MF 731 Corporate Risk Management
Final Exam, December 14, 2021
SOLUTIONS

This is the final exam. There are 4 questions for a total 100 points. Each question may contain multiple parts. You have between 9:00 and 11:00 AM to complete the exam

The exam is closed book, notes, cheat sheets, calculator, smart phone and smart watch.

You must upload your answers to Questrom Tools by 11:15 AM. If you type your answers or write them on a note-taking software program, upload your file to Questrom tools. If you write your answers on paper, take a picture of each page you would like to submit and upload the picture file to Questrom tools.

Write your name on every page of your exam (i.e. on every sheet of paper that you turn in)!

If you are stuck on a problem, MOVE ON to other parts of the exam and come back later. Also if you unsure of the answer, write as much as you can so that you can receive partial credit. Blank answers will receive 0 points. Also, please explain your reasoning/provide a derivation for your answers. Answers with no explanation will also receive no credit. Good luck!

Probability Distributions.

(1) Bernoulli: $X \sim B(p)$.

$$\mathbb{P}[X=1]=p; \quad \mathbb{P}[X=0]=1-p; \qquad 0<p<1.$$

Mean: p. Variance: $p(1-p)$. Skewness: $\frac{1-2p}{\sqrt{p(1-p)}}$.

(2) Normal: $X \sim N(\mu, \sigma^2)$.

$$F(x)=\mathbb{P}[X \le x]=\int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy; \qquad x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0.$$

Mean: μ. Variance: σ^2. Skewness: 0.

(3) Exponential: $X \sim \operatorname{Exp}(\lambda)$.

$$F(x)=1-e^{-\lambda x}; \qquad x \ge 0, \lambda > 0.$$

Mean: $\frac{1}{\lambda}$. Variance: $\frac{1}{\lambda^2}$. Skewness: 2.

(4) Poisson: $X \sim \operatorname{Poi}(\lambda)$.

$$p(x)=\mathbb{P}[X=x]=e^{-\lambda}\frac{\lambda^x}{x!}; \qquad x=0,1,2,..., \lambda>0.$$

Mean: λ. Variance: λ. Skewness $\lambda^{-1/2}$.

(5) Gamma: $X \sim \operatorname{Gamma}(\alpha, \beta)$.

$$F(x)=K(\alpha,\beta)\int_0^x y^{\alpha-1}e^{-\beta y}dy; \qquad x \ge 0, \alpha, \beta > 0.$$

Mean: $\frac{\alpha}{\beta}$. Variance: $\frac{\alpha}{\beta^2}$. Skewness: $\frac{2}{\sqrt{\alpha}}$.

(6) Pareto: $X \sim \operatorname{Pareto}(x_m, \alpha)$.

$$F(x)=1-\left(\frac{x_m}{x}\right)^{\alpha}; \qquad x > x_m > 0, \alpha > 0.$$

(7) Generalized Extreme Value: $X \sim \operatorname{GEV}(\xi, \mu, \sigma^2)$.

$$F(x)=H_{\xi,\mu,\sigma}(x)=\begin{cases} e^{-e^{-\frac{x-\mu}{\sigma}}} & \xi=0, x \in \mathbb{R} \\ e^{-\left(1+\xi\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}} & \xi \ne 0, 1+\xi\frac{x-\mu}{\sigma}>0 \end{cases}; \quad \mu \in \mathbb{R}, \sigma > 0.$$

(8) Generalized Pareto: $X \sim \operatorname{GP}(\xi, \beta)$.

$$F(x)=G_{\xi,\beta}(x)=\begin{cases} 1-e^{-\frac{x}{\beta}} & \xi=0, x \ge 0 \\ 1-\left(1+\frac{\xi x}{\beta}\right)^{-\frac{1}{\xi}} & \xi \ne 0, 1+\frac{\xi x}{\beta} \ge 0, x \ge 0 \end{cases}; \quad \beta > 0.$$

Abbreviations.

(1) pdf: probability density function.
(2) pmf: probability mass function.
(3) cdf: cumulative distribution function.
(4) iid: independent, identically distributed.
(5) rv: random variable.

1. (30 Points) Short Questions on Model Risk and Extreme Value Theory.

(a) (**15 Points**) Outline the method by which we account for uncertainty in our VaR estimates due to uncertainty in our estimation of the log return volatility σ. Here, you may assume that at time t, we are using linearized losses (for a given dollar position $\theta_t > 0$ in the asset), and given our information at t, the next day's log return is normally distributed with mean $\mu_{t+\Delta}$ and variance $\sigma^2_{t+\Delta}$. You may also assume we have realizations $\{x_n\}_{n=1}^N$ (either through historical data or Monte Carlo simulation) of the log returns.

(b) (**15 Points**) Outline the threshold exceedances method for estimating the VaR for a given (loss) random variable X. You may assume we have realizations $\{x_n\}_{n=1}^N$ (either through historical data or Monte Carlo simulation) of X which will yield $\xi > 0$ in the generalized Pareto fitting procedure.

Solution:

(a) Dropping all time subscripts we have $L = -\theta X$, so that $X \sim N(\mu, \sigma^2)$ implies $\text{VaR}_\alpha = -\theta\mu + \theta\sigma N^{-1}(\alpha)$. Now, to account for the uncertainty in VaR_α due to uncertainty in σ we first take our realizations $\{x_n\}_{n=1}^N$ and compute the sample mean $\overline{X}_N$ and variance S_N^2. We then write q_β as the β quantile of a chi-square random variable with $N-1$ degrees of freedom. Next, using the approximation $(N-1)S_N^2/\sigma^2 \approx \chi^2_{N-1}$ we state that with $100(1-\beta)\%$ certainty

$$q_{\beta/2} \leq \frac{(N-1)S^2N}{\sigma^2} \leq q_{1-\beta/2},$$

or equivalently

$$\sqrt{\frac{N-1}{q_{1-\beta/2}}}S_N \;=:\; \underline{\sigma}_\beta \leq \sigma \leq \overline{\sigma}_\beta \;:=\; \sqrt{\frac{N-1}{q_{\beta/2}}}S_N$$

This gives our $100(1-\beta)$ confidence interval for the VaR of

$$-\theta\mu + \theta\underline{\sigma}N^{-1}(\alpha) \leq \text{VaR}_\alpha \leq -\theta\mu + \theta\overline{\sigma}N^{-1}(\alpha).$$

(b) The threshold exceedance method makes the approximation that for $x > u$, where u is a given threshold, we have (here, F is the cdf of X)

$$F(x) \approx F(u) + (1 - F(u))G_{\xi,\beta}(x-u),$$

where $G_{\xi,\beta}$ is the generalized Pareto cdf. Solving this for a confidence $\alpha > F(u)$ we obtain the approximation

$$\text{VaR}_\alpha \approx u + G^{-1}_{\xi,\beta}\left(\frac{\alpha - F(u)}{1 - F(u)}\right).$$

When $\xi > 0$ calculation shows

$$G^{-1}_{\xi,\beta}(y) = \frac{\beta}{\xi}\left(\frac{1}{(1-y)^\xi} - 1\right), \quad y < 1,$$

and hence

$$\mathrm{VaR}_\alpha \approx u + \frac{\beta}{\xi}\left(\left(\frac{1-F(u)}{1-\alpha}\right)^{\xi} - 1\right).$$

2. (30 Points) CVA for a Defaultable Bond using Recovery at Treasury. Assume at time $t=0$ the buyer ("B") enters into the following OTC contract with a counter-party ("S"). For a given reference entity R, if R defaults prior to T, then S will give B $\delta^R \in (0,1)$ notional of a default free zero coupon bond (ZCB) maturing at T. In exchange, B will pay S a one time payment at time 0 to make the contract fairly valued.

Below, we assume the interest rate r and all default intensities $\gamma^R, \gamma^B, \gamma^S$ are constant.

(a) (**15 Points**) For $t \leq T$ and assuming R has not defaulted by t, identify the theoretical value of the OTC contract $V(t)$, ignoring default of B or S. In particular, find the time 0 value of the contract $V(0)$ which B must pay S at initiation.

(b) (**15 Points**) Assuming the default times of R, B and S are independent, identify the time 0 CVA of the contract. Be as explicit as possible.

Solution:

(a) Write τ^R as the default time of R and assume we are at $t \leq T$ with $\tau^R > t$. If $\tau^R > T$ the buyer receives nothing. If $t < \tau^R \leq T$ then the buyer receives δ^R at T. Thus, the time t price (here $\mathcal{F}_t$ is our information at t) is

$$\begin{aligned} V(t) &= E^{\mathbb{Q}}\left[1_{t<\tau^R\leq T}\delta^R e^{-r(T-t)} \mid \mathcal{F}_t\right], \\ &= 1_{\tau^R>t}\delta^R e^{-r(T-t)}\int_t^T \gamma^R e^{-\gamma^R(u-t)}du, \\ &= 1_{\tau^R>t}\delta^R e^{-r(T-t)}\left(1-e^{-\gamma^R(T-t)}\right). \end{aligned}$$

In particular the time 0 price is

$$V(0) = \delta^R e^{-rT}\left(1-e^{-\gamma^R T}\right).$$

(b) Write $\widetilde{V}(t) = \delta^R e^{-r(T-t)}\left(1-e^{-\gamma^R(T-t)}\right)$ so that above $V(t) = 1_{\tau^R>t}\widetilde{V}(t)$. Also write $\tau = \min\left\{\tau^R, \tau^B, \tau^S\right\}$ and ξ as the name of the first to default. To compute the CVA, assume $\tau \leq T$ and $\xi = S$. Here the theoretical amount S owes B is

$$V(\tau) = 1_{\tau^R>\tau}\widetilde{V}(\tau) = \widetilde{V}(\tau),$$

where the last equality follows because if $\xi = S$ then by definition $\tau^R > \tau = \tau^S$. The actual amount that B receives is $(1-\delta^S)V(\tau) = (1-\delta^S)\widetilde{V}(\tau)$. Thus, at τ there is a theoretical loss of

$$1_{\tau\leq T}1_{\xi=S}\delta^S\widetilde{V}(\tau).$$

The CVA at 0 is the time 0 price of this loss, given by the formula

$$\begin{aligned}
\mathrm{CVA}(0) &= E^{\mathbb{Q}}\left[1_{\tau\le T}1_{\xi=S}\delta^S\widetilde{V}(\tau)e^{-r\tau}\right],\\
&= \delta^S\frac{\gamma^S}{\gamma}\int_0^T\widetilde{V}(u)e^{-ru}\gamma e^{-\gamma u}du,\\
&= \delta^S\delta^R\frac{\gamma^S}{\gamma}e^{-rT}\int_0^T\left(1-e^{-\gamma^R(T-u)}\right)\gamma e^{-\gamma u}du,\\
&= \delta^S\delta^R\frac{\gamma^S}{\gamma}e^{-rT}\left(1-e^{-\gamma T}-\frac{\gamma}{\gamma-\gamma^R}e^{-\gamma^R T}\left(1-e^{-(\gamma-\gamma^R)T}\right)\right).
\end{aligned}$$

Here, we have used the independence of τ^R,τ^B,τ^S which implies $\gamma = \gamma^R+\gamma^B+\gamma^S$ is the intensity of τ.

3. (20 Points) A Better Way to Estimate Liquidity Adjusted Value at Risk. One problem with the way we computed the liquidity adjusted VaR over the horizon $[t,t+\Delta]$ was that we added the theoretical VaR (accounting for the log return over $[t,t+\Delta]$) to the cost of liquidating our position at t. This is inconsistent. We either liquidate at t and do not hold the position over $[t,t+\Delta]$, or we liquidate at $t+\Delta$ and should compute the liquidity cost then.

To remedy this, assume we hold $\lambda_t>0$ shares of the stock over $[t,t+\Delta]$ and then liquidate at $t+\Delta$, facing a proportional bid-ask spread of $s_{t+\Delta}$. Assume further that for the theoretical mid-price S, the log return $X_{t+\Delta}\overset{\mathcal{F}_t}{\sim}N(\mu_X,\sigma_X^2)$ is normally distributed given our information at t.

(a) (**10 Points**) Assume the spreads $s_t=s_{t+\Delta}=s$ are constant. Explicitly identify the VaR for linearized losses.

(b) (**10 Points**) Assume instead that $s_{t+\Delta}=s_t+\zeta_{t+\Delta}$ where $\zeta_{t+\Delta}\overset{\mathcal{F}_t}{\sim}N(\mu_s,\sigma_s^2)$ is independent of $X_{t+\Delta}$. Write down the main steps required to estimate the VaR for full losses including the liquidation costs in a Monte Carlo simulation.

Hint: The "value" of our portfolio at both t and $t+\Delta$ is the value assuming we liquidate. In particular to obtain V_t, we (pretend we) sold our shares at S_t^b.

Solution:

(a) With $\theta_t=\lambda_tS_t$ the theoretical dollar position, the beginning portfolio value is

$$V_t=\lambda_tS_t^b=\lambda_tS_t-\frac{1}{2}\lambda_tS_ts=\theta_t\left(1-\frac{1}{2}s\right).$$

The ending portfolio value is

$$V_{t+\Delta}=\lambda_tS_{t+\Delta}^b=\lambda_tS_{t+\Delta}-\frac{1}{2}\lambda_tS_{t+\Delta}s=\theta_te^{X_{t+\Delta}}\left(1-\frac{1}{2}s\right).$$

This gives the full loss

$$L_{t+\Delta} = -\theta_t\left(1-\frac{1}{2}s\right)\left(e^{X_{t+\Delta}}-1\right),$$

and hence the first order loss is

$$L^{lin}_{t+\Delta} = -\theta_t\left(1-\frac{1}{2}s\right)X_{t+\Delta}.$$

Given our assumptions

$$L^{lin}_{t+\Delta} \overset{\mathcal{F}_t}{\sim} -\theta_t\left(1-\frac{1}{2}s\right)\mu_t + \theta_t\left(1-\frac{1}{2}s\right)\sigma_X Z,$$

where $Z \overset{\mathcal{F}_t}{\sim} N(0,1)$. This gives the value at risk

$$\mathrm{VaR}_\alpha = -\theta_t\left(1-\frac{1}{2}s\right)\mu_t + \theta_t\left(1-\frac{1}{2}s\right)\sigma_X N^{-1}(\alpha).$$

(b) As in part (a) we have

$$V_t = \theta_t\left(1-\frac{1}{2}s_t\right),$$

$$V_{t+\Delta} = \theta_t e^{X_{t+\Delta}}\left(1-\frac{1}{2}s_{t+\Delta}\right) = \theta_t e^{X_{t+\Delta}}\left(1-\frac{1}{2}s_t-\frac{1}{2}\zeta_{t+\Delta}\right)$$

This gives

$$L_{t+\Delta} = -\theta_t\left(\left(1-\frac{1}{2}s_t\right)\left(e^{X_{t+\Delta}}-1\right)-\frac{1}{2}\zeta_{t+\Delta}e^{X_{t+\Delta}}\right).$$

To estimate the value at risk via simulation, for $m = 1, ..., M$ we sample $X^m_{t+\Delta} \sim N(\mu_X, \sigma_X^2)$, $\zeta^m_{t+\Delta} \sim N(\mu_s, \sigma_s^2)$ and compute

$$\ell_m = L^m_{t+\Delta} = -\theta_t\left(\left(1-\frac{1}{2}s_t\right)\left(e^{X^m_{t+\Delta}}-1\right)-\frac{1}{2}\zeta^m_{t+\Delta}e^{X^m_{t+\Delta}}\right).$$

We then estimate the value at risk by $\mathrm{VaR}_\alpha = \ell_{(\lceil M\alpha\rceil)}$ where the $\ell_{()}$ are the sorted (ascending) losses.

4. (20 Points) LDA with Exponential Distributions. Assume we have training data $\{(g_n, x_n)\}_{n=1}^N$ where for each n, $x_n \in \mathbb{R}^M_{++}$. In other words, each of the M data features has strictly positive numerical value. In this exercise, we ask what happens if use the LDA methodology, not with Gaussian conditional distributions, but rather with exponential conditional distribution. More precisely, let $\lambda_0, \lambda_1 \in \mathbb{R}^M_{++}$ and assume that given $G = k(k = 0, 1)$ the random variable X has pdf given by

$$f_k(x) = \prod_{m=1}^M \lambda_k^{(m)} e^{-\lambda_k^{(m)}x^{(m)}}, \qquad x \in \mathbb{R}^M_{++}.$$

Lastly, set $\pi = \mathbb{P}[G = 1]$ so that $1 - \pi = \mathbb{P}[G = 0]$.

(a) (**10 Points**) Set logit $(y) := \log(y/(1-y))$. If our classification rule is

$$G(x) = 1_{\text{logit}\left(\mathbb{P}\left[G=1 \;\middle|\; X=x\right]\right)\geq 0},$$

is the decision boundary linear? Justify your result.

(b) (**10 Points**) The optimal $\pi, \lambda_0, \lambda_1$ are found by maximum likelihood, where we maximize $\ell(\pi, \lambda_0, \lambda_1)$ defined by

$$\ell(\pi, \lambda_0, \lambda_1) := \log\left(\prod_{n=1}^{N} \mathbb{P}\left[G = g_n, X = x_n\right]\right),$$

where $\mathbb{P}\left[G = g_n, X = x_n\right]$ is the joint pmf/pdf of (G, X). Explicitly identify the optimal $\widehat{\pi}, \widehat{\lambda}_1, \widehat{\lambda}_2$, and argue why they are "natural".

Solution:

(a) As with the "regular" LDA methodology, Bayes rule gives

$$\mathbb{P}\left[G = 1 \;\middle|\; X = x\right] = \frac{\pi f_1(x)}{\pi f_1(x) + (1-\pi) f_0(x)},$$

so that

$$\begin{aligned}
&\text{logit}\left(\mathbb{P}\left[G = 1 \;\middle|\; X = x\right]\right) \\
&\quad = \log(\pi) + \log(f_1(x)) - \log(1-\pi) - \log(f_2(x)), \\
&\quad = \log(\pi) + \sum_{m=1}^{M} \log(\lambda_1^{(m)}) - \lambda_1^{\mathrm{T}} x - \log(1-\pi) - \sum_{m=1}^{M} \log(\lambda_0^{(m)}) + \lambda_0^{\mathrm{T}} x, \\
&\quad = \alpha^{\mathrm{T}} x + \alpha_0
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= \lambda_0 - \lambda_1, \\
\alpha_0 &= \log(\pi) + \sum_{m=1}^{M} \log(\lambda_1^{(m)}) - \log(1-\pi) - \sum_{m=1}^{M} \log(\lambda_0^{(m)})
\end{aligned}$$

Therefore, we see that the decision boundary is linear.

(b) If $g_n = 1$ then

$$\log(\mathbb{P}\left[G = g_n, X = x_n\right]) = \log(\pi f_1(x_n)) = g_n \log(\pi f_1(x_n)).$$

If $g_n = 0$ then

$$\log(\mathbb{P}\left[G = g_n, X = x_n\right]) = \log((1-\pi) f_0(x_n)) = (1-g_n) \log((1-\pi) f_0(x_n)).$$

This gives

$$\begin{aligned}
\ell(\pi, \lambda_0, \lambda_1) &= \sum_{n=1}^{N} g_n \left(\log(\pi) + \sum_{m=1}^{M} \left(\log(\lambda_1^{(m)}) - \lambda_1^{(m)} x_n^{(m)}\right)\right) \\
&\quad + \sum_{n=1}^{N} (1-g_n) \left(\log(1-\pi) + \sum_{m=1}^{M} \left(\log(\lambda_0^{(m)}) - \lambda_0^{(m)} x_n^{(m)}\right)\right)
\end{aligned}$$

The first order condition for π is

$$0=\frac{1}{\pi}\sum_{n=1}^{N}g_n-\frac{1}{1-\pi}\sum_{n=1}^{N}(1-g_n)$$

which has solution

$$\widehat{\pi}=\frac{1}{N}\sum_{n=1}^{N}g_n,$$

as one would expect since this is probability $G=1$ based off the empirical distribution. For $m=1,...,M$ the first order condition for $\lambda_1^{(m)}$ is

$$0=\sum_{n=1}^{N}g_n\left(\frac{1}{\lambda_1^{(m)}}-x_n^{(m)}\right),$$

which has solution

$$\widehat{\lambda}_1^{(m)}=\frac{\sum_{n=1}^{N}g_n}{\sum_{n=1}^{N}g_n x_n^{(m)}},\quad m=1,...,M,$$

which again is natural because $1/\lambda^{(m)}$ is the empirical mean of those $\left\{x_n^{(m)}\right\}$ with $g_n=1$. Similarly, for $\lambda_0^{(m)}$ we have

$$\widehat{\lambda}_0^{(m)}=\frac{\sum_{n=1}^{N}(1-g_n)}{\sum_{n=1}^{N}(1-g_n)x_n^{(m)}},\quad m=1,...,M,$$

which is natural for the same reason.