Weak convergence of CRR to BSM

<u>Goal</u> we will show

$$CRR(n, T, \sigma, r) \Longrightarrow BSM(T, \sigma, r)$$

<u>$CRR(n, T, \sigma, r)$</u>

① $S_0^n = 1$, $\ln S_{i\Delta}^n = m_{i\Delta}^n$, $\Delta = \frac{T}{n}$.

② $m_{i\Delta}^n = \sigma\sqrt{\Delta} \sum_{j=1}^{i} B_j^n$, where

$\{B_j^n : j = 1, \cdots n\}$ iid with

$$\mathbb{P}(B_j^n = 1) = 1 - \mathbb{P}(B_j^n = -1) = q^n$$

③ $$q^n = \frac{e^{r\Delta} - e^{-\sigma\sqrt{\Delta}}}{e^{\sigma\sqrt{\Delta}} - e^{-\sigma\sqrt{\Delta}}} = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta} + O(\Delta)$$

where $\mu = r - \frac{1}{2}\sigma^2$

<u>$BSM(T, \sigma, r)$</u>

$$S_t = S_0 \exp\{\underbrace{(r - \tfrac{1}{2}\sigma^2)}_{\mu} t + \sigma W_t\}, \quad S_0 = 1, \quad t \leq T$$

<u>Pf</u>. Note that

$$\mathbb{E}[B_j^n] = 2q^n - 1$$

thus, we set

$$\overline{m}_{i\Delta}^n = \sqrt{\Delta} \sum_{j=1}^{i} \overline{B}_j^n, \quad \text{where } \overline{B}_j^n = B_j^n - 2q^n + 1$$

Then

$$\{\overline{m}_{i\Delta}^n : i = 0, \cdots n\} \text{ is mtgl. with } Var(\overline{B}_j^n) = 1$$

By Donsker's invariance principle,

$$I(\overline{m}^n_{i\Delta} : i=0,\cdots n) \Rightarrow W$$

where $I(\cdot)$ means linear interpolation.

Note that

$$\{I(m^n_{i\Delta})(t)\} = \{\sigma I(\overline{m}^n_{i\Delta})(t) + \mu t + O(\tfrac{1}{\sqrt{n}})\}$$

$$\Rightarrow \{\mu t + \sigma W_t : t \le T\}$$

$$I(S^n_{i\Delta}) = I(\exp(m^n_{i\Delta})) \Rightarrow \{\exp(\mu t + \sigma W_t) : t \le T\}$$

$$= S.$$