Fall 2023 (Final)

Direction of Ratio

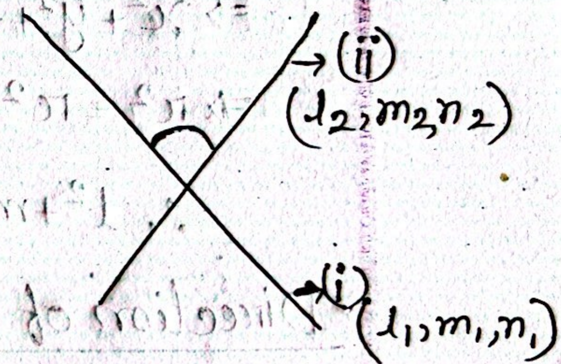
- direction of ratio (d.r)

  $(x_2-x_1), y_2-y_1, z_2-z_1)$

- direction of cosine (d.c)

$$\left(\frac{(x_2-x_1)}{\sqrt{(x_2-x)^2+(y_2-y_1)^2+(z_2-z_1)^2}}, \frac{(y_2-y_1)}{(\text{Same as prev})}, \frac{(z_2-z_1)}{(\text{Same as previous})}\right)$$

$$\cos\theta = l_1l_2+m_1m_2+n_1n_2$$



- Prove: $l^2+m^2+n^2=1$

  $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

  $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

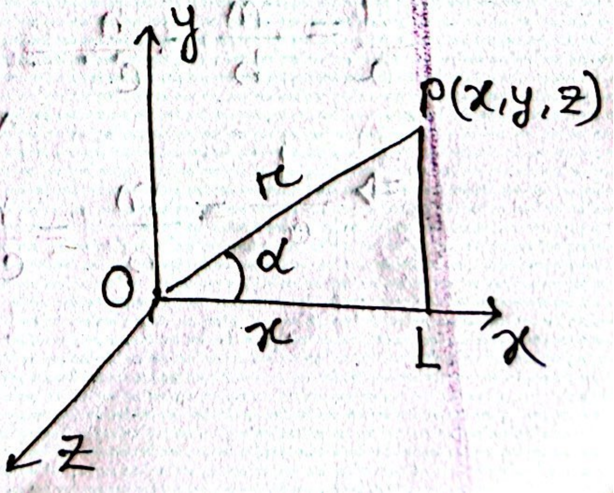
Direction of cosine (d.c)

*Any line makes angles with the positive direction of axis, these show that $l^2+m^2+n^2=1$ / The sum of square of the direction of cosine of every line is 1 / $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ / $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$

Solution:

$x \to \alpha$
$y \to \beta$
$z \to \gamma$

$\cos\alpha \to l$
$\cos\beta \to m$
$\cos\gamma \to n$



(l, m, n) be the direction of cosine

$$\cos\alpha = \frac{OL}{OP} = \frac{x}{r}$$

$$\therefore x = r\cos\alpha = lr$$

Similarly

$$y = mr$$

$$z = nr$$

We know, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$|r| = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow r^2 = x^2+y^2+z^2 \text{ ------ (i)}$$

$$\Rightarrow x^2+y^2+z^2 = r^2(l^2+m^2+n^2)$$

$$\Rightarrow r^2 = r^2(l^2+m^2+n^2)$$

$$\therefore l^2+m^2+n^2 = 1 \text{ (Proved)}$$

- **Direction of Ratio (d.r)**

(a, b, c)

$a \propto l$, $b \propto m$, $c \propto n$

$a = Kl$, $b = Km$, $c = Kn$

$$\frac{l}{a} = \frac{1}{K}, \quad \frac{m}{b} = \frac{1}{K}, \quad \frac{n}{c} = \frac{1}{K}$$

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow \frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{1}{\sqrt{a^2+b^2+c^2}}$$

So,

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}};$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}};$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}};$$

d.r (a, b, c)

d.c $\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$

• So, $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

d.r $(x_2-x_1, y_2-y_1, z_2-z_1)$

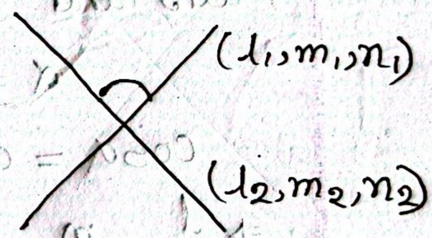
d.c $\left(\frac{x_2-x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}, \frac{y_2-y_1}{''}, \frac{z_2-z_1}{''}\right)$

Projection of AB $= (x_2-x_1)l + (y_2-y_1)m + (z_2-z_1)n$

Angle between two lines -

$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

Parallel হলে $\theta = 0°$



(1) Find the angle between the lines ~~which~~ whose direction of ratio are $(3, 1, 2)$ and $(2, -2, 4)$. Find the angle between the lines?

Ans: dc of dr(1) $= \left(\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$

dc of dr(2) $= \left(\frac{2}{\sqrt{24}}, \frac{-2}{\sqrt{24}}, \frac{4}{24}\right)$

$\cos\theta = \frac{\sqrt{3}}{\sqrt{7}}$

$\therefore \theta = \cos^{-1}\left(\sqrt{\frac{3}{7}}\right)$

(2) Given four points A, B, C, D whose co-ordinates are $(6, -6, 0)$, $(-1, -7, 6)$, $(3, -4, 4)$ and $(2, -9, 2)$. Show that, AB is Perpendicular to CD.

Ans: $\theta = 90°$

③ Show that the lines from $A(5,2,-3)$ to $B(6,1,4)$ and from $C(-3,-2,-1)$ to $D(-1,4,13)$ are parallel.

④ If the point, P and Q are given by $(2,3,4)$ and $(1,1,-1)$ respectedly, Find the angle between OP and OQ where $O(0,0,0)$. <u>Ans:</u> $\theta = \cos^{-1}\left(1/\sqrt{29}\sqrt{3}\right)$

⑤ Find the direction of cosine of the line which is equally inclined to the axis (সমান কোণ তৈরী করে).

<u>Ans:</u> Let the line OP make angles (α, β, γ) with X Axis, Y axis and Z axis. According to the question,

$$\alpha = \beta = \gamma$$

$$\cos\alpha = \cos\beta = \cancel{\cos}\cos\gamma$$

$$\Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{1} = \pm\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1^2+1^2+1^2}}$$

$$\Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{1} = \pm\frac{1}{\sqrt{3}} \quad \therefore l, m, n = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$$

⑥ A line makes angle $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube. Prove that, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$

⑦ Same question. Prove, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = 8/3$

⑧ Prove that, the angle between two diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

~~Show that, the lines~~ given 4 points, A, B, C, D whose co-ordinates are $(6,-6,0)$, $(-1,-7,6)$, $(3,-4,4)$, $(2,-9,2)$ respectively. Show that, AB is perpendicular to CD.

$\boxed{AB} \rightarrow A(6,-6,0),\ B(-1,-7,6)$

$dr(-1-6, -7+6, 6-0)$

$\Rightarrow \therefore dr(-7,-1,6)$

$$dc\left(\frac{-7}{\sqrt{49+1+36}}, \frac{-1}{\sqrt{49+1+36}}, \frac{6}{\sqrt{49+1+36}}\right) = \left(\frac{-7}{\sqrt{86}}, \frac{-1}{\sqrt{86}}, \frac{6}{\sqrt{86}}\right)$$

$\boxed{CD} \rightarrow C(2-3, -9+4, 2-4)$

$dr(-1,-5,-2)$

$$dc\left(\frac{-1}{\sqrt{1+25+4}}, \frac{-5}{\sqrt{1+25+4}}, \frac{-2}{\sqrt{1+25+4}}\right) = \left(\frac{-1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}, \frac{-2}{\sqrt{30}}\right)$$

θ be the angle between the lines AB and CD.

$$\cos\theta = (l_1 l_2 + m_1 m_2 + \cancel{m}\, n_1 n_2)$$

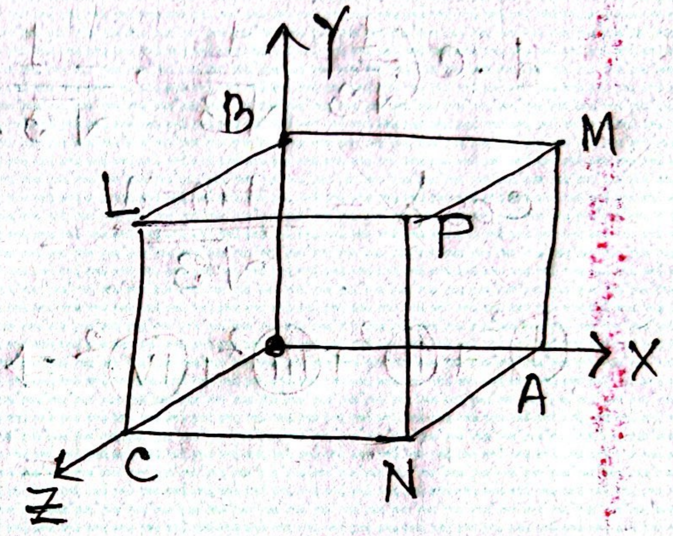
$$= \left(\frac{-7}{\sqrt{86}} \times \frac{-1}{\sqrt{30}}\right) + \left(\frac{-1}{\sqrt{86}} \times \frac{-5}{\sqrt{30}}\right) + \left(\frac{6}{\sqrt{86}} \times \frac{-2}{\sqrt{30}}\right)$$

$\cos\theta = 0$

$\therefore \theta = 90^\circ$, Since $\theta = 90^\circ$, AB ~~and~~ is perpendicular to CD.

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = 8/3$$

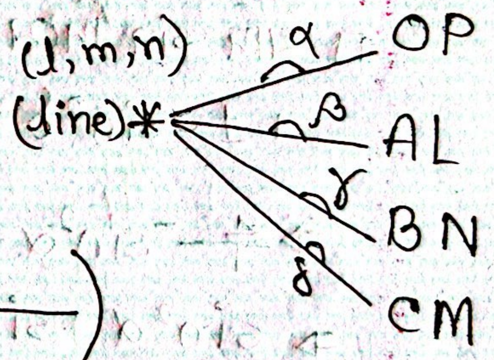
□ A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Prove that, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.



Diagonal,

OP, AL, BN, CM

$O(0,0,0)$, $P(a,a,a)$,
$A(a,0,0)$, $B(0,a,0)$,
$C(0,0,a)$, $L(0,a,a)$,
$M(a,a,0)$, $N(a,0,a)$



$\boxed{OP}$ d.r (a, a, a)

d.c $\left(\frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{a}{\sqrt{3}a}, \frac{a}{\sqrt{3}a}\right)$

$\therefore$ dc $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\cos\alpha = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{l+m+n}{\sqrt{3}} \text{ ---- (I)}$$

$\boxed{AL}$ d.r $(-a, a, a)$

d.c $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\cos\beta = \frac{-l+m+n}{\sqrt{3}} \text{ —— (II)}$$

$\boxed{BN}$ d.r $(a, -a, a)$

d.c $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\cos\gamma = \frac{l-m+n}{\sqrt{3}} \text{ —— (III)}$$

$\boxed{CM}$ $C(0,0,a)$, $M(a,a,0)$

$dr(a,a,-a)$

$d.c\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

$\cos\delta = \frac{l+m-n}{\sqrt{3}}$ ------ (IV)

(I)^2+(II)^2+(III)^2+(IV)^2 $\Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$

$= 4/3$

□ যদি sin দিয়ে থাকে তাহলে, $1-\sin^2\alpha+1-\sin^2\beta+1-\sin^2\gamma$

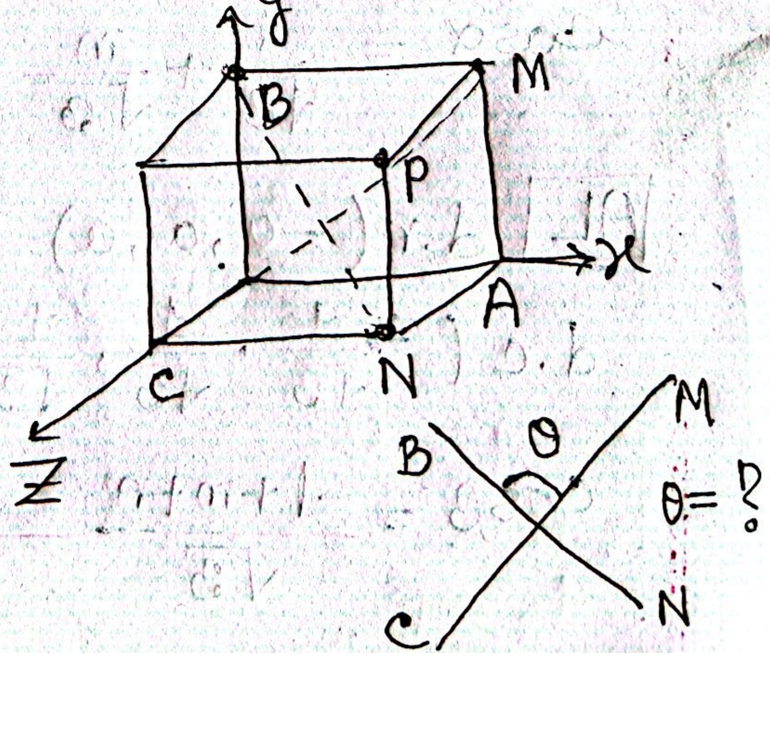
$+1-\sin^2\delta = \frac{4}{3}$

$\Rightarrow \cancel{4} - \sin^2\alpha-\sin^2\beta-\sin^2\gamma-\sin^2\delta = -4+\frac{4}{3}$

$\Rightarrow \sin^2\alpha+\sin^2\beta+\sin^2\gamma+\sin^2\delta = 4-\frac{4}{3}$

$\Rightarrow \sin^2\alpha+\sin^2\beta+\sin^2\gamma+\sin^2\delta = 8/3$ (Answer)

□ Prove that, the angle between two diagonals of a cube is $\cos^{-1} 1/3$



$B(0,a,0)$, $N(a,0,a)$

$C(0,0,a)$, $M(a,a,0)$

$\boxed{BN}$ $dr(a,-a,a)$

$d.c\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\boxed{CM}$ d.r $(a, a, -a)$

d.c $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

Let θ be the angle between the diagonals BN and CM

$\cos\theta = \left(-\frac{1}{3} - \frac{1}{3} - \frac{1}{3}\right)$

$\cos\theta = -\frac{1}{3}$ $\therefore \theta = \cos^{-1}\left(\frac{1}{3}\right)$