Homework 8: Portfolio Credit Risk

Credit Risk (MF772) Fall 2021 Instructor: Roza Galeeva

Due date: 8 am, Thursday November 18. Please, note that `late assignments will not be accepted.`

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1. Beta Distribution as Mixing Distribution

Consider a portfolio with m obligators, m is sufficiently large. Let Z be a random variable from a Beta distribution with density $f_Z^{a,b}(z)$

$$f_Z^{a,b}(z) = \frac{1}{\beta(a,b)} z^{a-1}(1-z)^{b-1}, z \in [0,1]$$

$$\beta(a,b) = \int_0^1 z^{a-1}(1-z)^{b-1} dz$$

Let the probability of default of each obligator conditioned on Z be given by $p(Z) = Z$.

Questions

a) Find the value of a and b so that the expected probability of default for each obligator $\bar{p} = 0.04$ and the correlation ρ between indicators of defaults X_i and X_j is $\rho = 0.2$. Give details of the derivation.

b) For calibrated a and b and using a Large Portfolio Approximation (LPA) find the probabilities that the percentage loss $\frac{N_m}{m}$ does not exceed 0.1 and 0.2.

2. **Simulations of Mixed Binomial Merton Model**

Set up

The goal is to check the quality of LPA approximations for the Mixed Binomial Merton (Vasicek) model. The algorithm is described on the third page. We consider the following inputs:

a. For all cases, the individual default probability for all obligators $\bar{p} = \mathbb{E}(p(Z))$ is fixed, $\bar{p} = 0.05$.

b. The number of obligators $m = 100$ and $m = 1000$

c. The parameter ρ in the Vasicek model takes values $\rho = 0.1$ and $\rho = 0.5$ [1]

d. The argument x takes values $x = [0.05, 0.15, 0.25...0.85]$

Questions

- For each combination of m and ρ (4 combinations) and for each x calculate the probability that $F_{sim}(x) = \mathbb{P}(L_m \leq x)$, where is percentage of defaults, $L_m = \frac{N_m}{m}$ by simulations and compare it with the LPA approximation $F(x) = \mathbb{P}(p(Z) \leq x)$.
- Plot $F_{sim}(x)$ and $F(x)$.

3. **Mapping parameter ρ in the Vasicek model to correlation of defaults, Optional**

Let the default probability $\bar{p} = 0.02$ and $\rho = 0.45$. Find the correlation between the indicators of defaults in the mixed binomial model. You will have to do it by numerical integration.

[1] Note, that this is not the correlation between indicators of defaults in the mixed binomial model

Simulations Algorithm

For fixed $x = x_k$ calculate the probability $F_{sim}(x_k) = \mathbb{P}(L_m \le x)$:

1. Draw standard normal Z

2. Calculate $p(Z)$

3. Simulate m uniform random numbers U_1,....U_m in $[0, 1]$

4. Define default indicators: $X_i = 1$ if $U_i \le p(Z)$ and 0 otherwise

5. Calculate number of defaults $N_m^{(k)} = \sum_{i=1}^m X_i$

6. Record $y_1^{(k)} = 1$ if $\frac{N_m^{(k)}}{m} \le x_k$ and 0 otherwise

7. Repeat all steps, record $y_2^{(k)}$, etc N times, N should be at least $10,000$

The average of $y_n^{(k)}$ is the desired probability $F_{sim}(x_k)$.