Due on Thursday morning

Problem 1
1. **Non-decreasing function of two variables**
Let function $H(x,y)$ be a function defined on $\mathbb{I}^2$ by $H(x,y) = (2x-1)(2y-1)$.
a) Show that $H(x,y)$ is 2-increasing function on $\mathbb{I}^2$
b) However show $H(x,y)$ is not non-decreasing function in each argument on $\mathbb{I}^2$.

Solution: **(a)** To show that $H(x,y)$ is 2-increasing functions on $\mathbb{I}^2$. We take two points (x_1,y_1) and (x_2,y_2) in $\mathbb{I}^2$ such that $x_1 < x_2$ and $y_1 < y_2$. Now, we need to show that $H(x_2,y_2)-H(x_1,y_2)-H(x_2,y_1)+H(x_1,y_1) > 0$. Now, we have

$$\begin{aligned}
&(x_2,y_2) - H(x_1,y_2) - H(x_2,y_1) + H(x_1,y_1)\\
&= (2x_2-1)(2y_2-1) - (2x_1-1)(2y_2-1) - (2x_2-1)(2y_1-1) + (2x_1-1)(2x_1-1)\\
&= 2(2x_2-1)(y_2-y_1) - 2(2x_1-1)(y_2-y_1)\\
&= 2(y_2-y_1)(2x_2-1-2x_1+1)\\
&= 4(y_2-y_1)(x_2-x_1) > 0 \quad (\because y_2 > y_1 \quad \text{and} \quad x_2 > x_1)
\end{aligned} \tag{1}$$

$\therefore H(x,y)$ is 2-increasing on $\mathbb{I}^2$. ■
(b) Consider

$$\begin{aligned}
&H(x+1,y) - H(x,y) = [2(x+1)-1](2y-1) - (2x-1)(2y-1)\\
&= (2y-1)(2x+1-2x+1) = 2(2y-1) < 0 \quad \text{if} \quad y \leq 0
\end{aligned} \tag{2}$$

Again,

$$\begin{aligned}
&H(x,y+1) - H(x,y) = (2x-1)[2(y+1)-1] - (2x-1)(2y-1)\\
&= (2x-1)(2y-1-2y+1)\\
&= 2(2x-1) < 0 \quad \text{if} \quad x \leq 0
\end{aligned} \tag{3}$$

$\therefore H(x,y)$ is not non-decreasing in each argument on $\mathbb{I}^2$. ■

Problem 2
Proposition 1
Let the function H from $S_1 \times S_2$ into $\mathbb{R}$ be 2-increasing and grounded.

- Prove that H is non-decreasing in each argument.

Solution: Let x_1, x_2 be in S_1 with $x_1 \leq x_2$, and let y_1, y_2 be in S_2 with $y_1 \leq y_2$. Then the function $t \to H(t,y_2) - H(t,y_1)$ is non-decreasing on S_1, and the function $t \to H(x_2,t) - H(x_1,t)$ is non-decreasing on S_2. Next, let a_1, a_2 denote the least elements of S_1, S_2, respectively, and set $x = a_1, y = a_2$. Given that H is grounded, thus $H(x,a_2) = 0 = H(a_1,y)$ for all (x,y) in $S_1 \times S_2$. Hence, proved. ■

Problem 3
Proposition 2
Let the function H from $S_1 \times S_2$ into $\mathbb{R}$ be 2-increasing, grounded and has margins $F(x)$ and $G(y)$. Let x_1, y_1 and x_2, y_2 be any points in $S_1 \times S_2$. Prove that

$$|H(x_2,y_2) - H(x_1,y_1)| \leq |F(x_2) - F(x_1)| + |G(y_2) - G(y_1)|$$

Hint: Add and substract for a mixed term, for example $H(x_1,y_2)$ and use triangle inequality.

Solution: Apply the triangle inequality, we have

$$|H(s_2,y_2) - H(x_1,y_1)| \leq |H(x_2,y_2) - H(x_1,y_2)| + |H(x_1,y_2) - H(x_1,y_1)|$$

Now assume $x_1 \leq x_2$. Since H is grounded, 2-increasing, and has margins, by previous proposition we have proven, we have $0 \leq H(x_2,y_2) - H(x_1,y_2) \leq F(x_2) - F(x_1)$. An similar inequality holds when $x_2 \leq x_1$, hence it follows that for any x_1, x_2 in S_1, $|H(x_2,y_2) - H(x_1,y_1)| \leq |F(x_2) - F(x_1)|$. Similarly, for any y_1, y_2 in S_2, $|H(x_1,y_2) - H(x_1,y_1)| \leq |G(y_2) - G(y_1)|$, which complete the proof. ■

Problem 4
Prove that if C is copula, then for every (u, v) in $\mathbb{I}^2$ we have

$$max(u+v-1,0) \leq C(u,v) \leq min(u,v)$$

Note You need to use the 2-increasing property of copula and proposition 1.

Solution: $C(u,0) = C(0,v) = 0$, $C(u,1) = u$, $C(1,v) = v$
$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$
Let $u_2 = v_2 = 1$, $u_1 \leq 1, v_1 \leq 1$
We have $C(u_1,v_1) \geq u_1 + v_1 - 1$, and by definition, $C \geq 0$, so
$C(u,v) \geq max(u+v-1,0)$
Let $u_2 = 1, v_1 = 0, u_1 \leq 1, v_2 \geq 0$
We have $C(u_1,v_2) \leq v_2$, by the same argument we can have $C(u_1,v_2) \leq u_1$, so
$C(u,v) \leq min(u,v)$
$\Rightarrow max(u+v-1,0) \leq C(u,v) \leq min(u,v)$ ■

Problem 5
Fundamental Copulas
Show that $W(u,v) = max(u+v-1,0)$ and $\Pi(u,v) = uv$ are copulas.

Solution: For every u, v in $\mathbb{I}$. Now, for $W(u,v)$, we have

$$W(0,v) = 0 = W(u,0) \quad \text{and} \quad W(u,1) = max(u+1-1,0) = u, W(1,v) = max(1+v-1,0) = v$$

Also, for every u_2, u_1, v_2, v_1 in $\mathbb{I}$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$\begin{aligned} &W(u_2,v_2) - W(u_2,v_1) - W(u_1,v_2) + W(u_1,v_1) \\ &= max(u_2+v_2-1,0) - max(u_2,v_1-1,0) - max(u_1+v_2-1,0) + max(u_1+v_1-1,0) \geq 0 \end{aligned} \tag{4}$$

Hence $W(u,v)$ is a copula.
Similarly,

$$\Pi(0,v) = \Pi(u,0) = 0$$
$$\Pi(1,v) = v \quad \text{and} \quad \Pi(u,1) = u$$

Then,

$$\begin{aligned} &\Pi(u_2,v_2) - \Pi(u_2,v_1) - \Pi(u_1,v_2) + \Pi(u_1,v_1) \\ &= u_2v_2 - u_2v_1 - u_1v_2 + u_1v_1 \\ &= u_2(v_2-v_1) - u_1(v_2-v_1) \geq 0 \quad (\because u_2 > u_1, v_2 > v_1) \end{aligned} \tag{5}$$

Therefore $\Pi(u,v)$ is a copula.

Problem 6
Convex Combination of Copulas
Let C_0 and C_1 be copulas and let θ be any number between $0 \leq \theta \leq 1$. Show that the weighted arithmetic average

$$C = (1-\theta)C_0 + \theta C_1$$

is also a copula. Hence any convex combination of copulas is a copula.

Solution: We express $C(u,v), C_0(u,v)$ and $C_1(u,v)$. Then $C(u,0) = C(0,v) = 0 + 0 = 0$
, and $C(u,1) = \theta u + (1-\theta)u = u$ and $C(1,v) = \theta v + (1-\theta)v = v$.

$$\begin{aligned} &C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \\ &= (1-\theta)(C_0(u_2,v_2) - C_0(v_2,v_1) - C_0(u_1,v_2) + C_0(u_1,v_1)) \\ &+ \theta(C_1(u_2,v_2) - C_1(u_2,v_1) - C_1(u_1,v_2) + C_1(u_1,v_1)) \geq 0 \end{aligned} \tag{6}$$

Thus, C is a copula. ■

Problem 7
Let d.f $H_\theta(x, y)$ be defined as

$$H_\theta(x, y) = (1 + e^{-x} + e^{-y} + (1 - \theta)e^{-x-y})^{-1}$$

for all x, y in $\overline{\mathbb{R}}$ and $\theta \in [-1, 1]$. Show that

- The margins are standard logistic distributions
- When $\theta = 1$ we have Gumbel' bivariate logistic distribution
- When $\theta = 0$ X and Y are independent
- Write down the expression for their copula $C_\theta(u, v)$
- Find and show figures of level sets
$$C_\theta(u, v) = const$$
for $\theta = 0, 0.5, 1$.

Solution: **(a)** $H_\theta(x, +\infty) = \frac{1}{1+e^{-x}}$, $H_\theta(x, +\infty) = \frac{1}{1+e^{-y}}$
(b) when $\theta = 1$, we have

$$H_\theta(x, y) = \frac{1}{1 + e^{-x} + e^{-y}}$$

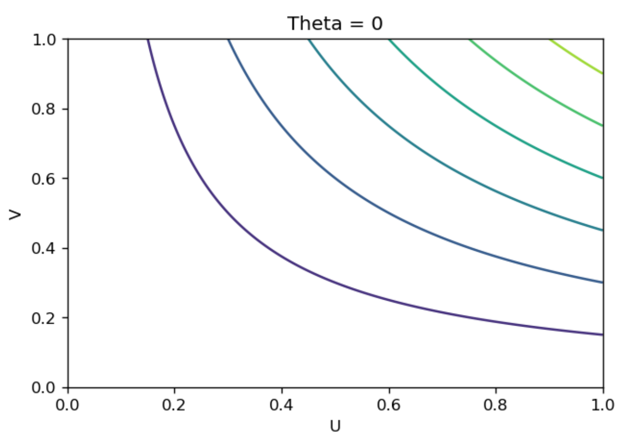
(c) when $\theta = 0$, we have,

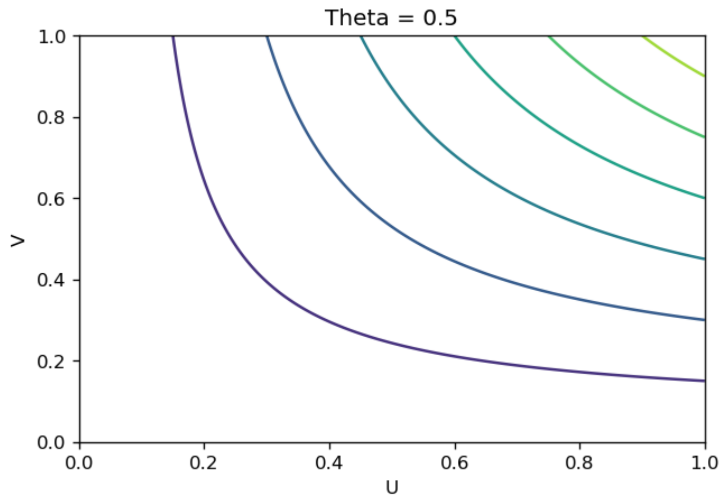
$$\begin{aligned} H_\theta(x, y) &= \frac{1}{1 + e^{-x} + e^{-y} + e^{-x-y}} \\ &= \frac{1}{1 + e^{-x}} \times \frac{1}{1 + e^{-y}} \\ &= H_\theta(x) \cdot H_\theta(y) \end{aligned} \tag{7}$$

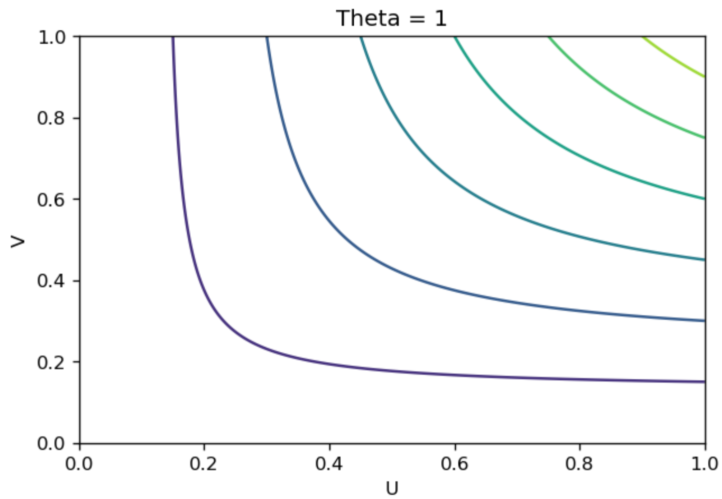
(d) $u = H(x, +\infty)$, $e^{-x} = \frac{1}{u} - 1$
$v = H(+\infty, y), e^{-y} = \frac{1}{v} - 1$

$$\begin{aligned} C(u, v) &= \frac{1}{1 + e^{-x} + e^{-y} + (1 - \theta)e^{-x}e^{-y}} \\ &= \frac{1}{\frac{1}{u} + \frac{1}{v} - 1 + (1 - \theta)(\frac{1}{u} - 1)(\frac{1}{v} - 1)} \\ &= \frac{uv}{1 - \theta(1 - u)(1 - v)} \end{aligned} \tag{8}$$

Figures and sample codes are as follows:








```python
import numpy as np
import matplotlib.pyplot as plt

def f(u,v,theta):
    return (u*v) / (1-theta*(1-u)*(1-v))

theta = [0,0.5,1]
u = np.linspace(0.00000000001, 1, 1000)
v = np.linspace(0.00000000001, 1, 1000)

U, V = np.meshgrid(u, v)
```