# Scenario Analysis and Stress Testing

MF 731 Corporate Risk Management

# Outline

Scenario Analysis.

Stress Testing.

# Scenario Analysis: Basic Idea

We want to measure the risk of our losses $L_{t+\Delta}$.

We have developed numerous methods to measure this risk based off the loss distribution given $\mathcal{F}_t$.

Volatility: $\sqrt{\mathrm{Var}\left[L_{t+\Delta}\right]}$.

Value at risk: $\mathrm{VaR}_\alpha\left(L_{t+\Delta}\right)$.

Coherent risk measure: $\varrho\left(L_{t+\Delta}\right)$.

# Basic Idea

However, there are complications. We had to either

Hypothesize the (conditional) distribution of losses.

Estimate the (conditional) distribution of losses.

Scenario analysis does “something simpler”.

Identifies losses under a certain (fixed) set of scenarios.

Rather than estimating or/and modeling the conditional dist.

# Method

To perform a scenario analysis, we

Step 1 Identify the losses as a function of the risk factor changes.

$$L_{t+\Delta} = \mathfrak{l}_{[t]}(X_{t+\Delta}).$$

$\mathfrak{l}_{[t]}$: full, linearized, quadratic loss operator.

Step 2 Choose a set of $\boxed{\text{scenarios}}$ $x_1, \dots, x_N$ for the risk factor changes $X_{t+\Delta}$.

E.g.: $n^{th}$ scenario is a $-20\%$ log return in $j^{th}$ stock.
$\Rightarrow x_n = (0, \dots, -.2, 0, \dots, 0)$.

Compute the losses under each scenario.

$$\ell_n = \mathfrak{l}_{[t]}(x_n).$$

# Method

Steps (continued):

Step 3 Weight each scenario according to its likelihood or importance.

$x_n$ has weight $w_n$.

E.g. (equal weighting): $w_n = 1, n = 1, ..., N$.

E.g. (first scenario twice as important): $w_1 = 2$, $w_2 = ... = w_N = 1$.

Step 4 Output the risk as the maximum weighted loss.

$$\Psi_{x,w} \; := \; \max\{w_1\ell_1, \ldots, w_N\ell_N\}.$$

# Recap and Discussion

Scenarios $x_1, \ldots, x_N$ with weights $w_1, \ldots, w_N$.

$\ell_n$: loss in $n^{th}$ scenario (full, linearized, etc).

Risk: $\Psi_{x,w} = \max\{w_1\ell_1, \ldots, w_N\ell_N\}$.

Why this is (seemingly) simpler:

No probabilistic model: $x_n$ are given scenarios. Losses easily calculated given $x_n$.

E.g. equities, log returns, linearized losses:

$l_{[t]}^{lin}(x_n) = -\theta_t^{\mathrm{T}} x_n$ for dollar positions $\theta_t$ at $t$.

# Discussion

Why this is not simpler:

How do we determine the scenarios?

We need to ensure scenarios are

- Relevant and plausible.
- Sufficiently non-overlapping.
- Cover our main risks.

How do we determine weights?

In essence: identifying risk factor changes and weights = building a probabilistic model!

# Example: Chicago Mercantile Exchange

Simplified formula for computing margin requirement for futures, options on futures.

14 scenarios with weight 1: 2 volatility movements $\times$ 7 futures price movements.

Volatility: "up", "down". Futures price: $0, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1$.

2 additional extreme scenarios with weight .35.

Margin capital required is $\Psi_{x,w}$.

# Scenario Types

Stylized movements in the risk factors:

- 20% drop in the S&P 500 index.
- 100 bp increase in the 5 year treasury par rate.
- 10% increase in stock volatility.

Historical events:

- The October 19, 1987 stock market crash.
- The Asian crisis of 1997.

Hypothetical events:

- Earthquake in California.
- Sovereign default.

# Scenario Types: Pros and Cons

Stylized movements:

Pros: easy to explain, methodology well known.

Cons: calculations may be un-manageable for a large number of scenarios.

Historical events:

Pros: these events actually happened!

Cons: which events to choose?

Hypothetical events:

Pros: need to account for these types of things somehow.

Cons: which ones to choose?

# Scenarios Analysis and Risk Measures

Scenario analysis predated use of risk measures.

$\mathrm{VaR}$ was considered an improvement.

However

- $\mathrm{VaR}$ is not sub-additive, which is a problem.
- Scenario analysis produces a coherent risk measure!

Thus

- Scenario analysis has made a (theoretical) comeback.

# Scenario Analysis Risk Measure

$$\Psi_{x,w} = \max\{w_n \ell_n\}_{n=1}^{N}.$$ can be viewed as coherent risk measure.
— a theoritical victory of VaR

$\ell_n = \mathfrak{l}_{[t]}(x_n)$: loss in $n^{th}$ scenario.

Full, linear, quadratic.

$0 < w_n \leq 1$ (normalized weights). $W_n$: weight of $n^{th}$ scenario

To make things simpler, assume $\mathfrak{l}_{[t]}(0) = 0$. No change. No loss

No factor change $\Rightarrow$ no loss.

E.g.: Equities, full losses $\Rightarrow$ $l_{[t]}(x) = -\theta_t^{\mathrm{T}}(e^x - 1)$, $\checkmark$.

One can modify the argument if $\mathfrak{l}_{[t]}(0) \neq 0$.

# $\Psi_{x,w} = \max \{w_n \ell_n\}_{n=1}^{N}$ as a Risk Measure

$\mathbb{P}_n(x_n) = w_n$ $\mathbb{P}_n(0) = 1 - w_n$

Consider a probability measure $\mathbb{P}_n$ which puts

prob space: $\Omega$ = {scenario space}

Weight $w_n$ on scenario $x_n$, and weight $1 - w_n$ on scenario 0.

- state space of scenarios

Consider the "identity" random variable

$X(x) = x$ for any scenario $x$ . $X(\omega) = \omega$

$\mathfrak{l}_{[t]}(0) = 0$ implies another prospective

$$\boxed{E^{\mathbb{P}_n}\left[\mathfrak{l}_{[t]}(X)\right] = w_n \mathfrak{l}_{[t]}(x_n) + (1 - w_n)\mathfrak{l}_{[t]}(0) = w_n \ell_n.}$$

Therefore

$$\begin{aligned}\Psi_{x,w} &= \max \left\{ w_n \ell_n \;\middle|\; n = 1, \ldots, N \right\} \\ &= \max \left\{ E^{\mathbb{P}_n}\left[\mathfrak{l}_{[t]}(X)\right] \;\middle|\; n = 1, \ldots, N \right\}\end{aligned}$$

# $\Psi_{x,w} = \max \{w_n \ell_n\}_{n=1}^N$ as a Risk Measure

family of measures

Therefore, with $\mathcal{P}_{x,w} = \{\mathbb{P}_n\}_{n=1}^N$ we have

$$\Psi_{x,w} = \max \left\{ E^{\mathbb{P}} \left[ \mathfrak{l}_{[t]}(X) \right] \;\middle|\; \mathbb{P} \in \mathcal{P}_{x,w} \right\}.$$

If $L = \ell_{(t)}(X)$

$\mathcal{P} = \{\mathbb{P}_n \mid n \in 1,2,3\ldots\}$

$\mathbb{P}_n \begin{cases} x_n & w_n \\ 0 & 1-w_n \end{cases}$

$\Psi_{x,w} = \varrho(L)$

Let's generalize this by setting

$\mathcal{P}$ be an arbitrary collection of probability measures.

$L$ be an arbitrary random variable. $\ell_{(t)}(X)$

Define the *scenario* risk measure $\varrho$ by

$$\varrho(L) := \sup \left\{ E^{\mathbb{P}} [L] \;\middle|\; \mathbb{P} \in \mathcal{P} \right\}.$$

scenario analysis risk measure

"sup" = supremum = least upper bound.

$$\varrho(L) = \sup \left\{ E^{\mathbb{P}}[L] \;\middle|\; \mathbb{P} \in \mathcal{P} \right\}$$

1) $L_1(w) \le L_2(w) \Rightarrow \varrho(L_1) \le \varrho(L_2)$

$\mathbb{E}^P(L_1) \le \mathbb{E}^P(L_2)$ — monotonicity

2) $\mathbb{E}^P(L-C) = \mathbb{E}^P[L] - C$

$\sup\{\mathbb{E}^P[L-C] \mid P \in \mathcal{P}\} = \sup\{\mathbb{E}^P[L]-C \mid P \in \mathcal{P}\} = \sup\{\mathbb{E}^P[L] \mid P \in \mathcal{P}\} - C$

Properties of $\varrho$:

Monotonic, cash-additive, positively homogenous: ✓. — cash-add

3) If $\lambda > 0$ $\mathbb{E}^P[\lambda L] = \lambda \mathbb{E}^P[L]$

$\sup\{\mathbb{E}^P[\lambda L] \mid P \in \mathcal{P}\} = \sup\{\lambda \mathbb{E}^P[L] \mid P \in \mathcal{P}\}$

$= \lambda \sup\{\mathbb{E}^P[L] \mid P \in \mathcal{P}\}$ since $\lambda > 0$

Sub-additive. For $\mathbb{P} \in \mathcal{P}$:

4)
$$\begin{aligned} E^{\mathbb{P}}[L_1 + L_2] &= E^{\mathbb{P}}[L_1] + E^{\mathbb{P}}[L_2]; \\ &\le \sup\left\{E^{\tilde{\mathbb{P}}}[L_1] \;\middle|\; \tilde{\mathbb{P}} \in \mathcal{P}\right\} + \sup\left\{E^{\tilde{\mathbb{P}}}[L_2] \;\middle|\; \tilde{\mathbb{P}} \in \mathcal{P}\right\}; \\ &= \varrho(L_1) + \varrho(L_2). \end{aligned}$$

Taking the sup over $\mathbb{P} \in \mathcal{P}$ of left hand side gives result.

$\varrho$ (and hence $\Psi_{x,w}$) is a coherent risk measure!

Has the superior (w.r.t. $\mathrm{VaR}$) sub-additivity property.

# Stress Test Measure

Think as new risk measure

Come back to

$$\Psi_{x,w} = \max\left\{w_n \ell_n\right\}_{n=1}^{N}. = \sup\{L(x) \mid x \in S\}$$

$L(x)$: loss in outcome $x$

$S$: set of outcomes

$\ell_n = \mathfrak{l}_{[t]}(x_n)$: loss in scenario $x_n$ (full, linear, etc).

$L(x) \triangleq \ell_{[t]}(x)$

$S \triangleq \{x_1, \cdots, x_n\}$

Assume equal weights ($w_n = 1$) so that

$$\Psi_{x,w} = \Psi_x = \max\left\{\mathfrak{l}_{[t]}(x_n)\right\}_{n=1}^{N}.$$

The risk measure analog of $\Psi_x$ is

$$\boxed{\varrho(L) = \sup\left\{L(x) \mid x \in \mathcal{S}\right\}}, \ \mathcal{S}\text{: set of outcomes.}$$

This is called the *stress test* risk measure.

As with the scenario risk measure, it is coherent.

# Stress Testing

## What is stress testing?

A procedure that attempts to gauge the vulnerability of our portfolio in crisis situations, where "normal" market relationships break down.

## What do we mean by breakdowns in "normal" market relationships?

Sudden increase in correlations or decrease in liquidity.

Macro-economic risks: shocks to employment and or domestic product; business relationships; relationships with foreign countries, etc..

# Stress Testing

When running a stress test, there is a huge variety in

- The event magnitude (bad, catastrophic, ...).
- The type of risk (market, liquidity, credit, ...).
- The risk factors (log prices, yields,...).
- The book type (trading, banking,...).
- The target (trading desk, business unit, corporation,...).

Stress testing is simple in principal but complex in practice.

# Stress Testing and Regulation

Stress testing is embedded into the regulatory framework.

BASEL I amendment: internal models should have a “rigourous and comprehensive” stress testing program.

Dodd-Frank: requires annual stress testing of large financial institutions.

# Benefits of Stress Testing

Stress testing can provide otherwise hidden information to risk managers.

As stress events fall beyond the $\mathrm{VaR}$ threshold, $\mathrm{VaR}$ gives no information and $\mathrm{ES}$ only provides an average. Stress tests see range of large losses.

Stress tests can cover long time horizons, illuminating after-effects of crises hidden from $\mathrm{VaR}$ analysis, which is usually carried out over a shorter time period.

# Stress Testing and Correlation

Stress testing is good at identifying consequences of changes in correlation.

Large adverse returns may drive correlations to nearly 1.

This is called being ”out of equilibrium”.

If not in recent data, our covariance estimation methods do not see this.

Stress testing can capture this effect

# Scenario Analysis Stress Testing

For scenario analysis stress tests:

Hypothetical scenarios $x_1, \ldots, x_N$ are proposed for the risk factor changes.

Just like before, these can be stylized, historical, one-off.

Given scenarios, the stress-test risk measure is computed.

$$\Psi_x = \max\left\{ \mathfrak{l}_{[t]}(x_n) \right\}_{n=1}^{N}.$$

# Scenario Analysis Stress Testing

In practice, the scenarios:

Involve time-series of risk factor changes.

I.e. $x_n$ is a hypothetical *path* of the risk factor changes over $[t, t + K\Delta]$.

$x_n = (x_n(t + \Delta), x_n(t + 2\Delta), ..., x_n(t + K\Delta))$.

We use the paths to compute the loss in scenario $x_n$.

Consider changes in broader macro-economic shocks.

E.g.: shocked-paths of GDP, house prices, unemployment rate.

A model must be used to translate these shocks into risk factor changes.

# Scenario Analysis Stress Testing

Difficulties of scenario analysis stress testing:

How do we choose a good set of scenarios?

Tracking shocks over time is very complex.

How do we account for interactions between risks?

How do we shock multiple factors at the same time?

E.g. do all log returns go down at once?

How do we compute stress test risk efficiently?

# Mechanical Stress Testing

"Mechanical" stress testing refers to

- Analyzing how a model will react to shocks.
- Shocking model parameters to see sensitivities.

We study three mechanical stress tests.

- Sensitivity of $\varrho$ with respect to volatility and correlation.
- Stressed $\varrho$.
- Sensitivity of $K$-day $\varrho$ with respect to an initial shock in log returns.

# Sensitivity Analysis of $\varrho$

Two stocks, normal conditional log returns.

We drop $t, t + \Delta$ subscripts: $X \sim N\left(\mu, \Sigma\right)$ where

$$\mu = \begin{pmatrix} \mu_{(1)} \\ \mu_{(2)} \end{pmatrix}; \qquad \Sigma = \begin{pmatrix} \sigma^2_{(1)} & \rho\sigma_{(1)}\sigma_{(2)} \\ \rho\sigma_{(1)}\sigma_{(2)} & \sigma^2_{(2)} \end{pmatrix}.$$

$\mu_{(1)}, \mu_{(2)}$: mean for stocks 1 and 2.

$\sigma_{(1)}, \sigma_{(2)}$: volatility for stocks 1 and 2.

$\rho$: correlation between stocks 1 and 2.

# Sensitivity Analysis of $\varrho$

Notation:

$\theta$: dollar position vector.

$\varrho$: cash-additive, positively homogenous risk measure.

Linearized returns:

$\sim N(-\theta^T\mu, \theta^T\Sigma\theta)$

$L^{lin} = -\theta^{\mathrm{T}} X \sim -\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta} \times Z$ where $Z \sim N(0,1)$.

$\varrho(L^{lin}) = -\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta}\,\varrho(Z)$.

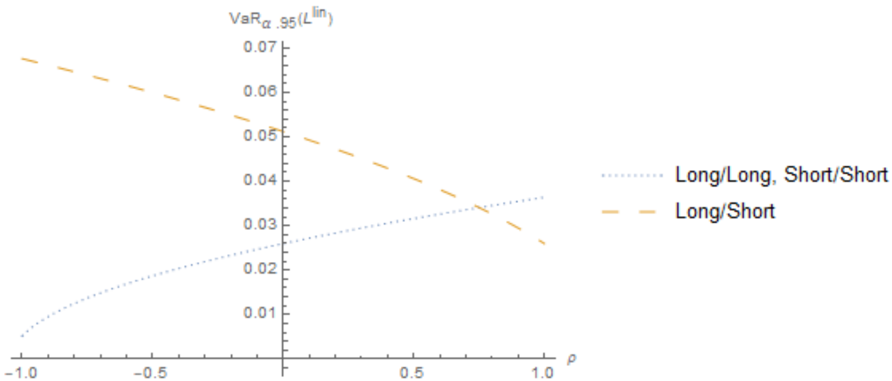
We consider the maps

$(\sigma_{(1)}, \sigma_{(2)}) \mapsto \varrho(L^{lin})$ and $\rho \mapsto \varrho(L^{lin})$.

Goal: study the sensitivity of $\varrho(L^{lin})$ w.r.t. $\sigma, \rho$.

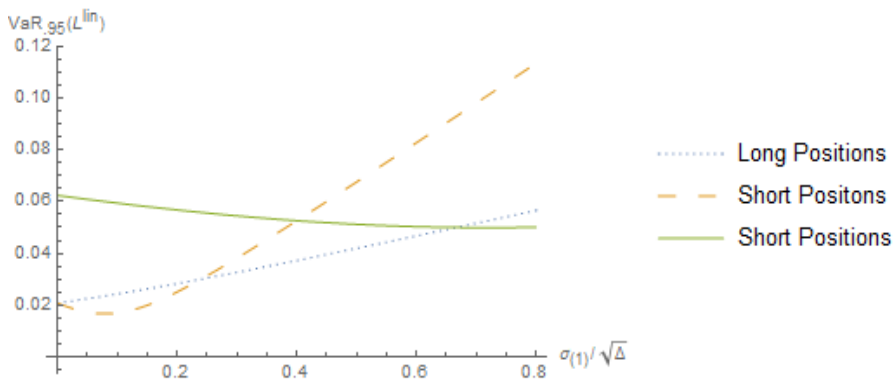
# Sensitivity Analysis of VaR

$\rho \mapsto \mathrm{VaR}_{.95}\left(L^{lin}\right)$:




ρ↑ risk↑

$\sigma_{(1)} \mapsto \mathrm{VaR}_{.95}\left(L^{lin}\right)$:



# Sensitivity Analysis of VaR

For both $\rho \mapsto \varrho(L^{lin})$ and $(\sigma_{(1)}, \sigma_{(2)}) \mapsto \varrho(L^{lin})$:

Effect of $\rho$ or $(\sigma_{(1)}, \sigma_{(2)})$ appears in $\theta^{\mathrm{T}}\Sigma\theta$ term.

$\partial_\rho \left(\theta^{\mathrm{T}}\Sigma\theta\right) = 2\sigma_{(1)}\sigma_{(2)}\theta_1\theta_2.$ $\theta^T\Sigma\theta = \theta_{(1)}^2\sigma_{(1)}^2 + \theta_{(2)}^2\sigma_{(2)}^2 + 2\theta_{(1)}\theta_{(2)}\sigma_{(1)}\sigma_{(2)}\rho$

Increasing in $\rho$ if either long-long or short-short.

Decreasing in $\rho$ if either long-short or short-long.

E.g.: $\partial_{\sigma_{(1)}} \left(\theta^{\mathrm{T}}\Sigma\theta\right) = 2\left(\sigma_{(1)}\theta_{(1)} + \rho\sigma_{(2)}\theta_{(2)}\right)\theta_{(1)}.$

A variety of relationships depending on $\theta_{(1)}, \theta_{(2)}$ and $\rho$.

$\partial_\rho \varrho(L^{lin}) = \frac{1}{2}\frac{1}{\sqrt{\theta^T\Sigma\theta}}\varrho(z) \times 2\theta_{(1)}\theta_{(2)}\sigma_{(1)}\sigma_{(2)}$ if $\theta_{(1)} > 0$, $\theta_{(2)} > 0$ then $\varrho(L^{lin}) > 0$.

$\varrho(z)$

$\partial_{\sigma_{(1)}} \rho(L^{lin}) = \frac{1}{2} \frac{1}{\sqrt{\theta^T \Sigma \theta}} (2\theta_{(1)}\sigma_{(1)} + 2\theta_1\theta_2\sigma_2\rho)$ → can be either "+" or "−"

# Sensitivity Analysis of $\varrho$: Conclusions

if $\theta_1 > 0$, $\theta_2 > 0 \Rightarrow \rho > 0$

$\partial_{\sigma_{(1)}} \rho(L^{lin}) > 0.$

Our risk measures are very sensitive to the correlation/volatility inputs.

In this analytic environment, we can answer

> "How would our one-day $\mathrm{VaR}$ change if correlations went from a small $\rho$ (normal) to $\rho \approx 1$ (stressed)?"

> "How would our one-day $\mathrm{VaR}$ change if volatilities went from a normal value of $\approx .2 - .4$ to a stressed value of $\approx .7 - .9$?"

# Stressed $\varrho$

The basic idea:

We wish to stress our losses, by assuming a large adverse value for one of the risk factor changes.

Question: what happens to the risk measure?

Subtlety:

How do we model changes in the other risk factors?

Identify how a shock to one risk factor to other risk factors.

# Stressed $\varrho$

Consider two stocks: $S^{(1)}$ and $S^{(2)}$.

Assume a large negative return for $S^{(2)}$.

How does $S^{(1)}$ react?

How does $\varrho$ change?

We assume $X \sim N(\mu, \Sigma)$ where

$$\mu = \begin{pmatrix} \mu_{(1)} \\ \mu_{(2)} \end{pmatrix}; \qquad \Sigma = \begin{pmatrix} \sigma_{(1)}^2 & \rho\sigma_{(1)}\sigma_{(2)} \\ \rho\sigma_{(1)}\sigma_{(2)} & \sigma_{(2)}^2 \end{pmatrix}.$$

For the stress test, we will assume

second log return

A shock (scenario) where $X_{(2)} = x_{(2)}$. 某个 fixed value $= \mu_{(2)} - S\sigma_{(2)}$

This shock does not affect $\Sigma$ (will change this later).

How does this affect $\varrho(L)$? First thing is to identify $X_{(1)}$

# Stressed $\varrho$

To identify how $S^{(1)}$ reacts, we must compute the distribution of $X_{(1)}$ given that $X_{(2)} = x_{(2)}$.

Claim. Given $X_{(2)} = x_{(2)}$:

reduce the variance

$$X_{(1)} \sim N\left(\mu_{(1)} + \tfrac{\rho\sigma_{(1)}}{\sigma_{(2)}}\left(x_{(2)} - \mu_{(2)}\right), \sigma^2_{(1)}(1 - \rho^2)\right).$$

Proof: straightforward, but too lengthy to give here.

# Stressed $\varrho$

$$X_{(1)} \sim N\left(\mu_{(1)} + \frac{\rho\sigma_{(1)}}{\sigma_{(2)}}\left(x_{(2)} - \mu_{(2)}\right), \sigma_{(1)}^2(1-\rho^2)\right)$$

Notes:

- If $\rho = 0$ (independence) then distribution unchanged.
- If $x_{(2)} = \mu_{(2)}$ then only variance changes. $X_{(1)} \sim N(\mu_{(1)}, \sigma_{(1)}^2(1-\rho^2))$
- As $|\rho| \to 1$: $X_{(1)} \to \mu_{(1)} + \frac{\rho\sigma_{(1)}}{\sigma_{(2)}}(x_{(2)} - \mu_{(2)})$.

  $X_{(1)}$ determined by $x_{(2)}$.

$L^{lin} = -\theta_{(1)} X_{(1)} - \theta_{(2)} x_{(2)};$

like const

$$\sim N\left(-\theta_{(1)}\left(\mu_{(1)} + \frac{\rho\sigma_{(1)}}{\sigma_{(2)}}(x_{(2)} - \mu_{(2)}\right) - \theta_{(2)} x_{(2)},\ \theta_{(1)}^2 \sigma_{(1)}^2 (1-\rho^2)\right)$$

$$= -\theta_{(1)}\left(\mu_{(1)} + \frac{\rho\sigma_{(1)}}{\sigma_{(2)}}(x_{(2)} - \mu_{(2)})\right) - \theta_{(2)} x_{(2)} + |\theta_{(1)}|\sigma_{(1)}\sqrt{1-\rho^2}\, z$$

$(\mu + \sigma z)$

$$\rho(L^{lin}) = -\theta_{(1)}\left(\mu_{(1)} + \frac{\rho\sigma_{(1)}}{\sigma_{(2)}}(x_{(2)} - \mu_{(2)})\right) - \theta_{(2)} x_{(2)} + |\theta_{(1)}|\sigma_{(1)}\sqrt{1-\rho^2}\, \rho(z)$$

— Explicit formula

# Stressed $\varrho$

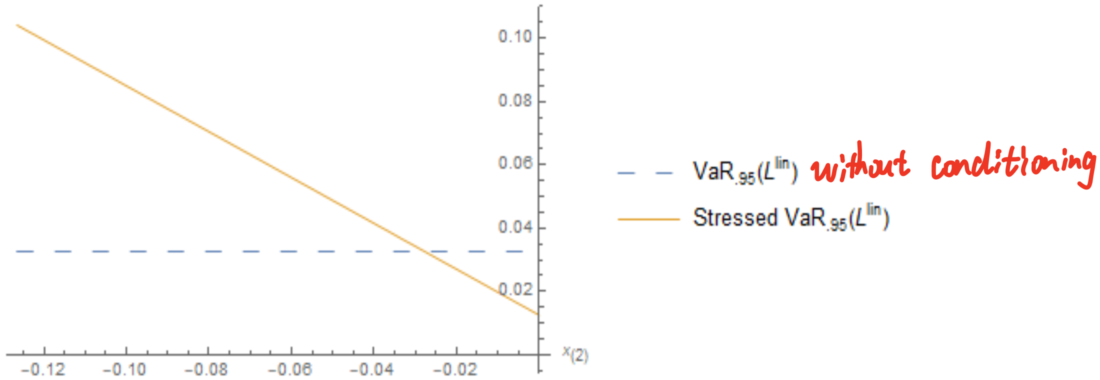
Since the distribution of $X_{(1)}$ given $X_{(2)} = x_{(2)}$ is known, obtaining $\varrho$ is easy.

E.g.: using linearized returns

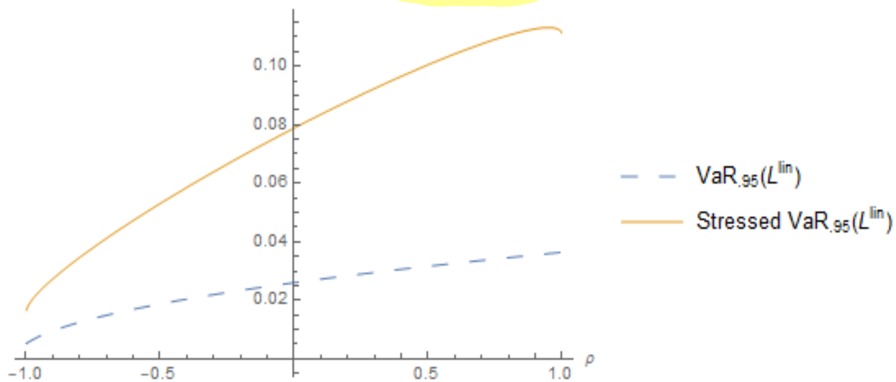
$$\begin{aligned}
L^{lin} &= -\theta_{(1)} X_{(1)} - \theta_{(2)} x_{(2)}; \\
&\sim N\left(-\theta_{(1)}\left(\mu_{(1)} + \frac{\rho\sigma_{(1)}}{\sigma_{(2)}}\left(x_{(2)} - \mu_{(2)}\right)\right) - \theta_{(2)} x_{(2)}, \theta_{(1)}^2 \sigma_{(1)}^2 (1 - \rho^2)\right); \\
&= -\theta_{(1)}\left(\mu_{(1)} + \frac{\rho\sigma_{(1)}}{\sigma_{(2)}}\left(x_{(2)} - \mu_{(2)}\right)\right) - \theta_{(2)} x_{(2)} + |\theta_{(1)}|\sigma_{(1)}\sqrt{1-\rho^2} Z. \\
\varrho(L^{lin}) &= -\theta_{(1)}\left(\mu_{(1)} + \frac{\rho\sigma_{(1)}}{\sigma_{(2)}}\left(x_{(2)} - \mu_{(2)}\right)\right) - \theta_{(2)} x_{(2)} \\
&\qquad + |\theta_{(1)}|\sigma_{(1)}\sqrt{1-\rho^2}\varrho(Z).
\end{aligned}$$

# Stressed $\varrho$

$x_{(2)} \mapsto \mathrm{VaR}_{.95}\left(L^{lin}\right)$:




$\rho \mapsto \mathrm{VaR}_{.95}\left(L^{lin}\right)$ for a $-5\sigma_{(2)}$ shock.



# Correlations under Stress

The above analysis misses one key point:

We assume return correlations unchanged by shock.

Folklore: "in times of crisis, correlations go to one."

How can we take this into account?

# “In times of crisis, correlations go to one.”

Last test: we want to initially shock one of the risk factors, and see how this propogates over time.

New issue: shock will also affect mean/covariance estimates.

Example: EWMA. During “normal” times

$$\Sigma = \begin{pmatrix} \sigma^2_{(1)} & 0 \\ 0 & \sigma^2_{(2)} \end{pmatrix}.$$

$\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ stylized fact

Idealization of “small” conditional correlations.

Now, assume over the period

Large negative return for stock 2: $X_{(2)} = x_{(2)} << 0$.

What is the estimated correlation at period end?

$$\Sigma_{new} = \lambda \Sigma_{old} + (1-\lambda) X X^T$$

$$= \left( \begin{array}{c|c} \lambda \sigma_{(1)}^2 + (1-\lambda) X_{(1)}^2 & (1-\lambda) X_{(1)} X_{(2)} \\ \hline (1-\lambda) X_{(1)} X_{(2)} & \lambda \sigma_{(2)}^2 + (1-\lambda) X_{(2)}^2 \end{array} \right)$$

↑

correlation introduced

$$\rho_{new} = \frac{(1-\lambda) X_{(1)} X_{(2)}}{\sqrt{(\lambda \sigma_{(1)}^2 + (1-\lambda) X_{(1)}^2)(\lambda \sigma_{(2)}^2 + (1-\lambda) X_{(2)}^2)}} \qquad \frac{Cov(X_1, X_2)}{\sigma_1 \sigma_2}$$

if $X_2 \to -\infty$, $\sqrt{\rho_{new}} = \dfrac{(1-\lambda)|X_{(1)}|}{\sqrt{\lambda \sigma_{(1)}^2 + (1-\lambda) X_{(1)}^2}}$ ($\approx 1$ if $X_{(1)} << 0$)

# "In times of crisis, correlations go to one."

Using $\Sigma_n = \lambda\Sigma_{old} + (1-\lambda)XX^{\mathrm{T}}$ ("n": new).

$$\rho_n^2 = \frac{(1-\lambda)x_{(1)}^2}{\lambda\sigma_{(1)}^2 + (1-\lambda)x_{(1)}^2} \times \frac{(1-\lambda)x_{(2)}^2}{\lambda\sigma_{(2)}^2 + (1-\lambda)x_{(2)}^2},$$

$$\rho_n^2 \nearrow \frac{(1-\lambda)x_{(1)}^2}{\lambda\sigma_{(1)}^2 + (1-\lambda)x_{(1)}^2} \text{ as } x_{(2)} \downarrow -\infty.$$

Correlations increase in magnitude to the maximal value based only on $\sigma_{(1)}, x_{(1)}$.

$|\rho_n| \approx 1$ to the extent $(1-\lambda)x_{(1)}^2$ dominates $\lambda\sigma_{(1)}^2$.

# Correlations under Stress

How can we capture changing correlations?

We will run a stress test where we

- Use historical data, EWMA, to estimate $\mu_{t+\Delta}, \Sigma_{t+\Delta}$ at $t$.
- Assume a one-time shock $X_{t+\Delta,(2)} = x_{(2)} << 0$.
- Update our mean, correlations using EWMA and sample new log returns.
- Estimate our portfolio losses over a holding period $[t, t + K\Delta]$.
- Compare our losses with the un-stressed case.

# A Stress Test Example

Market cap weighted port. of MSFT, AAPL.

$1$M$ portfolio. %43.8 MSFT, %56.2 AAPL.

"Normal" $\Sigma_{t+\Delta}, \mu_{t+\Delta}$ found via EWMA.

baseline $\rho$

Five years historical data. "Regular" correlation $\rho = 0.235$.

Assuming $X_{t+\Delta} \overset{\mathcal{F}_t}{\sim} N(\mu_{t+\Delta}, \Sigma_{t+\Delta})$ we first estimate

One day $\mathrm{VaR}_\alpha\left(L^{lin}_{t+\Delta}\right)$.

Regulatory capital charge: $3 \times \sqrt{10}\mathrm{VaR}_\alpha\left(L^{lin}_{t+\Delta}\right)$.

K day $\mathrm{VaR}_\alpha\left(L^{lin}_{t+K\Delta}\right) \approx \sqrt{K}\mathrm{VaR}_\alpha\left(L^{lin}_{t+\Delta}\right)$ for $K = 10$.

# The Stress Test

First,

Assume a $-5\sigma_{t+\Delta,(2)}$ shock for $X_{t+\Delta,(2)}$.

$$X_{t+\Delta,(2)} = x_{t+\Delta,(2)} = \mu_{t+\Delta,(2)} - 5\sigma_{t+\Delta,(2)}.$$

Sample $X_{t+\Delta,(1)}$ using the conditional distribution given $X_{t+\Delta,(2)} = x_{t+\Delta,(2)}$.

Update $\mu_{t+2\Delta}, \Sigma_{t+2\Delta}$ using EWMA.

Then, for days $k = 2, ..., K = 10$:

Sample $X_{t+k\Delta} \sim N(\mu_{t+k\Delta}, \Sigma_{t+k\Delta})$.

Update $\mu_{t+(k+1)\Delta}, \Sigma_{t+(k+1)\Delta}$ via EWMA (up to $K-1$).

# Stress Test Example

We compute the $K = 10$ day linearized losses and output

Average simulated $K$ day loss.

$K$ day Value at risk estimate.

Number of times $K$ day losses exceeded initial $K$ day $\mathrm{VaR}$.

Number of times $K$ day losses exceeded initial regulatory capital charge.

# Stress Test Example

Results: $100,000$ trials, $K = 10$ days.

Initial $\mathrm{VaR}_{.95}$ estimates.

$$\widehat{\mathrm{VaR}_{.95}}\left(L_{t+\Delta}^{lin}\right) = 14,384.$$

$$\sqrt{K}\widehat{\mathrm{VaR}_{.95}}\left(L_{t+\Delta}^{lin}\right) = 45,486.$$

$$3 \times \sqrt{K}\widehat{\mathrm{VaR}_{.95}}\left(L_{t+\Delta}^{lin}\right) = 136,457.$$

Results (shock; no shock)

Average loss: $39,823$; $-11,590$.

$\widehat{\mathrm{VaR}_{.95}}\left(L_{t+K\Delta}^{lin}\right)$: $105,405$; $44,102$.

% $\sqrt{K}\widehat{\mathrm{VaR}_{.95}}\left(L_{t+\Delta}^{lin}\right)$ exceedances: $43.91$%; $4.62$.%

% $3 \times \sqrt{K}\widehat{\mathrm{VaR}_{.95}}\left(L_{t+\Delta}^{lin}\right)$ exceedances: $0.897$%; $0.007$.%