Due on 2021/11/11.

Problem 1
Beta Distribution as Mixing Distribution.

Solution:

1. (a)

$$\begin{aligned}
\mathbb{P}(N_m = k) &= \int_{-\infty}^{\infty} \binom{m}{k} p(z)^k (1-p(z))^{m-k} f_Z(z)\,dz \\
&= \int_0^1 \binom{m}{k} p(z)^k (1-p(z))^{m-k} \frac{z^{a-1}(1-z)^{b-1}}{\beta(a,b)}\,dz \\
&= \binom{m}{k}\int_0^1 \frac{z^{k+a-1}(1-z)^{m-k+b-1}}{\beta(a,b)}\,dz \\
&= \binom{m}{k} \frac{\beta(k+a,\, m-k+b)}{\beta(a,b)} \int_0^1 \frac{z^{k+a-1}(1-z)^{m-k+b-1}}{\beta(k+a,\, m-k+b)}\,dz \\
&= \binom{m}{k} \frac{\beta(k+a,\, m-k+b)}{\beta(a,b)}
\end{aligned}$$

$$\mathbb{E}[X_i] = \bar{p} = \frac{a}{a+b} \qquad \rho = \frac{\mathrm{Var}(Z)}{p(1-p)}$$

This implies

$$a = p(\rho^{-1} - 1) = 0.16$$
$$b = (1-p)(\rho^{-1} - 1) = 3.84$$

(b) We first show the LPA distribution:

We know that $\mathbb{P}\left(\lim_{m\to\infty} \frac{N_m}{m} = p(z)\right) = 1$, so

$$F(x) := \mathbb{P}(p(Z) \le x) = \mathbb{P}(Z < x) = \int_{-\infty}^{x} f_Z(z)\,dz = \frac{1}{\beta(a,b)} \int_0^x z^{a-1}(1-z)^{b-1}\,dz$$

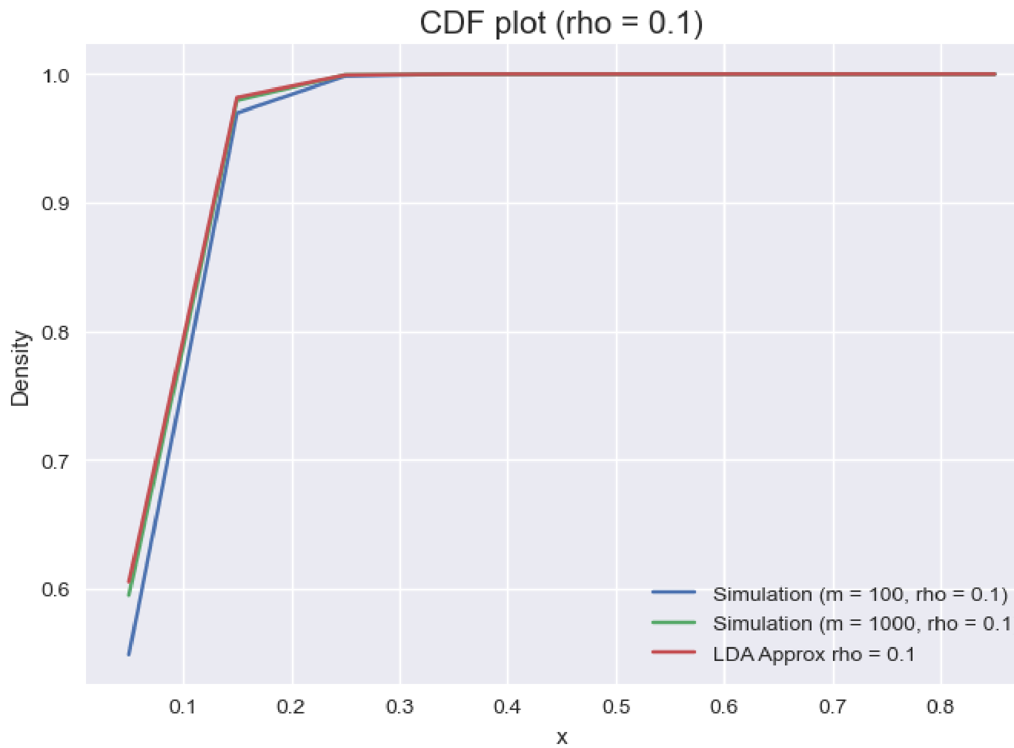
Thus,

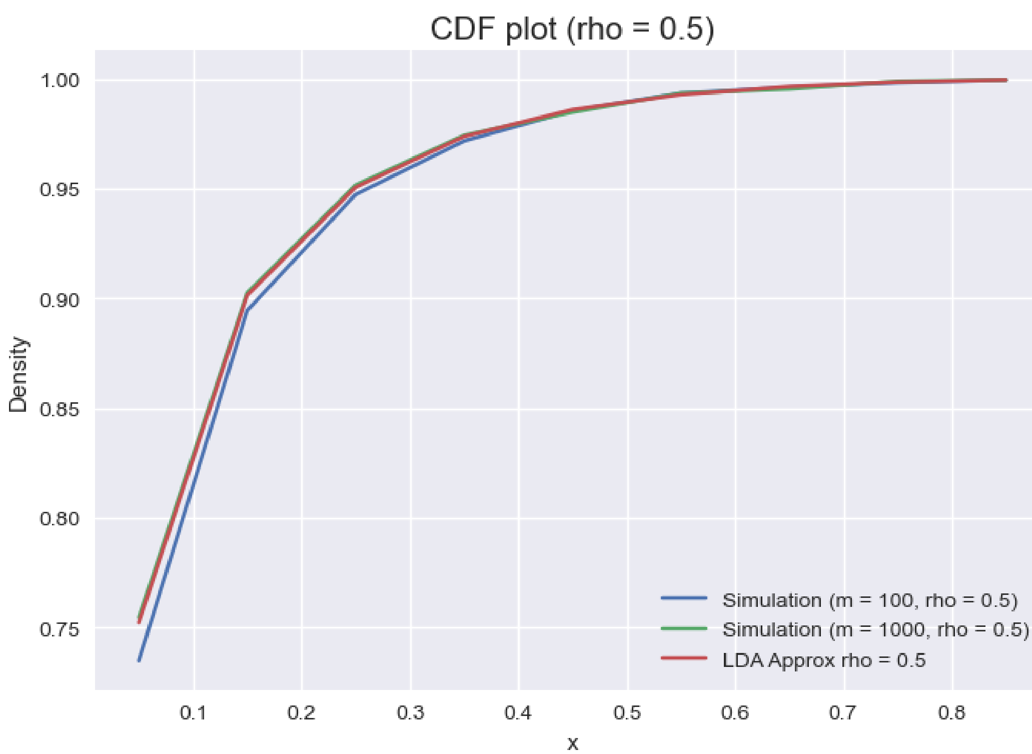
$$\mathbb{P}(p(Z) \le 0.1) = \frac{1}{\beta(a,b)} \int_0^{0.1} z^{0.16-1}(1-z)^{3.84-1}\,dz = \frac{4.207}{4.771} = 88.17\%$$

$$\mathbb{P}(p(Z) \le 0.2) = \frac{1}{\beta(a,b)} \int_0^{0.1} z^{0.16-1}(1-z)^{3.84-1}\,dz = \frac{4.578}{4.771} = 95.95\%$$

Problem 2
Simulations of Mixed Binomial Merton Model.

Solution: After exactly following the iteration procedure mentioned in class, the figures look like the following:


CDF plot (rho = 0.1)
Density
x
1.0
0.9
0.8
0.7
0.6
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Simulation (m = 100, rho = 0.1)
Simulation (m = 1000, rho = 0.1)
LDA Approx rho = 0.1


CDF plot (rho = 0.5)
Density
x
1.00
0.95
0.90
0.85
0.80
0.75
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
Simulation (m = 100, rho = 0.5)
Simulation (m = 1000, rho = 0.5)
LDA Approx rho = 0.5

Problem 3
Mapping parameter ρ in the Vasicek model to correlation of defaults, Optional

Solution:

3. $\bar{p} = 0.02$ $\rho = 0.45$

We already know that in Vasicek model:

$$p(z) = N\left(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}\, z}{\sqrt{1-\rho}}\right)$$

, and the formula for correlation is:

$$\hat{\rho} = Corr[X_i, X_j] = \frac{\mathbb{E}[p(z)^2] - \bar{p}^2}{\bar{p}(1-\bar{p})}$$

and by numerical integration, (See more detail in python)

$$\mathbb{E}[p(z)^2] = \int_{-\infty}^{+\infty} N^2\left(\frac{N^{-1}(\bar{p}) - \sqrt{\rho}\, z}{\sqrt{1-\rho}}\right) \phi(z)\, dz$$

We get $\hat{\rho} \approx 0.12$.

Part of the codes are shown below:

```
p = 0.02
rho = 0.45

def f(z, p = 0.02, rho = 0.45):
    d = (norm.ppf(p) - sqrt(rho) * z)
    n = sqrt(1 - rho)
    temp = pow(norm.cdf( d / n ), 2) *
    return temp

Epz2 = quad(f, -np.inf, np.inf)[0]

(Epz2 - p**2) / p * (1 - p)
```