KMV model

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Outline

Today we will discuss KMV model which is based on the Merton model

Main Learning Goals of the lecture:

- Revisit Merton (sensitivities, spreads behavior with time)
- Discuss the assumptions of KMV model
- The method of calculation of asset values and asset volatilities
- Distance to Default and mapping to EDF measure in KMV model

Random recovery in Merton Model

- In Merton model, at maturity T in case of default we recover the value V_T which is a random variable. Therefore we deal with a random recovery
- We may price the defaultable bond in Merton's model using the Simple Model of the previous sections: $N(-d_1) = \pi$

$$B = De^{-rT}N(d_2) + VN(-d_1)$$

$$= B_M(0,T) \;=\; DB(0,T)P(0,T) \;+\; D\pi\, B(0,T)P^{def}(0,T).$$

- It turns out that this is indeed the price of the defaultable bond if π is the expected recovery rate conditional on default.

$$V_0 N(-d_1) = D\pi^e e^{-rT} N(-d_2)$$

$$\pi^e = \frac{V_0 e^{rT}}{D} \frac{N(-d_1)}{N(-d_2)}$$

Proposition 1

The price of the defaultable debt with random recovery is:

$$B_M(0,T) \;=\; DB(0,T)P(0,T) \;+\; D\pi^e B(0,T)P^{def}(0,T)$$

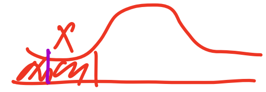
$$\pi^e = \frac{1}{\sqrt{2\pi}\, N(-d_2)} \int_{-\infty}^{-d_2} \frac{V_0}{D} e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}x + rT} e^{-\frac{1}{2}x^2} \, dx$$

where

$$\pi^e := E\big[\pi \mid V_T < D\big], \quad \pi := V_T / D.$$

$q_3 \longleftarrow$ exam

Structural Model in Practice



$V_T \le D$

$p(\frac{V_T}{D} \geq 0.5) = ?$

$0 \le \frac{V_T}{D} < 100\%$

$P_{50\%} = \frac{N(-d_2) - N(a^*)}{N(-d_2)}$

There is a number of industry models that descend from the Merton model. $\frac{V_T}{D} = \frac{1}{2}, V_0 = D, \quad -\frac{1}{2}\sigma^2 T + \sigma\sqrt{T} d^* + rT = \ln(\frac{1}{2})$

- An important example is so called public firm *EDF* model that is maintained by Moody's Analytics.
- "EDF" stands for *expected default frequency*; this is a specific estimate of the default probability of a given firm over one year horizon.
- The methodology is built on earlier by KMV, a private company named after its founders. [1]
- In KMV model the information contained in the firm's stock price and balance sheet can then be translated into an implied risk of default.

[1] The KMV model is named after Kealhofer, McQuown and Vasicek

EDF in Merton Model and DD

- In the classic Merton model the one year default probability of a given firm is given by the probability that the asset value in one year is below the threshold D (liabilities)
- Therefore

$$EDF_{Merton} = \mathcal{Q}(V_T \leq D) = 1 - \mathcal{N}\left(\frac{\ln V_0 - \ln D + (r - \sigma_V^2/2)}{\sigma_V}\right) \tag{1}$$

The important quantity of the KMV model is so called *Distance to Default*, DD:

$$\ln\frac{V_0}{D} = -\ln\left[\frac{D}{V_0}\right] = -\ln\left[\frac{D+V_0-V_0}{V_0}\right]$$

$$DD = \frac{V_0 - D}{\boxed{\sigma_V} V_0}$$

percentage

$$EDF_{Merton} \approx 1 - \frac{V_0 - D}{\sigma_V V_0}$$

It can be shown that the argument in normal CDF in eq 1

$$\approx -\ln\left[1 - \frac{V_0 - D}{V_0}\right] \approx \frac{V_0 - D}{V_0}$$

$$\ln(1+x) \approx x$$

$$A = \frac{\ln V_0 - \ln D + (r - \sigma_V^2/2)}{\sigma_V} \approx \frac{V_0 - D}{\sigma_V V_0}$$

since $r - \sigma_V^2/2$ is small.

$$p_{dd} \approx 1 - \mathcal{N}(DD)$$

First stage in KMV model

The derivation of the actual probabilities of default proceeds in several stages.

1) Estimation of the market value and volatility of the firm's assets.
2) Calculation of the **distance to default**, which is an index measure of default risk.
3) Scaling of the distance to default to actual probabilities of default using an **empirical** default database.

1) Estimation of the asset value V_0 and its volatility σ_V

Suppose we want to estimate V_0 and σ_V in the context of Merton's model.
Since

$$S_T = \max\{V_T - D, 0\},$$

we can use the price of a European call to find

$$S_0 = V_0 \mathcal{N}(d_1) - De^{-rT}\mathcal{N}(d_2)$$

We can find V_0 as

$$V_0 = \frac{S_0 + De^{-rT}\mathcal{N}(d_2)}{\mathcal{N}(d_1)} \qquad (2)$$

Since S_0, r, T, D are observable, we have one equation and two unknowns. V_0, σ_V

Jones method with Itô formula

On the other hand, considering $S_t = C^{BS}(V_t, t)$, an application of Ito's Lemma yields

$$dS_t = (\cdots)dt + \frac{\partial C^{BS}}{\partial V_t}\sigma_V V_t dW_t.$$

Or

$$dS_t/S_t = \frac{1}{S_t}(\cdots)dt + \frac{1}{S_t}\frac{\partial C^{BS}}{\partial V_t}\sigma_V V_t dW_t.$$

We assume that the equity price volatility σ_S is a constant that satisfies

$$\sigma_S = \frac{1}{S_t}\frac{\partial C^{BS}}{\partial V_t}\sigma_V V_t.$$

We assume that the volatility σ_S of the equity price can be observed from the market: being either the historical volatility, or an option-implied volatility.

The numerical procedure

Hence

$$\sigma_S = \frac{1}{S_0}\frac{\partial C^{BS}}{\partial V_0}\sigma_V V_0 = \frac{1}{S_0}\mathcal{N}(d_1)\sigma_V V_0.$$

Therefore, we have two equations and two unknowns:

$$\begin{cases} S_0 = C^{BS}(V_0, \sigma_V, r, T, D). \\ \sigma_S = \frac{1}{S_0}\mathcal{N}\left(\frac{\ln(V_0/D) + (r + \sigma_V^2/2)T}{\sigma_V\sqrt{T}}\right)\sigma_V V_0. \end{cases} \quad (3)$$

Numerical (root finding) methods can be used to find a solution (V, σ_V).

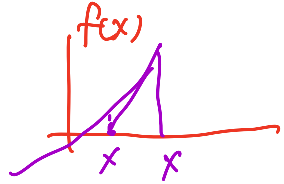
$f_1(V_0, \sigma_V) = 0 \quad f_2(V_0, \sigma_V) = 0$

$$\begin{cases} f_1 = C^{BS}(V_0, \sigma_V) - S = 0 \\ f_2 = \frac{1}{S}\mathcal{N}(d_1)\sigma_V V_0 - \sigma_S = 0 \end{cases}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial V_0} & \frac{\partial f_1}{\partial \sigma_V} \\ \frac{\partial f_2}{\partial V_0} & \frac{\partial f_2}{\partial \sigma_V} \end{bmatrix}$$

Example

Newton



$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

x

first guess

$$\vec{X} = \begin{pmatrix} V_0 \\ \sigma_V \end{pmatrix}$$

$V_0^{(1)}, \sigma_V^{(1)}$

$$V_0^{(1)} = S + De^{-rT}$$

	Billions
Equity Value	26.237
Equity Volatility	45.65%
Liabilities	51.662
Risk Free Rate	3.41%
Horizon	1 year

Figure 1: Example: Enron 2001

$$\sigma_V^{(1)} = \frac{V_0^{(1)}}{S} \times \sigma_S$$

$$\vec{X}^{(n)} = \vec{X}^{(n-1)} - J^{-1}\left[\vec{X}^{(n-1)}\right] \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

- There are various methods of solving a system of nonlinear equations for several variables

① S, σ_S

$$f_i(x_1, ...x_N) = 0$$

- We always start with initial guess and iterate until achieving a giving precision.
- Of the typical methods is Newton method

Iterative Method

- The above relationship 3 between equity volatility and asset volatility holds only instantaneously. ([Crosbie, Bohn 2003]). In practice market leverage moves around far too much. Overestimate (or underestimate) of asset vol will lead to wrong default probability
- Instead of estimating the instantaneous asset volatility, another more complex procedure is applied. We observe a time series of market equity values E_t (based on equity prices and number of shares) and liabilities D_t.
  1. Start with the initial estimate $\sigma_V^{(0)}$
  2. Using time series E_t , D_t and eq. 2 calculate asset values $V_t^{(0)}$.
  3. Using the calculated time series V_t calculate the new estimate of asset volatility $\sigma_V^{(1)}$.
  4. Construct a new series of asset values $V_t^{(1)}$, ...
  5. The procedure is iterated n times, until the volatility estimates $\sigma_V^{(n-1)}$ and $\sigma_V^{(n)}$ are sufficiently close.

Difference in the assumptions between Merton and KMV model

In the EDF model, the capital structure of the firm is modelled in a more sophisticated manner.

- There are several classes of liabilities:
  1. long and short term debt
  2. intermediate cash payouts and dividends are allowed
  3. in later editions of the model, convertible securities and preferred are included too
- Default occur at any time.
- Default point (the threshold value D^* such that the company defaults if V_t falls below D^*), is determined from a more detailed analysis of the term structure of the firm debt.
- The equity value is not given anymore by a simple Black Scholes formula but by a different function $g(t, V_t, \sigma_V)$ which has to be computed *numerically*.
- The general idea of the approach to estimate the asset value V_0 and the asset volatility is, however , exactly the same as described before.

Practical Approach to measure Default Probability

- There are three basic types of information available that are relevant to the default probability of a publicly traded firm:
  1. financial statements
  2. market prices of the firm's debt and equity
  3. subjective appraisals of the firm's prospects and risk.
- Investors form debt and equity prices as they anticipate the firm's future. In determining the market prices, investors use, amongst many other things, subjective appraisals of the firm's prospects and risk, financial statements, and other market prices.
- Market prices are the result of the combined willingness of many investors to buy and sell and thus prices embody the synthesized views and forecasts of many investors.
- The most effective default measurement, therefore, derives from models that utilize both market prices and financial statements.

Assumptions

(A) KMV uses a more sophisticated model for the capital structure of the firm than Merton's model,

$$S_0 = g(V_0, r, \sigma_V, d, T, c),$$

where

d = leverage ratio of the firm (short and long term liabilities)

c = average coupon payment.

Note that g is not a closed form and is not published.

(B) The second equation is:

$$\sigma_S \;=\; \frac{1}{S_0}\frac{\partial g}{\partial V}\,\sigma_V V_0.$$

More on the KMV equity model

- The model assumes the firm's equity is a perpetual option with the default point acting as the absorbing barrier for the firm's asset value
- When the asset value hits the default point, the firm is assumed to default.
- Multiple classes of liabilities are modeled: short-term liabilities, long-term liabilities, convertible debt, preferred equity, and common equity.
- When the firm's asset value becomes very large, the convertible securities are assumed to convert and dilute the existing equity.
- In addition, cash payouts such as dividends are explicitly used in the VK model

Default point

The KMV model replaces the simple structure of a zero coupon bond with maturity D, for a more complex structure.

In particular, D is replaced by D^*, defined below.

Definition 1

The **default point D^*** *is defined to be*

$$D^* \;=\; \textit{short-term debt} \;+\; \frac{1}{2}\;\textit{long-term debt}.$$

In addition, it is assumed $\mu_V - 1/2 \cdot \sigma_V^2 \approx 0$, claiming that in practice that difference is negligible.

In KMV model

$$\left(\frac{\ln \frac{V_0}{D^*} + (\mu_V - 1/2 \cdot \sigma_V^2)}{\sigma_V} \right)$$

is replaced by the distance to default, defined next.

Distance to Default

Definition 2

The **distance to default** DD *is defined by*

$$DD = \frac{V_0 - D^*}{V_0 \sigma_V}.$$

Now we can think of the default probability simply as

$$Q(V<D) = 1 - N(DD) = 1 - \mathcal{N}\left(\frac{V_0 - D^*}{V_0 \sigma_V}\right). \tag{4}$$

Clearly, the greater the distance to default DD, the higher the survival probability, and hence the lower the default probability.

Note that DD is an index specific to each reference entity, which is the input to compute the default probabilities.

Asset volatility by industry and size

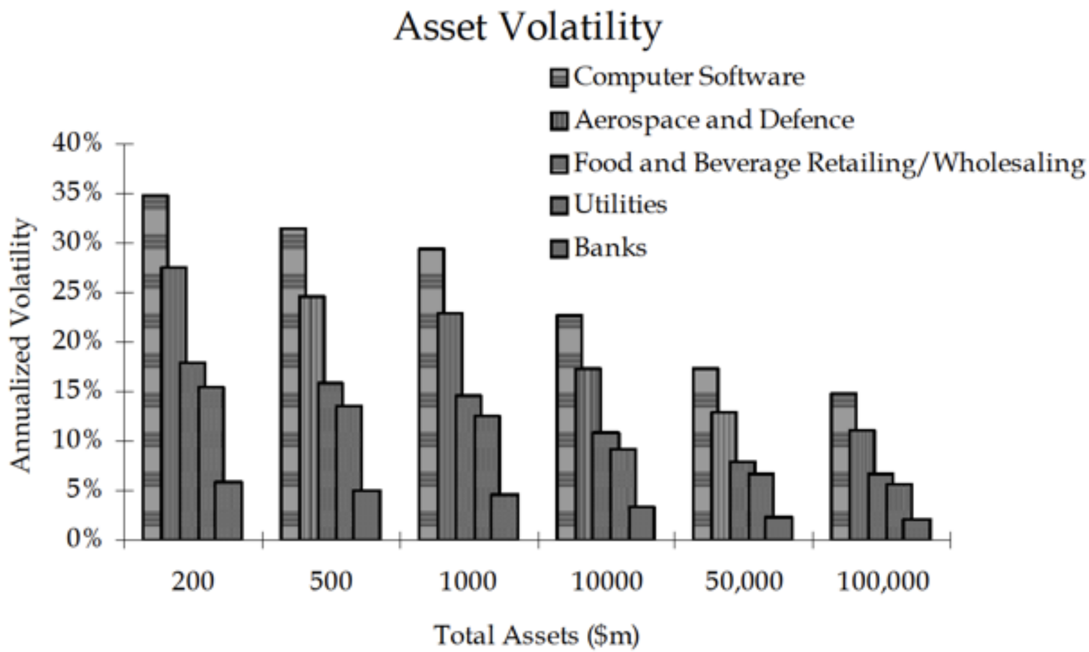


Figure 2: Asset volatility for several industries and asset sizes, [2]

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[2] Source: Modeling Default Risk, KMV document, Crosbie, Bohn 2003

Asset Volatility

- Industries with low asset volatility (for example, banking) tend to take on larger amounts of leverage,
- Industries with high asset volatility (for example, computer software) tend to take on less
- As a consequence of these compensatory differences in leverage, equity volatility is far less differentiated by industry and asset equity volatility is far less differentiated by industry and asset size than is asset volatility.
- Asset value, business risk and leverage can be combined into a single measure of default risk which compares the market net worth to the size of a one standard deviation move in the asset value. We called this ratio distance to default.

Variables

There are six variables that determine the default probability of a firm over some horizon, from now until time H

- The volatility of the future assets value at time H .
- The current asset value.
- The level of the default point, the book value of the liabilities.
- The distribution of the asset value at time H .
- The length of the horizon, H
- The expected rate of growth in the asset value over the horizon.

Calculate DD

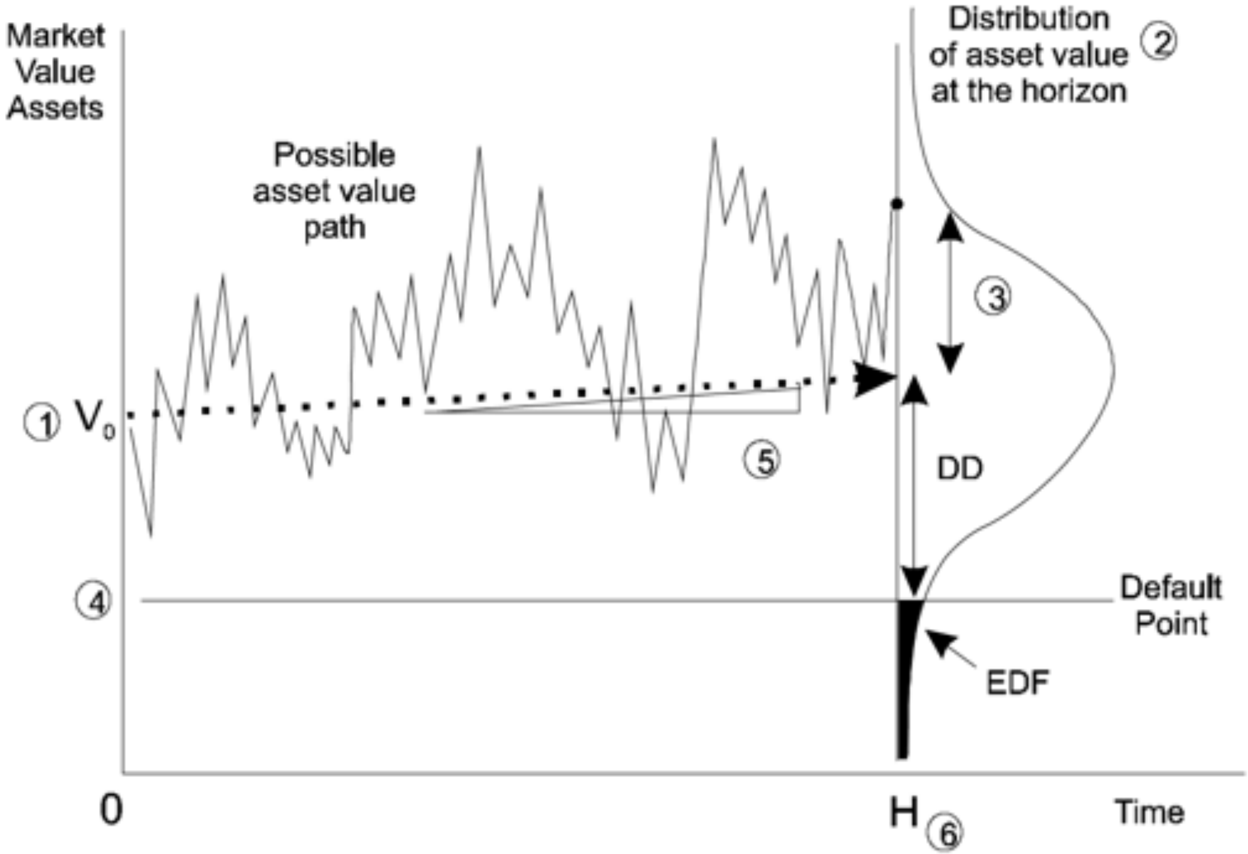

Figure 3:

Putting Together

TABLE 2

Variable	Value	Notes
Market value of equity	$ 110,688 MM1	(Share Price) x (Shares Outstanding).
Book Liabilities	$ 64,062 MM	Balance sheet.
Market value of assets	$ 170,558 MM	Option-pricing model.
Asset volatility	21%	Option-pricing model.
Default point	$ 47,499 MM	Liabilities payable within one year.
Distance-to-default	3.5	Ratio: $\frac{170-47}{170\times 21\%}$ (In this example we ignore the growth in the asset value between now and the end of the year.)

Figure 4: Table 1

3) Computing the default probability

This last step consists of mapping the distance to default to the actual probabilities of default. In KMV terminology, the default probabilities are called **Expected Default Frequencies** (EDF).

(A) In practice one cannot assume log-normal distributions for the assets returns. This is why equation (4) is not used by KMV.

(C) Instead, they use an extensive empirical data base, analyzing more than 2000 US companies that have defaulted or entered into bankruptcy over the last 20 years.

(C) *Example:* Suppose that Company X has distance to default $DD = 4$. Then the corresponding EDF is calculated as follows:

$$EDF = \frac{\#\text{ of firms with } DD = 4 \text{ that defaulted within one year}}{\#\text{ of total firms with } DD = 4}$$

Assuming that among a population of 1000 firms with a DD equal to 4, 20 defaulted one year later, then

$$EDF = 20/1000 = 0.02 = 2\%.$$

Recent changes to the methodology

We highlight some of major changes: Interested students can get familiar with more details reading Moody technical documents.

- Updated and improved DD-to-EDF measure mapping. The mapping for non-financial firms incorporates international defaults 1987-2014.
- Introduced a financial firm-specific DD-to-EDF mapping that uses international financial firm failures spanning 1987 – 2014. EDF values for financial firms represent the risk of either a default or a government bailout.
- Refinements to volatility calculations. These refinements fine-tune a volatility measure that is now implemented consistently across the globe, and granularity is now country-level. Additionally, the volatility measure better reflects the volatility trends of the specific country and industry that a firm operates within.

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