MF 731 Corporate Risk Management
Midterm Exam. October 26th 2021
SOLUTIONS

This is the midterm exam. There are 4 questions for a total 100 points. Each question may contain multiple parts. The exam will begin at 8:30 AM and end at 10:30 AM.

The exam is closed book, notes, cheat sheets, calculator, internet smart phone and smart watch.

You must upload your answers to Questrom Tools by 10:45 AM. If you type your answers or write them on a note-taking software program, upload your file to Questrom tools. If you write your answers on paper, take a picture of each page you would like to submit and upload the picture file to Questrom tools.

Write your name on every page of your exam (i.e. on every sheet of paper that you turn in)!

If you are stuck on a problem, MOVE ON to other parts of the exam and come back later. Also if you unsure of the answer, write as much as you can so that you can receive partial credit. Blank answers will receive 0 points. Also, please explain your reasoning/provide a derivation for your answers. Answers with no explanation will also receive no credit. Good luck!

1. (35 Points) Risk Measures from Historical Data Assume we have a portfolio of d stocks, and are estimating the risk of our portfolio using log returns for the risk factor changes. The portfolio is market cap weighted, which means each day, the fraction of wealth we hold in the i^{th} stock is its market capitalization divided by the total capitalization of all the stocks. To fix notation, at time t we write the market capitalization weights as $\left\{w_t^{(i),c}\right\}_{i=1}^{d}$.

(a) **(10 Points)** Express the full and linearized losses, $L_{t+\Delta}$ and $L_{t+\Delta}^{lin}$, over $[t, t+\Delta]$ in terms of the portfolio value at t, the weights at t, and the log-returns over $[t, t+\Delta]$.

(b) **(10 Points)** Now, assume we have historical data for the market capitalization weights and log returns

$$\left\{\widehat{w}_{t-(n-1)\Delta}^{c}\right\}_{n=1}^{N}; \qquad \left\{\widehat{x}_{t-(n-1)\Delta}\right\}_{n=1}^{N}.$$

Describe how you would estimate the Value at Risk for $L_{t+\Delta}, L_{t+\Delta}^{lin}$ using the historical empirical distribution.

(c) **(15 Points)** Using the historical data of part (b), describe how you would estimate the Expected Shortfall for $L_{t+\Delta}, L_{t+\Delta}^{lin}$ assuming $X_{t+\Delta} \overset{\mathcal{F}_t}{\sim} N\left(\mu_{t+\Delta}, \Sigma_{t+\Delta}\right)$ and EWMA updating. Here, you may take as given some initial estimate for the mean vector and covariance matrix, and where ever you may need it, you may always assume for a given (large) integer M that $M\alpha$ is also an integer.

Solution

(a) We have $V_t = V_t$ and

$$V_{t+\Delta} = \sum_{i=1}^{d} \lambda_t^{(i)} S_{t+\Delta}^{(i)},$$

where λ_t are the share positions. To find λ_t we recall that $\lambda_t^{(i)} S_t^{(i)} = \theta_t^{(i)} = V_t w_t^{(i),c}$. Along with $S_{t+\Delta}^{(i)} = S_t^{(i)} e^{X_{t+\Delta}^{(i)}}$ this gives

$$V_{t+\Delta} = V_t \sum_{i=1}^{d} w_t^{(i),c} e^{X_{t+\Delta}^{(i)}}$$

The full losses are then

$$L_{t+\Delta} = -V_t \left(\sum_{i=1}^{d} w_t^{(i),c} e^{X_{t+\Delta}^{(i)}} - 1 \right) = -V_t \sum_{i=1}^{d} w_t^{(i),c} \left(e^{X_{t+\Delta}^{(i)}} - 1 \right)$$

as the capitalization weights sum to 1. The first order approximation $e^x - 1 \approx x$ gives

$$L_{t+\Delta}^{lin} = -V_t \sum_{i=1}^{d} w_t^{(i),c} e^{X_{t+\Delta}^{(i)}}$$

(b) For the empirical loss distribution, we put a mass of $1/N$ at each loss ℓ_n where ℓ_n is determined by our loss operator convention (i.e. full or linear), as well as the current positions $\widehat{w}_t^c$ and historical log returns $\left\{\widehat{x}_{t-(n-1)\Delta}\right\}_{n=1}^N$. This gives

$$\ell_n = -V_t \sum_{i=1}^d \widehat{w}_t^{(i),c} \left(e^{\widehat{x}^{(i)}_{t-(n-1)\Delta}} - 1 \right); \qquad \text{(full)},$$

$$\ell_n = -V_t \sum_{i=1}^d \widehat{w}_t^{(i),c} \times \widehat{x}^{(i)}_{t-(n-1)\Delta}; \qquad \text{(linear)},$$

However we obtain the losses, once we have the loss data $\{\ell_n\}_{n=1}^N$ we estimate the Value at Risk by $\mathrm{VaR}_\alpha = \ell_{(\lceil N\alpha \rceil)}$ where the $\left\{\ell_{(n)}\right\}_{n=1}^N$ are the losses sorted in ascending order.

(c) Taking the initial estimates as fixed, we obtain our current estimate for the mean vector and covariance matrix through the following updating procedure. We first set $\mu^{old}_{t+\Delta}$ and $\Sigma^{old}_{t+\Delta}$ using some initial estimate. Then for $n = 1$ to N we update by

$$\mu^{new}_{t+\Delta} = \lambda \mu^{old}_{t+\Delta} + (1-\lambda)\widehat{x}_{t-(N-n)\Delta},$$
$$\Sigma^{new}_{t+\Delta} = \theta \Sigma^{old}_{t+\Delta} + (1-\theta)\left(\widehat{x}_{t-(N-n)\Delta} - \mu^{old}_{t+\Delta}\right)\left(\widehat{x}_{t-(N-n)\Delta} - \mu^{old}_{t+\Delta}\right)'.$$

We then set $\mu_{t+\Delta} = \mu^{new}_{t+\Delta}$ and $\Sigma_{t+\Delta} = \Sigma^{new}_{t+\Delta}$. To estimate the expected shortfall, we do different things for the full and linearized case. For the linearized case we use the fact that

$$L^{lin}_{t+\Delta} = -V_t (w_t^c)' X_{t+\Delta} = -V_t (w_t^c)' \mu_{t+\Delta} + V_t \sqrt{(w_t^c)' \Sigma_{t+\Delta} w_t^c} Z,$$

where $Z \overset{\mathcal{F}_t}{\sim} N(0,1)$. Since the expected shortfall of Z is $\varphi(N^{-1}(\alpha))/(1-\alpha)$ for the standard normal CDF N and pdf φ, we obtain from positive homogeneity and cash-additivity that

$$\mathrm{ES}_{L^{lin}_{t+\Delta}}(\alpha) = -V_t (w_t^c)' \mu_{t+\Delta} + V_t \sqrt{(w_t^c)' \Sigma_{t+\Delta} w_t^c} \frac{\varphi(N^{-1}(\alpha))}{1-\alpha}.$$

For the full losses we have to estimate the expected shortfall by simulation. Here, for $m = 1, ..., M$ we sample $X^m_{t+\Delta}$ off a $N(\mu_{t+\Delta}, \Sigma_{t+\Delta})$ distribution and compute the loss $\ell_m = -V_t \sum_{i=1}^d w_t^{(i),c}\left(e^{X^{(i),m}_{t+\Delta}} - 1\right)$. As we may assume $M\alpha$ is an integer, we then output the estimate

$$\mathrm{ES}_{L_{t+\Delta}}(\alpha) = \frac{1}{1-\alpha} \sum_{k=M\alpha+1}^{M} \ell_{(k)}.$$

2. (15 Points) Mini Questions on Stress Testing.

(a) **(5 Points)** Show that the stress test risk measure is coherent.

(b) **(10 Points)** In a two stock model, with log returns $(X^{(1)}, X^{(2)})$ over the period, assume that given $X^{(2)} = x$, we know $X^{(1)}$ is normally distributed with mean $a + bx$ and variance c^2 for certain constants a, b and $c > 0$. Derive an explicit expression for the stressed Value at Risk using full losses, in this case. Here, derive your answer using dollar positions θ.

Solution

(a) The stress test measure is $\varrho(L) = \sup\{L(x) \mid x \in \mathcal{S}\}$ where $\mathcal{S}$ is a set of scenarios. For monotonicity

$$L_1(x) \leq L_2(x) \ \forall\ x \in \mathcal{S} \Rightarrow \sup\{L_1(x) \mid x \in \mathcal{S}\} \leq \sup\{L_2(x) \mid x \in \mathcal{S}\}.$$

For cash-additivity, for a constant c

$$\sup\{L(x) - c \mid x \in \mathcal{S}\} = -c + \sup\{L(x) \mid x \in \mathcal{S}\}.$$

For positive homogeneity, if $\lambda > 0$

$$\sup\{\lambda L(x) \mid x \in \mathcal{S}\} = \lambda \times \sup\{L(x) \mid x \in \mathcal{S}\}.$$

Lastly, for sub-additivity, fix $x_0 \in \mathcal{S}$

$$L_1(x_0) + L_2(x_0) \leq \sup\{L_1(x) \mid x \in \mathcal{S}\} + \sup\{L_2(x) \mid x \in \mathcal{S}\}$$

As this holds for all $x_0 \in \mathcal{S}$ we have

$$\begin{aligned}
&\sup\{L_1(x_0) + L_2(x_0) \mid x_0 \in \mathcal{S}\} \\
&\quad \leq \sup\{\sup\{L_1(x) \mid x \in \mathcal{S}\} + \sup\{L_2(x) \mid x \in \mathcal{S}\} \mid x_0 \in \mathcal{S}\}, \\
&\quad = \sup\{L_1(x) \mid x \in \mathcal{S}\} + \sup\{L_2(x) \mid x \in \mathcal{S}\}
\end{aligned}$$

(b) Conditioning on $X^{(2)} = x$ and under the given normal assumption we have

$$\begin{aligned}
L &= -\theta^{(1)}\left(e^{X^{(1)}} - 1\right) - \theta^{(2)}\left(e^{x} - 1\right), \\
&= -\theta^{(1)}\left(e^{a+bx+cZ} - 1\right) - \theta^{(2)}\left(e^{x} - 1\right)
\end{aligned}$$

where $Z \sim N(0,1)$. We can write this

$$L = -\widetilde{a}e^{cZ} - \widetilde{b}$$

for

$$\widetilde{a} = \theta^{(1)}e^{a+bx}, \qquad \widetilde{b} = \theta^{(2)}(e^{x} - 1) - \theta^{(1)}.$$

From here, direct calculation shows

$$\alpha = \mathbb{P}\{L \leq \tau\} \quad \Leftrightarrow \tau = -\widetilde{a}e^{cN^{-1}(1-\alpha)} - \widetilde{b},$$

so that

$$\begin{aligned}
\mathrm{VaR}_\alpha &= -\widetilde{a}e^{cN^{-1}(1-\alpha)} - \widetilde{b} \\
&= -\theta^{(1)}(e^{a+bx+cN^{-1}(1-\alpha)} - 1) - \theta^{(2)}(e^{x} - 1).
\end{aligned}$$

3. (5 points each) True/False. Identify if each of the statements below is true or false. If it is true provide a short explanation or proof for why it is true. If it is false, provide a short explanation, proof, or counter-example for why it is false. Answers with no explanation will not be given any credit.

(a) In the Barings' Rogue Trader case study, it was market risk which played the primary role in the downfall of Barings bank.

(b) In the GARCH(1,1) model it is allowable to have the innovation, or "shock" random variables $\{Z_k\}$ take the form $Z_k = Y_k - 1$ where the $\{Y_k\}$ are iid exponential random variables with mean 1.

(c) Because VaR is not coherent, we have that $\mathcal{R}(\theta) \neq \sum_{i=1}^{d} \mathcal{R}_{\mathrm{M}}^{(i)}(\theta)\,\theta^{(i)}$. Here, we are considering an equity portfolio with d stocks, θ is the dollar position vector, $\mathcal{R}(\theta) = \mathrm{VaR}_{\alpha}(L)$ is the Value at Risk for the losses (as a function of θ), and $\mathcal{R}_M$ is the marginal Value at Risk.

(d) If the scenario weights sum to 1, and if there is a scenario where losses are non-negative, the stress test risk measure always produces a larger risk estimate than the scenario analysis risk measure.

(e) There is a spectral function ϕ such that $\varrho_{\phi}(Z) = 0$ where $Z \sim N(0,1)$ is a standard normal random variable.

Solution:

(a) FALSE. Barings was certainly negatively affected by market risk, in that the trader Nick Leeson bet on an increase in the NIKKEI index, but the index went down. However, the primary source of risk was operational; Nick Leeson had control over both the front and back offices, and management ignored an internal audit which warned of his power.

(b) TRUE. We only require the $\{Z_k\}$ to be iid with mean 0 and variance 1. Since Y_k has mean 1 and variance 1, $Z_k = Y_k - 1$ satisfies the given requirements.

(c) FALSE. The relationship $\mathcal{R}(\theta) = \sum_{i=1}^{d} \mathcal{R}_{\mathrm{M}}^{(i)}(\theta)\,\theta^{(i)}$ only requires positive homogeneity, which VaR satisfies.

(d) TRUE. The scenario analysis risk measure is $\Psi_{x,w} = \max\{w_1\ell_1, \ldots, w_n\ell_n\}$ where w_i is the weight of the i^{th} scenario, and ℓ_i is the loss in the i^{th} scenario. The stress test risk measure is $\Psi_x = \max\{\ell_1, ..., \ell_n\}$. Since $w_i \leq 1$ for each i we have $w_i\ell_i \leq \ell_i$ provided $\ell_i \geq 0$. It then follows that if there is a single scenario where losses are non-negative then the stress test measure outputs a larger value.

(e) TRUE. Take $\phi(u) = 1$ for all u. This is non-decreasing, right continuous with $\int_0^1 \phi(u)du = 1$. But

$$\varrho_{\phi}(Z) = \int_0^1 \mathrm{VaR}_Z(u)\,du = \int_0^1 N^{-1}(u)du = \int_{-\infty}^{\infty} z\varphi(w)dw = 0,$$

where N is the standard normal CDF and φ is the standard normal pdf.

4. (25 Points) Time Aggregated Risk Measures. As in problem 1, we consider an equity portfolio with d stocks and use log returns as the risk factor changes. However, rather than using market capitalizations as weights, we construct an equally weighted portfolio where $w_t^{(i)} = 1/d$ for $i = 1, ..., d$ and all t. For the full portfolio losses, we are interested in estimating both the Value at Risk (α confidence) and exponential spectral risk measure (spectral function $\phi_\gamma(u) = \frac{\gamma}{e^\gamma - 1} e^{\gamma u}$ for $\gamma > 0$). However, for regulatory purposes we must provide 10 day estimates, not one day estimates.

For each day t, we assume $X_{t+\Delta} \overset{\mathcal{F}_t}{\sim} N\left(0, \Sigma_{t+\Delta}\right)$ (i.e. we have conformed to the stylized fact that daily conditional returns are negligible), and to obtain $\Sigma_{t+\Delta}$ we use EWMA updating.

Starting at time t_0, and assuming you know V_{t_0} and already have an estimate for $\Sigma_{t_0+\Delta}$, write the pseudo-code for how you would estimate the 10 day value at risk, and exponential spectral risk measure

(a) **(10 Points)** Using the square root of time rule.

(b) **(15 Points)** Running a simulation over the 10 day period.

For the exponential spectral measure you may use without proof that

$$\int_a^b \phi_\gamma(u) du = \frac{e^{\gamma b} - e^{\gamma a}}{e^\gamma - 1}.$$

Solution

(a) First, we must estimate the risk measures for one day losses. To do this, for $m = 1, ..., M$ we sample $X_{t_0+\Delta}^m \sim N(0, \Sigma_{t_0+\Delta})$ and compute

$$\ell_m = -V_{t_0} \frac{1}{d} \sum_{i=1}^d \left(e^{X_{t+\Delta}^{(i),m}} - 1 \right).$$

We then estimate the one day risk measures

$$\begin{aligned}
\mathrm{VaR}_\alpha \left(L_{t_0+\Delta}\right) &= \ell_{(\lceil M\alpha \rceil)}, \\
\varrho_{\phi_\gamma}\left(L_{t_0+\Delta}\right) &= \int_0^1 \mathrm{VaR}_u \phi_\gamma(u) du = \int_0^1 \ell_{(\lceil Mu \rceil)} \phi_\gamma(u) du, \\
&= \sum_{m=1}^M \ell_{(m)} \int_{\frac{m-1}{M}}^{\frac{m}{M}} \phi_\gamma(u) du = \sum_{m=1}^M \ell_{(m)} \frac{e^{\gamma \frac{m}{M}} - e^{\gamma \frac{m-1}{M}}}{e^\gamma - 1}.
\end{aligned}$$

Lastly we estimate the 10 day risk measures by

$$\begin{aligned}
\mathrm{VaR}_\alpha \left(L_{t_0+10\Delta}\right) &= \sqrt{10} \times \mathrm{VaR}_\alpha \left(L_{t_0+\Delta}\right), \\
\varrho_{\phi_\gamma}\left(L_{t_0+10\Delta}\right) &= \sqrt{10} \times \varrho_{\phi_\gamma}\left(L_{t_0+\Delta}\right).
\end{aligned}$$

(b) For the simulation over the 10 day period, for $m = 1, ..., M$ we

(i) Set $\Sigma_m^0 = \Sigma_{t_0+\Delta}$ and $V_m^0 = V_{t_0}$.

(ii) For $k = 1, ..., 10$ we

(1) Sample $X_m^k \sim N(0, \Sigma_m^{k-1})$.

(2) Set $V_m^k = V_m^{k-1} \times \frac{1}{d} \sum_{i=1}^d e^{X_m^{(i),k}}$.

(3) Set $\Sigma_m^k = \lambda \Sigma_m^{k-1} + (1-\lambda) X_m^k (X_m^k)'$.

(iii) We then set $\ell_m = -\left(V_m^{10} - V_m^0\right)$

This gives our loss data. From here we estimate the risk measures using the same formulas as in part (a), but for 10 days.

$$\mathrm{VaR}_\alpha \left(L_{t_0+10\Delta}\right) = \ell_{(\lceil M\alpha \rceil)},$$

$$\varrho_{\phi_\gamma}(L_{t_0+10\Delta}) = \sum_{m=1}^{M} \ell_{(m)} \frac{e^{\gamma \frac{m}{M}} - e^{\gamma \frac{m-1}{M}}}{e^\gamma - 1}.$$