Risk Measure Components

MF 731 Corporate Risk Management

Outline

The effect of positions on portfolio risk.

Risk measure components.

Risk Measure Components

The basic idea (ϱ: a risk measure, e.g. VaR):

We use ϱ to measure the firm-wide market risk.

However, the firm has many trading desks.

We need to decompose the total risk into components representing the risk associated to each desk.

How can we do this?

Risk measure components

$\rho \to$ firm-wide or portfolio-wide risk

Goal: Decompose this risk into components, representing the risk due to individual desks, or individual securities.

Setting.

We focus on equity portfolio and log returns.

Window $[t, t+\Delta]$.

We condition on $\mathcal{F}_t$.

Drop "t", "$t+\Delta$" subscripts.

$$V_t = \sum_{i=1}^{d} \theta_t^{(i)} \qquad \theta_t^{(i)} = \lambda_t^{(i)} S_t^{(i)}$$

— use \$ positions

$$V_{t+\Delta} = \sum_{i=1}^{d} \theta_t^{(i)} e^{X_{t+\Delta}^{(i)}}$$

$$L = -\theta^T(e^X - 1) \qquad \text{full} \qquad (e^X)^{(i)} = e^{X^{(i)}} \qquad (1)^{(i)} = 1$$

$$L^{lin} = -\theta^T X$$

$$L^{quad} = -\theta^T\left(X + \tfrac{1}{2}X^2\right) \qquad (X^2)^{(i)} = (X^{(i)})^2$$

Losses in terms of Dollar Positions

We focus on equity portfolios and log returns.

Throughout

- The investment window is $[t, t+\Delta]$ and we condition on $\mathcal{F}_t$.
- We (mostly) drop "t" and "$t+\Delta$" subscripts.

We hold $\lambda^{(i)}$ shares in $S^{(i)}$ over $[t, t+\Delta]$, and set $\theta^{(i)} = \lambda^{(i)} S_t^{(i)}$ as the <u>dollar position</u> in $S^{(i)}$ at t.

- $L = L(\theta) = -\theta^{\mathrm{T}}(e^X - 1)$.
- $L^{lin} = L^{lin}(\theta) = -\theta^{\mathrm{T}} X$.
- $L^{quad} = L^{quad}(\theta) = -\theta' \left(X + \frac{1}{2} X^2 \right)$.
  - v^2: vector with i^{th} element $(v^{(i)})^2$, $i = 1, ..., d$.

Risk Measures using Dollar Positions

We use a risk measure ϱ satisfying:

Cash-additivity and positive homogeneity.

$$\varrho(A + B \times L) = A + B \times \varrho(L), \;\; A \in \mathbb{R}, B > 0.$$

We view ϱ as a function of the dollar position.

$$\mathcal{R}(\theta) \; := \; \varrho(\mathcal{L}(\theta)), \text{ for } \mathcal{L} \in \left\{L, L^{lin}, L^{quad}\right\}.$$

Only requirement on ρ

$\rho(A+B\times L) = A+B\times\rho(L), \; A\in\mathbb{R}, B>0$

prospective: think of ρ as a function of the dollar position θ.

Example: $X \sim N(\mu, \Sigma)$

Linearized loss, normal log returns.

$$L^{lin}(\theta) = -\theta^{\mathrm{T}} X = -\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta} \times Z$$

by cash-add and positive homo

$Z \sim N(0, 1)$.

$\varrho(L^{lin})$

$$\mathcal{R}^{lin}(\theta) = -\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta} \times \varrho(Z).$$

$\varrho(Z) = N^{-1}(\alpha)$ (Value at Risk).

$\varrho(Z) = \frac{1}{1-\alpha}\phi(N^{-1}(\alpha))$ (Expected Shortfall).

Marginal Risk

Marginal ϱ: change in ϱ from taking an additional dollar of exposure to a given position.

If $\theta^{(i)}$ goes to $\theta^{(i)} + 1$ how does ϱ change?

"\$"

This is just the partial derivative w.r.t. $\theta^{(i)}$.

$$\mathcal{R}_{\mathrm{M}}^{(i)}(\theta) := \partial_{\theta^{(i)}} \mathcal{R}(\theta), \text{ for } L, L^{lin}, L^{quad}. \ \forall i = 1, \dots, d$$

marginal

E.g: $X \sim N(\mu, \Sigma)$

$$\mathcal{R}^{lin}(\theta) = -\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta} \times \varrho(Z).$$

$$\mathcal{R}_{\mathrm{M}}^{lin,(i)}(\theta) = -\mu^{(i)} + \frac{(\Sigma\theta)^{(i)}}{\sqrt{\theta^{\mathrm{T}}\Sigma\theta}} \times \varrho(Z).$$

$$-\mu^{(i)} + \frac{1}{2}\frac{1}{\sqrt{\theta^{\mathrm{T}}\Sigma\theta}}\varrho(Z) \cdot 2(\Sigma\theta)^{(i)}$$

Marginal ϱ

Write $\nabla \mathcal{R}$ for the gradient so $\mathcal{R}_{\mathrm{M}}(\theta) = \nabla \mathcal{R}(\theta)$.

Typically no simple expression for $\mathcal{R}_{\mathrm{M}}(\theta)$.

Especially for non-equity examples.

Could do via simulation.

$R(\theta) = \rho(-\theta^T(e^X - 1))$

To obtain $\mathcal{R}_{\mathrm{M}}(\theta)$ numerically:

Issues computing $\nabla R(\theta)$ when we have data.

Assume a fictitious small trade in $S^{(i)}$ at t.

Linearized losses, simulated risk factor changes,

1) $\ell_i = -\theta^T X_i$ 2) $\rho = \ell_{(\lceil n\alpha \rceil)}$

$\Rightarrow R(\theta) = [\ell_{(\lceil n\alpha \rceil)}](\theta)$

Recalculate new portfolio ϱ.

Calculate the change in portfolio ϱ and divide by change in position size.

– triky to differentiate this

Incremental Risk

IR simply seeks for
– what's the change in Risk going from old position θ to new position ϕ.
$\mathcal{R}_I = \mathcal{R}(\phi) - \mathcal{R}(\theta)$

Incremental ϱ is similar to marginal ϱ but there is no restriction on

→ Changing the position of only one asset.

→ The size of the position change.

Definition: $\mathcal{R}_{\mathrm{I}}$ is the change in ϱ resulting from a change in portfolio positions.

$\mathcal{R}_{\mathrm{I}} = \mathcal{R}(\phi) - \mathcal{R}(\theta)$ if our original position is θ and our new position is ϕ.

Incremental ϱ

To compute IR, we can use brute force, compute R(θ), R(∅) and
R(I) = R(∅) − R(θ)

Interpretation:

— ok if we can have a specific ∅ in mind.
— not a good idea if we want optimal ∅
— Warning: optimal ≠ least risk

High $\mathcal{R}_\mathrm{I}$: new position adds significant risk.

Moderate $\mathcal{R}_\mathrm{I}$: new position adds some risk.

Negative $\mathcal{R}_\mathrm{I}$: new position reduces risk.

Risk (reducing) optimal hedge:

The new position which has the largest negative $\mathcal{R}_\mathrm{I}$.

Computing Incremental ϱ

Method one: brute force.

Compute $\mathcal{R}(\theta)$ (old), $\mathcal{R}(\phi)$ (new).

Set $\mathcal{R}_{\mathrm{I}} = \mathcal{R}(\phi) - \mathcal{R}(\theta)$.

Pro: straightforward and easy to explain.

Cons:

- Very computationally expensive.
- Not practical for identifying the optimal ϕ.

Approximating Incremental ϱ

Method two: first order Taylor approximation

$$\boxed{\mathcal{R}_{\mathrm{I}} \approx (\phi - \theta)^{\mathrm{T}} \nabla \mathcal{R}(\theta) = (\phi - \theta)^{\mathrm{T}} \mathcal{R}_{\mathrm{M}}(\theta).}$$

$R_I = R(\phi) - R(\theta) \approx \nabla R(\theta)^T (\phi - \theta)$

$= R_m(\theta)^T (\phi - \theta)$

Pros:

$R_m(\theta)$

Optimal to go in on a small changes.

Easy to implement for linear losses.

Easy to implement for full losses, provided good estimates for the partial derivatives.

Con:

Approximation only valid for small changes.

At this point, we can say how our risk changes with small changes in our dollar positions. But how can allocate/attribute our total risk across our individual position?

Positive Homogeneity Revisited

Positive homogeneity for risk measures:

$$\varrho(B \times L) = B \times \varrho(L) \text{ for } B > 0.$$

As a function of θ, losses have this property too.

$= B(-\theta^T(e^X-1))$

$$L(B \times \theta) = -(B \times \theta)^{\mathrm{T}}(e^X - 1) = B \times L(\theta).$$

$$L^{lin}(B \times \theta) = -(B \times \theta)^{\mathrm{T}} X = B \times L^{lin}(\theta).$$

Thus, $\mathcal{R}$ is positively homogeneous in θ.

$$\boxed{\mathcal{R}(B \times \theta) = \varrho(L(B \times \theta)) = \varrho(B \times L(\theta)) = B \times \varrho(L(\theta)) = B \times \mathcal{R}(\theta)}.$$

∂_B at $B = 1$ gives $\theta^{\mathrm{T}} \nabla \mathcal{R}(\theta) = \mathcal{R}(\theta)$. Thus

$$\mathcal{R}(\theta) = \nabla \mathcal{R}(\theta)^{\mathrm{T}} \theta = \sum_{i=1}^{d} \mathcal{R}_{\mathrm{M}}^{(i)}(\theta)\, \theta^{(i)}.$$

change perspective (again)

$B \mapsto R(B\theta)$

$(0,\infty) \mapsto \mathbb{R}$

$\frac{d}{dB} B R(\theta) = R(\theta)$ $\quad$ $\frac{d}{dB} R(B\theta) = \frac{d}{dB} R(B\theta^{(1)}, B\theta^{(2)}, \ldots)$

Component ϱ

$$= \sum_{i=1}^{d} \partial_{\theta^{(i)}} R(B\theta)\, \theta^{(i)}, \quad B>0$$

$$= R_M(B\theta)^T \theta$$

evaluate at $B=1$

$$R(\theta) = \nabla R(\theta)^T \theta$$

$$= R_M(\theta)^T \theta$$

$$= \sum_{i=1}^{d} R_M^{(i)}(\theta)\, \theta^{(i)}$$

Total Risk = Sum of component risks.

We thus define for $i = 1, \ldots, d$

$$\mathcal{R}_{\mathrm{C}}^{(i)}(\theta) := \mathcal{R}_{\mathrm{M}}^{(i)}(\theta)\, \theta^{(i)}.$$

With this, the total risk decomposes into

We interprete

Component of risk $R_M^{(i)}(\theta)\, \theta^{(i)}$ is the part of the total risk due to as dollar position in the i^{th} stock

$$\mathcal{R}(\theta) = \sum_{i=1}^{d} \mathcal{R}_{\mathrm{C}}^{(i)}(\theta).$$

We also think of percentage contributions to risk.

$$\mathcal{R}_{\mathrm{C},\%}^{(i)}(\theta) := 100 \times \frac{\mathcal{R}_{\mathrm{C}}^{(i)}(\theta)}{\mathcal{R}(\theta)} = 100 \times \frac{\mathcal{R}_{\mathrm{M}}^{(i)}(\theta)\theta^{(i)}}{\mathcal{R}(\theta)}.$$

$$= 100 \times \frac{R_M^{(i)}(\theta)\, \theta^{(i)}}{\sum_{i=1}^{d} R_M^{(i)}(\theta)\, \theta^{(i)}}$$

Component ϱ

$\mathcal{R}_{\mathrm{C}}(\theta)$ and $\mathcal{R}_{\mathrm{C},\%}(\theta)$ let us identify the contribution to portfolio risk due to each dollar position $\theta^{(i)}$.

E.g. our position in Stock A accounts for 35% of the portfolio risk.

This idea extends beyond equities, to firm-wide positions:

We can identify what fraction of firm-wide VaR comes from each underlying business unit or trading desk.

We can "drill-down" on our risk measure ϱ.

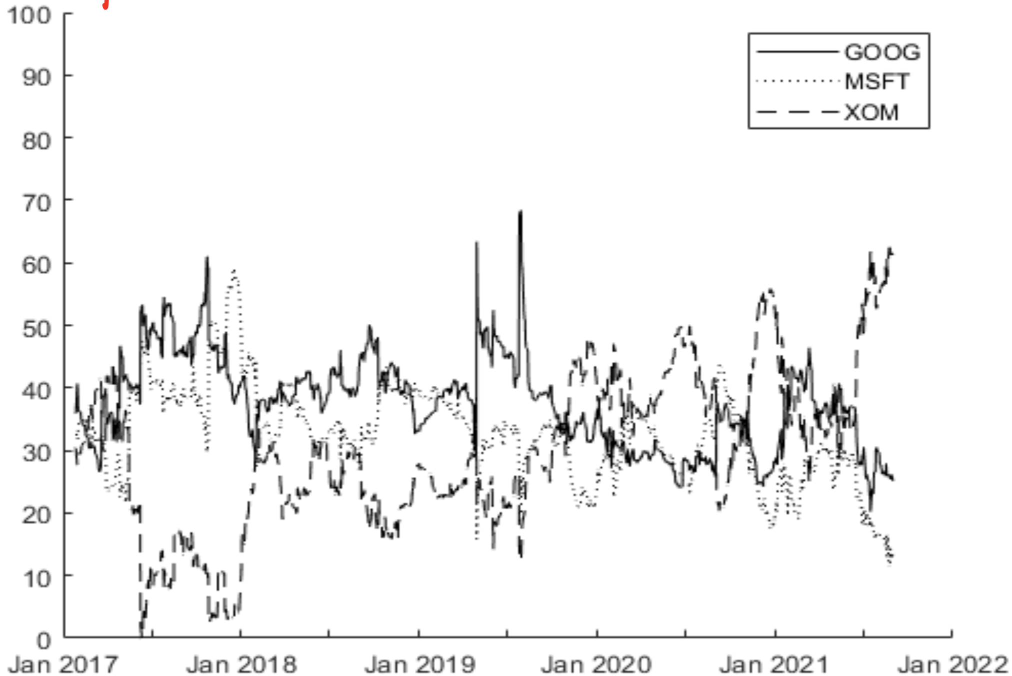
Example: GOOG,MSFT,XOM

Portfolio: \$1M, 33% in each stock.

Data: 8/31/16 – 8/31/21. Normal log ret. with EWMA.

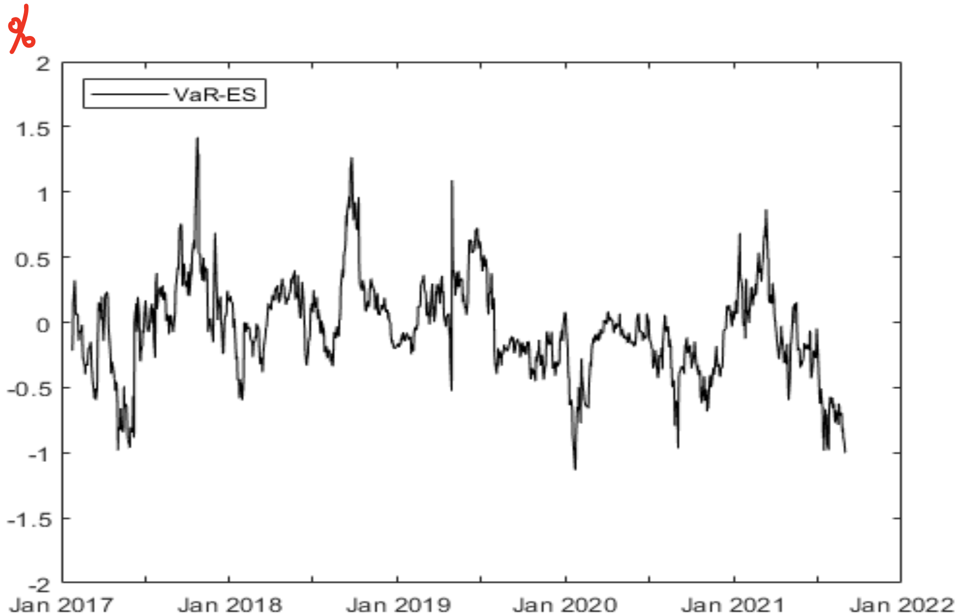
$\mathcal{R}^{lin}_{\mathrm{C},\%}$ through time for $\mathrm{VaR}_{.95}$.

VaR component



$\mathcal{R}^{lin}_{\mathrm{C},\%}$ for Different ϱ

Time series for $\mathcal{R}^{lin}_{\mathrm{C},\%}$ (using $\mathrm{VaR}_{.95}$) - $\mathcal{R}^{lin}_{\mathrm{C},\%}$ (using $\mathrm{ES}_{.95}$) for GOOG:



Why is this difference so small? Yes

We observe that our time-series estimates for $R_{c\%}$ are very similar across risk measures.

— e.g. series for VaR $\approx$ series for ES

Q: why?

Linearized / Normal setting

$$R(\theta) = -\theta'\mu + \sqrt{\theta'\Sigma\theta}\, \rho(z)$$

Stylized fact: $\mu \approx 0$.

$$R(\theta) = \sqrt{\theta'\Sigma\theta}\, \rho(z)$$

$$R_m(\theta) = \frac{\rho(z)}{\sqrt{\theta'\Sigma\theta}} \Sigma\theta$$

$$R_{c\%}^{(i)}(\theta) = 100 \cdot \frac{\frac{\cancel{\rho(z)}}{\sqrt{\theta'\Sigma\theta}} (\Sigma\theta)^{(i)}\theta^{(i)}}{\sqrt{\theta'\Sigma\theta}\, \cancel{\rho(z)}}$$

$$= 100 \cdot \underbrace{\frac{(\Sigma\theta)^{(i)}\theta^{(i)}}{\theta'\Sigma\theta}}_{\text{contribution of portfolio variance}}$$

independent of ρ!

— In reality, μ might not be 0

— we also might use full losses

$\Rightarrow$ "small" dependence on ρ in general.

$\mathcal{R}^{lin}_{\mathrm{C},\%}$ for Normal Log Returns

Recall: we assumed $X \sim N(\mu, \Sigma)$.

As before, by cash add. and pos. homogeneity

$$\mathcal{R}^{lin}(\theta) = -\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta} \times \varrho(Z) \text{ for } Z \sim N(0,1).$$

$$\mathcal{R}^{lin,(i)}_{\mathrm{M}}(\theta) = -\mu^{(i)} + \frac{(\Sigma\theta)^{(i)}}{\sqrt{\theta^{\mathrm{T}}\Sigma\theta}} \times \varrho(Z).$$

Therefore, $\mathcal{R}^{lin,(i)}_{\mathrm{C},\%}(\theta) = 100 \times \frac{\mathcal{R}^{lin,(i)}_{\mathrm{M}}(\theta)\theta^{(i)}}{\mathcal{R}^{lin}(\theta)}$ gives

$$\mathcal{R}^{lin,(i)}_{\mathrm{C},\%}(\theta) = 100 \times \left(\frac{-\theta^{(i)}\mu^{(i)} + \frac{\theta^{(i)}(\Sigma\theta)^{(i)}}{\sqrt{\theta^{\mathrm{T}}\Sigma\theta}}\varrho(Z)}{-\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta}\varrho(Z)} \right).$$

$$\mathcal{R}^{lin,(i)}_{\mathrm{C},\%}(\theta) = 100 \times \left(\frac{-\theta^{(i)}\mu^{(i)} + \frac{\theta^{(i)}(\Sigma\theta)^{(i)}}{\sqrt{\theta^{\mathrm{T}}\Sigma\theta}}\varrho(Z)}{-\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta}\,\varrho(Z)} \right)$$

Recall the stylized fact: $\mu = 0$.

Daily log returns have 0 conditional average.

If we set $\mu = 0$ then $\mathcal{R}^{lin,(i)}_{\mathrm{C},\%}(\theta) = 100 \times \left(\frac{\theta^{(i)}(\Sigma\theta)^{(i)}}{\theta^{\mathrm{T}}\Sigma\theta} \right)$.

Same for <u>all</u> risk measures ϱ !

$\mathcal{R}^{lin,(i)}_{\mathrm{C},\%}(\theta)$ is simply the percentage contribution of the i^{th} dollar position to portfolio variance !

For non-zero μ there is an effect due to ϱ.

Conclusions for $\mathcal{R}^{lin}_{\mathrm{C},\%}$

$\mathcal{R}_{\mathrm{C},\%}$ allows us to identify relative contributions of dollar positions to total risk.

BUT

No ϱ-specific effect if we use linearized losses, normal distribution for risk factor changes.

$\mathcal{R}^{lin}_{\mathrm{C},\%}$ just measures the % of variance.

$\mathcal{R}^{lin}_{\mathrm{C}}$ depends on ϱ only through the scalar $\varrho(Z)$.

If we use full losses, we typically lose analytic formulas.

Estimating $\mathcal{R}^{lin}_{\mathrm{C},\%}$ is cumbersome, as we have to compute partial derivatives.