Jump Diffusion Structural Models

Roza Galeeva
e-mail: groza@bu.edu

Department of Finance
Questrom School of Business

FALL 2021

Introduction

Drawbacks of structural models based on diffusion process

- One of the common assumptions of considered models Merton, Black-Cox is that the evolution of firm value follows a diffusion process.
- Under a diffusion process, because a sudden drop in firm value is impossible, firms never default unexpectedly (by "surprise")
- If firm cannot default unexpectedly and if it is not currently in financial distress, its probability of defaulting on very short debt is practically zero, therefore its short-term debt should have zero credit spreads Moreover, credit spreads should slope upward at the short end.
- However, reality can be different: credit spreads on short term bonds are not zero, they can be flat or even downward slopping
- Reduced form models treat default as unpredictable Poisson event involving a sudden loss in market value. However, there is no link between firm value and corporate default.

Jump diffusion approach

- In summary, a reduced form approach is usually more flexible to fit the observed credit spreads, while structural models generates more conceptual insight on default behavior.
- Of the solutions to combine advantages of both, is to valuing risky debt by modeling the evolution of firm value as a jump diffusion process
- Under a jump diffusion process, a default can happen expectedly because of slow but steady decline in firm value. A default can also happen unexpectedly because of sudden drop in firm value.
- Such flexibility allows different term shapes structure of credit spreads, Credit spreads on very short-term bonds of good quality firms can be larger than zero.
- We will consider a simplified version of structural models, with the same assumptions as in the classical Merton model, but based on jump diffusion evolution of asset values. The option pricing in jump diffusion model in fact was proposed by Merton himself in 1976. We will follow that model.

Poisson Processes

Poisson Processes in simple terms

A *Poisson* process describes the number of occurrences of a certain event before time t: such as

1. the number of cars arriving at a gas station until time t
2. the number of phone calls received on a certain day until time t
3. the number of visitors entering a museum on a certain day until time t
4. the number of earthquakes that occurred in Chile during the time interval $[0, t]$
5. the number of shocks in the stock market from the beginning of the year until time t
6. The number of twisters that might hit Alabama from the beginning of the century until time t

Poisson Process

Definition 1

A Poisson process is a stochastic process N_t, $t \geq 0$ which satisfies:

1. *The process starts at zero $N(0) = 0$*
2. *N_t has stationary independent increments* 独立增量
3. *The process is right continuous*
4. *The increments $N_t - N_s$ with $0 < s < t$ have a Poisson distribution with parameter $\lambda(t-s)$*

$$\mathbb{P}\left(N_t - N_s = k\right) = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}$$

Condition (4) implies that

$$\mathbb{P}\left(N_t - N_s = 1\right) = \lambda(t-s) + o(t-s)$$

$$\mathbb{P}\left(N_t - N_s \geq 2\right) = o(t-s), \lim_{h \to 0} o(h)/h = 0$$

It follows a jump of size 1 can occur in the infinitesimal interval dt with probability λdt, and zero jumps with probability $1 - \lambda dt$.

λ? derivative of probability of making a jump

$$\mathbb{P}(X(t+h) - X(t) = 1) = e^{-\lambda h} \frac{(\lambda h)^1}{1!}$$

$\sim poi(\lambda h)$

$$\lim_{h \to 0} \frac{\mathbb{P}(X(t+h) - X(t) = 1)}{h} = \lim_{h \to 0} \lambda e^{-\lambda h} = \lambda$$

$$\lim_{h \to 0} \frac{\mathbb{P}(X(t+h) - X(t) = 2)}{h} = \lim_{h \to 0} e^{-\lambda h} \frac{(\lambda^2 h)}{2!} = 0$$

jump more than 1 is 0

$$\mathbb{E}[N_t - N_s] = \sum_{k=1}^{\infty} k \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)} = \lambda(t-s)$$

$$\frac{\partial}{\partial \lambda} e^{\lambda(t-s)} = -(t-s) e^{\lambda(t-s)}$$

$$= \frac{\partial}{\partial \lambda} \sum \frac{\lambda^k (t-s)^k}{k!} = \sum \frac{k \lambda^{k-1} (t-s)^k}{k!}$$

$$\mathbb{E}[N_t - N_s] = \left[\frac{\partial}{\partial \lambda} \sum \frac{\lambda^k (t-s)^k}{k!}\right] \lambda \cdot e^{-\lambda(t-s)}$$

$$= e^{\lambda(t-s)} \lambda(t-s) e^{-\lambda(t-s)} = \lambda(t-s)$$

$$Var[N_t - N_s] = \sum k^2 \frac{\lambda^k (t-s)^k}{k!} = \lambda(t-s)$$

Properties of a Poisson process

1. Mean and Variance

$$\mathbb{E}(N_t - N_s) = \lambda(t-s), Var[N_t - N_s] = \lambda(t-s)$$

In particular,

$$\mathbb{E}[N_t] = \lambda t, Var[N_t] = \lambda t$$

2. Let N_t be $\mathcal{F}_t$ adapted. Then the process $M_t = N_t - \lambda t$ is $\mathcal{F}_t$-martingale.

$$\mathbb{E}[N_t|\mathcal{F}_s] = \mathbb{E}[N_s + (N_t - N_s)|\mathcal{F}_s] = N_s + \lambda(t-s)$$

We used the fact that N_s is $\mathcal{F}_s$ predictable and that the increment $N_t - N_s$ is independent of previous values of N_s therefore the information $\mathcal{F}_s$. Substracting λt yields

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s$$

Breakout room

Let $0 < s < t$. Since the increments are independent for $0 < s < t$ we have

$$\mathbb{E}[N_s N_t] = \mathbb{E}[(N_s - N_0)(N_t - N_s) + N_s^2] = \ldots$$

Calculate

1. $$\mathbb{E}[N_s N_t] = \lambda^2 st - \lambda^2 s^2 + \mathbb{E}[N_s^2]$$

2. $$Cov(N_s, N_t) = \lambda \min\{s, t\}$$

1. $Cov(N_t, N_s) = \mathbb{E}[(N_s - \mathbb{E}[N_s])(N_t - \mathbb{E}[N_t])]$

for poisson process, $\mathbb{E}[N_t] = \lambda t$ and $\mathbb{E}[N_s] = \lambda s$

$$\therefore \quad Cov(N_t, N_s) = \mathbb{E}[(N_s - \lambda s)(N_t - \lambda t)]$$
$$= \mathbb{E}[N_s N_t] - \lambda s \mathbb{E}[N_t] - \lambda t \mathbb{E}[N_s] + \lambda^2 st$$
$$= \mathbb{E}[N_s N_t] - \lambda^2 st$$

where $\mathbb{E}[N_s N_t] = \mathbb{E}[(N_t - N_s)]\mathbb{E}[(N_s - N_0)] + \mathbb{E}[N_s^2]$ $\quad (N_0 = 0)$

$$= \lambda^2 (t-s)s + \mathbb{E}[N_s^2] = \lambda^2 st - \lambda^2 s^2 + \mathbb{E}[N_s^2]$$

$$\therefore \quad Cov(N_t, N_s) = -\lambda^2 s^2 + \mathbb{E}[N_s^2]$$

$$\because \quad \mathbb{E}[X^2] = Var[X] - \mathbb{E}[X^2]$$

$$\therefore \quad Cov(N_s, N_t) = -\lambda^2 s^2 + Var(N_s) - \mathbb{E}[N_s]^2 = -\lambda^2 s^2 + \lambda s + \lambda^2 s^2 = \lambda s$$

If we have $t < s$, we would end up with λt, thus

$$Cov(N_s, N_t) = \lambda \min\{s, t\}$$

2. $Cor(N_s, N_t) = \dfrac{Cov(N_s, N_t)}{\sqrt{Var(N_s)}\sqrt{Var(N_t)}} = \dfrac{\lambda s}{\sqrt{\lambda s}\sqrt{\lambda t}} = \dfrac{\lambda s}{\lambda \sqrt{st}} = \dfrac{s}{\sqrt{st}}$

Interarrival times

- For each state of the world, ω the path $t \to N_t(\omega)$ is a step function that exhibits unit jumps. Each jump in the path corresponds to an occurrence of a new event.
- Let T_1 be the r.v. which describes the time of the first jump. Let T_2 be the time between the 1st and the second jumps... Let T_n be the times elapsed between $(n-1)$ and nth jumps.
- The r.v. T_n are called *interarrival times*

Proposition 1

The random times T_n are independent and exponentially distributed with mean

$$\mathbb{E}[T_n] = \frac{1}{\lambda}$$

Sketch of the proof

1. Events $T_1 > t$ and $N_t = 0$ are the same.

$$\mathbb{P}(T_1 > t) = \mathbb{P}[N_t = 0] = e^{-\lambda t}$$

   Hence the distribution function of T_1 is

$$\mathbb{P}(T_1 < t) = 1 - e^{-\lambda t}$$

   This is exponential distribution with $\mathbb{E}(T_1) = \frac{1}{\lambda}$.

2. Next, we need to show that r.v T_1 and T_2 are independent, so

$$\mathbb{P}(T_2 \leq t) = \mathbb{P}\left(T_2 \leq t | T_1 = s\right)$$

   which follows from the assumption of independence of the increments.

Merton Jump Diffusion Model for Asset Dynamics

Assumptions 1

1. Jumps are observed at random times $T_1 < T_2 < ...T_n$ with $T_1, T_2 - T_1$ drawn from an exponential distribution with intensity λ
2. The magnitudes of each log-asset jump is normally distributed with parameters (ν, δ^2)
3. Between jumps we assume a BS dynamics
4. The time of jumps, the magnitude of jumps and the Wiener process of the BS dynamics are independent

Therefore, we assume that the total change in the asset value can be of two types:

- Diffusion part : "normal" fluctuations in firm value due to gradual change in economical conditions and arriving of new information
- Jump component which describes sudden changes in firm value due to the arrival of new information which large effect on the firm value

Let $Y_1, Y_2, ...$ be a sequence of independent r.v. from a normal distribution with parameters (ν, δ^2).

Asset Dynamics

- Before the first jump , $t < T_1$

$$\frac{V(t)}{V(0)} = \exp[(\alpha - \sigma^2/2)t + \sigma W(t)]$$

- At the first jump $t = T_1$

jump part

$$\frac{V(t)}{V(0)} = \exp[(\alpha - \sigma^2/2)t + \sigma W(t) + Y_1]$$

- Between the first and the second jumps

$$\frac{V(t)}{V(0)} = \exp[(\alpha - \sigma^2/2)t + \sigma W(t) + Y_1]$$

...

- Up to time t we have accumulated $N(t)$ jumps :

$$\frac{V(t)}{V(0)} = \exp[(\alpha - \sigma^2/2)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i]$$

Risk Neutral Condition

a. We assume that the asset dynamics process follows the Merton model under $\mathbb{Q}$ the risk neutral probability. This implies

$$\mathbb{E}_{\mathbb{Q}}[V(t)] = V(0)e^{rt} \tag{1}$$

where r is the risk free rate. According to the Merton model

b.

$$V(t) = V(0)\exp[(\alpha - \sigma^2/2)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i] =$$

$$= V(0)e^{(\alpha-\sigma^2/2)t}e^{\sigma W(t)}e^{\sum_{i=1}^{N(t)} Y_i}$$

c. Taking expectations under $\mathbb{Q}$ and using the independence

$$\mathbb{E}_{\mathbb{Q}}[V(t)] = V(0)e^{(\alpha-\sigma^2/2)t}e^{\sigma^2/2t}\mathbb{E}_{\mathbb{Q}}\left(e^{\sum_{i=1}^{N(t)} Y_i}\right) \tag{2}$$

Derivations

(i) We start with calculating the last expectation in 2

$$\mathbb{E}_{\mathbb{Q}}\left(e^{\sum_{n=1}^{N(t)} Y_i}\right) = \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}}\left(1_{[N(t)=n]} e^{\sum_{i=1}^{n} Y_i}\right) =$$

$$= \sum_{n=0}^{\infty} \mathbb{E}_{\mathbb{Q}}[N(t) = n] \mathbb{E}_{\mathbb{Q}}\left(e^{\sum_{i=1}^{n} Y_i}\right)$$

(ii) Since $Y_1, ...Y_n$ are independent and drawn from the same normal distribution, we have

$$\mathbb{E}_{\mathbb{Q}} e^{Y_1} = e^{\nu+\delta^2/2}, \mathbb{E}_{\mathbb{Q}} e^{\sum_{i=1}^{n} Y_i} = e^{(\nu+\delta^2/2)n}$$

(iii) Since

$$\mathbb{E}_{\mathbb{Q}}[N(t) = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

(iv) After some simple manipulations, we will get:

$$\mathbb{E}_{\mathbb{Q}}\left(e^{\sum_{n=1}^{N(t)} Y_i}\right) = \exp[t\left(\lambda(e^{\nu+\delta^2/2} - 1)\right)]$$

Final condition

$\sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(e^{(r+\frac{1}{2}\sigma^2)}\right)^n$

$\mathbb{E}[e^{\sum Y_i}] = e^{-\lambda t} e^{\lambda t \cdot e^{r+\frac{1}{2}\sigma^2}} = \exp[\lambda t\, [e^{r+\frac{1}{2}\sigma^2} - 1]]$

$\exp(rt) = \exp[(\alpha - \frac{1}{2}\sigma^2)t + \frac{1}{2}\sigma^2 t + \lambda t(e^{r+\frac{1}{2}\sigma^2} - 1)] \to k$

Taking into account 2 and 1 , the risk neutral condition 1 becomes

$$\exp(rt) = \exp[t(\alpha - \sigma^2/2) + \sigma^2 t/2 + \lambda(e^{\nu+\delta^2/2} - 1)]$$

Denoting

$$k = e^{\nu+\delta^2/2} - 1 \tag{3}$$

we get

$$\alpha = r - \lambda k \tag{4}$$

Note, that for $\lambda = 0$ (no jumps), we recover the BS risk neutral evaluation condition

$$\alpha = r$$

Merton Jump Diffusion Model, MJD

Merton Jump Diffusion model under the risk neutral pricing is

$$V(t) = V(0)\exp[(r - \lambda k - \sigma^2/2)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i]$$

where:

1. r is the risk free rate
2. λ is the average number of jumps per unit of time (for example, a year)
3. σ is the volatility of the diffusion part
4. $W(t)$ is a Wiener process
5. $N(t)$ is a Poisson process with parameter λ
6. Y_i is the jump of the log-asset price, assumed to be normal with parameters (ν, δ^2)
7. $k = e^{\nu+\delta^2/2} - 1$

Option pricing under MJD

In Merton structural model we need to price equity at the maturity of the debt T as an European call option with strike =debt D.

a. The BS price of the option is given by

$$BS(V, D, T, r, \sigma) = V\mathcal{N}(d_1) - De^{-rT}\mathcal{N}(d_2)$$

b. We price according to the number of jumps in $[0, T]$. If we have n jumps in $[0, T]$ then

$$V_n(T) = V(0)\exp[(r - \lambda k - \sigma^2/2)T + \sigma W(T) + Y_1 + \ldots + Y_n]$$

c. Since $W(T)$ and the jumps are independent, the sum

$$\sigma W(T) + Y_1 + \ldots + Y_n$$

is normal with parameters $(n\nu, \underbrace{\sigma^2 T + n\delta^2}_{})$

$\sigma_n^2 T = \sigma^2 T + n\delta^2$

Merton's approach

$\Rightarrow \sigma_n^2 = \sigma^2 + \frac{n}{T}\delta^2$

Computing the option price as the expected reward under $\mathbb{Q}$, the option price (equity E) becomes

$$E = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(V(T) - D\right)^+ = \sum_{n=0}^{\infty} e^{-(\lambda+r)T}\frac{(\lambda T)^n}{n!}\mathbb{E}_{\mathbb{Q}}\left(V_n(T) - D\right)^+$$

Merton's proposal is to relate this price to the BS price, therefore one can define:

1. $$\sigma_n^2 = \sigma^2 + \frac{n}{T}\delta^2$$
2. $$r_n = r - \lambda k + (\nu + \delta^2/2)\frac{n}{T}$$

after some transformations, one can obtain:

$$E = \sum_{n=0}^{\infty} e^{-\lambda' T}\frac{(\lambda' T)^n}{n!} BS(V, D, T, r_n, \sigma_n) \tag{5}$$

$$\lambda' = \lambda(1 + k) = \lambda e^{\nu+\delta^2/2}$$

Survival probability

Proposition 2

Under the previous assumptions, the survival probability of the firm, within $[0, T]$, is given by

$$P(0,T) := \mathbb{E}_{\mathbb{Q}}(V_T \geq D) = \sum_{n=0}^{\infty} e^{-\lambda' T} \frac{(\lambda' T)^n}{n!} N(d_{2,n}),$$

where $d_{2,n}$ is given by the standard Black Scholes expression **mutadis mutandis**, *i.e. with corresponding r_n and σ_n*

Appendix: Exponential random variables

- In the study of continuous-time stochastic processes, the exponential distribution is usually used to model the time until something happens in the process.
- A positiive r.v. Y is said to follow an exponential distribution with parameter $\lambda > 0$ if it has a pdf given by

$$f(y) = \begin{cases} \lambda e^{-\lambda y} & \text{if } y > 0 \\ 0, & \text{othewise} \end{cases}$$

- The distribution function of Y is given by

$$\forall y \in [0, \infty), \mathbb{P}(Y \leq y) = 1 - \exp(-\lambda y)$$

- The exponential distribution has the following important property: If T is an exponential r.v. then it has "no memory"

$$t, s > 0, \mathbb{P}\left(T > t + s | T > t\right) = \frac{\exp(-\lambda(t+s))}{\exp(-\lambda t)} = \mathbb{P}(T > s) \quad (6)$$

- The exponential distribution is the only distribution with this property.

The Bernoulli process: a discrete version of Poisson process

- Classical Bernoulli scheme: "success" with probability p and "failure" with probability $q = 1 - p$. The probability of k successful events in a trial with n experiments is

$$b(k, n, p) = \binom{n}{k} p^k q^{n-k}$$

- For a big number of trials n and small p, the above becomes

$$b(k, n, p) \approx \frac{\lambda^k}{k!} e^{-\lambda}$$

- Suppose now we consider an interval $[0, t]$ and divide in n subintervals of length h. Suppose now each subinterval corresponds to an independent Bernoulli trial with probability of success being λh. If $B(t)$ is the number of successful events has approximately a Poisson distribution:

$$\mathbb{P}\left(B(t) = k\right) = \binom{n}{k} (\lambda h)^k (1 - \lambda h)^{n-k} \approx \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$