HW Problems for Assignment 4 - Part II
Due 6:30 PM Tuesday, November 13, 2018
SOLUTIONS

Note : this is the second part of HW 4, covering material from Lecture 9 on November 6th. The first part of HW 3 was posted on October 30th, and covered material from Lecture 8. Both parts are to be submitted by 6:30 PM on November 13th. There are 50 points possible on this part.

1. (10 Points) GEV distributions and Minima. Much like with the maximum of i.i.d. random variables, it is possible to use the GEV theory for the minimum of i.i.d. random variables, as we will now show. Let $\{X_n\}_{n=1,2,...}$ be a sequence of i.i.d. random variables with common c.d.f. F, and set $\tilde{M}_n := \min\{X_1, ..., X_n\}$ for $n = 1, 2, ...$.

(a) Show that $\tilde{M}_n$ has c.d.f. $\mathbb{P}\left[\tilde{M}_n \leq x\right] = 1 - (1 - F(x))^n$.

(b) Provided that the convergence takes place at all, show that

$$\mathbb{P}\left[\tilde{M}_n < c_n x - d_n\right] \longrightarrow 1 - H_{\xi,\mu,\sigma}(-x).$$

Thus, while for maxima we fit data to $H_{\xi,\mu,\sigma}$, for minima we fit data to (potentially different) $H_{\xi,\mu,\sigma}$.

Solution: Set $\hat{M}_n = \max\{-X_1, \dots, -X_n\} = -\tilde{M}_n$. Then for any $c_n > 0, d_n$

$$\mathbb{P}\left[\tilde{M}_n < c_n x - d_n\right] = 1 - \mathbb{P}\left[\hat{M}_n \leq c_n(-x) + d_n\right].$$

Thus, provided there is convergence, the right-hand side above goes to $H_{\xi,\mu,\sigma}(-x)$, and hence $\mathbb{P}\left[\tilde{M}_n < c_n x - d_n\right]$ converges to $1 - H_{\xi,\mu,\sigma}(-x)$.

2. (15 Points) Practice with GP Distributions. Let X have p.d.f. $f(x) = (k+1)x^k$ for $0 \leq x \leq 1$ and $k = 0, 1, 2, ...$ Show that, for all k, with $\xi = -1$ and $\beta(u) = 1 - u$ we have

$$\lim_{u \uparrow 1} \sup_{0 \leq x \leq 1-u} \left|\mathbb{P}\left[X - x \leq u \mid X > u\right] - G_{\xi,\beta(u)}(x)\right| = 0. \tag{0.1}$$

Hint: Note that $1 - u^{k+1} = (1-u)(1 + u + ... + u^k)$ and, since $u \leq 1$

$$1 + u + ... + u^k \leq 1 + k; \qquad 1 + u^{-1} + ...u^{-k} \geq 1 + k.$$

Extra Credit: (15 Points) Identify ξ and $\beta(u)$ such that (0.1) holds for $f(x) = 6x(1-x)$ for $0 \leq x \leq 1$. Make sure to prove your answer. This problem seems to be very hard.

Solution: Note that for $0 \le x \le 1-u$

$$\mathbb{P}\left[X - x \le u \;\middle|\; X > u\right] = \frac{\int_u^{u+x} f(y)dy}{\int_u^1 f(y)dy} = \frac{(x+u)^{k+1} - u^{k+1}}{1-u^{k+1}},$$

while $G_{-1,(1-u)} = x/(1-u)$ for $0 \le x \le 1-u$. Define

$$m(x,u) := \frac{(x+u)^{k+1} - u^{k+1}}{1-u^{k+1}} - \frac{x}{1-u},$$

so we must show $\lim_{u\uparrow 1} \sup_{0\le x\le 1-u} |m(x,u)| = 0$. For u fixed, we know $x \to m(x,u)$ is convex, with $m(0,u) = 0 = m(1-u,0)$. This gives $m(x,u) \le 0$ as the maximum of a convex function over a closed interval occurs at one of the end-points. As for the lower bound

$$\partial_x m(x,u) = \frac{(k+1)(x+u)^k}{(1-u)(1+u+...+u^k)} - \frac{1}{1-u}.$$

Using the hint, at $x=0$ this is

$$\frac{1}{1-u}\left(\frac{k+1}{1+u^{-1}+...+u^{-k}} - 1\right) < 0,$$

and at $x = 1-u$ this is

$$\frac{1}{1-u}\left(\frac{k+1}{1+u+...+u^k} - 1\right) > 0.$$

Therefore, by convexity there is a unique minimizer over $[0, 1-u]$ which satisfies the first order conditions

$$\hat{x}(u) = \left(\frac{1+u+...+u^k}{1+k}\right)^{1/k} - u.$$

Plugging in for $\hat{x}$ gives

$$\begin{aligned}
m(x,u) &\ge m(\hat{x}(u),u);\\
&= \frac{1}{1-u}\left(\frac{\left(\frac{1+u+...+u^k}{1+k}\right)^{(k+1)/k} - u^{k+1}}{1+u+...+u^k} - \left(\left(\frac{1+u+...+u^k}{1+k}\right)^{1/k} + u\right)\right);\\
&= \frac{1}{(1-u)(1+u+...+u^k)}\left(u(1+u+...+u^k) - u^{k+1} - k\left(\frac{1+u+...+u^k}{1+k}\right)^{(k+1)/k}\right);\\
&= \frac{1}{1-u^{k+1}}\left(u+...+u^k - k\left(\frac{1+u+...+u^k}{1+k}\right)^{(k+1)/k}\right);
\end{aligned}$$

The term within the parentheses goes to 0 as $u \uparrow 1$. Using l'Hospital's rule

$$\begin{aligned}\lim_{u\uparrow 1} m(\hat{x},u) &= \lim_{u\uparrow 1} \frac{1}{-(k+1)u^k} \\ &\quad \times \left(1+2u+...ku^{k-1}-(k+1)\left(\frac{1+u+...+u^k}{1+k}\right)^{1/k}\frac{1+2u+...+ku^{k-1}}{1+k}\right); \\ &= -\frac{1}{k+1}\left(k(k+1)/2-(k+1)\frac{k(k+1)/2}{k+1}\right); \\ &= 0.\end{aligned}$$

Therefore, the result holds.

3. (25 Points) Could we have predicted the 1987 **crash using extreme value theory?** The data file

"SP500_Log_Returns.csv"

contains daily log return data for the S%P 500 index from 6/10/1960 until 10/16/1987. Specifically, column 1 stores the date in numeric format (oldest to newest) and column 2 the log return.

As you may recall, 10/16/1987 was the day before the famous crash in the markets. In this exercise we will see if our extreme value theory-based risk measures could have predicted the crash.

Consider a hypothetical portfolio of $\$1,000,000$ in the S%P 500 index. The losses are estimated via the linearized returns so that $L' = -VX$ where V is the portfolio value, and X is the log return. For each of the methods described below, determine the VaR_α, as a function of α, for the portfolio losses.

(1) Using the empirical distribution for the log returns.
(2) Assuming that as of $t = 10/16/1987$ the log returns $X_{t+\Delta}$ over the next business day are normally distributed with mean $\mu_{t+\Delta}$ and variance $\sigma^2_{t+\Delta}$. To estimate $\mu_{t+\Delta}, \sigma^2_{t+\Delta}$ use an EWMA procedure with parameter $\lambda = \theta = .97$ and a 500 day initialization time (as we have done before).
(3) Assuming a GEV distribution for the maximum and the block maximum method. For the N days of data ($N = 6875$), break the data into blocks of $n = 125$ days for a total of $m = 55$ blocks. Estimate the resultant GEV parameters: in matlab this is done using the 'gevfit' command.
(4) Assume a GP excess loss distribution. Here, take $u = \text{VaR}_{.95}$ as estimated in (1) above. To estimate the GP parameters, you will have to select only those losses above the u threshold, and compute the loss minus u for these selected days. With this data you can then fit the GP distribution. For example, in matlab this is the 'gpfit' command. With the fitted parameters, estimate the VaR_α in terms of α and $F(u)$, which in this case is .95 by construction.

For each of the four methods, produce a plot of $\alpha \mapsto \text{VaR}_\alpha$ for α between .99 and .9999 in increments of .000099 (100 values). For $\alpha = .9999$ what are the four VaR_α values? Which is highest?

The actual log return over Monday October 19th, 1987 was $X = -0.099452258$, so that our linearized losses would have been $L' = \$99,452$. Could any of the above methods predicted such a loss via the VaR_α? If so, for what α?

Solution: See the Matlab file "SP_Crash.m".