Problem 1 (10 points)

Suppose that a bank has made a large number of loans. The one year probability of default of each loan is 2%. Use Vasicek model to estimate the default rate that we are 99.5% certain will not be exceeded. Assume that the parameter $\rho = 0.3$.

$$\mathbb{P}[Y_m \leq X_c] = \alpha = N\left[\frac{1}{\sqrt{\rho}}\sqrt{1-\rho}\,N^{-1}(x) - N^{-1}(\bar{p})\right] = \alpha$$

So, $$X = N\left(\frac{\sqrt{\rho}N^{-1}(\alpha) + N^{-1}(\bar{p})}{\sqrt{1-\rho}}\right)$$

$$\bar{p} = P(X_i = 1) = \mathbb{E}[X_i] = 0.02$$

$$\alpha = 0.995$$

$$\rho = 0.3$$

By using python, we have the result ≈ 0.22.

Problem 2 (10 points)

A five year credit default swap entered into on June 20 2018, requires quarterly payments at the rate of 600 basis points per year. The principal is $100 million. A default occurs after four years and 2 months. The action process finds the price of the cheapest deliverable bond to be 35% of its face value. List the cash flows and their timing for the seller of the credit default swap.

1. $1000000000 \times (600/10000) \times 0.25 = 1500000$

   Seller can receive 1500000 at times

   0.25, 0.5, 0.75, 1

   1.25, 1.5, 1.75, 2

   2.25, 2.5, 2.75, 3

   3.25, 3.5, 3.75, 4

   (quarterly payments for four years)

2. $100000000 \times (600/10000) \times (2/12) = 1000000$

   Seller can also receive 1000000 at time of default.

3. $100000000 \times (1-35\%) = 65000000$

   Seller pays 65000000 at the time of the default.

Problem 3 (10 points)

Let X and Y be continuous r.v. with copula C and univariate distribution function F and G respectively. The random variables $\max(X, Y)$ and $\min(X, Y)$ are *order statistics* for X and Y. Find their distribution functions:

$$F_{\max}(t) = \mathbb{P}[\max(X, Y) \leq t], F_{\min}(t) = \mathbb{P}[\min(X, Y) \leq t]$$

$$\begin{aligned}
F[\max(x,y) \leq t] &= \mathbb{P}[X \leq t, y \leq t] \\
&= C(F(t), G(t)) \\
\mathbb{P}[\min(x,y) \leq t] &= 1 - \mathbb{P}[X > t, y > t] \\
&= 1 - [1 - \mathbb{P}(X \leq t)][1 - \mathbb{P}(y \leq t)] \\
&= 1 - [1 - \mathbb{P}(X \leq t) - \mathbb{P}(y \leq t) + \mathbb{P}(X \leq t, y \leq t)] \\
&= \mathbb{P}(X \leq t) + \mathbb{P}(y \leq t) - \mathbb{P}(X \leq t, y \leq t) \\
&= F(t) + G(t) - C[F(t), G(t)]
\end{aligned}$$

Problem 4 (10 points)

Let X and Y be random variables with a joint d.f. given by

$$H_\theta(x,y) = \exp[-(e^{-\theta x} + e^{-\theta y})^{1/\theta}]$$

for all x, y in $\bar{\mathbb{R}}$, and $\theta \geq 1$. This distribution is called *bivariate extreme value distribution* Find their copula $C_\theta(u,v)$.

$H_\theta(x,y) = \exp[-(e^{-\theta x} + e^{-\theta y})^{1/\theta}]$ $\theta \geq 1$.

First we get marginal:

$$F_\theta(x) = H(x,\infty) = \lim_{y\to\infty} H_\theta(x,y)$$
$$= \lim_{y\to\infty} \exp[-(e^{-\theta x} + e^{-\theta y})^{1/\theta}]$$
$$= \exp[-e^{-x}]$$

$$F_\theta(y) = H(\infty,y) = \lim_{x\to\infty} H(x,y)$$
$$= \lim_{x\to\infty} \exp[-(e^{-\theta x} + e^{-\theta y})^{1/\theta}]$$
$$= \exp[-e^{-y}]$$

$$\Rightarrow F^{-1}(u) = -\ln(-\ln(u))$$
$$G^{-1}(v) = -\ln(-\ln(v))$$

$$\Rightarrow C_\theta(u,v) = H(F^{-1}(u), G^{-1}(v))$$
$$= \exp[-[(-\ln v)^\theta + (-\ln v)^\theta]^{\frac{1}{\theta}}]$$

Problem 5 (10 points)

Consider the Black Cox model under assumption of a constant safety covenant K observed during the life time of the bond. Assume:

a. The initial asset value $V_0 = 100$, the debt value at maturity $D = 70$

b. The maturity of the debt $T = 2$ (years), the risk free rate $r = 0.04$, the asset vol $\sigma_V = 0.3$

What should be the value of the safety covenant if the survival probability of the firm is $\mathcal{P}(0, T) = 0.7$? (the probability that safety covenant was not triggered, and the value of the firm at maturity exceeds the debt).

$K_1 = 63.14019 \leq De^{-rT} = 64.618$. See python for more details.

Problem 6 (20 points)

(a) (10 points) Consider the **mixed binomial model**. Provide **an example** of a random variable Z **and** a function $p(Z)$, such that the correlation of two indicator random variables X_i and X_j is equal to 0.3. Could we have an example where this correlation is **negative**? (please explain your answer).

(b) (10 points) Let X_A and X_B be default indicator variables for two firms over common time horizon and let $p_A = \mathbb{P}(X_A = 1) = 0.15$ and $p_B = \mathbb{P}(X_B = 1) = 0.1$. Find the maximum value of the **correlation** coefficient of X_A and X_B

Hint: Find the upper bound for covariance between indicators of default in terms of p_A and p_B for this case.

(a) From hw, let $P(Z) = e^{aZ^2}$ $\quad f_Z(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}}$

$$\rho = \frac{\mathbb{E}[P(Z)^2] - \bar{P}^2}{\bar{P}(1-\bar{P})}$$

$$\bar{P} = \mathbb{E}[P(Z)] = \int_{-\infty}^{+\infty} P(Z) f_Z(Z)\, dZ = \int_{-\infty}^{+\infty} e^{aZ^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}}\, dZ$$

$$= \frac{1}{\sqrt{1+2a}}$$

$$\mathbb{E}[P(Z)^2] = \int_{-\infty}^{+\infty} P(Z)^2 f_Z(Z)\, dZ = \int_{-\infty}^{+\infty} e^{2aZ^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}}\, dZ = \frac{1}{\sqrt{1+4a}}$$

$$\rho_{ij} = 0.3 = \frac{1/\sqrt{1+4a} - (1/\sqrt{1+2a})^2}{(1/\sqrt{1+2a})(1 - 1/\sqrt{1+2a})} \Rightarrow a = 0.306$$

So, the ρ can not be negative because $Cov(X_i, X_j) = Var[P(Z)] \geq 0$.

Therefore, ρ is positive strictly.

(b) $$Corr(X_i, X_j) = \frac{P_{AB} - P_A P_B}{\sqrt{P_A(1-P_A)P_B(1-P_B)}}$$

Thus, max $corr(X_A, X_B)$ is equivalent to max P_{AB}. So,

$$Corr(X_A, X_B) \leq \frac{\min(P_A, P_B) - P_A P_B}{\sqrt{P_A(1-P_A)P_B(1-P_B)}} = \frac{P_B - P_A P_B}{\sqrt{P_A(1-P_A)P_B(1-P_B)}} = 0.79$$

Problem 7 (15 points)

Consider a portfolio of 2 obligators with the same exposure, with the hazard rates $\lambda_1 = 0.04$ and $\lambda_2 = 0.05$ (per year), and with a Gaussian copula of default times, the correlation $\rho = 0.25$.

Evaluate a *first to default swap* by simulations

- The swap covers 5 years, and the interest rate is $r = 0.03\%$ and the recovery rate is $RR = 0.4$.
- The default event is the first default of any 2 obligators. The contract terminates after the first default (provided it happens during lifetime of the swap).
- Protection buyer pays a regular fee s, as percentage of the total exposure; the payments occur at the end of each year.
- In case of default, the protection seller covers the loss on one obligator.

Evaluate the fair price of the CDS spread s. Don't forget to include accrual payments. You don't need to make assumptions on timing of defaults, as you will use the actual times of defaults from simulations.

See python.

Problem 8 (20 points)

Suppose a bank A previously entered into a forward contract to buy 10,000 troy ounces of gold from a mining company B for the price $K = \$1,600$ per ounce.The current gold price today is 1,700 per ounce, and the forward contract expires in one year. The probability of the company defaulting in one year is 2%, and the probability of the bank defaulting in one year is 1.5%. The risk free rate is $r = 3\%$ per annum. We assume that either default can happen at the settlement of the contract (one year). The volatility of the forward contract is 20%. The recovery rate for the mining company is 30% and the recovery rate for the bank is 40%. There is no collateral from either site.

Questions:

1. (10 points) Calculate as of today the value of the transaction going to the books: The forward contract value, CVA and DVA. [1]
2. (5 points) What kind of risk is it for the bank A, wrong way or right way? (5 points) Give the rationale for your answer
3. (5 points) Assume that bank buys a CDS contract from a major bank to get protection. In theory, the seller of protection can default too - what kind of risk is it for the bank A? Give the rationale for your answer.

1. CVA: 25460.9
   DVA: 7633.7
   V: 952618.4
2. Right-Way risk.
   The credit quality will be benefited due to the raise of gold. Thus, the increase in exposure of A to the B is with the decrease in the probability of default by B.

3. Wrong-way risk.
   Since, once the bank get an protection, there exists an adverse effect, thus the probability of default increase. Also, due to CDS, the EAD also increase. Thus, this is wrong-way. Also, seller can default increased EAD.