- Course Title : Differential Equations and Special Functions
- Course Code: MAT102 (Section-3) // Lecture-13 (March 29, 2023)
- **Today's Lecture Topics:**

- **Ordinary Differential Equations**

- Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$
- Introduction of Partial differential equations (PDEs)
- Definition of PDE and Order and degree of PDEs
- Linear and non-linear PDEs, Notations used for PDEs and Classification of first order PDEs
- Rule: Derivation of a PDE by the elimination of arbitrary constants

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Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation**: $f(D)y = X = \phi(x) = F(x)$
- **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$ and $D \equiv \frac{d}{dx}$**

SIMPLE FORM OF THE METHOD

The *method of undetermined coefficients* is applicable only if $\phi(x)$ and *all* of its derivatives can be written in terms of **a** *finite* set of linearly independent functions, which we denote by $\{y_1(x), y_2(x), \ldots, y_n(x)\}$. The method is initiated by assuming a particular solution of the form

$$y_p(x) = A_1 y_1(x) + A_2 y_2(x) + \cdots + A_n y_n(x)$$

where $A_1, A_2, \ldots, A_n$ denote arbitrary multiplicative constants. These arbitrary constants are then evaluated by substituting the proposed solution into the given differential equation and equating the coefficients of like terms.

■ Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation**: $f(D)y = X = \phi(x) = F(x)$
- **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$ and $D \equiv \frac{d}{dx}$**

Case 1. $\phi(x) = p_n(x)$, an nth-degree polynomial in x. Assume a solution of the form

$$y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0 \tag{11.1}$$

where A_j ($j = 0, 1, 2, \ldots, n$) is a constant to be determined.

Case 2. $\phi(x) = ke^{\alpha x}$ where k and α are known constants. Assume a solution of the form

$$y_p = Ae^{\alpha x} \tag{11.2}$$

where A is a constant to be determined.

■ Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation**: $f(D)y = X = \phi(x) = F(x)$
- **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$ and $D \equiv \frac{d}{dx}$**

Case 3. $\phi(x) = k_1 \sin \beta x + k_2 \cos \beta x$ where k_1, k_2, and β are known constants. Assume a solution of the form

$$y_p = A \sin \beta x + B \cos \beta x \tag{11.3}$$

where A and B are constants to be determined.

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Non-homogeneous differential equation**: $f(D)y = X = \phi(x) = F(x)$

➢ **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$ and $D \equiv \frac{d}{dx}$**

GENERALIZATIONS

If $\phi(x)$ is the product of terms considered in Cases 1 through 3, take y_p to be the product of the corresponding assumed solutions and algebraically combine arbitrary constants where possible. In particular, if $\phi(x) = e^{\alpha x}p_n(x)$ is the product of a polynomial with an exponential, assume

$$y_p = e^{\alpha x}(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0) \tag{11.4}$$

where A_j is as in Case 1. If, instead, $\phi(x) = e^{\alpha x}p_n(x)\sin\beta x$ is the product of a polynomial, exponential, and sine term, or if $\phi(x) = e^{\alpha x}p_n(x)\cos\beta x$ is the product of a polynomial, exponential, and cosine term, then assume

$$y_p = e^{\alpha x}\sin\beta x\,(A_n x^n + \cdots + A_1 x + A_0) + e^{\alpha x}\cos\beta x\,(B_n x^n + \cdots + B_1 x + B_0) \tag{11.5}$$

where A_j and B_j $(j = 0, 1, \ldots, n)$ are constants which still must be determined.

If $\phi(x)$ is the sum (or difference) of terms already considered, then we take y_p to be the sum (or difference) of the corresponding assumed solutions and algebraically combine arbitrary constants where possible.

■ Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation: $f(D)y = X = \phi(x) = F(x)$**
- **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$ and $D \equiv \frac{d}{dx}$**

MODIFICATIONS

If any term of the assumed solution, disregarding multiplicative constants, is also a term of y_h (the homogeneous solution), then the assumed solution must be modified by multiplying it by x^m, where m is the smallest positive integer such that the product of x^m with the assumed solution has no terms in common with y_h.

LIMITATIONS OF THE METHOD

In general, if $\phi(x)$ is not one of the types of functions considered above, or if the differential equation *does not have constant coefficients*

Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation**: $f(D)y = X = \phi(x) = F(x)$
- **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$ and $D \equiv \frac{d}{dx}$**

Case 1. $\phi(x) = p_n(x)$**, an** n**th-degree polynomial in** x**.**

Assume a solution of the form $y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$ *(11.1)*

Example:1 Solve $y'' - y' - 2y = 4x^2$.

Solution:

$y_h = c_1 e^{-x} + c_2 e^{2x}$. Here $\phi(x) = 4x^2$, a second-degree polynomial. Using *(11.1)*, we assume that

$$y_p = A_2 x^2 + A_1 x + A_0 \tag{1}$$

Thus, $y_p' = 2A_2 x + A_1$ and $y_p'' = 2A_2$. Substituting these results into the differential equation, we have

$$2A_2 - (2A_2 x + A_1) - 2(A_2 x^2 + A_1 x + A_0) = 4x^2$$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation:** $f(D)y = X = \phi(x) = F(x)$

➢ **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

Example:1 Solve $y'' - y' - 2y = 4x^2$.

Solution: or, equivalently,

$$(-2A_2)x^2 + (-2A_2 - 2A_1)x + (2A_2 - A_1 - 2A_0) = 4x^2 + (0)x + 0$$

Equating the coefficients of like powers of x, we obtain

$$-2A_2 = 4 \qquad -2A_2 - 2A_1 = 0 \qquad 2A_2 - A_1 - 2A_0 = 0$$

Solving this system, we find that $A_2 = -2$, $A_1 = 2$, and $A_0 = -3$. Hence (*1*) becomes

$$y_p = -2x^2 + 2x - 3$$

and the general solution is

$$y = y_h + y_p = c_1e^{-x} + c_2e^{2x} - 2x^2 + 2x - 3$$

■ Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation:** $f(D)y = X = \phi(x) = F(x)$
- **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

Case 2. $\phi(x) = ke^{\alpha x}$ **where** k **and** α **are known constants.**

Assume a solution of the form $\quad y_p = Ae^{\alpha x} \quad$ (11.2)

Example 2: Solve $y'' - y' - 2y = e^{3x}$.

Solution:

$y_h = c_1 e^{-x} + c_2 e^{2x}$. Here $\phi(x)$ has the form displayed in Case 2 with $k = 1$ and $\alpha = 3$. Using (11.2), we assume that

$$y_p = Ae^{3x} \tag{1}$$

Thus, $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substituting these results into the differential equation, we have

$$9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} = e^{3x} \quad \text{or} \quad 4Ae^{3x} = e^{3x}$$

It follows that $4A = 1$, or $A = \frac{1}{4}$, so that (1) becomes $y_p = \frac{1}{4}e^{3x}$. The general solution then is

$$y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{4}e^{3x}$$

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Non-homogeneous differential equation:** $f(D)y = X = \phi(x) = F(x)$

➢ **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

Case 3. $\phi(x) = k_1 \sin \beta x + k_2 \cos \beta x$ **where** k_1, k_2, **and** β **are known constants.**

Assume a solution of the form $y_p = A \sin \beta x + B \cos \beta x \qquad (11.3)$

Example 3: Solve $y'' - y' - 2y = \sin 2x$.

Solution:

$y_h = c_1 e^{-x} + c_2 e^{2x}$. Here $\phi(x)$ has the form displayed in Case 3 with $k_1 = 1$, $k_2 = 0$, and $\beta = 2$. Using (11.3), we assume that

$$y_p = A \sin 2x + B \cos 2x \qquad (1)$$

Thus, $y_p' = 2A \cos 2x - 2B \sin 2x$ and $y_p'' = -4A \sin 2x - 4B \cos 2x$. Substituting these results into the differential equation, we have

$$(-4A \sin 2x - 4B \cos 2x) - (2A \cos 2x - 2B \sin 2x) - 2(A \sin 2x + B \cos 2x) = \sin 2x$$

or, equivalently,

$$(-6A + 2B) \sin 2x + (-6B - 2A) \cos 2x = (1) \sin 2x + (0) \cos 2x$$

- **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation:** $\boldsymbol{f(D)y = X = \phi(x) = F(x)}$

➢ **Method of undetermined coefficient for finding particular integral of** $\boldsymbol{f(D)y = X = \phi(x) = F(x)}$ **and** $\boldsymbol{D \equiv \frac{d}{dx}}$

Example 3: Solve $y'' - y' - 2y = \sin 2x$.

Solution:

Equating coefficients of like terms, we obtain

$$-6A + 2B = 1 \qquad -2A - 6B = 0$$

Solving this system, we find that $A = -3/20$ and $B = 1/20$. Then from (*1*),

$$y_p = -\frac{3}{20}\sin 2x + \frac{1}{20}\cos 2x$$

and the general solution is $$y = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{20}\sin 2x + \frac{1}{20}\cos 2x$$

Linear differential equations (second or higher order) with constant coefficients

- Non-homogeneous differential equation: $f(D)y = X = \phi(t) = F(t)$
- Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(t) = F(t)$ and $D \equiv \frac{d}{dt}$

Case 1. $\phi(x) = p_n(x)$, **an nth-degree polynomial in x.**

Assume a solution of the form $y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$ (11.1)

Example: 5 Solve $\ddot{y} - 6\dot{y} + 25y = 50t^3 - 36t^2 - 63t + 18.$

Solution:

$$y_h = c_1 e^{3t}\cos 4t + c_2 e^{3t}\sin 4t$$

Here $\phi(t)$ is a third-degree polynomial in t. Using (11.1) with t replacing x, we assume that

$$y_p = A_3t^3 + A_2t^2 + A_1t + A_0 \tag{1}$$

Consequently,

$$\dot{y}_p = 3A_3t^2 + 2A_2t + A_1$$

and

$$\ddot{y}_p = 6A_3t + 2A_2$$

Substituting these results into the differential equation, we obtain

$$(6A_3t + 2A_2) - 6(3A_3t^2 + 2A_2t + A_1) + 25(A_3t^3 + A_2t^2 + A_1t + A_0) = 50t^3 - 36t^2 - 63t + 18$$

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Non-homogeneous differential equation**: $f(D)y = X = \phi(t) = F(t)$

➢ **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(t) = F(t)$ and $D \equiv \frac{d}{dt}$**

Example: 5 Solve $\ddot{y} - 6\dot{y} + 25y = 50t^3 - 36t^2 - 63t + 18.$

Solution:

or, equivalently,

$$(25A_3)t^3 + (-18A_3 + 25A_2)t^2 + (6A_3 - 12A_2 + 25A_1) + (2A_2 - 6A_1 + 25A_0) = 50t^3 - 36t^2 - 63t + 18$$

Equating coefficients of like powers of t, we have

$$25A_3 = 50; \quad -18A_3 + 25A_2 = -36; \quad 6A_3 - 12A_2 + 25A_1 = -63; \quad 2A_2 - 6A_1 + 25A_0 = 18$$

Solving these four algebraic equations simultaneously, we obtain $A_3 = 2$, $A_2 = 0$, $A_1 = -3$, and $A_0 = 0$, so that (*1*) becomes

$$y_p = 2t^3 - 3t$$

The general solution is

$$y = y_h + y_p = c_1e^{3t} \cos 4t + c_2e^{3t} \sin 4t + 2t^3 - 3t$$

Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation**: $f(D)y = X = \phi(x) = F(x)$
- **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$ and $D \equiv \frac{d}{dx}$**

$$\boxed{y_p = e^{\alpha x}(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0) \qquad (11.4)}$$

Example: 6 Solve $y''' - 6y'' + 11y' - 6y = 2xe^{-x}$.

Solution: $y_h = c_1e^x + c_2e^{2x} + c_3e^{3x}$. Here $\phi(x) = e^{\alpha x}p_n(x)$, where $\alpha = -1$ and $p_n(x) = 2x$, a first-degree polynomial. Using Eq. (*11.4*), we assume that $y_p = e^{-x}(A_1x + A_0)$, or

$$y_p = A_1xe^{-x} + A_0e^{-x} \qquad (1)$$

Thus,

$$y_p' = -A_1xe^{-x} + A_1e^{-x} - A_0e^{-x}$$

$$y_p'' = A_1xe^{-x} - 2A_1e^{-x} + A_0e^{-x}$$

$$y_p''' = -A_1xe^{-x} + 3A_1e^{-x} - A_0e^{-x}$$

Substituting these results into the differential equation and simplifying, we obtain

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation:** $f(D)y = X = \phi(x) = F(x)$

➢ **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

Example: 6 Solve $y''' - 6y'' + 11y' - 6y = 2xe^{-x}$.

Solution:

$$-24A_1xe^{-x} + (26A_1 - 24A_0)e^{-x} = 2xe^{-x} + (0)e^{-x}$$

Equating coefficients of like terms, we have

$$-24A_1 = 2 \quad 26A_1 - 24A_0 = 0$$

from which $A_1 = -1/12$ and $A_0 = -13/144$.

Equation (*1*) becomes

$$y_p = -\frac{1}{12}xe^{-x} - \frac{13}{144}e^{-x}$$

and the general solution is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{1}{12}xe^{-x} - \frac{13}{144}e^{-x}$$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation:** $f(D)y = X = \phi(x) = F(x)$

➢ **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

$$\boxed{y_p = e^{\alpha x}(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0) \qquad (11.4)}$$

Example 7: Solve $y' - 5y = x^2e^x - xe^{5x}$.

Solution:

$y_h = c_1e^{5x}$. Here $\phi(x) = x^2e^x - xe^{5x}$, which is the difference of two terms, each in manageable form. For x^2e^x we would assume a solution of the form

$$e^x(A_2x^2 + A_1x + A_0) \tag{1}$$

For xe^{5x} we would try initially a solution of the form

$$e^{5x}(B_1x+B_0) = B_1xe^{5x} + B_0e^{5x}$$

But this supposed solution would have, disregarding multiplicative constants, the term e^{5x} in common with y_h. We are led, therefore, to the modified expression

$$xe^{5x}(B_1x + B_0) = e^{5x}(B_1x^2 + B_0x) \tag{2}$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation:** $f(D)y = X = \phi(x) = F(x)$
- **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

Example 7: Solve $y' - 5y = x^2e^x - xe^{5x}$.

Solution:

We now take y_p to be the sum of (1) and (2):

$$y_p = e^x(A_2x^2 + A_1x + A_0) + e^{5x}(B_1x^2 + B_0x) \tag{3}$$

Substituting (3) into the differential equation and simplifying, we obtain

$$\begin{aligned} &e^x[(-4A_2)x^2 + (2A_2 - 4A_1)x + (A_1 - 4A_0)] + e^{5x}[(2B_1)x + B_0] \\ &= e^x[(1)x^2 + (0)x + (0)] + e^{5x}[(-1)x + (0)] \end{aligned}$$

Equating coefficients of like terms, we have

$$-4A_2 = 1 \quad 2A_2 - 4A_1 = 0 \quad A_1 - 4A_0 = 0 \quad 2B_1 = -1 \quad B_0 = 0$$

from which $\quad A_2 = -\frac{1}{4} \quad A_1 = -\frac{1}{8} \quad A_0 = -\frac{1}{32} \quad B_1 = -\frac{1}{2} \quad B_0 = 0$

Equation (3) then gives $\quad y_p = e^x\left(-\frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32}\right) - \frac{1}{2}x^2e^{5x}$

and the general solution is $\quad y = c_1e^{5x} + e^x\left(-\frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32}\right) - \frac{1}{2}x^2e^{5x}$

- Linear differential equations (second or higher order) with constant coefficients
- Non-homogeneous differential equation: $f(D)y = X = \phi(x) = F(x)$
- **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$ and $D \equiv \frac{d}{dx}$**

Exercise-17

Solve, using the method of undetermined coefficients :

1. $y'' + y = \sin x$ **Ans.** $y = c_1 \cos x + c_2 \sin x - (x/2) \cos x$

1. $(D^2 + 2D + 1)\, y = x^2 - \cos x$ **Ans.** $y = (c_1 + c_2 x)\, e^{-x} + x^2 - 4x + 6 - (1/6) \sin x\; e^{-2x}$

3. $(D^2 - D - 2)\, y = 4x^2$ **Ans.** $y = c_1\, e^{2x} + c_2 e^{-x} - 3 + 2x - 2x^2$

4. $(D^2 - 1)\, y = e^x \sin 2x.$ **Ans.** $y = c_1 e^x + c_2 e^{-x} - e^x (\sin 2x + \cos 2x)/8$

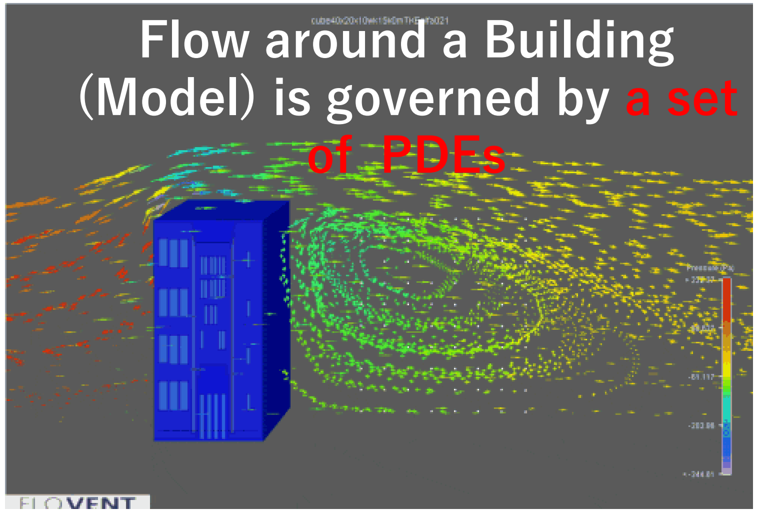
➢ Partial differential equation

➢ Introduction of Partial differential equations (PDEs)

INTRODUCTION

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables.

Applications of PDE's in Science & Engineering including our real-life oriented problems



Navier-Stokes Equations

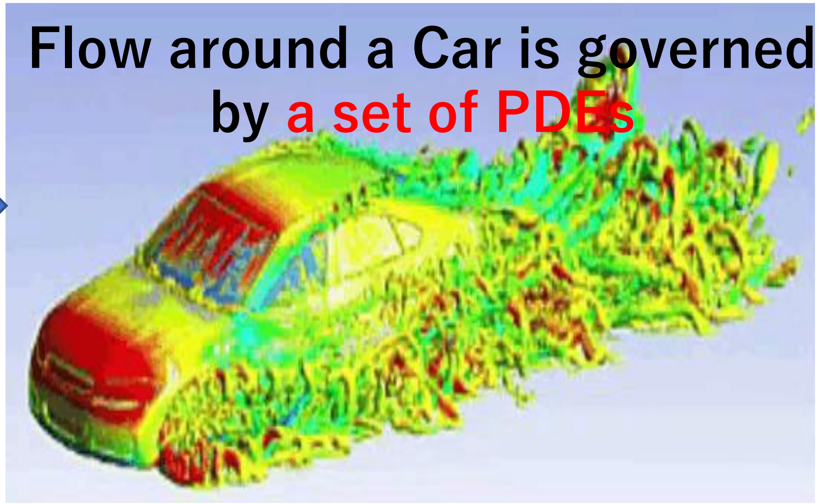
(PDEs)

Continuity Equation

$$\nabla \cdot \vec{V} = 0$$

Momentum Equations

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$



$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

➢ ☞ Instructions:

The following slides are given to all of you in advance for your convenience, and please check and try to understand the given slides (lecture martials) before joining the next class.

If you do so, then it would be easier for you to understand the next class. Besides, you will feel comfortable in asking relevant questions in the classroom, if any!

- **Partial differential equation**
- **Definition of Partial differential equations (PDEs)**

PARTIAL DIFFERENTIAL EQUATION (P.D.E.)

Definition. An equation containing one or more partial derivatives of an unknown frunction of two or more independent variables is known as a *partial differential equation.*

For examples of partial differential equations we list the following:

$$\partial z/\partial x + \partial z/\partial y = z + xy \quad \ldots (1)$$

$$(\partial z/\partial x)^2 + \partial^3 z/\partial y^3 = 2x(\partial z/\partial x) \quad \ldots (2)$$

$$z(\partial z/\partial x) + \partial z/\partial y = x \quad \ldots (3)$$

$$\partial u/\partial x + \partial u/\partial y + \partial u/\partial z = xyz \quad \ldots (4)$$

$$\partial^2 z/\partial x^2 = (1 + \partial z/\partial y)^{1/2} \quad \ldots (5)$$

$$y\left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2\right\} = z(\partial z/\partial y) \quad \ldots (6)$$

- **Partial differential equation**
- **Definition of the order of a partial differential equation (PDE)**

ORDER OF A PARTIAL DIFFERENTIAL EQUATION

Definition. The *order* of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.

For examples of partial differential equations we list the following:

$$\partial z/\partial x + \partial z/\partial y = z + xy \quad \ldots (1)$$

$$(\partial z/\partial x)^2 + \partial^3 z/\partial y^3 = 2x(\partial z/\partial x) \quad \ldots (2)$$

$$z(\partial z/\partial x) + \partial z/\partial y = x \quad \ldots (3)$$

$$\partial u/\partial x + \partial u/\partial y + \partial u/\partial z = xyz \quad \ldots (4)$$

$$\partial^2 z/\partial x^2 = (1 + \partial z/\partial y)^{1/2} \quad \ldots (5)$$

$$y\left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2\right\} = z(\partial z/\partial y) \quad \ldots (6)$$

Equations (1), (3), (4) and (6) are of the first order, (5) is of the second order and (2) is of the third order.

- **Partial differential equation**
- **Definition of the degree of a partial differential equation (PDE)**

DEGREE OF A PARTIAL DIFFERENTIAL EQUATION

The *degree* of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, *i.e.*, made free from radicals and fractions so far as derivatives are concerned.

For examples of partial differential equations we list the following:

$$\partial z/\partial x + \partial z/\partial y = z + xy \qquad \text{... (1)}$$

$$(\partial z/\partial x)^2 + \partial^3 z/\partial y^3 = 2x(\partial z/\partial x) \qquad \text{... (2)}$$

$$z(\partial z/\partial x) + \partial z/\partial y = x \qquad \text{... (3)}$$

$$\partial u/\partial x + \partial u/\partial y + \partial u/\partial z = xyz \qquad \text{... (4)}$$

$$\partial^2 z/\partial x^2 = (1 + \partial z/\partial y)^{1/2} \qquad \text{... (5)}$$

$$y\left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2\right\} = z(\partial z/\partial y) \qquad \text{... (6)}$$

Equations (1), (2), (3) and (4) are of first degree while equations (5) and (6) are of second degree.

- **Partial differential equation**
- **Definition of linear and non-linear of a partial differential equation (PDE)**

LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definitions. A partial differential equation is said to be *linear* if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a *non-linear* partial differential equation.

For examples of partial differential equations we list the following:

$$\partial z / \partial x + \partial z / \partial y = z + xy \quad \dots (1)$$

$$(\partial z / \partial x)^2 + \partial^3 z / \partial y^3 = 2x(\partial z / \partial x) \quad \dots (2)$$

$$z(\partial z / \partial x) + \partial z / \partial y = x \quad \dots (3)$$

$$\partial u / \partial x + \partial u / \partial y + \partial u / \partial z = xyz \quad \dots (4)$$

$$\partial^2 z / \partial x^2 = (1 + \partial z / \partial y)^{1/2} \quad \dots (5)$$

$$y\left\{(\partial z / \partial x)^2 + (\partial z / \partial y)^2\right\} = z(\partial z / \partial y) \quad \dots (6)$$

Equations (1) and (4) are linear while equations (2), (3), (5) and (6) are non-linear.

- **Partial differential equation**
- **Notations for partial differential equations (PDEs)**

NOTATIONS

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable.

We adopt the following notations throughout the study of partial differential equations

$$p = \partial z / \partial x, \qquad q = \partial z / \partial y, \qquad r = \partial^2 z / \partial x^2,$$

$$s = \partial^2 z / \partial x \partial y \qquad \text{and} \qquad t = \partial^2 z / \partial y^2$$

- **Partial differential equation**
- **Notations for partial differential equations (PDEs)**

In case there are n independent variables, we take them to be x_1, x_2....., x_n and z is then regarded as the dependent variable. In this case we use the following notations :

$$p_1 = \partial z/\partial x_1, \qquad p_2 = \partial z/\partial x_2, \qquad p_3 = \partial z/\partial x_3, \text{ and } \qquad p_n = \partial z/\partial x_n.$$

Sometimes the partial differntiations are also denoted by making use of suffixes.

Thus we write $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$,

$u_{xx} = \partial^2 u/\partial x^2$, $u_{xy} = \partial^2 u/\partial x \partial y$ and so on.

- **Partial differential equation**
- **Classification of partial differential equations (PDEs)**

Linear equation. *A first order equation* $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form $P(x, y)\, p + Q(x, y)\, q = R(x, y)\, z + S(x, y)$.

For examples, $yx^2p + xy^2q = xyz + x^2y^3$

and

$$p + q = z + xy$$

are both first order linear partial differential equations.

- **Partial differential equation**
- **Classification of partial differential equations (PDEs)**

Non-linear equation. *A first order partial differential equation $f(x, y, z, p, q) = 0$ which does* **not meet the criterion of linear equation is** *known as a non-liner equation.*

For examples, $p^2 + q^2 = 1$, $p\,q = z$

and $x^2 p^2 + y^2 q^2 = z^2$

are all non-linear partial differential equations.

Extra Slide: Recapitulations of Implicit Partial differential equation for PDEs formation

IMPLICIT PARTIAL DIFFERENTIATION

Suppose z is a function of x, y defined implicitly by $F(x,y,z)=0$ where F is a differentiable function of three variables. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

Apply $\frac{\partial}{\partial x}$ $\quad F(x, y, z(x,y)) = 0$ as functions of x, y

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{1} + \cancel{\frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{0}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

— what we want to solve for

If $\frac{\partial F}{\partial z} \neq 0$ then $\boxed{\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}}$

- **Partial differential equation**
- **Rule: Derivation of a PDE by the elimination of arbitrary constants**

Rule: Derivation of a partial differential equation by the elimination of arbitrary constants.

Consider an equation $$F(x, y, z, a, b) = 0, \qquad ...(1)$$

where a and b denote arbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (1) with respect to x and y partially in turn, we get

$$\partial F / \partial x + p(\partial F / \partial z) = 0 \quad \text{and} \quad \partial F / \partial y + q(\partial F / \partial z) = 0 \qquad ..(2)$$

Eliminating two constants a and b from three equations of (1) and (2), we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0, \qquad ...(3)$$

which is partial differential equation of the first order.

In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to partial differential equations of higher order than the first.

- **Partial differential equation**
- **Working rules for solving problems**

Working rule for solving problems:

For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a, b, the relation is differentiated partially with respect to independent variables x and y.

Finally arbitrary constants a and b are eliminated from the relations

$F(x, y, z, a, b) = 0,$ $\quad \partial F / \partial x = 0 \quad$ and $\quad \partial F / \partial y = 0.$

The equation free from a and b will be the required partial differential equation

- **Partial differential equation**
- **Working rules for solving problems**

Three situations may arise :

Situation I. *When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.*

For example, consider $$z = ax + y, \qquad \text{... (1)}$$
where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. 'x', we get $$\partial z / \partial x = a \qquad \text{... (2)}$$

Differentiating (1) partially w.r.t. 'y', we get $$\partial z / \partial y = 1 \qquad \text{... (3)}$$

Eliminating a between (1) and (2) yields $$z = x(\partial z / \partial x) + y \qquad \text{... (4)}$$

Since (3) does not contain arbitrary constant, so (3) is also partial differential under consideration. Thus, we get two partial differential equations (3) and (4).

- **Partial differential equation**
- **Working rules for solving problems**

Situation II. *When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial differential equation of order one.*

Example: *Eliminate a and b from* $az + b = a^2x + y$ ... (1)

Differentiating (1) partially w.r.t 'x' and 'y', we have

$a(\partial z / \partial x) = a^2$ ... (2) $a(\partial z / \partial y) = 1$ ... (3)

Eliminating a from (2) and (3), we have $(\partial z / \partial x)\,(\partial z / \partial y) = 1$,

which is the unique partial differential equation of order one.

- **Partial differential equation**
- # Working rules for solving problems

Situation III. *When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one.*

Example: *Eliminate a, b and c from* $z = ax + by + cxy$ ... (1)

Differentiating (1) partially w.r.t., 'x' and 'y', we have

$\partial z / \partial x = a + cy$ ... (2) $\partial z / \partial y = b + cx$ ... (3)

From (2) and (3), $\partial^2 z / \partial x^2 = 0,$ $\partial^2 z / \partial y^2 = 0$ ... (4)

and $\partial^2 z / \partial x \partial y = c$ ... (5)

Now, (2) and (3) $\Rightarrow$ $x(\partial z / \partial x) = ax + cxy$ and $y(\partial z / \partial y) = by + cxy$

$\therefore$ $x(\partial z / \partial x) + y(\partial z / \partial y) = ax + by + cxy + cxy$

or $x(\partial z / \partial x) + y(\partial z / \partial y) = z + xy(\partial^2 z / \partial x \partial y),$ using (1) and (5) ... (6)

Thus, we get three partial differential equations given by (4) and (6), which are all of order two.

- **Partial differential equation**
- **Working rules for solving problems**

Ex. 1. *Find a partial differential equation by eliminating a and b from* $z = ax + by + a^2 + b^2$.

Sol. Given $\quad z = ax + by + a^2 + b^2. \quad$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a \quad$ and $\quad \partial z/\partial y = b.$

Substituting these values of a and b in (1) we see that the arbitrary constants a and b are eliminated and we obtain,

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\partial x)^2 + (\partial z/\partial y)^2,$$

which is the required partial differential equation.

- **Partial differential equation**
- **Working rules for solving problems**

Ex. 2. *Eliminate arbitrary constants a and b from* $z = (x - a)^2 + (y - b)^2$ *to form the partial differential equation.*

Sol. Given $$z = (x - a)^2 + (y - b)^2. \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = 2(x - a)$ and $\partial z/\partial y = 2(y - b)$.

Squaring and adding these equations, we have

$$(\partial z/\partial x)^2 + (\partial z/\partial y)^2 = 4(x - a)^2 + 4(y - b)^2 = 4\,[(x - a)^2 + (y - b)^2]$$

or $(\partial z/\partial x)^2 + (dz/\partial y)^2 = 4z$, using (1), **which is required partial differential equation.**

- **Partial differential equation**
- **Working rules for solving problems**

Ex. 4 *Form the partial differential equation by eliminating h and k from the equation* $(x-h)^2+(y-k)^2+z^2=\lambda^2$.

Sol. Given $(x-h)^2+(y-k)^2+z^2=\lambda^2$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$2(x-h)+2z(\partial z/\partial x)=0 \quad \text{or} \quad (x-h)=-z(\partial z/\partial x) \quad ...(2)$$

and $$2(y-k)+2z(\partial z/\partial y)=0 \quad \text{or} \quad (y-k)=-z(\partial z/\partial y). \quad ...(3)$$

Substituting the values of $(x-h)$ and $(y-k)$ from (2) and (3) in (1) gives

$$z^2(\partial z/\partial x)^2+z^2(\partial z/\partial y)^2+z^2=\lambda^2 \quad \text{or} \quad z^2[(\partial z/\partial x)^2+(\partial z/\partial y)^2+1]=\lambda^2,$$

which is the required partial differential equation.

- **Partial differential equation**
- **Working rules for solving problems**

HW

Ex. 5 . *Form differential equation by eliminating constants A and p from* $z = A\, e^{pt} \sin px$.

Sol. Given $z = A\, e^{pt} \sin px$. ...(1)

Differentiating (1) partially with respect to x and t, we get

$\partial z/\partial x = Ap\, e^{pt} \cos px$ ...(2) $\partial z/\partial t = Ap\, e^{pt} \sin px$. ...(3)

Differentiating (2) and (3) partially with respect to x and t respectively gives

$\partial^2 z/\partial x^2 = -\,Ap^2 e^{pt} \sin px$. ...(4) $\partial^2 z/\partial t^2 = Ap^2 e^{pt} \sin px$. ...(5)

Adding (4) and (5), $\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0$,

which is the required partial differential equation.

- **Partial differential equation**
- **Working rules for solving problems**

Ex. 6 *Form the partial differential equation by eliminating the arbitrary constants a and b from log (az – 1) = x + ay + b.*

Sol. Given $$\log (az - 1) = x + ay + b \qquad \text{... (1)}$$

Differentiating (1) partially w.r.t. 'x', we get $$\frac{a}{az-1}\frac{\partial z}{\partial x} = 1 \qquad \text{... (2)}$$

Differentiating (1) partially w.r.t. 'y', we get $$\frac{a}{az-1}\frac{\partial z}{\partial y} = a \qquad \text{... (3)}$$

From (3), $az - 1 = \dfrac{\partial z}{\partial y}$ so that $$a = \frac{1 + (\partial z / \partial y)}{z} \qquad \text{... (4)}$$

Putting the above values of $az - 1$ and a in (2), we have

$$\frac{1 + (\partial z / \partial y)}{z(\partial z / \partial y)}\frac{\partial z}{\partial x} = 1 \quad \text{or} \quad \left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial y}.$$

- **Partial differential equation**
- **Working rules for solving problems**

Ex. 7 . *Find a partial differential equation by eliminating a, b, c, from* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Sol. Given $$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1. \qquad \ldots (1)$$

Differentiating (1) partially with respect to x and y, we get

HW

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{dz}{dx} = 0 \quad \text{or} \quad c^2x + a^2z\frac{dz}{dx} = 0 \qquad \ldots (2)$$

and $$\frac{2y}{b^2} + \frac{2x}{c^2}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2y + b^2z\frac{\partial z}{\partial y} = 0. \qquad \ldots (3)$$

Differentiating (2) with respect to x and (3) with respect to y, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0 \qquad \ldots (4)$$

$$c^2 + b^2\left(\frac{\partial z}{\partial x}\right)^2 + b^2z\frac{\partial^2 z}{\partial y^2} = 0. \qquad \ldots (5)$$

- **Partial differential equation**
- **Working rules for solving problems**

Ex. 7 . *Find a partial differential equation by eliminating a, b, c, from* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Sol. Given $$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1. \quad \ldots (1)$$

Hw

From (2), $$c^2 = -(a^2 z/x) \times (\partial z/\partial x) \quad \ldots (6)$$

Putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0 \quad \text{or} \quad zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0. \quad \ldots (7)$$

Similarly, from (3) and (5), $$zy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0. \quad \ldots (8)$$

Differentiating (2) partially w.r.t. y, $$0 + a^2\left\{(\partial z/\partial y)\,(\partial z/\partial x) + z(\partial^2 z/\partial x\partial y)\right\} = 0$$

or $$(\partial z/\partial x)\,(\partial z/\partial y) + z(\partial^2 z/\partial x\partial y) = 0 \quad \ldots (9)$$

(7), (8) and (9) are three possible forms of the required partial differential equations.

- **Partial differential equation**
- **Working rules for solving problems**

Exercise 01 (PDE)

Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding partial differential equations.

1. $z = A\,e^{pt}\sin px$, (p and A). **Ans.** $\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0.$

2. $z = A\,e^{-p^2 t}\cos px$, (p and A) **Ans.** $\partial^2 z/\partial x^2 = dz/\partial t$

3. $z = ax^3 + by^3$; (a, b) **Ans.** $x(\partial z/\partial x) + y(\partial z/\partial y) = 3z$

4. $4z = [ax + (y/a) + b]^2$; (a, b). **Ans.** $z = (dz/\partial x)\,(\partial z/\partial y)$

5. $z = ax^2 + bxy + cy^2$, (a, b,c) **Ans.** $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) = 2z$

Thank you for your attendance and attention