NYU Tandon Bridge **Spring 2024**

Homework 8

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Question 7

(a) Exercise 6.1.5

section b

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\cdot 4^2}{\binom{52}{5}} \approx 0.0211$$

section c

$$\frac{\binom{13}{5}\binom{4}{1}}{\binom{52}{5}} \approx 0.00198$$

section d

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\cdot 4^3}{\binom{52}{5}} \approx 0.4226$$

(b) Exercise 6.2.4

section a

$$1 - \frac{\binom{39}{5}}{\binom{52}{5}} \approx 0.7785$$

section b

$$1 - \frac{\binom{13}{5}\cdot 4^5}{\binom{52}{5}} \approx 0.4929$$

section c

$$\frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}} + \frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}} - \frac{\binom{13}{1}\binom{13}{1}\binom{26}{3}}{\binom{52}{5}} \approx 0.6538$$

section d

$$1 - \frac{\binom{26}{5}}{\binom{52}{5}} \approx 0.9747$$

Question 8

(a) Exercise 6.3.2

section a

$$p(A) = \frac{P(6,6)}{P(7,7)} = \frac{1}{7}$$

$$p(B) = \frac{\binom{7}{2}P(5,5)}{P(7,7)} = \frac{1}{2}$$

$$p(C) = \frac{P(5,5)}{P(7,7)} = \frac{1}{42}$$

section b

$$\begin{aligned} p(A|C) &= \frac{|A \cap C|}{|C|} \\ &= \frac{P(2,1)P(3,3)}{P(5,5)} \\ &= \frac{1}{10} \end{aligned}$$

section c

$$\begin{aligned} p(B|C) &= \frac{|B \cap C|}{|C|} \\ &= \frac{\binom{5}{2}P(3,3)}{P(5,5)} \\ &= \frac{1}{2} \end{aligned}$$

section d

$$\begin{aligned} p(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \frac{3 \cdot P(5,5)}{\binom{7}{2}P(5,5)} \\ &= \frac{1}{7} \end{aligned}$$

section e

If two events X and Y are independent, then $p(X|Y) = p(X)$, for event A, B, C, we have:

$$\begin{aligned} &p(A|B) = P(A) = \frac{1}{7} \\ &p(A|C) \neq P(A) \\ &p(B|C) = P(B) = \frac{1}{2} \end{aligned}$$

Therefore, (A, B) and (B, C) are independent.

(b) Exercise 6.3.6

section b

$$\left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^5$$

section c

$$\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^9$$

(c) Exercise 6.4.2

section a

0.4038.

Let F be the event that you choose a fair die. Let X be the event that the outcome you get after 6 rolls is 4, 3, 6, 6, 5, 5. We have:

$$\begin{aligned}
p(F) &= \frac{1}{2} \\
p(\overline{F}) &= \frac{1}{2} \\
p(X|F) &= (\frac{1}{6})^6 \\
p(X|\overline{F}) &= 0.15^4 \cdot 0.25^2
\end{aligned}$$

To calculate the probability that the die you choose is fair under the condition that the outcome is 4, 3, 6, 6, 5, 5, we need to calculate $p(F|X)$:

$$\begin{aligned}
p(F|X) &= \frac{p(X|F)p(F)}{p(X|F)p(F) + p(X|\overline{F})p(\overline{F})} \\
&= \frac{(\frac{1}{6})^6 \cdot \frac{1}{2}}{(\frac{1}{6})^6 \cdot \frac{1}{2} + 0.15^4 \cdot 0.25^2 \cdot \frac{1}{2}} \\
&\approx 0.4038
\end{aligned}$$

Question 9

(a) Exercise 6.5.2

section a

$A = \{0, 1, 2, 3, 4\}$

section b

$\{(0, 0.6588), (1, 0.2995), (2, 0.03993), (3, 0.00174), (4, 0.00001847)\}$

$$p(A = 0) = \frac{\binom{48}{5}}{\binom{52}{5}} \approx 0.6588$$

$$p(A = 1) = \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}} \approx 0.2995$$

$$p(A = 2) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} \approx 0.03993$$

$$p(A = 3) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}} \approx 0.00174$$

$$p(A = 4) = \frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} \approx 0.00001847$$

(b) Exercise 6.6.1

section a

1.4.

There could be either 0, 1, or 2 girls been chosen.

$$\text{For } G=0,\ p(G=0) = \frac{\binom{3}{2}}{\binom{10}{2}}$$

$$\text{For } G=1,\ p(G=1) = \frac{\binom{7}{1}\binom{3}{2}}{\binom{10}{2}}$$

$$\text{For } G=2,\ p(G=1) = \frac{\binom{7}{1}\binom{3}{2}}{\binom{10}{2}}$$

$$\begin{aligned} E[G] &= p(G=0)\cdot 0 + p(G=1)\cdot 1 + p(G=2)\cdot 2 \\ &= 0\cdot\frac{\binom{3}{2}}{\binom{10}{2}} + 1\cdot\frac{\binom{7}{1}\binom{3}{2}}{\binom{10}{2}} + 2\cdot\frac{\binom{7}{1}\binom{3}{2}}{\binom{10}{2}} \\ &= 1.4 \end{aligned}$$

(c) Exercise 6.6.4

section a

$\frac{91}{6}$.

Since X denotes the square of number on the die. X can be 1, 4, 9, 16, 25, or 36.

$$\begin{aligned} &\text{For } X=1,\ p(X=1)=1/6 \\ &\text{For } X=4,\ p(X=4)=1/6 \\ &\text{For } X=9,\ p(X=9)=1/6 \\ &\text{For } X=16,\ p(X=16)=1/6 \\ &\text{For } X=25,\ p(X=25)=1/6 \\ &\text{For } X=36,\ p(X=36)=1/6 \end{aligned}$$

$$\begin{aligned} [E] &= p(X=1)\cdot 1 + p(X=4)\cdot 4 + p(X=9)\cdot 9 + p(X=16)\cdot 16 + p(X=25)\cdot 25 + p(X=36)\cdot 36 \\ &= \frac{1}{6}\cdot(1+4+9+16+25+36) \\ &= \frac{91}{6} \end{aligned}$$

section b

3.

Since there can be 0, 1, 2, or 3 heads, Y can be 0, 1, 4, or 9.

$$
\begin{aligned}
&\text{For } Y = 0,\ p(Y = 0) = 1/8 \\
&\text{For } Y = 1,\ p(Y = 1) = 3/8 \\
&\text{For } Y = 4,\ p(Y = 4) = 3/8 \\
&\text{For } Y = 9,\ p(Y = 9) = 1/8
\end{aligned}
$$

$$
\begin{aligned}
E[Y] &= 0 \cdot p(Y = 0) + 1 \cdot p(Y = 1) + 4 \cdot p(Y = 4) + 9 \cdot p(Y = 9) \\
&= 0 \cdot 1/8 + 1 \cdot 3/8 + 4 \cdot 3/8 + 9 \cdot 1/8 \\
&= 3
\end{aligned}
$$

(d) Exercise 6.7.4

section a

1.

Let C be random variable denoting the number of children get who get their coats. Define random variable C_j to be 1 if j^{th} child get his/her coat and 0 otherwise, (for $j = 1, 2, ...10$). Then C is the sum of C_j's.

For all C_j, they have the same expectation. We can first calculate the expectation of C_1:

$$
\begin{aligned}
E[C_1] &= 1 \cdot p(C = 1) + 0 \cdot p(C = 0) \\
&= 1 \cdot \frac{1}{10} + 0 \cdot \frac{9}{10} \\
&= \frac{1}{10}
\end{aligned}
$$

We then calculate the expectation of C:

$$
\begin{aligned}
E[C] &= E[C_1] + E[C_2] ... + E[C_{10}] \\
&= 10 \cdot E[C_1] \\
&= 10 \cdot \frac{1}{10} \\
&= 1
\end{aligned}
$$

So the expected number of children who get their coats is 1.

Question 10

(a) Exercise 6.8.2

section a

The probability that exactly 2 circuit boards have defects:

$$b(2; 100, 0.01) = \binom{100}{2}(0.01)^2(0.99)^{98} \approx 0.185$$

section b

The probability that at least 2 circuit boards have defects:

$$\begin{aligned} p &= 1 - b(0; 100, 0.01) - b(1; 100, 0.01) \\ &= 1 - 0.99^{100} - \binom{100}{1}(0.01)(0.99)^{99} \approx 0.264 \end{aligned}$$

section c

Let X be the random variable denoting the number of circuit boards have defects, the expected number of circuit boards with defects is:

$$E[X] = 100 \times 0.01 = 1$$

section d

If at least 2 circuit boards have defects, this means at least 1 batch has defects. We can first calculate the probability that no batch has defects and subtract this result from 1 to get the probability that at least 1 batch (2 circuit boards) has defects.

The probability that no batch has defects:

$$b(0; 50, 0.01) = 0.99^{50}$$

The probability that at least 1 batch has defects:

$$p = 1 - 0.99^{50} \approx 0.395$$

Therefore, the probability at least 2 circuit boards have defects is 0.395.

Let X be the random variable denoting the number of circuit boards with defects and Y be the random variable denoting the number of batches with defects, then we can express their relationship as:

$$X = 2Y$$

The expected number of batches with defects:

$$E[Y] = 0.1 \times 50 = 0.5$$

The expected number of circuit boards with defects:

$$E[X] = E[2Y] = 2E[Y] = 2 \times 0.5 = 1$$

Compared to the situation in which each circuit boards is made separately, the probability that at least 2 circuit boards have defects increases while the expected number of circuit boards with defects does not change.

(b) Exercise 6.8.3

section b

0.3504.

The probability you reach an incorrect conclusion if the coin is biased equals the probability the number of heads is at least 4 if the coin is biased.

To calculate the probability the number of heads is at least 4 if the coin is biased, we first calculate the probability that the number of heads is 0, 1, 2, 3 and then subtract these results from 1.

$$\begin{aligned}
p(H = 0) &= b(0; 10, 0.3) = \binom{10}{0}(0.3)^0 0.7^{10} \approx 0.0282 \\
p(H = 1) &= b(1; 10, 0.3) = \binom{10}{1}(0.3)^1 (0.7)^9 \approx 0.1211 \\
p(H = 2) &= b(2; 10, 0.3) = \binom{10}{2}(0.3)^2 (0.7)^8 \approx 0.2335 \\
p(H = 3) &= b(3; 10, 0.3) = \binom{10}{3}(0.3)^3 (0.7)^7 \approx 0.2668
\end{aligned}$$

We then calculate the probability that the number of heads is at least 4:

$$p(H \geq 4) = 1 - 0.0282 - 0.1211 - 0.2335 - 0.2668 \approx 0.3504$$