EOM of 2-DOF Pendularm

Shichao Zeng
shichaoz@umich.edu

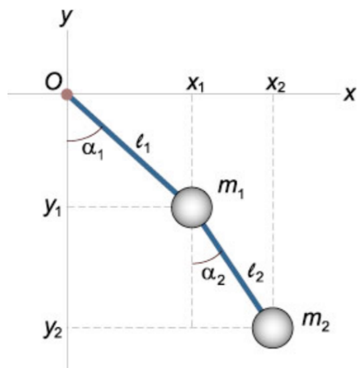

Fig. 1. Model of 2-DOF Pendularm

A pendularm with 2-DOF can be modeled as shown in figure 1. The positions of mass points are shown below based on the model in figure 1:

$$x_1 = l_1 \sin\alpha_1 \tag{1}$$

$$x_2 = l_1 \sin\alpha_1 + l_2 \sin\alpha_2 \tag{2}$$

$$y_1 = -l_1 \cos\alpha_1 \tag{3}$$

$$y_2 = -l_1 \cos\alpha_1 - l_2 \cos\alpha_2 \tag{4}$$

According to Lagrangian,

$$T_1 = \frac{m_1 v_1^2}{2} = \frac{m_1(\dot{x}_1^2 + \dot{y}_1^2)}{2} \tag{5}$$

$$T_2 = \frac{m_2 v_2^2}{2} = \frac{m_2(\dot{x}_2^2 + \dot{y}_2^2)}{2} \tag{6}$$

$$V_1 = m_1 g y_1 \tag{7}$$

$$V_2 = m_2 g y_2 \tag{8}$$

Also, we have

$$L = T - V = T_1 + T_2 - (V_1 + V_2) \tag{9}$$

Thus,

$$L = \frac{m_1(\dot{x}_1^2 + \dot{y}_1^2)}{2} + \frac{m_2(\dot{x}_2^2 + \dot{y}_2^2)}{2} - m_1 g y_1 - m_2 g y_2 \tag{10}$$

calculate the derivatives,

$$\dot{x}_1 = l_1 \cos\alpha_1 \cdot \dot{\alpha}_1 \tag{11}$$

$$\dot{x}_2 = l_1 \cos\alpha_1 \cdot \dot{\alpha}_1 + l_2 \cos\alpha_2 \cdot \dot{\alpha}_2 \tag{12}$$

$$\dot{y}_1 = -l_1 \sin\alpha_1 \cdot \dot{\alpha}_1 \tag{13}$$

$$\dot{y}_2 = -l_1 \sin\alpha_1 \cdot \dot{\alpha}_1 - l_2 \sin\alpha_2 \cdot \dot{\alpha}_2 \tag{14}$$

From Equation 5 to 14,

$$\begin{aligned} L = {} & \frac{(m_1+m_2)}{2} l_1^2 \dot{\alpha}_1^2 + \frac{m_2}{2} l_2^2 \dot{\alpha}_2^2 + m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \cos(\alpha 1 - \alpha 2) \\ & + (m_1+m_2) g l_1 \cos\alpha_1 + m_2 g l_2 \cos\alpha_2 \end{aligned} \tag{15}$$

Lagrangian equations of motion shown in Equation 16 and 17

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_1} - \frac{\partial L}{\partial \alpha_1} = \tau_1 - \tau_2 \tag{16}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_2} - \frac{\partial L}{\partial \alpha_2} = \tau_2 \tag{17}$$

Then,

$$\frac{\partial L}{\partial \dot{\alpha}_1} = (m_1+m_2) l_1^2 \dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) \tag{18}$$

$$\frac{\partial L}{\partial \dot{\alpha}_2} = m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2) \tag{19}$$

$$\frac{\partial L}{\partial \alpha_1} = -m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - (m_1+m_2) g l_1 \sin\alpha_1 \tag{20}$$

$$\frac{\partial L}{\partial \alpha_2} = m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - m_2 g l_2 \sin\alpha_2 \tag{21}$$

Combine Equation 18 to 21 with 16 and 17,

$$\begin{aligned} & l_2 \ddot{\alpha}_2 + l_1 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - l_1 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) \\ & + g \sin\alpha_2 - \frac{\tau_2}{m_2 l_2} = 0 \end{aligned} \tag{22}$$

$$\begin{aligned} & (m_1+m_2) l_1 \ddot{\alpha}_1 + m_2 l_2 \ddot{\alpha}_2 \cos(\alpha_1 - \alpha_2) \\ & + m_2 l_2 \dot{\alpha}_2^2 \sin(\alpha_1 - \alpha_2) \\ & + (m_1+m_2) g \sin\alpha_1 - \frac{\tau_1 - \tau_2}{l_1} = 0 \end{aligned} \tag{23}$$

Combine Equation 22 and 23,

$$\begin{aligned} \ddot{\alpha}_1 = {} & \left(\frac{\tau_1 - \tau_2}{l_1} - \frac{\tau_2}{l_2} \cos(\alpha_1 - \alpha_2)\right) \\ & - m_2 l_1 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) \cos(\alpha_1 - \alpha_2) \\ & + m_2 g \sin\alpha_2 \cos(\alpha_1 - \alpha_2) - m_2 l_2 \dot{\alpha}_2^2 \sin(\alpha_1 - \alpha_2) \\ & - (m_1+m_2) g \sin\alpha_1 / [(m_1+m_2) l_1 \\ & \qquad - m_2 l_1 \cos(\alpha_1 - \alpha_2)^2] \end{aligned} \tag{24}$$

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\begin{aligned}
\ddot{\alpha}_2 = & -\frac{\tau_1 - \tau_2}{l_1}\cos(\alpha_1 - \alpha_2) \\
& + (m_1 + m_2) g \sin\alpha_1 \cos(\alpha_1 - \alpha_2) \\
& + m_2 l_2 \dot{\alpha}_2^2 \sin(\alpha_1 - \alpha_2)\cos(\alpha_1 - \alpha_2) \\
& + \frac{(m_1 + m_2)\tau_2}{m_2 l_2} - (m_1 + m_2) g \sin\alpha_2 + (m_1 \\
& \; + m_2) l_1 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) / l_2 [m_1 + m_2 \sin(\alpha_1 - \alpha_2)^2]
\end{aligned} \tag{25}
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