Credit Risk: Modeling Default Dependence with Copulas

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Introduction and Outline of the Lecture

- The word *copula* is a Latin noun that means a link or tie that connects two different things. It is used in grammar to describe that part of a proposition which connects the subject and predicate.
- In Quantitative Risk Management: it is used to find good joint models.
- The idea of copulas is to go from individual models to joint model. The idea , study and applications of copulas is rather a modern phenomenon.
- In this lecture we will introduce copulas, discuss their properties and look at their applications in Credit Risk.

General Notion of Copula

Copulas are functions that join or "couple" multivariate distribution functions to their one-dimensional marginal distribution functions

- Consider a pair of r.v. X and Y with distribution functions

$$F(x) = \mathbb{P}[X \leq x] \text{ and } G(y) = \mathbb{P}[Y \leq y]$$

- Consider a joint distribution function

$$H(x, y) = \mathbb{P}[X \leq x, Y \leq y]$$

- To each pair of real numbers (x, y) we can associate three numbers $F(x)$, $G(y)$ and $H(x, y)$. Each of these numbers lies are in the interval $[0, 1]$. Thus, each pair (x, y) of real numbers leads to a point $(F(x), G(y))$ in the unit square $[0, 1] \times [0, 1]$ and this ordered pair in turn corresponds to a number $H(x, y)$ in $[0, 1]$.
- This correspondence which assigns the value of the joint distributions function is a function and called *copula*

Properties of Distribution Function of two Random Variables

1. $$H(-\infty, y) = H(x, -\infty) = 0, H(+\infty, +\infty) = 1$$

2. $$H(x, \infty) = F(x), H(\infty, y) = G(y)$$

3. For $x_1 \leq x_2$ and $y_1 \leq y_2$
$$\mathbb{P}(x_1 \leq X \leq x_2, Y \leq y) = H(x_2, y) - H(x_1, y)$$
$$\mathbb{P}(X \leq x, y_1 \leq Y \leq y_2) = H(x, y_2) - H(x, y_1)$$

4. $$\mathbb{P}(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) = H(x_2, y_2) - H(x_1, y_2) -$$
$$-H(x_2, y_1) + H(x_1, y_1)$$

Preliminaries

(i) Let $\mathbb{R}$ denote the real line $(-\infty, \infty)$, $\bar{\mathbb{R}}$ is the extended real line $[-\infty, \infty]$ and $\bar{\mathbb{R}}^2 = \bar{\mathbb{R}} \times \bar{\mathbb{R}}$ is extended real plane.

(ii) A rectangle in $\bar{\mathbb{R}}^2$ is the Cartesian product B of two closed intervals $B = [x_1, x_2] \times [y_1, y_2]$. The unit square $\mathbb{I}^2$ is the product $\mathbb{I} \times \mathbb{I}$ where $\mathbb{I} = [0, 1]$

(iii) Let H be a function defined on a subset $DomH = S_1 \times S_2$, S_1, S_2 be nonempty subsets of $\bar{\mathbb{R}}$. The H-volume of B is given by

$$H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1) \quad (1)$$

(iv) $V_H(B)$ is also the H-mass of the rectangle B. We can define the first order difference H on the rectangle B as

$$\Delta_{x_1}^{x_2} = H(x_2, y) - H(x_1, y) \text{ and } \Delta_{y_1}^{y_2} = H(x, y_2) - H(x, y_1)$$

(v) It follows that

$$V_H(B) = \Delta_{y_1}^{y_2} \Delta_{x_1}^{x_2} H(x, y)$$

Notion of non-decreasing function of two variables

Definition 1

The function H *defined on subset* $DomH$ *is called* $2-$ *increasing if* $V_H(B) \geq 0$ *for all rectangles* B *whose vertices lie in domain* $DomH$*.*

a.
b. The statement the function is $2-$ increasing neither implies or implied by the facts that H is non-decreasing in each argument.
    1. Let function $H(x, y)$ be a function defined on $\mathbb{I}^2$ by $H(x, y) = \max(x, y)$.
    2. Let function $H(x, y)$ be a function defined on $\mathbb{I}^2$ by $H(x, y) = (2x - 1)(2y - 1)$.

Grounded functions

- If S_1 and S_2 are nonempty subsets of $\bar{\mathbb{R}}$. Suppose S_1 has a least element a_1, and and S_2 has a least element a_2.

Definition 2

The function H from $S_1 \times S_2$ into $\mathbb{R}$ is grounded, if

$$H(a_1, y) = 0 = H(x, a_2)$$

Proposition 1

Let H be defined as before, it is 2-increasing and grounded. Then H is nondecreasing in each argument. (Prove it)

- Now assume S_1 has a greatest element b_1 and S_2 has a greatest element b_2.

Definition 3

We say that function H has margins defined:

$$F(x) = H(x, b_2), \forall x \in S_1, G(y) = H(b_1, y), \forall y \in S_2$$

Example 1

Let H be a function with domain $[-1,1] \times [0,\infty]$ given by

$$H(x,y) = \frac{(x+1)(e^y - 1)}{x + 2e^y - 1}$$

Check if H is grounded and find its margins, $F(x) = H(x,\infty)$ and $G(y) = H(1,y)$.

$S_1 = \{-1, 0\}$ $S_2 : \{1, \infty\}$

$$H(0,y) = \frac{(-1+1)(e^y-1)}{-1+2e^y-1} = 0$$

grounded!

$$H(x,1) = 0$$

$$F(x) = H(x,\infty) = \lim_{y\to\infty} \frac{(x+1)(e^y-1)}{x+2e^y-1} = \frac{x+1}{2}$$

$$G(x) = H(1,y) = \frac{2(e^y-1)}{2e^y} = \frac{e^y-1}{e^y} = 1-e^{-y}$$

Example and some properties

Example 1

Let H be a function with domain $[-1,1]\times[0,\infty]$ given by

$$H(x,y)=\frac{(x+1)(e^y-1)}{x+2e^y-1}$$

Check if H is grounded and find its margins, $F(x)=H(x,\infty)$ and $G(y)=H(1,y)$.

Proposition 2

Let H is as defined in Proposition 1, and it has margins $F(x)$ and $F(y)$. Let x_1,y_1 and x_2,y_2 be any points in $S_1\times S_2$. Then

$$\mid H(x_2,y_2)-H(x_1,y_1)\mid\leq\mid F(x_2)-F(x_1)\mid+\mid G(y_2)-G(y_1)\mid$$

Definition of copula in two-dimensional case

Definition 4

A two -dimensional copula is a function C from $\mathbb{I}^2$ to $\mathbb{I}$ with the following properties:

1. *For every u, v in $\mathbb{I}$*

$$C(u, 0) = 0 = C(0, v)$$

$$C(u, 1) = u \text{ and } C(1, v) = v$$

2. *For every u_1, u_2, v_1, v_2 in $\mathbb{I}$ such that $u_1 \le u_2$ and $v_1 \le v_2$*

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \ge 0$$

Theorem 3

If C is a copula, then for every (u, v) in $\mathbb{I}^2$ we have

$$\max(u + v - 1, 0) \le C(u, v) \le \min(u, v)$$

Notion of Comonotonicity and Countermonotonicity

Perfect dependence (or Comonotonicity) can be defined in many equivalent ways. Either by a copula $M(u,v) = \min(u,v)$, or

Definition 5

R.v. X_1, X_2 are called comonotonic, if they can be presented as monotonically increasing function of a single r.v. (one single source of risk):

$$(X_1, X_2) =_d (v_1(Z), v_2(Z))$$

where v_1, v_2 are increasing functions.

Countermonotonicity can defined in a similar fashion. We can define it through a copula $W(u,v) = \max(u+v-1, 0)$, or

Definition 6

R.v. X_1, X_2 are called countercomonotonic, if

$$(X_1, X_2) =_d (v_1(Z), v_2(Z))$$

where v_1 is increasing function, and v_2 is decreasing (or vice versa)

.

Fundamental Copulas

1. *Comonotonicity* copula: $M(u,v) = \min(u,v)$
2. *Countermonotonicity* copula: $W(u,v) = \max(u+v-1,0)$
3. According to the theorem, the two above copulas give the bounds for any copula, they are called "Fréchet- Hoeffding bounds.
4. The *independence copula:* the third important copula is the product copula
$$\Pi(u,v) = uv$$

A simple but useful way to present the graph of a copula with a *contour diagram* , with graphs of its level sets in $\bar{\mathbb{I}}^2$ given by $C(u,v) = const$

Questions

1. Prove that $M(u,v)$ is a copula
2. Give contour diagrams for all three copulas

Distribution Functions

Definition 7

A distribution function is a function F with domain $\bar{\mathbb{R}}$ such that

1. F *is non-decreasing*
2. $F(-\infty) = 0$ *and* $F(\infty) = 1$

Definition 8

A joint distribution function H with domain $\bar{\mathbb{R}}^2$ such that

1. *H is* 2*-increasing*
2. $H(x, -\infty) = H(-\infty, 0) = 0$ *and* $H(\infty, \infty) = 1$

Sklar's Theorem, 1959

Theorem 4

(a) *Let H be a joint distribution function with margins F and G. Then, there exists a copula C such that for all x, y in $\bar{\mathbb{R}}$*

$$H(x,y) = C(F(x), G(y)) \tag{2}$$

If F and G are continuous, then C is unique.

(b) *If C is a copula function and F, G are distribution functions, then the function H defined by 2 is a joint distribution function with margins F and G.*

Notion of quasi-inverse

Definition 9

Let F be a distribution function with range $RanF$. Then a quasi-inverse of F is a function $F^{(-1)}(t)$ with domain $\mathbb{I}$ such that

1. *If t is in $RanF$ then $F^{(-1)}(t)$ is a number x in $\bar{\mathbb{R}}$ such that $F(x) = t$, so*
$$F(F^{(-1)}(t)) = t$$
2. *If t is not in $RanF$, then*
$$F^{(-1)}(t) = \inf[x|F(x) \geq t] = \sup[x|F(x) \leq t]$$

If F is strictly increasing, then it has an unique quasi-inverse, which coincides with the usual inverse F^{-1}.

Copula from distribution

$-\infty < x, y < \infty$

1) $x = \infty$, $H(-\infty, y) = \frac{1}{1+e^{-y}}$

$y = \infty$, $H(x, \infty) = \frac{1}{1+e^{-x}}$

Example 2

Gumbel's bivariate logistic distribution Let X and Y be r.v. with joint d.f.

$1+e^{-y} = \frac{1}{v}$ $\quad e^{-y} = \frac{1-v}{v}$

$$H(x,y) = (1 + e^{-x} + e^{-y})^{-1}$$

$1+e^{-x} = \frac{1}{u}$ $\quad e^{-x} = \frac{1-u}{u}$

for all x, y in $\bar{\mathbb{R}}$

$H(x,y) = \left[1 + \frac{1-u}{u} + \frac{1-v}{v}\right]^{-1} = C(u,v)$

1. *Find d.f. of X, Y (margins)*

$\Rightarrow C(u,v) = \frac{uv}{uv + v - uv + u - uv} = \frac{uv}{u+v-uv}$

2. *Write down copula $C(u,v)$ for X and Y.*

Gumbel's bivariate logistic distribution lacks parameter which limits its use. This can be corrected in different ways, for example, one can define the following family and answer the same questions:

Example 3

Dependence control

$$H_\theta(x,y) = (1 + e^{-x} + e^{-y} + (1-\theta)e^{-x-y})^{-1}$$

for all x, y in $\bar{\mathbb{R}}$ and $\theta \in [-1, 1]$.

The Gaussian Copula

If X, Y belong to bivariate normal distribution with correlation ρ then their copula is so-called Gauss (or Gaussian) copula. Without loss of generality we can assume that X, Y are standard normal [1] Then margins are $N(x)$ and $N(y)$, $u = N^{-1}(x)$, $v = N^{-1}(y)$ and the bivariate Gaussian copula is given by

$$C_\rho(u, v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{N^{-1}(u)} \int_{-\infty}^{N^{-1}(v)} e^{\left(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right)} dx_1 dx_2.$$

The independence and comonotonicity copulas are special cases : $\rho = 0$ corresponds to independence, $\rho = 1$ to comonotonicity, and $\rho = -1$ to countermonotonicity.

[1] A copula is invariant under monotone transformations

References

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