Credit Risk: Portfolio Credit Risk

Roza Galeeva

e-mail: groza@bu.edu

Department of Finance

Questrom School of Business

FALL 2021

Outline

Today we start discussing the modeling of the portfolio credit risk

Main Learning Goals of the lecture:

- Define portfolio loss and risk measures , as VaR and Expected Shortfall and Economic Capital
- Define coherent risk measures
- Define default correlations
- Introduce Binomial Model for portfolio credit risk
- Binomial Expansion Model
- Discuss methodology to map given portfolio to a homogenous portfolio

Portfolio Loss

PDs, EADS, and LGDs

We consider a Bank with m obligors, a fixed time period $[0, T]$ and let τ_i be the (random) time of default for obligor i. The default indicator X_i is Bernoulli random variable defined by $X_i = \mathbf{1}_{\{\tau_i \leq T\}}$.

Definition 1

For each obligor i, the bank assigns

- *The* **Probability of Default** $PD_i := \mathcal{P}(X_i = 1)$.
- *The* **Loss Given Default** LGD_i*: the portion of loss that the bank will suffer in case of default.*
- *The* **Exposure At Default** EAD_i*: it is the quantity that specifies the exposure that the bank has to obligor i.*

Example

$EAD_i = \$10M$, $LGD_i = 20\%$, *then the potential loss is* $\$2M$.

Expected Loss

The loss of any obligor i is defined by a **loss variable**

$$Loss_i = EAD_i \times LGD_i \times X_i.$$

Definition 2

The **expected loss** *(EL_i) of an obligor i is the expectation of its loss variable. That is,*

$$EL_i = \mathbf{E}[Loss_i] = EAD_i \times LGD_i \times PD_i.$$

A bank covers the *expected losses* by charging them to its obligors in the form of risk premium. This happens in loan pricing where obligors pay premiums that are determined by its creditworthiness. With bonds the coupon payments are the implicit risk premiums.

Portfolio Loss

Definition 3

*The **portfolio loss** that considers only losses caused by default is*

$$Porfolio\ Loss\ = L = \sum_{i=1}^{M} Loss_i = \sum_{i=1}^{M} EAD_i \times LGD_i \times X_i.$$

The **expected loss of the portfolio** that considers only losses caused by default is then just the sum of the individual expected losses. That is,

$$EL = \sum_{i=1}^{M} \mathbf{E}[Loss_i].$$

Remark 1

In terms of the notation of previous chapters,

$$LGD_i\ = 1\ -\ \pi_i,$$

$$LGD_i \times EAD_i\ = EAD_i\ -\ R_i.$$

where π_i is the recovery rate i and R_i the recovery i.

The Value-at-Risk

To measure the full potential loss we need to consider the whole loss distribution and apply a variety of risk measures to this. The most popular risk measures like **Value-at-Risk** (VaR) and **Expected Shortfall** (ES) describe the right tail of the portfolio loss distribution.

Definition 4 (VaR)

Let $\alpha \in (0,1)$ *and* F_L *the distribution function of* L. *The* $VaR_\alpha(L)$ *is given by*

$$VaR_\alpha(L) \ = = \ \inf\{y \in \mathbb{R} \mid F_L(y) \geq \alpha\} \ = \ q_\alpha(L).$$

That is, the $VaR_\alpha(L)$ *is given by the smallest number* y *such that the probability that the losses does not exceed* y *is at least than* α. *Here* $q_\alpha(L)$ *is the quantile* α *of the portfolio loss. Therefore,* $VaR_\alpha(L)$ *represents the worst loss that could happen in probabilistic sense (with a given confidence)*

Economic Capital

The measure of risk that is most often used in an economic capital model of credit risk is the **economic capital** (EC).

Definition 5

The **economic capital** *is defined as*

$$EC_\alpha := VaR_\alpha(L) - \mathbf{E}[L].$$

The rationale is that because the expected loss is already priced into the business of lending, this measure gives a better indication of the **risk capital required** to ensure solvency over the time horizon with a given probability.

Coherent Risk Measures

Risk Measures are real valued functions defined on a linear space of r.v. $\mathcal{M}$, which includes constants. R.v could be future asset values of portfolios V, or they could be future losses $L = -(V - V_0)$, where V_0 is the current value of the asset value. We interpret the risk measure as the total amount of equity capital that is necessary to back a position with loss L, denoted as $\rho(L)$. A risk measure is call *coherent* if the following properties are true:

1. *Monotonicity: Positions leading to a higher loss in every state of the world, require higher capital*: For $L_1, L_2 \in \mathcal{M}$ such that $L_1 \le L_2$ almost surely, we have $\rho(L_1) \le \rho(L_2)$
2. *Translation Invariance* : $\rho(L + C) = \rho(L) + C$, where C is a real constant
3. *Positive Homogeneity* $\rho(\lambda L) = \lambda\rho(L)$, $\lambda > 0$
4. *Sub-additivity: a merger does not create extra risk:*

$$\rho(L_1 + L_2) \le \rho(L_1) + \rho(L_2)$$

Expected shortfall

Because VaR is not a coherent risk measure, [1], an alternative is a risk capital based on *Expected Shortfall* (ES) or *tail conditional expectation*.

Definition 6

Let $\alpha \in (0,1)$ *and* F_L *the distribution function of* L. *The* **Expected Shortfall** *is given by*

$$ES_\alpha(L) := \mathbf{E}[L \mid L \geq VaR_\alpha(L)].$$

This is also known as conditional VaR, denoted $CVaR$. It is the expectation of the losses that are greater than the VaR.

[1] The VaR risk measure is not subadditive.

Illustration of the risk quantities

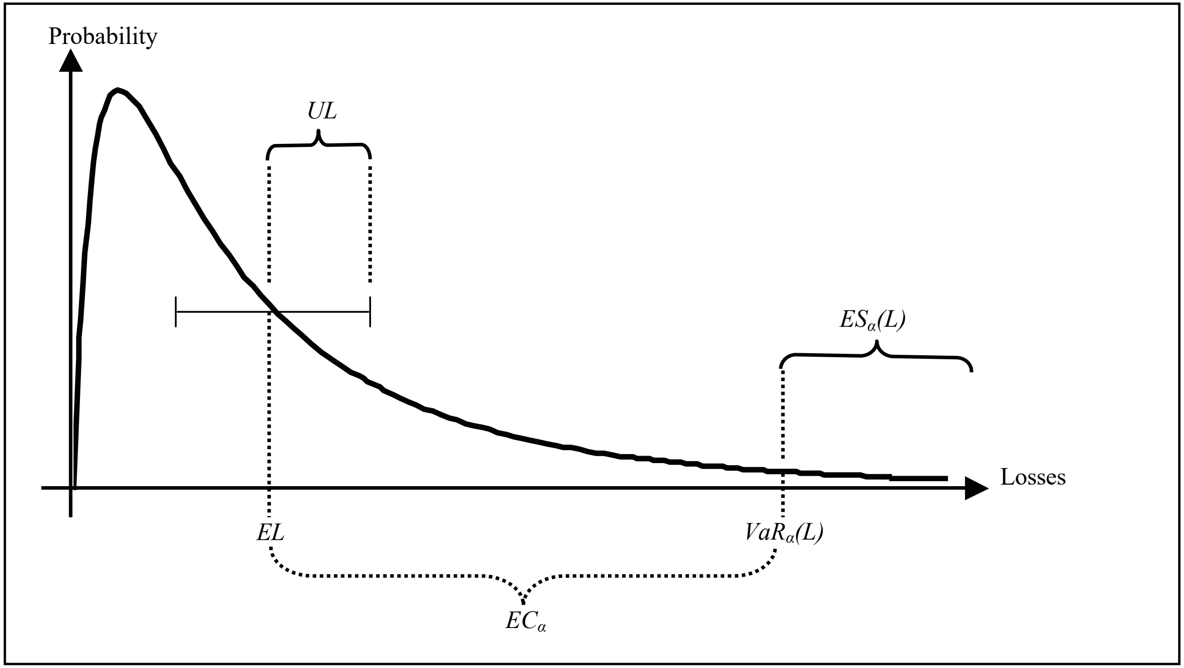


Figure 1: The portfolio Loss distribution

Breakout Room 1 question

Consider a portfolio of two defaultable bonds. Assume the following:

- Each bond could default with probability 4%. PD = 4%
- Defaults are independent $\perp\!\!\!\perp$
- In case of no default, each bonds pays $\$100$ and 0 otherwise (zero recovery) LGD = 100%

Question

1. Calculate 95% VaR and ES for each bond and for the total portfolio.
2. Check the property for sub-additivity for both measures

$$V_{t+\Delta}^{(A)} = \begin{cases} 0 & P = 0.04 \\ 100 & P = 0.96 \end{cases} \qquad V_{t+\Delta}^{(B)} = \begin{cases} 0 & P = 0.04 \\ 100 & P = 0.96 \end{cases}$$

$$L_{t+\Delta}^{(A)} = \begin{cases} 100 & P = 0.04 \\ 0 & P = 0.96 \end{cases} \qquad L_{t+\Delta}^{(B)} = \begin{cases} 100 & P = 0.04 \\ 0 & P = 0.96 \end{cases}$$

$$VaR_\alpha^{(A)} + VaR_\alpha^{(B)} = 0 + 0 = 0 \qquad VaR = \begin{cases} 100 & 0.96 \le u \le 1 \\ 0 & 0.95 \le u \le 0.96 \end{cases}$$

$$ES^{(B)} = ES^{(A)} = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\alpha(u)\,du = \frac{1}{1-0.95}(100 \times 0.04 + 0) = 20 \times 4 = 80$$

$ES(A) + ES(B) = 160$

$$V_{t+\Delta}^{combined} = \begin{cases} 0 & P = 0.04^2 \\ 100 & P = 0.96 \times 0.04 \\ 200 & P = 0.96^2 \end{cases} \qquad Loss^{combined} = \begin{cases} 200 & P = 0.04^2 \\ 100 & P = 0.96 \times 0.04 \times 2 \\ 0 & P = 0.96^2 = 0.9216 \end{cases}$$

$$VaR^{combined} = \begin{cases} 200 & 0.9984 \le u \le 1 \\ 100 & 0.95 \le u \le 0.9984 \\ 0 & 0.9216 \le u \le 0.95 \end{cases}$$

$$ES^{combined} = \frac{1}{1-0.95} VaR = 103.2$$

$103.2 = ES(A+B) \le ES(A) + ES(B) = 160$

$100 = VaR(A+B) \nleq VaR(A) + VaR(B) = 0$ ← not sub-additive

Default probabilities

Consider two obligors A and B.

Definition 7

The default probability for each obligor is:

$$p_A := \mathcal{P}(X_A = 1),$$

$$p_B := \mathcal{P}(X_B = 1).$$

Knowing the above two quantities is not yet sufficient to determine the following quantities.

$$p_{AB} := \mathcal{P}(X_B = 1,\ X_A = 1), \qquad \text{[joint default probability]}$$

$$p_{A|B} := \mathcal{P}(X_A = 1 \mid X_B = 1), \qquad \text{[conditional default probability]}$$

$$p_{B|A} := \mathcal{P}(X_B = 1 \mid X_A = 1). \qquad \text{[conditional default probability]}$$

Default correlation

Definition 8

The **linear default correlation** *between obligor A and B is defined as*

$$\rho_{AB} = \frac{\mathbf{C}ov[X_A\,,\,X_B]}{\sqrt{\mathbf{V}ar(X_A)\cdot\mathbf{V}ar[X_B]}}.$$

Given the above definition, we establish the relation between the different default probabilities.

Proposition 1

$$(i) \qquad \rho_{AB} = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1-p_A)p_B(1-p_B)}}\,,$$

$$(ii) \qquad p_{AB} = p_A p_B + \rho_{AB}\sqrt{p_A(1-p_A)p_B(1-p_B)}\,,$$

$$(iii) \qquad p_{A|B} = p_A + \rho_{AB}\sqrt{p_A(1-p_A)(1-p_B)/p_B}\,.$$

The impact of linear correlation

Then next result allows us to get an intuitive feeling for the size and effect of the linear correlation coefficient.

Proposition 2

Assuming $p_A = p_B = p$, *such that* $p << 1$ *very small; and* $\rho_{AB} >> 0$.

$$(i) \qquad p_{AB} \quad \approx \quad p \cdot \rho_{AB}\,,$$

$$(ii) \qquad p_{A|B} \quad \approx \quad p \;+\; \rho_{AB}.$$

We note that $p^2 \approx 0$ and $(1 - p) \approx 1$.

$$(i) \qquad p_{AB} \;=\; p^2 \;+\; p\rho_{AB} \quad \approx \quad p \cdot \rho_{AB}\,,$$

$$(ii) \qquad p_{A|B} \;=\; p \;+\; \rho_{AB}(1 - p) \quad \approx \quad p \;+\; \rho_{AB}.$$

Example: If $\rho_{AB} = 60\%$, $p = 1\%$, then $p_{AB} =?$, $p_{A|B} = 6?$. The joint default probability and the conditional default probability are dominated by the correlation coefficient ρ.

Binomial Model

Homogenous portfolio, Independent defaults

We consider a **homogeneous** credit portfolio model with m obligors and **independent defaults**. Without loss of generality we make the following assumptions about the portfolio.

1. We consider default and survival of a portfolio until a fixed horizon T. Interest rates are set to zero.
2. We have a portfolio of m exposures to m different obligators
3. The exposures are of identical size l and have identical recovery π
4. The defaults of the obligators happen independently of each other. Each obligator defaults with probability p before the time horizon T.

Bernoulli trials

Assumption 1

Let X_i be a random variable such that

$$X_i = 1 \quad \textit{if obligor } i \textit{ defaults before } T,$$

$$X_i = 0 \quad \textit{if obligor } i \textit{ survives up to time } T.$$

- *The random variables $X_1, X_2, \cdots, X_m$ are i.i.d.*
- *and $\mathcal{P}(X_i = 1) = p$, $\mathcal{P}(X_i = 0) = 1 - p$, for $p \in (0, 1)$.*

Thus, X_i is a Bernoulli random variable with parameter p.

Total credit loss

Definition 9

The **total credit loss** *in the portfolio,* L_m*, at time* T *is given by*

$$L_m := \sum_{i=1}^{m} l \cdot X_i = l \sum_{i=1}^{m} X_i = l \cdot N_m$$

where

loss amount

$$N_m := \sum_{i=1}^{m} X_i, \; l = L(1-\pi)$$

Thus, N_m *is the* **number of defaults** *in the portfolio up to time* T*.*

Remark 2

N_m *is binomially distributed with parameters* m *and* p*, that is* $N_m \sim Binomial(m,p)$*.*

Binomial Model

Proposition 3

In the binomial model, we have m里k个default

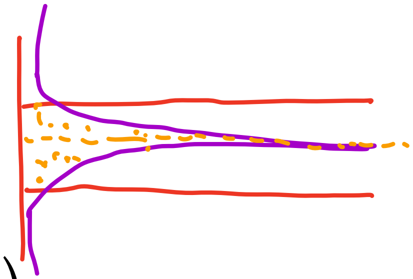
$$(i) \qquad \mathcal{P}(L_m = k \cdot l) = \binom{m}{k} p^k \cdot (1-p)^{m-k}, \quad k \in \{0, 1, 2, \cdots\},$$

$$(ii) \qquad \mathbf{E}[N_m] = mp, \qquad \mathbf{V}ar[N_m] = mp(1-p),$$

$$(iii) \qquad \mathbf{E}[L_m] = lmp, \qquad \mathbf{V}ar[L_m] = l^2 mp(1-p).$$

Use the fact that $N_m \sim \text{Binomial}\,(m, p)$.

Some facts



Proposition 4

In the binomial model, we have

$$(i) \qquad \mathcal{P}\left(\left|\frac{N_m}{m} - p\right| \geq \epsilon\right) \to 0, \quad as \;\; m \to \infty,$$

weak

$$(ii) \qquad \mathcal{P}\left(\lim_{m\to\infty} \frac{N_m}{m} = p\right) = 1,$$

stronge

$$(iii) \qquad N_m \approx mp \quad for \; m \; large.$$

(i) and (ii) follow from the weak law of large numbers (WLLN) and the strong law of large numbers (SLLN), respectively. On the other hand, (iii) is a consequence of (ii).

Example

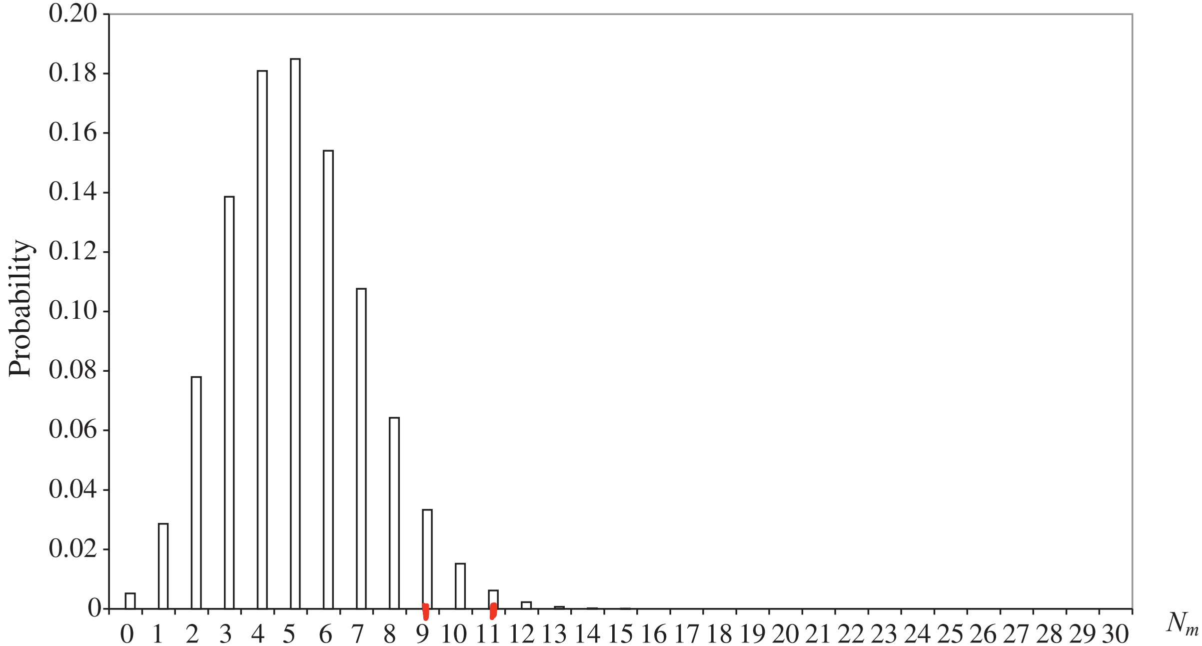


Figure 2: The portfolio credit loss distribution in the binomial model with $m = 100$ and $p = 0.05$.

The loss distribution has "thin" tails, 95% VaR is for 9 defaults, and 99% VaR is 11 defaults.

The other extreme: perfectly dependent defaults

The extreme case of default correlation is given when defaults are perfectly dependent. This means:

- Either all obligators default (in our example with $p = 5\%$ probability
$$\mathbb{P}[N_m > 0] = p = \mathbb{P}[N_m = N]$$
- Or none of the obligators defaults (in the example, with 95% probability.
$$\mathbb{P}[N_m = 0] = 1 - p$$

1. This can be also represented as a binomial distribution function with only one draw and much higher stakes.
2. The independence can be represented by m obligators with loss amount $l = L(1 - \pi)$ per each.
3. The perfect dependence case with just *one* obligator and a big loss $mL(1 - \pi)$

The Binomial Expansion Method

The method is used by the ratings agency Moody's to assess the default risk in bond and loan portfolio

Definition 10

- *Consider a portfolio of m obligators with a total notional amount LT and the average default probability p.*
- *The BET loss distribution with score D is given by binomial distribution of a portfolio of D obligators and loss $\frac{LT}{D}$.*
- *The parameter D is call the diversity score of the portfolio.*

(i) *Example* The same example: $m = 100$ obligators $p = 0.05$. Now instead of 100 independent obligators, we consider only 50 independent obligators with potential loss $2L(1 - \pi)$.

(ii) Equivalently, we group the original obligators into D groups and assume that all obligators in one group default at the same time (perfect dependence) and all groups default independently of each other.

Mapping to an uniform portfolio in practice

- In general the portfolio is not uniform neither in exposure nor in the ratings
  of individual obligators.
- Our goal is to outline a methodology to map it to a uniform as accurately as possible. We start with the calculation with diversity score.

Mapping correlation to the number of obligators

1. We consider a sample portfolio of m bonds with same probability of default p and same face value
2. We also assume that the pairwise default correlation is the same and equals r.
3. We can construct an equivalent portfolio of $n(r)$ independent obligators by matching the first two moments of portfolio (expectation and variance)
4. As a result we get

$$n(r) = \frac{m}{1 + r(m-1)}$$

Calculation of diversity score methodology

Diversity Score D starts with an initial score $D = N$, and then is adjusted downwards according to the following criteria:

1. Exposure sizes: Large exposures represent concentration and carry a diversity score penalty. The new diversity score can be calculated as
$$D = \sum_{n=1}^{N} \min[\frac{l_n}{LT/N}, 1]$$
2. Industry group diversification: if several of the obligators belong to the same industry group, the diversity score is adjusted downwards:
    a. Allocate bonds to their respective industry group
    b. The industry total within each group is the sum of relative contributions of all companies in that group (exposure weighted)
    c. The diversity score is the sum of the industry group diversity scores (correlation $\approx 30\%$)

Calculation of the weighted average probability

1. There is a table of weights for each rating provided by Moody, so that weight increase for lower ratings are much higher (for example for Aaa is 1, for Aaa is 10m Baa is 260, and the lowest could be a high as $10,000$
2. First we sum all the ratings weighted by the exposure in the portfolio in this rating clas
3. This number is divided the total exposure to yield the average number
4. Then by the weighted average rating is being looked up from the table.