Black Cox Model

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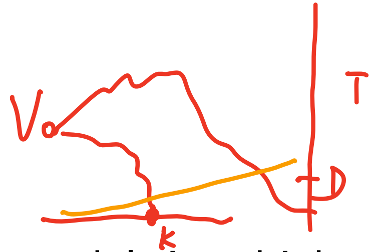
Assumptions and general setup

Black and Cox model



In order to improve Merton's model, in which default can occur at maturity, and allow for **premature default**, the default time can be modeled as a stopping time.

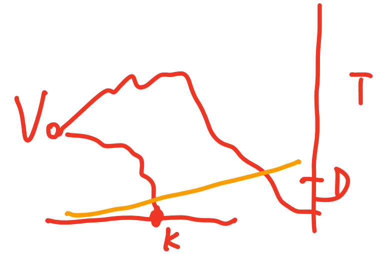
In particular, the time of default can be defined as the first-passage time of the assets value to a deterministic barrier v.

Black and Cox [1976] use the same setting as in Merton (debt is a zero coupon bond with face value D) and include a **safety covenant**. Safety covenants provide the firm's bondholders with the right to force the firm to bankruptcy or reorganization if the firm is doing poorly according to a set standard.

Assumption 1 on the Default Barrier

$t=0 \quad v_t = Ke^{-rT}$

$t=T \quad V_t = K$



Assumption 1

- *The standard for a poor performance is set in terms of a time dependent deterministic barrier*

$$v_t = Ke^{-r(T-t)}, \quad \forall t \in [0,T), \quad K > 0.$$

As soon as the value of the assets crosses this lower threshold v_t *during* $[0,T)$, *the debtholders take-over the firm, liquidate the assets and recover the amount* v_t.

- *Otherwise, default occurs at maturity or not depending on whether* $V_T < D$ *or not. In case of default, the debtholders recover an amount* V_T.

Assumption 2 on the Default Barrier

We set the default barrier for $t \in [0, T]$ to be:

$$v_t = \begin{cases} v_t, & \text{if } t < T, \\ D, & \text{if } t = T. \end{cases}$$

Assumption 2

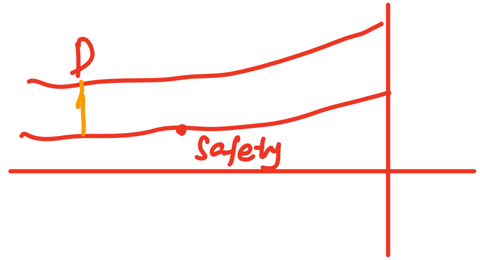
The following inequalities hold:

1. $V_0 > v_0$,
2. $K \leq D$.

The first condition is imposed to avoid default at inception date. The second one ensures that the payoff to the bondholders at the default time τ never exceeds the face value of debt discounted with risk free rate.

$$\boxed{v_\tau = Ke^{-r(T-\tau)} \leq De^{-r(T-\tau)},} \quad \tau \in [0, T).$$

The default time



- the early default time $\overline{\tau}$ equals

$$\overline{\tau} = \inf\{t \in [0,T) : V_t < v_t\},$$

  and $\hat{\tau}$ stands for Merton's default time

$$\hat{\tau} = T\mathbb{1}_{\{V_T < D\}} + \infty\mathbb{1}_{\{V_T \geq D\}}.$$

- The amount recovered in the two default situations is:

$$V_T, \quad \tau = \hat{\tau} = T, \text{ default at T}$$

$$V_{\overline{\tau}}, \quad \tau = \overline{\tau} < T, \text{ early default.}$$

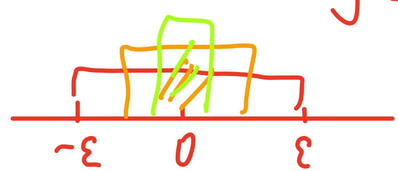
- The default time τ :

$$\boxed{\tau = \overline{\tau} \wedge \hat{\tau}}$$

Dirac function

$\int = 1$

$\varepsilon \to 0$



Valuation equations for stock and debt values

1. Recall that in Merton model stock S is a call option on asset value V with strike = debt D. Under GBM, we have a BS problem for S:

$$\frac{\partial S}{\partial t} + rV\frac{\partial S}{\partial V} + \frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 S}{\partial V^2} - rS = 0 \tag{1}$$

with boundary conditions:

Barier

$$S(V,T) = \max(V_T - D, 0), S(Ke^{-r(T-t)}, t) = 0 \tag{2}$$

2. Similarly, for debt value B

$$\frac{\partial B}{\partial t} + rV\frac{\partial B}{\partial V} + \frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 B}{\partial V^2} - rB = 0 \tag{3}$$

$$B(V,T) = \min(V_T, D), B(Ke^{-r(T-t)}, t) = Ke^{-r(T-t)} \tag{4}$$

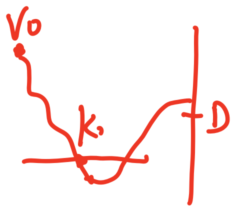
get whole Debt or remaining V.

knock-out

Case of constant default threshold: equity value

$k_1 = const \leq De^{-rT}$

- If a default threshold (safety covenant) is assumed to be constant $v_t = K_1 \leq De^{-rT}$ then the previous problem for stock value is a classical knock out or "down-and-out" call option.
- It can be priced by solving PDE and using the method of *images*. In this case $K_1 < D$ and we also assume that $V_0 > K_1$. The value of knock out call is given by



$$S(V,t) = C(V,t,D) - \left(\frac{V}{K_1}\right)^{-(m_1-1)} C\left(\frac{K_1^2}{V}, t, D\right) \quad (5)$$

where $C(V,t,D)$ is a BS call option with underlying V, strike D, expiration T, and $m_1 = \frac{r}{1/2\sigma_V^2}$

- It can be checked that at the boundary $V = K_1$ we satisfy the condition 2

$$S(K_1, t, D) = 0$$

$\frac{1}{De^{-rT}} \times \left(\frac{V_0}{K_1}\right)^{-(m_1-1)} \times$

Case of constant default threshold: bond value

- The bond value $B(V,t) = V - S(V,t)$ will be given by

$$B(V,t) = De^{-rT}\mathcal{N}(d_2) + V\mathcal{N}(-d_1) + \left(\frac{V}{K_1}\right)^{-(m_1-1)} C\left(\frac{K_1^2}{V}, t, D\right) \tag{6}$$

where d_1 , d_2 are usual arguments in BS

$$d_1 = \frac{\ln(V/D) + \left(r + \sigma_V^2/2\right)T}{\sigma_V\sqrt{T}}, \quad d_2 = \frac{\ln(V/D) + \left(r - \sigma_V^2/2\right)T}{\sigma_V\sqrt{T}}. \tag{7}$$

- It can be checked that at the boundary $V = K_1$ we satisfy the condition 4

$$B(K_1, t) = K_1$$

Default Probability for constant default threshold

1. The survival probability of the firm is given by

$$\mathbb{P}(0,T) = \mathcal{Q}(\tau > T) = \mathcal{Q}(\overline{\tau} = \infty, V_T \geq D) \tag{8}$$

2. To calculate the survival probability we can consider a digital down-and-out call option, which pays at expiry one dollar if
   - The firm value never falls below the safety covenant K_1
   - At expiry T, the firm value $V \geq D$

   which replicates exactly the definition 8
3. Its valuation can be done in a similar fashion as it was done for a knock out call with replacement of payoff $\max(V_T - D, 0)$ by binary payoff $1_{V_T \geq D}$, and knock out feature at the barrier K_1. Using the image method, we get

$$\mathbb{P}(0,T) = \mathcal{N}(d_2) - \left(\frac{V_0}{K_1}\right)^{-(m_1-1)} \mathcal{N}(\tilde{d}_2) \tag{9}$$

$$\tilde{d}_2 = \frac{\ln(\frac{K_1^2}{V_0 D}) + (r - \sigma_V^2/2)T}{\sigma_V \sqrt{T}}$$

   and d_2 is given in 7

Moving default threshold

In case the safety covenant is time dependent, things become more complicated. However, for our specific choice $v_t = Ke^{-r(T-t)}$ the same approach works and gives a closed form solution. We will indicate only the key steps, but not the whole derivation. [1]

1. We consider the same PDE 1, 2 for equity value S, but with a time dependent boundary :

$$S(Ke^{-r(T-t)}, t) = 0$$

2. In order to make boundary time independent, we can change variable as

$$x = \ln\left(\frac{V}{v_t}\right), u(x(V), t) = S(V, t)$$

which will change the PDE and the boundary conditions:

$$u(x, T) = \max(e^x v_T - D, 0), u(0, t) = 0$$

[1] The problem can be also solved by using the First Passage Time ("FPM") models. Both derivations would be lengthy. I choose PDE

Next Key Steps

As before, the goal is to reduce the problem to the classical problem of heat equation. As indicated it works out in our case (but not in general). [2]
We introduce functions of time $a(t)$, $b(t)$

$$u(x,t) = U(x,t)e^{a(t)x+b(t)}$$

We also change time to

$$\tau = 1/2\sigma_V^2(T-t)$$

With the "right" $a(t)$ and $b(t)$ we reduce the problem to

$$\boxed{\begin{aligned} \frac{\partial U}{\partial \tau} &= \frac{\partial^2 U}{\partial x^2} && (10) \\ U(x,0) &= \max e^{-a(T)x}\left(e^x v_T - D, 0\right), \mathsf{U}(0,\tau) = 0 && (11) \end{aligned}}$$

[2] Note that both firm value V and boundary v_t "grow" with risk free rate r

Appendix: PDE approach

Reducing the problem to the heat equation

The main idea is to transform the BS equation 1 to the classical heat equation and solve a boundary value problem.

- We use the change of variables:

$$V = De^{x}, t = T - \frac{\tau}{\frac{1}{2}\sigma_V^2}, c(t,V) = Dv(x,\tau), v = e^{\alpha x + \beta\tau}u(\tau,x)$$

$$\alpha = -\frac{1}{2}(m_1 - 1), \beta = -\frac{1}{4}(m_1+1)^2, m_1 = \frac{r}{\frac{1}{2}\sigma^2} \tag{12}$$

- In the new variables the barrier transforms to $x_0 = \log(K_1/D)$ and the problem becomes

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{13}$$

$$u(x,0) = u_0(x) = \max\left(e^{\frac{1}{2}(m_1+1)x} - e^{\frac{1}{2}(m_1-1)x}, 0\right) = u_0(x), x \geq x_0 \tag{14}$$

$$u(x_0,t) = 0 \tag{15}$$

Fundamental Solution of the Heat Equation

Consider the problem

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty \tag{16}$$

with initial condition

$$u(x,0) = u_0(x)$$

$$u \to 0, \text{ as } x \to \pm\infty$$

A special solution of the equation 17 is the following one:

$$u_\delta(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-\frac{x^2}{4\tau}} \tag{17}$$

It satisfies the heat equation 17 (check it) . For $\tau > 0$ it is a smooth Gaussian curve, but at $\tau = 0$ it is so called *delta* function: it vanishes for $x \neq 0$ and at $x = 0$ it is infinite, and its integral is still one.

Solving the heat equation

1. Since we can represent the initial condition as

$$u_0(x) = \int_{-\infty}^{\infty} u_0(\xi)\delta(\xi - x)d\xi$$

   where $\delta(.)$ is the Dirac delta function (see Appendix).

2. The fundamental solution 18:

$$u_\delta(s - x, \tau) = u_\delta(x - s, \tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-\frac{(x-s)^2}{4\tau}}$$

   is a solution of 17 (in x or s), and its initial value is

$$u_\delta(s - x, 0) = \delta(s - x)$$

3. Thus for each s the function

$$u_0(s)\ u_\delta(s - x, 0) \tag{18}$$

   satisfies 17 and has initial condition $u_0(s)\delta(s - x)$.

Superposition Principle

Since the diffusion equation is linear, we can superpose solutions 19:

The initial temperature is decomposed into a continuum of impulses 19 at each point. The resulting temperature is found by integrating them:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds \tag{19}$$

which has initial condition

$$u(x,0) = \int_{-\infty}^{\infty} u_0(s)\delta(s-x)ds = u_0(x)$$

Method of Images for Knock-Out

1. In the method of images, we solve a semi-infinite problem by first solving an infinite problem made up of two semi-infinite problems with equal and opposite temperature distributions: one half is "cold" and another one is hot.
2. We can apply this method to the barrier option problem. We reflect the initial data about the point x_0, at the same time changing its sign. Thus, instead of solving the problem 14 - 16 on the interval $x_0 < x < \infty$, we solve 14 for all x but subject to

$$u(x,0) = u_0(x) - u_0(2x_0 - x)$$

$$u(x,0) = \begin{cases} \max\left(e^{\frac{1}{2}(m_1+1)x} - e^{\frac{1}{2}(m_1-1)x}, 0\right) & \text{for } x > x_0 \\ -\max\left(e^{(m_1+1)(x_0-\frac{1}{2}x)} - e^{(m_1-1)(x_0-\frac{1}{2}x)}, 0\right) & \text{if } x < x_0 \end{cases}$$

Pricing of a knock out option

1. The value of a call option (same strike and expiry, no barrier)
$$C(V,t) = De^{\alpha x + \beta\tau} u_1(x,\tau) \tag{20}$$
2. Next we write the solution to the barrier option value as
$$U(V,t) = De^{\alpha x + \beta\tau}\left(u_1(x,\tau) + u_2(x,\tau)\right)$$
$u_2(x,\tau)$ is the solution with antisymmetric initial data.
3. $u_2(x,\tau)$ can be found in terms of u_1
$$u_2(x,\tau) = -u_1(2x_0 - x, \tau)$$
4. Replacing x by $2x_0 - x$ is equivalent to replacing V by K_1^2/V.
5. Finally, making all the calculations we get
$$U(V,t) = C(V,t) - \left(\frac{V}{K_1}\right)^{-(m_1-1)} C(K_1^2/V, t)$$
6. The final condition is satisfied only for $V > K_1$, for $V < K_1$ the option is worthless

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