Credit Risk: Portfolio Credit Risk: Vasicek Model

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Randomization of default probability

- Randomize the default probability in the standard binomial model,
- The economic intuition behind this randomization of the default probability $p(Z)$ is that Z should be a common background variable affecting all obligors in the portfolio.

Definition 1

Let Z *be a random variable with density function* $f_Z(z)$ *and, let* $p(Z) \in [0,1]$ *be a random variable with distribution function* F *and mean* $\overline{p}$. *That is,*

$$F(x) := \mathcal{P}(p(Z) \leq x),$$

$$\overline{p} := E[p(Z)] = \int_{-\infty}^{\infty} p(z) f_Z(z) dz.$$

The variable Z is called **mixing variable**,

Mixed binomial model

Suppose that

1. Z, $p(Z)$ and F are defined as above,
2. conditional on Z, each obligator i has default probability $p(Z)$, that is,

$$\mathcal{P}(X_i = 1 \mid Z) = p(Z) \quad \text{and} \quad \mathcal{P}(X_i = 0 \mid Z) = 1 - p(Z);$$

3. conditional on Z, the indicator random variables $X_1, X_2, \cdots, X_m$ are i.i.d.

Then we say that $X_1, \cdots, X_m$ define a mixed binomial model with mixing variable Z and distribution F.

Remark 1

Given Z, the random variables $X_1, \cdots, X_m$ are i.i.d, $Ber(p(Z))$, and hence $N_m := \sum_{i=1}^{m} X_i \sim Bin(m; p(Z))$.

Properties

Proposition 2

In a mixed binomial model, we have

$$(i) \qquad \mathcal{P}(X_i = 1) = \overline{p} = E[X_i],$$

$$(ii) \qquad \mathbf{V}ar[X_i] = \overline{p}(1-\overline{p}),$$

$$(iii) \qquad \mathbf{C}ov[X_i, X_j] = E[p(Z)^2] - \overline{p}^2 = \mathbf{V}ar[p(Z)] > 0, \quad i \neq j.$$

$$(iv) \qquad \rho_{i,j} := \mathbf{C}orr[X_i, X_j] = \frac{E[p(Z)^2] - \overline{p}^2}{\overline{p}(1-\overline{p})} > 0, \qquad i \neq j.$$

Proof (Sketch).

From (i), X_i is $\mathrm{Bernoulli}\,(\overline{p})$. Hence (ii) follows. To show (iii)

$$\begin{aligned}\mathbf{C}ov[X_i, X_j] &= E[E[X_i X_j \mid Z]] - E[X_i]E[X_j] \\ &= E[E[X_i \mid Z]E[X_j \mid Z]] - \overline{p}^2 \\ &= E[p(Z)p(Z)] - \overline{p}^2, \quad i \neq j.\end{aligned}$$

Finite Portfolio Distribution

In a mixed binomial model, we have

$$(i) \qquad \mathcal{P}(L_m = k \cdot l \mid Z) = \binom{m}{k} p(Z)^k \cdot (1 - p(Z))^{m-k},$$
$$k \in \{0, 1, 2, \cdots, m\},$$

$$(ii) \qquad \mathcal{P}\big(N_m = k\big) = \int_{-\infty}^{\infty} \binom{m}{k} p(z)^k \cdot (1 - p(z))^{m-k} f_Z(z) dz,$$
$$k \in \{0, 1, 2, \cdots, m\},$$

$$(iii) \qquad \mathbf{V}ar[N_m] = m\mathbf{V}ar[X_1] + m(m-1)\mathbf{C}ov(X_1, X_2)$$

N_m: # of defaults

$$= m\overline{p}(1 - \overline{p}) + m(m-1)\left(E[p(Z)^2] - \overline{p}^2\right),$$

$$(iv) \qquad \mathbf{V}ar\left(\underbrace{\frac{N_m}{m}}_{\text{proportion of default}}\right) \to \left(E[p(Z)^2] - \overline{p}^2\right), \qquad m \to \infty.$$

Examples of mixing distributions

The only requirement of mixing distribution is that $p(Z) \in [0,1]$, with Z some r.v. Popular examples include

1. Beta distribution: $p(Z) = Z$, and Z is from beta distribution $\beta(a,b)$ with density

$$f_Z(z) = \frac{1}{\beta(a,b)} z^{a-1}(1-z)^{b-1}, z \in [0,1]$$

$$\beta(a,b) = \int_0^1 z^{a-1}(1-z)^{b-1} dz$$

2. Logit-normal distribution :

$$p(Z) = \frac{1}{1+e^{-(\mu+\sigma Z)}}$$

and Z is standard normal.

Mixed binomial Merton model (Vasicek (1987))

Consider a homogeneous credit portfolio model with m obligors.

Assumption 1

Each obligor i (think of a firm named i) follows the Merton's model, in the sense that the value of the assets $V_{t,i}$ of obligor i follows the dynamics

$$dV_{t,i} \ = \ \mu V_{t,i} dt \ + \ \sigma V_{t,i} dB_{t,i}, \quad V_{0,i} = V_0 > 0,$$

where

$$B_{t,i} \ := \ \sqrt{\rho}\, W_{t,0} \ + \ \sqrt{1-\rho}\, W_{t,i} \quad \rho \in (0,1).$$

Here $W_{t,0}, W_{t,1}, \cdots, W_{t,m}$ are independent standard Brownian motions.

Note that $B_{t,i}$ is a standard Brownian motion, and hence $V_{t,i}$ is a geometric Brownian motion.

The idea

Remark 1

The intuition behind assumption is that the asset of each obligor i is driven by a systematic risk factor $W_{t,0}$ representing the state of the business cycle, and idiosyncratic factor $W_{t,i}$ specific to each firm.

Proposition 3

For $i \neq j$,

$$\mathbf{C}orr(B_{t,i}, B_{t,j}) = \rho$$

Definition 2

Let the principal value of debt $D_i = D$. For $i = 1, 2, \cdots, m$, we define

$$X_i := \mathbf{1}_{\{V_{T,i} < D\}}$$

Mixed Binomial

Proposition 4

Let Z be a standard normally distributed random variable. Define

$$p(Z) := N\left(\frac{-C - \sqrt{\rho}\,Z}{\sqrt{1-\rho}}\right),$$

$$\bar{p} := \mathbb{E}(p(Z)) = N(-C)$$

$$F(x) := N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\,N^{-1}(x) + C\right)\right)$$

where

$$C := \frac{\ln(V_0/D) + (\mu - 1/2\sigma^2)T}{\sigma\sqrt{T}}.$$

Then $X_1, \cdots, X_m$ determines a mixed binomial model with mixing variable Z and distribution F.

Sketch of derivation of the first equation

1. Solving SDE, we get the following expression for the value $V_{t,i}$:

$$V_{t,i} = V_0 e^{[(\mu - 1/2\sigma^2)T + \sigma B_{t,i}]}$$

2. The default happens if at expiry T the value falls below the debt:

$$V_{T,i} < D$$

3. Introduce $Y_0 = \frac{W_{T,0}}{\sqrt{T}}$ and $Y_i = \frac{W_{T,i}}{\sqrt{T}}$, $Y_0 \sim N(0,1)$, $Y_i \sim N(0,1)$

4. Solving for Y_i and using the definition

$$V_{T,i} = V_0 e^{[\mu - \frac{1}{2}\sigma^2]T + \sigma\sqrt{T}[\sqrt{\rho} Z + \sqrt{1-\rho} Y_i]} < D$$

$$p(z) = \mathbb{P}((X_i = 1)|Z) = \mathbb{P}(V_{T,I} < D)$$

$$(\mu - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}(\sqrt{\rho} Z + \sqrt{1-\rho} Y_i) < -\ln\frac{V_0}{D}$$

leads us to the first expression.

$$Y_i < \frac{-\ln\frac{V}{D} - [\mu - \frac{1}{2}\sigma^2]T - \sigma\sqrt{T}\sqrt{\rho} Z}{\sqrt{1-\rho}\,\sigma\sqrt{T}} = y^* = \frac{-C - \sqrt{\rho} Z}{\sqrt{1-\rho}}$$

Derivation of the second equation

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y^*} e^{-\frac{1}{2}x^2}dx = N[y^*]$

(i) Expected default $\bar{p}$: The condition of default $V_{T,i} < D$ can be equivalently written as

$$\rho Z + \sqrt{1-\rho}Y_i \equiv X_i < -C$$

$\bar{p} = N(-C)$

where X_i is standard normal. The statement follows. C can be written through $\bar{p}$: [1]

$$C = -N^{-1}(\bar{p})$$

(ii) We can work with $\bar{p}$ and rewrite $p(Z)$ as

$$p(Z) = N\left(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right)$$

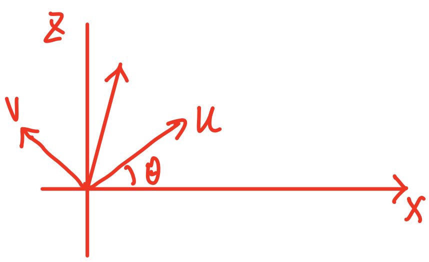
[1] Another (longer) way is through direct calculation of $\mathbb{E}(p(Z))$.

$$\bar{P} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P(z)\, e^{-\frac{1}{2}z^2}\, dz$$

$$\frac{B_{t,i}}{\sqrt{T}} = X_i \in \mathcal{N}(0,1)$$

$$\sqrt{\rho}\, W_{t,1} + \sqrt{1-\rho}\, W_{t,2} = W_t$$

$$\bar{P} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y^*} e^{-\frac{1}{2}y^2}\, dy\; e^{-\frac{1}{2}z^2}\, dz \qquad y^* = \frac{-C - \sqrt{\rho} z}{\sqrt{1-\rho}}$$



$\sqrt{\rho} = \cos\theta$

$u = \cos\theta\, x + \sin y$

$v = -\sin\theta\, x + \cos y$

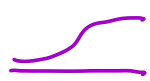
$u = \sqrt{\rho} z + \sqrt{1-\rho}\, y$

$z^2 + y^2 = u^2 + v^2$

$\Rightarrow N(-C)$

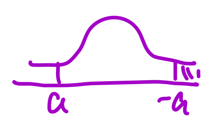
$$\begin{bmatrix} \cos\theta & \sin y \\ -\sin y & \cos\theta \end{bmatrix} = 1$$

$$F(x) = \mathbb{P}(P(x) < x) = \mathcal{N}\left[\frac{\sqrt{1-\rho}\, \mathcal{N}^{-1}(x) + C}{\sqrt{\rho}}\right]$$



$$\mathcal{N}\left[\frac{-C - \sqrt{\rho} z}{\sqrt{1-\rho}}\right] \leq x \quad \Rightarrow \quad \frac{-C - \sqrt{\rho} z}{\sqrt{1-\rho}} \leq \mathcal{N}^{-1}(x)$$

$$\Rightarrow -z \leq \frac{\sqrt{1-\rho}\, \mathcal{N}^{-1}(x) + C}{\sqrt{\rho}}$$



$z \geq -a$

$L_T = e \cdot m, \quad L = N_m \cdot e \cdot \underset{1}{\underbrace{LGD}}$

$\frac{N_m}{m} = \#$ of loss

$P[L_m \leq x]$

Large Portfolio Approximation

We derive the closed form approximation for a loss distribution and VaR value in the case when the number of obligators m is large.[2]. *It is widely used by financial institutions to manage risk in large credit portfolios, as well as for Basel calculations.*

1. Define $F(x)$ as

$$F(x) = \mathbb{P}[p(Z) \le x] = \mathbb{P}\left[N\left(\frac{-C - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \le x\right]$$

2. Next, we can rewrite $F(x)$ as

$$F(x) = \mathbb{P}\left[-Z \le \frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) + C\right)\right]$$

[2] see Vasicek "Limiting Loan Distribution", KMV Working Paper, 1991

Loss Distribution

3. Using the last expression we arrive at the following formula with $C = -N^{-1}(\bar{p})$:

$$F(x) = N\left[\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(x) + C\right)\right]$$

4. Now for large number of obligators we have

$$\mathbb{P}\left(\lim_{m\to\infty}\frac{N_m}{m} = p(Z)\right) = 1$$

therefore number of defaults $N_m \approx mp(Z)$.

5. We can derive the approximate distribution of the percentage portfolio loss $L_m = N_m/m$ for large m [3]

$$\mathbb{P}(L_m \leq X) \approx N\left[\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}(X) - N^{-1}(\bar{p})\right)\right] \qquad (1)$$

[3] For simplicity we assume that LGD =loss given default is 100%

Calculation of VaR

$\mathcal{N}\left[\frac{1}{\sqrt{\rho}}\left[\sqrt{1-\rho}\,\mathcal{N}^{-1}(x)-\mathcal{N}^{-1}(\bar{p})\right]\right]=\alpha \Rightarrow \frac{1}{\sqrt{\rho}}\left[\sqrt{1-\rho}\,\mathcal{N}^{-1}(x)-\mathcal{N}^{-1}(\bar{p})\right]=\mathcal{N}(\alpha)$

$\mathcal{N}^{-1}(x)=\frac{\sqrt{\rho}\,\mathcal{N}^{-1}(\alpha)+\mathcal{N}^{-1}(\bar{p})}{\sqrt{1-\rho}}$ $\quad X=\mathcal{N}\left[\frac{\sqrt{\rho}\,\mathcal{N}^{-1}(\alpha)+\mathcal{N}^{-1}(\bar{p})}{\sqrt{1-\rho}}\right]$

Using the derived approximation for loss distribution we can find a closed form solution for VaR:

6. For a given confidence level α for VaR, we have

$$\mathbb{P}(L_m \le X) = \alpha$$

7. Using the loss distribution 1 we can find VaR for the total loss LT (in dollars):

$$VaR_\alpha(LT) = lmN\left[\frac{\sqrt{\rho}N^{-1}(\alpha) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right]$$

where (l is the exposure per obligator) [4]

8. If LGD is not 100%, in the distribution of $\%$ losses, we will need to replace X by X/LGD.

[4] In previous slide, we dealt with percentage of the total loss LT