# Credit Risk: Modeling Default Dependence with Copulas

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# Invariance of copulas

A useful property of the copula is invariance under *strictly increasing* transformations of marginals.

## Proposition 1

*Let $(X, Y)$ be a random vector with continuous margins and copula $C$ and let $T_1, T_2$ be strictly increasing functions. Then $(T_1(X_1), T_2(X_2))$ also has copula $C$.*

*Sketch of the proof*

1. $(T_1(X_1), T_2(X_2))$ is a random vector with continuous margins
2. Since transformations $T_i$ are increasing, the following sets are equivalent:

$$\{X \leq x\} = \{T_1(X) \leq T_1(x)\}, \{Y \leq y\} = \{T_2(Y) \leq T_2(y)\}$$

(3) If $H(x, y)$ is a joint d.f. then

$$C(u, v) = H(F^{(-1)}(u), G^{(-1)}(v)) = \mathbb{P}\left(X \le F^{(-1)}(u), Y \le G^{(-1)}(v)\right)$$

$$= \mathbb{P}\left(T_1(X) \le T_1 \circ F^{(-1)}(u), T_2(Y) \le T_2 \circ G^{(-1)}(v)\right) =$$

$$= \mathbb{P}\left(T_1(X) \le F^{(-1)}_{T_1(X)}(u), T_2(Y) \le F^{(-1)}_{T_2(Y)}(v)\right)$$

(4) The last equality follows from the fact that for a r.v. $X$ and increasing continuous function $T$ we have: [1]

$$F^{(-1)}_{T(X)}(\alpha) = T\left(F^{(-1)}(\alpha)\right)$$

(prove it).

[1] More general it is enough $T$ is left continuous

# Revisit the fundamental copulas

- Comonotonicity: From the proposition for comonotonic r.v. $X$ and $Y$ their copula is the same as copula of $(X, X)$, which is d.f of $(F(x), F(y))$ or $(U, U)$. Then

$$H(x,y) = C(F(x), F(y)) = \mathbb{P}(U \leq u, U \leq v) = \min(u, v)$$

- Countermonotonicity: By the same arguments, if $X$ and $Y$ are countermonotonic, their copula is the same as of $(X, -X)$ which is d.f. $(F(x), 1 - F(-y))$ or $(U, 1 - U)$. Then

$$H(x,y) = C(F(x), 1 - F(-y)) = \mathbb{P}(U \leq u, 1 - U \leq v) =$$

$$= max(u + v - 1, 0)$$

we need to use $\max$ since the probability is non-negative.

$$\bar{H}(x,y) = 1 - F(x) - G(y) + H(x,y) = \bar{F}(x) + \bar{G}(y) - 1 + C[F(x), G(y)]$$

$1-\bar{F}(x)$ , $1-\bar{G}(y)$

$\bar{F}(x) = u \qquad \bar{G}(y) = v$

$$\bar{C}(u,v) = u + v - 1 + C[1-u, 1-v]$$

# Survival Copulas

1. In credit we often talk about survival: the probability of default occurring after some time $T$. Therefore, it is useful to define survival function $\bar{F}(x) = \mathbb{P}(X \geq x) = 1 - F(x)$, where $F$ is the d.f. of r.v $X$
2. For a pair of r.v. $(X, Y)$ with joing d.f. $H$, the joint survival is given by $\bar{H}(x,y) = \mathbb{P}(X \geq x, Y \geq y)$. Now lets assume that the copula of $X$ and $Y$ is $C$ Then we have: [2]

$$\bar{H}(x,y) = 1-F(x)-G(y)+H(x,y) = \bar{F}(x)+\bar{G}(y)-1+C(F(x),\overset{G}{\cancel{H}}(y))$$

$$\cancel{= 1-F(x)-G(y)+H(x,y)} = \bar{F}(x)+\bar{G}(y)-1+C(1-\bar{F}(x),\underset{1-\bar{G}(y)}{\cancel{\bar{H}(y)}})$$

3. We can define a survival copula $\bar{C} : \mathbb{I}^2 \to \mathbb{I}$ by

$$\bar{C}(u,v) = u + v - 1 + C(1-u, 1-v)$$

4. $\bar{C}$ is a copula (check), and it is called a *survival* copula.

[2] volume of rectangles with vertices $x, y$, $x, \infty$, $\infty, y$, $\infty, \infty$

# Example

- Consider Gumbel's bivariate distribution function $\theta \in [0,1]$ and write its survival copula

$$H_\theta(x,y) = \left(1 - e^{-x} - e^{-y} + e^{-(x+y+\theta xy)}\right), x \backslash geq 0, y \geq 0$$

$$H(x,y) = 0 \text{ otherwise}$$

- Write its copula and the survival copula.

$$H_\theta(x,+\infty) = 1 - e^{-x} = u = F(x)$$
$$\Rightarrow 1-u = e^{-x}$$
$$\ln(1-u) = -x$$
$$F^{-1}(u) = x = -\ln(1-u)$$

$F^{(-1)}(u) = -\ln(1-u) \qquad G^{(-1)}(v) = -\ln(1-v)$

$$C_\theta(u,v) = u+v-1+(1-u)(1-v)e^{-\theta\ln(1-u)\ln(1-v)}$$

$$\bar{C}_\theta(u,v) = uve^{-\theta\ln u\ln v}$$

$$\bar{C}_\theta(u,v) = (1-u)+(1-v)-1+C(u,v)$$

$$= \cancel{1}-\cancel{u}+\cancel{1}-\cancel{v}-\cancel{1}+\cancel{u}+\cancel{v}-\cancel{1}+uve^{-\theta\ln u\ln v}$$

$$= uve^{-\theta\ln u\ln v}$$

# Volume in $n$-dimensional space

1. For any positive integer $n$ we define $\bar{\mathbb{R}}^n$. Let $\mathbf{a} = (a_1, ...a_n)$ be a vector. We write $\mathbf{a} \leq \mathbf{b}$ if $a_k \leq b_k$ for all $k$
2. For $\mathbf{a} \leq \mathbf{b}$ let $[\mathbf{a}, \mathbf{b}]$ denote $n$-box

$$B = [a_1, b_1] \times [a_2, b_2] \times ..[a_n, b_n]$$

3. Vertices of an $n$-box $B$ are the points $\mathbf{c} = (c_1, c_2, ...c_n)$.

## Definition 1

*Let $S_1$,..$S_n$ be non-empty sets of $\bar{\mathbb{R}}$ and let $H$ be real function defined in $DomH = S_1 \times S_2 ... S_n$. Let $B = [\mathbf{a}, \mathbf{b}]$ be $n$=box with vertices in $DomH$. Then $H$-volume of $B$ is given by*

time of using a

$$V_H(B) = \sum sgn(\mathbf{c})\mathbf{H}(\mathbf{c})$$

*where the sum is taken over all vertices $\mathbf{c}$ and $sgn(\mathbf{c})$ is given by*

$$sgn(\mathbf{c}) = \begin{cases} 1, & \text{if } c_k = a_k \text{ for even number of } k's. \\ -1, & \text{if } c_k = a_k \text{ for odd number of } k's \end{cases} \quad (1)$$

# Copula function in $n$-dimensional case

## Definition 2

*A function $C : \mathbb{I}^d \to \mathbb{I}$ is a copula if*

1. *$C$ in $d$-increasing function ( so the generalized volume defined by $C$ is positive for any box in $\mathbb{I}^d$.)*
2. $$C(0, .., u_i, 0..) = 0, \qquad \forall i = 1, 2, \ldots, d.$$
3. $$C(1, \cdots, 1, u_i, 1, \cdots, 1) = u_i, \qquad \forall i = 1, 2, \ldots, d.$$

## Theorem 2

(a) *Let $F$ be a joint distribution function with margins $F_i$. There exists a copula $C : [0, 1]^d \to [0, 1]$ such that*

$$C\big(F_1(x_1), \cdots, F_d(x_d)\big) \ = \ F(x_1, \cdots, x_d).$$

*If the margins are continuous, then $C$ is unique.*

(b) *And vice versa.*

# A copula model for dependent defaults

## Proposition 3

*Assume we are given:*

- *A term structure of default probabilities* $F_i(T) := \mathcal{P}(\tau_i \le T)$ *for each obligor* $i \in \{1, \cdots, m\}$.
- *A copula function* $C : [0,1]^m \to [0,1]$.

*Then,* $C\Big(F_1(T_1), \cdots, F_m(T_m)\Big)$ *gives a joint probability function for* $\tau_i$.

$$\mathcal{P}\Big(\tau_i \le T_i, \;\; \forall i \in \{1, \cdots, m\}\Big) = C\Big(F_1(T_1), \cdots, F_m(T_m)\Big),$$

# The Gaussian Copula

- If $X, Y$ belong to bivariate normal distribution with correlation $\rho$ then their copula is so-called Gauss (or Gaussian) copula.
- By Proposition 1 we can assume $X$, $Y$ are standard normal
- Then margins are $N(x)$ and $N(y)$, $u = N^{-1}(x)$ , $v = N^{-1}(y)$ and the bivariate Gaussian copula is given by

$$C_\rho(u, v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{N^{-1}(u)} \int_{-\infty}^{N^{-1}(v)} e^{\left(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right)} dx_1 dx_2.$$

- The independence and comonotonicity copulas are special cases : $\rho = 0$ corresponds to independence, $\rho = 1$ to comonotonicity, and $\rho = -1$ to countermonotonicity.
- The Gaussian copula is *implicit*, it does not have a simple closed form, differently from explicit copulas (like Gumbel).

# Gaussian copula: simulation

Let $C_G$ be the Gaussian copula function, and $N_d$ be a $d$-dimensional normal distribution function for a vector $(X_1, \cdots, X_d)$ of standard normal random variables with covariance matrix $\Sigma$.

(a) Draw $(X_1, \cdots, X_d)$ with distribution $N_d(0, \Sigma_d)$.

(b) Define $U_i := N(X_i), \quad i \in \{1, \cdots, d\}$.

*Example*: Two obligators for which we are given hazard rates (or intensities), $\lambda_1$ and $\lambda_2$

1. Generate two correlated variables $x_i$ , $y_i$ with correlation coefficient $\rho$ (Cholevsky)
2. Calculate their CDF, generating series of uniform r.v. $u_i$, $v_i$:
$$u_i = N(x_i), v_i = N(y_i)$$
3. Using $u_i$ and $v_i$ generate random default times $\tau_i^{(1)}, \tau_i^{(2)}$:
$$\tau_i^{(1)} = -\frac{\ln(1-u_i)}{\lambda_1}, \tau_i^{(2)} = -\frac{\ln(1-v_i)}{\lambda_2}$$

# General Procedure for simulations from a copula

- By virtue of Sklar's theorem, we need to generate a pair of $(u, v)$ observations from of uniform $(0,1)$ r.v. $(U, V)$ whose joint distribution function is $C$
- One procedure of generating such a pair $(u, v)$ is the conditional distribution method. For this method we need the conditional distribution function for $V$ given $U = u$, which we denote $c_u(v)$: [3].

$$c_u(v) = \mathbb{P}[V \leq v | U = u] = \lim_{\delta u \to 0} \frac{C(u + \Delta u, v) - C(u, v)}{\Delta u} = \frac{\partial C(u, v)}{\partial u}$$

- Therefore to generate a pair $(u, v)$ we proceed as follows:
  1. Generate two independent uniform $(0,1)$ variates $u$ and $t$
  2. Set $v = c_u^{(-1)}(t)$, where $c_u^{(-1)}$ denotes a quasi-inverse of $c_u$
  3. The desired pair is $(u, v)$

[3] The existence of partial derivatives and their properties , a non-decreasing function, taking values in $[0,1]$ follows from the definition of a copula

# $t$-distributions

An alternative candidates in lieu of normal distribution for a copula is $t$-distribution. It has heavier tails comparing with a normal distribution and better properties at extremes for the copula.

1. The $t$-distributions are very important in classical statistics because of their use in testing and confidence interval.
2. The classical definition of $t$-distribution with $\nu = N - 1$ degrees of freedom is
$$X = \frac{\bar{r} - \mu}{s_r/\sqrt{N}}$$
where $\bar{r}$ and $s_r$ are estimates of the mean and standard deviations, and $r_i, i = 1, ..N$ are drawn from a normal distribution.
3. The variance of $t$-distribution is finite if $\nu > 2$ and equal $\sigma_{t_\nu} = \nu/(\nu - 2)$, the kurtosis is
$$Kurt = 3 + \frac{6}{\nu - 4}$$
4. $t$-distributions have polynomial tails, so heavy tails.

# Mixture Models

Another class of models with heavy tails distributions are *mixture* models. To illustrate, we consider a simple example:

- Consider a distribution which is $90\%$ $N(0,1)$ and $10\%$ $N(0,25)$.
- To generate a r.v. $Y$ from this distribution, we can generate a uniform r.v. $U \in [0,1]$ and a normal r.v. $X$ with mean and variance $1$. If $Y \leq 0.9$, $Y = X$ and if $U \geq 0.9$ , $Y = 5X$.
- These two cases can represent two regimes: one is "normal volatility", and the second is "high volatility", the first one happens $90\%$ of the time.
- A r.v $Y$ from such distribution is very different from normal r.v.$Z$ which matches the variance

Questions:

1. Calculate and compare the probabilities that $|Y| > 6$ and $|Z| > 6$
2. Calculate kurtosis for $Y$ and $Z$.

# Continuous mixtures

A $t_\nu$-distribution is an example of a continuous normal *mixture* distribution.

B The general definition of a normal mixture is given by the following distribution:

$$\mu + \sqrt{W} Z$$

where $\mu$ is a constant equal to the mean, $Z$ is $N(0,1)$, $U$ is a positive r.v. giving the variance of each component; $Z$ and $W$ are independent.

C If $W$ can assume only a finite number of values then it is a discrete (or finite) mixture distribution (previous example)

D If $W$ is continuous, then we have have a continuous mixture distribution. For $t$-distribution the mixing variable $W$ is

$$W = \frac{\nu}{Y}$$

where $Y$ is chi-squared with $\nu$ degrees of freedom.

# $t$-copula

1. To create a multivariate $t$-distribution, we can start with multivariate Gaussian vector $(X_1, \cdots, X_d)$ of standard normal random variables with covariance matrix $\Sigma$
2. Randomizing (or mixing) the normal components with $\sqrt{W} = \sqrt{\frac{\nu}{Y}}$, we will get multivariate $t$-distribution. Note that the correlation matrix stays the same $(\mathbb{E}(W) < \infty$ [4]
3. Let $F_\nu$ be a $t$- d.f. with $\nu$ degrees of freedom, and let $F_\nu^{(-1)}$ be its inverse. Then we can write down a $t$-copula function as

$$C_\nu(u_1, ...u_d) = H_\nu\left(F_\nu^{(-1)}(u_1), ....F_\nu^{(-1)}(u_d)\right)$$

where $H_\nu$ is the multivariate $t$-distribution function with $\nu$ degrees of freedom (and given covariance matrix)

[4] However lack of correlation does imply independence for multivariate $t$ components