < x + u]}{\mathbb{P}[X > u]}$

$$= \frac{F(x+u) - F(u)}{1 - F(u)} = \frac{1 - F(u) - (1 - F(x+u))}{1 - F(u)}$$

Recall that $G_{0,1/u}(x) = 1 - e^{-xu}$ for $x > 0$. Thus, from $\circledast$ and since $u > 0$ is fixed, we deduce for all $x > 0$ that

$$F_u(x) - G_{0,1/u}(x) = \frac{1-F(u) - (1-F(x+u))}{1-F(u)} - 1 + e^{-xu}$$

$$\geq \frac{\frac{u}{1+u^2} e^{-u^2/2} - \frac{1}{x+u} e^{-(x+u)^2/2}}{\frac{1}{u} e^{-u^2}} - 1 + e^{-xu}$$

$$= \frac{1}{u^2} + e^{-xu} \left(\frac{x+u(1-e^{-x^2/2})}{x+u} \right)$$

$$\geq -\frac{1}{u^2}$$

This gives $\liminf_{u \uparrow \infty} \inf_{x > 0} F_u(x) - G_{0,1/u}(x) = 0$

For the upper bound we obtain for $u > 0$ fixed and all $x \geq \delta$:

$$F_u(x) - G_{0,1/u}(x) = \frac{1-F(u) - (1-F(x+u))}{1-F(u)} - 1 + e^{-xu}$$

$$\leq \frac{\frac{1}{u} e^{-u^2/2} - \frac{x+u}{1+(x+u)^2} e^{-(x+u)^2/2}}{\frac{u}{1+u^2} e^{-u^2}} - 1 + e^{-xu}$$

$$= \frac{1}{u^2} + e^{-xu} \left(1 - \frac{(1+u^2)(x+u)}{u(1+(x+u)^2)} e^{\frac{-x^2}{2}} \right)$$

$$\leq \frac{1}{u^2} + e^{-xu} \leq \frac{1}{u^2} + e^{-\delta u}$$

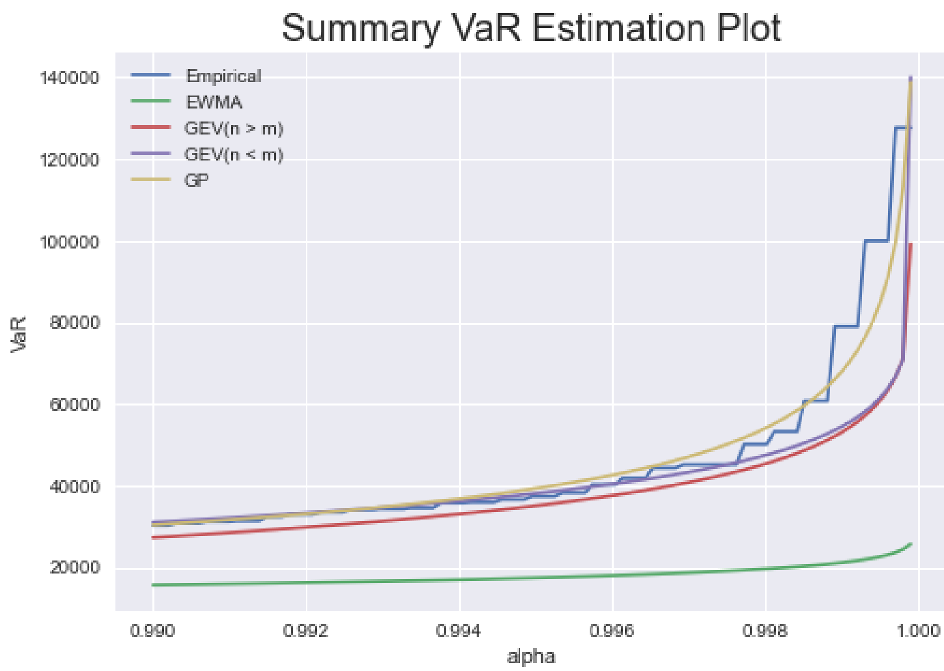
Thus, we see that

$$\limsup_{u \uparrow \infty} \sup_{x \geq \delta} F_u(x) - G_{0,1/u}(x) = 0$$

Put the lower and upper bounds together gives the result.

Problem 5
(20 Points). How bad can things get?
For each method, produce a plot of $\alpha \to VaR_\alpha$ for between .99 and .9999 in increments of .000099 (100 values). For $\alpha = .9999$ what are theVaR values? Which is highest? How do the two EVT estimates compare (is there a meaningful difference)

Solution: The plots of VaR for different method and VaR values have been shown below.
Clearly, the one with highest VaR at 0.9999 level is GP and the second one is empirical method. It is not surprised since what we learned in the class has been presented that GP will achieve the best fit. Since it not only, unlike expected shortfall, captured the distribution for the large loss at tail, also provided more information about the distribution. However, ES only gives the conditional average.
For the EVT, both n or m case is reasonable, but $m > n$ case gives a better fit, the difference is not that obvious but still exists.

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The VaR for alpha = 0.9999 (Empirical) is 127652.0
The VaR for alpha = 0.9999 (EWMA) is 25662.0
The VaR for alpha = 0.9999 (GEV, n = 73 m = 35) is 82677.0
The VaR for alpha = 0.9999 (GEV, n = 35 m = 73) is 102609.0
The VaR for alpha = 0.9999 (GP) is 138782.0
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