Operational and Liquidity Risk

MF 731 Corporate Risk Management

Outline

Operational risk.

Liquidity risk.

Operational Risk

Operational risk (BASEL (2001)):

"The risk of losses resulting from inadequate or failed internal processes, people and systems, or from external events."

Seven broad categories:

Internal fraud.

External fraud.

Employment practices and workplace safety.

Clients, products, and business practices.

Damage to physical assets.

Business disruption and system failures.

Execution, delivery, and process management.

Examples of Operational Risk

Internal fraud: Barings lost $1 billion due to fraudulent trading.

External fraud: Republic New York lost $611 million because of fraud committed by a custodial client.

Employment practices: Merrill Lynch lost $250 million is a legal settlement regarding gender discrimination.

System failures: Salomon Brothers lost $303 million from a change in computing technology.

Uniqueness/Importance of Operational Risk

Some risks can be exploited strategically.

- E.g.: developing derivatives to mitigate market risk.

However, operational risk

- Has no upside for a bank.
  - Cannot be bought, sold, or transferred.
  - (recently: some operational risk insurance contracts)
- Can be hard to measure/predict.
  - Lack of publicly available operational loss data.
  - (recently: some industry wide efforts to gather data)
- Is, despite all of this, very important to measure.
  - Capital charges built into the BASEL accords.

Measuring Operational Risk

Three main approaches:

Basic indicator approach.

Standardized approach.

Advanced measurement approach.

Basic Indicator (BI) Approach

Simplest method for measuring operational risk.

Capital charge is a fixed percentage of average positive annual gross income over the last three years.

Idea: annual GI is a proxy for scale of operations.
"Bigger GI Bigger operations"

$$\mathrm{RC}_{\mathrm{BI}}^{t} = \alpha \times \frac{\sum_{i=1}^{3} \max\left\{\mathrm{GI}^{t-i}, 0\right\}}{\sum_{i=1}^{3} 1_{\mathrm{GI}^{t-i}>0}}.$$

percent

only count years with positive GI

GI^{t-i}: annual gross income in year $t-i$.

BASEL suggests $\alpha = 15\%$.

Ideas behind the charge:

Annual gross income is a proxy for the scale of operations.

Charge α percent if annual gross income is positive. Else charge nothing. Take average over the last three years.

BI Approach: Examples

Positive annual gross income all three years.

$$\mathrm{RC}_{\mathrm{BI}}^{t} = \alpha \times \frac{1}{3}\left(\mathrm{GI}^{t-1} + \mathrm{GI}^{t-2} + \mathrm{GI}^{t-3}\right).$$

Positive annual gross income only in years $t-1, t-3$:

$$\mathrm{RC}_{\mathrm{BI}}^{t} = \alpha \times \frac{1}{2}\left(\mathrm{GI}^{t-1} + \mathrm{GI}^{t-3}\right).$$

Standardized Approach

Splits firm activities into 8 business lines.

Corporate finance; trading & sales; retail banking; commercial banking; payment & settlement; agency services; asset management; retail brokerage.

Charge found by weighing business line gross incomes.

Allows for losses in one line to offset gain in another.

$$\mathrm{RC}_S^t = \frac{1}{3}\sum_{i=1}^{3} \underbrace{\max\left\{\sum_{j=1}^{8} \beta_j \mathrm{GI}_j^{t-i}, 0\right\}}_{\neq \sum_{j=1}^{8} \beta_j \max[GI_j^{t-i}, 0]}.$$

大一点

e.g: 两个 Business line, +1 and -1 case.

GI_j^{t-i}: annual gross income of line j in year $t-i$.

β_j: factor which relates income to scale.

Business Line not equally risky.

$$\mathrm{RC}_S^t = \frac{1}{3}\sum_{i=1}^{3} \max\left\{\sum_{j=1}^{8} \beta_j \mathrm{GI}_j^{t-i}, 0\right\}$$

Negative gross income in certain lines offset by positive gross income in other lines.

"Netting" effect: induce firms to prefer standardized.

β's (BASEL):

Business Line	Beta factors
corporate finance	18%
trading & sales	18%
retail banking	12%
commercial banking	15%
payment & settlement	18%
agency services	15%
asset management	12%
retail brokerage	12%

Advanced Measurement Approach (AMA)

BASEL allows banks to use internal systems when determining operational risk capital charges.

- Expects AMA to become the industry standard.

To use AMA, the bank must:

- Have a well-documented operational risk management system, which captures severe tail loss events.
- Keep track of, and report on, operational risk losses.
- Allow for independent review of operational risk management systems.
- Allocate operational risk capital in a way which reduces future operational risk losses.

- AMA
- expected to become industry standard

outline: window $[t, t+\Delta]$, t: now, Δ: one year ← op risk losses occur at a low frequency.

Loss: (modeling)

label losses $\{X_k^{b,\ell}\}$

"b": business line, $1, \dots, 8$

"ℓ": loss category, fraud, physical damage ...

unknown → $N^{b\ell}$: # of op risk losses in business line b in category ℓ over $[t, t+\Delta]$.

$\{X_k^{b\ell}\}_{k=1,\dots,N^{b\ell}}$: losses in line b, category ℓ.

Losses in line b

$$L^b = \sum_{\ell=1}^{7} \sum_{k=1}^{N^{b,\ell}} X_k^{b\ell}$$

Risk of losses in line b, $\rho(L^b)$ ρ: risk measure

Basel recommand: $VaR_{0.999} = \rho$

— Very high, EVT will be used here.

Total risk: $\sum_{b=1}^{8} \rho(L^b)$ → charge

AMA Outline

Our period of interest is $[t, t+\Delta]$.

For operational risk, $\Delta = 1\text{yr}$ typically.

Label operational losses $\left\{X_k^{b,\ell}\right\}$:

$b = 1, ..., 8$: business line.

$\ell = 1, ..., 7$: operational loss category.

$N^{b,\ell}$: number of such losses over $[t, t+\Delta]$.

For $k = 1, ..., N^{b,\ell}$:

$X_k^{b,\ell}$ is the k^{th} such loss over $[t, t+\Delta]$.

AMA Outline

Losses by business line b: $L^b = \sum_{\ell=1}^{7} \sum_{k=1}^{N^{b,\ell}} X_k^{b,\ell}$.

Risk of such losses (i.e. the capital charge): $\varrho(L^b)$.

BASEL recommends $\varrho = \mathrm{VaR}_{.999}$.

Very high α: EVT may be helpful.

Aggregate, firm wide risk:

$\mathrm{RC}_{\mathrm{AM}} = \sum_{b=1}^{8} \varrho(L^b)$. ← independent

Alternatively: $\mathrm{RC}_{\mathrm{AM}} = \sqrt{\sum_{b,c=1}^{8} \varrho(L^b)\varrho(L^c)\rho_{bc}}$. ← like weighted avg.

ρ_{bc}: correlation between business line losses, which somehow must be estimated.

AMA Outline

$$\mathrm{RC}_{\mathrm{AM}} = \sum_{b=1}^{8} \varrho(L^b) = \sum_{b=1}^{8} \varrho\left(\sum_{\ell=1}^{7}\sum_{k=1}^{N^{b,\ell}} X_k^{b,\ell}\right)$$

$X_k^{b,\ell}$: r.v. which represent the $\boxed{k^{th}}$ operational loss in line b of type ℓ over $[t, t+\Delta]$. typically one year.

The number of such losses $N^{b,\ell}$ is also random.

Abstractly: we wish to estimate

$$\varrho\left(S_N := \sum_{k=1}^{N} X_k\right).$$ Random sum of R.V.

N: (random) number of losses.

X_k: (random) size of k^{th} loss.

Compound Sums: $S_N = \sum_{k=1}^{N} X_k$

S_N is a random sum of random variables.

$$S_N(\omega) = \sum_{k=1}^{N(\omega)} X_k(\omega).$$

We will assume

1) $\{X_k\}_{k=1,2,\ldots}$ are non-negative and i.i.d. with c.d.f. G.

   Typically use log-normal distribution. G is log-normal dist.

2) N and $\{X_k\}_{k=1,2,\ldots\infty}$ are independent. The # of losses $\perp\!\!\!\perp$ loss size

3) N has p.m.f. p_N. $S_0 \triangleq 0$

   $N \sim G$N.

   Typically use Poisson distribution.

S_N is called a "compound sum".

To generate samples from S_N, we first sample
1) sample $N \sim p_N$
2) If $N=0$, set $S_N = 0$

no losses is OK.

If $N=n$, $n=1,2,\cdots$ for $k=1,2,\dots,n$ we sample $X_k \sim G$ output $S_N=\sum_{k=1}^N X_k$.

$\{X_k\}$ i.i.d $\sim G$
$N \perp\!\!\!\perp X_k$
$N \sim$ poi

$$S_N = \sum_{k=1}^{N} X_k$$

We want $\rho(S_N)$,
$\Leftrightarrow$ We want to estimate cdf of S_N.
$VaR_\alpha(S_N) = F^{-1}(\alpha)$
cdf $= S_N$.

The c.d.f. of S_N:
Law of total prob

$$F_{S_N}(x) = \mathbb{P}[S_N \le x] = \sum_{k=0}^{\infty} \mathbb{P}[S_N \le x \mid N = k]\,\mathbb{P}[N = k];$$

$$= \sum_{k=0}^{\infty} \mathbb{P}\left[\sum_{j=1}^{k} X_j \le x\right] p_N(k).$$

e.g. $N=0,1$, $X=10,20$

$N=0$, $S_N=0$, P_0
$N=1$ $\begin{cases} S_N = 10 \; (1-P_0)P_{10} \\ S_N = 20 \; (1-P_0)P_{20} \end{cases}$

Except trivial case.

Even if we know p_N, G, the sum is hard to evaluate. no explicit form

$\sum_{j=1}^{k} X_j \sim G^{(k)}$ where $G^{(k)}$ is the k-fold convolution of G.

e.g. $\{X_k\}$ i.i.d $N(0,1)$.
$\sum_{k=1}^{n} X_k \sim N(0,n) \sim \sqrt{n}N(0,1)$

$P_N \sim \text{poi}(\lambda)$, $\Rightarrow$ $\underbrace{1_{0\le x} + \sum_{n=1}^{\infty} N(x/\sqrt{n})\,\text{poisson}}$

We have to approximate.

$\mathbb{P}(\sum_{k=1} X_k \leq x) = N(?, ?)$

$$S_N = \sum_{k=1}^{N} X_k$$

even if nice condition, cannot solve.

How can we estimate the dist. of S_N?

We use the following facts:

$$\boxed{E[S_N] = E[X]\,E[N]. \qquad \mu_{S_N} = \mu_X \mu_N.}$$

$X \sim G$

$\mathbb{E}[\sum_{k=1}^{N} X_k] = \mathbb{E}[\mathbb{E}[\sum_{k=1}^{N} X_k \mid N]] = \mathbb{E}[N\,\mathbb{E}[X]]$
$= \mathbb{E}[N]\,\mathbb{E}[X]$

$$\boxed{\mathrm{Var}[S_N] = \mathrm{Var}[X]\,E[N] + E[X]^2\,\mathrm{Var}[N]. \qquad \sigma^2_{S_N} = \sigma^2_X \mu_N + \mu^2_X \sigma^2_N.}$$

— also follows by conditioning

Typically, we also assume

$N \sim \mathrm{Poi}(\lambda)$ (compound *Poisson* random variable).

$$\text{Skewness} := \frac{E\left[(S_N - \mu_{S_N})^3\right]}{\sigma^3_{S_N}} = \frac{E[X^3]}{\sqrt{\lambda E[X^2]^3}}.$$

$\parallel$ $\mathbb{E}[(S_N - \mathbb{E})^3]$

$X \sim LN(\mu, \sigma^2)$.

Approximations to $S_N = \sum_{k=1}^N X_k$

First approximation: "CLT". $S_N \sim \mu_{S_N} + \sigma_{S_N} N(0,1)$

$\sim \mu_X \mu_N + \sqrt{\mu_N \sigma_X^2 + \mu_X^2 \sigma_N^2}\, N(0,1)$

$$\frac{S_N - \mu_{S_N}}{\sigma_{S_N}} = \frac{S_N - \mu_N \mu_X}{\sqrt{\mu_N \sigma_X^2 + \mu_X^2 \sigma_N^2}} \approx N(0,1).$$

PRETEND N is large.

This holds under some conditions.

Not obvious because N is not necessarily large.

Second approximation: Gamma. Translated Gamma Approximation.

constant

$$S_N \approx k + \mathrm{Gamma}(\alpha, \beta) \text{ if } N \sim \mathrm{Poi}(\lambda).$$

$k + \frac{\alpha}{\beta} = \lambda \mathbb{E}[X_1]$

Gamma(α, β): p.d.f. $\overset{k(\alpha,\beta)}{K} x^{\alpha-1} e^{-\beta x}$.

$\frac{\alpha}{\beta^2} = \lambda \mathbb{E}[X_1^2]$ $\quad \frac{2}{\sqrt{\alpha}} = \frac{\mathbb{E}[X_1^3]}{\sqrt{\lambda (\mathbb{E}(X_1^2))^3}}$

k, α, β found my matching mean, variance, skewness.

Approx. to $S_N = \sum_{k=1}^{N} X_k$

Third approximation: Generalized Pareto.

Idea: It is easy to simulate S_N.

Sample N, Given $N = k$, sample $X_1, \dots, X_k$ and output $X_1 + \dots + X_k$.

Given sampled data $S_N^1, \dots, S_N^M$ and a threshold u. only keep above u

$S_N - u \mid S_u > u$

1) sample $S_N^1, \dots, S_N^m$

2)

We fit a generalized Pareto distribution $G_{\xi,\beta}$ to $\left\{ \tilde{S}_N^m - u \right\}_{m=1}^{M_u}$ where $\tilde{S}_N^m$ are samples which exceed u (M_u total).

$M_u = \#$ of $S_N > u$.

$(\tilde{S}_N^m)_{m \geq 1}^{M_u}$: samples greater than u.

$\mathbb{P}(S_N \leq x) = \mathbb{P}(S_N \leq x \mid S_N > u)\mathbb{P}(S_N > u) + \mathbb{P}(S_N \leq x \mid S_N \leq u)\mathbb{P}(S_N \leq u)$

$$F_{S_N}(x) \approx G_{\xi,\beta}(x-u)(1-F(u)) + F(u), x > u.$$

Need to also approximate $F(u) \approx 1 - \frac{M_u}{M}$.

Approx. to $S_N = \sum_{k=1}^{N} X_k$

Other approximations:

Panjer recursion.

Poisson mixtures.

Homogenous Poisson processes.

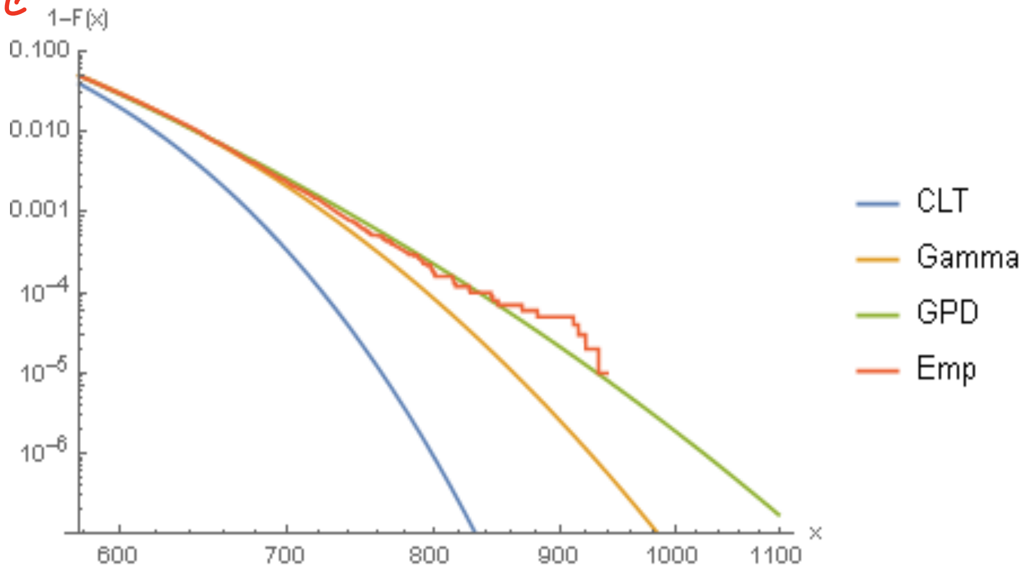
Simple discrete distributions.

E.g.: $N = 0, 1, 2$. $X = 1000, 10000, 100000$ with given probabilities.

Manually compute distribution of S_N by enumerating outcomes and computing probabilities.

E.g.: $N \sim \mathrm{Poi}(\lambda = 100)$, $\log(X) \sim N(1,1)$

around 0.95 threshold.



Plot of $1 - F(x)$ vs. x. $u = 576$.

Empirical distribution found with new sample.

Fit ordering: GPD $>$ Gamma $>$ Normal.

Operational Risk Measures

Approximations allows us to estimate risk measures.

For each method, approximate $F_{S_N}(x) = \mathbb{P}[S_N \leq x]$.

Estimate the risk measure based on the approximation.

E.g. $S_N \sim N\left(\mu_N \mu_X, \mu_N \sigma_X^2 + \mu_X^2 \sigma_N^2\right)$:

$$\mathrm{VaR}_\alpha = \mu_N \mu_X + \sqrt{\mu_N \sigma_X^2 + \mu_X^2 \sigma_N^2}\, N^{-1}(\alpha).$$

E.g. $F_{S_N}(x) \sim G_{\xi,\beta}(x-u)(1-F(u)) + F(u)$:

$$\mathrm{VaR}_\alpha(S_N) = u + \frac{\beta}{\xi}\left(\left(\frac{1-F(u)}{1-\alpha}\right)^{\xi} - 1\right).$$

Gamma: $S_N \sim k + \mathrm{Gamma}(\alpha, \beta)$

$\mathrm{VaR}_\alpha = k + F^{-1}_{\mathrm{Gamma}(\alpha,\beta)}(\alpha)$

Operational Risk Conclusions

Operational risk: inadequate or failed internal processes, people and systems, external events.

BASEL mandates operational risk capital charges.

Basic (standard) charge: a fraction α of average positive gross income (split along business lines) over the most recent three years.

Advanced methods model operational losses as $S_N = \sum_{k=1}^{N} X_k$; a random number of losses (N) with random severities ($\{X_k\}$).

Operational Risk Conclusions

Capital charges computed using risk measures on S_N.

The distribution of S_N is only explicitly known for very simple discrete examples.

In general, we approximate the distribution of S_N using asymptotic results.

To estimate the distributions for N, X, we use historical data along with modeling assumptions.

Challenge: operational risk loss data is scarce.

Liquidity Risk

We now to turn to liquidity risk.

Liquidity refers to the ability of a

- Trader to execute a trade or liquidate a position.
- Company to make cash payments as they come due.

Liquidity depends on

- Number of traders in a market.
- The frequency and size of trades.
- The time it takes to carry out a trade.
- The economic environment.

Liquidity Risk

Traditional assumption: markets are liquid.

- Convenient, but highly questionable.
- Large markets (equities, FX) are liquid.
- "Thin" markets (MBS) are not always liquid.

How do we measure liquidity costs and risk?

- Bid-ask spread.
- Ability to liquidate positions.

We think liquidity risk as a cost added to market risk.

Numerous methods for estimating liquidity risk.

- Our focus: simple methods which add liquidity risk to market risk.

Liquidity and Spreads

There is no "one" price for an asset.

- Bid price: that at which "the market" is willing to buy.
- Ask price: that at which "the market" is willing to sell.

We always have bid $<$ ask.

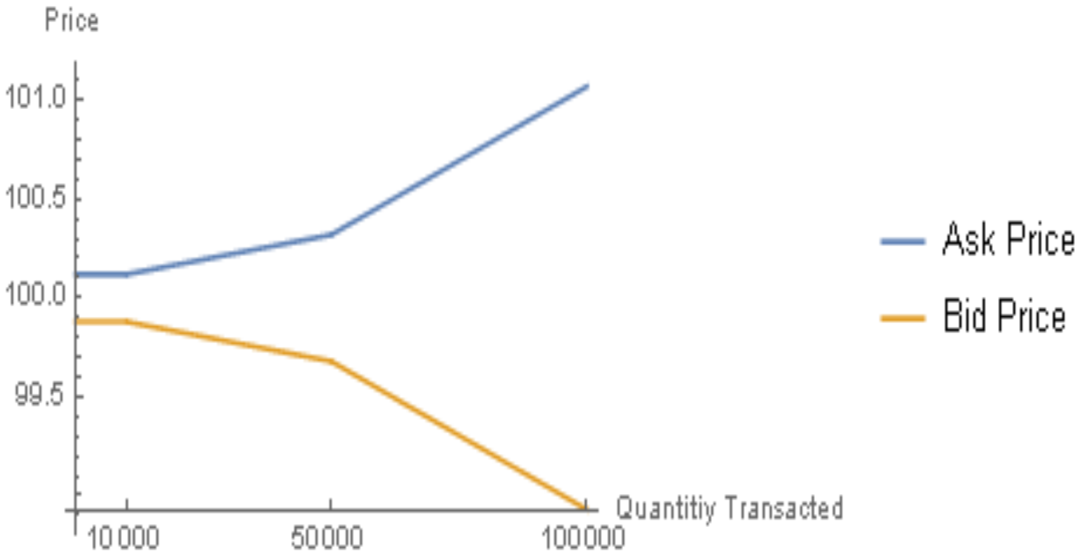
- We sell at the bid. We buy at the ask.

To complicate matters:

- The price also depends on the size of the order.
  - Selling a lot may drive the bid price down.
  - Buying a lot may drive the ask price up.

Liquidity and Spreads

Impact of transaction size on prices.



Stylized figure - in reality impact may not be linear.

Liquidity Costs

We begin by ignoring the position size effect.

S^b: bid price; S^a: ask price.

$\boxed{S = \frac{1}{2}\left(S^a + S^b\right)}$: mid price.

S: "theoretical" price which we model.

Bid-ask spread: $p = S^a - S^b$.

Proportional bid-ask spread: $s = \frac{S^a - S^b}{S}$.

$s = \frac{S^a - S^b}{S}$ ← theoretical price

For U.S. equities: $s \approx .2\%$. Range: $s \in (.05\%, 5\% - 10(+)\%)$.

Compute Liquidity Cost

- Assume we want to sell $\lambda_t > 0$ shares.

Theoritical value of doing this : $\lambda_t S_t$

Actual value : $\lambda_t S_t^b$

$= \lambda_t S_t - \frac{1}{2}\lambda_t S_t \, S'$

$= \theta_t - \frac{1}{2}\theta \, S'$

$LC = \theta_t - \theta_t + \frac{1}{2}\theta_t S' = \frac{1}{2}\theta_t S'$

multiply positions : $\sum_{i=1}^{d} \frac{1}{2}\theta_t^{(i)} S'^{(i)}$

$S = \frac{S^a + S^b}{2}$, $S^a = 2S - S^b$

$S' = \frac{S^a - S^b}{S} = \frac{2S - 2S^b}{S}$

$\Rightarrow S^b = S - \frac{1}{2}S\,S'$

Liquidity Costs

Assume s is constant over $[t, t+\Delta]$.

For a share-position $\lambda_t > 0$ at t:

Theoretical liquidation value: $\lambda_t S_t$.

Actual liquidation value: $\lambda_t S_t^b = \lambda_t S_t - \frac{1}{2}\lambda_t S_t s$.

Liquidity Cost (LC):

$$\mathrm{LC} = \underbrace{\lambda_t S_t}_{\text{Theoretical}} - \underbrace{\lambda_t S_t^b}_{\text{Actual}} = \frac{1}{2} s\theta_t.$$ θ_t: dollar position.

For multiple positions:

$$\mathrm{LC} = \frac{1}{2}\sum_{i=1}^{n} s^{(i)}\theta_t^{(i)}.$$ $s^{(i)}$: spread of i^{th} position.

Liquidity Adjusted VaR

VaR: found using the theoretical price S.

Full losses, linearized losses; Normal, empirical dist. Etc.

For constant s, the liquidity adjusted VaR is

$$\mathrm{LVaR}_\alpha := \mathrm{VaR}_\alpha + \mathrm{LC} = \mathrm{VaR}_\alpha + \frac{1}{2}s\theta_t.$$

however we compute this

The percentage increase in the VaR_α is

$$\left(\frac{\mathrm{LVaR}_\alpha}{\mathrm{VaR}_\alpha} - 1\right)\% := 100\frac{s\theta_t}{2\mathrm{VaR}_\alpha}.$$

$$= 100\frac{LC}{VaR} = 100\left(\tfrac{1}{2}s\,\theta_t / VaR_\alpha\right)$$

LVaR for Constant Spreads

E.g.: equities, full losses, $X_{t+\Delta} \overset{\mathcal{F}_t}{\sim} N(\mu_{t+\Delta}, \sigma^2_{t+\Delta})$.

$$L_{t+\Delta} = -\theta_t(e^{X_{t+\Delta}} - 1).$$

$$\mathrm{VaR}_\alpha = \theta_t \left(1 - e^{\mu_{t+\Delta} + \sigma_{t+\Delta} N^{-1}(1-\alpha)}\right).$$

$$\left(\frac{\mathrm{LVaR}_\alpha}{\mathrm{VaR}_\alpha} - 1\right)\% = \frac{100s}{2\left(1 - e^{\mu_{t+\Delta} + \sigma_{t+\Delta} N^{-1}(1-\alpha)}\right)}.$$

Ex: $\Delta = \frac{1}{252}$, $S_t = 59$, $s = .2\%$, $\mu_{t+\Delta} = 0$, $\sigma_{t+\Delta} = \frac{.3}{\sqrt{252}}$, $\alpha = .99$.

$$\mathrm{VaR}_\alpha = 2.538,\ \mathrm{LC} = 0.059,\ \mathrm{LVaR}_\alpha = 2.597.$$

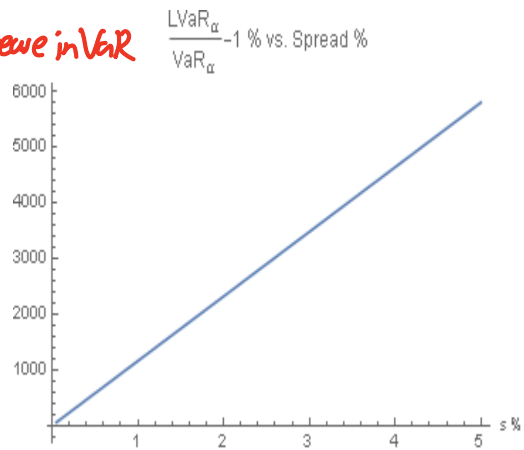
$$\frac{\mathrm{LVaR}_\alpha}{\mathrm{VaR}_\alpha} - 1\ \% = 2.325\%.$$

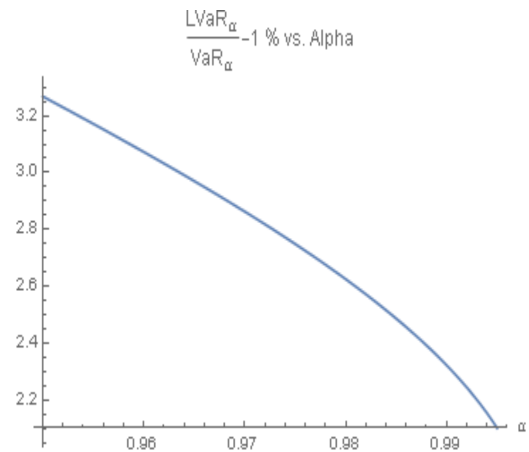
2.325% increase in VaR_α due to liquidity.

LVaR for Constant Spreads

Numerical example (con't).

% increase in VaR





Linearly increasing in spread: can be very high for large spreads. Decreasing in α.

LVaR for Random Spreads

In reality, spreads vary (randomly) over time.

Assume random spreads, $\perp\!\!\!\perp$ of other risk factors.

E.g.: Equities. $s \overset{\mathcal{F}_t}{\sim} N\left(\nu_{t+\Delta}, \zeta^2_{t+\Delta}\right)$, $s \perp\!\!\!\perp X_{t+\Delta}$.

Equity example
- for $m=1,\dots,n$
  Sample $X^m_{t+\Delta} \sim N(\mu_{t+\Delta}, \sigma_{t+\Delta})$
  $L^m_{t+\Delta} = -\theta_t(e^{X^m_{t+\Delta}} - 1)$
  Sample $s^m = N(\nu_{t+\Delta}, \zeta^2_{t+\Delta})$
  $LC^m = \frac{1}{2}\theta_t s^m$
  $\ell^m = L^m_{t+\Delta} + LC^m$
  $LVaR_\alpha = \ell_{(\lceil m\alpha \rceil)}$

Here, the liquidity cost is random.

$$\mathrm{LC} = \tfrac{1}{2}\theta_t s \overset{\mathcal{F}_t}{\sim} N\left(\tfrac{1}{2}\theta_t \nu_{t+\Delta}, \tfrac{1}{4}\theta_t^2 \zeta^2_{t+\Delta}\right).$$

Options to estimate LVaR:

1) Simulate spreads and log returns, compute LVaR.
2) Approximate: $\mathrm{LC} = \frac{1}{2}\theta_t\left(\nu_{t+\Delta} + k\zeta_{t+\Delta}\right)$. $k \approx 1-3$.

Industry Approximation
rather than sampling s, they assume
$s = \nu_{t+\Delta} + k\zeta_{t+\Delta}$
$k = 1, 3, \dots$
$LVaR_\alpha = VaR_\alpha + \frac{1}{2}\theta_t(\nu_{t+\Delta} + k\zeta_{t+\Delta})$
- very little theoretical justification for this.

2) is used in practice. Limited theoretical support.

LVaR for Random Spreads

Industry approximations. $\mathrm{LVaR}_\alpha \approx$:

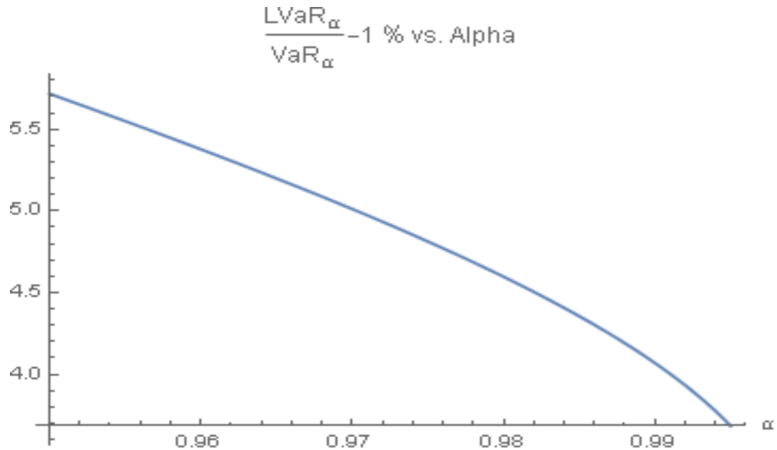
$\mathrm{VaR}_\alpha + \frac{1}{2}\theta_t\left(\nu_{t+\Delta} + k\zeta_{t+\Delta}\right)$: single position.

$\mathrm{VaR}_\alpha + \sum_{i=1}^{n} \frac{1}{2}\theta_t^{(i)}\left(\nu_{t+\Delta}^{(i)} + k\zeta_{t+\Delta}^{(i)}\right)$: multiple positions.

Continuing our example.

$\mathrm{VaR}_\alpha = \theta_t\left(1 - e^{\mu_{t+\Delta}+\sigma_{t+\Delta}N^{-1}(1-\alpha)}\right)$.

For $\nu_{t+\Delta} = .2\%$, $\zeta_{t+\Delta} = .05\%$, $k = 3$:



Price Impact

The first two liquidity risk adjustments do not assume the bid-ask spread depends on our position size.

$\lambda_t S_t^b(\lambda_t)$ ← function of λ_t

E.g.: we liquidate our position λ_t at S_t^b, but S_t^b itself does not depend on λ_t.

$\lambda_t \longmapsto S_t^b(\lambda_t)$

We now incorporate the risk of our position size affecting the bid/ask price.

Price Impact

A simple price impact model.

λ_t^M: total outstanding shares in the market.

Our position relative to the market is λ_t / λ_t^M. fractional shares we want to sell

Assume we want to liquidate our position.

The proportional spread s is modelled by:

$$\frac{1}{2}s = \frac{S_t - S_t^b}{S_t} := \eta \frac{\lambda_t}{\lambda_t^M}.$$

$\eta > 0$: "price elasticity" of demand.

$\rightarrow$ High $\eta > 0$ (illiquid): price moves a lot for large trades.

$\rightarrow$ Low $\eta > 0$ (liquid): prices insensitive to trade size.

$\lambda_t S_t^b(\lambda_t) = \lambda_t S_t - \frac{1}{2}\lambda_t S_t s(\lambda_t)$

$= \lambda_t S_t - \frac{1}{2}\lambda_t \eta \left(\frac{\lambda_t}{\lambda_t^M}\right) S$

$LC = \eta \frac{\lambda_t^2 S_t}{\lambda_t^M}$

$= \eta S_t \lambda_t^M (\lambda_t / \lambda_t^M)^2 = \eta \theta_t^M (\lambda_t/\lambda_t^M)^2 \rightarrow$ cost is quadratic in λ_t.

Price Impact

With this formulation the liquidity cost is

$$\mathrm{LC} = \frac{1}{2}\theta_t s = \eta \theta_t^M \left(\frac{\lambda_t}{\lambda_t^M}\right)^2.$$

$\theta_t^M = \lambda_t^M S_t$: total market capitalization.

Cost is quadratic in our relative position.

The liquidity adjusted Value at Risk is

$$\mathrm{LVaR}_\alpha = \mathrm{VaR}_\alpha + \eta \theta_t^M \left(\frac{\lambda_t}{\lambda_t^M}\right)^2.$$

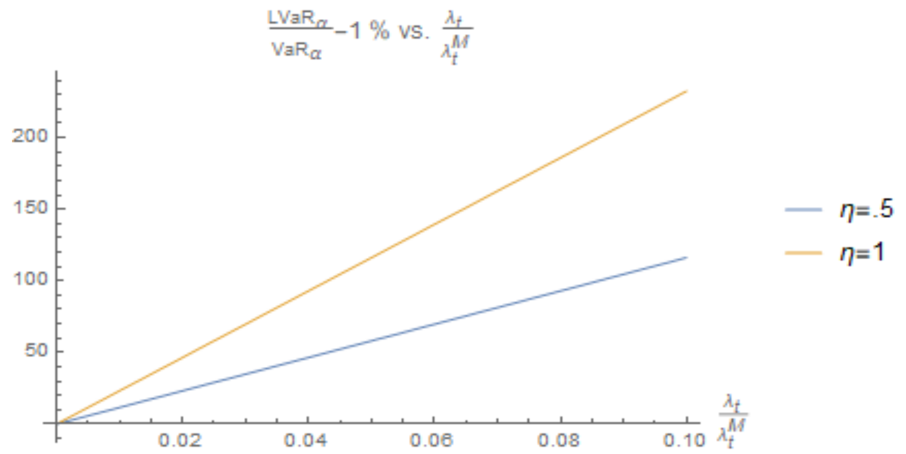
Price Impact

Continuing our example

$$\mathrm{VaR}_\alpha = \theta_t \left(1 - e^{\mu_{t+\Delta}+\sigma_{t+\Delta}N^{-1}(1-\alpha)}\right).$$

$$\mathrm{LVaR}_\alpha = \mathrm{VaR}_\alpha + \eta\theta_t^M \left(\lambda_t/\lambda_t^M\right)^2.$$

Parameters: $\mu_{t+\Delta} = 0$, $\sigma_{t+\Delta} = \frac{.3}{\sqrt{252}}$, $\alpha = .99$.



For $\eta = 1$, $\frac{\lambda_t}{\lambda_t^M} = 5\%$ we have to double the VaR_α!

Price Impact

There are many interesting price impact problems.

Multiple securities:

$$\frac{1}{2}s^i = \eta^i \frac{\lambda_t^i}{\lambda_t^{M,i}},\ i = 1, \dots, d.$$

Non-linear impact functions: $\frac{1}{2}s = \eta \left(\frac{\lambda_t}{\lambda_t^M}\right)^p,\ p \neq 1.$

Optimal liquidation: assume we must sell λ_t shares by $t + K\Delta$. What are the optimal liquidation amounts $\Psi_{t+k\Delta}$ at $(k = 1, \dots, K)$ so that $\sum_{k=1}^{K} \Psi_{t+k\Delta} = \lambda_t$ and the total cost is minimized?

每天出多少最后出完然后cost是minimized

拉格朗日乘子

We have covered a basic situation to communicate the main ideas.

Liquidity Risk Conclusions

Liquidity refers to the ability to make cash payments as they come due.

Liquidity risk is affected by

- The size of the market and of the trade.
- The time it takes to carry out a trade.

We saw three liquidity risk models, focusing on:

- Constant bid-ask spreads.
- Random bid-ask spreads.
- Linear price impact in bid-ask spreads.

Liquidity Risk Conclusions

For each method, we computed the associated liquidity cost LC.

We then added the liquidity cost onto our market based risk measure.

E.g.: $\mathrm{LVaR}_\alpha = \mathrm{VaR}_\alpha + \mathrm{LC}$.

There are many variants on liquidity risk:

Liquidity funding risk.

Non-linear price impact models.

Optimal execution problems.