- Course Title : Differential Equations and Special Functions
- Course Code: MAT102
- Section-3 // Lecture-10 (March 20, 2023)

Today's Lecture Topics:

Ordinary Differential Equations

- Linear differential equations (second or higher order) with constant coefficients: The general form of non-homogeneous equations and its solution procedures
- Method of Variation of Parameters for finding solution of $f(D)y = X$ (for Second order linear differential equation) where $D \equiv \frac{d}{dx}$

- Course Instructor: Dr. Akter Hossain, Assistant Professor of MPS Department, EWU, Dhaka, BD

- **Linear differential equations (second or higher order) with constant coefficients**
- **Working rule II for finding particular integral , $y_p = \frac{1}{f(D)} X$**

where X is x^m or a polynomial of degree m, m being positive integer.

Short method of finding P.I. when $X = x^m$, m being a positive integer.

Working rule for evaluating $\{1/f(D)\}\, x^m$.

Step I. Bring out the lowest degree term from $f(D)$ so that the remaining factor in the denominator is of the form $[1 + \phi(D)]^n$ or $[1 - \phi(D)]^n$, n being a positive integer.

Step II. We take $[1 + \phi(D)]^n$ or $[1 - \phi(D)]^n$ in the numerator so that it takes the form $[1 + \phi(D)]^{-n}$ or $[1 - \phi(D)]^{-n}$.

Step III. We expand $[1 \pm \phi(D)]^{-n}$ by the binomial theorem, namely

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

In particular the following binomial expansions should be remembered.

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$; $\qquad (1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

$(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$; $\qquad (1 - x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$

In any case, the expansion is to be carried upto D^m, since $D^{m+1} x^m = 0$, $D^{m+2} x^m = 0$, and all the higher differential coefficients of x^m vanish.

- **Linear differential equations (second or higher order) with constant coefficients**
- **Working rule II for finding particular integral , $y_p = \frac{1}{f(D)} X$**

  where X is x^m or a polynomial of degree m, m being positive integer.

- **Example:2 Solve** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6$

➢ **Solution: The given differential equation is**

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6 \quad -------------(1)$$

Let $y = e^{mx}$ be a trial solution of the homogeneous equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 0 \quad -------------------(2)$$

Then the auxiliary equation of (2) is $m^2 + 4m - 2 = 0$

$$\Rightarrow m^2 + 2.m.2 + 4 - 6 = 0 \Rightarrow (m+2)^2 = 6 \Rightarrow m + 2 = \pm\sqrt{6}$$

$$\therefore m = -2 \pm \sqrt{6}$$

- **Linear differential equations (second or higher order) with constant coefficients**
- **Working rule II for finding particular integral , $y_p = \frac{1}{f(D)}X$**

  where X is x^m or a polynomial of degree m, m being positive integer.

- **Example:2 Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6$**

➢ **Solution: The given differential equation is**

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6 \quad ------------ -(1)$$

Therefore, the complementary solution of (1) is

$$y_c = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{-(2-\sqrt{6})x}$$

when the roots of auxiliary equation of

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 0$ **are $-2 + \sqrt{6}$ and $-2 - \sqrt{6}$.**

- **Linear differential equations (second or higher order) with constant coefficients**
- **Working rule II for finding particular integral , $y_p = \frac{1}{f(D)}X$**

  where X is x^m or a polynomial of degree m, m being positive integer.

- **Example:2 Solve** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6$

➢ **Solution:**

Now the particular integral is $y_p = \frac{1}{D^2+4D-2}\left(2x^2 - 3x + 6\right)$

$$= \frac{1}{-2\left(1 - 2D - \frac{1}{2}D^2\right)}\left(2x^2 - 3x + 6\right)$$

$$= -\frac{1}{2}\left(1 - 2D - \frac{1}{2}D^2\right)^{-1}\left(2x^2 - 3x + 6\right)$$

$$= -\frac{1}{2}\left\{1 - \left(2D + \frac{1}{2}D^2\right)\right\}^{-1}\left(2x^2 - 3x + 6\right)$$

- **Linear differential equations (second or higher order) with constant coefficients**
- **Working rule II for finding particular integral , $y_p = \frac{1}{f(D)} X$**

  where X is x^m or a polynomial of degree m, m being positive integer.

- **Example:2 Solve** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6$

➢ **Solution:**

Now the particular integral is

$$y_p = -\frac{1}{2}\left\{1 + \left(2D + \frac{1}{2}D^2\right) + \left(2D + \frac{1}{2}D^2\right)^2 + \cdots\right\}(2x^2 - 3x + 6)$$

$$\left[\text{We know that } (1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots\right]$$

$$= -\frac{1}{2}\left(1 + 2D + \frac{1}{2}D^2 + 4D^2\right)(2x^2 - 3x + 6)$$

$$= -\frac{1}{2}\left(1 + 2D + \frac{9}{2}D^2\right)(2x^2 - 3x + 6) \;\; ------- -(2)$$

- Linear differential equations (second or higher order) with constant coefficients
- Working rule II for finding particular integral , $y_p = \frac{1}{f(D)} X$

  where X is x^m or a polynomial of degree m, m being positive integer.
- Example:2 Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6$

➢ Solution: Now the particular integral is

$$y_p = -\frac{1}{2}\left(2x^2 - 3x + 6 + 8x - 6 + 18\right)$$

$$\left[\text{by using } Eq. (2): -\frac{1}{2}\left(1 + 2D + \frac{9}{2}D^2\right)\left(2x^2 - 3x + 6\right)\right]$$

$$= -\frac{1}{2}\left(2x^2 + 5x + 18\right) = -x^2 - \frac{5}{2}x - 9$$

Therefore, the general solution is $y = y_c + y_p$

$$= c_1 e^{(-2+\sqrt{6})x} + c_2 e^{-(2-\sqrt{6})x} - x^2 - \frac{5}{2}x - 9 \text{ (Ans.)}$$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation:** $f(D)y = X$ where X is x^m or a polynomial of degree m, m being positive integer.

Exercise-14

Ex. 1. *Solve* $(D^4 - D^2)\, y = 2.$

Ans. $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} - x^2$

Ex. 2. *Solve* $(D^4 - a^4)\, y = x^4.$

Ans. $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax - (1/a^4)\,(x^4 + 24/a^4).$

Ex. 3 *Solve* $(D^2 + 2D + 2)\, y = x^2.$

Ans. $y = e^{-x}\,(c_1 \cos x + c_2 \sin x) + (x^2 - 2x + 1)/2.$

Ex. 4 *Solve* $(D^3 - D^2 - D - 2)\, y = x.$

Ans. $y = c_1 e^{2x} + e^{-x/2}\, \{c_3 \cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)\} - (1/8) \times (4x - x^2).$

- **Linear differential equations (second or higher order) with constant coefficients**

HW (It is a must event)

- **Non-homogeneous differential equation**

- **Suppose** $y_p = \frac{1}{f(D)}X$ **where** X **is the form of** $\cos ax$ **or** $\sin ax$

- **Working rule for finding** y_p **when** X **is the form of** $\cos ax$ **or** $\sin ax$

- **Formulae III : We have to express** $f(D)$ **as the function of** D^2**, say** $\phi\left(D^2\right)$ **and then replace** D^2 **by** $-a^2$**.**

  If $\phi\left(-a^2\right) \neq 0$**, then we have to use following result.**

- $If\ y_p = \frac{1}{f(D)}\cos ax = \frac{1}{\phi\left(D^2\right)}\cos ax$ **then** $y_p = \frac{1}{\phi\left(-a^2\right)}\cos ax$

- **Example: Find** $y_p = \frac{1}{D^4+D^2+1}\cos 2x$ (Only particular integral is calculated)

- Solution: Given $y_p = \frac{1}{D^4+D^2+1}\cos 2x = \frac{1}{\left(D^2\right)^2+D^2+1}\cos 2x = \frac{1}{\left(-2^2\right)^2-2^2+1}\cos 2x$

$$= \frac{1}{\left(-2^2\right)^2-2^2+1}\cos 2x = \frac{1}{13}\cos 2x$$

- **Linear differential equations (second or higher order) with constant coefficients**

HW (It is a must event)

- **Non-homogeneous differential equation**
- **Suppose $y_p = \frac{1}{f(D)} X$ where X is the form of $\cos ax$ or $\sin ax$**
- **Working rule for finding y_p when X is the form of $\cos ax$ or $\sin ax$**
- **Formulae III : We have to express $f(D)$ as the function of D^2, say $\phi\left(D^2\right)$ and then replace D^2 by $-a^2$.**

  If $\phi\left(-a^2\right) \neq 0$, then we have to use following result.
- $If\ y_p = \frac{1}{f(D)} \cos ax = \frac{1}{\phi\left(D^2\right)} \cos ax$ then $y_p = \frac{1}{\phi\left(-a^2\right)} \cos ax$
- **Case: 1: Sometimes, we cannot form $\phi\left(D^2\right)$. Then we will try to get $\phi\left(D^2, D\right)$ that is a function of $D\ and\ D^2$.**

In such cases, we have to use the procedures as given below:

$$y_p = \frac{1}{D^2-2D+1} cos\ 3x = \frac{1}{-3^2-2D+1} cos\ 3x = \frac{1}{-8-2D} cos\ 3x$$

- **Linear differential equations (second or higher order) with constant coefficients**

HW (It is a must event)

- **Non-homogeneous differential equation**

- **Case: 1: Sometimes, we cannot form $\phi\left(D^2\right)$. Then we will try to get $\phi\left(D^2, D\right)$ that is a function of $D\ and\ D^2$.**

In such case, we have to use the following procedures:

$$y_p = \frac{1}{-8-2D} \cos 3x = \frac{1}{-2}\frac{1}{D+4} \cos 3x$$

$$= \frac{1}{-2}\frac{D-4}{D^2-4^2} \cos 3x$$

$$= -\frac{1}{2}\ \frac{D-4}{-3^2-4^2} \cos 3x$$

$$= \frac{1}{50}(D-4)\cos 3x$$

- **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation**

- **Case: 1: Sometimes, we cannot form $\phi\left(D^2\right)$. Then we will try to get $\phi\left(D^2, D\right)$ that is a function of $D\ and\ D^2$.**

In such case, we have to use the following procedures:

$$y_p = \frac{1}{50}(D-4)cos\ 3x = \frac{1}{50}\left(-3sin3x - 4\ cos3x\right)$$ **(Ans.)**

➢ **Case 2: If $\phi\left(-a^2\right) = 0,$ then we will use the following formulae $\frac{1}{f(D)}e^{ax} = e^{ax}\frac{1}{f(D+a)}e^{0x}$ and we have to follow the procedure as presented in the next slide.**

- ☞ **Instructions:**

The following slides are given to all of you in advance for your convenience, and please check and try to understand the given slides (lecture martials) before joining the next class.

If you do so, then it would be easier for you to understand the next class. Besides, you will feel comfortable in asking relevant questions in the classroom, if any!

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**
- **Case 2: Let** $y_p = \frac{1}{D^2+a^2} \sin ax$

$$= \text{Imaginary Part of } \left(\frac{1}{D^2+a^2} \cos ax + i \frac{1}{D^2+a^2} \sin ax\right)$$

$$= \text{Imaginary Part of } \frac{1}{D^2+a^2} (\cos ax + i \sin ax$$

$$\therefore\ y_p = \text{Imaginary Part of } \frac{1}{D^2+a^2} e^{iax} ------ -(1)$$

But $\frac{1}{D^2+a^2} e^{iax} = e^{iax} \frac{1}{(D+ia)^2+a^2} e^{0x}$

$$= e^{iax} \frac{1}{D^2+2Dai} e^{0x} = e^{iax} \frac{1}{D(D+2ai)} e^{0x}$$

$$= e^{iax} \frac{1}{D} \frac{1}{(D+2ai)} e^{0x}$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**
- **Case 2: If $\phi\left(-a^2\right)=0$, then** $\frac{1}{f(D)}e^{ax}=e^{ax}\frac{1}{f(D+a)}e^{0x}$

Now, $\frac{1}{D^2+a^2}\,e^{iax}=e^{iax}\,\frac{1}{D}\,\frac{1}{(D+2ai)}\,e^{0x}=\;e^{iax}\,\frac{1}{D}\,\frac{1}{(0+2ai)}\,e^{0x}$

$$=\;e^{iax}\frac{1}{D}\,\frac{1}{2ai}\,1=\;e^{iax}\,\frac{x}{2ai}$$

$$=\frac{x}{2ai}\;e^{iax}=\frac{x}{2ai}\,(\cos ax+i\sin ax)$$

$$=\frac{x}{2ai}\cos ax+\frac{x}{2a}\sin ax$$

$$=-\frac{xi}{2a}\cos ax+\frac{x}{2a}\sin ax$$

Therefore, imaginary part of $\frac{1}{D^2+a^2}\,e^{iax}=-\frac{x}{2a}\cos ax$

Hence, we get from (1), $y_p=\;\frac{1}{D^2+a^2}\,sin\,ax=\;-\frac{x}{2a}\cos ax$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation**

➢ **Formulae IV:** $(a)\ y_p = \frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax$

$$(b)\ y_p = \frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax$$

Example: Solve $\frac{d^2y}{dx^2} + 4y = 8\cos 2x$

Solution: Let $y = e^{mx}$ **the trail solution of** $\frac{d^2y}{dx^2} + 4y = 0$. **Therefore, the corresponding auxiliary equation is** $m^2 + 4 = 0\ \therefore m = \pm 2i$

Therefore, the complementary solution of given equation is

$$y_c = c_1\cos 2x + c_2\sin 2x ------------------(1)$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**

Example: Solve $\frac{d^2y}{dx^2} + 4y = 8\,cos\,2x$

Solution: The particular integral, $y_p = \frac{1}{D^2+4} 8\,cos\,2x$

$$\Rightarrow y_p = 8\frac{1}{D^2+4}\,cos\,2x$$

$$[We\ know\ that, y_p = \frac{1}{D^2+a^2}\,cos\,ax = \frac{x}{2a}\sin \mathrm{ax}]$$

$$\therefore\ y_p = 8\frac{x}{2.\,2} sin\,2x = 2x\,sin\,2x$$

Therefore, the general solution of given equation is

$$y = y_c + y_p = c_1 cos2x + c_2 \sin 2x + 2\mathrm{x} \sin 2\mathrm{x}\ (\mathrm{Ans.})$$

- **Linear differential equations (second or higher order) with constant coefficients**
- **Non-homogeneous differential equation**

Exercise-15

Solve the following differential equations :

1. $(D^2 + 9)\, y = \cos 2x + \sin 2x.$ **Ans.** $y = c_1 \cos 3x + c_2 \sin 3x + (1/5)(\cos 2x + \sin 2x)$

2. $(D^3 + D^2 - D - 1)\, y = \cos 2x.$ **Ans.** $y = c_1 e^x + (c_2 + c_3 x)\, e^{-x} - (1/25)(2 \sin 2x + \cos 2x)$

3. $(D^2 - 5D + 6)\, y = \sin 3x.$ **Ans.** $y = c_1 e^{2x} + c_2 e^{3x} + (1/78)(5 \cos 3x - \sin 3x)$

4. $(D^2 + D + 1)\, y = \sin 2x.$ **Ans.** $y = e^{-x/2} \left\{ c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2) \right\} - (1/13)(2\cos 2x + 3\sin 2x)$

- Linear differential equations (second or higher order) with constant coefficients

➢ Non-homogeneous differential equation

- Suppose $y_p = \frac{1}{f(D)} X$ where X is the form of $e^{ax}\,V$ where V is any function of x.

➢ Working rule for finding y_p when X is the form of of $e^{ax}\,V$

- Formulae V : $y_p = \frac{1}{f(D)} e^{ax}\,V = e^{ax}\,\frac{1}{f(D+a)} V$ where $\frac{1}{f(D+a)} V$ can be calculated by the methods which have already discussed in the previous section.

➢ Example: Solve : $\left(D^2 + 3D + 2\right)y = e^{2x} sinx$

Solution: Let $y = e^{mx}$ the trail solution of $\left(D^2 + 3D + 2\right)y = 0$. Then the corresponding auxiliary is $m^2 + 3m + 2 = 0$

$$\Rightarrow m^2 + 2m + \mathrm{m} + 2 = 0$$

$$\Rightarrow m(m + 2) + 1(m + 2) = 0 \Rightarrow (m + 2)\,(m + 1) = 0 \;\therefore m = -1, -2$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**
- **Example: Solve :** $(D^2 + 3D + 2)y = e^{2x} sinx$

Solution: Since $m = -1, -2,$ the complementary solution of the given equation is

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

Now, the particular integral, $y_p = \frac{1}{D^2+3D+2} e^{2x} sinx$

$$= e^{2x} \frac{1}{(D+2)^2+3(D+2)+2} sinx$$

$$= e^{2x} \frac{1}{D^2+4D+4+3D+6+2} sinx$$

$$= e^{2x} \frac{1}{-1^2+4D+4+3D+6+2} sinx$$

$$= e^{2x} \frac{1}{11+7D} sinx$$

Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation**
- **Example: Solve :** $\left(D^2+3D+2\right)y = e^{2x} sinx$

Solution: Now, the particular integral, $y_p = e^{2x}\dfrac{1}{11+7D} sinx$

$$= e^{2x}\frac{11-7D}{(11+7D)(11-7D)} sinx$$

$$= e^{2x}\frac{11-7D}{(121-49D^2)} sinx$$

$$= e^{2x}\frac{11-7D}{(121+49)} sinx$$

$$= \frac{e^{2x}}{170}\,(11-7D)\, sinx$$

$$= \frac{e^{2x}}{170}\,(11sinx-7cosx)$$

Therefore, the general solution is $y = y_c + y_p$

$$= C_1e^{-x} + C_2e^{-2x} + \frac{e^{2x}}{170}\,(11sinx-7cosx)$$

- Linear differential equations (second or higher order) with constant coefficients
- Non-homogeneous differential equation

Exercise-16

Solve *the following differential equations:*

1. $(D^2 - 2D + 4)\, y = e^x \cos x$ **Ans.** $y = e^x\,[c_1 \cos(x\sqrt{3}\,) + c_2 \sin(x\sqrt{3}\,)] + (1/2)\, e^x \cos x$
2. $(D + 1)^3\, y = x^2 e^{-x}$ **Ans.** $y = (c_1 + c_2 x + c_3 x^2)\, e^{-x} + (x^5/60) \times e^{-x}$
3. $(D^2 - 2D + 5)\, y = e^{2x} \sin x$ **Ans.** $y = e^x\,(c_1 \cos 2x + c_2 \sin 2x) - (1/10)\, e^{2x}\,(\cos x - 2 \sin x)$
4. $(D^2 - 1)\, y = e^x \cos x.$ **Ans.** $y = c_1 e^x + c_2 e^{-x} + (1/5)\, e^x\,(2 \sin x - \cos x)$

No need to solve.

- Linear differential equations (second or higher order) with constant coefficients

- Non-homogeneous differential equation: $f(D)y = X\ [X = \phi(x) = F(x)]$

- Method of Variation of Parameters for finding solution of $f(D)y = X$ (for Second order linear differential equation) where $D \equiv \frac{d}{dx}$

- Working rule for solving $f(D)y = X$ by the Method of Variation of Parameters

<u>Step 1</u>: Reduce the general form of non-homogeneous linear differential equation with constant coefficients: $f(D)y = X$ to 2nd order differential equation and re-write it as $y_2 + Py_1 + Qy = R$ ———————————— $-(1)$

(where P and Q are constant and R is a function of x) in which the coefficient of y_2 must be unity.

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving: $y_2 + Py_1 + Qy = R$ *by the* Method of Variation of Parameters**

Step 2. Consider $$y_2 + Py_1 + Qy = 0 \qquad \text{... (2)}$$

which is obtained by taking $R = 0$ in (1). Solve (2) by methods **discussed in the previous lectures**

Let the general solution of (2) *i.e.*, C.F. of (1) be

$$\text{C.F.} = C_1 u + C_2 v, \; C_1, C_2 \text{ being arbitrary constants} \qquad \text{... (3)}$$

Step 3. General solution of (1) is $$y = \text{C.F.} + \text{P.I.} \qquad \text{... (4)}$$

where $$\text{C.F.} = C_1 u + C_2 v, \; C_1, C_2 \text{ being arbitrary constants} \qquad \text{... (5)}$$

and $$\text{P.I.} = u\, f(x) + v\, g(x) \qquad \text{... (6)}$$

where $$f(x) = -\int \frac{vR}{W}\, dx \quad \text{and} \quad g(x) = \int \frac{uR}{W}\, dx, \qquad \text{... (7)}$$

where $$W = \text{Wronskian of } u \text{ and } v = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = uv_1 - u_1 v \qquad \text{... (8)}$$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters**

Example 1. *Apply the method of variation of parameters to solve*

$(i)\ y_2+n^2y=sec\ nx$;

$(ii)\ y_2+y=sec\ x$

$(iii)\ y_2+4y=sec\ 2x$;

$(iv)\ y_2+9y=sec\ 3x$

Solution: (*i*) Given $y_2+n^2y=\sec nx$ ... (1)

Comparing (1) with $y_2+Py_1+Q\,y=R$, we have $R=\sec nx$

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Working rule for solving:** $y_2 + Py_1 + Qy = R$ *by the* **Method of Variation of Parameters**

Solution $(i)\ y_2 + n^2 y = sec\ nx$

Consider $y_2 + n^2 y = 0$ or $(D^2 + n^2)\, y = 0$, where $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $m^2 + n^2 = 0$ so that $m = \pm\, in$.

C.F. of (1) $= C_1 \cos nx + C_2 \sin nx$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos nx$, $v = \sin nx$ Also, here $R = \sec nx$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos nx & \sin nx \\ -n \sin nx & n \cos nx \end{vmatrix} = n \neq 0 \qquad \text{... (5)}$$

Then, P.I. of (1) $= u\, f(x) + v\, g(x)$ ... (6)

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving: $y_2 + Py_1 + Qy = R$ *by the* Method of Variation of Parameters**

Solution $(i)\ y_2 + n^2 y = sec\ nx$

Then, P.I. of (1) $= u\, f(x) + v\, g(x)$ ... (6)

$u = \cos nx$
$v = \sin nx$
$R = \sec nx$

where $f(x) = -\int \frac{vR}{W} dx = -\int \frac{\sin nx \sec nx}{n} dx = \frac{1}{n^2} \log \cos nx$, by (4) and (5)

$g(x) = \int \frac{uR}{W} dx = \int \frac{\cos nx \quad \sec nx}{n} dx = \frac{x}{n}$, *by* (4) and (5)

$\therefore$ P.I. of (1) $= (\cos nx) \times (1/n^2) \log \cos nx + (\sin nx) \times (x/n)$, by (6)

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $y = C_1 \cos nx + C_2 \sin nx + (1/n^2) \times \cos nx \log \cos nx + (x/n) \times \sin nx$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters**

Solution: (*ii*) $y_2+y=sec\ x$

(*iii*) $y_2+4y=sec\ 2x$

(*iv*) $y_2+9y=sec\ 3x$

The general solution of (1) is $y=$ C.F. + P.I.

$y=C_1\cos nx+C_2\sin nx+(1/n^2)\times\cos nx\log\cos nx+(x/n)\times\sin nx$

$\boxed{(i)\ y_2+n^2y=sec\ nx}$

(*ii*) Compare it with part (*i*). Here $n=1$. Now do as in part (*i*).

The required solution is $y=C_1\cos x+C_2\sin x+\cos x\log\cos x+x\sin x$.

(*iii*) Proceed as in part (*i*). Note that here $n=2$.

Ans. $y=C_1\cos 2x+C_2\sin 2x+(1/4)\times\cos 2x\log\cos 2x+(x/2)\times\sin 2x$

(*iv*) Proceed as in part (*i*). Note that here $n=3$.

Ans. $y=C_1\cos 3x+C_2\sin 3x+(1/9)\times\cos 3x\log\cos 3x+(x/3)\times\sin 3x$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving: $y_2 + Py_1 + Qy = R$ *by the* Method of Variation of Parameters**

Example 2. *Apply the method of variation of parameters to solve*

(*i*) $y_2 + a^2 y = cosec\ ax$

(*ii*) $y_2 + y = cosec\ x$

(*iii*) $y_2 + 9y = cosec\ 3x$

Solution: (*i*) Given $y_2 + a^2 y = \operatorname{cosec} ax$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\,y = R$, we have $R = \operatorname{cosec} ax$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters**

Solution: (i) $y_2 + a^2y = cosec\ ax$

Consider $y_2 + a^2y = 0$ or $(D^2 + a^2)\,y = 0,\quad D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $m^2 + a^2 = 0$ so that $m = \pm\, ai$

$\therefore$ C.F. of (1) $= C_1 \cos ax + C_2 \sin ax$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos ax$, $v = \sin ax$. Also, here $R = \operatorname{cosec} ax$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \neq 0 \qquad \text{... (5)}$$

Then, P.I. of (1) $= u\,f(x) + v\,g(x)$, ... (6)

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters**

Solution: $(i)\ y_2 + a^2 y = cosec\ ax$

where

$u = \cos ax$
$v = \sin ax$
$R = \text{cosec } ax$

$$f(x) = -\int \frac{vR}{W} dx = -\int \frac{\sin ax \text{ cosec } ax}{a} dx = -\frac{x}{a}, \text{ by (4) and (5)}$$

$$g(x) = \int \frac{uR}{W} dx = \int \frac{\cos ax \text{ cosec } ax}{a} dx = (1/a^2) \times \log \sin ax, \text{ by (4) and (5)}$$

$\therefore$ P.I. of (1) $= (\cos ax) \times (-x/a) + (\sin ax) \times (1/a^2) \times \log \sin ax$, by (6)

P.I. of (1) $= u\,f(x) + v\,g(x)$.

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $y = C_1 \cos ax + C_2 \sin ax - (x/a) \times \cos ax + (1/a^2) \times \sin ax \log \sin ax$

Linear differential equations (second or higher order) with constant coefficients

- **Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters**

$\boxed{(i)\ y_2+a^2y=cosec\ ax}$

The general solution of (1) is $y=\text{C.F.}+\text{P.I.}$

i.e., $y=C_1\cos ax+C_2\sin ax-(x/a)\times\cos ax+(1/a^2)\times\sin ax\log\sin ax$

Solution: $(ii)\ y_2+y=cosec\ x$

$(iii)\ y_2+9y=cosec\ 3x$

(*ii*) Proceed as in part (*i*). Note that here $a=1$.

Ans. $y=C_1\cos x+C_2\sin x-x\cos x+\sin x\log\sin x$

(*iii*) Proceed as in part (*i*). Note that have $a=3$

Ans. $y=C_1\cos 3x+C_2\sin 3x-(x/3)\times\cos 3x+(1/9)\times\sin 3x\log\sin 3x$

- **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters**

Example:4 *Solve $d^2y/dx^2 - 2(dy/dx) = e^x \sin x$ by using the method of variation of parameters.*

Solution: Given $(D^2 - 2D)\,y = e^x \sin x$, where $D \equiv d/dx$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\,y = R$, here $R = e^x \sin x$

Consider $(D^2 - 2D)\,y = 0$ ... (2)

Auxiliary equation of (2) is $\mathbf{m}^2 - 2 = 0$ so that $\mathbf{m} = 0, 2.$

C.F. of (1) $= C_1 + C_2\,e^{2x}$, C_1 and C_2 being arbitrary constants. ... (3)

Let $u = 1$ and $v = e^{2x}$. Also, here $R = e^x \sin x$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x} \neq 0$$ ... (5)

Then, P.I. of (1) $= u\,f(x) + v\,g(x)$, ... (6)

■ Linear differential equations (second or higher order) with constant coefficients: Working rule for solving: $y_2 + Py_1 + Qy = R$ *by the* Method of Variation of Parameters

Example:4 *Solve* $d^2y/dx^2 - 2(dy/dx) = e^x \sin x$ *by using the method of variation of parameters.*

Solution: where

$$\boxed{u = 1 \quad v = e^{2x}. \quad R = e^x \sin x}$$

$$f(x) = -\int \frac{vR}{W}dx = -\int \frac{e^{2x}e^x \sin x}{2e^{2x}} = -\frac{1}{2}\int e^x \sin x\, dx, \text{ by (4) and (5)}$$

$$= -\frac{1}{2}\frac{e^x}{1^2+1^2}(\sin x - \cos x), \text{as} \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)$$

$$= -(1/4) \times e^x (\sin x - \cos x)$$

and

$$g(x) = \int \frac{uR}{W}dx = \int \frac{e^x \sin x}{2e^{2x}}dx = \frac{1}{2}\int e^{-x} \sin dx, \text{by (4) and (5)}$$

$$= \frac{1}{2}\frac{e^{-x}}{(-1)^2+1^2}\{(-1)\sin x - \cos x\} = -\frac{e^{-x}}{4}(\sin x + \cos x)$$

$$\boxed{\text{P.I. of (1)} = u\, f(x) + v\, g(x).}$$

$\therefore$ P.I. of (1) $= -(1/4) \times e^x(\sin x - \cos x) + e^{2x} \times (-1/4) \times e^{-x}(\sin x + \cos x)$, by (6)

$= -(1/4) \times e^x \{(\sin x - \cos x) + (\sin x + \cos x)\} = -(1/2) \times e^x \sin x$

Hence the required general solution is $y = \text{C.F} + \text{P.I.}$

i.e., $y = C_1 + C_2 e^{2x} - (1/2) \times e^x \sin x$, C_1, C_2 being arbitrary constants.

■ **Linear differential equations (second or higher order) with constant coefficients: Working rule for solving: $y_2+Py_1+Qy=R$ by the Method of Variation of Parameters**

Exercise-18

1. *Apply the method of variation of parameters to solve the equations:*

(*i*) $y_2-2y_1+y=e^x$ **Ans.** $y=(C_1+C_2x)\,e^x+(x^2/2)\times e^x$

(*ii*) $y_2-2y_1+y=(1/x^3)\,e^x$ **Ans.** $y=(c_1+c_2x)\,e^x-(1/2x)\times e^x$

(*iii*) $y_2+a^2y=\cos ax$ **Ans.** $y=C_1\cos ax+C_2\sin ax+(x/2a)\times\sin ax$

(*iv*) $y_2+4y=\sin x$ **Ans.** $y=C_1\cos 2x+C_2\sin 2x+(1/3)\times\sin x$

No need to solve

Thank you for your attendance and attention