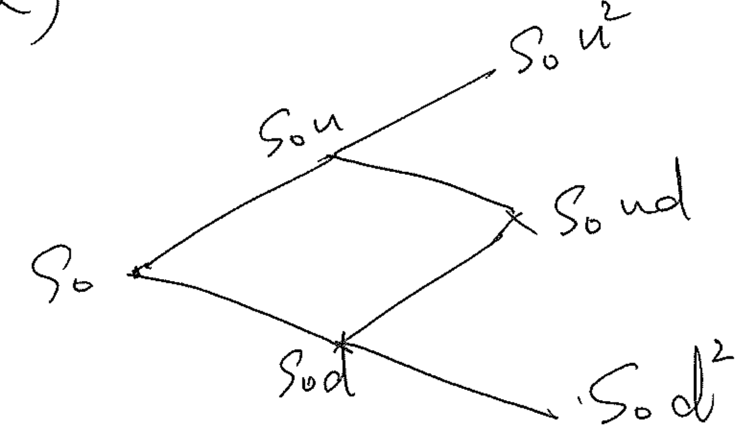
§ BinTree + CRR.

Recall

☆ BinTree (S_0, N, T, u, d, r)

EuCall
EuPut
AmCall
AmPut

$\left.\begin{matrix}\text{EuCall}\\ \text{EuPut}\\ \text{AmCall}\\ \text{AmPut}\end{matrix}\right\} (T, K)$



☆ CRR(S_0, N, T, σ, r) $\to$ $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$

prop.

~~CRR~~

$$\text{CRR}(S_0, N, T, \sigma, r) + \text{EuCall}(T, K) \xrightarrow{N\to\infty}$$

$$\text{BSM}(S_0, T, \sigma, r) + \text{EuCall}(T, K).$$

Homework (etra point)
prove the above prop!

Q. For $Bin(S_0, N, T, u, d, r)$

① Find $\mathbb{Q}$, where $\mathbb{Q}$ is EMM.

② Find $\mathbb{E}^{\mathbb{Q}}[S_{t+\Delta t} \mid S_t] = ?$

$Var^{\mathbb{Q}}[S_{t+\Delta t} \mid S_t] = ?$

Sol) ①

$$q = \mathbb{Q}(S_{t+\Delta t} = u S_t \mid S_t) = \frac{e^{r\Delta t} - d}{u - d}$$

$$1 - q = \mathbb{Q}(S_{t+\Delta t} = d S_t \mid S_t).$$

②

$$\mathbb{E}^{\mathbb{Q}}[S_{t+\Delta t} \mid S_t]$$

$$= S_t u \cdot q + S_t \cdot d \cdot (1 - q)$$

$$= S_t \left(\frac{u(e^{r\Delta t} - d)}{u - d} + \frac{d(u - e^{r\Delta t})}{u - d} \right)$$

$$= S_t e^{r\Delta t}$$

$$Var^{\mathbb{Q}}[S_{t+\Delta t} \mid S_t]$$

$$\cancel{= \mathbb{E}^{\mathbb{Q}}[S_{t+\Delta t} \quad \mathbb{E}^{\mathbb{Q}}[S_t}$$

$$= \mathbb{E}^{\mathbb{Q}}[(S_{t+\Delta t} - S_t e^{r\Delta t})^2 \mid S_t]$$

$$= S_t^2 (u - e^{r\Delta t})(e^{r\Delta t} - d).$$

Q Consider $CRR(S_0, N, T, \sigma, r)$

① Find $\mathbb{Q}$

② Find $\mathbb{E}^{\mathbb{Q}}[\ln S_{t+\Delta t} \mid S_t]$

③ Find $Var^{\mathbb{Q}}[\ln S_{t+\Delta t} \mid S_t]$

Soln

$$\begin{array}{ccc} & & S_t \cdot e^{\sigma\sqrt{\Delta t}} \\ & \nearrow & \\ S_t & & \\ & \searrow & \\ & & S_t e^{-\sigma\sqrt{\Delta t}} \end{array}$$

①
$$q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$$1-q = \frac{e^{\sigma\sqrt{\Delta t}} - e^{r\Delta t}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

② $\mathbb{E}^{\mathbb{Q}}[\ln S_{t+\Delta t} \mid S_t]$

$$= \left(\ln S_t e^{\sigma\sqrt{\Delta t}}\right) \cdot q + \left(\ln S_t e^{-\sigma\sqrt{\Delta t}}\right)(1-q)$$

$$= \ln S_t + (2q-1)\sigma\sqrt{\Delta t}$$

$$= \ln S_t + \frac{2e^{r\Delta t} - e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \cdot (\sigma\sqrt{\Delta t})$$

$$\mathrm{Var}^{Q}\left[\ln S_{t+\Delta t} \,\middle|\, S_t\right]$$

$$\begin{tikzcd}[column sep=large]
 & \ln S_t + \sigma\sqrt{\Delta t} \\
\ln S_t \arrow[ur, "q"] \arrow[dr, "1-q"'] & \\
 & \ln S_t - \sigma\sqrt{\Delta t}
\end{tikzcd}$$

$$= \left(\sigma\sqrt{\Delta t} - (2q-1)\sigma\sqrt{\Delta t}\right)^2 \cdot q + \left(-\sigma\sqrt{\Delta t} - (2q-1)\sigma\sqrt{\Delta t}\right)^2 \cdot (1-q)$$

$$= \left(\sigma\sqrt{\Delta t}\right)^2 q \cdot 4(1-q)^2 + \left(\sigma\sqrt{\Delta t}\right)^2 \cdot (1-q) \cdot 4q^2$$

$$= \boxed{4\sigma^2 \Delta t \cdot q(1-q)}$$

ex Prove that, $\ln \frac{S_T^N}{S_0}$ converges to $N\left((r-\frac{1}{2}\sigma^2)T, \sigma^2 T\right)$ $\cancel{N(rT, \sigma^2 T)}$ ~~Normal di~~ in distribution as $N \to \infty$

where S_T^N is the stock price at T from $CRR(S_0, N, T, \sigma, r)$.

pf

$$\ln S^N_{t+\Delta t} = \ln S^N_t + \sigma\sqrt{\Delta t}\, B_t$$

$$\Downarrow$$

$$\ln \frac{S_T^N}{S_0} = \frac{\sigma\sqrt{T}}{\sqrt{n}}\left(B_0 + \cdots + B_{(n-1)\Delta t}\right)$$

where

$$B_{i\Delta t} = \begin{cases} 1, & q \\ -1, & 1-q. \end{cases}$$

ex Given $S_t = S_0 \cdot \exp((\cancel{r} - \frac{1}{2}\sigma^2)t + \sigma W_t)$

$$A_1 = \frac{S_{0.5} + S_1}{2}$$

$$G_1 = \sqrt{S_{0.5} \cdot S_1}$$

$$f(x) = (x - K)^+$$

Q1 Is it true?

$$\mathbb{E}[f(G_1)] \le \underline{\mathbb{E}[f(A_1)] \le \mathbb{E}[f(S_1)]}$$

Sol) ① $\mathbb{E}[f(G_1)] \le \mathbb{E}[f(A_1)]$

b/c ⓐ Geom. Aver. $\le$ Arem Aver.

ⓑ f is increasing

②

By Jensen's thm.

$$\mathbb{E}[f(A_1)] \le \frac{1}{2}\mathbb{E}[f(S_{0.5})] + \frac{1}{2}\mathbb{E}[f(S_1)] \overset{?}{\le} \mathbb{E}[f(S_1)]$$

Thus It's enough to show

$$\mathbb{E}[f(S_{0.5})] \underset{?}{\le} \mathbb{E}[f(S_1)]$$

S_t is mtgl. b/c $r = 0$

$$dS_t = S_t \cdot \sigma dW_t$$

$$\mathbb{E}[S_1 \| S_{\frac{1}{2}}] = S_{\frac{1}{2}}$$

$$f(\mathbb{E}[S_1 \| S_{\frac{1}{2}}]) = f(S_{\frac{1}{2}})$$

Tower $\mathbb{E}[f(S_1) | S_{\frac{1}{2}}] \ge \cancel{\mathbb{E}[f(S_1}$

$$\mathbb{E}[f(S_1)] = \mathbb{E}[\mathbb{E}[f(S_1) | S_{\frac{1}{2}}]] \ge \mathbb{E}[f(S_{\frac{1}{2}})]$$

$\square$