NYU Tandon Bridge **Spring 2024**

Homework 7

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Question 3

(a) Exercise 8.2.2

section b

Proof:

In order to show $f = \Theta(n^3)$, we need to show $f = \Omega(n^3)$ and $f = O(n^3)$.

If we let $n \geq 1$, we can get:

$$3n^2 + 4 \geq 0 \text{ for all } n \geq 1$$

Adding n^3 on both sides to get:

$$n^3 + 3n^2 + 4 \geq n^3 \text{ for all } n \geq 1$$

If we let $n_0 = 1$ and $c = 1$, we can get:

$$n^3 + 3n^2 + 4 \geq c \cdot n^3 \text{ for all } n \geq n_0$$

Therefore, $f = \Omega(n^3)$.

If we let $n \geq 1$, we can get:

$$4 \leq 4n^2 \text{ for all } n \geq 1$$

Adding $3n^2$ on both sides to get:

$$3n^2 + 4 \leq 7n^2 \text{ for all } n \geq 1$$

If $n \geq 1$, $n^2 \leq n^3$, $7n^2 \leq 7n^3$, therefore,

$$n^3 + 3n^2 + 4 \leq 7n^3 \text{ for all } n \geq 1$$

If we let $n_0 = 1$ and $c = 7$, we can get:

$$n^3 + 3n^2 + 4 \leq c \cdot n^3 \text{ for all } n \geq n_0$$

Therefore, $f = O(n^3)$.

Since $f = \Omega(n^3)$ and $f = O(n^3)$, $f = \Theta(n^3)$.

(b) Exercise 8.3.5

section a

We first search for the leftmost number in the sequence that is greater than or equal to p and mark it as a_i. We then search for the rightmost number in the sequence that is less than p and mark it as a_j. If a_i and a_j exist and a_i appears before a_j, then we swap a_i and a_j. And we keep doing this search until $a_i = a_j$. Eventually, the sequence would have all the numbers less than p on the left and all the numbers greater than or equal to p on the right.

section b

It does not depend on the actual values of the numbers in the sequence. It only depends on the length of the sequence. Eventually, the loop would end when $i = j$. Suppose when the loop ends, "$i : i + 1$" runs k times and "$j : j - 1$" runs l times. we have:

$$i + k = j - l$$

To calculate the total number of times the lines "$i : i + 1$" or "$j : j - 1$" are executed, we need to calculate $k + l$, according to the equation above, we can get:

$$\begin{aligned} k + l &= j - i \\ &= n - 1 \end{aligned}$$

Therefore, the total number of times these two lines are executed on a sequence of length n is $n - 1$.

section c

The total number of times of the swap operation depends on the numbers in the sequence. If all the numbers in the sequence is less than p, i would keep increase until it is equal to j. In this case, the swap operation would be executed 0 times. If all the numbers on the left half of the sequence is greater than or equal to p and all the numbers on the right half of the sequence is less than p, the swap operation would be executed $\lfloor \frac{n}{2} \rfloor$ times. So the minimum number of times is 0 and the maximum number of times is $\lfloor \frac{n}{2} \rfloor$.

section d

According to the previous calculation, the first two loops would be executed $n - 1$ times in total and the third loop would be executed $\lfloor \frac{n}{2} \rfloor$ times max and 0 times min. So the asymptotic lower bound would be $\Omega(n)$. It is not important to consider the worst case input since the asymptotic lower bound would still be $\Omega(n)$.

section e

According to the previous calculation, the max runtime for the loops would be $n - 1 + \lfloor \frac{n}{2} \rfloor$ in total. So the upper bound is $O(n)$.

Question 4

(a) Exercise 5.1.2

section b

$40^7 + 40^8 + 40^9$.

Total numbers of characters:

$$10 + 26 + 4 = 40$$

Strings of length 7, 8, 9:

$$40^7 + 40^8 + 40^9 = 2,688,614,400,000,000$$

section c

$14 \times (40^6 + 40^7 + 40^8)$.

$$14 \times 40^6 + 14 \times 40^7 + 14 \times 40^8 = 940,344,704,000,000$$

(b) Exercise 5.3.2

section a

1536.

There are 3 options for the 1st character: a, b or c. For the 2nd to the 10th character, there are only 2 options for each because they cannot be the same as the previous character. so the total would be $3 * 2^9 = 1536$.

(c) Exercise 5.3.3

section b

$26^4 \times P(10,3)$.

$$26^4 \times P(10,3) = 328,821,120$$

section c

$P(26,4) \times P(10,3)$.

$$P(26,4) \times P(10,3) = 258,336,000$$

(d) Exercise 5.2.3

section a

Define a function $f : B^9 \to E_{10}$. f is obtained by add a 0 to the end of the string if it has even number of 1's and add a 1 to the end of the string if it has odd number of 1's.

For each element in B^9, if it has even number of 1's, we add a 0 to the end of the string; if it has odd number of 1's, we add a 1 to the end of the string. For example, f(100000000) = 1000000001. The function is a bijection because it has a well defined inverse $f^{-1} : E_{10} \to B^9$. f^{-1} is obtained by removing the last bit.

section b

512.

Since there is a bijection between B^9 and E_{10}. $|B^9| = |E_{10}| = 2^9 = 512$.

Question 5

(a) Exercise 5.4.2

section a

20000.

$$2 \times 10^4 = 20000$$

section b

10080.

$$2 \times 10 \times 9 \times 8 \times 7 = 10080$$

(b) Exercise 5.5.3

section a

$2^{10} = 1024$.

section b

$2^7 = 128$.

section c

$2^7 + 2^8 = 384$.

section d

$2^8 = 256$.

section e

$\binom{10}{6} = 210$.

section f

$\binom{9}{6} = 84$.

section g

$\binom{5}{1}\binom{5}{3} = 50$.

(c) Exercise 5.5.5

section a

$\binom{35}{10}\binom{30}{10}$.

(d) Exercise 5.5.8

section c

65780.

$$\begin{aligned}\binom{26}{5} &= \frac{26!}{5!(26-5)!} \\ &= 65780\end{aligned}$$

section d

624.

$$\begin{aligned}\binom{13}{1}\binom{48}{1} &= \frac{13!}{1!(13-1)!} \times \frac{48!}{1!(48-1)!} \\ &= 624\end{aligned}$$

section e

3744.

$$\begin{aligned}\binom{4}{2}\binom{13}{1}\binom{4}{3}\binom{12}{1} &= \frac{4!}{2!(4-2)!} \times \frac{13!}{1!(13-1)!} \times \frac{4!}{1!(4-3)!} \times \frac{12!}{1!(12-1)!} \\ &= 6 \times 13 \times 4 \times 12 \\ &= 3744\end{aligned}$$

section f

1,317,888.

$$\begin{aligned}4^5 \times \binom{13}{5} &= 1024 \times \frac{13!}{5!(13-5)!} \\ &= 1,317,888\end{aligned}$$

(e) Exercise 5.6.6

section a

4,148,350,734,528.

$$\begin{aligned}\binom{44}{5}\binom{56}{5} &= \frac{44!}{5!(44-5)!} \times \frac{56!}{5!(56-5)!} \\ &= 4,148,350,734,528\end{aligned}$$

section b

5,827,360.

$$\begin{aligned}P(44,2) \times P(56,2) &= \frac{44!}{(44-2)!} \times \frac{56!}{(56-2)!} \\ &= 5,827,360\end{aligned}$$

Question 6

(a) Exercise 5.7.2

section a

2,023,203.

Numbers of 5-card hands with no club: $\binom{39}{5}$

Total 5-card hands: $\binom{52}{5}$

Numbers of 5-card hands with at least one club: $\binom{52}{5} - \binom{39}{5} = 2,023,203$

section b

$\binom{52}{5} - \binom{13}{5} \cdot 4^5 = 1,280,112$

(b) Exercise 5.8.4

section a

5^{20}.

section b

305,781,735,000.

$$\begin{aligned}\binom{20}{4}\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4} &= \frac{20!}{4!4!4!4!4!} \\ &= 305,781,735,000\end{aligned}$$

Question 7

(a)

0.

There is no one-to-one function from a set with 5 elements to a set with 4 elements. There would always be 2 inputs mapping to the same output for any well defined functions.

(b)

120.

In order to create a one-to-one function, we need to choose 5 elements from the target so that each of them can be mapped to exactly one element in the domain and the order of these 5 elements matters. So we have $P(5,5)$ ways to choose and arrange these 5 elements.

$$P(5,5) = 120$$

(c)

720.

In order to create a one-to-one function, we need to choose 5 elements from the target so that each of them can be mapped to exactly one element in the domain and the order of these 5 elements matters. So we have $P(6,5)$ ways to choose and arrange these 5 elements.

$$P(6,5) = 720$$

(d)

2520.

In order to create a one-to-one function, we need to choose 5 elements from the target so that each of them can be mapped to exactly one element in the domain and the order of these 5 elements matters. So we have $P(7,5)$ ways to choose and arrange these 5 elements.

$$P(7,5) = 2520$$