Nov 22

§2. IS

Goal Compute

$$\alpha = \mathbb{E}[h(X)] = \int h(x) f(x) dx, \quad X \sim f.$$

ex Let $h(x) = 100\, I_{(0, \frac{1}{100})}(x)$

$$f(x) = I_{(0,1)}(x)$$

Then $\alpha = 1$.

OMC $(n=10)$:

$$\hat{\alpha}_{10} = \frac{1}{10} \sum_{i=1}^{10} h(X_i) \quad X_i \sim f.$$

Rk $\text{Prob}(\hat{\alpha}_{10} = 0) = \left(\frac{99}{100}\right)^{10} = \text{Big}$

IS $(n=10)$:

Let $g(x) = b\, I_{(0, \frac{1}{100})} + \frac{100-b}{99} I_{(\frac{1}{100}, 1)}$

$$\hat{\alpha}_{10} = \frac{1}{10} \sum_{i=1}^{10} h(X_i) \frac{f(X_i)}{g(X_i)} \quad X_i \sim g$$

likelyhood

Radon-Nikodym

$$\alpha = \mathbb{E}^f[h(X)] = \mathbb{E}^g\left[h(X)\frac{f(X)}{g(X)}\right]$$

where $\frac{h(x)f(x)}{g(x)} = \begin{cases} 100/b & \text{on } (0, \frac{1}{100}) \\ 0 & \text{on } (\frac{1}{100}, 1) \end{cases}$

Q Fix $b = 99$, what $P_r(\hat{\alpha}_{10} = 0) = ?$

A $P_r^g\left(X_i \in (\frac{1}{100}, 1)\right) = \frac{1}{100}$

$$P_r(\hat{\alpha}_{10} = 0) = \left(\frac{1}{100}\right)^{10} = \text{tiny}$$

Def (IS)

$$\mathbb{E}^f[h(X)] = \mathbb{E}^g\left[h(X)\frac{f(X)}{g(X)}\right]$$

~~MC~~ (OMC) $\mathbb{E}^f[h^2(X)] = \int_0^1 h^2(x) f(x)\, dx$

$$= \int_0^{\frac{1}{100}} 100^2 \cdot 1\, dx = 100$$

(IS) $\mathbb{E}^g\left[h^2(X)\frac{f^2(X)}{g^2(X)}\right] = \int_0^{1/100} 100^2 \cdot \frac{1^2}{99^2} \cdot 99\, dx$

$$= \frac{100}{99} \approx 1$$

ex Asset

$$S_t = S_0 \exp(\hat{\mu} t + \sigma W_t) \quad ; \mathbb{Q}$$

Option payoff,

$$h(S_T) = I(S_T < S_0 e^{-b}).$$

for some constant b.

Fwd Price,

$$v = \mathbb{E}^{\mathbb{Q}} h(S_T).$$

[para]

$r = 0.03, \quad \sigma = 0.2,$

$\hat{\mu} = r - \frac{1}{2}\sigma^2 = 0.01$

$T = 1, \quad b = 0.39.$

(1)
$$
\begin{aligned}
v &= \mathbb{E}^{\mathbb{Q}}\left[I\left(\ln \frac{S_T}{S_0} < -b\right)\right] \\
&= \mathbb{E}^{\mathbb{Q}}[I(\hat{\mu} T + \sigma \underbrace{W_T}_{\sqrt{T} Z} < -b)] \\
&= \mathbb{E}^{\mathbb{Q}}\left[I\left(Z < \frac{-b - \hat{\mu} T}{\sigma \sqrt{T}}\right)\right] \\
&= \mathbb{E}^{\mathbb{Q}}[I(Z < -2)] = \mathbb{P}(Z < -2) \\
&= \Phi(-2) = 0.02275
\end{aligned}
$$

<u>OMC</u>

$$v = \mathbb{E}^{Q}\left[I(Z<-2)\right]$$

$$\hat{v}_{10} = \frac{1}{10}\sum_{i=1}^{10} I(Z_i<-2) \quad \text{where } Z_i \sim N(0,1)$$

<u>IS</u>

$$v = \int_{-\infty}^{-2} \varphi(x)\,dx \quad \varphi \to \text{p.d.f. of } N(0,1)$$

$$= \int_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= \int_{-\infty}^{-2} \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\varphi_b(x)}\,\varphi_b(x)\,dx$$

$$\varphi_b(x) \sim -b+Z \sim N(-b,1)$$

$$\varphi_b(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+b)^2}{2}}$$

$$= \int_{-\infty}^{-2} \frac{e^{-\frac{x^2}{2}}}{e^{-\frac{(x+b)^2}{2}}}\,\varphi_b(x)\,dx$$

$$= \int_{-\infty}^{-2} e^{-\frac{1}{2}(x^2-x^2-b^2-2bx)}\,\varphi_b(x)\,dx$$

$$= \int_{-\infty}^{-2} e^{\frac{1}{2}b^2+bx}\,\varphi_b(x)\,dx$$

$$= \mathbb{E}^{\varphi_b}\left[e^{\frac{1}{2}b^2+bX}\cdot I(X<-2)\right]$$

$$= e^{\frac{1}{2}b^2}\,\mathbb{E}^{\varphi_b}\left[e^{bX} I(X<-2)\right]$$

$$v = e^{\frac{1}{2}b^2} \mathbb{E}^{\varphi_b}[e^{bX} I(X<-2)]$$

Algo $(n=10)$

~~① Sampling from~~

① Take n samples from $\varphi_b \sim b+Z$

$\{X_1, X_2 \cdots X_{10}\}$

② Compute

$$\hat{v}_{10} = e^{\frac{1}{2}b^2} \frac{1}{10} \cdot \sum_{i=1}^{10} e^{10X_i} I(X_i<-2)$$

ex

① find $\mathbb{E}(\hat{v}_{10})^2$ for ome

② find $\mathbb{E}(\hat{v}_{10})^2$ for IS$(b=3)$

③ (open) what b makes IS most efficient

§3. IS on Discretely Monitored Barrier Option

Ref [Glas 03] ex 4.6.4.

Asset: $GBM(S_0, r, \sigma)$.

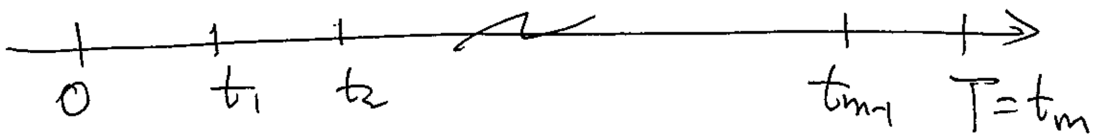
$$S_t = S_0 e^{L_t}$$

where log price L_t is

$$L_t = (r - \tfrac{1}{2}\sigma^2)t + \sigma W_t.$$

for constants b, c.

Payoff: $I(S_T > S_0 e^c)\, I(\min_{1 \le k \le m} S(t_k) < S_0 e^{-b})$



F Price

$$\mathbb{E}[\text{payoff}].$$

para: $r = 5\%$, $\sigma = 0.15$, $S_0 = 95$

$T = 0.25$ $m = 50$,

$S_0 e^{-b} = 85$,

$S_0 e^{c} = 96$

Idea

$$\text{Payoff} = I(S_T > S_0 e^c)\, I\left(\min_{1\le k\le m} S(t_k) < S_0 e^{-b}\right)$$

$$= I(L_T > c)\, I\left(\min_{1\le k\le m} L(t_k) < -b\right)$$

$$= I(L_T > c) \cdot I(\tau \le \cancel{m}T)$$

where
$$\tau = \inf\{ i : L(t_i) < -b \}$$

~~Set~~ $\cancel{\hat{r} = r - \frac{1}{2}\sigma^2}$

$\cancel{L_t = \hat{r} t + \sigma W_t}$

$\cancel{\int_0^t \mu_s \, ds}$

~~Set~~

$$L_t = \left(r - \tfrac{1}{2}\sigma^2\right) t + \sigma W_t$$

$$= \int_0^t \left(\hat{r}_s - \tfrac{1}{2}\sigma^2\right) ds + \sigma \hat{W}_t$$

$$\sigma W_t + r t = \int_0^t \hat{r}_s \, ds + \sigma \hat{W}_t$$

$$W_t = \int_0^t \underbrace{\frac{\hat{r}_s - r}{\sigma}}_{-\theta_s} ds + \hat{W}_t$$

By Girsanov Thm.

$$\frac{d\mathbb{P}}{d\hat{\mathbb{P}}} = \mathcal{E}\left(\int_0^{\cdot} \theta_s \, d\hat{W}_s\right)_T$$

$$= \exp\left(\int_0^T \theta_s \, d\hat{W}_s - \frac{1}{2}\int_0^T \theta_s^2 \, ds\right) = \gamma(\hat{W})$$

$$F\,\text{Price} = \mathbb{E}[\text{payoff}]$$

$$= \hat{\mathbb{E}}[\text{payoff} \cdot \gamma(\hat{W})]$$

$$= \hat{\mathbb{E}}[I(L_T > c) I(\tau \leq T) \gamma(\hat{W})]$$

where $L_t = \int_0^t (\hat{r}_s - \frac{1}{2}\sigma^2) ds + \sigma \hat{W}_t$

(*2) $\tau = \min\{i : L_t < -b\}$

Let's take

$$\hat{r}_s = \begin{cases} -r_1 & s < h \\ r_2 & s \geq h \end{cases}$$

then

(*1) $$L_t = \begin{cases} (-r_1 - \frac{1}{2}\sigma^2)t + \sigma d\hat{W}_t & \text{if } t < h \\ L_h + (r_2 - \frac{1}{2}\sigma^2)(t-h) + \sigma \hat{W}_{h,t} & \text{if } t \geq h \end{cases}$$

$(\hat{W}_t - \hat{W}_h)$

$$\gamma(\hat{W}) = \exp\left(\int_0^T \theta_s \, d\hat{W}_s - \frac{1}{2}\int_0^T \theta_s^2 \, ds\right)$$

Set

$$\theta_s = \begin{cases} \theta_1 = \dfrac{-r_1 - r}{\sigma} & \text{for } s < h \\ \theta_2 = \dfrac{r_2 - r}{\sigma} & \text{for } s > h \end{cases}$$

$$(*3) \quad \gamma = \exp\Big(\theta_1 \hat{W}_h + \theta_2 \underbrace{(\hat{W}_T - \hat{W}_h)}_{\hat{W}_{h,T}} - \frac{1}{2}\theta_1^2 h - \frac{1}{2}\theta_2^2 (T-h)\Big).$$

<u>Algo</u> ($m = 50$, $n = 1000$, $k = 100$)

m: Time steps for each path

n: No of simulations for each MC

k: No. MC Implement.

Repeat k times MC computation:

~~Repeat n time~~

Generate n ~~times~~ paths:

1. Generate $\hat{W}$ on $(t_1 \leq t_2 \cdots < t_m = T)$
2. Generate L on $(t_1 < t_2 \cdots < t_m = T)$. use (*1)
3. Find τ, use (*2)
4. Find Payoff
5. Find γ, use (*3)

Average $(\text{payoff} \cdot \gamma)_{(1,2\ldots n)} \rightarrow$ price appr

Find MSE, Mean of k prices