Merton's Structural Models

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Outline

Today we will discuss the classical asset value model introduced by Merton

Main Learning Goals of the lecture:

- Structural Models of default
- The foundations of the classical Merton model
- The probability of default and recovery in the Merton model
- Estimate of the asset value and volatility
- Refer to chapter 3 "Asset Value Models" in the textbook.

Balance Sheet

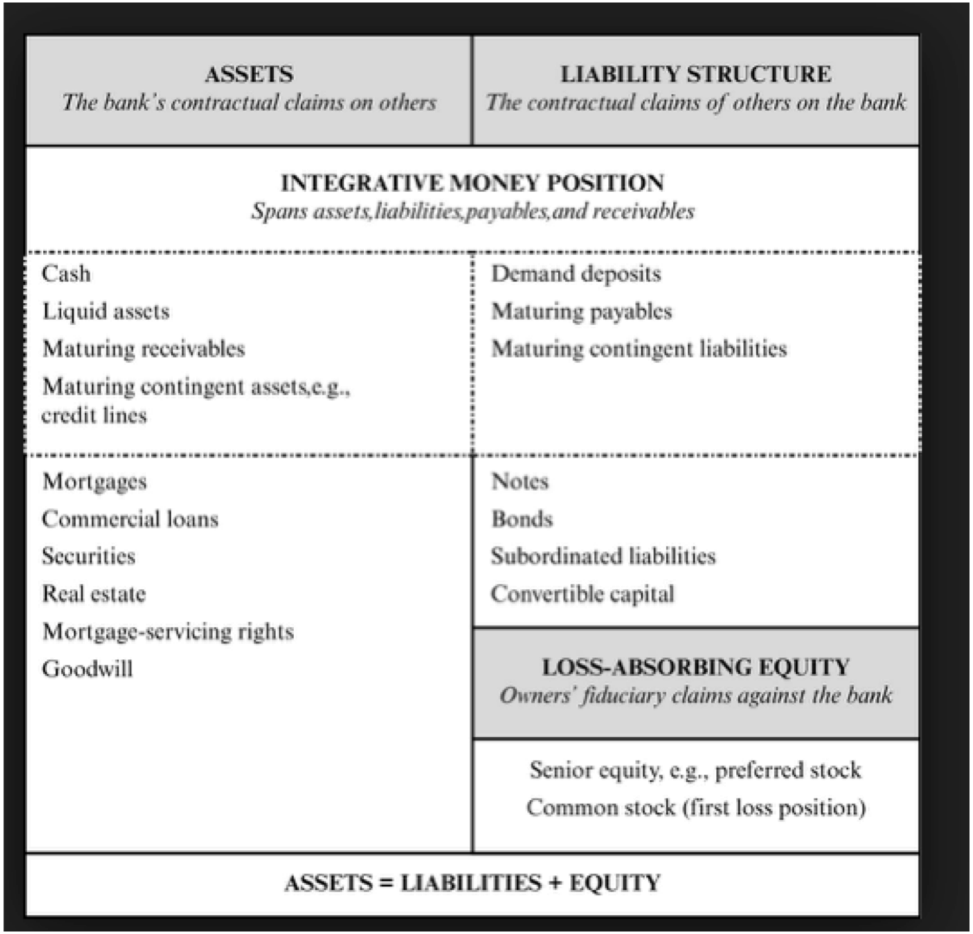


Figure 1: The stylized balance sheet of a typical bank

Structural Models of Default

- A hazard rate approach does not give any fundamental reasons for arrival of defaults, only a consistent description of the market implied distribution of the default arrival times
- In structural or firm value models one postulates a mechanism for the default of a firm in terms of the relationship between its asset and liabilities.
- Default occurs whenever a stochastic variable generally representing an asset value falls below a threshold representing liabilities.
- These models have been very influential in the analysis of credit risk and in the development of industry solutions.

We begin with analysis of Merton's model [Merton, 1974] is the prototype of all firm value models. It is an influential benchmark even today.

Modeling Approach

Consider a firm value whose asset follows some stochastic process V_t

- The firm finances itself by *equity* (issuing shares) and by *debt*.
- In Merton model, debt consists of zero coupon bonds with common maturity T. The nominal value of debt at maturity is given by the constant D. (debt) The firm does not pay dividends, and can't issue a new debt
- Default occurs if the firm misses a payment to its debtholders, which in the Merton model can occur only at the maturity T.
- The values at time t of the equity and debt denoted by S_t and B_t.

Cases at maturity

1. $V_T > D$ The value of the firm's assets exceeds the nominal value of the liabilities. In that case the debtholders receive D; the shareholders receive the residual value $S_T = V_T - D$; there is no default.
2. $V_T \leq D$ The value of the firm is less than its liabilities and the firm cannot meet its financial obligations. In that case case shareholders let the firm go into default. Control over the firm assets is passed to the bondholders, who distribute the proceeds. $S_T = 0$.

Summarizing, we have the relationships:

1 $S_T = \max(V_T - D, 0) = (V_T - D)^+$, European call option with strike D

2 $B_T = \min(V_T, D) = D - (D - V_T)^+$. Nominal value of liabilities - put option with strike D.

Limitations of the model

1. The model is a stylized description of default, the reality is much more complex, default can happen on many different dates
2. Default does not automatically imply bankruptcy (liquidation).

Nonetheless, Merton's model is a useful starting point for modeling credit risk and pricing securities subject to default.

The option explanation of equity and debt is useful in explaining potential conflicts of interest between shareholders and debtholders.

Question 1 : How the values of equity and bonds change with the volatility of the assets? what is the financial interpretation?

Modeling of Default

Assumption 1

- *Continuously compounded interest rate is deterministic* $r \geq 0$.
- *The firm asset value* V_t *is independent of the way the firm is financed, in particular it is independent of the debt level* D.
- *In the classical Merton model default can occur only at maturity of the debt* T [1]
- *The assets values* V_t *can be traded on a frictionless market, and the asset value dynamics is given by the GBM.*

Comments

- High debt level, hence high probability of default may adversely affect the ability of a firm to generate business.

[1] This assumption will be later relaxed

The dynamic of the firm's value

At time 0, we consider a firm whose value at each time $t \in [0, T]$ is represented by V_t.

Assumption 2

- *Under the risk neutral measure $\mathcal{Q}$, the value of the firm's assets is currently $V_0 > 0$ and follows a geometric Brownian motion*

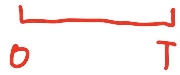
$$dV_t = rV_t dt + \sigma_V V_t dW_t, \qquad V_0 > 0,$$

  where W is a (standard) Brownian motion under $\mathcal{Q}$.

- *We endow the probability space with the Brownian filtration $\{\mathcal{F}_t\}_{t \geq 0}$.*

$$S = \max[V_T - D, 0]$$

$$S_0 = C^{BS}[V_0, \sigma_V, r, T, D]$$



$$S_0 = V_0 N(d_1) - De^{-rT} N(d_2)$$

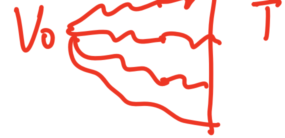
Sensitivity to price

$$\frac{\partial C^{BS}}{\partial S} = N(d_1)$$

probability of exercise

$N(d_2) = \mathbb{P}(V_T \geq D)$ same as survival

The survival probability



$V_T = V_0 e^{-\frac{1}{2}\sigma_V^2 T + rT + \sigma\sqrt{T} X}$, $X \sim N(0,1)$

Proposition 1

The survival probability of the firm is:

prob of survival

$$P(0,T) := \mathcal{Q}\left(V_T \geq D\right) = \mathcal{N}\left(d_2\right)$$

where

prob of Default $= 1 - \mathcal{N}(d_2)$

$$d_2 = \frac{\ln\left(V_0/D\right) + \left(r - \sigma_V^2/2\right)T}{\sigma_V\sqrt{T}}.$$

The notation $\mathcal{N}$ stands for the standard Gaussian cumulative distribution function:

e.g: d_2

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du, \quad \forall x \in \mathbb{R}.$$

Pricing

put:

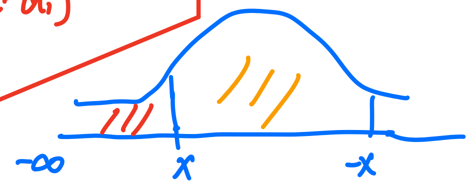
$B_0 \longleftarrow B_T = D - (D - V_T)^+$

$B_0 = De^{-rT} - [De^{-rT}\mathcal{N}(-d_2) - V_0\mathcal{N}(-d_1)]$

$= De^{-rT}[1 - \mathcal{N}(-d_2)] + V_0\mathcal{N}(-d_1)$

$= De^{-rT}\mathcal{N}(d_2) + V_0\mathcal{N}(-d_1)$

$\mathcal{N}(x) + \mathcal{N}(-x) = 1$



Proposition 2

The price of the defaultable debt is:

$$B_M(0,T) = De^{-rT}\mathcal{N}(d_2) + V_0\mathcal{N}(-d_1),$$

with

$$d_1 = \frac{\ln(V_0/D) + \left(r + \sigma_V^2/2\right)T}{\sigma_V\sqrt{T}}, \quad d_2 = \frac{\ln(V_0/D) + \left(r - \sigma_V^2/2\right)T}{\sigma_V\sqrt{T}}.$$

Proof.

Question 2: Can be derived directly using the known results for a put option in the Black-Scholes model. □

Yield to maturity

Definition 1

The continuous **yield to maturity** $y(T)$ *for the defaultable debt is defined as the solution to:*

$$B_M(0,T) = De^{-y(T)T}.$$

Therefore,

$$y(T) = \frac{\ln\Big(D/\,B_M(0,T)\Big)}{T}.$$

Credit spreads

Definition 2

The **credit spread** *is defined as the risk premium with respect to the risk-free rate:*

$$s(T) := \underbrace{y(T)}_{\text{yield to maturity}} - r.$$

In this model, we can obtain analytical expression for the credit spread.

$B(0) = De^{-rT}\mathcal{N}(d_2) + V_0\mathcal{N}(-d_1) = e^{-rT}D$

$y = \frac{\ln[D/B]}{T}$ $\qquad$ $S = y - r$

Proposition 3

The credit spread is given by:

$$\boxed{s(T) = -\frac{1}{T}\ln\left(\mathcal{N}(d_2) + \frac{V_0}{De^{-r\cdot T}}\mathcal{N}(-d_1)\right).}$$

$$y = \frac{\ln\left[\frac{D}{De^{-rT}\mathcal{N}(d_2) + V_0\mathcal{N}(-d_1)}\right]}{T} = \ln\left[\frac{D}{De^{-rT}}\left[\frac{1}{\mathcal{N}(d_2) + \frac{V_0}{De^{-rT}}\mathcal{N}(-d_1)}\right]\right] / T$$

Debt ratio

$y = r - \frac{1}{T}\ln(N(d_2) + \frac{V_0}{De^{-rT}} N(-d_1))$

$s = y - r = -\frac{1}{T}\ln(N(d_2) + \frac{V_0}{De^{-rT}} N(-d_1))$

Definition 3

The **debt ratio** *is defined as*

$$L = \frac{De^{-r \cdot T}}{V_0}.$$

The quantity L represents a **measure of leverage**.

d_1, d_2, can be written as functions of L, $\sigma_V\sqrt{T}$.

$d_1 = \frac{\ln(V_0/D) + (r + \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}$ $\quad d_2 = d_1 - \sigma_V\sqrt{T}$

$L = \frac{De^{-rT}}{V_0}$ $\quad \sigma \approx \frac{1}{\sqrt{T}}$

$d_1, d_2 \rightarrow f(L, \sigma_V\sqrt{T})$

$\sigma = 0.2$, $T = 1$

$0.2 \times \sqrt{1} = 0.2$

$\sigma = 0.1$, $T = 4$

$0.1 \times \sqrt{4} = 0.2$

$$l := \frac{1}{L} = \frac{V_0}{e^{-rT}D}$$

$$d_1 = \frac{\ln(l + \frac{\sigma_V^2 T}{2})}{\sigma_V\sqrt{T}}, d_2 = d_1 - \sigma_V\sqrt{T}$$

Therefore, the same is true for the credit spread s.

Behavior of the credit spread

σ_V decreases as bond value,
Less bond value means default
is more likely, so we have
wide spread.

- Based on these representations we can investigate the behavior of the credit spread $s(T)$ as a function of leverage L and the asset volatility $\sigma_V\sqrt{T}$.
- For that matter we can calculate "Greeks", sensitivities of the credit spread w.r.t to L and $\Sigma_V = \sigma_V\sqrt{T}$: [2]

1. $$\frac{\partial s}{\partial \Sigma_V} = ?$$ Positive

2. $$\frac{\partial s}{\partial L} = ?$$

[2] Hint: find the relationship between d_1^2 and d_2^2 , and between $n(d_1)$ and $n(d_2)$, where $n(x)$ is the standard normal pdf

A simple example

Consider the following set up:

(a) The initial value of assets $V_0 = 120$ MM, yearly volatility $\sigma = 0.25$, maturity $T = 4$ years, interest rate $r = 0.05$

(b) Let the debt D =40, 100 , 180 MM.

For each debt value calculate

1. the credit spread s
2. The probability of default at time T, which is the probability that the call option won't be exercised.

Random recovery and pricing

$\frac{D}{V_0} = 1 \quad \frac{V_0}{D} = 1$

Remark 1

We may price the defaultable bond in Merton's model using the Simple Model of the previous sections:

$$B_M(0,T) = DB(0,T)P(0,T) + D\pi B(0,T)P^{def}(0,T).$$

It turns out that this is indeed the price of the defaultable bond if π is the expected recovery rate conditional on default.

Proposition 4

$\mathbb{P}(R > 50\% \mid V_T < D) = \frac{\mathbb{P}(R > 50\%, V_T < D)}{\mathbb{P}(V_T < D)} =$

The price of the defaultable debt with random recovery is:

$$B_M(0,T) = DB(0,T)P(0,T) + D\pi^e B(0,T)P^{def}(0,T)$$

where

$$\pi^e := E\left[\pi \mid V_T < D\right], \quad \pi := V_T/\,D.$$

$\mathbb{P}(R > 50\%) = \mathbb{P}(\frac{V_T}{D} > 0.5)$

$= \mathbb{P}(V_T > 0.5D)$

$\mathbb{P}(R < 50\%) = 1 - \mathbb{P}(V_T > 0.5D)$

Estimation of the asset value V_0 and its volatility σ_V

1. The general problem with asset value models is that asset value processes are not observable. Instead one can see every day in the stock market are equity values. Now we discuss how one can estimate V_0 and σ_V in the context of Merton's model. Since

$$S_T = \max\{V_T - D, 0\},$$

we can use the price of a European call to find

$$S_0 = C^{BS}(V_0, \sigma_V, r, T, D).$$

$$S_0 = V_0 \mathcal{N}(d_1) - De^{-rT}\mathcal{N}(d_2)$$

2. We can find V_0 as

$$V_0 = \frac{S_0 + De^{-rT}\mathcal{N}(d_2)}{\mathcal{N}(d_1)} \tag{1}$$

3. Since S_0, r, T, D are observable, we have one equation and two unknowns.

Jones method with Itô formula

On the other hand, considering $S_t = C^{BS}(V_t, t)$, an application of Ito's Lemma yields

$$dS_t = (\cdots)dt + \frac{\partial C^{BS}}{\partial V_t}\sigma_V V_t dW_t.$$

Or

$$dS_t/S_t = \frac{1}{S_t}(\cdots)dt + \frac{1}{S_t}\frac{\partial C^{BS}}{\partial V_t}\sigma_V V_t dW_t.$$

We assume that the equity price volatility σ_S is a constant. It can be found as

$$\sigma_S = \frac{1}{S_0}\frac{\partial C^{BS}}{\partial V_0}\sigma_V V_0 = \frac{1}{S_0}\mathcal{N}(d_1)\sigma_V V_0.$$

We also assume that the volatility σ_S of the equity price can be observed from the market: being either the historical volatility, or an option-implied volatility.

Iterative Method

- The above relationship 1 between equity volatility and asset volatility holds only instantaneously. ([Crosbie, Bohn 2003]). In practice market leverage moves around far too much. Overestimate (or underestimate) of asset vol will lead to wrong default probability
- Instead of estimating the instantaneous asset volatility, another more complex procedure is applied. We observe a time series of market equity values E_t (based on equity prices and number of shares) and liabilities D_t.
  1. Start with the initial estimate $\sigma_V^{(0)}$
  2. Using time series E_t , D_t and eq. 1 calculate asset values $V_t^{(0)}$.
  3. Using the calculated time series V_t calculate the new estimate of asset volatility $\sigma_V^{(1)}$.
  4. Construct a new series of asset values $V_t^{(1)}$, ...
  5. The procedure is iterated n times, until the volatility estimates $\sigma_V^{(n-1)}$ and $\sigma_V^{(n)}$ are sufficiently close.