Oct 25.

[KP 9.1] Euler approximation

[KP]: Kloeden & Platen 1999, "Numerical Solutions of SDE"

SDE

$$dX_t = a(t, X_t)\,dt + b(t, X_t)\,dW_t$$

ex GBM is when

$$a(t,x) = rx$$
$$b(t,x) = \sigma x$$

Suppose $F: \mathbb{R}^d \mapsto \mathbb{R}$ is payoff function, then we are interested in

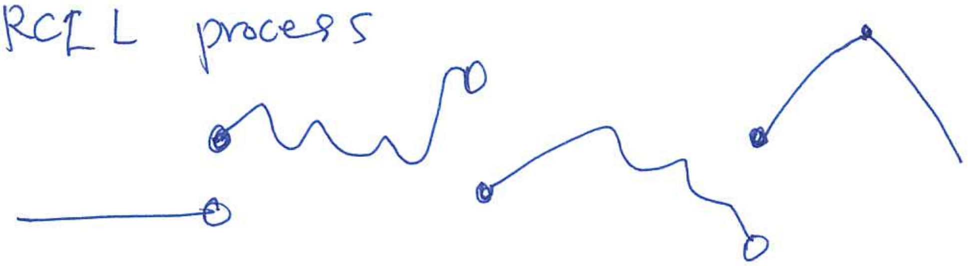
$$\text{price} = \mathbb{E}[F(X_T)] \leftarrow \text{path indep.}$$

Suppose $F: D[0,T] \mapsto \mathbb{R}$ is payoff, then we are interested in

$$\text{price} = \mathbb{E}[F(X)] \leftarrow \text{path dep.}$$

where $D[0,T]$ is set of RCLL (right cont. left limit exits) ~~processed~~ processes.

ex of RCLL process



ex of $F: D[0,T] \Rightarrow \mathbb{R}$ is

$$F(\omega) = \max_{0 \le t \le T} (W_t - K)^+$$

Euler-Maruyma (EM)

Discrete process $\{(Y_n, \tau_n) : n \in \mathbb{N}\}$ is EM, if

① $0 = \tau_0 < \tau_1 < \tau_2 < \cdots \cdots < \tau_N = T$

② $Y_0 = X_0$

③ $Y_{n+1} = Y_n + a(\tau_n, Y_n)\, \Delta_n + b(\tau_n, Y_n)\, \Delta_n W$

where

$\Delta_n = \tau_{n+1} - \tau_n$

$\Delta_n W = W_{n+1} - W_n \sim \sqrt{\Delta_n}\, Z_n$ ← iid std normal

Goal $Y_n \approx X(\tau_n) \quad \forall n$

(in some sense?)

ex Demonstrate $\left(Y_N \approx X_{t_N} = X_T\right)$ by following example.

For $GBM(r = 0.0475,\ \sigma = 0.2,\ X_0 = 100)$,

Evaluate $Put(T = 5,\ K = 110)$ by Euler.

step1 Generate 1000 Euler trajectories.

$$\{Y^{(1)}, Y^{(2)}, \ldots, Y^{(1000)}\}$$

step2

$$Put \approx e^{-rT} \boxed{\hat{\mathbb{E}}[(Y_T - K)^-]}$$

$$\cancel{\approx e^{-rT} \frac{Y_N^{(1)} + Y_N^{(2)} + \cdots + Y_N^{(1000)}}{1000}}$$

$$\approx e^{-rT} \frac{\sum_{i=1}^{1000} (Y_N^{(i)} - K)^-}{1000}$$

True

$$Put = e^{-rT} \mathbb{E}[(Y_T - K)^-]$$

$=$ computed By BSM formula.

Prelim

① max time step
$$\delta = \max_{1 \le n \le N} \Delta_n$$

② If $\delta = \frac{T}{N} = \Delta_n \quad \forall n$, then EM is <u>equidistant</u>.

③ $\mathbb{E}[\Delta_n W] = 0 \qquad \mathbb{E}[(\Delta_n W)^2] = \Delta_n$

i.e. $\Delta_n W \sim \mathcal{N}(0, \Delta_n) = \sqrt{\Delta_n}\,\mathcal{N}(0,1)$.

<u>ex</u> ① plot EM of BM
② plot EM of GBM

§ [KP 9.6]. strong convergence.

Given an approximation $\{Y^\delta : \delta \geq 0\}$ to a process X, we say

<u>Absolute Error</u>
$$\varepsilon(\delta) = \sup_{0 \le s \le T} \mathbb{E}\left| Y^\delta(s) - X(s) \right|^2$$

<u>Def</u>
① If $\lim_{\delta \to 0} \varepsilon(\delta) = 0$, then $Y^\delta \to X$ (strong)
② If $\varepsilon(\delta) \le c\,\delta^{2\gamma}$, then covergence order is γ.

Consider 1-d/homogeneous EM of equidistance process with

$$\begin{cases} ① \ \tau_n = \frac{T}{N}(n-1) \\ ② \ Y_0 = X_0 \\ ③ \ Y_{n+1} = Y_n + a(Y_n)\delta + b(Y_n)\Delta_n W \end{cases}$$

for approximation of X of

$$(*_1) \begin{cases} dX_t = a(X_t)\,dt + \sigma(X_t)\,dW_t \\ X_0. \end{cases}$$

[Assumption] $a(\cdot)$, $b(\cdot)$ are Lipschitz cont.

<u>Fact</u> Under [A], $(*_1)$ has unique soln.

<u>prop</u> $\exists K$ constant. s.t.

① $\mathbb{E}\left[\sup_{0 \le s \le t} |X_s|^2\right] \le K|X_0|^2 (1 + t e^{Kt})$

② $\mathbb{E}\left[\sup_{0 \le s \le t} |X_t - X_0|^2\right] \le K e^{Kt} |X_0|^2 \cdot t$.

<u>Ref</u> [KP1999], [FS06]

Fleming

Soner

~~Let~~

Interpolation

Let $n_t = \max\{n \in \mathbb{N} : \tau_n \le t\}$

Interpolation of $\left(Y_n^\delta : n = 0, 1 \cdots N\right)$ is

$$\cancel{\bar{Y}^\delta(s) = X_0 + \int_0^{\tau_{n_s}} a(\bar{Y}_s^\delta)\,ds + b}$$

$$\bar{Y}^\delta(s) = Y_{n_s}^\delta$$

~~Then~~

Thm 9.6.2 With [A].

$\bar{Y}^\delta \to X$ with $\gamma = \frac{1}{2}$, which means (convergence order)

$$\sup_{0 \le s \le T} \mathbb{E}\left|\bar{Y}_s^\delta - X_s\right|^2 \le K\delta$$