MF795: Capstone Homework Project

Due on Thursday, December 17, 2020

POSTED ON: OCTOBER 26, 2020

One randomly chosen problem from this project assignment will be included in the third exam in MF795. The assignment will not be graded and will not be collected.

Throughout this problem set the probability space $(\Omega, \mathcal{F}, P)$ and the filtration $\mathscr{F} \setminus\!\subset \mathcal{F}$, assumed to satisfy the usual conditions, are fixed. All notions of local martingale, semimartingale, etc., are understood relative to the filtration $\mathscr{F}$ and the measure P. The token W stands for a (1-dimensional) Brownian motion relative to $\mathscr{F}$ and P.

Problem 1: A *continuous semimartingale* is any $\mathscr{F}$-adapted process, S, that can be written as the sum, $S = M + A$, of a continuous local martingale M and a continuous and $\mathscr{F}$-adapted process A that starts from 0 and has sample paths that a.s. have finite variation on finite intervals. If a decomposition of this form exists, the local martingale component, M, and the finite variation component, A, are unique, and the identification $S = M + A$ is known as *the canonical semimartingale decomposition of* S.

Assuming that S is a semimartingale with canonical decomposition $S = M + A$, show that every one of the following processes is also a semimartingale and identify its canonical semimartingale decomposition:

(a) $e^S \equiv (e^{S_t})_{t\in\mathbb{R}_+}$; (b) $S^2 \equiv (S_t^2)_{t\in\mathbb{R}_+}$; (c) $S^n \equiv (S_t^n)_{t\in\mathbb{R}_+}$ for some $n \in \mathbb{N}_{++}$; (d) $S\bullet\imath \equiv \big(\int_0^t S_u\,\mathrm{d}u\big)_{t\in\mathbb{R}_+}$;

(e) $S\bullet W + S\bullet\imath \equiv \big(\int_0^t S_u\,\mathrm{d}W_u + \int_0^t S_u\,\mathrm{d}u\big)_{t\in\mathbb{R}_{++}}$; and (f) $f(S,\imath) \equiv \big(f(S_t,t)\big)_{t\in\mathbb{R}_+}$,

for some function $f\colon \mathbb{R}\times\mathbb{R}_+ \mapsto \mathbb{R}$ such that derivatives $\frac{\partial^2}{\partial x^2} f(x,t)$ and $\frac{\partial}{\partial t} f(x,t)$ exist and are continuous. ○

Problem 2: Answer the questions in Problem 1 in the special case where S is an Itô process of the form $S = S_0 + \sigma\bullet W + b\bullet\imath$, i.e.,

$$S_t = S_0 + \int_0^t \sigma_u\,\mathrm{d}W_u + \int_0^t b_u\,\mathrm{d}u\,, \quad t \in \mathbb{R}_+\,,$$

and explain what conditions for the processes $\sigma(\omega,t)$ and $b(\omega,t)$ can guarantee that the two integrals in the rights side of the last identity are well defined. ○

Problem 3: Consider the functions $\sigma\colon \mathbb{R}\times\mathbb{R}_+ \mapsto \mathbb{R}$ and $b\colon \mathbb{R}\times\mathbb{R}_+ \mapsto \mathbb{R}$ and give the conditions under which the following stochastic differential equation has a unique strong solution S:

$$S = S_0 + \sigma(S,\imath)\bullet W + b(S,\imath)\bullet\imath\,, \quad \text{i.e.,} \quad S_t = S_0 + \int_0^t \sigma(S_u,u)\,\mathrm{d}W_u + \int_0^t b(S_u,u)\,\mathrm{d}u\,, \quad t \in \mathbb{R}_+\,.$$

Explain what it means for a stochastic differential equation to have a strong solution, and what it means for a strong solution to be unique. Answer the questions in Problem 1 in the special case where the semimartingale S is the solution to the above equation and, finally, write a stochastic equation of diffusion type that the process $X = e^S$, i.e., $X_t = e^{S_t}$, $t \in \mathbb{R}_+$, can be claimed to satisfy (S is again the solution to the above equation). ○

Problem 4: Answer the questions in Problem 3 in the special case where S is a geometric Brownian motion, i.e., $\sigma(S_u,u) = \sigma S_u$ and $b(S_u,u) = bS_u$ for some constants $\sigma, b \in \mathbb{R}$. ○

Problem 5: Give the solution, X, to every one of the following linear stochastic differential equations, in which σ, b, m, and k are fixed scalars:

(a) $X = X_0 + b\imath + \sigma X\bullet W$; (b) $X = X_0 + bX\bullet\imath + \sigma X\bullet W$; (c) $X = X_0 + (bX - m)\bullet\imath + \sigma X\bullet W$;

$$\text{(d) } X = X_0 + (bX - m)\bullet\imath + (\sigma X + k)\bullet W\,. \quad \circ$$

Problem 6: Give the solution, X, to every one of the following linear stochastic differential equations, in which σ, b, m, and k are predictable (for the filtration $\mathscr{F}$) stochastic processes with a.s. locally bounded sample paths (i.e., sample paths that are bounded on finite intervals):

$$\text{(a) } X = X_0 + b\bullet\imath + \sigma X\bullet W\,;\ \text{(b) } X = X_0 + bX\bullet\imath + \sigma X\bullet W\,;\ \text{(c) } X = X_0 + (bX - m)\bullet\imath + \sigma X\bullet W\,;$$
$$\text{(d) } X = X_0 + (bX - m)\bullet\imath + (\sigma X + k)\bullet W\,. \quad \circ$$

Problem 7: Let θ be some jointly measurable and adapted $\mathbb{R}^d$-valued stochastic process, such that the sample paths of the process $|\theta|^2\bullet\imath$, i.e., the process $\int_0^t |\theta_s|^2\,\mathrm{d}s$, $t \in \mathbb{R}_+$, do not explode (remain finite) in finite time ($|\theta_s|$ stands for the Euclidean norm of $\theta_s \in \mathbb{R}^d$). Assuming that W is a d-dimensional Brownian motion, and that r is some jointly measurable and adapted $\mathbb{R}$-valued process with locally integrable sample paths, consider the process

$$X = e^{-\theta^\intercal\bullet W - (r + \frac{1}{2}\theta^\intercal\theta)\bullet\imath}\,, \quad \text{i.e.,} \quad X_t = e^{-\int_0^t \theta_s^\intercal\,\mathrm{d}W_s - \int_0^t (r_s + \frac{1}{2}\theta_s^\intercal\theta_s)\,\mathrm{d}s}\,, \quad t \in \mathbb{R}_+\,,$$

and show that this process is a semimartingale by identifying its canonical decomposition into a continuous local martingale and a continuous process of finite variation. Write down a stochastic differential equation that the process X satisfies. $\circ$