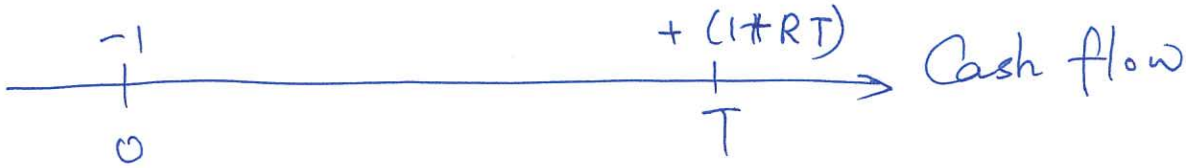
§ Basic terminologies in term structure

Def A simple rate R over $[0,T]$ means



ex Today, you have $X_0 = 1\$$

You're allowed to invest repeatly at

a simple rate R over $[\frac{k}{n}, \frac{k+1}{n}]$

for $\forall k \in \mathbb{N}$.

whats the maximum value you can make

over $[0,T]$?

Sol $\left(1+\frac{R}{n}\right)^{nT}$.

Def n-compound rate R over $[0,T]$ means

ex $\lim_{n\to\infty}\left(1+\frac{R}{n}\right)^{nT} = \cancel{1+RT}\ e^{RT}$

Def

A continuously compounded rate R over $[0,T]$.

means

Def

Zero-coupon bond is a security with

$$\text{payoff}\Big|_T = 1$$

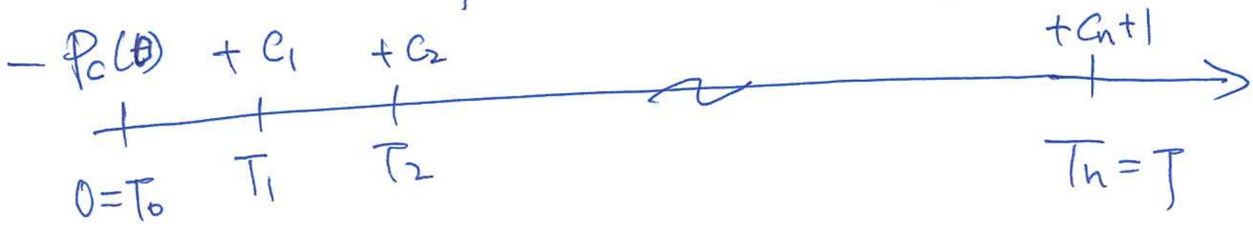
If we use $P(t,T)$ to denote its price,

then

Def A coupon bond with coupon structure

$$C = \{(C_i, T_i) : i = 1, 2, \cdots n\}$$

is a security with cash flow



where $P_c(t)$ is Bond price at t

ex If $T_{i-1} < t \le T_i$, then prove

$$P_c(t) = P(t, T) + \sum_{k=i}^{n} C_k P(t, T_k)$$

Rk coupon Bond price can be obtained from a series of zero bond prices.

Def

spot rate $R(t,T)$ is the yield to maturity of $P(t,T)$, i.e.

$$P(t,T) = e^{-R(t,T)\cdot(T-t)}$$

or
$$R(t,T) = -\frac{\ln P(t,T)}{T-t}$$

Def

For $t < T < S$,

forward rate at t on $[T, S]$ is

$$F(t,T,S) = \frac{1}{S-T} \ln \frac{P(t,T)}{P(t,S)}$$

or
$$\frac{P(t,T)}{P(t,S)} = e^{F(t,T,S)\cdot(S-T)}$$

ex $F(t, t, s) = R(t, s)$

Def The instantaneous forward rate at time t and T, $(T > t)$ is

$$f(t, T) = \lim_{s \to T} F(t, T, s).$$

ex show that

$$P(t, T) = \exp\left\{ - \int_t^T f(t, u)\, du \right\}.$$

Pf
$$f(t, T) = \lim_{s \to T} F(t, T, s)$$
$$= \lim_{s \to T} \frac{\ln \frac{P(t,T)}{P(t,s)}}{s - T}$$
$$= - \boxed{\lim_{s \to T} \frac{-\ln P(t,T) + \ln P(t,s)}{s - T}}$$
$$= - \frac{\partial}{\partial T} \ln P(t, T) = - \partial_T \ln P(t,T)$$

$$\ln P(t, T) - \underline{\ln P(t, t)} = \int_t^T - f(t, u)\, du$$
$$\ln P(t, T) = - \int_t^T f(t, u)\, du$$
$$P(t, T) = e^{- \int_t^T f(t, u)\, du}.$$

$\boxed{S}$

ex $f(t,T) = -\frac{\partial}{\partial T} \ln P(t,T)$

$$= \frac{-\partial_T P(t,T)}{P(t,T)}$$

$\left(\ln f(x)\right)' = \frac{f'(x)}{f(x)}$

$(\ln x)' = \frac{1}{x}$

Def <u>short rate $r(t)$</u> is

$$r(t) = \lim_{T \to t} R(t,T)$$

<u>Facts</u>

① $r(t) = f(t,t) = \lim_{T \to t} R(t,T)$

② $P(t,T) = \exp\left\{-\int_t^T f(t,u)\,du\right\}$

$$= \mathbb{E}^{Q}\left[\exp\left\{-\int_t^T r(s)\,ds\right\} \Big| \mathcal{F}_t\right]$$

$$= \exp\left\{-R(t,T)(T-t)\right\}$$

ex

Given continuously compounded forward rates as

T	1	2	3	4	5
$F(0,T-1,T)$	0.042	0.05	0.055	0.056	0.053

Find the following values.

T	1	2	3	4	5
$P(0,T)$	$e^{-0.042}$				
$R(0,T)$	0.042				

$F(0,0,1) = 0.042 = R(0,1)$ $\quad 1 \longrightarrow 1 \cdot e^{0.042 \cdot 1}$

$P(0,1) = e^{-0.042}$ $\quad \square \longrightarrow \underline{1}$

$$F(0,1,2) = \frac{\ln \frac{P(0,1)}{P(0,2)}}{2-1}$$

$P(0,2) = P(0,1)\, e^{-F(0,1,2)} \longrightarrow ?$

$$\cancel{P(0,3)}\ F(0,2,3) = \frac{\ln \frac{P(0,2)}{P(0,3)}}{3-2}$$

$P(0,3) = P(0,2)\, e^{-F(0,2,3)} \longrightarrow ?$

⋮

Boot Strapping

§ SDE models

~~§~~ General SDE

$$\begin{cases} dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \\ X_0 \end{cases}$$

or

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s.$$

§1. GBM

Def $\quad dX_t = X_t(\mu\,dt + \sigma\,dW_t)$

Explicit soln

$$X_t = X_0 \cdot \exp\{\hat{\mu} t + \sigma W_t\}$$

where $\hat{\mu} = \mu - \frac{1}{2}\sigma^2$

Rewrite it by

$$X_{t+\delta} = X_t \cdot \exp\{\hat{\mu}\delta + \sigma W_{t,t+\delta}\}$$

where $\quad W_{t,t+\delta} = W_{t+\delta} - W_t.$

~~Def~~

<u>Def</u> An approx. $\hat{X}$ on $(0 = \tau_0 < \tau_1 < \cdots < \tau_N = T)$ is called <u>exact</u> if

$$\text{Law}(\hat{X}_{\tau_0}, \hat{X}_{\tau_1}, \cdots \hat{X}_{\tau_N}) = \text{Law}(X_{\tau_0}, X_{\tau_1}, \cdots, X_{\tau_N})$$

i.e. for any $\underset{\text{bounded}}{\underline{\text{b.d}}}$ continuous function

$$f : \mathbb{R}^{N+1} \to \mathbb{R}$$

it shall satisfy

$$\hat{\mathbb{E}} f(\hat{X}_{\tau_0}, \cdots \hat{X}_{\tau_N}) = \mathbb{E} f(X_{\tau_0}, \cdots X_{\tau_N})$$

<u>Exact simulation $\hat{X}$ on $[0, T]$</u>

① Let $\tau_i = \frac{T}{N} i$, $i = 0, 1, \cdots N$

② $\hat{X}_0 = X_0$

③ Repeat for $n = 0, 1, 2 \cdots N-1$ by

$$\hat{X}_{n+1} = \hat{X}_n \exp\{\hat{\mu} \delta + \sigma \sqrt{\delta} Z_n\}$$

<u>ex</u> prove that Euler approx is not exact approximation.

§2. Gaussian short rate models

§2.1 Vasicek model

__def__ (vasicek)

$$d r_t = \alpha (b - r(t))\, dt + \sigma dW_t$$

(Ho-Lee)

$$d r_t = g(t)\, dt + \sigma dW_t$$

(Hull-White)

$$d r_t = [g(t) + h(t) r(t)]\, dt + \sigma(t) dW_t$$

__Rk__

All above models are __Gaussian processes.__

A process $r(t)$ is Gaussian if it has $(r(t_1), r(t_2) \dots r(t_n))$ being normal distribution for any $\{t_1, t_2 \cdots t_n\}$

__explicit soln__ (HW)

$$r(t) = e^{H_t} r_0 + \int_0^t e^{H_{s,t}} g(s)\, ds + \int_0^t e^{H_{s,t}} \sigma(s)\, dW_s$$

where $H_{s,t} = \int_s^t h(u)\, du$, $H_t = H_{0,t}$

exact simulation of HW

$$r_{t+\delta} = e^{H_{t,t+\delta}} r_t + \int_t^{t+\delta} e^{H_{s,t+\delta}} g(s)\,ds + \underbrace{\int_t^{t+\delta} e^{H_{s,t+\delta}} \sigma(s)\,dW_s}_{I}$$

<u>Fact</u> $\int_t^s \sigma(r)\,dW_r \sim N\left(0, \int_t^s \sigma^2(r)\,dr\right)$

if σ is a deterministic function

So. $I \sim N(0, \hat{\sigma}^2) = \hat{\sigma} \cdot Z$

where $\hat{\sigma}^2 = \int_t^{t+\delta} e^{2H_{s,t+\delta}} \sigma^2(s)\,ds.$

<u>ex</u> ① write $\hat{\sigma}^2$ for vasicek model,

② Run exact simulation.

<u>Fact</u> For vasicek

$$\ln P(t,T) = A(t,T) - B(t,T)\,\underline{r(t)}$$

where

$$B(t,T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha}$$

$$A(t,T) = \left(B(t,T) - (T-t)\right)\left(b - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2}{4\alpha} B^2(t,T)$$

<u>Rk</u> B/c $\ln P(t,T)$ is affine in $r(t)$,

we call this as affine rate model.

ex Use Monte-Carlo on

$$P(t,T) = \mathbb{E}\left[\exp\left\{-\int_t^T r(s)ds\right\} \Big| \mathcal{F}_t\right]$$

① run Euler

② run Milstein

③ run exact simulation

~~to~~ compute $P(t,T)$, compare with exact formula for vasicek model. find weak convergence rate.

§2.2. Multifactor models

$$dX_t = c(b - X_t)dt + D\,dW_t$$

where $b, X \in \mathbb{R}^d$, $W \in \mathbb{R}^d$

$c, D \in \mathbb{R}^{d \times d}$

Explicit soln is available.