Recommended HW Problems for Assignment 6 - Part I
SOLUTIONS

Note : this is the first half of HW 6, covering material from Lecture 12 on November 27th. The second half of HW 6 will be posted by December 4th, and covers material from Lecture 13, also on the 4th. You do not have to turn in these problems, but you should definitely try them, as they will help you study for the final!

1. Credit Spreads for Defaultable Bonds. Assume the spot and hazard rates are deterministic, continuous, functions of t, given by $r = \{r(t)\}_{t\geq 0}$ and $\gamma = \{\gamma(t)\}_{t\geq 0}$ respectively. For a maturity T and current time $t \leq T$, let $p_0(t,T)$, $p_1(t,T)$, $p_1^{RT}(t,T)$ and $p_1^{RF}(t,T)$ be the time-t prices for a default-free ZCB, defaultable ZCB with 0 recovery, defaultable ZCB with RT recovery and defaultable ZCB with RF recovery respectively. Assume the loss fraction given default δ is constant.

Define the credit spread for 0 recovery as

$$c(t,T) \triangleq -\frac{1}{T-t}\left(\log(p_1(t,T)) - \log(p_0(t,T))\right),$$

with similar formulas for $c^{RT}(t,T)$ and $c^{RF}(t,T)$ replacing p_1 with p_1^{RT}, p_1^{RF} respectively. On the set $\{\tau > t\}$, compute:

(a) $c(t,T)$, $c^{RT}(t,T)$ and $c^{RF}(t,T)$. Be as explicit as possible.

(b) The limits

$$\lim_{T\downarrow t} c(t,T); \quad \lim_{T\downarrow t} c^{RT}(t,T); \quad \lim_{T\downarrow t} c^{RF}(t,T).$$

These are the *instantaneous credit spreads*. Here, you should get very simple expressions. Can you explain *why* they are what they are?

Solution:

(a) On the set $\{\tau > t\}$:

$$\begin{aligned}
p_0(t,T) &= e^{-\int_t^T r(u)du};\\
p_1(t,T) &= e^{-\int_t^T (r(u)+\gamma(u))du};\\
p_1^{RT}(t,T) &= \delta e^{-\int_t^T (r(u)+\gamma(u))du} + (1-\delta)e^{-\int_t^T r(u)du};\\
p_1^{RF}(t,T) &= e^{-\int_t^T (r(u)+\gamma(u))du} + (1-\delta)\int_t^T \gamma(s)e^{-\int_t^s (r(u)+\gamma(u))du}ds.
\end{aligned}$$

From here it follows that

$$
\begin{aligned}
c(t,T) &= -\frac{1}{T-t}\left(\log(p_1(t,T)) - \log(p_0(t,T))\right);\\
&= -\frac{1}{T-t}\left(-\int_t^T (r(u)+\gamma(u))du + \int_t^T r(u)du\right);\\
&= \frac{1}{T-t}\int_t^T \gamma(u)du.
\end{aligned}
\tag{0.1}
$$

For the RT method:

$$
\begin{aligned}
c^{RT}(t,T) &= -\frac{1}{T-t}\left(\log(p_1^{RT}(t,T)) - \log(p_0(t,T))\right);\\
&= -\frac{1}{T-t}\log\left(\frac{\delta e^{-\int_t^T (r(u)+\gamma(u))du} + (1-\delta)e^{-\int_t^T r(u)du}}{e^{-\int_t^T r(u)du}}\right);\\
&= -\frac{1}{T-t}\log\left(\delta e^{-\int_t^T \gamma(u)du} + (1-\delta)\right).
\end{aligned}
\tag{0.2}
$$

For the RF method:

$$
\begin{aligned}
&c^{RF}(t,T)\\
&= -\frac{1}{T-t}\left(\log(p_1^{RF}(t,T)) - \log(p_0(t,T))\right);\\
&= -\frac{1}{T-t}\log\left(\frac{e^{-\int_t^T (r(u)+\gamma(u))du} + (1-\delta)\int_t^T \gamma(s)e^{-\int_t^s ((r(u)+\gamma(u))du)}ds}{e^{-\int_t^T r(u)du}}\right);\\
&= -\frac{1}{T-t}\log\left(e^{-\int_t^T \gamma(u)du} + (1-\delta)e^{\int_t^T r(u)du}\int_t^T \gamma(s)e^{-\int_t^s ((r(u)+\gamma(u))du)}ds\right).
\end{aligned}
\tag{0.3}
$$

(b) Since γ is continuous we see from (0.1) that

$$\lim_{T\downarrow t} c(t,T) = \gamma(t).$$

Using l'Hospital's rule and the mean value theorem we see from (0.2) that

$$\lim_{T\downarrow 0} c^{RT}(t,T) = \lim_{T\downarrow t} -\frac{-\delta\gamma(T)e^{-\int_t^T \gamma(u)du}}{\delta e^{-\int_t^T \gamma(u)du} + (1-\delta)} = \delta\gamma(t).$$

Again, using l'Hospital's rule and the mean value theorem in (0.3) gives

$$\lim_{T\downarrow t} c^{RF}(t,T) = \delta\gamma(t).$$

So, the RF and RT recovery methods yield an instantaneous credit spread of $\delta\gamma(t)$ where as the 0 recovery ($\delta = 1$) gives $\gamma(t)$.

2. Credit Default Swap Spreads and Accrued Interest. Assume the hazard rate process is a deterministic function of t given by $\gamma = \{\gamma(t)\}_{t\geq 0}$. Here we will assume 0 interest rates (i.e. $r(t) = 0$ for all $t \geq 0$). In class we derived the time t credit default swap spread for entering into a N year swap as

$$x_t = \delta \frac{\int_t^{t+N} \gamma(s) e^{-\int_t^s \gamma(u)du} ds}{\frac{1}{\Delta}\sum_{n=1}^{N\Delta} e^{-\int_t^{t+n/\Delta}\gamma(u)du}}, \tag{0.4}$$

ignoring the accrued interest component paid by the buyer to the seller upon default. In this exercise we will not ignore accrued interest, and see how things change. Below, we set $t_n = t + n/\Delta$ for $n = 1, ..., N\Delta$.

(a) Recall that if $t_{n-1} < \tau \leq t_n$ then upon default, the buyer must pay the seller accrued interest in the amount of $x(\tau - t_{n-1})$ where x is the swap spread. As the interest rate is 0, the fair value of these payments is

$$E^{\mathbb{Q}}\left[\sum_{n=1}^{N\Delta} 1_{t_{n-1}<\tau\leq t_n} x(\tau - t_{n-1}) \;\Big|\; \mathcal{H}_t\right].$$

Simplify this formula to show that, including accrued interest, the time t swap rate is

$$x_t^{ai} = \delta \frac{\int_t^{t+N} \gamma(s) e^{-\int_t^s \gamma(u)du} ds}{\int_t^{t+N} e^{-\int_t^s \gamma(u)du} ds}, \tag{0.5}$$

which admits a simple interpretation as a weighted average of the intensity function.

(b) Write $x_0(N)$ and $x_0^{ai}(N)$ for the time 0 swap rates from (0.4), (0.5) respectively, to stress the dependence on the maturity N. For a Weibull intensity $\gamma(t) = \lambda\alpha t^{\alpha-1}$ write a program which produces a plot of the maps

$$N \mapsto 10000 x_0(N); \qquad N \mapsto 10000 x_0^{ai}(N),$$

i.e. give the spread in basis points. For parameter values use $\delta = .5$, $\Delta = 4$, and $\lambda = 2\%$. Take $N = K/4$ for $K = 1, 2, \ldots, 40$ (i.e. quarterly increments out ten years). Produce plots for $\alpha = .5, \alpha = 1$ and $\alpha = 1.5$. How do the spreads change with the maturity? How do the spreads change with α?

(c) As you might have seen in your plot, for $\alpha = 1$ the spread is constant. Prove this fact analytically for general values of λ, Δ and δ.

Solution:

(a) Using integration by parts

$$
\begin{aligned}
&E^{\mathbb{Q}}\left[1_{t_{n-1}<\tau\le t_n} x(\tau - t_{n-1}) \mid \mathcal{H}_t\right] \\
&\qquad = x \times 1_{\tau>t} \int_{t_{n-1}}^{t_n} (s - t_{n-1})\gamma(s) e^{-\int_t^s \gamma(u)du} ds \\
&\qquad = x \times 1_{\tau>t} \left(-(s - t_{n-1}) e^{-\int_t^s \gamma(u)du} \Big|_{s=t_{n-1}}^{s=t_n} + \int_{t_{n-1}}^{t_n} e^{-\int_t^s \gamma(u)du} ds \right) \\
&\qquad = x \times 1_{\tau>t} \left(-\frac{1}{\Delta} e^{-\int_t^{t_n} \gamma(u)du} + \int_{t_{n-1}}^{t_n} e^{-\int_t^s \gamma(u)du} ds \right)
\end{aligned}
$$

Summing from 1 to $N\Delta$

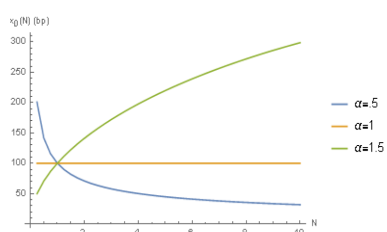
$$
\begin{aligned}
&E^{\mathbb{Q}}\left[\sum_{n=1}^{N\Delta} 1_{t_{n-1}<\tau\le t_n} x(\tau - t_{n-1}) \mid \mathcal{H}_t\right] \\
&\qquad = x \times 1_{\tau>t} \left(\int_t^{t+N} e^{-\int_t^s \gamma(u)du} ds - \frac{1}{\Delta} \sum_{n=1}^{n\Delta} e^{-\int_t^{t+n/\Delta} \gamma(u)du} \right).
\end{aligned}
$$

Therefore, the value of the payments made by the buyer is

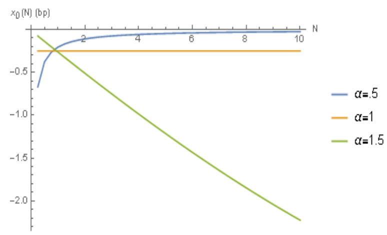
$$
\begin{aligned}
&E^{\mathbb{Q}}\left[\sum_{n=1}^{N\Delta} \frac{x}{\Delta} 1_{\tau_n<\tau} \mid \mathcal{H}_t\right] + E^{\mathbb{Q}}\left[\sum_{n=1}^{N\Delta} 1_{t_{n-1}<\tau\le t_n} x(\tau - t_{n-1}) \mid \mathcal{H}_t\right] \\
&= x \times 1_{\tau>t} \left(\frac{1}{\Delta} \sum_{n=1}^{n\Delta} e^{-\int_t^{t+n/\Delta} \gamma(u)du} + \int_t^{t+N} e^{-\int_t^s \gamma(u)du} ds - \frac{1}{\Delta} \sum_{n=1}^{n\Delta} e^{-\int_t^{t+n/\Delta} \gamma(u)du} \right) \\
&= x \times 1_{\tau>t} \int_t^{t+N} e^{-\int_t^s \gamma(u)du} ds
\end{aligned}
$$

From here the result follows just as in class.

(b) See the Mathematica notebook
"CDS_Calculations_Det.nb"
As for the plot, I obtained the following ignoring accrued interest.



So, for $\alpha = .5$ the spread decreases with maturity. For $\alpha = 1$ the spread is constant and for $\alpha = 1.5$ the spread is increasing with the maturity. The plots including accrued interest are similar, so I instead plot $x_0^{ai}(N) - x_0(N)$



So, it looks like ignoring accrued interest in this case ($r \equiv 0$, Weibull intensities) does not make much of a difference.

(c) For $\alpha = 1$, $\gamma(t) = \lambda$ is constant. We first evaluate the numerator in (0.4):

$$\int_t^{t+N} \gamma(s) e^{-\int_t^s \gamma(u)du} ds = \int_t^{t+N} \lambda e^{-\lambda(s-t)} ds = 1 - e^{-\lambda N}.$$

For the denominator in (0.4):

$$\begin{aligned} \frac{1}{\Delta} \sum_{n=1}^{N\Delta} e^{-\int_t^{t_n} \gamma(u)du} &= \frac{1}{\Delta} \sum_{n=1}^{N\Delta} e^{-\lambda \frac{n}{\Delta}}; \\ &= \frac{1}{\Delta} \sum_{n=1}^{N\Delta} \left(e^{-\lambda/\Delta} \right)^n; \\ &= \frac{1}{\Delta} \frac{e^{-\lambda/\Delta}}{1 - e^{-\lambda/\Delta}} \left(1 - e^{-N\lambda} \right). \end{aligned}$$

Here, we have used the identity that if $0 < p < 1$ then $\sum_{n=1}^{L} p^k = \frac{p}{1-p}(1 - p^L)$. Putting them together gives

$$x_t(N) = \delta \frac{1 - e^{-\lambda N}}{\frac{1}{\Delta} \frac{e^{-\lambda/\Delta}}{1 - e^{-\lambda/\Delta}} (1 - e^{-N\lambda})} = \delta \Delta \left(e^{\lambda/\Delta} - 1 \right).$$

Thus, the credit spread ignoring accrued interest does not depend on the maturity N. For the accrued interest case, from (0.5) it is clear for constant $\gamma(t) = \lambda$ that $x_t^{ai}(N) = \delta\lambda$ does not depend on N.

3. Cox Models and Stochastic Recovery. Recall the doubly-stochastic hazard rate model where

$$\mathbb{Q}\left[\tau > t \mid \mathcal{F}_t\right] = e^{-\int_0^t \gamma(u)du},$$

with γ an $\mathbb{F}$ adapted process. Recall also that we allowed for stochastic interest rates r. Now, in class we considered constant loss given default δ but in fact, we can allow δ to be $\mathbb{F}$ adapted (varying in both time and scenario) as well, as we will now see.

Let $\delta = \{\delta(t)\}_{t \geq 0}$ be an $\mathbb{F}$ adapted process such that for each $\omega \in \Omega$ the map $t \to \delta(t)(\omega)$ is smooth with derivative $\dot{\delta}(t)(\omega)$. In the stochastic recovery case, under the RT convention, the bond holder receives $1 - \delta(\tau)(\omega)$ units of a default-free ZCB upon default prior to t and in scenario ω.

Using integration by parts, show that the time 0 defaultable ZCB price with recovery at treasury is

$$
\begin{aligned}
p_1^{RT}(0,T) &= E^{\mathbb{Q}}\left[\delta(T)e^{-\int_0^T (r(u)+\gamma(u))du}\right] + (1-\delta(0))p_0(0,T) \\
&\quad - E^{\mathbb{Q}}\left[e^{-\int_0^T r(u)du}\int_0^T \dot{\delta}(s)e^{-\int_0^s \gamma(u)du}ds\right],
\end{aligned} \tag{0.6}
$$

where p_1 is the defaultable ZCB price with 0 recovery and p_0 is the default free ZCB price. Thus, we recover the formula in class in the case of constant loss rates, since $\dot{\delta}(s) = 0$ and $p_1(0,T) = E^{\mathbb{Q}}\left[e^{-\int_0^T (r(u)+\gamma(u))du}\right]$.

Solution: We have

$$
\begin{aligned}
p_1^{RT}(0,T) &= E^{\mathbb{Q}}\left[\left(1_{\tau>T} + (1-\delta(\tau))1_{\tau\le T}\right)e^{-\int_0^T r(u)du}\right]; \\
&= E^{\mathbb{Q}}\left[e^{-\int_0^T (r(u)+\gamma(u))du} + e^{-\int_0^T r(u)du}\int_0^T (1-\delta(s))\gamma(s)e^{-\int_0^s \gamma(u)du}ds\right]; \\
&= E^{\mathbb{Q}}\left[e^{-\int_0^T (r(u)+\gamma(u))du}\right] \\
&\quad + E^{\mathbb{Q}}\left[e^{-\int_0^T r(u)du}\left(-(1-\delta(s))e^{-\int_0^s \gamma(u)du}\Big|_{s=0}^{s=T} - \int_0^T \dot{\delta}(s)e^{-\int_0^s \gamma(u)du}ds\right)\right]; \\
&= E^{\mathbb{Q}}\left[\delta(T)e^{-\int_0^T (r(u)+\gamma(u))du}\right] + (1-\delta(0))E^{\mathbb{Q}}\left[e^{-\int_0^T r(u)du}\right] \\
&\quad - E^{\mathbb{Q}}\left[e^{-\int_0^T r(u)du}\int_0^T \dot{\delta}(s)e^{-\int_0^s \gamma(u)du}ds\right].
\end{aligned}
$$

This gives the result, since $p_0(0,T) = E^{\mathbb{Q}}\left[e^{-\int_0^T r(u)du}\right]$.