Problem 1

$$\mathrm{L_m} = \mathrm{l}\sum_{i=1}^{m} X_i = 40N_m, \qquad N_m \sim B(10000, 0.04) \sim N\left(400, \sqrt{384}\right)$$

$$\mathrm{ES}_\alpha\left(N(0,1)\right) = \frac{1}{1-\alpha}\phi\left(N^{-1}(\alpha)\right), \qquad \mathrm{ES}_{0.99}\left(N(0,1)\right) = 2.665214220345808$$

If D=1000

$$\mathrm{L_m} = 400\sum_{i=1}^{1000} X_i = 400N_{1000}, \qquad N_{1000} \sim B(1000, 0.04) \sim N\left(40, \sqrt{38.4}\right) \sim 40 + \sqrt{38.4}N(0,1)$$

$$\mathrm{ES_L}(\mathrm{D} = 1000) = 400\left(40 + \sqrt{38.4} * 2.665214\right) = 22606.29138526368$$

If D=500

$$\mathrm{L_m} = 800\sum_{i=1}^{500} X_i = 800N_{500}, \qquad N_{500} \sim B(500, 0.04) \sim N\left(20, \sqrt{19.2}\right) \sim 20 + \sqrt{19.2}N(0,1)$$

$$\mathrm{ES_L}(\mathrm{D} = 500) = 800\left(20 + \sqrt{19.2} * 2.665214\right) = 25342.706874028438$$

If D=200

$$\mathrm{L_m} = 2000\sum_{i=1}^{200} X_i = 2000N_{200}, \qquad N_{200} \sim B(200, 0.04) \sim N\left(8, \sqrt{7.68}\right) \sim 8 + \sqrt{7.68}N(0,1)$$

$$\mathrm{ES_L}(\mathrm{D} = 200) = 2000\left(8 + \sqrt{7.68} * 2.665214\right) = 30772.116616620842$$

Problem 2

(a)

$$\bar{p} = \int_{-\infty}^{+\infty} e^{-ax^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(2a+1)x^2}{2}}dx$$

$$= \sqrt{\frac{1}{2a+1}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{\frac{1}{2a+1}}\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x}{\sqrt{\frac{1}{2a+1}}}\right)^2}dx = \sqrt{\frac{1}{2a+1}}$$

$$E[p(Z)^2] = \int_{-\infty}^{+\infty} e^{-2ax^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = \sqrt{\frac{1}{4a+1}}$$

$$\rho_{1,2} = \frac{E[p(Z)^2] - \bar{p}^2}{\bar{p}(1-\bar{p})} = \frac{\sqrt{\frac{1}{4a+1}} - \frac{1}{2a+1}}{\sqrt{\frac{1}{2a+1}}\left(1 - \sqrt{\frac{1}{2a+1}}\right)} = 0.25$$

Solve it by python scipy.optimize, we have a=0.21828008

$\bar{p} = 0.8343304450412693$

(b)

standard deviation of the number of defaults with p=p(Z)

$\mathrm{Var}[\mathrm{N_m}] = \mathrm{m}\bar{p}(1-\bar{p}) + m(m-1)(E[p(Z)^2] - \bar{p}^2) = 355.9246188214332$
$\mathrm{stdev}_1 = 18.865964561119934$

stdev of portfolio with m = 100 independent defaults with probability of default $\mathrm{p} = \bar{p}$
$\mathrm{stdev}_2 = \sqrt{m\bar{p}(1-\bar{p})} = 3.7178374563515657$

The 2 stdev are very far from each other, that means we cannot ignore the randomness of p(Z) when calculating variance and stdev of number of defaults.

Problem 3

(a)

$\mathrm{Loss} = 1 * \mathrm{N_m} = N_m = (X_1 + X_2 + \cdots + X_m)$
$\mathrm{Var}[\mathrm{Loss}] = \mathrm{Var}[\mathrm{N_m}] = \mathrm{mVar}[\mathrm{X_1}] + \mathrm{m}(\mathrm{m}-1)\mathrm{Cov}[\mathrm{X_1}, \mathrm{X_2}]$
$= \mathrm{mp}(1-\mathrm{p}) + \mathrm{m}(\mathrm{m}-1)\rho\mathrm{p}(1-\mathrm{p}) = 99.00980100000001$
Stdev[Loss]= 9.95036687765833

(b)

$$\mathrm{Loss} = \frac{1000}{\mathrm{D}} * N_D, \qquad N_D \sim B(D, 0.01)$$

$$\mathrm{Var}[\mathrm{Loss}] = \frac{1000000}{\mathrm{D}^2} D * 0.01 * 0.99 = 99.00980100000001$$

$\Rightarrow \mathrm{D} = 99.99010098000298 \approx 100$