Nov - 1

§1 Numerical method for SDE

Consider 1-Dim SDE, time-homogeneous.

$$\begin{cases} dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t \\ X(t_0) = x_0 \end{cases}$$

Integral form is

$$X(t) = x_0 + \int_{t_0}^{t} \mu(X_s)\,ds + \sigma(X_s)\,dW_s$$

In small interval $[t, t+\delta]$, this can be rewritten as.

$$X_{t+\delta} - X_t = \int_t^{t+\delta} \mu(X_s)\,ds + \sigma(X_s)\,dW_s .$$

Let's denote

$$X_{t,s} = X_s - X_t$$

Thus

$$X_{t,\,t+\delta} = \int_t^{t+\delta} \mu(X_s)\,ds + \sigma(X_s)\,dW_s .$$

Ito says

$$\mu(X_s) = \mu(X_t) + \int_t^s \left(\mu'(X_r)\mu(X_r) + \frac{1}{2}\mu''(X_r)\sigma^2(X_r)\right)dr + \int_t^s \mu'(X_r)\sigma(X_r)\,dW_r$$

$$\sigma(X_s) = \sigma(X_t) + \int_t^s \left(\sigma'(X_r)\mu(X_r) + \frac{1}{2}\sigma''(X_r)\sigma^2(X_r)\right)dr + \int_t^s \sigma'(X_r)\sigma(X_r)\,dW_r.$$

§2, Euler method.

Assume $\delta > 0$ is a small constant, then

$$\mu(X_s) \approx \mu(X_t) \quad \text{and} \quad \sigma(X_s) \approx \sigma(X_t)$$

Thus

$$X_{t,t+\delta} \approx \int_t^{t+\delta} \mu(X_t)\,ds + \int_t^{t+\delta} \sigma(X_t)\,dW_s = \mu(X_t)\cdot\delta + \sigma(X_t)\cdot W_{t,t+\delta}$$

and we know

$$W_{t,t+\delta} \sim N(0,\delta) = \sqrt{\delta}\,Z.$$

(*) Euler $\hat{X}$ on $[0, T]$ with step size $\delta = \frac{T}{N}$

① partition $[0, T]$ equally, denoted by

$$\{0 = \tau_0 < \tau_1 < \cdots < \tau_N = T\}$$

where $\tau_i = \frac{T}{N} \cdot i$

② $\hat{X}_0 = X_0$

③ $\hat{X}_{n+1} = \hat{X}_n + \mu(\hat{X}_n)\delta + \sigma(\hat{X}_n)\sqrt{\delta}\, Z_n$

where $\{Z_n \sim N(0, 1) \quad \text{iid}\}$

④ Interpolation by piecewise constants.

$$I_t^\delta = I_t^\delta(\hat{X}) = \hat{X}_n \quad \text{if} \quad \tau_n \le t < \tau_{n+1}$$

$$= \sum_{n=0}^{N} \hat{X}_n \cdot I_{[\tau_n, \tau_{n+1})}(t)$$

§3. strong convergence of Euler.

Strong Error

Given an approximated process Y^δ to a process X,

we define

$$\varepsilon(\delta) = \left(\sup_{0\le s\le T} \mathbb{E} \left| Y^\delta(s) - X(s) \right|^2 \right)^{\frac{1}{2}}$$

① If $\lim_{\delta\to 0} \varepsilon(\delta) = 0$, then we say

$$Y^\delta \to X \quad \text{(strongly)}$$

② If $\varepsilon(\delta) \le c\,\delta^\gamma$ for some constants $c, \gamma > 0$, then we say γ is the convergence order.

Thm $I^\delta(\hat{X}) \to X$ with $\gamma = \frac{1}{2}$, i.e.

$$\sup_{0\le s\le T} \mathbb{E} \left| I^\delta_s(\hat{X}) - X_s \right|^2 \le k\delta$$

Pf See [KP], section 9.6.

§4. Weak convergence of Euler

Weak error

Given Y^δ approxiamting to a process X, we define

$$\varepsilon^g(\delta) = \left| \mathbb{E}[g(X_T)] - \mathbb{E}[g(Y_T^\delta)] \right|$$

① we say

$$Y_T^\delta \Rightarrow X_T \text{ (weakly)}$$

if $\lim_{\delta \to 0} \varepsilon^g(\delta) = 0 \quad \forall g \in C_b$

② we say

$$Y_T^\delta \Rightarrow X_T \quad \text{with order } \gamma,$$

if $\exists K_g > 0$ s.t.

$$\varepsilon^g(\delta) \le K_g \delta^\gamma$$

for any $g \in C_b$

Thm $I^\delta(\hat{x}) \Rightarrow X$ with $\gamma = 1$

Pf see Section 9.7. of [KP]

§5. Milstein scheme

Assume $\delta > 0$ is a small constant.

From Ito on μ and σ in §1. we have

$$\mu(X_s) \approx \mu(X_t)$$

$$\sigma(X_s) \approx \sigma(X_t) + \int_t^s \underbrace{\sigma'(X_r)\sigma(X_r)}\, dW_r$$

$$\approx \sigma(X_t) + \sigma'(X_t)\sigma(X_t) \int_t^s dW_r$$

$$\sigma(X_s) \approx \sigma(X_t) + \sigma'(X_t)\sigma(X_t) W_{t,s}$$

~~ex prove $\int_t^s W_r \, dr = \frac{1}{2}(W_s^2 - W_t^2) -$~~

ex prove that

$$\int_t^s W_r \, dW_r = \frac{1}{2}(W_s^2 - W_t^2) - \frac{1}{2}(s-t)$$

Continued from $X_{t,t+\delta}$ of §1.

$$X_{t,t+\delta} \approx \int_t^{t+\delta} \mu(X_t)\, ds + \int_t^{t+\delta} \sigma(X_t) + \sigma'(X_t)\sigma(X_t) W_{t,s} \, dW_s$$

$$= \mu(X_t)\cdot\delta + \int_t^{t+\delta} \underline{\sigma(X_t)}\, dW_s + \int_t^{t+\delta} \underline{\sigma'(X_t)\sigma(X_t)}\, W_{t,s}\, dW_s$$

$$= \underbrace{\mu(X_t)\delta + \sigma(X_t)\, W_{t,t+\delta}}_{\text{Euler}} + \sigma'(X_t)\sigma(X_t) \int_t^{t+\delta} W_{t,s}\, dW_s$$

Note that

$$\int_t^{t+\delta} W_{t,s}\, dW_s$$

$$= \int_t^{t+\delta} (W_s - W_t)\, dW_s$$

$$= \int_t^{t+\delta} W_s\, dW_s - \int_t^{t+\delta} W_t\, dW_s$$

$$= \frac{1}{2}\left(W_{t+\delta}^2 - W_t^2\right) - \frac{1}{2}\delta - W_t \cdot W_{t,t+\delta}$$

$$= \frac{1}{2}\left((W_t + W_{t,t+\delta})^2 - W_t^2\right) - \frac{1}{2}\delta - W_t \cdot W_{t,t+\delta}$$

$$= \frac{1}{2}\left(2 W_t \cdot W_{t,t+\delta} + W_{t,t+\delta}^2\right) - \frac{1}{2}\delta - W_{t,t+\delta}$$

$$= \frac{1}{2} W_{t,t+\delta}^2 - \frac{1}{2}\delta$$

Thus,

$$X_{t,t+\delta} \approx \text{Euler} + \frac{1}{2}\sigma'(X_t)\sigma(X_t)\left(W_{t,t+\delta}^2 - \delta\right).$$

⊛ Milstein $\hat{X}$ on $[0,T]$ with $\delta = \frac{T}{N}$

① partition $[0,T]$ equally.

$$\{0 = \tau_0 < \tau_1 < \cdots < \tau_N = T\}$$

where $\tau_n = \frac{T}{N} \cdot n$.

② $\hat{X}_0 = X_0$

③ $\hat{X}_{n+1} = \hat{X}_n +$

$$\mu(\hat{X}_n)\delta + \sigma(\hat{X}_n)\sqrt{\delta}\, Z_n \quad \leftarrow \text{Euler change}$$

$$+ \frac{1}{2}\sigma'(\hat{X}_n)\sigma(\hat{X}_n)(\delta Z_n^2 - \delta)$$

where $\{Z_n \sim N(0,1), \text{ iid}\}$

④ Interpolation

$$I_t^\delta = \hat{X}_n \quad \text{if} \quad \tau_n \le t < \tau_{n+1}$$

Thm Milstein has convergence

strong order $= 1$

weak order $= 1$.

Pf See section 10.3 [KP 99]