1. $ES_\alpha(L) = \mathbb{E}[L \mid L \geq VaR_\alpha(L)]$

$$= \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)\,du$$

$$= \frac{1}{1-\alpha}\int_\alpha^1 (\mu + \sigma\cdot\Phi^{-1}(u))\,du$$

$$= \frac{1}{1-\alpha}[(1-\alpha)\mu + \sigma\cdot\varphi(\Phi^{-1}(\alpha))] \quad (u = \Phi(y)) \to y = \Phi^{-1}(u) \Rightarrow du = \Phi(y)dy$$

$$= \mu + \frac{\sigma}{1-\alpha}\varphi(\Phi^{-1}(\alpha))$$

$\varphi$ and $\Phi$ are PDF, CDF of standard normal distribution.

2. $m = 10000$ $\quad EAD = 100$ $\quad LGD = 40\%$ $\quad P = 0.04$ $\quad D = 1000, 500, 200$

Binomial distribution can be approximated as normal distribution

$$\mathbb{E}(L) = \frac{m\cdot EAD\cdot LGD}{D}\cdot D\cdot P \approx \mu$$

$$Var(L) = \left(\frac{m\cdot EAD\cdot LGD}{D}\right)^2\cdot D\cdot P\cdot(1-P) \approx \sigma^2$$

$$\ell = \frac{LGD\cdot EAD\cdot m}{D}$$

$$L_m = \ell\sum_{i=1}^{m} X_i = 40 N_m \quad , \quad N_m \sim Bin(10000, 0.04) \sim N(400, \sqrt{384})$$

$$ES_\alpha = \frac{1}{1-\alpha}\phi(N^{-1}(\alpha)) \quad , \quad ES_{0.99}(N(0,1)) = 2.665214$$

$$\ell = \frac{LGD\cdot EAD\cdot m}{D}$$

If $D = 1000$,

$$L_m = 400\sum_{i=1}^{1000} X_i = 400 N_{1000} \quad , \quad N_{1000} \sim N(40, \sqrt{38.4}) \sim 40 + \sqrt{38.4}\,N(0,1)$$

$$ES_L(D=1000) = 400(40 + \sqrt{38.4}\times 2.665214) = 22606.2913835\,6$$

If $D = 500$,

$$L_m = 800\sum_{i=1}^{500} X_i = 800 N_{500} \quad , \quad N_{500} \sim Bin(500, 0.04) \sim N(20, \sqrt{19.2}) \sim 20 + \sqrt{19.2}\,N(0,1)$$

$$ES_L(D=500) = 800(20 + \sqrt{19.2}\times 2.665214) = 25342.7068$$

If $D = 200$,

$$L_m = 2000\sum_{i=1}^{200} X_i = 2000 N_{200} \quad , \quad N_{200} \sim Bin(200, 0.04) \sim N(8, \sqrt{7.68}) \sim 8 + \sqrt{7.68}\,N(0,1)$$

$$ES_L(D=200) = 2000(8 + \sqrt{7.68}\times 2.665214) = 30772.1166$$

3. (a) $P_{X_1} = P_{X_2} = \int_{-\infty}^{+\infty} P(z) f_Z(z)\, dz = \int_{-\infty}^{+\infty} e^{-az^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$

$$= \frac{1}{\sqrt{1+2a}} \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{1+2a}}$$

$$P_{X_1X_2} = \int_{-\infty}^{+\infty} P(z)^2 f_Z(z)\, dz = \frac{1}{\sqrt{1+4a}}$$

$$\rho_{X_1X_2} = \frac{P_{X_1X_2} - P_{X_1}P_{X_2}}{\sqrt{P_{X_1}(1-P_{X_1})P_{X_2}(1-P_{X_2})}} = 0.4$$

$$1+2a - \sqrt{1-4a} = 0.4\sqrt{1+2a}\sqrt{1+4a} - 0.4\sqrt{1+4a}$$

$$1+2a = 0.4\sqrt{1+2a}\sqrt{1+4a} - 0.6\sqrt{1+4a}$$

$$a = 0.602$$

(b) $P(z) = e^{-az^2}$, $a>0$, $m=2$, $X_1$ and $X_2$ are the indicators of default.

$P(X_i=1) = P(z)$, $i=1,2$.

$$N_m = \sum_{i=1}^{m} X_i$$

$$Var(N_m) = Var\left(\sum_{i=1}^{m} X_i\right)$$

$$= \sum_{i=1}^{m} Var(X_i) + \sum_{i=1}^{m}\sum_{j\neq i}^{m} Cov(X_i, X_j)$$

$$Var(X_i) = \bar{p}(1-\bar{p})$$

$$Cov(X_i, X_j) = \rho\sqrt{Var(X_i)} \cdot \sqrt{Var(X_j)} = \rho\bar{p}(1-\bar{p})$$

$$Var(N_m) = m\bar{p}(1-\bar{p}) + m(m-1)\rho\bar{p}(1-\bar{p})$$

$m=100$, $\rho=0.4$, $\bar{p} = \mathbb{E}[P(z)] = \frac{1}{\sqrt{1+2a}} = 0.6737$

$\Rightarrow Std(N_m) = 29.87$ $\qquad Cov(X_i, X_j) = 0$

$$Std(N_m) = \sqrt{m\bar{p}(1-\bar{p})} = 4.69$$

Thus, the standard deviation is smaller for independent default case.