NYU Tandon Bridge	Spring 2024

Homework 2

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Question 5

(a)

1. Exercise 1.12.2

section b

(1)	$p \to (q \land r)$	Hypothesis
(2)	$\neg q$	Hypothesis
(3)	$\neg q \lor \neg r$	Addition, 2
(4)	$\neg(q \land r)$	De Morgan's law, 3
(5)	$\neg p$	Modus tollens, 4

section e

(1)	$p \lor q$	Hypothesis
(2)	$\neg p \lor r$	Hypothesis
(3)	$\neg q$	Hypothesis
(4)	p	Disjunctive syllogism, 1, 3
(5)	$\neg\neg p$	Double negation law, 4
(6)	r	Disjunctive syllogism, 2, 5

2. Exercise 1.12.3

section c

(1)	$p \lor q$	Hypothesis
(2)	$\neg p$	Hypothesis
(3)	$\neg\neg p \lor q$	Double negation law, 1
(4)	$\neg p \to q$	Conditional identity, 3
(5)	q	Modus ponens, 2, 4

3. Exercise 1.12.5

section c

$p =$ I will buy a new car
$q =$ I will buy a new house

$r =$ I will get a job

The argument can be formed as,

$$\begin{array}{l} (p \wedge q) \to r \\ \neg r \\ \hline \therefore \neg p \end{array}$$

The argument is invalid.

Proof:

When p is true, q is false, and r is false, both hypotheses $(p \wedge q) \to r$ and $\neg r$ are true and the conclusion $\neg p$ is false. Therefore, the argument is invalid.

section d

$p =$ I will buy a new car
$q =$ I will buy a new house
$r =$ I will get a job

The argument can be formed as,

$$\begin{array}{l} (p \wedge q) \to r \\ \neg r \\ q \\ \hline \therefore \neg p \end{array}$$

The argument is valid.

Proof:

(1)	$(p \wedge q) \to r$	Hypothesis
(2)	$\neg r$	Hypothesis
(3)	q	Hypothesis
(4)	$\neg(p \wedge q)$	Modus tollens, 1, 2
(5)	$\neg p \vee \neg q$	De Morgan's law, 4
(6)	$\neg\neg q$	Double negation law, 3
(7)	$\neg p$	Disjunctive syllogism, 5, 6

(b)

1. Exercise 1.13.3

section b

	P	Q
a	F	F
b	F	T

In the domain set $\{a, b\}$, the two hypotheses, $\exists x(P(x) \vee Q(x))$ and $\exists x \neg Q(x)$ are both true for the values for P and Q on element a and b given in the table. However, the conclusion $\exists x P(x)$ is false. Therefore, the argument is invalid.

2. Exercise 1.13.5

section d

Define the following three predicates:

$M(x)$: x missed class.
$D(x)$: x got a detention.

The form of this argument is:

$$\begin{array}{l} \forall x(M(x) \to D(x)) \\ \text{Penelope is an element} \\ \neg M(Penelope) \\ \hline \therefore \neg D(Penelope) \end{array}$$

The argument is invalid.

Suppose the domain of x is the set $\{Penelope, b\}$,

	M	D
$Penelope$	F	T
b	F	T

In the domain set $\{Penelope, b\}$, the three hypotheses, $\forall x(M(x) \to D(x))$, Penelope is a particular element, $\neg M(Penelope)$ are all true for the values for M and D on element a and b given in the table. However, the conclusion $\neg D(Penelope)$ is false. Therefore, the argument is invalid.

section e

Define the following three predicates:

$M(x)$: x missed class.
$D(x)$: x got a detention.
$A(x)$: x got an A.

The form of this argument is:

$$
\begin{array}{l}
\forall x((M(x) \vee D(x)) \to \neg A(x)) \\
\text{Penelope is an element} \\
A(Penelope) \\
\hline
\therefore \neg D(Penelope)
\end{array}
$$

The argument is valid.

Proof:

(1)	$\forall x((M(x) \vee D(x)) \to \neg A(x))$	Hypothesis
(2)	Penelope is an element	Hypothesis
(3)	$A(Penelope)$	Hypothesis
(4)	$(M(Penelope) \vee D(Penelope)) \to \neg A(Penelope)$	Universal instantiation, 1, 2
(5)	$\neg\neg A(Penelope)$	Double negation law, 3
(6)	$\neg(M(Penelope) \vee D(Penelope))$	Modus tollens, 4, 5
(7)	$\neg M(Penelope) \wedge \neg D(Penelope)$	De Morgan's law, 6
(8)	$\neg D(Penelope)$	Simplification, 7

Question 6

Exercise 2.4.1

section d

Proof:

Let x and y be two odd integers.

Since x is an odd integer, $x = 2k + 1$, for some integer k.

Since y is an odd integer, $y = 2n + 1$, for some integer n.

Plug $x = 2k + 1$ and $y = 2n + 1$ into xy to get:

$$\begin{aligned} xy &= (2k+1)(2n+1) \\ &= 4kn + 2k + 2n + 1 \\ &= 2(2kn + k + n) + 1 \end{aligned}$$

Since n and k are integers, $2kn + k + n$ is also an integer.

Since $xy = 2m + 1$, where $m = 2kn + k + n$ is an integer, xy is odd.

Therefore, the product of two odd integers is an odd integer.

Exercise 2.4.3

section b

Assume x is a real number and $x \leq 3$, we will prove $12 - 7x + x^2 \geq 0$.

Since $x < 3$, we can subtract 3 from both sides to get,

$$x - 3 \leq 0$$

we can further subtract 1 from both sides to get,

$$x - 4 \leq -1$$

Since both $x - 3$ and $x - 4$ are less than or equal to 0, their product would be greater than or equal to 0,

$$(x-3)(x-4) \geq 0$$

Therefore,

$$x^2 - 7x + 12 \geq 0$$

Question 7

Exercise 2.5.1

section d

Assume n is even, we will prove $n^2 - 2n + 7$ is odd.

For any integer n, if n is even, then $n = 2k$ for some integer k.

Plug $n = 2k$ into the expression $n^2 - 2n + 7$ to get,

$$\begin{aligned} n^2 - 2n + 7 &= (2k)^2 - 2(2k) + 7 \\ &= 4k^2 - 4k + 7 \\ &= 2(2k^2 - 2k + 3) + 1 \end{aligned}$$

Since $n^2 - 2n + 7 = 2m + 1$, where $m = 2k^2 - 2k + 3$ is an integer, $n^2 - 2n + 7$ is odd.

Exercise 2.5.4

section a

Assume $x > y$, we will prove $x^3 + xy^2 > x^2y + y^3$, for every pair of real number x and y.

Since x and y are real numbers,

$$\begin{aligned} x^2 \geq 0 \\ y^2 \geq 0 \end{aligned}$$

Add them to get,

$$x^2 + y^2 \geq 0$$

If $x^2 + y^2 = 0$, then $x = 0$, $y = 0$, and $x = y$. But $x > y$, so $x^2 + y^2 \neq 0$, therefore,

$$x^2 + y^2 > 0$$

Since $x^2 + y^2$ is positive, we can multiply $x^2 + y^2$ on both sides of the inequality $x > y$ to get,

$$\begin{aligned} x(x^2 + y^2) > y(x^2 + y^2) \\ x^3 + xy^2 > x^2y + y^3 \end{aligned}$$

section b

Assume $x \leq 10$ and $y \leq 10$, we will prove $x + y \leq 20$, for every pair of real number x and y.

Add $x \leq 10$ and $y \leq 10$ to get,

$$\begin{aligned} x + y &\leq 10 + 10 \\ x + y &\leq 20 \end{aligned}$$

Exercise 2.5.5

section c

Assume $\frac{1}{r}$ is rational, we will prove r is rational.

If $\frac{1}{r}$ is rational, then $\frac{1}{r} = \frac{a}{b}$ for some integers a and b, where $b \neq 0$.

Multiply br on both sides of the equation to get,

$$b = ar$$

If $a = 0$, then $ar = 0$, but $ar = b$ and $b \neq 0$, so $a \neq 0$.

Since $b = ar$ and $a \neq 0$, we can divide both sides by a to get,

$$r = \frac{b}{a}$$

Since both a and b are integers, r is rational.

Question 8

Exercise 2.6.6

section c

Assume the negation of the statement that the average of three real numbers is less than any of these three numbers.

Define the three real numbers as x, y and z, then we have,

$$\frac{x+y+z}{3} < x$$
$$\frac{x+y+z}{3} < y$$
$$\frac{x+y+z}{3} < z$$

Add the three inequalities to get,

$$x + y + z < x + y + z$$

Since the negation of the statement leads to a contradiction, the statement that the average of three real numbers is greater than or equal to at least one of the numbers must be true.

section d

Assume the negation of the argument that there is a smallest integer.

Define the smallest integer as x, we have,

$$x - 1 < x$$

Since $x - 1$ is less than x, there is an integer that is smaller than x and x is not the smallest integer. The negation of this statement leads to a contradiction. Therefore, the statement that there is no smallest integer must be true.

Question 9

Exercise 2.7.2

section b

Proof:

Consider the following two cases:

Case 1: x and y are both even, then $x = 2k$ and $y = 2j$, where k and j are both some integers.

$$\begin{aligned} x + y &= 2k + 2j \\ &= 2(k + j) \end{aligned}$$

Since $x + y = 2m$, where $m = k + j$ is an integer, $x + y$ is even.

Case 2: x and y are both odd, then $x = 2k + 1$ and $y = 2j + 1$, where k and j are both some integers.

$$\begin{aligned} x + y &= 2k + 1 + 2j + 1 \\ &= 2(k + j + 1) \end{aligned}$$

Since $x + y = 2m$, where $m = k + j + 1$ is an integer, $x + y$ is even.

In both cases, $x + y$ is even. Therefore, if x and y have the same parity, then $x + y$ is even.