HW Problems for Assignment 5 - Part 2
Due 6:30 PM Wednesday, November 23, 2021

1. (20 Points) Calibrating to At the Money Options. In this exercise we will reproduce the calibration results given at the end of the Model Risk lecture.

Assume 0 interest rate, and the market prices for the call options are

$$C^m(1, S_0) = \frac{\lambda_1 S_0}{4}; \qquad C^m(2, S_0) = \frac{\lambda_2 S_0}{4}, \qquad 0 < \lambda_1 < \lambda_2 < 1.$$

We express the options prices in terms of $S_0/4$ because it makes the algebra much simpler, and we assume $\lambda_1 < \lambda_2 < 1$ to ensure that we can calibrate. The assumption $\lambda_1 < \lambda_2$ reflects the call price is more valuable the longer the remaining time to maturity, and $\lambda_2 < 1$ is a very mild condition saying the time 2 call price cannot be more than 25% of the stock price.

(a) (**10 Points**) Show that $\sigma_0 = \lambda_1/2$ and that σ_1^u, σ_1^d satisfy the equality

$$\begin{aligned} \lambda_2 - \lambda_1 = {} & \frac{1}{2}\left((2+\lambda_1)\sigma_1^u - \lambda_1\right) 1_{\frac{\lambda_1}{2+\lambda_1} < \sigma_1^u < 1} \\ & + \frac{1}{2}\left((2-\lambda_1)\sigma_d^1 - \lambda_1\right) 1_{\frac{\lambda_1}{2-\lambda_1} < \sigma_1^d < 1}. \end{aligned} \tag{0.1}$$

(b) (**10 Points**) Show that (0.1) has no solution if $\sigma_1^u > \frac{2\lambda_2 - \lambda_1}{2+\lambda_1}$. However, for $0 < \sigma_1^u \leq \frac{2\lambda_2-\lambda_1}{2+\lambda_1}$ if we set

$$\sigma_1^d = \begin{cases} \frac{2\lambda_2 - \lambda_1}{2-\lambda_1} & 0 < \sigma_1^u \leq \frac{\lambda_1}{2+\lambda_1} \\ \frac{2\lambda_2 - (2+\lambda_1)\sigma_1^u}{2-\lambda_1} & \frac{\lambda_1}{2+\lambda_1} < \sigma_1^u < \frac{2\lambda_2-\lambda_1}{2+\lambda_1} \\ \text{any value in } \left(0, \frac{\lambda_1}{2-\lambda_1}\right] & \sigma_1^u = \frac{2\lambda_2 - \lambda_1}{2+\lambda_1} \end{cases}$$

then (σ_1^u, σ_1^d) solve (0.1). A plot of this function is on slide 38 (you DO NOT have to plot the function). **Note:** $\lambda_2 < 1$ ensures $(2\lambda_2 - \lambda_1)/(2 + \lambda_1) < 1$.

2. (30 Points) VaR with Unknown Parameters. The file

"AAPL_Data.xlsx",

contains daily price data for Apple, Inc from November 1, 2016 through October 31, 2018. Column A contains the date, and column B the closing price. Data is sorted oldest to newest.

In this exercise, you will estimate the VaR for the losses of a $\$1,000,000$ portfolio in Apple index over $10/31/18 - 11/1/2018$. In addition to estimating VaR, you will output confidence intervals for your estimate based off the methodologies described in class.

Assume $X_{t+\Delta} \overset{\mathcal{F}_t}{\sim} N\left(\mu_{t+\Delta}, \sigma^2_{t+\Delta}\right)$ where $\mu_{t+\Delta}, \sigma^2_{t+\Delta}$ are the (unknown) true values of the mean and variance. Using the full loss operator

$$L_{t+\Delta} = -V_t\left(e^{X_{t+\Delta}} - 1\right),$$

we know the (theoretical) VaR is

$$\mathrm{VaR}_\alpha\left(L_{t+\Delta}\right) = V_t\left(1 - e^{\mu_{t+\Delta} + \sigma_{t+\Delta} N^{-1}(1-\alpha)}\right). \tag{0.2}$$

Write a simulation to estimate $\mathrm{VaR}_\alpha\left(L_{t+\Delta}\right)$ as well as the confidence intervals in the two ways listed below. For each, note that from the data file, we can obtain the number of days of data n, along with the standard estimators for the log return mean $\bar{X}_n$ and variance S^2_n.

(a) (**15 Points**) Assuming $\mu_{t+\Delta}$ is known (and given by $\bar{X}_n$) but $\sigma_{t+\Delta}$ is unknown. In this setting, first estimate $\mathrm{VaR}_\alpha\left(L_{t+\Delta}\right)$ using 1) the empirical distribution of log returns and 2) the theoretical formula in (0.1) with the sample mean and variance. Next, produce a $100(1-\beta)\%$ confidence interval for your theoretical estimate, following the methodology of the lecture.

(b) (**15 Points**) Assuming both $\mu_{t+\Delta}$ and $\sigma_{t+\Delta}$ are unknown. Here, write a simulation to estimate $\mathrm{VaR}_\alpha\left(L_{t+\Delta}\right)$ using the methodology described in class. Note that the simulation will produce samples $Y_m = \mathrm{VaR}_\alpha\left(L_{t+\Delta}\right)^m$ for $m = 1, ..., M$. With these simulated values, output the average

$$\bar{Y}_M = \frac{1}{M}\sum_{m=1}^{M} Y_m,$$

as well as the confidence interval (A, B) where A is the $\beta/2$ quantile of the empirical distribution of the $\{Y_m\}$ and B is the $1-\beta/2$ quantile of the empirical distribution of the $\{Y_m\}$.

For parameter values use $\alpha = .97$, $\beta = .02$, and $M = 125,000$ trials for your simulation in part (b). How do your VaR estimates and confidence intervals compare?