MF795: STOCHASTIC METHODS IN ASSET PRICING I

Fall Semester 2020

Assignment № 2

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HW 2

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2.2 ① S is a σ-field, so $\phi \in S$. $X^{-1}(\phi) = \{\omega \in \Omega : X(\omega) \in \phi\} = \phi$, where $B = \phi \Rightarrow \phi \in X^{-1}(S)$.

② If $A \in X^{-1}(S)$, then A can be intepreted as $A = X^{-1}(B)$ for a fixed B. $B \in S \Rightarrow B^c = X \setminus B \in S$ $\therefore A^c = X^{-1}(B^c) \in X^{-1}(S)$.

③ For any $A_i \in X^{-1}(S)$ $(i \in \mathbb{N}_{++})$. $A_i = X^{-1}(B_i) = \{\omega \in \Omega, X(\omega) \in B_i\}$ $(i \in \mathbb{N}_{++})$. S is a σ-field. So for $\forall B_i \in S$, we get $\cup B_i \in S$. Thus, $\cup A_i = \cup X^{-1}(B_i) = \{\omega \in \Omega, X(\omega) \in \cup B_i\} = X^{-1}(\cup B_i)$ $\therefore \cup A_i \in X^{-1}(S)$.

Therefore, we have $X^{-1}(S) = \{X^{-1}(B), B \in S\}$ is a σ-field.

2.5 For $f^{-1}(A \times B) \in \mathcal{F}$, we know that $A \in S$ and $B \in \mathcal{T}$ and the σ-field $S \otimes \mathcal{T}$ is generated by $A \times B$. So, for $f^{-1}(S \otimes \mathcal{T}) \subseteq \mathcal{F}$, f is $(S \otimes \mathcal{T}) / \mathcal{F}$-meaurable. If f is $(S \otimes \mathcal{T}) / \mathcal{F}$-measurable, there is $f^{-1}(S \otimes \mathcal{T}) \subseteq \mathcal{F}$ and we know that $S \times \mathcal{T}$ is a σ-field genrated by $A \times B$ for $A \in S$ and $B \in \mathcal{T}$. Therefore, we have $f^{-1}(A \times B) \in \mathcal{F}$, for ~~every~~ every $A \in S$ and $B \in \mathcal{T}$.

2.6 If $X^{-1}(]-\infty, a]) \in \mathcal{F}$ for every $a \in \mathbb{R}$, $X : \Omega \to \mathbb{R}$ we want to prove that for any $B \in \mathcal{B}(\mathbb{R})$, $X^{-1}(B) \in \mathcal{F}$, $(\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. From the theorem 2.4, if $]-\infty, a]$ is a family set that generates $\mathcal{B}(\mathbb{R})$, and for every $]-\infty, a]$, $X^{-1}(]-\infty, a]) \in \mathcal{F}$. We can get that $X^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{B}(\mathbb{R})$. By definition: $A =]a_1, b_1] \cup]a_2, b_2] \cup \cdots \cup]a_n, b_n]$. $\mathcal{B}(\mathbb{R})$ is a σ-field generated by A. $(-\infty < a_1 < b_1 \le a_2 \le b_2 \cdots < +\infty)$ and $]-\infty, a]$ for every $a \in \mathbb{R}$ can generate all kind of A. Therefore, $X^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{B}(\mathbb{R})$. X is Borel measurable. If X is Borel-measurable. It implies that for every $B \in \mathcal{B}(\mathbb{R})$, $X^{-1}(B) \in \mathcal{F}$. By definition, $\forall B \in \mathcal{B}(\mathbb{R})$ can be generated by $]-\infty, a]$ for every $a \in \mathbb{R}$. Thus, $X^{-1}(]-\infty, a]) \in \mathcal{F}$ for every $a \in \mathbb{R}$.

2.7 $\because$ for measurable space $(\Omega, \mathcal{F})$, $(X, \mathcal{S})$ and $(Y, \mathcal{T})$, we have $X: \Omega \to X$ and $Y: X \to Y$, X is $\mathcal{S}/\mathcal{F}$-measurable and Y is $\mathcal{T}/\mathcal{S}$-measurable.

$\therefore Y^{-1}(a) \in \mathcal{S}$ for every $a \in \mathcal{T}$ and $X^{-1}(b) \in \mathcal{F}$ for every $b \in \mathcal{S}$.

So, $X^{-1}(Y^{-1}(a)) \in \mathcal{F}$ for every $a \in \mathcal{T}$ and $Z^{-1}(a) = (Y \circ X)^{-1}(a) = X^{-1}(Y^{-1}(a)) \in \mathcal{F}$.

Thus, Z is $\mathcal{T}/\mathcal{F}$-measurable.

2.8 Given $f: \Omega \times X \mapsto Y$ is $\mathcal{T}/(\mathcal{F} \otimes \mathcal{S})$ measurable, we can create a $h: \Omega \to \Omega \times X$, $(\omega, x^*) \in \Omega \times X$ and x^* is fixed. So, $h^{-1}(\omega \times x^*) \in \mathcal{F}$. $\therefore h$ is $\mathcal{F} \otimes \mathcal{S}/\mathcal{F}$-measurable function. The mapping $Z: \omega \rightsquigarrow f(\omega, x^*)$ can be writtern as $\omega \rightsquigarrow (\omega, x^*) \rightsquigarrow f(\omega, x^*)$ as $f(h(\omega))$.

$\because f$ is $\mathcal{T}/(\mathcal{F} \otimes \mathcal{S})$ measurable, h is $(\mathcal{F} \otimes \mathcal{S})/\mathcal{F}$-measurable. From the 2.7, the mapping $f(h(\omega)): \omega \rightsquigarrow f(\omega, x^*)$ is $\mathcal{T}/\mathcal{F}$-measurable.

2.19 For the topological space $(X, \mathcal{T})$, $(Y, \mathcal{V})$, $f: X \to Y$ is continuous. We have $f^{-1}(A) \in \mathcal{T}$, A is any open set in $\mathcal{V}$ and $f^{-1}(A)$ is any open sets in $\mathcal{T}$. $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ are the smallest σ-field that includes all open sets in $\mathcal{T}$ and $\mathcal{V}$: $\mathcal{B}(X) \subseteq \mathcal{T}$, $\mathcal{B}(Y) \subseteq \mathcal{V}$. Thus, $f: X \to Y$ can be based on σ-field $\mathcal{B}(X)$ and $\mathcal{B}(Y)$. From the theorem 2.4. $\mathcal{B}(Y)$ is the smallest σ-field that contains all open sets A and $\mathcal{B}(X)$ is the smallest σ-field that contains all open sets $f^{-1}(A)$.

Therefore, we can conclude that $f: X \to Y$ is measurable on $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, where $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ are two Borel σ-field that these two space generate.

2.23 $Z_{10} \in \{-10, -8, -6 \cdots 0 \cdots 6, 8, 10\}$

$P(Z_{10}=-10) = C_{10}^{0} (\frac{1}{2})^{10} (\frac{1}{2})^{0} = C_{10}^{0} \frac{1}{2^{10}}$

$P(Z_{10}=-8) = C_{10}^{1} (\frac{1}{2})^{9} (\frac{1}{2})^{1} = C_{10}^{1} \frac{1}{2^{10}}$

$\vdots$

$P(Z_{10}=10) = C_{10}^{0} (\frac{1}{2})^{0} (\frac{1}{2})^{10} = C_{10}^{0} \frac{1}{2^{10}}$

$$\therefore P(Z_{10}=k) = \begin{cases} C_{10}^{(10-k)/2} \cdot \frac{1}{2^{10}}, & k \in \{-10, -8, \cdots 0, \cdots 8, 10\} \\ 0, & \text{others} \end{cases}$$

$$\therefore F_{Z_{10}}(x) = \begin{cases} 0 & x < -10 \\ C_{10}^{0} \frac{1}{2^{10}} & -10 \le x \le -8 \\ (C_{10}^{0} + C_{10}^{1}) \frac{1}{2^{10}} & -8 \le x \le -6 \\ \vdots & \vdots \\ (C_{10}^{0} + \cdots + C_{10}^{9}) \frac{1}{2^{10}} & 8 \le x < 10 \\ 1 & x \geq 10 \end{cases}$$

2.24 i) $X: \Omega \to \mathbb{R}$; suppose $x<y$, $F_X(x) = P(X \in]-\infty, x])$. $F_X(y) = P(X \in]-\infty, y]) = P(X \in]-\infty, x] \cup]-\infty, y])$. $F_X(y) \geq P(X \in]-\infty, x]) = F_X(x)$.

$\therefore F_X(x)$ is increasing function.

ii) càdlàg: suppose $X_n (n \geq 1)$ is a decreasing sequence in $\mathbb{R}$: $X_1 > X_2 \cdots > X_n$.

we need to prove $X_n \to X_0$ when $n \to \infty$. $\lim F(x_n) = \lim F(x_0)$ $X_n \to X_0 (n \to \infty)$.

then $]-\infty, X_0] = \bigcap_{n \in \mathbb{N}}]-\infty, X_n]$. $F_X(X_0) = \mathcal{L}_X(]-\infty, X_0]) = \mathcal{L}_X(\bigcap_{n \in \mathbb{N}}]-\infty, X_n])$

$= \lim_{X_n \to X_0} \mathcal{L}_X(]-\infty, X_n]) = \lim_{X_n \to X_0} F_X(X_n)$ $\therefore \lim_{X_n \to X_0} (F(x_n)) = F(x_0)$, it is right continuous.

iii) $\lim_{X \to +\infty} F_X(x) = \mathcal{L}_X(]-\infty, +\infty]) = 1$, $\lim_{X \to -\infty} F_X(x) = \mathcal{L}_X(]-\infty, -\infty]) = 0$.

iv) Left continuous $\Rightarrow$ non-atomic

$\mathcal{L}_X(]X_n, x]) = \mathcal{L}_X(]-\infty, x]) - \mathcal{L}_X(]-\infty, X_n])$

$= F_X(X) - F_X(X_n)$

$\mathcal{L}_X(\{x\}) = \lim_{x_n \to x} \mathcal{L}_X(]x_n - x])$

$= \lim_{x_n \to x} (F_X(x) - F_X(x_n))$

$= F_X(x) - \lim_{x_n \to x} F_X(x_n) = 0$

It is non-atomic. non-atomic $\Rightarrow$ left continuous if the distribution law $\mathcal{L}_X$ is non-atomic. $F_X(x) - \lim_{x_n \to x} F_X(x_0) = \mathcal{L}_X(]-\infty, x]) - \lim_{x_n \to x} F_X(x_n)$

$= \lim_{x_n \to x} \mathcal{L}_X(]x_n, x]) = \mathcal{L}_X(\bigcap_{n \in \mathbb{N}}]x_n, x]) = \mathcal{L}_X(\{x\}) = 0$

v) In fact, it is possible for the distribution law $\mathcal{L}_X$ and the Lebesgue measure Λ to be singular to one another ($\mathcal{L}_X \perp \Lambda$) if the distribution function is continuous. let C be cantor set. P_∞ can be treated as a measure on $C \cap \mathcal{B}([0,1])$, since P_∞ is non-atomic, $\mathcal{L}_\infty$ is continuous. $\mathcal{L}_\infty(C) = 1$, $\Lambda(C) = 0$. Thus, $\mathcal{L}_\infty \perp \Lambda$.

2.29. $F_X(x) = \mathcal{L}_X(]-\infty, x]) = \mathbb{P}(X(\omega) \in]-\infty, x]) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in]-\infty, x]\}) = \mathbb{P}(\{\omega \in [0, x]\}) = \Lambda([0,x]) = x$, $\forall x \in [0,1]$. $\therefore$ X is uniformly distributed in $[0,1]$.

$F_Y(y) = \mathcal{L}_Y(]-\infty, y]) = \mathbb{P}(Y(\omega) \in]-\infty, y]) = \mathbb{P}(\{\omega \in \Omega : Y(\omega) \in]-\infty, y]\}) = \mathbb{P}(\{\omega \in [1-y, 1]\}) = \Lambda([1-y, 1]) = y$, $\forall y \in [0,1]$. $\therefore$ Y is uniformly distributed in $[0,1]$.

$F_Z(z) = \mathcal{L}_Z(]-\infty, z]) = \mathbb{P}(Z(\omega) \in]-\infty, z]) = \mathbb{P}(\{\omega \in \Omega : Z(\omega) \in]-\infty, z]\}) = \mathbb{P}(\{\omega \in [0, \frac{z-a}{b-a}]\}) = \Lambda([0, \frac{z-a}{b-a}]) = \frac{z-a}{b-a}$, $\forall z \in [a,b]$ $\therefore$ Z is uniformly distributed in $[a,b]$.

2.35 ① ~~$\int_{\mathbb{R}} f(x) \Lambda(dx)$~~ (on the back!) last page

~~$= \int_{\{x \in \mathbb{R} : f(x) = 0\}} f(x)(=0) \Lambda(dx) + \int_{\{x \in \mathbb{R} : f(x) > 0\}} f(x) \Lambda(dx)(=0) = 0 + 0 = 0$~~

~~② $\int_{\mathbb{R}} f(x) \Lambda(dx)$~~

~~$= \int_{\{x \in \mathbb{R} : f(x) = +\infty\}} f(x)(=+\infty) \Lambda(dx)(>0) + \int_{\{x \in \mathbb{R} : f(x) \in \mathbb{R}_+\}} f(x)(>0) \Lambda(dx) = +\infty + a(a>0) = +\infty$~~

~~③ $\int_{\mathbb{R}} f(x) \Lambda(dx)$~~

~~$= \int_{\{x \in \mathbb{R} : f(x) > 0\}} f(x)(>0) \Lambda(dx)(>0) + \int_{\{x \in \mathbb{R} : f(x) = 0\}} f(x)(=0) \Lambda(dx) = a + 0 = a > 0$~~

2.40 For any $B \in \mathcal{B}(\mathbb{R})$

$$f^{-1}(B) = \begin{cases} \phi, & \{0\}, \{1\} \notin B \\ Q, & \{1\} \in B,\ \{0\} \notin B \\ \mathbb{R}/Q, & \{1\} \notin B,\ \{0\} \in B \\ \mathbb{R}, & \{0\}, \{1\} \in B \end{cases}$$

There four sets all belong to $\mathcal{B}(\mathbb{R})$, so $f(x)$ is measurable.

Define a partition on $[0,1]$. $0 = X_0 < X_1 < X_2 < \cdots < X_n = 1$

let $\lambda = \max_{i=0}^{n-1} \{(X_{i+1} - X_i)\}$, $\Delta X_i = X_i - X_{i-1}$

let m be the Riemann summation of $f(x)$, then there exists $\mathcal{E}_i \in]X_{i-1}, X_i[$,

$\Delta_i m = f(\mathcal{E}_i) \cdot \Delta X_i$, $m = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\mathcal{E}_i) \Delta X_i$. If $\mathcal{E}_i \in Q$, $m=1$; If $\mathcal{E}_i \notin Q$,

$m=0$. So, Riemann integral does not exist.

$$\int_{[0,1]} f(x) \Lambda(dx) = 1 \times \Lambda(Q \cap [0,1]) + 0 \times \Lambda([0,1] \setminus Q) = 0$$

$$\int_{\mathbb{R}} f(x) \Lambda(dx) = 1 \times \Lambda(Q) + 0 \times \Lambda(\mathbb{R} \setminus Q) = 0$$

2.42 $\int_X f(x) \mu(dx) = \int_X f^+(x) \mu(dx) - \int_X f^-(x) \mu(dx)$

$\int_X |f(x)| \mu(dx) = \int_X f^+(x) \mu(dx) + \int_X f^-(x) \mu(dx)$

$f^+(x) \geq 0 \quad f^-(x) \geq 0$

$\left|\int_X f(x) \mu(dx)\right| \leq \max\left(\int_X f^+(x) \mu(dx), \int_X f^-(x) \mu(dx)\right) \leq \int_X f^+(x) \mu(dx) + \int_X f^-(x) \mu(dx)$

$= \int_X |f(x)| \mu(dx)$

$\therefore \left|\int_X f(x) \mu(dx)\right| \leq \int_X |f(x)| \mu(dx)$

2.48 $P(X=k) = P(1-P)^k$, $\mathbb{E}[X] = \sum_{k=0}^{\infty} k \cdot P(1-P)^k = P\sum_{k=0}^{\infty} k(1-P)^k$

let $S_k = \sum_{k=0}^{\infty} k(1-P)^k \Rightarrow (1-P)S_k = \sum_{k=0}^{\infty} k(1-P)^{k+1}$

$S_k - (1-P)S_k = (1-P) + (1-P)^2 + \cdots = \frac{1}{P}$ $\therefore S_k = \frac{1}{P^2}$ $\therefore \mathbb{E}[X] = P \cdot S_k = \frac{1}{P}$

$P(X=1) = \frac{1}{2}$ $P(X=2) = \frac{1}{4}$

$\mathbb{E}[X] = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} \cdots$ ①

$\frac{1}{2}\mathbb{E}[X] = 1 \times \frac{1}{4} + 2 \times \frac{1}{8} + \cdots$ ②

①-②: $\frac{1}{2}\mathbb{E}[X] = \frac{1}{2} + (\frac{1}{4} + \frac{1}{8} + \cdots) = 1$ $\therefore \mathbb{E}[X] = 2$

2.49 X is poisson distrubtion , so $P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$

$\mathbb{E}[X] = \sum_{k=0}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k\in N_{++}} k\frac{\lambda^k}{k!} = \lambda e^{-\lambda}\sum_{k\in N_{++}}\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\sum_{k\in N_{+}}\frac{\lambda^k}{k!}$

$\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\cdots = \sum_{i\in N_{+}}\frac{x^k}{k!}$

$\therefore e^{\lambda} = \sum_{k\in N}\frac{\lambda^k}{k!}$ $\therefore \mathbb{E}[X] = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$

2.50 R.V $X = X(w)$ $\Omega = \{w_1, w_2 \cdots w_n \cdots\}$ $X(w_i) = i$ for $i \in N_{++}$

$\therefore 0 < X(w_i) < \infty$ for every $w_i \in \Omega$

$P(X(w_i) = i) = \frac{C}{i^2}$ when $C = \frac{6}{\pi^2}$

~~$\sum_{i=1}^{\infty} P(X(w_i) = i) = 1$ $i \in N_{++}$~~

$\mathbb{E}[X] = \sum_{i\in N_{++}} \frac{C}{i} = \frac{6}{\pi^2} \sum_{i\in N_{++}} \frac{1}{i} \geq \frac{6}{\pi^2}(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots)$

$= \lim_{N\to\infty} \frac{6}{\pi^2}(1 + \frac{N}{2}) = \infty$

2.60 ① $\int_{\mathbb{R}_+} (\int_{\mathbb{R}_+} e^{-2x-3y} dy) dx = \int_0^{\infty} (\int_0^{\infty} e^{-2x-3y} dy) dx = \int_0^{\infty} -\frac{1}{3} e^{-2x-3y} \Big|_0^{+\infty} dx$

$= \int_0^{\infty} \frac{1}{3} e^{-2x} dx = -\frac{1}{6} e^{-2x} \Big|_0^{\infty} = \frac{1}{6}$

② $\int_{\mathbb{R}_+} (\int_{\mathbb{R}} e^{-2x-3y} dx) dy = \int_0^{\infty} (\int_0^{\infty} e^{-2x-3y} dx) dy = \int_0^{\infty} -\frac{1}{2} e^{-2x-3y} \Big|_0^{+\infty} dy$

$= \int_0^{\infty} e^{-3y} \cdot [e^{-2x} \cdot (-\frac{1}{2})] \Big|_0^{\infty} dy = \int_0^{\infty} e^{-3y} \cdot \frac{1}{2} dy = \frac{1}{2} [e^{-3y} \cdot (-\frac{1}{3})] \Big|_0^{\infty} = \frac{1}{6}$

e^{-2x-3y} is a positive function, so we can apply Tonelli-Fubini's theorem.

uniform dist 2.68 a) $E[X] = \int_{]a,b[} x \phi(x) dx = \int_a^b x \frac{1}{b-a} dx = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2} = \frac{a+b}{2}$

$E[X^2] = \int_{]a,b[} x^2 \phi(x) dx = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+2ab+b^2}{3}$

Ex dist b) $E[X] = \int_{\mathbb{R}_+} x c e^{-cx} dx = -\frac{1}{c} \int_{\mathbb{R}_+} cx \, d e^{-cx} = -\frac{1}{c} (cx e^{-cx} |_{\mathbb{R}_+} - \int_{\mathbb{R}_+} e^{-cx} dx)$

$= -\frac{1}{c} (e^{-cx} |_{\mathbb{R}_+}) = -\frac{1}{c} \times -1 = \frac{1}{c}$

$E[X^2] = \int_{\mathbb{R}_+} c x^2 e^{-cx} dx = -x^2 e^{-cx} |_{\mathbb{R}_+} 2x e^{-cx} dx = \frac{1}{c} \times \frac{2}{c} = \frac{2}{c^2}$

standard norm c) $E[X] = \int_{\mathbb{R}} \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x e^{-\frac{x^2}{2}} dx = 0 \quad \because f(x) = x e^{-\frac{x^2}{2}} = -f(-x) \therefore E[X] = 0$

$E[X^2] = \int_{\mathbb{R}} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = -\int_{\mathbb{R}} \frac{x}{\sqrt{2\pi}} d e^{-\frac{x^2}{2}} = -\frac{1}{\sqrt{2\pi}} (x e^{-\frac{x^2}{2}} |_{\mathbb{R}} - \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx)$

$= -\frac{1}{\sqrt{2\pi}} (0 - \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} d\frac{x}{\sqrt{2}} = \frac{1}{\sqrt{\pi}} \times \frac{2\sqrt{\pi}}{2} = 1$

normal dist d) $E[X] = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot x \, dx = \mu$

$E[X^2] = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot x^2 dx = \mu^2 + \sigma^2$

Cauchy e) $E[X] = \int_{\mathbb{R}} \frac{x}{\pi(1+x^2)} dx = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+x^2} d(x^2+1) = \frac{1}{2\pi} \ln(1+x^2) \Big|_{-\infty}^{+\infty} = \infty - \infty \quad \therefore$ not defined

$E[X^2] = \int_{\mathbb{R}} \frac{x^2}{\pi(1+x^2)} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^2 x \, dx = +\infty$

Gamma f) $E[X] = \int_0^{+\infty} \frac{c^p}{\Gamma(p)} x^{p-1} e^{-cx} dx = \frac{c^p}{\Gamma(p)} \int_0^{+\infty} x^p e^{-cx} dx \quad \{cx = t\} \quad \frac{c^p}{\Gamma(p)} \int_0^{+\infty} \frac{t^p e^{-t}}{c^p} \cdot \frac{1}{c} dt$

$= \frac{c^p}{\Gamma(p)} \cdot \frac{1}{c^{p+1}} \int_0^{+\infty} t^p e^{-t} dt = \frac{c^p}{\Gamma(p)} \cdot \frac{\Gamma(p+1)}{c^{p+1}} = \frac{p}{c}$ (Given property of gamma function $\Gamma(p+1) = p\Gamma(p)$ for $p>0$)

$E[X^2] = \int_0^{+\infty} x^2 \frac{c^p}{\Gamma(p)} x^{p-1} e^{-cx} dx = \frac{c^p}{\Gamma(p)} \int_0^{+\infty} x^{p+1} e^{-cx} dx = \frac{c^p}{\Gamma(p)} \cdot \frac{1}{c^{p+2}} \int_0^{\infty} t^{p+1} e^{-t} dt = \frac{p(p+1)}{c^2}$

Inverse Gamma g) $E[X]=\int_0^{+\infty}\frac{xc^p}{\Gamma(p)}x^{-p-1}e^{-\frac{c}{x}}dx$; $x=\frac{1}{y}$, $\frac{1}{x}=y$ $dy=-\frac{dx}{x^2}$;

$E[X]=\int_0^{+\infty}\frac{-x^2c^p}{\Gamma(p)}y^pe^{-cy}dy=\int_0^{+\infty}-\frac{c^p}{\Gamma(p)}y^{p-2}e^{-cy}dy=\frac{c}{p-1}$

$E[X^2]=\int_0^{+\infty}\frac{c^p}{\Gamma(p)}y^{p-3}e^{-cy}dy=\frac{c^2+cp-2c}{(p-2)(p+1)^2}$

beta dist h) $E[X]=\int_0^1 x^{\alpha}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}dx=\frac{\alpha}{\alpha+\beta}$

$E[X^2]=\int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}dx=\frac{\alpha^3+\alpha^2\beta+\alpha^2+\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}=\frac{\alpha+\alpha^2}{(\alpha+\beta+1)(\alpha+\beta)}$

chi-squard i) $E[X]=\int_0^{+\infty}x\cdot e^{-\frac{1}{2}(x+\lambda)}\frac{x^{\frac{k}{4}-\frac{1}{2}}}{2\lambda^{\frac{k}{4}-\frac{1}{2}}}I_{\frac{k}{2}-1}(\sqrt{\lambda x})$, for $x\in\mathbb{R}_{++}$, $=k+\lambda$

$E[X^2]=k^2+\lambda^2+2k\lambda+2(k+2\lambda)$

2.69 a) $E[e^{-X}]=\int_{\mathbb{R}_+}ce^{-x}e^{-cx}dx=\int_{\mathbb{R}_+}ce^{-(c+1)x}dx=-\frac{c}{c+1}e^{-(c+1)x}\Big|_{\mathbb{R}_+}=\frac{c}{c+1}$

b) $E[e^{-X}]=\int_{\mathbb{R}}\frac{e^{-x}}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$, let $z=\frac{x-\mu}{\sigma\sqrt{2}}$ $x=\sqrt{2}\sigma z+\mu$,

$=\int_{\mathbb{R}}\frac{e^{-\sqrt{2}\sigma z}\cdot e^{-\mu}}{\sqrt{\pi}}e^{-z^2}dz=\frac{e^{-\mu}}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-\sqrt{2}\sigma z}e^{-z^2}dz=\frac{e^{-\mu+\frac{\sigma^2}{2}}}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-(z-\frac{\sigma\sqrt{2}}{2})^2}d(z-\frac{\sigma\sqrt{2}}{2})$

$=e^{-\mu+\frac{\sigma^2}{2}}$

2.71 a) uniform: $Var(x)=\frac{4a^2+4ab+4b^2}{12}-\frac{3a^2+6ab+3b^2}{12}=\frac{(a-b)^2}{12}=\frac{(b-a)^2}{12}$

$Std(x)=\frac{b-a}{2\sqrt{3}}$. The higher the b, the longer the std.

The shape is a horizontal line, and density is closer to 0.

b) Ex dist: $Var(x)=\frac{1}{c^2}$ $std(x)=\frac{1}{c}$. The higher c the lower std and the density gets deeper and deeper.

c) Standard norm: $Var(X)=1$, $std(x)=1$, The shape is like a bell.

d) normal: $Var(X)=\sigma^2$ $std(x)=\sigma$. The higher the σ, the higher the std, and the density is more spread over x.

e) Cauchy dist:

$Var(x)=\infty^2-(\infty-\infty)^2$ not defined.

f) Gamma dist: $Var(x) = \frac{p}{c^2}$ $Std(x) = \frac{\sqrt{p}}{c}$, the higher the p, the larger the std. when p gets larger and larger, the density function move from a exponential dist shape to bell shape. when c gets larger, the density spread wider over x axis.

g) Inverse gamma: $Var(x) = \frac{c^2}{(p-2)(p-1)^2}$ $Std(x) = \frac{c}{(p-1)\sqrt{p-2}}$, the larger the P, the smaller the std. When c gets larger and lerger, the density spread over x axis wider and bell-shaped.

h) $Var(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ $Std(x) = \sqrt{\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha-\beta)^2}}$

The std decreases with α and β getting larger. The function goes more concentrated when α and β become smaller.

i) chi-squrd: $Var(x) = 2(k+2\lambda)$ $std = \sqrt{2(k+2\lambda)}$. The std increases with k and λ. The function goes concentrted when k and λ lerger.

~~2.75 By using python:~~ $\left(\int_{-1}^{1} (1-x^2)^{1000} dx\right)^{1/1000} = 0.99712 \Rightarrow 1$

(on the back)
last page

$\left(\int_{-\infty}^{\infty} (e^{-1000x^2}) dx\right)^{1/1000} \cancel{= 0.99712 \Rightarrow 1}$

2.79 $\phi_X(x) = \begin{cases} \frac{2}{x^3} & x > 1 \\ 0 & x \le 1 \end{cases}$ $\int_{\mathbb{R}} \phi_X(x) dx = -\frac{1}{x^2}\Big|_1^{+\infty} = 1$

$E[X] = \int_1^{+\infty} \frac{2}{x^2} dx = -\frac{2}{x}\Big|_1^{\infty} = 2$

$E[X^2] = \int_1^{+\infty} \frac{2}{x} dx = 2\ln|x|\Big|_1^{+\infty} = +\infty$

2.80 $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $E[f_p(Y)] = E[Y^p] = E[(e^x)^p] = E[e^{px}] = \int_{\mathbb{R}} e^{px} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$= \frac{1}{\sqrt{2\pi}} e^{\frac{p^2}{2}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-p)^2} d(x-p) = e^{\frac{p^2}{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-p)^2} d(x-p) = e^{\frac{p^2}{2}}$

$E[Y] = E[e^x]$ when $p=1$. $E[e^x] = e^{\frac{1}{2}}$.

For Jensen inequality : $f_p(E[Y]) \le E[f_p(Y)] \Rightarrow e^{\frac{p}{2}} \le e^{\frac{p^2}{2}} \Rightarrow \frac{p}{2} \le \frac{p^2}{2} \Rightarrow p \ge 1$ or $p=0$ it holds.

when $0<p<1$, Jensen's inequality doesn't hold. $f_p(x)$ is convex only if $p>1$, which is required for $f(x)$ in Jensen Inequality. However, when $p=0$ or $p=1$. $f_p[E(Y)] = E[f_p[Y]]$ and $f_p(x)$ is concave when $0<p<1$, thus inequality does not hold.

2.81 $P(-3\sigma \le X \le 3\sigma) = \int_{-3\sigma}^{3\sigma} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{\sqrt{\pi}} \int_{-3\sigma}^{3\sigma} e^{-\frac{x^2}{2\sigma^2}} d\frac{x}{\sigma\sqrt{2}} = 0.9973002039 = a$

For Chebyshev : $P(-3\sigma \le x \le 3\sigma) \ge 1 - \frac{\sigma^2}{9\sigma^2} = \frac{8}{9} = 0.888888889 = b$

$a>b$ significantly.

2.88 a) $F_X^{-1}(y)$ $X \sim N(0,1)$

$F_X^{-1}(y) = \inf\{x \in \mathbb{R}, F_X(x) \ge y\}$ $F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2}} d\mu = \frac{1}{2} + \int_0^x \frac{1}{\sqrt{\pi}} e^{-\frac{\mu^2}{2}} d\frac{\mu}{\sqrt{2}}$

$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\mu^2} du$ $\therefore F_X(x) = y = \frac{1}{2} + \frac{1}{2}\mathrm{erf}(x)$ $2y-1 = \mathrm{erf}(x)$

$F_X^{-1}(y) = \mathrm{erf}^{-1}(2y-1)$

b) $F_X^{-1}(y) = \begin{cases} 0 & 0 \le y \le \frac{1}{8} \\ 1 & \frac{1}{8} < y \le \frac{4}{8} \\ 2 & \frac{4}{8} < y \le \frac{7}{8} \\ 3 & \frac{7}{8} < y \le 1 \end{cases}$

2.89 We know that $F^{-1}(y) = \inf\{x \in \mathbb{R} : F(x) \geq y\}$

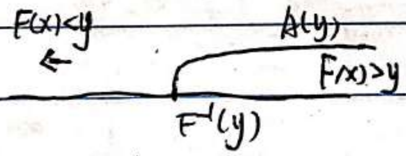
① $F(x) \geq y \Rightarrow x \geq F^{-1}(y)$. Let $A = \{x \in \mathbb{R} : F(x) \geq y\}$, according to the def, $F^{-1}(y) = \inf A$. So, there will be $x \geq F^{-1}(y)$ for any x in A.

② $x \geq F^{-1}(y) \Rightarrow F(x) \geq y$. Define a decreasing sequence x_n in A: $\lim x_n = \inf A = F^{-1}(y)$, then $F(x_n) \geq y$, $\lim F(x_n) \geq y$ and given F is an increasing càdlàg function. So, $F(\lim x_n) \geq y$, $F(F^{-1}(y)) \geq y$ with the condition $x \geq F^{-1}(y)$.

$F(x) \geq F(F^{-1}(y)) \geq y \Rightarrow F(x) \geq y$.

$\therefore x \geq F^{-1}(y) \Rightarrow F(x) \geq y$

2.91 $F: I \to \mathbb{R}$ is increasing and càdlàg and F is continuous, so F^{-1} always exists. Assume I is $[l, \,[$, $F(l) = \inf(\mathrm{span}(F))$. If $F^{-1}(y) = l$ for some $y \in \mathrm{span}(F)$, then l must be that $y = F(l)$. $y \in \mathrm{span}(F)$, $y > F(l) \Rightarrow$ $A(y) = \{x \in I : F(x) \geq y\} \to [F^{-1}(y), \,[$. $F^{-1}(y) \geq I$, $x \geq F^{-1}(y)$, $F(x) \geq y$.



F is continuous, so we know that $F(F^{-1}(y)) \geq y$ $F(F^{-1}(y) - \varepsilon) < y$. So $F(F^{-1}(y)) = y$ $\forall y \in \mathrm{span}(F)$. $\{y \in \mathrm{Span}(F) : y \geq F(x)\} \geq \{y \in \mathrm{span}(F) : F^{-1}(y) \geq x\}$.

$F(y) = \mathrm{Sup}(\mathrm{span}(F))$ if $\{y \in \mathrm{span}(F) : F^{-1}(y) \geq x\}$ is $\emptyset$.

We know $\inf\{y \in \mathrm{span}(F) : F^{-1}(y) \geq x\} \geq \inf\{y \in \mathrm{span}(F) : y \geq F(x)\}$ ①

$\inf\{y \in \mathrm{span}(F) : F^{-1}(y) \geq x\} > y' > F(x)$ | $y' \in \mathbb{R}$. F-continuous $y' \in \mathrm{span}(F)$.

$F^{-1}(y') < x$ and $F(x) < y'$ contradiction. So in ①, the two must be equal. (according to 2.88)

3.3 $F(Y<y) = F(e^x < y) = F(x < \log y) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\log y - a)^2}{2\sigma^2}} \cdot \frac{1}{y} dy$

$= \int_{-\infty}^{+\infty} \frac{1}{y\sqrt{2\pi}\sigma} e^{-\frac{(\log y - a)^2}{2\sigma^2}} dy$, $\quad f(y) = F_Y'(y) = \frac{1}{y\sqrt{2\pi}\sigma} e^{-\frac{(\log y - a)^2}{2\sigma^2}} dy$

3.6 $\underset{①}{E[e^{-\theta\frac{X-a}{\sigma^2} - \frac{1}{2}\frac{|\theta|^2}{\sigma^2}}]} = \underset{②}{E[e^{\theta\frac{X-a}{\sigma^2} - \frac{1}{2}\frac{|\theta|^2}{\sigma^2}}]} = 1$

$① = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-a)^2}{2\sigma^2} - \frac{2\theta(x-a)}{2\sigma^2} - \frac{\theta^2}{2\sigma^2}} dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{-(x-a+\theta)^2}{2\sigma^2}} dx - a + \theta$

$= \frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-\frac{(x-a+\theta)^2}{2\sigma^2}} d\frac{x-a+\theta}{\sqrt{2}\sigma} = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = 1$

$② = \int_{\mathbb{R}} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-a)^2}{2\sigma^2} + \frac{2\theta(x-a)}{2\sigma^2} - \frac{\theta^2}{2\sigma^2}} dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(x-a-\theta)^2}{2\sigma^2}} dx - a - \theta = 1$

3.7 ① Gussian dist:

$m(t) = E[e^{tx}] = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}((x-\mu)^2 - 2t\sigma^2 x)} dx$

$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}((x-(\mu+t\sigma^2))^2 - 2\mu t\sigma^2 - t^2\sigma^4)} dx$

$= e^{\frac{2\mu t\sigma^2 + t^2\sigma^4}{2\sigma^2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-(\mu+t))^2}{2\sigma^2}} dx = e^{\mu t + \frac{1}{2}t^2\sigma^2}$

② poission dist:

$m(t) = E[e^{tx}] = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k e^{-\lambda}}{k!}$

$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t\lambda)^k}{n!}$

$= e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t - 1)}$

3.16 a) $P(R) = \frac{2}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{4} = \frac{1}{3} + \frac{1}{12} = \frac{5}{12}$

b) $P(R_1|R) = \frac{\frac{1}{3}}{\frac{5}{12}} = \frac{4}{5}$

3.17 $P = \frac{2}{3}\times\cancel{1}\times\frac{1}{2}\times\frac{3}{5} = \frac{1}{5}$

3.18 (a) $P(C) = P(C|K)\times P(K) + P(C|NK)\times P(NK) = 1\times 0.7 + \frac{1}{n}\times 0.3 = 0.7 + \frac{0.3}{n}$

(b) $P(K|C) = \frac{P(C|K)\times P(K)}{P(C)} = \frac{0.7}{0.7+\frac{0.3}{n}}$

(c) $n\to+\infty$ $\quad \lim_{n\to\infty} P(C) = 0.7 \quad \lim_{n\to\infty} P(K|C) = 1$

3.26 $E[|XY|] = \int_{\mathbb{R}^2} |xy|\, \mathcal{L}_{X,Y}(x,y) = \int_{\mathbb{R}^2} |xy| \mathcal{L}_X(x)\mathcal{L}_Y(y)$ {independent} $= \int_{\mathbb{R}^2} |x||y| \mathcal{L}_X(x)\mathcal{L}_Y(y)$

$= \int_{\mathbb{R}} |x| \mathcal{L}_X(x) \int_{\mathbb{R}} |y| \mathcal{L}_Y(y) = E[|X|]E[|Y|] < \infty \quad \therefore XY \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$

$E[XY] = \int_{\mathbb{R}^2} xy\, \mathcal{L}_{X,Y}(x,y) = \int_{\mathbb{R}^2} xy\, \mathcal{L}_X(x)\mathcal{L}_Y(y) = \int_{\mathbb{R}} x\mathcal{L}_X(x) \int_{\mathbb{R}} y\mathcal{L}_Y(y) = E[X]E[Y]$

3.34 $Var(S_n) = E[X_1 + X_2 + \cdots + X_n - E(X_1 + X_2 + \cdots + X_n)]^2$

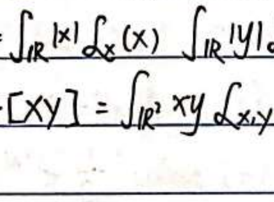
$= E[(X_1 - E(X_1)) + (X_2 - E(X_2)) + \cdots + (X_n - E(X_n))]^2$

$= E(X_1 - E(X_1))^2 + E(X_2 - E(X_2))^2 + \cdots + E(X_n - E(X_n))^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} E[X_i - E(X_i)][X_j - E(X_j)]$

$= Var(X_1) + Var(X_2) + Var(X_n) + 2\sum_{i=1}^{n-1}\sum_{j=i}^{n} cov(X_i, X_j)$

$= \sum_{i=1}^{n} Var(X_i) + 2\sum_{i=1}^{n-1}\sum_{j=i}^{n} cov(X_i, X_j)$

3.37



Suppose that $A_i \in \mathcal{G}'$ $\quad B \in \mathcal{G}$ and

$B = A_1 + A_2 + \cdots + A_n$

$E[X|\mathcal{G}'] = \frac{E[1_A X]}{P(A)} \cdot 1_A$

$E[E[X|\mathcal{G}']|\mathcal{G}] = \frac{E\left[\sum_{i=1}^{n}\left(\frac{E[1_{A_i}X]}{P(A_i)} 1_{A_i}\right)\right]}{P(B)} \cdot 1_B = \frac{\sum_{i=1}^{n} E[1_{A_i} X]}{P(B)} \cdot 1_B$

$= \frac{E[1_B X]}{P(B)} \cdot 1_B = E[X|\mathcal{G}]$

3.44 $\because E[Z|Y] = E[Z|\sigma(Y)] = \phi(Y)$, X is F-measurable.

$\therefore$ A S_Z can be found that S_Z is F-measurable and satisfies $E[ZX] = E[S_Z X]$

$E[\phi(Y)g(Y)] = \int_{\mathbb{R}^2} \phi(Y)g(Y)\,\phi_{X,Y}(x,y)\,dx\,dy = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\int_{\mathbb{R}} f(x,y)\phi_{X,Y}(x,y)dx}{\int_{\mathbb{R}} \phi_{X,Y}(x,y)dx} g(y)\,\phi_{X,Y}(x,y)\,dx\,dy$

$= \int_{\mathbb{R}} \frac{\int_{\mathbb{R}} f(x,y)\,\phi_{X,Y}(x,y)dx}{\int_{\mathbb{R}} \phi_{X,Y}(x,y)\,dx} \int_{\mathbb{R}} \phi_{X,Y}(x,y)dx \cdot g(y)\,dy = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\phi_{X,Y}(x,y)dx \cdot g(y)dy$

$= E[f(x,y)\cdot g(y)]$. As a result, $E[\phi(Y)g(Y)] = E[f(x,y)\cdot g(Y)]$

Since $g(Y)$ is Y-measurable, we get that $E[f(x,y)|Y] = \phi(Y)$

4.6 Consider A sequence $\{1_{B_i}\}_{i\in\mathbb{N}}$, $B_i = \left[\frac{i-2^m}{2^m}, \frac{i+1-2^m}{2^m}\right]$, $m = [\log_2(i)]$

$P(|X_i - X_*| > \varepsilon) = \frac{1}{2^m}$

$(P(|X_i - X_*| > \varepsilon))_{i\in\mathbb{N}} = (\frac{1}{2^m})_{i\in\mathbb{N}}$ $\forall \varepsilon' \in \mathbb{R}_{++}$ $\exists\, i' = 2^{m'}$ s.t.

$2^{m'} > \frac{1}{\varepsilon}'$ when $i > i'$

$\left(\frac{1}{2^{[\log_2(i)]}}\right) < \left(\frac{1}{2^m}\right) < \varepsilon$

$\therefore (P(|X_i - X_*| > \varepsilon))_{i\in\mathbb{N}}$ converges to 0 $\therefore \{1_{B_i}\}_{i\in\mathbb{N}}$ converges to

probability $\sup_{i,j \geq n} |X_i - X_j| = 1$ $\lim_{n\to\infty} \sup_{i,j \geq n} |X_i - X_j| = 1$ for all $\omega \in \Omega$

$\therefore \{1_{B_i}\}_{i\in\mathbb{N}}$ does not converge almost surely.

4.11 $\because \{X_i\}_{i\in\mathbb{N}}$ $\{Y_i\}_{i\in\mathbb{N}}$ are statistically indistinguishable.

$\therefore |X_j - X_i| = |Y_j - Y_i|$ $\therefore P(|X_j - X_i| > \varepsilon) = P(|Y_j - Y_i| > \varepsilon)$

$\therefore d_P(X_j, X_i) = \inf\{\varepsilon \in \mathbb{R}_+ : P(|X_j - X_i| > \varepsilon) \leq \varepsilon\} = \inf\{\varepsilon \in \mathbb{R}_+ : P(|Y_j - Y_i| > \varepsilon) \leq \varepsilon\}$
$= d_P(Y_j, Y_i)$

$\therefore$ If $\{X_i\}_{i\in\mathbb{N}} = X$ converges in probability P

$\sup_{j\geq i} d_P(X_j, X_i) = \sup_{j\geq i} d_P(Y_j, Y_i)$

$\therefore \lim_{i\to\infty} \sup_{j\geq i} d_P(X_j, X_i) = 0 = \lim_{i\to\infty} \sup_{j\geq i} d_P(Y_j, Y_i)$

$\therefore Y = \{Y_i\}_{i\in\mathbb{N}}$ is also convergent in probability P.

If $X=\{x_i\}_{i\in\mathbb{N}}$ converges P a.s. $|x_j - x_i| = |y_j - y_i|$

$\therefore \sup_{j\geq i}|x_j - x_i| = \sup_{j\geq i}|y_j - y_i|$

$\therefore P(\sup_{j\geq i}|x_j - x_i| \geq \varepsilon) = P(\sup_{j\geq i}|y_j - y_i| \geq \varepsilon)$

$\because$ X is convergent P a.s.

$\therefore \lim_i P(\sup_{j\geq i}|x_j - x_i| \geq \varepsilon) = 0 = \lim_i P(\sup_{j\geq i}|y_j - y_i| \geq \varepsilon)$

$\therefore Y = \{y_i\}_{i\in\mathbb{N}}$ is convergence P a.s.

4.21 $\because \Xi$ is uniformly integrable then $\lim_{c\to\infty}\sup_{\xi\in\Xi} E[|\xi|1_{\{|\xi|\geq c\}}]=0$ which means $\forall \varepsilon\in\mathbb{R}_{++}$ $\exists C'$ when $C\geq C'$ $\sup_{\xi\in\Xi} E[|\xi|1_{\{\xi\geq C\}}]<\varepsilon$ which means $\forall \xi\in\Xi$ $E[|\xi|1_{\{|\xi|\geq c'\}}]\leq\varepsilon$

$E[|\xi|1_{\{|\xi|<c'\}}]\leq c'$ $\therefore$ Set $C=c'+\varepsilon$ $E[|\xi|1_{\{|\xi|\geq c'\}}]+E[|\xi|1_{\{|\xi|<c'\}}]=E[|\xi|]$

$\leq c'+\varepsilon=C$ for any $\xi\in\Xi$.

4.25 $\because |\eta|$ is an integrable random variable. $\therefore \lim_{c\to\infty}E[|\eta|1_{\{|\eta|\geq c\}}]=0$

$\because |\xi|\leq|\eta|$ P-a.s $\therefore \lim_{c\to\infty}E[|\xi|1_{\{|\xi|\geq c\}}]=0$ for all $\xi\in\mathcal{A}$

$\therefore \forall\varepsilon>0, \varepsilon\in\mathbb{R}_{++}$ $\exists C$ when $c'>c$ $E[|\xi|1_{\{|\xi|\geq c'\}}]<\varepsilon$ for all $\xi\in\mathcal{A}$

$\therefore \sup_{\xi\in\mathcal{A}}E[|\xi|1_{\{|\xi|\geq c'\}}]<\varepsilon$

which means $\forall\varepsilon>0, \varepsilon\in\mathbb{R}_{++}$ $\exists C$ when $C'>C$ $\sup_{\xi\in\mathcal{A}}E[|\xi|1_{\{|\xi|\geq c'\}}]<\varepsilon$

$\therefore \lim_{c\to\infty}\sup_{\xi\in\mathcal{A}}E[|\xi|1_{\{|\xi|\geq c\}}]=0$

$\therefore$ family $\mathcal{A}$ is uniformly integrable.

4.30 $\omega = \{\varepsilon_i \in \{-1,+1\}\}_{i\in\mathbb{N}_{++}} \in \Omega_\infty$

A run of size $k \geq 2$ inside ω is any string of k tokens, which makes $A_i = \varepsilon_j = \varepsilon_{j+1} = \cdots = \varepsilon_{j+k-1}$. Thus, we have $P(A_i) = \frac{1}{2^k}$

Thus, $\sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} \frac{1}{2^k} = +\infty$, for ω have infinitely many of sequences of size k.

According to Second Borel-CANTELLI LEMMA.

$\sum_{i\in\mathbb{N}} P(A_i) = \infty \Rightarrow P(A_i, i.o) = 1$

Thus, we can conclude that a randomly chosen sequence to contain infinitly many runs of any possible finite size K is equal to 1

4.31 $\sum_{i\in\mathbb{N}} E[|X_i|] < \infty$

According to Chebyshev's Inequality

$P(|X_i| > \varepsilon) \leq \frac{E[|X_i|]}{\varepsilon}$ $(\forall \varepsilon \in \mathbb{R}_+) \Rightarrow \sum P(|X_i| > \varepsilon) \leq \sum \frac{E[|X_i|]}{\varepsilon} < \infty$ (Given condition)

According to first Borel-Cantelli lemma $\Rightarrow P(\lim\sup_i \{|X_i| > \varepsilon\}) = 0$

$\Rightarrow P(\{|X_i| > \varepsilon\}\ i.o) = 0$ $\therefore \lim X_i = 0$ P-a.s.

4.36 1) $X_i = \frac{X_i(\omega)}{i}$ $i \in \mathbb{N}_{++}$, $|X_i| < 1$ when $i \geq 2$.

$\therefore \sum_{i\in\mathbb{N}} P(|X_i| \geq 1) = P(|X_1| \geq 1) = 1$

2) $\because |X_i| \leq 1$ $i \in \mathbb{N}_{++}$ $E[X_i \mathbb{1}\{|X_i| \leq 1\}] = E[X_i] = \frac{E[X_i(\omega)]}{i} = 0$

$\therefore \sum E[X_i \mathbb{1}\{|X_i| \leq 1\}]$ converges

3) $Var(X_i \mathbb{1}\{|X_i| \leq 1\}) = Var(X_i) = E[X_i^2] = \frac{1}{i^2}$

$\sum_{i\in\mathbb{N}} Var(X_i \mathbb{1}\{|X_i| \leq 1\}) = \frac{\pi^2}{6}$

$\therefore$ The series $\sum_{i\in\mathbb{N}} X_i$ converges a.s.

$\therefore \sum_{i\in\mathbb{N}} \frac{X_i(\omega)}{i}$ converges a.s for every $\omega \in \Omega$

$\because \sum_{i\in\mathbb{N}} \frac{X_i(\omega)}{i}$ converges to a finite limit for $\omega \in \Omega$

$\therefore A = \{\omega \in \Omega : \sum_{i\in\mathbb{N}} \frac{X_i(\omega)}{i}$ converges to a finite limit for $\omega \in \Omega\}$ $P(A) = 1$

4.45 $\hat{\sigma}_n^2 \overset{def}{=} \frac{1}{n-1}((x_1-\bar{x}_n)^2+\cdots+(x_n-\bar{x}_n)^2)$, suppose that $E[X]=a$.

$E[\hat{\sigma}_n^2] = E[\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x}_n)^2] = E[\frac{1}{n-1}\sum_{i=1}^{n}[(x_i-a)-(\bar{x}_n-a)]^2$

$= E[\frac{1}{n-1}\sum_{i=1}^{n}[(x_i-a)^2+(\bar{x}_n-a)^2-2(x_i-a)(\bar{x}_n-a)]]$

$= E[\frac{1}{n-1}(n(\bar{x}_n-a)^2-2(\bar{x}_n-a)\sum_{i=1}^{n}(x_i-a)+\sum_{i=1}^{n}(x_i-a)^2)]$

$= E[\frac{1}{n-1}(n(\bar{x}_n-a)^2-2n(\bar{x}_n-a)^2+\sum_{i=1}^{n}(x_i-a^2)]$

$= \frac{-n}{n-1}E[(\bar{x}_n-a)^2]+\frac{1}{n-1}E[\sum_{i=1}^{n}(x_i-a)^2]$

$(E[(\bar{x}_n-a)^2]=\frac{\sigma^2}{n}$ and $E[\sum_{i=1}^{n}(x_i-a)^2]=n\sigma^2)$

$= -\frac{n}{n-1}\cdot\frac{\sigma^2}{n}+\frac{1}{n-1}n\sigma^2 = \frac{-1+n}{n-1}\sigma^2 = \sigma^2$

So, it could be concluded that $E[\hat{\sigma}_n^2]=\sigma^2$, so the $\hat{\sigma}_n^2$ is an unbiased estimation.

2.35 $f:\mathbb{R}\to\bar{\mathbb{R}}^+ \therefore Ran(f)=[0,+\infty]$

a) $\int_{\mathbb{R}}f(x)\Lambda(dx) = 0\times\Lambda(f^{-1}\{0\})+\int_{f^{-1}(x>0)}f(x)\Lambda(dx) = \int_{f^{-1}(x>0)}f(x)\Lambda(dx)$

$\because \Lambda(\{x\in\mathbb{R}: f(x)>0\})=0$

$\therefore \int_{\mathbb{R}}f(x)\Lambda(dx) = \int_{f^{-1}(x>0)}f(x)\Lambda(dx) = \int f(x)\Lambda(f^{-1}(x>0)) = 0$

$\therefore \int_{\mathbb{R}}f(x)\Lambda d(x) = 0$

b) When $f(x)<+\infty$ Def $A=\{x\mid x\in\mathbb{R}, f(x)<+\infty\}$ $\because f(x)\geq 0$

$\int_{\mathbb{R}}f(x)\Lambda(dx) = \int_A f(x)\Lambda(dx)+\int_{A^c}f(x)\Lambda(dx) \geq 0\cdot\Lambda(A)+\int_{A^c}f(x)\Lambda(dx) = \int_{A^c}f(x)\Lambda(dx)$ (e1)

$\because$ when $x\in A^c$ $f(x)=+\infty$

$\therefore e1 = +\infty\cdot\Lambda(A^c)$ $\because \Lambda(\{x\in\mathbb{R}: f(x)=+\infty\}) = a>0$ $\therefore e_1 = a\cdot+\infty = +\infty$

$\therefore$ If $\Lambda(\{x\in\mathbb{R}: f(x)=+\infty\})>0 \Rightarrow \int_{\mathbb{R}}f(x)\Lambda(dx)=\infty$

c) Def $A=\{x\in\mathbb{R}: f(x)=0\}$

$\Lambda(A^c)>0$

$\int_{\mathbb{R}}f(x)\Lambda(dx) = \int_A f(x)\Lambda(dx)+\int_{A^c}f(x)\Lambda(dx) = 0+\int_{A^c}f(x)\Lambda(dx)$

$\because \Lambda(dx)$ on A^c are positive, $f(x)$ on A^c are positive.

$\therefore f(x)\Lambda(dx)$ on A^c are positive.

$\therefore \int_{\mathbb{R}} f(x)\Lambda(dx) = \int_{A^c} f(x)\Lambda(dx) > 0$

$\therefore$ If $\Lambda(\{x\in\mathbb{R}: f(x)>0\}) > 0 \Rightarrow \int_{\mathbb{R}} f(x)\Lambda(dx) > 0$

2.75 $\int_{-1}^{1}(1-x^2)^{1000}dx$ let $x=\sin t$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2000}t \, d\sin t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2001}t\,dt = 2\int_0^{\frac{\pi}{2}} \cos^{2001}t\,dt = \frac{2000}{2001}\times\frac{1998}{1999}\times\cdots\times\frac{2}{3}\times 2$$

$$\left(\frac{2}{1001}\right)^{\frac{1}{1000}} \le \left(\int_{-1}^{1}(1-x^2)^{1000}dx\right)^{\frac{1}{1000}} < \sqrt[1000]{2}$$

$$\int_{-1}^{1}(1-x^2)^{1000}dx \geqslant \frac{2000}{2002}\times\frac{1998}{2000}\times\frac{1996}{1998}\times\cdots\times\frac{2}{4}\times 2 = \frac{2}{1001}$$

$0.9938 \le \left(\int_{-1}^{1}(1-x^2)^{1000}dx\right)^{\frac{1}{1000}} \le 1.0007$

$\therefore \left(\int_{-1}^{1}(1-x^2)^{1000}dx\right)^{\frac{1}{1000}}$ is closed to 1.

$$\int_{-\infty}^{+\infty} e^{-1000x^2}dx = 2\int_0^{+\infty} e^{-1000x^2}dx = \frac{2}{10\sqrt{10}}\int_0^{+\infty} e^{-1000x^2}\, d10\sqrt{10}x = \frac{\sqrt{\pi}}{10\sqrt{10}}$$

$\therefore \left(\frac{\sqrt{\pi}}{10\sqrt{10}}\right)^{\frac{1}{1000}} \approx 0.9971$

$\therefore \left(\int_{-\infty}^{+\infty} e^{-1000x^2}dx\right)^{\frac{1}{1000}}$ is close to 1.

Max $(1-x^2)$ for $[-1,1]$ is when $x=0$ $1-x^2=1$

Max e^{-x^2} for $[-\infty,+\infty]$ is when $x=0$ $e^{-x^2}=1$