Problem 1 (16 points)

Consider a defaultable bond with maturity of three years , annual coupon $c = 0.05$, paid semi-annually and recovery 0.4. Assume the following:

a. Under the risk-neutral measure $\mathbb{Q}$, the default time τ follow the Weibull distribution with parameters $\lambda = 0.04$ and $p = 0.6$. Use continuous hazard rates.

b. The risk-free rate r (continuously compounding) is constant, $r = 0.03$

c. If default happens in $(T_{n-1}, T_n]$ the payment is made at T_n.

Work on the following questions:

1. Calculate the price of the defaultable bond (5 points).
2. Calculate the yield y of the bond (so that the discounted cashflows match the price). (6 points)
3. Define the coupon rate so that the bond trades at par (value 1). (5 points).

See Excel.

Question 2 (15 points)

Consider a Poisson process N_t with intensity λ.

- Let T_1 be the r.v. which describes the time of the first jump, T_2 be the time between the 1st and the second jumps, ...
- Consider the "waiting time" until the n-th jump:

$$S_n = T_1 + T_2 + .. + T_n$$

Calculate

1. the expected value of S_n , $\mathbb{E}[S_n]$ (10 points)
2. $Corr(N_t, N_s)$ (correlation), $0 < s < t$. (5 points)

1. $S_n = \sum_{i=1}^{n} T_i = T_1 + T_2 + \cdots + T_n$, we need first calculate the distribution then expection. We try to get the density,

$$F_{W_n}(t) = \mathbb{P}(W_n \le t) \leftarrow \text{prob of } n\text{th arrival before time } t$$

$$= \mathbb{P}(\underbrace{X(t)}_{\text{\# of arrivals between 0 and t}} \ge n)$$

$$= \sum_{k=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

$$f_{W_n}(t) = \frac{d}{dt}\left[\sum_{k=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!}\right] \quad , \quad t \ge 0$$

$$= \sum_{k=n}^{\infty}\left[-\lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!} + e^{-\lambda t} \frac{k\lambda}{k!} (\lambda t)^{t-1}\right]$$

$$= \sum_{k=n}^{\infty}\left[-\lambda e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!} + \lambda e^{-\lambda t} \frac{1}{(k-1)!} (\lambda t)^{k-1}\right]$$

$$= -\lambda e^{-\lambda t} \cancel{\frac{(\lambda t)^n}{n!}} + \lambda e^{-\lambda t} \frac{1}{(n-1)!} (\lambda t)^{n-1} \qquad \lim \frac{t^k}{k!} = 0$$

$$- \lambda e^{-\lambda t} \cancel{\frac{(\lambda t)^{n+1}}{(n+1)!}} + \lambda e^{-\lambda t} \cancel{\frac{(\lambda t)^n}{n!}}$$

$$= \lambda e^{-\lambda t} \frac{1}{(n-1)!} (\lambda t)^{n-1}$$

$$= \frac{e^{-\lambda t} \lambda^n t^{n-1}}{(n-1)!}$$

$$\begin{cases} \frac{e^{-\lambda t} \lambda^n t^{n-1}}{(n-1)!} & t \geq 0 \\ 0 & t < 0 \end{cases}$$ arrival time follows gamma dist

Thus, the $f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \quad t \geq 0$

$$\mathbb{E}[S_n] = \int_0^{+\infty} t \times \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} = \frac{1}{\lambda (n-1)!} \int_0^{+\infty} e^{-x} x^n dx \ ,$$

this integral is a Gamma function with value $n!$.

Therefore, the final result is

$$\mathbb{E}[S_n] = \frac{n}{\lambda}.$$

2. $Cov(N_t, N_s) = \mathbb{E}[(N_s - \mathbb{E}[N_s])(N_t - \mathbb{E}[N_t])]$

for poisson process, $\mathbb{E}[N_t] = \lambda t$ and $\mathbb{E}[N_s] = \lambda s$

$\therefore \; Cov(N_t, N_s) = \mathbb{E}[(N_s - \lambda s)(N_t - \lambda t)]$

$= \mathbb{E}[N_s N_t] - \lambda s\,\mathbb{E}[N_t] - \lambda t\,\mathbb{E}[N_s] + \lambda^2 st$

$= \mathbb{E}[N_s N_t] - \lambda^2 st$

where $\mathbb{E}[N_s N_t] = \mathbb{E}[(N_t - N_s)]\mathbb{E}[(N_s - N_0)] + \mathbb{E}[N_s^2]$ $\quad (N_0 = 0)$

$= \lambda^2(t-s)s + \mathbb{E}[N_s^2] = \lambda^2 st - \lambda^2 s^2 + \mathbb{E}[N_s^2]$

$\therefore \; Cov(N_t, N_s) = -\lambda^2 s^2 + \mathbb{E}[N_s^2]$

$\because \; \mathbb{E}[X^2] = Var[X] - \mathbb{E}[X^2]$

$\therefore \; Cor(N_s, N_t) = -\lambda^2 s^2 + Var(N_s) - \mathbb{E}[N_s]^2 = -\lambda^2 s^2 + \lambda s + \lambda^2 s^2 = \lambda s$

If we have $t < s$, we would end up with λt, thus

$Cov(N_s, N_t) = \lambda \min\{s, t\}$

$$Cor(N_s, N_t) = \frac{Cov(N_s, N_t)}{\sqrt{Var(N_s)}\sqrt{Var(N_t)}} = \frac{\lambda s}{\sqrt{\lambda s}\sqrt{\lambda t}} = \frac{\lambda s}{\lambda\sqrt{st}} = \frac{s}{\sqrt{st}}$$

Question 3 (18 points)

Consider Merton model. Answer the following questions.

1. Derive an expression connecting the debt volatility σ_B and the asset volatility σ_V. Show details of your derivation (7 points)

2. Consider a particular case: the debt $D = 105$, the equity value $S = 52.18$, the maturity of debt $T = 2$, the risk free rate $r = 0.03$, the equity vol $\sigma_S = 0.5443$. Find:

   a. The asset value V, the asset volatility σ_V and the debt volatility σ_B (5 points)

   b. Calculate the probability that the recovery is between 30% and 60% (conditioned on default). (6 points)

1. Considering $B(0,T) = De^{-rT}N(d_2) + V_0N(-d_1)$. By Itô's Lemma

$$dB_T = \frac{\partial(D-P_0)}{\partial t}dt + \frac{\partial(D-P_0)}{\partial V}dV + \frac{1}{2}\frac{\partial^2(D-P_0)}{\partial V^2}\langle dV, dV\rangle$$

$$= \frac{\partial(D-P_0)}{\partial t}dt + \frac{\partial(D-P_0)}{\partial V}\sigma_V V\, dW_t$$

then $\frac{\partial(D-P_0)}{\partial V} = N(-d_1)$ $\qquad \sigma_V = \frac{S_0}{V_0N(d_1)}$

$\therefore \sigma_B = N(-d_1)\sigma_V\frac{V_0}{B_0}$ $\qquad S = VN(d_1) - De^{-rT}N(d_2)$ $\qquad \sigma_S = \frac{1}{S_0}N(d_1)\sigma_V V_0$

2. (a) $V = 150.00056$ $\quad \sigma_V = 0.19997$ $\quad \sigma_B \approx 0.015$ $\quad$ See python

(b) $\mathbb{P}(30\% \leq RR \leq 60\% \mid V_T < D)$

$$= \frac{\mathbb{P}(30\% \leq RR \leq 60\%,\ V_T < D)}{\mathbb{P}(V_T < D)}$$

$$= \frac{N(-d_2(\frac{V}{D}, 0.6, T)) - N(-d_2(\frac{V}{D}, 0.3, T))}{N(-d_2(\frac{V}{D}, 1, T))}$$

$= 0.0093$

Question 4 (10 points)

The spread between the yield on a five-year bond issued by a company and the yield on a similar risk-free bond is 100 basis points (b.p.). If the spread is 95 b.p. for a three-year bond, and the recovery rate is 30%, what is the average hazard rate in years 4 and 5?

Hazard rate for 3 year bond = Spread / (1 − RR)

= 0.0095 / (1 − 0.3)

= 0.01357

Hazard rate for 5 year bond = spread / (1 − RR)

= 0.01 / (1 − 0.3)

= 0.01429

For 4 and 5 average: $\frac{0.01429 \times 5 - 0.01357 \times 3}{5-3} = 0.01536$

Question 5 (10 points)

"The position of a buyer of a credit default swap is similar to the position of someone who is long a risk-free bond and short a corporate bond". Give the rationale of the statement.

CDS: It is a derivative contract that allows investor to swap his/her credit risk with other invertor. It is used to transfer the credit risk between two or more parties. In this buyer of the swap pays to seller of the swap until the maturity data of contract. So, the credit default swap insures a corporate bond, issued by the company against default. CDS mitigates the risk by converting corporate bond into risk free bonds. Buyer of a CDS choosese to exchange a corporate bond for a risk free bond that is why the buyer is long with a risk free bond and short with a corporate bond.

Problem 6 (16 points)

Consider Merton model under jump diffusion. Assume the following :

- The asset value $V_0 = 100$, the maturity of debt $T = 1$, the volatility of diffusion part $\sigma = 0.2$, the risk free rate $r = 0.03$.
- The jump component: the average number of jumps per year is $\lambda = 0.4$, the parameters of the normal distribution (ν, δ) of jumps of log asset price are $\nu = -0.04$, $\delta = 0.3$.

Calculate the equity price and the implied BS volatility (i.e. the BS vol implied by the jump diffusion call price), for the cases

1. $D = 55$ (5 points)
2. $D = 80$. (5 points)
3. For either case calculate the sensitivity (delta) of the equity value w.r.t V_0 $\frac{\partial S}{\partial V}$. (6 points)

1. price = 46.787 implied vol $\approx$ 0.31

2. price = 24.015 implied vol $\approx$ 0.24

See python

3. $D=55.$

$$\frac{S(V_0+\frac{1}{2}\Delta V)-S(V_0-\frac{1}{2}\Delta V)}{\Delta V}, \Delta V = 0.1 = \frac{46.8906 - 46.7918}{\Delta V}$$

$$= 0.988$$

$$D=80 : = \frac{S(V_0+\frac{1}{2}\Delta V)-S(V_0-\frac{1}{2}\Delta V)}{\Delta V}, \Delta V = 0.1 = \frac{24.6815 - 24.5937}{\Delta V} = 0.878$$

Problem 7 (16 points)

A company has issued a four-year coupon bond.

- The bond has a coupon of 5% per year, paid semi-annually
- The yield on the bond is 6% per annum (with continuous compounding)
- Risk-free interest yield curve is flat at 4% (with continuous compounding).
- Defaults can take place at the end of each year (immediately before a coupon or principal payment) and the recovery rate is 30%.
- We assume that the unconditional probability of default per year is the same each year and equals Q.

Calculate Q based on the given information.

$Q \approx 2.7\%$. See Excel