APPENDIX A

Properties of Several Important Univariate Pdf's

Here we provide properties of several important univariate pdf's which have appeared at various points in the text.

A.1 UNIVARIATE NORMAL (UN) PDF

A random variable, $\tilde{x}$, is normally distributed if, and only if, its pdf has the following form:

$$\text{(A.1)} \quad p(x|\theta, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2\sigma^2}(x-\theta)^2\right], \qquad -\infty < x < \infty.$$

This pdf has two parameters: a location parameter θ, $-\infty < \theta < \infty$, and a scale parameter σ, $0 < \sigma < \infty$. It also has a single mode at $x = \theta$ and is symmetric about the point. Thus θ is the median and the mean of the UN pdf.[1] Given symmetry about $x = \theta$, the odd-order moments about θ are all zero; that is,

(A.2)

$$\mu_{2r-1} \equiv E(\tilde{x} - \theta)^{2r-1} = \int_{-\infty}^{\infty} (x-\theta)^{2r-1} p(x|\theta, \sigma)\, dx = 0, \quad r = 1, 2, 3, \ldots;$$

for example, $\mu_1 = 0$ or $E\tilde{x} = \theta$.

The even-order moments about the mean are given by

$$\text{(A.3)} \quad \begin{aligned} \mu_{2r} \equiv E(\tilde{x} - \theta)^{2r} &= \int_{-\infty}^{\infty} (x-\theta)^{2r} p(x|\theta, \sigma)\, dx \\ &= \frac{2^r \sigma^{2r}}{\sqrt{\pi}} \Gamma(r + \tfrac{1}{2}), \qquad r = 1, 2, \ldots, \end{aligned}$$

where $\Gamma(r + \frac{1}{2})$ denotes the gamma function with argument $r + \frac{1}{2}$ [see (A.6) for a definition of this well-known and important function].

[1] That $x = \theta$ is the modal value of the UN pdf can be seen by inspection from (A.1). That $x = \theta$ is also median follows from the fact that (A.1) is a normalized pdf [i.e., $\int_{-\infty}^{\infty} p(x|\theta, \sigma)\, dx = 1$] and that it is symmetric about the single modal value $x = \theta$.

Further, we note that by a change of variables $z = (x - \theta)/\sigma$, (A.1) can be brought into the standardized UN form[2]:

$$p(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad -\infty < z < \infty. \tag{A.4}$$

The moments of this pdf are easily obtained from the expressions for the moments shown in (A.2) and (A.3).

Proofs of Properties of the UN Pdf

First, we establish that (A.1) is a proper normalized pdf; that is $\int_{-\infty}^{\infty} p(x|\theta, \sigma)\, dx = 1$. Note that $p(x|\theta, \sigma) > 0$ for all x such that $-\infty < x < \infty$. Now make the change of variable $z = (x - \theta)/\sigma$ to obtain (A.4) and note that $\int_{-\infty}^{\infty} p(x|\theta, \sigma)\, dx = \int_{-\infty}^{\infty} p(z)\, dz$. Now, letting $u = z^2/2$, we have $0 < u < \infty$ and $du = z\, dz$, or $dz = du/\sqrt{2u}$. Then[3]

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz = \frac{1}{\sqrt{\pi}} \int_0^{\infty} u^{-1/2} e^{-u}\, du. \tag{A.5}$$

The integral on the rhs of (A.5) is the gamma function with argument $\frac{1}{2}$; that is, the gamma function, denoted by $\Gamma(q)$, is defined as

$$\Gamma(q) = \int_0^{\infty} u^{q-1} e^{-u}\, du, \qquad 0 < q < \infty, \tag{A.6}$$

and thus the rhs of (A.5) is $(1/\sqrt{\pi})\, \Gamma(\frac{1}{2})$. Since it is shown in the calculus that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the rhs of (A.5) is indeed equal to one.

Second, we show that the odd-order moments of $x - \theta$ are all zero, as shown in (A.2). This is equivalent to showing that $E\tilde{z}^{2r-1} = 0$, $r = 1, 2, \ldots$, since $\tilde{z} = (\tilde{x} - \theta)/\sigma$. We have

$$E\tilde{z}^{2r-1} = \int_{-\infty}^{0} z^{2r-1} p(z)\, dz + \int_0^{\infty} z^{2r-1} p(z)\, dz.$$

Since z^{2r-1}, with $-\infty < z < 0$, is negative and $p(-z) = p(z)$ from the symmetrical form of (A.4), the first integral on the rhs can be expressed as the negative of the second and thus their sum is zero, as was to be shown.

Third, we derive the expression for the even-order moments shown in (A.3). We shall obtain the even-order moments of $p(z)$ in (A.4) and from

[2] Note from $z = (x - \theta)/\sigma$, $dz = dx/\sigma$ and thus the factor $1/\sigma$ in (A.1) does not appear in (A.4).

[3] Remember that in changing variables by $u = \frac{1}{2}z^2$ we have $\int_{-\infty}^{\infty} p(z)\, dz = 2\int_0^{\infty} p(u)\, du/\sqrt{2u}$, with the factor 2 appearing to take account of the area under $p(z)$ for *both* positive and negative values of z.

them obtain the expression in (A.3). Let $u = \frac{1}{2}z^2$; then

$$\begin{aligned} \int_{-\infty}^{\infty} z^{2r} p(z)\, dz &= 2\int_0^{\infty} (2u)^r p(\sqrt{2u}) \frac{du}{\sqrt{2u}} \\ &= \frac{2^r}{\sqrt{\pi}} \int_0^{\infty} u^{r-1/2} e^{-u}\, du \\ &= \frac{2^r}{\sqrt{\pi}} \Gamma(r + \tfrac{1}{2}), \qquad r = 1, 2, \ldots, \end{aligned} \tag{A.7}$$

where $\Gamma(r + \frac{1}{2})$ is the gamma function in (A.6) with argument $q = r + \frac{1}{2}$. Since $z = (x - \mu)/\sigma$, the even-order moments, denoted μ_{2r}, in (A.3), are just σ^{2r} times the expression given in (A.7) and the result shown in (A.3) is established.

For convenience we provide explicit expressions for the second and fourth moments shown in (A.3)[4]:

$$\mu_2 = E(\tilde{x} - \theta)^2 = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma(1 + \tfrac{1}{2}) = \sigma^2 \tag{A.8}$$

and

$$\mu_4 = E(\tilde{x} - \theta)^4 = \frac{2^2\sigma^4}{\sqrt{\pi}} \Gamma(2 + \tfrac{1}{2}) = 3\sigma^4. \tag{A.9}$$

Explicit expressions for higher order even moments can be obtained in a similar fashion.

As regards measures other than moments to characterize properties of univariate pdf's, we shall just take up measures of skewness and kurtosis. Measures of skewness, that is, of departures from symmetry, include K. Pearson's measure, shown in (A.10), and two others:

$$Sk = \frac{\text{mean-mode}}{\sigma}, \tag{A.10}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}, \tag{A.11}$$

and

$$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}}. \tag{A.12}$$

Of course, for the symmetric UN pdf all of these measures have a zero value. With respect to kurtosis, the following measure, the "excess," is often employed:

$$\gamma_2 \equiv \beta_2 - 3, \tag{A.13}$$

[4] Use is made of the following two properties of the gamma function: (i) $\Gamma(q + 1) = q\Gamma(q)$, for $q > 0$, and (ii) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. In (A.9) property (i) is used twice.

where $\beta_2 = \mu_4/\mu_2^2$. The measure γ_2 assumes a value of zero for the UN pdf; pdf's for which $\gamma_2 = 0$ are called mesokurtic, those for which $\gamma_2 > 0$ are leptokurtic, and those for which $\gamma_2 < 0$, platykurtic. As already pointed out,

"... it was thought that leptokurtic curves were more sharply peaked, and platykurtic curves more flat-topped, than the normal curve. This, however, is not necessarily so and although the terms are useful they are best regarded as describing the sign of γ_2 rather than the shape of the curve."[5]

A.2 THE UNIVARIATE STUDENT t (US t) Pdf

A random variable, $\tilde{x}$, is distributed in the US t form if, and only if, it has the following pdf:

(A.14)
$$p(x|\theta, h, \nu) = \frac{\Gamma[(\nu + 1)/2]}{\Gamma(1/2)\,\Gamma(\nu/2)}\left(\frac{h}{\nu}\right)^{1/2}\left[1 + \frac{h}{\nu}(x-\theta)^2\right]^{-(\nu+1)/2}, \qquad -\infty < x < \infty,$$

where $-\infty < \theta < \infty$, $0 < h < \infty$, $0 < \nu$ and where Γ denotes the gamma function. This pdf has three parameters, θ, h, and ν. From inspection of (A.14) it is seen that the US t pdf has a single mode at $x = \theta$ and is symmetric about the modal value $x = \theta$. Thus $x = \theta$ is the median and mean (which exists for $\nu > 1$—see below) of the US t pdf. The following expressions give the odd- and even-order moments about the mean:

(A.15)
$$\left.\mu_{2r-1} \equiv E(\tilde{x}-\theta)^{2r-1} = \int_{-\infty}^{\infty}(x-\theta)^{2r-1}p(x|\theta, h, \nu)\,dx = 0\right\} \qquad \begin{array}{l} r = 1, 2, \ldots, \\ \nu > 2r-1, \end{array}$$

and

(A.16)
$$\begin{aligned}\mu_{2r} \equiv E(\tilde{x}-\theta)^{2r} &= \int_{-\infty}^{\infty}(x-\theta)^{2r}p(x|\theta, h, \nu)\,dx \\ &= \left.\frac{\Gamma(r+1/2)\Gamma(\nu/2 - r)}{\Gamma(1/2)\,\Gamma(\nu/2)}\left(\frac{\nu}{h}\right)^r\right\} \qquad \begin{array}{l} r = 1, 2, \ldots, \\ \nu > 2r. \end{array}\end{aligned}$$

As shown below, for the existence of the $(2r-1)$st moment about θ we must have $\nu > 2r-1$. Similarly, for the existence of the $2r$th-order moment about θ, ν must satisfy $\nu > 2r$. Given the symmetry of the US t pdf about $x = \theta$, all existing odd-order moments in (A.15) are zero. In particular, $E(\tilde{x}-\theta) = 0$ so that $E\tilde{x} = \theta$ which exists for $\nu > 1$.[6] With respect to the even-order

[5] M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*, Vol. I. London: Griffin, 1958. New York: Hafner, p. 86. However, for many distributions encountered in practice a positive γ_2 does mean a sharper peak with higher tails than if the distribution were normal.

[6] For $\nu = 1$ the US t pdf is identical to the Cauchy pdf for which the first- and higher-order moments do not exist.

moments in (A.16) the second- and fourth-order moments are given by[7]

(A.17)
$$\mu_2 \equiv E(\tilde{x}-\theta)^2 = \frac{1}{\nu - 2}\frac{\nu}{h}, \qquad \text{for } \nu > 2,$$

and

(A.18)
$$\mu_4 \equiv E(\tilde{x}-\theta)^4 = \frac{3}{(\nu-2)(\nu-4)}\left(\frac{\nu}{h}\right)^2, \qquad \text{for } \nu > 4.$$

Given that $\nu > 2$, the variance μ_2 exists and is seen to depend on ν and h. When $\nu > 4$, the fourth-order moment μ_4 exists and, as with μ_2, depends on just ν and h.

Since the US t pdf is symmetric about $x = \theta$, the measures of skewness, discussed in connection with the UN pdf, are all zero, provided, of course, that moments on which they depend exist. With respect to kurtosis, we have from (A.17) and (A.18)

(A.19)
$$\begin{aligned}\gamma_2 &= \frac{\mu_4}{\mu_2^2} - 3 \\ &= 3\left(\frac{\nu-2}{\nu-4} - 1\right) = \frac{6}{\nu - 4}, \qquad \nu > 4.\end{aligned}$$

Thus, for finite ν the US t pdf is leptokurtic ($\gamma_2 > 0$), probably because it has fatter tails than a UN pdf with mean θ and variance $1/h$. As ν gets large, the US t pdf assumes the shape of a UN pdf with mean θ and variance $\mu_2 = 1/h$.[8]

We can obtain the standardized form of the US t pdf from (A.14) by the following change of variable

(A.20)
$$t = \sqrt{h}\,(x-\theta), \qquad -\infty < t < \infty.$$

Using (A.20), we obtain

(A.21)
$$p(t|\nu) = \frac{\Gamma[(\nu+1)/2]}{\sqrt{\nu}\,\Gamma(1/2)\,\Gamma(\nu/2)}\left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2}, \qquad -\infty < t < \infty,$$

which, with $\nu > 0$, is the proper normalized standard US t pdf. This pdf has a single mode at $t = 0$ and is symmetric about this point. From (A.20) the moments of $\tilde{t}$ can be obtained from those of $\tilde{x} - \theta$ when they exist.

Proofs of Properties of the US t Pdf

To establish properties of the US t pdf we need the following results from the calculus:

[7] In obtaining these results, repeated use is made of the relationship $\Gamma(q+1) = q\Gamma(q)$, $q > 0$, a fundamental property of the gamma function.

[8] For ν about 30 these two pdf's are similar; (A.17) and (A.18) show explicitly how μ_2 and μ_4 for the US t are related to the corresponding moments of the limiting normal pdf; that is, $\mu_2 = 1/h$ and $\mu_4 = 3/h^2$.

1. If $f(v)$ is continuous for $a \le v < \infty$ and $\lim_{v\to\infty} v^r f(v) = A$, a finite constant, for $r > 1$, then $\int_a^\infty |f(v)|\, dv < \infty$; that is, the integral converges absolutely.[9]

2. If $g(v)$ is continuous for $-\infty < v \le b$ and $\lim_{v\to-\infty} (-v)^r g(v) = c$, a constant, for $r > 1$, then $\int_{-\infty}^b |g(v)|\, dv < \infty$.

3. The relation connecting the beta function, denoted $B(u, v)$, and the gamma function is[10]

$$B(u, v) = \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)}, \qquad 0 < u, v < \infty, \tag{A.22}$$

where

$$B(u, v) \equiv \int_0^1 x^{u-1}(1-x)^{v-1}\, dx, \qquad 0 < u, v < \infty. \tag{A.23}$$

With these results stated, we first show that for $\nu > 0$ the US t pdf in (A.21) is a proper normalized pdf.[11] We note that $p(t|\nu) > 0$ for $-\infty < t < \infty$, and letting $t' = t/\sqrt{\nu}$ we can write (A.21) as

$$p(t'|\nu) = \left[B\left(\frac{1}{2}, \frac{\nu}{2}\right)\right]^{-1} (1 + t'^2)^{-(\nu+1)/2}, \qquad -\infty < t' < \infty, \tag{A.24}$$

with

$$B\left(\frac{1}{2}, \frac{\nu}{2}\right) = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)\Big/\Gamma\left(\frac{\nu+1}{2}\right)$$

from the result given in (A.22). Now make the change of variable $z = 1/(1 + t'^2)$, $0 < z < 1$, to obtain[12]

$$p(z|\nu) = \left[B\left(\frac{1}{2}, \frac{\nu}{2}\right)\right]^{-1} z^{\nu/2-1}(1-z)^{-1/2}\frac{dz}{2}, \qquad 0 < z < 1. \tag{A.25}$$

Noting that $\int_{-\infty}^\infty p(t'|\nu)\, dt' = 2\int_0^1 p(z|\nu)\, dz$, we have

$$\int_{-\infty}^\infty p(t'|\nu)\, dt' = \left[B\left(\frac{1}{2}, \frac{\nu}{2}\right)\right]^{-1} \int_0^1 z^{\nu/2-1}(1-z)^{-1/2}\, dz = 1, \tag{A.26}$$

since the integral on the rhs of (A.26) is just $B(1/2, \nu/2)$, *provided that* $\nu > 0$. This condition is required in order to have $\int_{-\infty}^\infty p(t'|\nu)\, dt' < \infty$ [see (1) and (2)].

[9] Note that $\int_a^\infty |f(v)|\, dv < \infty$ implies $\int_a^\infty f(v)\, dv < \infty$; that is, absolute convergence implies simple convergence. See, for example, D. V. Widder, *Advanced Calculus*. New York: Prentice-Hall, 1947, p. 271. See also p. 273 ff. for proofs of (1) and (2).

[10] Equation A.22 implies $B(u, v) = B(v, u)$.

[11] Since (A.21) is obtained from (A.14) by the change of variable in (A.20), showing that (A.21) is a proper normalized pdf will imply that (A.14) also has this property.

[12] From $z = 1/(1 + t'^2)$, $|dz|dt'| = 2t'/(1 + t'^2)^2$ and $t'^2 = (1-z)/z$; thus $|dt'/dz| = z^{-3/2}(1-z)^{-1/2}/2$.

The results for the odd-order moments in (A.15) can be established easily by considering

$$\int_{-\infty}^\infty t'^{2r-1} p(t'|\nu)\, dt', \qquad -\infty < t' < \infty, \tag{A.27}$$

with $t' = t/\sqrt{\nu} = \sqrt{h/\nu}\,(x - \theta)$. For the integral in (A.27) to converge we need $2r - 1 < \nu$ by application of (1) and (2). Thus, if $2r - 1 < \nu$, the $(2r-1)$st odd-order moment exists and is zero by virtue of the symmetry of $p(t'|\nu)$ about $t' = 0$.

The expression for the even-order moments in (A.16) is most easily obtained by evaluating

$$\int_{-\infty}^\infty t'^{2r} p(t'|\nu)\, dt', \qquad -\infty < t' < \infty, \tag{A.28}$$

with $t' = \sqrt{h/\nu}\,(x - \theta)$. For the integral in (A.28) to converge we need $\nu > 2r$. If this condition is satisfied, we use the transformation $z = 1/(1 + t'^2)$, or $t'^2 = (1-z)/z$ [see (A.24)] to obtain

$$\begin{aligned}\int_{-\infty}^\infty t'^{2r} p(t'|\nu)\, dt' &= \left[B\left(\frac{1}{2}, \frac{\nu}{2}\right)\right]^{-1} \int_0^1 \left(\frac{1-z}{z}\right)^r z^{\nu/2-1}(1-z)^{-1/2}\, dz \\ &= \left[B\left(\frac{1}{2}, \frac{\nu}{2}\right)\right]^{-1} \int_0^1 z^{\nu/2-r-1}(1-z)^{r-1/2}\, dz \\ &= \frac{B(\nu/2 - r, r + 1/2)}{B(1/2, \nu/2)} = \frac{\Gamma(r+1/2)\,\Gamma(\nu/2 - r)}{\Gamma(1/2)\,\Gamma(\nu/2)}, \qquad \nu > 2r.\end{aligned} \tag{A.29}$$

This gives the $2r$th moment of $p(t'|\nu)$. From $x - \theta = \sqrt{\nu/h}\, t'$ we obtain the $2r$th moment of $p(x|\theta, h, \nu)$,

$$\mu_{2r} \equiv E(\tilde{x} - \theta)^{2r} = \frac{\Gamma(r+1/2)\,\Gamma(\nu/2 - r)}{\Gamma(1/2)\,\Gamma(\nu/2)}\left(\frac{\nu}{h}\right)^r, \qquad \nu > 2r,$$

which is just (A.16).

A.3 THE GAMMA (G) AND χ^2 Pdf's

As the name implies, the G pdf is closely linked to the gamma function. A random variable, $\tilde{x}$, is distributed according to the G distribution if, and only if, its pdf is given by

$$p(x|\gamma, \alpha) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\gamma^\alpha} e^{-x/\gamma}, \qquad 0 < x < \infty, \tag{A.30}$$

where α and γ are strictly positive parameters; that is, $\alpha, \gamma > 0$. From the form of (A.30) it is seen that γ is a scale parameter. When $\alpha \ge 1$, the pdf has a single mode[13] at $x = \gamma(\alpha - 1)$. For small values of α the pdf has a long tail to the right. As α grows in size for any given value of γ, the pdf becomes more symmetric and approaches a normal form.

The G pdf can be brought into a standardized form by the change of variable $z = x/\gamma$, which results in

$$\text{(A.31)} \qquad p(z|\alpha) = \frac{1}{\Gamma(\alpha)} z^{\alpha-1}e^{-z}, \qquad 0 < z < \infty.$$

From the definition of the gamma function it is obvious that (A.31) is a proper normalized pdf and that moments of all orders exist. The moments about zero, denoted by μ_r', are given by

$$\text{(A.32)} \qquad \mu_r' = \int_0^\infty z^r p(z|\alpha)\, dz = \frac{\Gamma(r+\alpha)}{\Gamma(\alpha)}, \qquad r = 1, 2, \ldots.$$

From (A.32) we have for the first four moments about zero[14]

$$\text{(A.33)} \qquad \mu_1' = \alpha; \qquad \mu_2' = (1+\alpha)\alpha; \qquad \mu_3' = (2+\alpha)(1+\alpha)\alpha;$$
$$\text{and} \quad \mu_4' = (3+\alpha)(2+\alpha)(1+\alpha)\alpha.$$

From these results it is seen that α is the mean of the G pdf and, surprisingly, also its variance.[15] Further, for the third and fourth moments about the mean we have $\mu_3 = 2\alpha$ and $\mu_4 = 3\alpha(2+\alpha)$. Collecting these results, we have

$$\text{(A.34)} \qquad \mu_1' = \alpha, \qquad \mu_2 = \alpha, \qquad \mu_3 = 2\alpha \quad \text{and} \quad \mu_4 = 3\alpha(2+\alpha).$$

Given that $0 < \alpha < \infty$, the skewness is always positive. Since the mode is located at $z = \alpha - 1$ for $\alpha \geq 1$, we have for Pearson's measure of skewness $Sk = (\text{mean-mode})/\sqrt{\mu_2} = 1/\sqrt{\alpha}$. Clearly, as α grows in size, this measure of skewness approaches zero. As regards kurtosis, $\gamma_2 = \mu_4/\mu_2^2 - 3 = 6/\alpha$, which also approaches zero as α grows large. That $Sk \to 0$ and $\gamma_2 \to 0$ as $\alpha \to \infty$ is, of course, connected with the fact that the G pdf assumes a normal form as $\alpha \to \infty$.[16]

The χ^2 pdf is a special case of the G pdf (A.30) in which $\alpha = \nu/2$ and $\gamma = 2$; that is, the χ^2 pdf has the following form[17]:

$$\text{(A.35)} \qquad p(x|\nu) = \frac{x^{\nu/2-1}e^{-x/2}}{2^{\nu/2}\,\Gamma(\nu/2)}, \qquad 0 < x < \infty,$$

[13] For $0 < \alpha < 1$ the pdf has no mode.

[14] In obtaining the expressions for the moments below, we use the relation $\Gamma(1+q) = q\,\Gamma(q)$ repeatedly.

[15] The variance μ_2 is in general related to moments about zero by the following relationship: $\mu_2 = \mu_2' - \mu_1'^2$. For the G pdf $\mu_2 = (1+\alpha)\alpha - \alpha^2 = \alpha$.

[16] Moments, etc., for the unstandardized G pdf in (A.30) are readily obtained from those for the standardized pdf (A.31) by taking note of $z = x/\gamma$.

[17] Often (A.35) is written with $x = \chi^2$.

where $0 < \nu$. The parameter ν is usually referred to as the "number of degrees of freedom." Since (A.35) is a special case of (A.30), the standardized form is given by (A.31) with α replaced by $\nu/2$. Also, the moments of the standardized form are given by (A.33) and (A.34), again with $\alpha = \nu/2$. Since the standardized variable z is related to the unstandardized variable x by $z = x/\gamma = x/2$, moments of the unstandardized χ^2 pdf in (A.35) can be obtained easily from moments of the standardized χ^2 pdf. For the reader's convenience we present the moments associated with (A.35)[18]:

$$\text{(A.36)} \qquad \mu_1' = \nu; \qquad \mu_2 = 2\nu; \qquad \mu_3 = 8\nu; \quad \text{and} \quad \mu_4 = 24\nu\left(2 + \frac{\nu}{2}\right).$$

A very important property of the χ^2 pdf is that any sum of squared independent, standardized normal random variables has a pdf in the χ^2 form; that is, if $\tilde{x} = \tilde{z}_1^2 + \tilde{z}_2^2 + \cdots + \tilde{z}_n^2$, where the $\tilde{z}_i$'s are independent, standardized normal random variables, then $\tilde{x}$ has a pdf in the form of (A.35) with $\nu = n$.[19]

A.4 THE INVERTED GAMMA (IG) Pdf's

The IG pdf is obtained from the G pdf in (A.30) by letting y equal the positive square root of $1/x$; that is, $y = |\sqrt{1/x}|$ and thus $y^2 = 1/x$. With this change of variable the IG pdf is[20]

$$\text{(A.37}a\text{)} \qquad p(y|\gamma, \alpha) = \frac{2}{\Gamma(\alpha)\gamma^\alpha y^{2\alpha+1}} e^{-1/\gamma y^2}, \qquad 0 < y < \infty,$$

where $\gamma, \alpha > 0$. Since this pdf is encountered frequently in connection with prior and posterior pdf's for a standard deviation, we rewrite (A.37a), letting $\sigma = y$, $\alpha = \nu/2$, and $\gamma = 2/\nu s^2$ to obtain

$$\text{(A.37}b\text{)} \qquad p(\sigma|\nu, s) = \frac{2}{\Gamma(\nu/2)}\left(\frac{\nu s^2}{2}\right)^{\nu/2} \frac{1}{\sigma^{\nu+1}} e^{-\nu s^2/2\sigma^2}, \qquad 0 < \sigma < \infty,$$

where $\nu, s > 0$. The pdf in (A.37b) has a single mode at the following value[21] of σ:

$$\text{(A.38)} \qquad \sigma_{\text{mod}} = s\left(\frac{\nu}{\nu+1}\right)^{1/2}.$$

Clearly, as ν gets large, $\sigma_{\text{mod}} \to s$.

[18] These are obtained from (A.34) with $\alpha = \nu/2$ and $z = x/2$, where x is the χ^2 variable in (A.35), z is the standardized variable in (A.31), and α is the parameter in (A.31).

[19] See M. G. Kendall and A. Stuart, *op. cit.*, pp. 246–247, for a proof of this result.

[20] Since (A.37a) is obtained from a proper normalized pdf by a simple one-to-one differentiable change of variable, it is a proper normalized pdf.

[21] This is easily established by taking the log of both sides of (A.37b) and then finding the value of σ for which $\log p(\sigma|\nu, s)$ achieves its largest value.

The moments of (A.37b), when they exist, are obtained by evaluating the following integral:

$$\mu_r' = c\int_0^\infty \sigma^{r-(\nu+1)} e^{-\nu s^2/2\sigma^2}\, d\sigma, \tag{A.39}$$

where

$$c = \frac{2}{\Gamma(\nu/2)}\left(\frac{\nu s^2}{2}\right)^{\nu/2}.$$

On letting $y = \nu s^2/2\sigma^2$, (A.39) can be expressed as

$$\mu_r' = \tfrac{1}{2}c\left(\frac{\nu s^2}{2}\right)^{(r-\nu)/2}\int_0^\infty y^{(\nu-r)/2-1} e^{-y}\, dy. \tag{A.40}$$

The integral in (A.40) is the gamma function. For it to converge we must have

$$\nu - r > 0, \tag{A.41}$$

which is the condition for existence of a moment of order r. Inserting the definition of c in (A.40), we have

$$\mu_r' = \frac{\Gamma[(\nu - r)/2]}{\Gamma(\nu/2)}\left(\frac{\nu s^2}{2}\right)^{r/2}, \qquad \nu > r, \tag{A.42}$$

a convenient expression for the moments about zero. The first four moments are

$$\mu_1' = \frac{\Gamma[(\nu-1)/2]}{\Gamma(\nu/2)}\left(\frac{\nu}{2}\right)^{1/2} s, \qquad \nu > 1, \tag{A.43}$$

$$\mu_2' = \frac{\Gamma(\nu/2 - 1)}{\Gamma(\nu/2)}\left(\frac{\nu}{2}\right) s^2 = \frac{\nu s^2}{\nu - 2}, \qquad \nu > 2, \tag{A.44}$$

$$\mu_3' = \frac{\Gamma[(\nu-3)/2]}{\Gamma(\nu/2)}\left(\frac{\nu}{2}\right)^{3/2} s^3, \qquad \nu > 3, \tag{A.45}$$

and

$$\mu_4' = \frac{\Gamma(\nu/2 - 2)}{\Gamma(\nu/2)}\left(\frac{\nu}{2}\right)^2 s^4 = \frac{\nu^2}{(\nu-2)(\nu-4)} s^4, \qquad \nu > 4. \tag{A.46}$$

It is seen from (A.43) that the mean μ_1' is intimately related to the parameter s. As ν gets large, $\mu_1' \to s$,[22] which is also approximately the modal value for large ν (see above).

[22] In the calculus it is shown that with $0 < q < \infty$, as $q \to \infty$, $q^{b-a}\Gamma(q + a)/\Gamma(q + b) \to 1$ for a and b finite.

As regards moments about the mean,[23] we have

$$\begin{aligned}\mu_2 &= \mu_2' - \mu_1'^2, \qquad \nu > 2,\\ &= \frac{\nu s^2}{\nu - 2} - \mu_1'^2,\end{aligned} \tag{A.47}$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2 - \mu_1'^3, \qquad \nu > 3, \tag{A.48}$$

and

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3 - 6\mu_1'^2\mu_2 - \mu_1'^4, \qquad \nu > 4. \tag{A.49}$$

These formulas are useful when we have to evaluate higher order moments of the IG pdf.

The Pearson measure of skewness for the IG pdf is given by

$$\begin{aligned} Sk &= \frac{\text{mean} - \text{mode}}{\sqrt{\mu_2}}\\ &= \left[\frac{\Gamma[(\nu-1)/2]}{\Gamma(\nu/2)}\left(\frac{\nu}{2}\right)^{1/2} - \left(\frac{\nu}{\nu+1}\right)^{1/2}\right]\frac{s}{\sqrt{\mu_2}}, \qquad \nu > 2.\end{aligned} \tag{A.50}$$

Since this measure is generally positive, the IG pdf is skewed to the right. Clearly, as ν gets large $Sk \to 0$. For small to moderate ν the IG pdf has a rather long tail to the right.[24]

A.5 THE BETA (B) Pdf

A random variable, $\tilde{x}$, is said to be distributed according to beta distribution if, and only if, its pdf has the following form:

$$p(x|a, b, c) = \frac{1}{cB(a, b)}\left(\frac{x}{c}\right)^{a-1}\left(1 - \frac{x}{c}\right)^{b-1}, \qquad 0 \le x \le c, \tag{A.51}$$

where $a, b, c > 0$ and $B(a, b)$ denotes the beta function, shown in (A.23), with arguments a and b. It is seen that the range of the B pdf is 0 to c.

By a change of variable, $z = x/c$, we can obtain the standardized B pdf,

$$p(z|a, b) = \frac{1}{B(a, b)} z^{a-1}(1 - z)^{b-1}, \qquad 0 \le z \le 1, \tag{A.52}$$

which has a range zero to one. Some properties of (A.52) follow.

That (A.52) is a proper normalized pdf is established by observing that the pdf is non-negative in $0 \le z \le 1$ and that $B(a, b) = \int_0^1 z^{a-1}(1 - z)^{b-1}\, dz$,

[23] See M. G. Kendall and A. Stuart, *op. cit.*, p. 56, for formulas connecting moments about zero with moments about the mean.

[24] Since it is rather straightforward to obtain the pdf for σ^n, $n = 2, 3, \ldots$, from that for σ in (A.37b) and to establish its properties, we do not provide these results.

which converges for all $a, b > 0$. If $a > 2$, (A.52) is tangential to the abscissa at $z = 0$, and if $b > 2$ it is tangential at $z = 1$. For $a, b > 1$,[25] the mode is

$$z_{\text{mod}} = \frac{a-1}{a+b-2}. \tag{A.53}$$

The first and higher moments about zero for the standardized B pdf in (A.52) are

$$\begin{aligned} \mu_r' &= \frac{1}{B(a,b)} \int_0^1 z^{r+a-1}(1-z)^{b-1}\,dz \\ &= \frac{B(r+a,b)}{B(a,b)} = \frac{\Gamma(r+a)}{\Gamma(r+a+b)}\frac{\Gamma(a+b)}{\Gamma(a)} \\ &= \frac{a(a+1)(a+2)\cdots(a+r-1)}{(a+b)(a+b+1)\cdots(a+b+r-1)}, \qquad r = 1, 2, \ldots, \end{aligned} \tag{A.54}$$

where (A.22) and the recurrence relation for the gamma function, $\Gamma(q+1) = q\,\Gamma(q)$, have been employed. Thus the first three moments from (A.54) are

$$\mu_1' = \frac{a}{a+b}, \tag{A.55}$$

$$\mu_2' = \frac{a(a+1)}{(a+b)(a+b+1)}, \tag{A.56}$$

$$\mu_3' = \frac{a(a+1)(a+2)}{(a+b)(a+b+1)(a+b+2)}, \tag{A.57}$$

and so forth. The mean and higher moments are seen to depend simply on the parameters a and b. Further, the variance is given by

$$\mu_2 = \frac{ab}{(a+b)^2(a+b+1)}. \tag{A.58}$$

As regards skewness, Pearson's measure for $a, b > 1$ is

$$\begin{aligned} Sk &= \frac{a/(a+b) - (a-1)/(a+b-2)}{\sqrt{\mu_2}} \\ &= \frac{(b-a)/(a+b)(a+b-2)}{\sqrt{\mu_2}}. \end{aligned} \tag{A.59}$$

Thus, if $b = a$, $Sk = 0$ and the pdf is symmetric. If $b > a$, there is positive skewness, whereas if $b < a$ there is negative skewness.[26]

[25] For $0 < a, b < 1$ the pdf approaches ∞ as $z \to 0$ or 1.

[26] Since the variable of the unstandardized B pdf in (A.51) is related to that of the standardized B pdf in (A.52) by $x = cz$, the moments, etc., associated with (A.51) are easily obtained.

A useful and important result connecting the standardized gamma (G) and beta (B) pdf's follows. Let $\tilde{z}_1$ and $\tilde{z}_2$ be two independent random variables, each with a standardized G pdf with parameters α_1 and α_2, respectively [see (A.31)]. Then the random variable $\tilde{z} = \tilde{z}_1/(\tilde{z}_1 + \tilde{z}_2)$ has a standardized B pdf with parameters α_1 and α_2; that is, the pdf for $\tilde{z} = \tilde{z}_1/(\tilde{z}_1 + \tilde{z}_2)$ is $z^{\alpha_1-1}(1-z)^{\alpha_2-1}/B(\alpha_1, \alpha_2)$.[27] This result is often used when $\tilde{z}_1$ and $\tilde{z}_2$ are independent sums of squares of independent standardized normal random variables. Further, since the B pdf can be transformed to the F pdf, as shown below, we can state on this basis that $\tilde{z} = \tilde{z}_1/(\tilde{z}_1 + \tilde{z}_2)$ has a pdf transformable to an F pdf.

A pdf closely associated with the B pdf is the beta prime or inverted beta (IB) pdf.[28] Its standardized form is obtained from the standardized B pdf in (A.52) by letting $z = 1/(1+u)$ to obtain the IB pdf:

$$p(u|a,b) = \frac{1}{B(a,b)}\frac{u^{b-1}}{(1+u)^{a+b}}, \qquad 0 \le u < \infty, \tag{A.60}$$

with $a, b > 0$. The moments of this pdf are given by

$$\begin{aligned} \mu_r' &= \frac{1}{B(a,b)} \int_0^\infty \frac{u^{r+b-1}}{(1+u)^{a+b}}\,du \\ &= \frac{B(b+r, a-r)}{B(a,b)}, \qquad r < a, \end{aligned} \tag{A.61}$$

a result that is obtained by a change of variable $u = 1/(1+z)$ in (A.61) and noting that the result is in the form of a standardized unnormalized B pdf. Then, from (A.61),

$$\mu_1' = \frac{b}{a-1}, \qquad a > 1, \tag{A.62}$$

$$\mu_2' = \frac{b(b+1)}{(a-1)(a-2)}, \qquad a > 2, \tag{A.63}$$

and so on. The variance is

$$\mu_2 = \frac{b(a+b-1)}{(a-1)^2(a-2)}, \qquad a > 2. \tag{A.64}$$

[27] The joint pdf for $\tilde{z}_1$ and $\tilde{z}_2$ is the product of their individual pdf's, since they are assumed to be independent; that is, $p(z_1, z_2|\alpha_1, \alpha_2)\,dz_1\,dz_2 = [\Gamma(\alpha_1)\Gamma(\alpha_2)]^{-1} z_1^{\alpha_1-1} \times z_2^{\alpha_2-1}e^{-(z_1+z_2)}\,dz_1\,dz_2$. Now change variables to $v = z_1 + z_2$ and $z = z_1/(z_1+z_2)$, with $dz_1\,dz_2 = v\,dz\,dv$, and integrate with respect to v from zero to infinity to obtain the result above.

[28] See, for example, J. F. Kenney and E. S. Keeping, *Mathematics of Statistics: Part Two* (2nd ed.). Princeton, N.J.: Van Nostrand, 1951, pp. 95–96, and H. Raiffa and R. Schlaifer, *Applied Statistical Decision Theory*, Boston: Graduate School of Business Administration, Harvard University, 1961, pp. 220–221.

The IB pdf has a single mode if $b > 1$ at

(A.65) $$u_{\text{mod}} = \frac{b-1}{a+1}.$$

Then Pearson's measure of skewness for the IB pdf is

(A.66)
$$\begin{aligned} Sk &= \frac{b/(a-1) - (b-1)/(a+1)}{\sqrt{\mu_2}} \\ &= \frac{2b+a-1}{a+1}\left[\frac{a-2}{b(a+b-1)}\right]^{1/2}, \qquad b > 1 \quad \text{and} \quad a > 2, \end{aligned}$$

which is positive and shows that the IB pdf usually has a long tail to the right.

Last, we can obtain an important alternative form of (A.60) by letting $u = y/c$, with $0 < c$, to yield

(A.67) $$p(y|a, b, c) = \frac{1}{cB(a, b)} \frac{(y/c)^{b-1}}{(1 + y/c)^{a+b}}, \qquad 0 \le y < \infty,$$

where $a, b, c > 0$. Since $y = cu$ and we have already found the moments associated with (A.60), the moments for (A.67) are directly available from (A.61) to (A.64), that is the rth moment about zero is $c^r\mu_r'$ with μ_r' given in (A.61). It will be seen in the next section that the F and US t pdf's are special cases of (A.67).

A.6 THE FISHER-SNEDECOR F PDF

A random variable, $\tilde{x}$, is said to have an F distribution if, and only if, it has a pdf in the following form:

(A.68)
$$p(x|\nu_1, \nu_2) = \left[B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)\right]^{-1}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \frac{x^{\nu_1/2-1}}{(1 + (\nu_1/\nu_2)x)^{(\nu_1+\nu_2)/2}}, \qquad 0 < x < \infty,$$

where $\nu_1, \nu_2 > 0$. It is seen that (A.68) is a special case of the IB pdf in (A.67), where $a = \nu_2/2$, $b = \nu_1/2$, and $c = \nu_2/\nu_1$. The parameters ν_1 and ν_2 are usually referred to as degrees of freedom and (A.68) is called the F pdf with ν_1 and ν_2 degrees of freedom.

If $\nu_1/2 > 1$, the F pdf has a single mode at

(A.69) $$x_{\text{mod}} = \frac{\nu_2}{\nu_1}\frac{\nu_1/2 - 1}{\nu_2/2 + 1}.$$

The moments of the F pdf can, of course, be obtained directly from those associated with the IB pdf shown in (A.62) to (A.64). For easy reference we

list the moments of the F pdf:

(A.70) $$\mu_1' = \frac{\nu_2}{\nu_1}\frac{\nu_1/2}{\nu_2/2 - 1}, \qquad \frac{\nu_2}{2} > 1,$$

(A.71) $$\mu_2' = \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{(\nu_1/2)(\nu_1/2 + 1)}{(\nu_2/2 - 1)(\nu_2/2 - 2)}, \qquad \frac{\nu_2}{2} > 2,$$

and so on. The variance of the F pdf is

(A.72) $$\mu_2 = \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{(\nu_1/2)[(\nu_1 + \nu_2)/2 - 1]}{(\nu_2/2 - 1)^2(\nu_2/2 - 2)}, \qquad \frac{\nu_2}{2} > 2.$$

We now review relations of the F pdf to several other well-known pdf's.

1. If in the F pdf in (A.68) $\nu_1 = 1$ and we let $t^2 = x$, the F pdf is transformed to a standardized US t pdf with ν_2 degrees of freedom.
2. If $\tilde{z}_1$ and $\tilde{z}_2$ are independent random variables with χ^2 pdf's that have ν_1 and ν_2 degrees of freedom, respectively, then $\tilde{x} = (\tilde{z}_1/\nu_1)/(\tilde{z}_2/\nu_2)$ has an F pdf with ν_1 and ν_2 degrees of freedom, provided $\nu_1, \nu_2 > 0$.
3. If $\tilde{x}$ has the F pdf in (A.68), then, when $\nu_2 \to \infty$, the random variable $\nu_1\tilde{x}$ will have a χ^2 pdf with ν_1 degrees of freedom.
4. If $\tilde{x}$ has the F pdf in (A.68), then, when $\nu_1 \to \infty$, the random variable $\nu_2/\tilde{x}$ will have a χ^2 pdf with ν_2 degrees of freedom.
5. If $\tilde{x}$ has the F pdf in (A.68), then, when $\nu_2 \to \infty$ with $\nu_1 = 1$, the random variable $\sqrt{\tilde{x}}$ will have a standardized UN pdf.
6. If $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are independent random variables with IG pdf's in the form of (A.37b) and parameters ν_1, s_1, and ν_2, s_2, respectively, the random variable $\tilde{x} = (\tilde{\sigma}_1^2/s_1^2)/(\tilde{\sigma}_2^2/s_2^2)$ will have an F pdf with ν_2 and ν_1 degrees of freedom.

Proposition (1) is established by making the change of variable $t^2 = x$ in (A.68) and noting that with $\nu_1 = 1$ the resulting pdf is precisely the standardized US t pdf with ν_2 degrees of freedom.

Proposition (2) is established by noting that the joint pdf for $\tilde{z}_1$ and $\tilde{z}_2$ is

$$p(z_1, z_2|\nu_1, \nu_2) = kz_1^{\nu_1/2-1}z_2^{\nu_2/2-1}e^{-(z_1+z_2)/2}, \qquad 0 < z_1, z_2 < \infty,$$

with $1/k = 2^{(\nu_1+\nu_2)/2}\Gamma(\nu_1/2)\Gamma(\nu_2/2)$. Now let $v = z_1/z_2$ and $y = (z_1 + z_2)/2$, which implies that $z_1 = 2vy/(v + 1)$ and $z_2 = 2y/(v + 1)$. The Jacobian of this transformation is $4y/(v + 1)^2$ and thus the pdf for v and y is

$$p(v, y|\nu_1, \nu_2) = 2^{(\nu_1+\nu_2)/2}k\frac{v^{\nu_1/2-1}}{(1 + v)^{(\nu_1+\nu_2)/2}}y^{(\nu_1+\nu_2)/2-1}e^{-y}, \qquad 0 < v, y, <\infty.$$

On integrating with respect to y, the result is

$$p(v|\nu_1, \nu_2) = \frac{\Gamma[(\nu_1 + \nu_2)/2]}{\Gamma(\nu_1/2, \nu_2/2)}\frac{v^{\nu_1/2-1}}{(1 + v)^{(\nu_1+\nu_2)/2}}, \qquad 0 < v < \infty.$$

Finally, on letting $x = (\nu_2/\nu_1)v$, we obtain the F pdf in (A.68).

To prove Proposition (3) let $z = \nu_1 x$ in (A.68) to obtain

$$p(z|\nu_1, \nu_2) = \frac{\Gamma[(\nu_1 + \nu_2)/2]}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \frac{1}{\nu_2^{\nu_1/2}} \frac{z^{\nu_1/2-1}}{(1 + z/\nu_2)^{(\nu_1+\nu_2)/2}}, \qquad 0 < z < \infty,$$

and as $\nu_2 \to \infty$ with ν_1 fixed the limit is

$$\lim_{\nu_2 \to \infty} p(z|\nu_1, \nu_2) = \left[2^{\nu_1/2}\,\Gamma\left(\frac{\nu_1}{2}\right)\right]^{-1} z^{\nu_1/2-1} e^{-z/2}, \qquad 0 < z < \infty,$$

which is a χ^2 pdf with ν_1 degrees of freedom. Propositions (4) and (5) can be established by using similar methods.

Proposition (6) is established by writing the joint pdf for $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$:

$$p(\sigma_1, \sigma_2|\nu_1, \nu_2, s_1, s_2) = k(\sigma_1^{\nu_1+1}\sigma_2^{\nu_2+1})^{-1} \exp\left(-\frac{\nu_1 s_1^2}{2\sigma_1^2} - \frac{\nu_2 s_2^2}{2\sigma_2^2}\right), \qquad 0 < \sigma_1, \sigma_2 < \infty,$$

with

$$k = \frac{4}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \left(\frac{\nu_1 s_1^2}{2}\right)^{\nu_1/2} \left(\frac{\nu_2 s_2^2}{2}\right)^{\nu_2/2}.$$

Now make the following change of variables $\lambda = \sigma_1^2/\sigma_2^2$ and $\phi = \sigma_1$ to obtain

$$p(\lambda, \phi|\nu_1, \nu_2, s_1, s_2) = \frac{k}{2} \frac{\lambda^{\nu_2/2-1}}{\phi^{\nu_1+\nu_2+1}} \exp\left(-\frac{1}{2\phi^2}(\nu_1 s_1^2 + \lambda\nu_2 s_2^2)\right), \qquad 0 < \lambda, \phi, < \infty.$$

On integrating with respect to ϕ, the result is

$$\begin{aligned} p(\lambda|\nu_1, \nu_2, s_1, s_2) &= \frac{k}{4}\Gamma\left(\frac{\nu_1 + \nu_2}{2}\right) \frac{2^{(\nu_1+\nu_2)/2}\lambda^{\nu_2/2-1}}{(\nu_1 s_1^2 + \lambda\nu_2 s_2^2)^{(\nu_1+\nu_2)/2}} \\ &= \frac{\Gamma[(\nu_1 + \nu_2)/2]}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \left(\frac{\nu_2 s_2^2}{\nu_1 s_1^2}\right)^{\nu_2/2} \frac{\lambda^{\nu_2/2-1}}{(1 + \nu_2 s_2^2/\nu_1 s_1^2 \lambda)^{(\nu_1+\nu_2)/2}}. \end{aligned}$$

If we change variables by $x = s_2^2\lambda/s_1^2$, the pdf for x will be precisely the F pdf in (A.68).

APPENDIX B

Properties of Some Multivariate Pdf's

B.1 THE MULTIVARIATE NORMAL (MN) Pdf

The elements of a random vector $\tilde{\mathbf{x}}' = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m)$ are said to be jointly normally distributed if, and only if, they have a pdf in the following form:

(B.1) $$p(\mathbf{x}|\boldsymbol{\theta}, \Sigma) = \frac{|\Sigma|^{-1/2}}{(2\pi)^{m/2}} \exp\{-\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\theta})'\,\Sigma^{-1}(\mathbf{x} - \boldsymbol{\theta})\},$$
$$-\infty < x_i < \infty, \; i = 1, 2, \ldots, m,$$

where $\mathbf{x}' = (x_1, x_2, \ldots, x_m)$, $\boldsymbol{\theta}' = (\theta_1, \theta_2, \ldots, \theta_m)$, with $-\infty < \theta_i < \infty$, $i = 1, 2, \ldots, m$, and Σ is an $m \times m$ positive definite symmetric (PDS) matrix. For convenience the MN pdf is often written

(B.2) $$p(\mathbf{x}|\boldsymbol{\theta}, V) = \frac{|V|^{1/2}}{(2\pi)^{m/2}} \exp\{-\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\theta})'V(\mathbf{x} - \boldsymbol{\theta})\},$$
$$-\infty < x_i < \infty, \; i = 1, 2, \ldots, m,$$

where

(B.3) $$V \equiv \Sigma^{-1}$$

is PDS.

That (B.2) is a proper normalized pdf can be shown by observing that the pdf is positive over the region for which it is defined. Also,

(B.4) $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(\mathbf{x}|\boldsymbol{\theta}, V)\, d\mathbf{x} = 1,$$

where $d\mathbf{x} = dx_1\, dx_2 \cdots dx_m$; (B.4) can be shown easily by employing the following change of variables:

(B.5) $$\mathbf{x} - \boldsymbol{\theta} = C\mathbf{z}$$

with C an $m \times m$ nonsingular symmetric matrix such that $C'VC = I_m$.[1]

[1] Since V is PDS, there exists an orthogonal matrix, say P, such that $P'VP = D$, where D is an $m \times m$ diagonal matrix with the positive roots of V on the diagonal. Then $C = PD^{-1/2}$ is a nonsingular symmetric matrix such that $C'VC = I_m$.

The Jacobian of the transformation in (B.5) is $|C|$ and thus (B.2) can be expressed as[2]

$$p(\mathbf{z}) = \frac{|C|\,|V|^{1/2}}{(2\pi)^{m/2}} \exp\{-\tfrac{1}{2}\mathbf{z}'C'VC\mathbf{z}\} \tag{B.6}$$
$$= \frac{1}{(2\pi)^{m/2}} \exp\{-\tfrac{1}{2}\mathbf{z}'\mathbf{z}\}, \qquad -\infty < z_i < \infty,\ i = 1, 2, \ldots, m.$$

The pdf in (B.6) is in the form of a product of m standardized, proper normalized UN pdf's. Thus (B.6), integrated with respect to z_i, $-\infty < z_i < \infty$, $i = 1, 2, \ldots, m$, is equal to 1, which establishes (B.4).

The pdf in (B.6) is usually referred to as a standardized multivariate normal (SMN) pdf.[3] If the elements of an $m \times 1$ random vector $\tilde{\mathbf{z}}$ have a SMN pdf, it is clear from the form of (B.6) that they are independently distributed with

$$E\tilde{\mathbf{z}} = \mathbf{0} \quad \text{and} \quad E\tilde{\mathbf{z}}\tilde{\mathbf{z}}' = I_m; \tag{B.7}$$

that is, each $\tilde{z}_i$ has a zero mean and unit variance and all covariances, $E\tilde{z}_i\tilde{z}_j$, $i, j = 1, 2, \ldots, m$, $i \neq j$, are zero.

From $\tilde{\mathbf{x}} - \boldsymbol{\theta} = C\tilde{\mathbf{z}}$ and the results in (B.7) we have

$$E(\tilde{\mathbf{x}} - \boldsymbol{\theta}) = CE\tilde{\mathbf{z}} = \mathbf{0} \tag{B.8}$$

and

$$E(\tilde{\mathbf{x}} - \boldsymbol{\theta})(\tilde{\mathbf{x}} - \boldsymbol{\theta})' = CE\tilde{\mathbf{z}}\tilde{\mathbf{z}}'C' \tag{B.9}$$
$$= CC' = V^{-1} = \Sigma.^4$$

The result in (B.8) gives $E\tilde{\mathbf{x}} = \boldsymbol{\theta}$, as the mean vector of the MN pdf, whereas (B.9) yields V^{-1} (or Σ) as its covariance matrix.

We now derive the conditional pdf for $\tilde{\mathbf{x}}_1$, given $\tilde{\mathbf{x}}_2$, where $\tilde{\mathbf{x}}' = (\tilde{\mathbf{x}}_1'\tilde{\mathbf{x}}_2')$ has a MN pdf given by (B.2). Partitioning $\mathbf{x} - \boldsymbol{\theta}$ and V to correspond to the partitioning of $\tilde{\mathbf{x}}$, that is

$$\mathbf{x} - \boldsymbol{\theta} = \begin{pmatrix} \mathbf{x}_1 - \boldsymbol{\theta}_1 \\ \mathbf{x}_2 - \boldsymbol{\theta}_2 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

[2] Note that $|C|\,|V|^{1/2} = |C'VC|^{1/2} = 1$.

[3] Often the term "spherical" is used rather than "standardized" to emphasize the fact that the contours of (B.6) are spherical (or, with $m = 2$, circular).

[4] From $C'VC = I_m$ we have $V = (C')^{-1}C^{-1}$ and thus $V^{-1} = CC'$.

we can express the quadratic form in the exponent of (B.2) as

$$\begin{aligned}(\mathbf{x} - \boldsymbol{\theta})'V(\mathbf{x} - \boldsymbol{\theta}) &= (\mathbf{x}_1 - \boldsymbol{\theta}_1)'V_{11}(\mathbf{x}_1 - \boldsymbol{\theta}_1) + 2(\mathbf{x}_1 - \boldsymbol{\theta}_1)'V_{12}(\mathbf{x}_2 - \boldsymbol{\theta}_2) \\ &\quad + (\mathbf{x}_2 - \boldsymbol{\theta}_2)'V_{22}(\mathbf{x}_2 - \boldsymbol{\theta}_2) \\ &= [\mathbf{x}_1 - \boldsymbol{\theta}_1 + V_{11}^{-1}V_{12}(\mathbf{x}_2 - \boldsymbol{\theta}_2)]'V_{11} \\ &\quad \times [\mathbf{x}_1 - \boldsymbol{\theta}_1 + V_{11}^{-1}V_{12}(\mathbf{x}_2 - \boldsymbol{\theta}_2)] \\ &\quad + (\mathbf{x}_2 - \boldsymbol{\theta}_2)'(V_{22} - V_{21}V_{11}^{-1}V_{12})(\mathbf{x}_2 - \boldsymbol{\theta}_2)\end{aligned} \tag{B.10}$$

by completing the square on $\mathbf{x}_1$. Further, we have

$$|V| = \begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} = |V_{11}|\,|V_{22} - V_{21}V_{11}^{-1}V_{12}|. \tag{B.11}$$

On substituting from (B.10) and (B.11) into (B.2), we can write (B.2) as the product of two factors

$$\begin{aligned}p(\mathbf{x}_1, \mathbf{x}_2|\boldsymbol{\theta}, V) &= \Big(\frac{|V_{11}|^{1/2}}{(2\pi)^{m_1/2}} \exp\{-\tfrac{1}{2}[\mathbf{x}_1 - \boldsymbol{\theta}_1 + V_{11}^{-1}V_{12}(\mathbf{x}_2 - \boldsymbol{\theta}_2)]'V_{11} \\ &\qquad \times [\mathbf{x}_1 - \boldsymbol{\theta}_1 + V_{11}^{-1}V_{12}(\mathbf{x}_2 - \boldsymbol{\theta}_2)]\}\Big) \\ &\quad \times \Big\{\frac{|V_{22} - V_{21}V_{11}^{-1}V_{12}|^{1/2}}{(2\pi)^{m_2/2}} \\ &\qquad \times \exp[-\tfrac{1}{2}(\mathbf{x}_2 - \boldsymbol{\theta}_2)'(V_{22} - V_{21}V_{11}^{-1}V_{12})(\mathbf{x}_2 - \boldsymbol{\theta}_2)]\Big\},\end{aligned} \tag{B.12}$$

where m_1 and m_2 are the number of elements in $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, with $m_1 + m_2 = m$. Both factors in (B.12) are in the form of normalized MN pdf's. The first factor on the rhs of (B.12) is the conditional pdf for $\mathbf{x}_1$, given $\mathbf{x}_2$, since in general we can write $p(\mathbf{x}_1, \mathbf{x}_2|\boldsymbol{\theta}, V) = p(\mathbf{x}_1|\mathbf{x}_2, \boldsymbol{\theta}, V)\,p(\mathbf{x}_2|\boldsymbol{\theta}, V)$. It is a MN pdf with mean vector

$$E(\tilde{\mathbf{x}}_1|\tilde{\mathbf{x}}_2) = \boldsymbol{\theta}_1 - V_{11}^{-1}V_{12}(\mathbf{x}_2 - \boldsymbol{\theta}_2) \tag{B.13a}$$

and covariance matrix

$$\operatorname{Cov}(\tilde{\mathbf{x}}_1|\tilde{\mathbf{x}}_2) = V_{11}^{-1}. \tag{B.14a}$$

Since $V = \Sigma^{-1}$, we can express (B.13) and (B.14) in terms of submatrices of Σ[5]:

$$E(\tilde{\mathbf{x}}_1|\tilde{\mathbf{x}}_2) = \boldsymbol{\theta}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \boldsymbol{\theta}_2) \tag{B.13b}$$

[5] That is, we partition Σ^{-1} to correspond to the partitioning of V,

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix}.$$

Then $V_{11}^{-1} = (\Sigma^{11})^{-1}$ and $V_{12} = \Sigma^{12}$. If we partition Σ correspondingly as

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

and

$$\text{(B.14b)} \qquad \operatorname{Cov}(\tilde{\mathbf{x}}_1|\tilde{\mathbf{x}}_2) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

The marginal pdf for $\mathbf{x}_2$ can be obtained from (B.12) by integrating with respect to the elements of $\mathbf{x}_1$. Since $\mathbf{x}_1$ appears just in the first factor on the rhs of (B.12), and this is in the form of a normalized MN pdf, integrating the first factor with respect to the elements of $\mathbf{x}_1$ yields 1. Thus the second factor on the rhs of (B.12) is the marginal pdf for $\mathbf{x}_2$. It is a MN pdf with mean vector $\boldsymbol{\theta}_2$ and covariance matrix[6]

$$\text{(B.15)} \qquad \begin{aligned}\operatorname{Cov}(\tilde{\mathbf{x}}_2) &= (V_{22} - V_{21}V_{11}^{-1}V_{12})^{-1}\\ &= \Sigma_{22}.\end{aligned}$$

By similar operations the marginal pdf for $\tilde{\mathbf{x}}_1$ is found to be MN, with mean $\boldsymbol{\theta}_1$ and covariance matrix $(V_{11} - V_{12}V_{22}^{-1}V_{21})^{-1} = \Sigma_{11}$.

Last, consider linear combinations of the elements of $\tilde{\mathbf{x}}$; that is

$$\text{(B.16)} \qquad \tilde{\mathbf{w}}_1 = L_1\tilde{\mathbf{x}},$$

where $\tilde{\mathbf{x}}$ is an $m \times 1$ vector of normal random variables with a MN pdf, as shown in (B.2), and L_1 is a $n \times m$ matrix of given quantities with $n \leq m$ of rank n. Then $\tilde{\mathbf{w}}_1$ is an $n \times 1$ vector whose elements are linear combinations of the elements of $\tilde{\mathbf{x}}$. If $n < m$, write

$$\text{(B.17)} \qquad \tilde{\mathbf{w}} = \begin{pmatrix}\tilde{\mathbf{w}}_1\\ \tilde{\mathbf{w}}_2\end{pmatrix} = \begin{pmatrix}L_1\\ L_2\end{pmatrix}\tilde{\mathbf{x}} = L\tilde{\mathbf{x}}$$

with the matrix L an $m \times m$ nonsingular matrix. Then $E\tilde{\mathbf{w}} = LE\tilde{\mathbf{x}} = L\boldsymbol{\theta}$, and we can write

$$\text{(B.18)} \qquad \tilde{\mathbf{w}} - L\boldsymbol{\theta} = L(\tilde{\mathbf{x}} - \boldsymbol{\theta}).$$

Then, on noting that the Jacobian of the transformation from $\tilde{\mathbf{x}}$ to $\tilde{\mathbf{w}}$ in (B.18) is $|L^{-1}|$, we can obtain the pdf for $\tilde{\mathbf{w}}$:

$$\text{(B.19)} \quad p(\mathbf{w}|\boldsymbol{\theta}, \Sigma, L) = \frac{|L'\Sigma L|^{-1/2}}{(2\pi)^{m/2}} \exp\left[-\tfrac{1}{2}(\mathbf{w} - L\boldsymbol{\theta})'L^{-1\prime}\Sigma^{-1}L^{-1}(\mathbf{w} - L\boldsymbol{\theta})\right],$$

we have $\Sigma^{12} = -\Sigma^{11}\Sigma_{12}\Sigma_{22}^{-1}$ and $(\Sigma^{11})^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Thus $V_{11}^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ and $V_{11}^{-1}V_{12} = -\Sigma_{12}\Sigma_{22}^{-1}$.

[6] Since $V = \Sigma^{-1}$, we have

$$\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\begin{pmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix} = \begin{pmatrix}I & 0\\ 0 & I\end{pmatrix},$$

and thus $\Sigma_{21}V_{12} + \Sigma_{22}V_{22} = I$ and $\Sigma_{21}V_{11} + \Sigma_{22}V_{21} = 0$. The second of these relations yields $\Sigma_{21} = -\Sigma_{22}V_{21}V_{11}^{-1}$ which, when substituted in the first, yields $\Sigma_{22} = (V_{22} - V_{21}V_{11}^{-1}V_{12})^{-1}$.

which is a MN pdf with mean vector $L\boldsymbol{\theta}$ and covariance matrix $L\Sigma L'$. Thus $\tilde{\mathbf{w}} = L\tilde{\mathbf{x}}$ has a MN pdf. If we partition $\mathbf{w}$, as shown in (B.17), the marginal pdf for $\mathbf{w}_1$ will be multivariate normal with mean vector $L_1\boldsymbol{\theta}$ and covariance matrix $L_1\Sigma L_1'$, an application of the general result associated with (B.12) which gives the marginal pdf for a MN pdf.

B.2 THE MULTIVARIATE STUDENT (MS) *t* Pdf

A random vector $\tilde{\mathbf{x}}' = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m)$ has elements distributed according to the MS t distribution if, and only if, they have the following pdf:

$$\text{(B.20)} \quad \begin{aligned}p(\mathbf{x}|\boldsymbol{\theta}, V, \nu, m) &= \frac{\nu^{\nu/2}\Gamma[(\nu + m)/2]|V|^{1/2}}{\pi^{m/2}\Gamma(\nu/2)}[\nu + (\mathbf{x} - \boldsymbol{\theta})'V(\mathbf{x} - \boldsymbol{\theta})]^{-(m+\nu)/2},\\ &\qquad -\infty < x_i < \infty,\ i = 1, 2, \ldots, m,\end{aligned}$$

where $\nu > 0$, V is an $m \times m$ PDS matrix, and $\boldsymbol{\theta}' = (\theta_1, \theta_2, \ldots, \theta_m)$, with $-\infty < \theta_i < \infty$, $i = 1, 2, \ldots, m$. Since the quadratic form $(\mathbf{x} - \boldsymbol{\theta})'V(\mathbf{x} - \boldsymbol{\theta})$ is PD, the MS t pdf has a single mode at $\mathbf{x} = \boldsymbol{\theta}$. Further, since the pdf is symmetric about $\mathbf{x} = \boldsymbol{\theta}$, $\boldsymbol{\theta}$ is the mean of the MS t pdf which exists for $\nu > 1$, as shown below. The symmetry about $\boldsymbol{\theta}$ implies that odd-order moments about $\boldsymbol{\theta}$, when they exist, will all be zero. The matrix of second-order moments about the mean exists for $\nu > 2$ and is given by $V^{-1}[\nu/(\nu - 2)]$.

To establish that (B.20) is a proper normalized pdf we note that it is positive in the region for which it is defined. If we let

$$\text{(B.21)} \qquad \mathbf{x} - \boldsymbol{\theta} = C\mathbf{z},$$

where C is an $m \times m$ nonsingular matrix such that $C'VC = I_m$, the pdf for $\mathbf{z}$, an $m \times 1$ vector, is[7]

$$\text{(B.22)} \quad p(\mathbf{z}|\nu, m) = \frac{\nu^{\nu/2}\Gamma[(\nu + m)/2]}{\pi^{m/2}\Gamma(\nu/2)}(\nu + \mathbf{z}'\mathbf{z})^{-(m+\nu)/2}, \qquad -\infty < z_i < \infty,\ i = 1, 2, \ldots, m,$$

the standardized form of the MS t pdf. We show that (B.22) is a normalized pdf[8] by making the following change of variables from $z_1, z_2, \ldots, z_m$ to

[7] Note from $C'VC = I_m$, $|V|^{1/2} = |C|^{-1}$. The Jacobian associated with the transformation in (B.21) is $|C|$ and thus $|V|^{1/2}$ times the Jacobian is equal to 1.

[8] Another way of showing this is to observe that $p(\mathbf{z}|\nu)$ can be written as $p(\mathbf{z}|\nu) = p(z_m|\mathbf{z}_{(m-1)}, \nu)\,p(z_{m-1}|\mathbf{z}_{(m-2)}, \nu)\cdots p(z_1|\nu)$, where $\mathbf{z}_{m-j}' = (z_1, z_2, \ldots, z_{m-j})$, $j = 1, 2, \ldots, m - 1$. Each factor is in the form of a US t pdf and can be integrated by using the results of Appendix A.

$u, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}$, given by

$$
\begin{aligned}
z_1 &= u^{1/2} \cos\alpha_1 \cos\alpha_2 \cdots \cos\alpha_{m-1} \\
z_2 &= u^{1/2} \cos\alpha_1 \cos\alpha_2 \cdots \cos\alpha_{m-2} \sin\alpha_{m-1} \\
&\cdots\cdots\cdots\cdots \\
z_j &= u^{1/2} \cos\alpha_1 \cos\alpha_2 \cdots \cos\alpha_{m-j} \sin\alpha_{m-j+1} \\
&\cdots\cdots\cdots\cdots \\
z_m &= u^{1/2} \sin\alpha_1,
\end{aligned}
\tag{B.23}
$$

where $0 < u < \infty$, $-\pi/2 < \alpha_i < \pi/2$ for $i = 1, 2, \ldots, m-2$, and $0 < \alpha_{m-1} < 2\pi$. From trigonometry (B.23) yields

$$u = z_1^2 + z_2^2 + \cdots + z_m^2 = \mathbf{z}'\mathbf{z}. \tag{B.24}$$

Also, the Jacobian of the transformation in (B.23) is[9] $\frac{1}{2}u^{m/2-1}\cos^{m-2}\alpha_1 \times \cos^{m-3}\alpha_2 \cdots \cos\alpha_{m-2}$. Thus the pdf in (B.22) becomes

$$
\begin{aligned}
p(u, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}|\nu, m) &= \frac{1}{2}\frac{\nu^{\nu/2}\Gamma[(\nu+m)/2]}{\pi^{m/2}\Gamma(\nu/2)}\frac{u^{m/2-1}}{(\nu+u)^{(\nu+m)/2}} \\
&\quad \times \cos^{m-2}\alpha_1 \cos^{m-3}\alpha_2 \ldots \cos\alpha_{m-2}.
\end{aligned}
\tag{B.25}
$$

Now (B.25) can be integrated with respect to u and the α's by using

$$\int_0^\infty \frac{u^{m/2-1}}{(\nu+u)^{(\nu+m)/2}}\,du = \frac{1}{\nu^{\nu/2}}B\left(\frac{\nu}{2}, \frac{m}{2}\right) = \frac{\Gamma(\nu/2)\Gamma(m/2)}{\nu^{\nu/2}\,\Gamma[(\nu+m)/2]}, \qquad \nu > 0, \tag{B.26}$$

from (A.60),

$$\int_{-\pi/2}^{\pi/2} \cos^{m-j-1}\alpha_j\,d\alpha_j = \pi^{1/2}\frac{\Gamma[(m-j)/2]}{\Gamma[(m-j-1)/2+1]}, \qquad j = 1, 2, \ldots, m-2, \tag{B.27}$$

and

$$\int_0^{2\pi} d\alpha_{m-1} = 2\pi. \tag{B.28}$$

On substituting from (B.26), (B.27), and (B.28) into

$$\int \cdots \int p(u, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}|\nu, m)\,du\,d\alpha_1 \cdots d\alpha_{m-1}, \tag{B.29}$$

with the integrand given by (B.25), the integral in (B.29) has a value of 1. Thus (B.25) and (B.20) are normalized pdf's.

Note from (B.25) and (B.26) that the normalized marginal pdf for $u = \mathbf{z}'\mathbf{z}$ is

$$p(u|\nu, m) = \frac{\nu^{\nu/2}}{B(\nu/2, m/2)}\frac{u^{m/2-1}}{(\nu+u)^{(\nu+m)/2}}, \qquad 0 < u < \infty. \tag{B.30}$$

[9] See, for example, M. G. Kendall and A. S. Stuart, *The Advanced Theory of Statistics*, Vol. I. London: Griffin, 1958, p. 247.

Letting $u = my$, we have

$$p(y|\nu, m) = \left[B\left(\frac{\nu}{2}, \frac{m}{2}\right)\right]^{-1}\left(\frac{m}{\nu}\right)^{m/2}\frac{y^{m/2-1}}{(1 + (m/\nu)y)^{(\nu+m)/2}}, \qquad 0 < y < \infty, \tag{B.31}$$

which is an F pdf with m and ν degrees of freedom [see (A.68)]. Thus the random variable $\tilde{y} = \tilde{u}/m = \tilde{\mathbf{z}}'\tilde{\mathbf{z}}/m$, with $\tilde{\mathbf{z}}$ having a pdf given by the standardized MS t pdf in (B.22), has an F pdf with m and ν degrees of freedom. Further, by writing (B.21) to connect random variables we have

$$
\begin{aligned}
\tilde{y} = \frac{\tilde{\mathbf{z}}'\tilde{\mathbf{z}}}{m} &= \frac{(\tilde{\mathbf{x}} - \boldsymbol{\theta})'(C^{-1})'C^{-1}(\tilde{\mathbf{x}} - \boldsymbol{\theta})}{m} \\
&= \frac{(\tilde{\mathbf{x}} - \boldsymbol{\theta})'V(\tilde{\mathbf{x}} - \boldsymbol{\theta})}{m},
\end{aligned}
\tag{B.32}
$$

and thus the quadratic form $(\tilde{\mathbf{x}} - \boldsymbol{\theta})'V(\tilde{\mathbf{x}} - \boldsymbol{\theta})/m$ has an F pdf with m and ν degrees of freedom when $\tilde{\mathbf{x}}$'s pdf is the MS t pdf shown in (B.20).

To obtain expressions for the first and second moments associated with (B.20) we determine moments associated with (B.22) and then use (B.21) to find moments for (B.20). As regards existence of moments, consider the rth moment about zero. To evaluate this moment we have to consider the integral

$$\int_{-\infty}^{\infty} \frac{z_1^r\,dz_1}{(a + z_1^2)^{(\nu+m)/2}}, \qquad m = 1, 2, \ldots, \tag{B.33}$$

with $a \equiv \nu + \sum_{i=2}^m z_i^2$. Using the tests for convergence described in Appendix A, we see that (B.33) will converge for any m if $r + 1 < \nu + 1$ or $\nu > r$. Thus for the first moment to exist we need $\nu > 1$, for the second, $\nu > 2$, and so on. From the symmetry of (B.22) we have

$$E\tilde{\mathbf{z}} = \mathbf{0}, \qquad \nu > 1, \tag{B.34}$$

and from $\tilde{\mathbf{x}} - \boldsymbol{\theta} = C\tilde{\mathbf{z}}$, $E\tilde{\mathbf{x}} = \boldsymbol{\theta}$ for $\nu > 1$.

To evaluate the second moments associated with (B.22) consider (B.33) with $r = 2$. By letting $u = z_1^2/a$ (B.33) can be brought into the following form:

$$\frac{1}{a^{(\nu+m-3)/2}}\int_0^\infty \frac{u^{1/2}}{(1+u)^{(\nu+m)/2}}\,du = \frac{1}{a^{(\nu+m-3)/2}}B\left(\frac{\nu+m-3}{2}, \frac{3}{2}\right). \tag{B.35}$$

Now (B.35) has to be integrated with respect to $z_2, z_3, \ldots, z_m$, which appear in the quantity $a = \nu + \sum_{i=2}^m z_i^2$; that is, using (B.35),

$$E\tilde{z}_1^2 = kB\left(\frac{\nu+m-3}{2}, \frac{3}{2}\right)\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{dz_2 \cdots dz_m}{(\nu + \sum_{i=2}^m z_i^2)^{(\nu+m-3)/2}}, \tag{B.36}$$

with $k = \nu^{\nu/2}\Gamma[(\nu + m)/2]/\pi^{m/2}\Gamma(\nu/2)$, the normalizing constant of (B.22). The integrand of (B.36) can be brought into the form of a standardized MS t pdf and integrated[10] to yield

$$E\tilde{z}_1^2 = \frac{\nu}{\nu - 2}. \tag{B.37}$$

Since this argument can be applied separately to $\tilde{z}_2, \tilde{z}_3, \ldots, \tilde{z}_m$ and $E\tilde{z}_i\tilde{z}_j = 0$ for $i \neq j$, the covariance matrix for $\tilde{\mathbf{z}}$ is

$$E\tilde{\mathbf{z}}\tilde{\mathbf{z}}' = \frac{\nu}{\nu - 2} I_m, \qquad \nu > 2. \tag{B.38}$$

From $\tilde{\mathbf{x}} - \boldsymbol{\theta} = C\tilde{\mathbf{z}}$ the covariance matrix for $\tilde{\mathbf{x}} - \boldsymbol{\theta}$ is

$$\begin{aligned} E(\tilde{\mathbf{x}} - \boldsymbol{\theta})(\tilde{\mathbf{x}} - \boldsymbol{\theta})' &= CE\tilde{\mathbf{z}}\tilde{\mathbf{z}}C' \\ &= \frac{\nu}{\nu - 2} CC' = \frac{\nu}{\nu - 2} V^{-1}, ^{11} \qquad \nu > 2. \end{aligned} \tag{B.39}$$

We now consider the marginal and conditional pdf's associated with the MS t pdf in (B.20). To accomplish this conveniently we let[12] $H \equiv V/\nu$ and rewrite (B.20) as

(B.40)
$$p(\mathbf{x}|\boldsymbol{\theta}, H, \nu) = \frac{\Gamma[(\nu + m)/2]}{\pi^{m/2}\Gamma(\nu/2)} |H|^{1/2}[1 + (\mathbf{x} - \boldsymbol{\theta})'H(\mathbf{x} - \boldsymbol{\theta})]^{-(m+\nu)/2}, \qquad \nu > 0.$$

Now partition $(\mathbf{x} - \boldsymbol{\theta})' = [(\mathbf{x}_1 - \boldsymbol{\theta}_1)'(\mathbf{x}_2 - \boldsymbol{\theta}_2)']$, where $\mathbf{x}_1 - \boldsymbol{\theta}_1$ is an $m_1 \times 1$ vector and $\mathbf{x}_2 - \boldsymbol{\theta}_2$ is an $m_2 \times 1$ vector with $m_1 + m_2 = m$. Also let

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

[10] The integrand of (B.36) can be written as

$$\left[\left(\frac{\nu}{\nu'}\right)^{(\nu+m-3)/2}\left(\nu' + \frac{\nu'}{\nu}\sum_{i=2}^{m} z_i^2\right)^{(\nu'+m-1)/2}\right]^{-1}, \qquad -\infty < z_i < \infty,\ i = 2, 3, \ldots, m,$$

with $\nu' = \nu - 2$. Then letting $w_i = \sqrt{\nu'/\nu}\, z_i$, (B.36) becomes

$$\begin{aligned} kB\left(\frac{\nu + m - 3}{2}, \frac{3}{2}\right)\left(\frac{\nu'}{\nu}\right)^{\nu/2-1} \int\cdots\int \frac{dw_2\cdots dw_m}{(\nu' + \sum_{i=2}^m w_i^2)^{(\nu'+m-1)/2}} \\ = kB\left(\frac{\nu + m - 3}{2}, \frac{3}{2}\right)\left(\frac{\nu'}{\nu}\right)^{\nu/2-1} \frac{\pi^{(m-1)/2}\Gamma(\nu'/2)}{(\nu')^{\nu'/2}\Gamma[(\nu' + m - 1)/2]} = \frac{\nu}{\nu - 2}, \\ -\infty < w_i < \infty,\ i = 2, 3, \ldots, m. \end{aligned}$$

[11] From $C'VC = I_m$, $V = (C')^{-1}C^{-1}$ or $V^{-1} = CC'$.

[12] With $\nu > 0$, H is PDS.

where the partitioning of H corresponds to that of $\mathbf{x} - \boldsymbol{\theta}$. Then

$$\begin{aligned} 1 + (\mathbf{x} - \boldsymbol{\theta})'H(\mathbf{x} - \boldsymbol{\theta}) &= 1 + (\mathbf{x}_1 - \boldsymbol{\theta}_1)'H_{11}(\mathbf{x}_1 - \boldsymbol{\theta}_1) \\ &\quad + 2(\mathbf{x}_1 - \boldsymbol{\theta}_1)'H_{12}(\mathbf{x}_2 - \boldsymbol{\theta}_2) \\ &\quad + (\mathbf{x}_2 - \boldsymbol{\theta}_2)'H_{22}(\mathbf{x}_2 - \boldsymbol{\theta}_2) \\ &= 1 + [\mathbf{x}_1 - \boldsymbol{\theta}_1 + H_{11}^{-1}H_{12}(\mathbf{x}_2 - \boldsymbol{\theta}_2)]'H_{11} \\ &\quad \times [\mathbf{x}_1 - \boldsymbol{\theta}_1 + H_{11}^{-1}H_{12}(\mathbf{x}_2 - \boldsymbol{\theta}_2)] \\ &\quad + (\mathbf{x}_2 - \boldsymbol{\theta}_2)'(H_{22} - H_{21}H_{11}^{-1}H_{12})(\mathbf{x}_2 - \boldsymbol{\theta}_2) \\ &= 1 + Q_{1\cdot 2} + Q_2, \end{aligned} \tag{B.41}$$

where $Q_{1\cdot 2}$ and Q_2 denote the first and second quadratic forms, respectively, on the rhs of the second line of (B.41). Then, noting $|H| = |H_{11}|\,|H_{22} - H_{21}H_{11}^{-1}H_{12}|$, we can express (B.40) as

$$\begin{aligned} p(\mathbf{x}_1, \mathbf{x}_2|\boldsymbol{\theta}, H, \nu) &= \frac{\Gamma[(\nu + m)/2]|H|^{1/2}}{\pi^{m/2}\Gamma(\nu/2)}[(1 + Q_2 + Q_{1\cdot 2})^{(m+\nu)/2}]^{-1} \\ &= \left[\frac{k_1|H_{22} - H_{21}H_{11}^{-1}H_{12}|^{1/2}}{(1 + Q_2)^{(m_2+\nu)/2}}\right]\left[\frac{k_2(1 + Q_2)^{-m_1/2}|H_{11}|^{1/2}}{[1 + Q_{1\cdot 2}/(1 + Q_2)]^{(m+\nu)/2}}\right], \end{aligned} \tag{B.42}$$

where

$$k_1 = \frac{\Gamma[(\nu + m_2)/2]}{\pi^{m_2/2}\Gamma(\nu/2)} \quad \text{and} \quad k_2 = \frac{\Gamma[(m + \nu)/2]}{\pi^{m_1/2}\Gamma[(\nu + m_2)/2]}.$$

The second line of (B.42) gives explicit expressions for the marginal and conditional pdf's; that is

$$p(\mathbf{x}_1, \mathbf{x}_2|\boldsymbol{\theta}, H, \nu) = p(\mathbf{x}_2|\boldsymbol{\theta}, H, \nu)\, p(\mathbf{x}_1|\mathbf{x}_2, \boldsymbol{\theta}, H, \nu), \tag{B.43}$$

with

$$p(\mathbf{x}_2|\boldsymbol{\theta}, H, \nu) = \frac{k_1|H_{22} - H_{21}H_{11}^{-1}H_{12}|^{1/2}}{(1 + Q_2)^{(m_2+\nu)/2}}, \tag{B.44}$$

and[13]

$$p(\mathbf{x}_1|\mathbf{x}_2, \boldsymbol{\theta}, H, \nu) = \frac{k_2(1 + Q_2)^{-m_1/2}|H_{11}|^{1/2}}{[1 + Q_{1\cdot 2}/(1 + Q_2)]^{(m+\nu)/2}}, \tag{B.45}$$

with $Q_{1\cdot 2}$ and Q_2 defined in connection with (B.41); (B.44) and (B.45) show that in general the marginal and conditional pdf's associated with a MS t pdf have the forms of MS t pdf's.[14] From (B.44) we have for the mean and covariance matrix of the marginal pdf for $\mathbf{x}_2$

$$E\tilde{\mathbf{x}}_2 = \boldsymbol{\theta}_2, \qquad \nu > 1, \tag{B.46}$$

[13] The expression for the conditional MS t pdf, given in H. Raiffa and R. Schlaifer, *Applied Statistical Decision Theory*, Boston: Graduate School of Business Administration, Harvard University, 1961, p. 258 is erroneous.

[14] If $\mathbf{x}_2$ in (B.44) and $\mathbf{x}_1$ in (B.45) are scalars, these pdf's are US t pdf's.

and

$$\text{(B.47)}\quad \begin{aligned}\operatorname{Var}(\tilde{\mathbf{x}}_2) &= \frac{1}{\nu - 2}(H_{22} - H_{21}H_{11}^{-1}H_{12})^{-1} \\ &= \frac{\nu}{\nu - 2}(V_{22} - V_{21}V_{11}^{-1}V_{12})^{-1}, \qquad \nu > 2,\end{aligned}$$

since $H \equiv V/\nu$. For the conditional pdf in (B.45) we have

$$\text{(B.48)}\quad \begin{aligned}E(\tilde{\mathbf{x}}_1|\tilde{\mathbf{x}}_2) &= \boldsymbol{\theta}_1 - H_{11}^{-1}H_{12}(\mathbf{x}_2 - \boldsymbol{\theta}_2) \\ &= \boldsymbol{\theta}_1 - V_{11}^{-1}V_{12}(\mathbf{x}_2 - \boldsymbol{\theta}_2),\end{aligned} \qquad m_2 + \nu > 1,$$

and

$$\text{(B.49)}\quad \begin{aligned}\operatorname{Var}(\tilde{\mathbf{x}}_1|\tilde{\mathbf{x}}_2) &= \frac{1}{m_2 + \nu - 2}(1 + Q_2)H_{11}^{-1} \\ &= \frac{\nu}{m_2 + \nu - 2}(1 + Q_2)V_{11}^{-1}, \qquad m_2 + \nu > 2,\end{aligned}$$

where $Q_2 = (\mathbf{x}_2 - \boldsymbol{\theta}_2)'(H_{22} - H_{21}H_{11}^{-1}H_{12})(\mathbf{x}_2 - \boldsymbol{\theta}_2)$.

Next consider linear combinations of the elements of a random vector $\tilde{\mathbf{x}}$ with a MS t pdf, as shown in (B.40):

$$\text{(B.50)}\qquad \tilde{\mathbf{w}} = L\tilde{\mathbf{x}},$$

where L is an $m \times m$ nonsingular matrix. Then $E\tilde{\mathbf{w}} = L\boldsymbol{\theta}$. The Jacobian of the transformation in (B.50) is $|L^{-1}|$ and thus from (B.40) the pdf for $\tilde{\mathbf{w}}$ is

$$\text{(B.51)}\quad p(\mathbf{w}|\boldsymbol{\theta}, F, \nu) = \frac{\Gamma[(\nu + m)/2]}{\pi^{m/2}\Gamma(\nu/2)}|F|^{1/2}[1 + (\mathbf{w} - L\boldsymbol{\theta})'F(\mathbf{w} - L\boldsymbol{\theta})]^{-(m+\nu)/2},$$

where $F \equiv L^{-1\prime}HL^{-1}$. Thus the elements of $\tilde{\mathbf{w}}$ have a MS t pdf with mean $L\boldsymbol{\theta}$ and covariance matrix equal to $(\nu - 2)^{-1}LH^{-1}L' = [\nu/(\nu - 2)]LV^{-1}L'$. The marginal and conditional pdf's associated with (B.51) are easily obtained by using (B.44) and (B.45) and, of course, will be in the MS t form. A single linear combination of the elements of $\tilde{\mathbf{x}}$, say $\tilde{w}_1$, the first element of $\tilde{\mathbf{w}}$, will have a marginal US t pdf.

Last, as is apparent from many examples cited, the MS t is related to the MN and IG pdf's. Consider the joint pdf

$$\text{(B.52)}\qquad p(\mathbf{x}, \sigma|\boldsymbol{\theta}, V, \nu) = g(\mathbf{x}|\boldsymbol{\theta}, \sigma, V)\, h(\sigma|\nu),$$

where $g(\mathbf{x}|\boldsymbol{\theta}, \sigma, V)$ denotes an m-dimensional MN pdf with mean $\boldsymbol{\theta}$ and covariance matrix $V^{-1}\sigma^2$ and $h(\sigma|\nu)$ denotes an IG pdf with parameters $\nu > 0$ and $s = 1$ [see (A.37b)]; that is,

$$\text{(B.53)}\quad p(\mathbf{x}, \sigma|\boldsymbol{\theta}, V, \nu) = k\frac{|V|^{1/2}}{\sigma^{m+\nu+1}}\exp\left\{-\frac{1}{2\sigma^2}[\nu + (\mathbf{x} - \boldsymbol{\theta})'V(\mathbf{x} - \boldsymbol{\theta})]\right\},$$

where k is the normalizing constant. Then, on integrating (B.53), with respect to σ, 0 to ∞, we have

$$\text{(B.54)}\qquad p(\mathbf{x}|\boldsymbol{\theta}, V, \nu) = k'|V|^{1/2}[\nu + (\mathbf{x} - \boldsymbol{\theta})'V(\mathbf{x} - \boldsymbol{\theta})]^{-(m+\nu)/2},$$

which is precisely in the form of (B.20), with k' the normalizing constant.

B.3 THE WISHART (W) Pdf

The $m(m + 1)/2$ distinct elements of an $m \times m$ PDS random matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ are distributed according to the W distribution if, and only if, they have the following pdf:

$$\text{(B.55)}\quad p(A|\Sigma, \nu, m) = k\frac{|A|^{(\nu-m-1)/2}}{|\Sigma|^{\nu/2}}\exp\{-\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}A\}, \qquad |A| > 0,$$

where $k^{-1} = 2^{\nu m/2}\pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma[(\nu + 1 - i)/2]$, $m \leq \nu$, and $\Sigma = [\sigma_{ij}]$, an $m \times m$ PDS matrix. The pdf in (B.55) is defined for the region given by $|A| > 0$. We denote the pdf in (B.55) by $W(\Sigma, \nu, m)$. Some properties of the W pdf are listed below.

1. If $\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2, \ldots, \tilde{\mathbf{z}}_\nu$ are $m \times 1$ mutually independent random vectors, each with a MN pdf, zero mean vector, and common PDS $m \times m$ covariance matrix Σ, the distinct elements of $\tilde{A} = \tilde{Z}\tilde{Z}'$, where $\tilde{Z} = (\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2, \ldots, \tilde{\mathbf{z}}_\nu)$, have a $W(\Sigma, \nu, m)$ pdf. Note that $\tilde{Z}\tilde{Z}' = \sum_{i=1}^{\nu}\tilde{\mathbf{z}}_i\tilde{\mathbf{z}}_i'$ has diagonal elements given by $\sum_{i=1}^{\nu}\tilde{z}_{ij}^2$, $j = 1, 2, \ldots, m$, and off-diagonal elements given by $\sum_{i=1}^{\nu}\tilde{z}_{ij}\tilde{z}_{ik}$, $j \neq k = 1, 2, \ldots, m$. Thus $\tilde{Z}\tilde{Z}'/\nu = \tilde{S}$, the sample covariance matrix, and the distinct elements of $\tilde{S}$ have a $W[(1/\nu)\Sigma, \nu, m]$ pdf.
2. The distinct elements of a random matrix $\tilde{A}$, with the $W(\Sigma, \nu, m)$ pdf in (B.55), have the following means, variances, and covariances:

$$\text{(B.56)}\qquad E\tilde{a}_{ij} = \nu\sigma_{ij},$$

$$\text{(B.57)}\qquad \operatorname{Var}\tilde{a}_{ij} = \nu(\sigma_{ij}^2 + \sigma_{ii}\sigma_{jj}),$$

and

$$\text{(B.58)}\qquad \operatorname{Cov}(\tilde{a}_{ij}, \tilde{a}_{kl}) = \nu(\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}).$$

Let us partition A and Σ correspondingly as

$$A = \begin{array}{c c} & \begin{array}{cc} m_1 & m - m_1 \end{array} \\ \begin{array}{c} m_1 \\ m - m_1 \end{array} & \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{array}, \qquad A^{-1} = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix},$$

$$\Sigma = \begin{array}{c c} & \begin{array}{cc} m_1 & m - m_1 \end{array} \\ \begin{array}{c} m_1 \\ m - m_1 \end{array} & \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \end{array}, \qquad \Sigma^{-1} = \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix},$$

where, in each instance, the (1, 1)th submatrix is of size $m_1 \times m_1$ and the (2, 2)th submatrix is of size $m - m_1 \times m - m_1$. Further, let

$$A_{11\cdot 2}^{-1} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = A^{11}$$

and

$$\Sigma_{11\cdot 2}^{-1} = (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} = \Sigma^{11}.$$

Then the following properties of the $W(\Sigma, \nu, m)$ pdf are known to hold:[15]

3. The joint pdf for the distinct elements of A_{11} is $W(\Sigma_{11}, \nu, m_1)$.
4. The joint pdf for the distinct elements of $A_{11\cdot 2}$ is $W[\Sigma_{11\cdot 2}, \nu - (m - m_1), m_1]$.
5. The marginal pdf for $r_{12} = a_{12}/(a_{11}a_{22})^{1/2}$ is

$$h(r_{12}|\rho_{12}, \nu) = k_1(1 - r_{12}^2)^{(\nu-3)/2}(1 - \rho_{12}^2)^{\nu/2}\, I_\nu(\rho_{12}r_{12}), \tag{B.59}$$

with $k_1 = [(\nu - 1)/\sqrt{2\pi}][\Gamma(\nu)/\Gamma(\nu + \frac{1}{2})]$, $\rho_{12} = \sigma_{12}/(\sigma_{11}\sigma_{22})^{1/2}$, and[16]

$$I_\nu(\rho r) = \int_0^\infty \frac{dy}{(\cosh y - \rho r)^\nu}; \tag{B.60}$$

(B.59) gives the pdf for a sample correlation coefficient based on ν pairs of observations drawn independently from a bivariate normal pdf with zero means and 2×2 PDS covariance matrix Σ.

Property 1 is a fundamental relationship between the MN pdf and the W pdf. It is established as follows.[17] The joint pdf for ν normal, mutually independent $m \times 1$ vectors, $\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2, \ldots, \tilde{\mathbf{z}}_\nu$, each with zero mean vector and common PDS covariance matrix, Σ, is

$$\begin{aligned} p(Z|\Sigma, \nu, m) &= \frac{|\Sigma|^{-\nu/2}}{(2\pi)^{\nu m/2}} \exp\left(-\tfrac{1}{2}\textstyle\sum_{i=1}^{\nu} \mathbf{z}_i'\Sigma^{-1}\mathbf{z}_i\right) \\ &= \frac{|\Sigma|^{-\nu/2}}{(2\pi)^{\nu m/2}} \exp\{-\tfrac{1}{2}\, tr\, \Sigma^{-1}ZZ'\}, \end{aligned} \tag{B.61}$$

where $Z = (\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_\nu)$, an $m \times \nu$ matrix with $m \le \nu$. Now make the following transformation $Z = TK$, where K is an $m \times \nu$ matrix such that $KK' = I_m$; that is, K is a semiorthogonal matrix, and T is an $m \times m$ lower

[15] The following well known properties have been listed in S. Geisser, "Bayesian Estimation in Multivariate Analysis," *Ann. Math. Statistics*, **36**, 150–159 (1965).

[16] The "cosh" function is defined by $\cosh u = (e^u + e^{-u})/2$.

[17] The derivation follows that presented in S. N. Roy, *Some Aspects of Multivariate Analysis*. New York: Wiley, 1957, p. 33.

triangular matrix:

$$T = \begin{pmatrix} t_{11} & 0 & \cdots & 0 \\ t_{21} & t_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mm} \end{pmatrix}, \tag{B.62}$$

with $|T| = \prod_{i=1}^m t_{ii} > 0$. Note that $KK' = I_m$ places $m(m + 1)/2$ constraints on the elements of K. Thus there are really only $\nu m - m(m + 1)/2$ independent elements in K. Choose an independent set of elements from those in K, say $(k_{11}, k_{12}, \ldots, k_{1,\nu-1}), (k_{21}, k_{22}, \ldots, k_{2,\nu-2}), \ldots, (k_{m1}, k_{m2}, \ldots, k_{m,\nu-m})$, and call this set K_I. Thus we can regard the transformation $Z = TK$ subject to $KK' = I_m$ as equivalent to the transformation from the νm elements of Z to the $m(m + 1)/2$ elements of T and the $\nu m - m(m + 1)/2$ elements of K_I. Then, on substituting $Z = TK$ in (B.61), we have

$$p(T, K_I|\Sigma, \nu, m) = \frac{J|\Sigma|^{-\nu/2}}{(2\pi)^{\nu m/2}} \exp\{-\tfrac{1}{2}\, tr\, \Sigma^{-1}TT'\}, \tag{B.63}$$

where J denotes the Jacobian of the transformation from the elements of Z to those of T and K_I. To obtain an explicit expression for J we use the following result[18]:

If $y_i = f_i(x_1, x_2, \ldots, x_p, x_{p+1}, \ldots, x_{p+q})$ for $i = 1, 2, \ldots, p$, where the x_j's, $j = 1, 2, \ldots, p + q$, are subject to q constraints, $f_i(x_1, x_2, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}) = 0$ for $i = p + 1, p + 2, \ldots, p + q$, then[19] the Jacobian J associated with the transformation from $x_1, x_2, \ldots, x_p$ to $y_1, y_2, \ldots, y_p$ is

$$J = \left|\frac{\partial(f_1, f_2, \ldots, f_p, f_{p+1}, \ldots, f_{p+q})}{\partial(x_1, x_2, \ldots, x_p, x_{p+1}, \ldots, x_{p+q})}\right| \div \left|\frac{\partial(f_{p+1}, \ldots, f_{p+q})}{\partial(x_{p+1}, \ldots, x_{p+q})}\right|. \tag{B.64}$$

In applying this result to the present problem, $Z = TK$ takes the place of $y_i = f_i$ and $KK' - I_m = 0$ takes the place of $f_i = 0$. Further, the elements of K_I are to be associated with $x_1, x_2, \ldots, x_p$, whereas the remaining elements of K, denoted K_D, are to be associated with $x_{p+1}, \ldots, x_{p+q}$. Then the Jacobian in (B.60) is

$$J = \left|\frac{\partial(Z, KK')}{\partial(T, K)}\right|_{T,K_I} \div \left|\frac{\partial(KK' - I_m)}{\partial(K_D)}\right|_{K_I}. \tag{B.65}$$

The explicit expression for the numerator of (B.65) is[20]

$$\left|\frac{\partial(Z, KK')}{\partial(T, K)}\right|_{T,K_I} = 2^m \prod_{i=1}^m t_{ii}^{\nu-i}. \tag{B.66}$$

[18] This result from the calculus is presented in S. N. Roy, *op. cit.*, p. 165.

[19] It is assumed that the usual conditions for the existence of the Jacobian, including the nonvanishing of the numerator and denominator in (B.64), are satisfied.

[20] Cf. Roy, *op. cit.*, pp. 170–174.

Thus (B.63) becomes

$$\text{(B.67)} \quad p(T, K_I|\Sigma, m, \nu) = \frac{2^m \prod_{i=1}^m t_{ii}^{\nu-i}}{(2\pi)^{\nu m/2}|\Sigma|^{\nu/2}} e^{-\frac{1}{2}\operatorname{tr}\Sigma^{-1}TT'} \div \left|\frac{\partial(KK')}{\partial(K_D)}\right|_{K_I}.$$

Since[21] $\int dK_I \div |\partial(KK')/\partial(K_D)|_{K_I}$ integrated over the region $KK' = I_m$, equals $\pi^{\nu m/2-[m(m-1)]/4}/\prod_{i=1}^m \Gamma[(\nu - i + 1)/2]$, the marginal distribution of the elements of T is

$$\text{(B.68)} \qquad p(T|\Sigma, m, \nu) = c\frac{\prod_{i=1}^m t_{ii}^{\nu-i}}{|\Sigma|^{\nu/2}} \exp\{-\tfrac{1}{2}\, tr\, \Sigma^{-1}TT'\}$$

with

$$c^{-1} = 2^{\nu m/2-m}\pi^{m(m-1)/4}\prod_{i=1}^m \Gamma\left(\frac{\nu - i + 1}{2}\right).$$

Now the sample covariance matrix S, with $m(m + 1)/2$ distinct elements, is given by $\nu S = ZZ' = TKK'T' = TT'$. Transform (B.68), which involves $m(m + 1)/2$ elements of T, to a pdf for the distinct elements of S. This yields[22]

$$\begin{aligned} p(S|\Sigma, \nu, m) &= c\frac{\prod_{i=1}^m t_{ii}^{\nu-i}}{|\Sigma|^{\nu/2}} \frac{\nu^{m(m+1)/2}}{2^m \prod_{i=1}^m t_{ii}^{m-i+1}} \exp\{-\tfrac{1}{2}\operatorname{tr}\nu\Sigma^{-1}S\} \\ \text{(B.69)} \qquad &= \frac{c\nu^{m(m+1)/2}}{2^m}\frac{|T|^{\nu-m-1}}{|\Sigma|^{\nu/2}} \exp\{-\tfrac{1}{2}\operatorname{tr}\nu\Sigma^{-1}S\} \\ &= c_1\frac{|S|^{(\nu-m-1)/2}}{|\Sigma|^{\nu/2}} \exp\{-\tfrac{1}{2}\operatorname{tr}\nu\Sigma^{-1}S\}, \end{aligned}$$

where

$$c_1^{-1} = \frac{\pi^{[m(m-1)]/4}}{(\nu/2)^{\nu m/2}}\prod_{i=1}^m \Gamma\left(\frac{\nu - i + 1}{2}\right).$$

The pdf in (B.69) is $W[(1/\nu)\Sigma, \nu, m]$, as was to be shown. By a simple change of variables the pdf for $A = \nu S$ can be obtained from (B.69) and is $W(\Sigma, \nu, m)$, the pdf given in (B.55).[23]

[21] Cf. Roy, *op. cit.*, p. 197.

[22] The Jacobian of the transformation from the elements of T to the distinct elements of S is $\nu^{m(m+1)/2} \div (2^m \prod_{i=1}^m t_{ii}^{m-i+1})$. Also, in the second line of (B.69), from $\nu S = TT'$, we use $|T| = |\nu S|^{1/2} = \nu^{m/2}|S|^{1/2}$.

[23] Note that we have defined $\tilde{A} = \tilde{Z}\tilde{Z}' = \nu\tilde{S}$, with the ν columns of $\tilde{Z}$ assumed to be independent. Frequently we have an $n \times m$ random matrix $\hat{U} = [I_n - \iota(\iota'\iota)^{-1}\iota']\tilde{Y}$, where $\iota' = (1 \ldots 1)$, a $1 \times n$ vector of ones and $\tilde{Y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_m)$, with each column of $\tilde{Y}$ an $n \times 1$ vector. The *rows* of $\tilde{Y}$ are assumed to be independently and normally distributed, each with a $1 \times m$ mean vector μ' and common $m \times m$ PDS covariance matrix Σ. Although the rows of $\tilde{Y}$ are independent, the rows of the residual matrix $\hat{U}$ are not. Writing $\tilde{Y} = \iota\mu' + \tilde{U}$, where $\tilde{U}$ is an $n \times m$ matrix whose rows are independently and normally distributed, each with zero mean vector and common PDS covariance matrix Σ, we have $\hat{U} = (I_n - \iota(\iota'\iota)^{-1}\iota')\tilde{U}$ and $\hat{U}'\hat{U} = \tilde{U}'[I_n - \iota(\iota'\iota)^{-1}\iota']\tilde{U}$,

The formulas for the moments, (B.56) to (B.58), have been obtained in the literature[24] from those of the elements of $\tilde{Z}\tilde{Z}'$, since $\tilde{A} = \tilde{Z}\tilde{Z}'$, with $\tilde{Z}$ having the MN pdf in (B.61):

$$E\tilde{a}_{ij} = E\sum_{\alpha=1}^{\nu} \tilde{z}_{\alpha i}\tilde{z}_{\alpha j} = \nu\sigma_{ij}$$

and

$$\begin{aligned} E\tilde{a}_{ij}\tilde{a}_{kl} &= E\left(\sum_{\alpha=1}^{\nu} \tilde{z}_{\alpha i}\tilde{z}_{\alpha j}\right)\left(\sum_{\alpha'=1}^{\nu} \tilde{z}_{\alpha' k}\tilde{z}_{\alpha' l}\right) \\ &= E\sum_{\alpha=1}^{\nu} \tilde{z}_{\alpha i}\tilde{z}_{\alpha j}\tilde{z}_{\alpha k}\tilde{z}_{\alpha l} + E\sum_{\substack{\alpha,\alpha'=1 \\ \alpha\neq\alpha'}}^{\nu} \tilde{z}_{\alpha i}\tilde{z}_{\alpha j}\tilde{z}_{\alpha' k}\tilde{z}_{\alpha' l} \\ &= \nu(\sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}) + \nu(\nu - 1)\sigma_{ij}\sigma_{kl} \\ &= \nu^2\sigma_{ij}\sigma_{kl} + \nu\sigma_{ik}\sigma_{jl} + \nu\sigma_{il}\sigma_{jk}. \end{aligned}$$

Then

$$\begin{aligned} \operatorname{Cov}(\tilde{a}_{ij}, \tilde{a}_{kl}) &= E(\tilde{a}_{ij} - \nu\sigma_{ij})(\tilde{a}_{kl} - \nu\sigma_{kl}) \\ &= \nu(\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}), \end{aligned}$$

and, for $i = k$ and $j = l$,

$$\operatorname{Var}(\tilde{a}_{ij}) = \nu(\sigma_{ii}\sigma_{jj} + \sigma_{ij}^2).$$

Property 3 is most easily shown by partitioning $\tilde{Z}' = (\tilde{Z}_1'\tilde{Z}_2')$ where $\tilde{Z}_1$ is an $m_1 \times \nu$ random matrix with columns independently and normally distributed with zero vector mean and $m_1 \times m_1$ PDS covariance matrix Σ_{11}. Then, using (1), $\tilde{A}_{11} = \tilde{Z}_1\tilde{Z}_1'$ has a $W(\Sigma_{11}, \nu, m_1)$ pdf. In the case that $m_1 = 1$, $W(\sigma_{11}, \nu, 1)$ is in the form of a univariate gamma (G) pdf, which, of course, can be transformed to a χ^2 pdf. Thus the W pdf can be viewed as a multivariate generalization of the univariate G pdf.

To prove Property 4 let us write $\tilde{A} = \tilde{V}'\tilde{V}$, where $\tilde{V}$ is an $\nu \times m (m \leq \nu)$ random matrix with *rows* independently and normally distributed, each with

where $I_n - \iota(\iota'\iota)^{-1}\iota'$ is idempotent, with rank $n - 1$. Now let $\tilde{U} = L\tilde{V}$, where L is an $n \times n$ nonsingular orthogonal matrix such that

$$L'[I_n - \iota(\iota'\iota)^{-1}\iota']L = \left(\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0' & 0 \end{array}\right).$$

Then $\hat{U}'\hat{U} = \tilde{V}'L'[I_n - \iota(\iota'\iota)^{-1}\iota']L\tilde{V} = \tilde{V}_1'\tilde{V}_1$, where $\tilde{V}' = (\tilde{V}_1' \vdots \tilde{v}_n)$; that is, $\tilde{V}_1$ is an $(n - 1) \times m$ matrix formed from $\tilde{V}$ by deleting the last row. Since the $n - 1$ rows of $\tilde{V}_1$ are independently and normally distributed, each with zero vector mean and common covariance matrix Σ (Note: $E\tilde{V}'\tilde{V} = E\tilde{U}'L'L\tilde{U} = E\tilde{U}'\tilde{U} = \Sigma \otimes I_n$), it satisfies the conditions of Property 1. Thus $\hat{U}'\hat{U} = \tilde{V}_1'\tilde{V}_1$ has a $W(\Sigma, \nu, m)$ pdf with $\nu = n - 1$.

[24] See, for example, T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*. New York: Wiley, 1958, p. 161. The fourth-order moments required in the derivation are given on p. 39 of Anderson's book.

zero vector mean and common $m \times m$ PDS covariance matrix Σ. Then, partitioning $\tilde{V} = (\tilde{V}_1 \vdots \tilde{V}_2)$, we have

$$\tilde{A} = \tilde{V}'\tilde{V} = \begin{pmatrix} \tilde{V}_1'\tilde{V}_1 & \tilde{V}_1'\tilde{V}_2 \\ \tilde{V}_2'\tilde{V}_1 & \tilde{V}_2'\tilde{V}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix},$$

where $\tilde{V}_1'\tilde{V}_1$ and $\tilde{V}_2'\tilde{V}_2$ are of size $m_1 \times m_1$ and $m_2 \times m_2$, respectively, where $m_1 + m_2 = m$. Then

$$\begin{aligned} \tilde{A}_{11\cdot 2} &= \tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21} \\ &= \tilde{V}_1'\tilde{V}_1 - \tilde{V}_1'\tilde{V}_2(\tilde{V}_2'\tilde{V}_2)^{-1}\tilde{V}_2'\tilde{V}_1 \\ &= \tilde{V}_1'[I_\nu - \tilde{V}_2(\tilde{V}_2'\tilde{V}_2)^{-1}\tilde{V}_2']\tilde{V}_1. \end{aligned} \tag{B.70}$$

Now, given $\tilde{V}_2$, if we let $\tilde{V}_1 = L\tilde{Z}_1$, where L is a $\nu \times \nu$ orthogonal matrix such that[25]

$$L'[I_\nu - \tilde{V}_2(\tilde{V}_2'\tilde{V}_2)^{-1}\tilde{V}_2']L = \begin{bmatrix} I_{\nu - m_2} & 0 \\ 0 & 0 \end{bmatrix},$$

we have from the third line of (B.70)

$$\begin{aligned} \tilde{A}_{11\cdot 2} &= \tilde{Z}_1'L'[I_\nu - \tilde{V}_2(\tilde{V}_2'\tilde{V}_2)^{-1}\tilde{V}_2']L\tilde{Z}_1 \\ &= \tilde{Z}_{1a}'\tilde{Z}_{1a}, \end{aligned}$$

where $\tilde{Z}_{1a}$ is a $\nu - m_2 \times m_1$ submatrix of $\tilde{Z}_1$; that is, $\tilde{Z}_1' = (\tilde{Z}_{1a}'\tilde{Z}_{1b}')$. Therefore $\tilde{A}_{11\cdot 2}$ can be expressed as $\tilde{Z}_{1a}'\tilde{Z}_{1a}$, where the rows of $\tilde{Z}_{1a}$ are independently and normally distributed, each with zero mean vector and covariance matrix $\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$.[26] Thus, using Property 1, $\tilde{A}_{11\cdot 2}$ has a $W(\Sigma_{11\cdot 2}, \nu - m_2, m_1)$ pdf, where $m - m_1 = m_2$.

Property 5 is derived from the pdf for $\tilde{A}_{11} = \{\tilde{a}_{ij}\}$ $i, j = 1, 2$; that is, $p(a_{11}, a_{22}, a_{12}|\Sigma_{11}, \nu)$, which is $W(\Sigma_{11}, 2, \nu)$, by expressing it in terms of a_{11}, a_{22}, and $r = a_{12}/(a_{11}a_{22})^{1/2}$ and integrating out a_{11} and a_{22}.[27]

[25] For given $\tilde{V}_2$, $I_\nu - \tilde{V}_2(\tilde{V}_2'\tilde{V}_2)^{-1}\tilde{V}_2'$ is an idempotent matrix of rank $\nu - m_2$. Thus it has $\nu - m_2$ roots equal to 1 and m_2 equal to 0.

[26] From $\tilde{V}_1 = L\tilde{Z}_1$, $E\tilde{Z}_1 = 0$. Further, given $\tilde{V}_2$, $E\tilde{Z}_1'L'L\tilde{Z}_1 = E\tilde{Z}_1'\tilde{Z}_1 = E\tilde{V}_1'\tilde{V}_1 = \Sigma_{11\cdot 2} \otimes I_\nu$, where $\Sigma_{11\cdot 2}$ is the covariance matrix of the m_1 elements of any row of $\tilde{V}_1$, given the m_2 elements in the corresponding row of $\tilde{V}_2$.

[27] See, for example, T. W. Anderson, *op. cit.*, pp. 68–69, for details. Since the function $I_\nu(\rho r)$ in (5) can be expressed as a hypergeometric function, the pdf for r_{12} can be expressed as a rapidly converging series; that is

$$h(r_{12}|\rho_{12}, \nu) = \frac{(\nu - 1)\Gamma(\nu)}{\sqrt{2\pi}\,\Gamma(\nu + \frac{1}{2})} \frac{(1 - r_{12}^2)^{(\nu - 3)/2}(1 - \rho^2)^{\nu/2}}{(1 - r_{12}\rho)^{(2\nu - 1)/2}} S_\nu(\rho r_{12}),$$

where $S_\nu(\rho r_{12})$ denotes the hypergeometric function, $F(\frac{1}{2}, \frac{1}{2}; \nu + \frac{1}{2}; (1 + r_{12}\rho)/2)$, which

B.4 THE INVERTED WISHART (IW) Pdf

The $m(m + 1)/2$ distinct elements of an $m \times m$ PDS random matrix $\tilde{G}$ follow the IW distribution if, and only if, they have the following pdf[28]:

$$p(G|H, \nu, m) = k \frac{|H|^{\nu/2}}{|G|^{(\nu + m + 1)/2}} \exp\{-\tfrac{1}{2} \operatorname{tr} G^{-1}H\}, \qquad |G| > 0, \tag{B.71}$$

where $k^{-1} = 2^{\nu m/2}\pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma[(\nu + 1 - i)/2]$, $\nu \geq m$, and H is an $m \times m$ PDS matrix. The IW pdf is defined as (B.71) in the region $|G| > 0$ and zero elsewhere. If the elements of $\tilde{G}$ have the pdf (B.71), we say that they have an $IW(H, \nu, m)$ pdf. Some properties of (B.71) follow:

1. The joint pdf for the $m(m + 1)/2$ distinct elements of $G^{-1} = A$ is a $W(H^{-1}, \nu, m)$ pdf.
2. Let G and H be partitioned correspondingly as

$$G = \begin{array}{c} \\ m_1 \\ m_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} m_1 & m_2 \end{array} \\ \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \end{array}, \qquad H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where $m_1 + m_2 = m$. Then the joint pdf for the $m_1(m_1 + 1)/2$ distinct elements of G_{11} is an $IW(H_{11}, \nu - m_2, m_1)$ pdf.

3. When in (2) G_{11} is a scalar, say g_{11}, the pdf for g_{11} is[29]

$$p(g_{11}|h_{11}, \nu', m_1) = \frac{k_1}{g_{11}^{(\nu' + 2)/2}} e^{-h_{11}/2g_{11}}, \qquad 0 < g_{11}, \tag{B.72}$$

with $k_1 = (h_{11}/2)^{\nu'/2}/\Gamma(\nu'/2)$, with $\nu' = \nu - m + 1$.

4. By virtue of (B.72) the moments of the diagonal elements of $\tilde{G}$ can be obtained from those associated with a univariate inverted gamma pdf.

Property 1 is a fundamental one connecting the W and IW pdf's. To establish it we require the Jacobian of the transformation of the $m(m + 1)/2$ distinct elements of G to the $m(m + 1)/2$ distinct elements of $A = G^{-1}$. The

is given by

$$\begin{aligned} S_\nu(\rho r_{12}) &= \frac{\Gamma(\nu + \frac{1}{2})}{[\Gamma(\frac{1}{2})]^2} \sum_{i=0}^{\infty} \frac{[\Gamma(\frac{1}{2} + i)]^2}{i!\,\Gamma(\nu + \frac{1}{2} + i)} \left(\frac{1 + r_{12}\rho}{2}\right)^i \\ &= 1 + \frac{1}{\nu + \frac{1}{2}} \frac{1 + r_{12}\rho}{8} + \frac{1^2 \cdot 3^2}{2!\,(\nu + \frac{1}{2})(\nu + \frac{3}{2})} \left(\frac{1 + r_{12}\rho}{8}\right)^2 + \cdots. \end{aligned}$$

See, for example, H. Jeffreys, *Theory of Probability* (3rd ed.), Oxford: Clarendon, 1961, p. 175, for the details of expressing $I_\nu(\rho r)$ in terms of a hypergeometric function.

[28] In the analysis of the multivariate regression model with a diffuse prior pdf in Chapter 8 we found that the posterior pdf for the disturbance covariance matrix is given by $p(\Sigma|y) \propto |\Sigma|^{-\nu'/2} \exp(-\frac{1}{2} \operatorname{tr} \Sigma^{-1}S)$. This is in the form of (B.71), as can be seen by letting $G = \Sigma$, $H = S$, and $\nu' = \nu + m + 1$. Thus $p(\Sigma|y)$ is an IW pdf.

[29] Equation B.72 can be obtained from (A.37*b*) by letting $g_{11} = \sigma^2$ and $h_{11} = \nu s^2$. Of course, the positive square root of g_{11} will have a pdf precisely in the form of (A.37*b*).

Jacobian of the transformation is $|A|^{-(m+1)}$[30] and thus (B.71) can be expressed in terms of $A = G^{-1}$:

$$
\text{(B.73)} \quad \begin{aligned} p(A|H, \nu, m) &= k \frac{|H|^{\nu/2}|A|^{-(m+1)}}{|A|^{-(\nu+m+1)/2}} \exp\{-\tfrac{1}{2}\operatorname{tr} AH\}, \qquad |A| > 0. \\ &= k|H|^{\nu/2}|A|^{(\nu-m-1)/2} \exp\{-\tfrac{1}{2}\operatorname{tr} AH\}, \qquad |A| > 0, \end{aligned}
$$

with k given in connection with (B.71). If, in (B.73), we define $\Sigma^{-1} = H$, it is seen that (B.73) is in precisely the form of the $W(\Sigma, \nu, m)$ pdf shown in (B.55).

Property 2 can be established by noting that $G_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = A_{11\cdot2}^{-1}$. As shown in the preceding section, if A has a W pdf, $A_{11\cdot2}$ also has a W pdf. Then Property 1 of the IW pdf can be employed to obtain the pdf for $G_{11} = A_{11\cdot2}^{-1}$ from that for $A_{11\cdot2}$. Given this result, the special case in which G_{11} is a scalar, g_{11}, leads to the result in (B.72) which is Property 3. Property 4 is an immediate consequence of Property 3.

B.5 THE GENERALIZED STUDENT t (GSt) Pdf[31]

The pq elements of a $p \times q$ random matrix, $\tilde{T} = \{\tilde{t}_{ij}\}$, have the GSt distribution if, and only if, they have the following pdf[32]:

$$
\text{(B.74)} \qquad p(T|P, Q, n) = k \frac{|Q|^{(n-p)/2}|P|^{q/2}}{|Q + T'PT|^{n/2}}, \qquad -\infty < t_{ij} < \infty,
$$

where $k^{-1} = \pi^{pq/2} \prod_{i=1}^{q} \Gamma[(n - p - i + 1)/2]/\prod_{i=1}^{q} \Gamma[(n - i + 1)/2]$, $n > p + q - 1$, and P and Q are PDS matrices of sizes $p \times p$ and $q \times q$, respectively. For convenience we denote the pdf in (B.74) as $T(P, Q, 0, n)$, where

[30] To show that the Jacobian is $|A|^{-(m+1)}$ write $AG = I$. Then $(\partial A/\partial\theta)G + A(\partial G/\partial\theta) = 0$ or $\partial G/\partial\theta = -G(\partial A/\partial\theta)G$. If $\theta = a_{ij}$, we have $\partial g_{\alpha\beta}/\partial a_{ij} = -g_{\alpha i}g_{\beta j}$ for α, β, i, and $j = 1, 2, \ldots, m$, with $\beta \le \alpha$ and $j \le i$, since G and A are symmetric matrices and the transformation from the elements of G to those of A involves just $m(m + 1)/2$ distinct elements of G. On forming the Jacobian matrix and taking its determinant, we have $|G|^{m+1} = |A|^{-(m+1)}$. See T. W. Anderson, *op. cit.*, p. 162, for a derivation of this Jacobian which relies on properties of the W pdf. Also on pp. 348 to 349 Anderson provides the Jacobian of the transformation $G \to A = G^{-1}$ for the general case in which G and A are not symmetric, a result not applicable to the present case in which G and A are symmetric.

[31] Some call this pdf the matrix t pdf. See J. M. Dickey, "Matricvariate Generalizations of the Multivariate t Distribution and the Inverted Multivariate t Distribution," *Ann. Math. Statistics*, 38, 511–518 (1967); S. Geisser, "Bayesian Estimation in Multivariate Regression," *cit. supra*; G. C. Tiao and A. Zellner, "On the Bayesian Estimation of Multivariate Regression," *cit. supra*; and the references cited in these works for further analysis of this pdf.

[32] The pdf in (B.74) was encountered in Chapter 8 in connection with the analysis of the multivariate regression model. With a diffuse prior pdf the posterior pdf for the regression coefficients was found to be $p(B|Y) \propto |S + (B - \hat{B})'X'X(B - \hat{B})|^{-n/2}$. If we let $S = Q$, $P = X'X$, and $T = B - \hat{B}$, $p(B|Y)$ is exactly in the form of (B.74).

0 appears to denote that the mean of (B.74), by symmetry, is a zero matrix. Some properties of (B.74) follow:

1. The pdf in (B.74) can be obtained as the marginal distribution of $p(G, T) = p_1(G)\,p_2(T|G)$, where $p_1(G)$ denotes an inverted Wishart pdf and $p_2(T|G)$ denotes a multivariate normal pdf.

Let

$$
Q = \begin{array}{c} \\ q_1 \\ q_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \end{array} \quad \text{and} \quad P = \begin{array}{c} \\ p_1 \\ p_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \end{array},
$$

with $q_1 + q_2 = q$ and $p_1 + p_2 = p$ and

$$
Q_{11\cdot2} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}, \qquad P_{22\cdot1} = P_{22} - P_{21}P_{11}^{-1}P_{12}.
$$

These quantities appear in the properties of (B.74).[33]

2. If $T = (T_1, T_2)$, the conditional pdf for T_1, given T_2, is a GSt pdf with parameters $(P^{-1} + T_2Q_{22}^{-1}T_2')^{-1}$, $Q_{11\cdot2}$, $T_2Q_{22}^{-1}Q_{21}$, n. The mean is $T_2Q_{22}^{-1}Q_{21}$.
3. If $T' = (X_1, X_2)$, the conditional pdf for X_1, given X_2, is a GSt pdf with parameters P_{11}, $Q + X_2'P_{22\cdot1}X_2$, $P_{11}^{-1}P_{12}X_2$, n. The mean is $P_{11}^{-1}P_{12}X_2$.
4. If $T = (T_1, T_2)$, with $T_1 p \times q_1$ and $T_2 p \times q_2$, the marginal pdf for T_2 is GSt with parameters P, Q_{22}, 0, $n - q_1$.
5. If $T' = (X_1, X_2)$, with $X_1 p_1 \times q$ and $X_2 p_2 \times q$, the marginal pdf for X_2 is GSt, with parameters $P_{22\cdot1}$, Q, 0, $m - p_1$.
6. If in (2) T_1 is a $p \times 1$ vector, the conditional pdf for T_1, given T_2, is in the multivariate Student t form. Similarly, if in (4) T_2 is a $p \times 1$ vector, it has a marginal pdf in the multivariate Student t form.
7. With $T = (\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_q)$,

$$
\text{(B.75)} \qquad p(T) = p(\mathbf{t}_1)\,p(\mathbf{t}_2|\mathbf{t}_1)\,p(\mathbf{t}_3|\mathbf{t}_1, \mathbf{t}_2) \cdots p(\mathbf{t}_q|\mathbf{t}_1, \ldots, \mathbf{t}_{q-1})
$$

and each of the pdf's on the rhs of (B.75) is in the form of a multivariate Student t pdf.

To establish Property 1 write the IW pdf for the $q(q + 1)/2$ distinct elements of G as

$$
\text{(B.76)} \qquad \mathrm{IW}(G|Q, \nu, p) = k_1 \frac{|Q|^{\nu/2}}{|G|^{(\nu+q+1)/2}} \exp\{-\tfrac{1}{2}\operatorname{tr} G^{-1}Q\},
$$

[33] Some of the following properties established in Chapter 8 draw on the results in papers by J. M. Dickey, S. Geisser, and G. C. Tiao and A. Zellner, cited in the preceding footnote.

where Q is a $q \times q$ PDS matrix and k_1 is the normalizing constant, and the multivariate normal pdf for the elements of T, a $p \times q$ matrix, given G, as[34]

$$\text{(B.77)} \qquad \mathrm{MN}(T|G, P) = k_2|P|^{q/2}|G|^{-p/2} \exp\{-\tfrac{1}{2}\operatorname{tr} T'PTG^{-1}\},$$

where P is a $p \times p$ PDS matrix and k_2 is a normalizing constant. Then the joint pdf for the distinct elements of G and those of T, $p(G, T)$, is the product of (B.76) and (B.77); that is,

$$\text{(B.78)} \qquad p(G, T) = k_1 k_2 \frac{|Q|^{\nu/2}|P|^{q/2}}{|G|^{(\nu+p+q+1)/2}} \exp\{-\tfrac{1}{2}\operatorname{tr}(Q + T'PT)G^{-1}\}.$$

Now note from the properties of the IW pdf that

$$\text{(B.79)} \qquad \int_{|G|>0} \frac{|Q + T'PT|^{(\nu+p)/2}}{|G|^{(\nu+p+q+1)/2}} \exp\{-\tfrac{1}{2}\operatorname{tr}(Q + T'PT)G^{-1}\}\, dG = \frac{1}{k_3},$$

where k_3 is the normalizing constant of the IW pdf, $p(G|Q + T'PT, \nu + p, q)$. Using (B.79), we find that the integration of (B.78) with respect to the elements of G over the region $|G| > 0$ yields

$$\text{(B.80)} \qquad p(T) = \frac{k_1 k_2}{k_3} \frac{|Q|^{\nu/2}|P|^{q/2}}{|Q + T'PT|^{(\nu+p)/2}}, \qquad -\infty < t_{ij} < \infty.$$

If we let $\nu = n - p$, we see that (B.80) is in precisely the form of (B.74), a GSt pdf.[35]

Property 2 is most easily established if we note that the GSt pdf can be written in the following alternative form[36]:

$$\text{(B.81)} \qquad p(T|P, Q, n) = k \frac{|P|^{-[(n-q)/2]}|Q|^{-p/2}}{|P^{-1} + TQ^{-1}T'|^{n/2}}, \qquad -\infty < t_{ij} < \infty.$$

Then

$$\begin{aligned}
P^{-1} + TQ^{-1}T' &= P^{-1} + (T_1 T_2)\begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}\begin{pmatrix} T_1' \\ T_2' \end{pmatrix} \\
&= P^{-1} + T_1 Q^{11} T_1' + T_2 Q^{21} T_1' + T_1 Q^{12} T_2' + T_2 Q^{22} T_2' \\
&= P^{-1} + T_2[Q^{22} - Q^{21}(Q^{11})^{-1}Q^{12}]T_2' \\
&\quad + [T_1 + T_2 Q^{21}(Q^{11})^{-1}]Q^{11}[T_1 + T_2 Q^{21}(Q^{11})^{-1}]' \\
&= P^{-1} + T_2 Q_{22}^{-1} T_2' + (T_1 - T_2 Q_{22}^{-1} Q_{21}) Q_{11\cdot 2}^{-1} \\
&\quad \times (T_1 - T_2 Q_{22}^{-1} Q_{21})',
\end{aligned}$$

[34] Note that with $T = (\mathbf{t}_1 \cdots \mathbf{t}_q)$, we can write the MN pdf for the elements of T, given G as $\mathrm{MN}(T|G, P) = k_2|G^{-1} \otimes P|^{1/2} \exp[-\frac{1}{2}(\mathbf{t}_1', \mathbf{t}_2', \ldots, \mathbf{t}_q')G^{-1} \otimes P(\mathbf{t}_1', \mathbf{t}_2', \cdots, \mathbf{t}_q')']$ and that $|G^{-1} \otimes P|^{1/2} = |P|^{q/2}|G|^{-p/2}$; see, for example, T. W. Anderson, *op. cit.*, p. 348.
[35] We could transform (B.76) to a Wishart pdf for $A = G^{-1}$ and obtain (B.80) as the marginal pdf of a Wishart pdf times a conditional MN pdf for T given A.
[36] See J. M. Dickey, *op. cit.*, p. 512.

where Q^{ij}, $i, j = 1,2$, are submatrices of Q^{-1} and $Q^{21}(Q^{11})^{-1} = -Q_{22}^{-1}Q_{21}$. On substituting this result in (B.81), we have

$$\begin{aligned}
&p(T_1, T_2|Q, P, n) = k \\
&\text{(B.82)} \qquad \times \frac{|P|^{-(n-q)/2}|Q|^{-p/2}}{|P^{-1} + T_2 Q_{22}^{-1} T_2' + (T_1 - T_2 Q_{22}^{-1} Q_{21}) Q_{11\cdot 2}^{-1} \times (T_1 - T_2 Q_{22}^{-1} Q_{21})'|^{n/2}}
\end{aligned}$$

From (B.82) it is apparent that the conditional pdf for T_1, given T_2, is in the GSt form (B.81) with parameters $(P^{-1} + T_2 Q_{22}^{-1} T_2')^{-1}$, $Q_{11\cdot 2}$, and $T_2 Q_{22}^{-1} Q_{21}$, n, where the conditional mean is given by $T_2 Q_{22}^{-1} Q_{21}$. Further, the marginal pdf for T_2 is obtained from (B.82) by integrating with respect to the elements of T_1. Note that (B.82) can be expressed as

$$\begin{aligned}
&p(T_1, T_2|Q, p, n) \propto [|P^{-1} + T_2 Q_{22}^{-1} T_2'|^{(n-q_1)/2}]^{-1} \\
&\text{(B.83)} \qquad \times \frac{|P^{-1} + T_2 Q_{22}^{-1} T_2'|^{(n-q_1)/2}|Q_{11\cdot 2}|^{-p/2}}{|P^{-1} + T_2 Q_{22}^{-1} T_2' + (T_1 - T_2 Q_{22}^{-1} Q_{21}) Q_{11\cdot 2}^{-1} \times (T_1 - T_2 Q_{22}^{-1} Q_{21})'|^{n/2}}
\end{aligned}$$

and that on integrating with respect to the elements of T_1 the second factor integrates to a numerical factor. Thus the marginal pdf for the elements of T_2 is proportional to the first factor on the rhs of (B.83) which is in the GSt form with parameters P, Q_{22}, 0, $n - q_1$. This is Property 4.

Properties 3 and 5 can be established in the same way as Properties 2 and 4. However, in proof of the former two properties, it is convenient to use the form of the GSt pdf in (B.74). Property 6 follows from Property 4. Also, it and Property 7 have been demonstrated in the text of Chapter 8.