NYU Tandon Bridge **Spring 2024**

Homework 11

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Question 5

a.

Proof:

Base case: $n = 1$.

When $n = 1$,

$$n^3 + 2n = 1^3 + 2 \times 1 = 3$$

Therefore, 3 divides $n^3 + 2n$ when $n = 1$.

Inductive step: For all $k \geq 1$, if 3 divides $k^2 + 2k$, we shall prove 3 divides $(k+1)^2 + 2(k+1)$.

Since 3 divides $k^2 + 2k$, we can get:

$$k^3 + 2k = 3m,\ m \in \mathbb{Z}^+$$

$$\begin{aligned}
(k+1)^3 + 2 \cdot (k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
&= (k^3 + 2k) + (3k^2 + 3k + 3) \\
&= 3m + 3(k^2 + k + 1) \\
&= 3(m + k^2 + 2k + 1)
\end{aligned}$$

Since $m + k^2 + 2k + 1$ is an positive integer, we prove 3 divides $(k+1)^2 + 2(k+1)$.

b.

Proof:

Base case: $n = 2$. Since 2 is a prime number, it is a product of the prime number 2.

Inductive step: Assume for all k ≥ 2, every integer from 2 to k can be written as a product of primes, we shall prove $k+1$ can also be written as a product of primes.

If $k+1$ is prime, then it is a product of the prime number $k+1$.

If $k+1$ is composite, then it can be written as a product of two integers a and b such that a and b both are in the range from 2 to k:

$$k + 1 = a \cdot b \quad 2 \leq a, b \leq k$$

Since we assume that every integer from 2 to k can be written as a product of primes, a and b can both be written as products of primes:

$$
\begin{aligned}
a &= p_1 \cdot p_2 \cdot p_3 ... p_n \\
b &= q_1 \cdot q_2 \cdot q_3 ... q_m
\end{aligned}
$$

Therefore, their product $k + 1$ can also be written as a product of primes:

$$
\begin{aligned}
k + 1 &= a \cdot b \\
&= p_1 \cdot p_2 \cdot p_3 ... p_n \cdot q_1 \cdot q_2 \cdot q_3 ... q_m
\end{aligned}
$$

Question 6

(a) Exercise 7.4.1

section a

when n = 3, the left side of the equation is:

$$\begin{aligned}\sum_{j=1}^{3} j^2 &= 1^2 + 2^2 + 3^2 \\ &= 14\end{aligned}$$

The right side of the equation is:

$$\frac{3 \times (3+1)(2 \times 3 + 1)}{6} = 14$$

Therefore, $P(3)$ is true.

section b

$$\sum_{j=1}^{k} k^2 = \frac{k(k+1)(2k+1)}{6}$$

section c

$$\sum_{j=1}^{k+1} (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

section d

$P(1)$ is true.

section e

For all $k \geq 1$, $P(k)$ implies $P(k+1)$.

section f

$P(k)$.

section g

Proof:

Base case: $n = 1$.

when n = 1, the left side of the equation is:

$$\sum_{j=1}^{1} k^2 = 1^2 = 1$$

The right side of the equation is:

$$\frac{1 \times (1+1) \times (2 \times 1 + 1)}{6} = \frac{1 \times 2 \times 3}{6} = 1$$

Since $1 = 1$, $P(1)$ is true.

Inductive step: For all $k \geq 1$, if $P(k)$ is true, we shall prove $P(k+1)$ is true.

Since $P(k)$ is true, we have:

$$\sum_{j=1}^{k} k^2 = \frac{k(k+1)(2k+1)}{6}$$

$$\begin{aligned}
\sum_{j=1}^{k+1} (k+1)^2 &= \sum_{j=1}^{k} k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)(k+1)}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

Therefore, for all $k \geq 1$, if $P(k)$ is true, $P(k+1)$ is true.

(b) Exercise 7.4.3

section c

Proof:

Base case: $n = 1$.

when n = 1, the left side of the equation is:

$$\sum_{j=1}^{n} \frac{1}{j^2} = \frac{1}{1^2} = 1$$

The right side of the equation is:

$$2 - \frac{1}{1} = 1$$

Since $1 \leq 1$, $P(1)$ is true.

Inductive step: For all $k \geq 1$, if $P(k)$ is true, we shall prove $P(k+1)$ is true.

Since $P(k)$ is true, we have:

$$\sum_{j=1}^{k} \frac{1}{k^2} \leq 2 - \frac{1}{k}$$

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{(k+1)^2} &= \sum_{j=1}^{k} \frac{1}{k^2} + \frac{1}{(k+1)^2} \\
&\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\
&\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} \\
&\leq 2 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)} \\
&\leq 2 - \frac{k}{k(k+1)} \\
&\leq 2 - \frac{1}{k+1}
\end{aligned}$$

Therefore, for all $k \geq 1$, if $P(k)$ is true, $P(k+1)$ is true.

(c) Exercise 7.5.1

section a

Proof:

Base case: $n = 1$.

when n = 1,

$$3^{2\times 1} - 1 = 8$$

Since 4 evenly divides 8, the base case $P(1)$ is true.

Inductive step: For all $k \geq 1$, if $P(k)$ is true, we shall prove $P(k+1)$ is true.

Since $P(k)$ is true, we have:

$$3^{2k} - 1 = 4m,\ m \in \mathbb{Z}^+$$

$$\begin{aligned}
3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\
&= 3^2 \cdot 3^{2k} - 1 \\
&= 9 \cdot 3^{2k} - 9 + 8 \\
&= 9 \cdot (3^{2k} - 1) + 8 \\
&= 9 \cdot 4m + 8 \\
&= 36m + 8 \\
&= 4 \cdot (9m + 2)
\end{aligned}$$

Since $9m + 2$ is a positive integer, 4 evenly divides $3^{2(k+1)} - 1$. Therefore, for all $k \geq 1$, if $P(k)$ is true, $P(k+1)$ is true.