§ Improving MC

§1. OMC

⊛ Given radom estimator $\hat{\alpha}$ of α (unknown)

$$\text{Bias} = \mathbb{E}[\hat{\alpha}] - \alpha$$

$$\begin{aligned} MSE &= \mathbb{E}|\hat{\alpha} - \alpha|^2 \\ &= |\text{Bias}|^2 + Var(\hat{\alpha}). \end{aligned}$$

⊛ The smaller MSE, the better MC.

<u>ex</u> To find

$$\alpha = \mathbb{E}[X], \qquad \underbrace{\mathbb{P}X^{-1}}_{\text{Law}(X)} \sim \mu$$

<u>OMC</u>

$$\hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad X_i \sim \mu, \text{ iid}$$

① $\hat{\alpha}_n$ is <u>unbiased</u>

② $MSE = Var(\hat{\alpha}_n) = E|\hat{\alpha}_n|^2 - (E\hat{\alpha}_n)^2$

$$= E|\hat{\alpha}_n|^2 - \alpha^2$$

" Smaller 2^{nd} Moment of $\hat{\alpha}_n$, Better MC "

③ $E(\hat{\alpha}_n)^2$

$$= E\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right)^2$$

$$= \frac{1}{n^2} E\left(\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n} \mathbb{E} X^2 + \frac{n-1}{n} \alpha^2$$

"Smaller EX^2, the better MC"

§2 Control variates

Assumption

① want $E[X] = \alpha$?

② $E[Y] = 0$, $E[XY]$ is known

$E[Y^2] > 0$, is known

③ $\mathbb{P}X^{-1}$, $\mathbb{P}Y^{-1}$ sampling available

Think

$$\alpha = E[X]$$

$$= \mathbb{E}[X - bY] \quad \text{for any constant } b.$$

$$\approx \frac{1}{n} \sum_{i=1}^{n} (X_i - bY_i) \triangleq \hat{\alpha}_n(b)$$

ex $\forall b \in \mathbb{R}$, $\hat{\alpha}_n(b)$ is unbiased.

we want choose smart b^* s.t.

$$\min_b \mathbb{E}[(X - bY)^2] = \mathbb{E}[(X - b^*Y)^2].$$

i.e.

$$b^* = \underset{b}{\operatorname{argmin}}\ \mathbb{E}[(X - bY)^2].$$

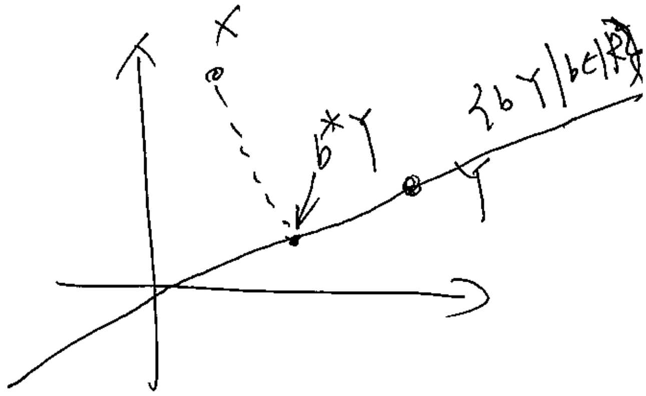
Let $L^2(\mathbb{P})$ be the space, defined by

$$L^2(\mathbb{P}) = \{X \mid \mathbb{E}[X^2] < \infty\}$$

Then $L^2(\mathbb{P})$ is a Hilbert space with

inner product $\langle X, Y\rangle \triangleq \mathbb{E}[XY]$

norm $\|X\|^2 \triangleq \langle X, X\rangle = \mathbb{E}[X^2]$



We shall find b^* s.t.

$$(X - b^*Y) \perp \underbrace{\{bY \mid b \in \mathbb{R}^2\}}_{\text{line}}$$

$$(X - b^*Y) \perp Y,$$

$$E(X - b^*Y)\cdot Y = 0$$

$$E(XY) - b^* E Y^2 = 0$$

$$b^* = \frac{E(XY)}{EY^2} = \frac{\langle X, Y\rangle}{\|Y\|}$$

Algo

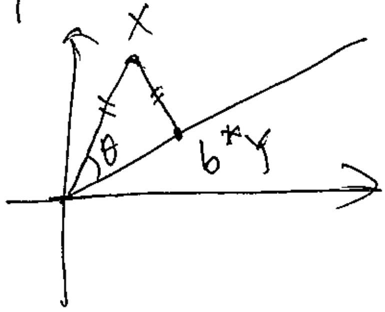
① Compute

$$b^* = \frac{E(XY)}{EY^2}$$

② Compute

$$\hat{\alpha}_n(b^*) = \frac{1}{n}\sum_{i=1}^{n}(X_i - b^* Y_i).$$

Q How much does $\hat{\alpha}_n(b^*)$ improve from $\hat{\alpha}_n(0)$?



A

$$\frac{\|X - b^*Y\|^2}{\|X\|^2} = \sin^2\theta$$

$$= 1 - \cos^2\theta$$

$$= 1 - \frac{\langle X, b^*Y\rangle^2}{\|X\|^2 \cdot \|b^*Y\|^2}$$

$$= 1 - \left(\frac{\langle X, Y\rangle}{\|X\| \cdot \|Y\|}\right)^2$$

$$= 1 - \rho_{XY}^2$$

correlation

RK

~~"Control variate" improves "QMC"~~

~~for~~

Rk

The Bigger $|\rho_{XY}|$, The better improvement by "control variates" from "OMC"

* Extension for $E[Y] = a \neq 0$

we shall use $Y - a \longrightarrow Y$.

b/c $E[Y-a] = 0$

So,

$$b^* = \frac{E[X(Y-a)]}{E[(Y-a)^2]} = \frac{E[X(Y-a)]}{E[(Y-a)^2]}$$

$$= \frac{E(XY) - E[X]\cdot E[Y]}{Var(Y)}$$

$$\boxed{b^* = \frac{Cov(X,Y)}{Var(Y)}}$$

Algo (If $Cov(X,Y)$, $Var(Y)$ is known)

Then

① Compute b^*

② Sampling (X_i, Y_i),

③ Compute

$$\hat{\alpha}_n^* = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - b^*(Y_i - a)\right)$$

Extension

If $\operatorname{cov}(X, Y)$, $\operatorname{var}(Y)$ is not known, then

use
$$\operatorname{Cov}(X,Y) \approx \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})$$
$$\operatorname{Var}(Y) \approx \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

where
$\bar{X} = \sum X_i / n$ $\quad$ $\bar{Y} = \sum Y_i / n$

PrJ Compute AAC with GAC as its control variates.

(Para):

$S_0 = 100$ $\quad$ $K = 110$. $\quad$ $r = 0.0475$

$\sigma = 0.2$, $\quad$ $T = 1$, $\quad$ $n = 5$

§2, IS (Importance Sampling)

Goal

$\alpha = E[h(X)]$ where $X \sim f$

$= \int h(x) f(x) dx$.

ex Let

$h(x) = 100 \cdot I_{(0, \frac{1}{100})}$

$f(x) = I_{(0,1)}$

$\alpha = E[h(X)] = 1$

⊛ ① MC $(n=10)$

$\hat{\alpha}_{10} = \frac{1}{10} \sum_{i=1}^{10} h(X_i)$, $X_i \sim f$

Q. $\text{Prob}(\hat{\alpha}_{10} = 0) = ?$

A. $\left(\frac{99}{100}\right)^{10} = \text{Big}$

⊛ IS$(n=10)$:

Let $g(x) = b\, I_{(0,\frac{1}{100})}$ $\cancel{+ \left(1 - \frac{b}{100}\right) I_{(\frac{1}{100},1)}}$

$+ \frac{100-b}{99} I_{(\frac{1}{100},1)}$

ex verify g is pdf.

$\hat{\alpha}_{10} =$