NYU Tandon Bridge **Spring 2024**

Homework 5

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Question 3

(a) Exercise 4.1.3

section b

$f(x)$ is not a function since for $x = 2$ and $x = -2$, they do not map to any element in $\mathbb{R}$.

section c

$f(x)$ is a function because for every x, there is exactly one $f(x)$. The range of the function is $[0, \infty)$.

(b) Exercise 4.1.5

section b

range of $f = \{4, 9, 16, 25\}$

section d

range of $f = \{0, 1, 2, 3, 4, 5\}$

section h

range of $f = \{(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (1, 3), (2, 3), (3, 3)\}$

section i

range of $f = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3)(3, 4)\}$

section l

range of $f = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$

Question 4

I.

(a) Exercise 4.2.2

section c

The function is one-to-one but not onto.

For $h(x) = 2$, there does not exist an $x \in \mathbb{Z}$ such that $h(x) = 2$.

section g

The function is one-to-one but not onto.

For $f(x, y) = (0, 1)$, there does not exist a pair of $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $f(x, y) = (0, 1)$ since $2y$ would always be an even integer.

section k

The function is neither one-to-one nor onto.

$$f(1,3) = 2^1 + 3 = 5$$
$$f(2,1) = 2^2 + 1 = 5$$

Since $(1, 3) \neq (2, 4)$, but $f(1, 3) = f(2, 1)$, the function is not one-to-one.

Since x and y are both positive integers greater than or equal to 1. $f(x, y)$ would be greater than or equal to $2^1 + 1 = 3$. Therefore, there does not exist a pair of $(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $f(x, y) = 1$ or $f(x, y) = 2$, which means the function is not onto.

(b) Exercise 4.2.4

section b

The function is neither one-to-one nor onto.

$$f(000) = 100$$
$$f(100) = 100$$

Since $000 \neq 100$, but $f(000) = f(100)$, the function is not one-to-one.
For $f = 000$, there does not exist a string $x \in \{0, 1\}^3$ such that $f(x) = 000$. Therefore, the function is not onto.

section c

The function is one-to-one and onto.

section d

The function is one-to-one but not onto.

For $f = 0001$, there does not exist a string $\in \{0,1\}^3$ such that $f = 0001$.

section g

The function is neither one-to-one nor onto.

For $X_1 = \{2\}, X_2 = \{1,2\}, X_1 \neq X_2$ but $F(X_1) = F(X_2) = \{2\}$. Therefore, the function is not one-to-one.

The function is not onto because $\{1\}$ is an element in the target but there is no X in the domain that would map to $\{1\}$ since the function $F(X)$ would subtract $\{1\}$ from X.

II.

section a

$$f : \mathbb{Z} \to \mathbb{Z}^+,\ f(x) = \begin{cases} 2x+3 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

This function is one-to-one but not onto.
For example, there does not exist an $x \in \mathbb{Z}$ such that $f(x) = 1$.

section b

$$f : \mathbb{Z} \to \mathbb{Z}^+,\ f(x) = |x| + 1$$

This function is onto but not one-to-one.
For all $f(x) \in \mathbb{Z}^+$, there exists an x such that $f(x)$, so the function is onto. Since $-1 \neq 1$ and $f(-1) = f(1) = 2$, the function is not one-to-one.

section c

$$f : \mathbb{Z} \to \mathbb{Z}^+,\ f(x) = \begin{cases} 2x+1 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

section d

$$f : \mathbb{Z} \to \mathbb{Z}^+,\ f(x) = |x| + 2$$

This function is neither one-to-one nor onto.
Since $-1 \neq 1$ and $f(-1) = f(1) = 3$, the function is not one-to-one. Since there does not exists an x such that $f(x) = 1$, the function is not onto.

Question 5

(a) Exercise 4.3.2

section c

The function has a well-defined inverse.

$$f^{-1} : \mathbb{R} \to \mathbb{R}.\ f^{-1}(x) = \frac{x-3}{2}$$

section d

The function does not have a well-defined inverse because it is not one-to-one.

Since $\{1\} \neq \{2\}$, $f(\{1\}) = f(\{2\}) = 1$, the function is not one-to-one and therefore does not have a well-defined inverse.

section g

The function has a well-defined inverse.

$f^{-1} : \{0,1\}^3 \to \{0,1\}^3$. The output of $f^{-1}$is obtained by taking the input string and reversing the bits.

section i

The function has a well-defined inverse.

$$f^{-1} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}.\ f^{-1}(x,y) = (x-5, y+2)$$

(b) Exercise 4.4.8

section c

$$\begin{aligned} f \circ h &= f(h(x)) \\ &= f(x^2+1) \\ &= 2 \cdot (x^2+1) + 3 \\ &= 2x^2 + 5 \end{aligned}$$

section d

$$\begin{aligned} h \circ f &= h(f(x)) \\ &= h(2x+3) \\ &= (2x+3)^2 + 1 \\ &= 4x^2 + 12x + 10 \end{aligned}$$

(c) Exercise 4.4.2

section b

$$\begin{aligned}
(f \circ h)(52) &= f(h(52)) \\
&= f(\left\lceil \frac{52}{5} \right\rceil) \\
&= f(11) \\
&= 11^2 \\
&= 121
\end{aligned}$$

section c

$$\begin{aligned}
(g \circ h \circ f)(4) &= g(h(f(4))) \\
&= g(h(16)) \\
&= g(4) \\
&= 16
\end{aligned}$$

section d

$$\begin{aligned}
h \circ f(x) &= h(f(x)) \\
&= h(x^2) \\
&= \left\lceil \frac{x^2}{5} \right\rceil
\end{aligned}$$

(d) Exercise 4.4.6

section c

$$\begin{aligned}
(h \circ f)(010) &= h(f(010)) \\
&= h(110) \\
&= 111
\end{aligned}$$

section d

{101,111}

The range of f is $\{100, 101, 110, 111\}$ since the first digit would be replaced by 1 and the second and third digits can be either 1 or 0.

The range of $h \circ f$ would be {101,111} since the last bit would be replaced by the first bit, which is 1. So the first and third bits both have to be 1, and the second digit can be either 1 or 0.

section e

{001,101,011,111}

The range of f is $\{100, 101, 110, 111\}$ since the first digit would be replaced by 1 and the second and third digits can be either 1 or 0.

The range of $g \circ f$ would be {001,101,011,111} since the g is obtained by reversing all the bits.

(e) Exercise 4.4.4

section c

No.
If f is not one-to-one, that means there would be two different inputs x_1 and x_2 that map to the same output y such that $f(x_1) = f(x_2) = y$.

$$g \circ f(x_1) = g(f(x_1)) = g(y)$$
$$g \circ f(x_2) = g(f(x_2)) = g(y)$$

Since g is a function, input y can only map to one output $g(y)$. This would mean that the function $g \circ f$ would map two different inputs x_1 and x_2 to the same output $g(y)$, so $g \circ f$ is not one-to-one.

section d

Yes.
Here is an example: