Homework : Copulas

Credit Risk (MF772) Fall 2021 Instructor: Roza Galeeva

Due date: 8 am, Thursday Dec 2 . Please, note that `late assignments will not be accepted.`

In this homework you will need to prove some of the propositions from the last lecture. Note that frequently proofs can be based on the previous propositions, so the best is to attempt to do them all. For this homework I **strongly** *prefer you submit your answers in pdf or word format (not handwritten please), there is enough time to do that*

1. Non-decreasing functions of two variables

Let function $H(x, y)$ be a function defined on $\mathbb{I}^2$ by $H(x, y) = (2x - 1)(2y - 1)$.

a) Show that $H(x, y)$ is 2-increasing function on $\mathbb{I}^2$

b) However show $H(x, y)$ is not non-decreasing function in each argument on $\mathbb{I}^2$.

2. Proposition 1

Let the function H from $S_1 \times S_2$ into $\mathbb{R}$ be 2-increasing and grounded.

- Prove that H is nondecreasing in each argument.

3. Proposition 2

Let the function H from $S_1 \times S_2$ into $\mathbb{R}$ be 2-increasing, grounded and has margins $F(x)$ and $G(y)$. Let x_1, y_1 and x_2, y_2 be any points in $S_1 \times S_2$. Prove that

$$\mid H(x_2, y_2) - H(x_1, y_1) \mid \leq \mid F(x_2) - F(x_1) \mid + \mid G(y_2) - G(y_1) \mid$$

Hint: Add and subtract for a mixed term, for example $H(x_1, y_2)$ and use triangle inequality.

4. Fréchet- Hoeffding bounds

Prove that if C is a copula, then for every (u, v) in $\mathbb{I}^2$ we have

$$\max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v)$$

Note You need to use the 2-increasing property of copula and proposition 1.

5. Fundamental Copulas

Show that $W(u, v) = \max(u + v - 1, 0)$ and $\Pi(u, v) = uv$ are copulas.

6. Convex Combination of copulas

Let C_0 and C_1 be copulas and let θ be any number between $0 \leq \theta \leq 1$. Show that the weighted arithmetic average

$$C = (1 - \theta)C_0 + \theta C_1$$

is also a copula. Hence any convex combination of copulas is a copula.

7. Generalized Gumbel' distribution

Let d.f. $H_\theta(x, y)$ be defined as

$$H_\theta(x, y) = \left(1 + e^{-x} + e^{-y} + (1 - \theta)e^{-x-y}\right)^{-1}$$

for all x, y in $\bar{\mathbb{R}}$ and $\theta \in [-1, 1]$. Show that

a) The margins are standard logistic distributions

b) When $\theta = 1$ we have Gumbel' bivariate logistic distribution

c) When $\theta = 0$ X and Y are independent

d) Write down the expression for their copula $C_\theta(u, v)$

e) Find and show figures of level sets

$$C_\theta(u, v) = const$$

for $\theta = 0, 0.5, 1$.