- Course Title : Differential Equations and Special Functions
- Course Code: MAT102 (Section-3) // Lecture-12 (March 27, 2023)
- **Today's Lecture Topics:**

- **Ordinary Differential Equations**

- Linear differential equations (second or higher order) with constant coefficients: The general form of non-homogeneous equations and its solution procedures
- Method of Variation of Parameters for finding solution of $f(D)y = X$ (for Second order linear differential equation) where $D \equiv \frac{d}{dx}$
- Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$

- Course Instructor: Dr. Akter Hossain, Assistant Professor of MPS Department, EWU, Dhaka, BD

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**
- **Case 2: Let** $y_p = \frac{1}{D^2+a^2} \sin ax$

$$= \text{ Imaginary Part of } \left(\frac{1}{D^2+a^2} \cos ax + i \frac{1}{D^2+a^2} \sin ax\right)$$

$$= \text{Imaginary Part of } \frac{1}{D^2+a^2} (\cos ax + i \sin ax$$

$$\therefore \; y_p = \text{Imaginary Part of } \frac{1}{D^2+a^2} e^{iax} ------ -(1)$$

But $\frac{1}{D^2+a^2} e^{iax} = e^{iax} \frac{1}{(D+ia)^2+a^2} e^{0x}$

$$= e^{iax} \frac{1}{D^2+2Dai} e^{0x} = e^{iax} \frac{1}{D(D+2ai)} e^{0x}$$

$$= e^{iax} \frac{1}{D} \frac{1}{(D+2ai)} e^{0x}$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**
- **Case 2: If $\phi\left(-a^2\right) = 0$, then** $\frac{1}{f(D)} e^{ax} = e^{ax} \frac{1}{f(D+a)} e^{0x}$

Now, $\frac{1}{D^2+a^2} e^{iax} = e^{iax} \frac{1}{D} \frac{1}{(D+2ai)} e^{0x} = e^{iax} \frac{1}{D} \frac{1}{(0+2ai)} e^{0x}$

$$= e^{iax} \frac{1}{D} \frac{1}{2ai} 1 = e^{iax} \frac{x}{2ai}$$

$$= \frac{x}{2ai} e^{iax} = \frac{x}{2ai} (\cos ax + i \sin ax)$$

$$= \frac{x}{2ai} \cos ax + \frac{x}{2a} \sin ax$$

$$= -\frac{xi}{2a} \cos ax + \frac{x}{2a} \sin ax$$

Therefore, imaginary part of $\frac{1}{D^2+a^2} e^{iax} = -\frac{x}{2a} \cos ax$

Hence, we get from (1), $y_p = \frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$

■ Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation**
- **Formulae IV:** $(a)\ y_p = \frac{1}{D^2+a^2}\ \sin ax = -\frac{x}{2a}\cos ax$

$$(b)\ y_p = \frac{1}{D^2+a^2}\ \cos ax = \frac{x}{2a}\sin ax$$

Example: Solve $\frac{d^2y}{dx^2} + 4y = 8\cos 2x$

Solution: Let $y = e^{mx}$ the trail solution of $\frac{d^2y}{dx^2} + 4y = 0$. Therefore, the corresponding auxiliary equation is $m^2 + 4 = 0\ \therefore m = \pm 2i$

Therefore, the complementary solution of given equation is

$$y_c = c_1 \cos 2x + c_2 \sin 2x \text{ ------------------ } (1)$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**

Example: Solve $\frac{d^2y}{dx^2} + 4y = 8\, cos\, 2x$

Solution: The particular integral, $y_p = \frac{1}{D^2+4} 8\, cos\, 2x$

$$\Rightarrow y_p = 8\frac{1}{D^2+4}\, cos\, 2x$$

$$[We\ know\ that, y_p = \frac{1}{D^2+a^2}\, cos\, ax = \frac{x}{2a} \sin \mathrm{ax}]$$

$$\therefore\ y_p = 8\frac{x}{2.\,2} sin\, 2x = 2x\, sin\, 2x$$

Therefore, the general solution of given equation is

$$y = y_c + y_p = c_1 cos 2x + c_2 \sin 2x + 2\mathrm{x} \sin 2\mathrm{x}\ (\mathrm{Ans.})$$

- Linear differential equations (second or higher order) with constant coefficients
- Non-homogeneous differential equation

Exercise-15

Solve the following differential equations :

1. $(D^2 + 9)\, y = \cos 2x + \sin 2x.$ **Ans.** $y = c_1 \cos 3x + c_2 \sin 3x + (1/5)(\cos 2x + \sin 2x)$

2. $(D^3 + D^2 - D - 1)\, y = \cos 2x.$ **Ans.** $y = c_1 e^x + (c_2 + c_3 x)\, e^{-x} - (1/25)(2 \sin 2x + \cos 2x)$

3. $(D^2 - 5D + 6)\, y = \sin 3x.$ **Ans.** $y = c_1 e^{2x} + c_2 e^{3x} + (1/78)(5 \cos 3x - \sin 3x)$

4. $(D^2 + D + 1)\, y = \sin 2x.$ **Ans.** $y = e^{-x/2} \left\{ c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2) \right\} - (1/13)(2\cos 2x + 3\sin 2x)$

- Linear differential equations (second or higher order) with constant coefficients

➢ Non-homogeneous differential equation

- Suppose $y_p = \frac{1}{f(D)} X$ where X is the form of $e^{ax}\, V$ where V is any function of x.

➢ Working rule for finding y_p when X is the form of of $e^{ax}\, V$

- Formulae V : $y_p = \frac{1}{f(D)} e^{ax}\, V = e^{ax} \frac{1}{f(D+a)} V$ where $\frac{1}{f(D+a)} V$ can be calculated by the methods which have already discussed in the previous section.

➢ Example: Solve : $\left(D^2 + 3D + 2\right)y = e^{2x} sinx$

Solution: Let $y = e^{mx}$ the trail solution of $\left(D^2 + 3D + 2\right)y = 0$. Then the corresponding auxiliary is $m^2 + 3m + 2 = 0$

$$\Rightarrow m^2 + 2m + \mathrm{m} + 2 = 0$$

$$\Rightarrow m(m+2) + 1(m+2) = 0 \Rightarrow (m+2)\,(m+1) = 0 \;\therefore m = -1, -2$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**
- **Example: Solve :** $(D^2+3D+2)y = e^{2x}\sin x$

Solution: Since $m=-1,-2,$ the complementary solution of the given equation is

$$y_c = C_1e^{-x} + C_2e^{-2x}$$

Now, the particular integral, $y_p = \frac{1}{D^2+3D+2}e^{2x}\sin x$

$$= e^{2x}\frac{1}{(D+2)^2+3(D+2)+2}\sin x$$

$$= e^{2x}\frac{1}{D^2+4D+4+3D+6+2}\sin x$$

$$= e^{2x}\frac{1}{-1^2+4D+4+3D+6+2}\sin x$$

$$= e^{2x}\frac{1}{11+7D}\sin x$$

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Non-homogeneous differential equation**

➢ **Example: Solve :** $(D^2 + 3D + 2)y = e^{2x} sinx$

Solution: Now, the particular integral, $y_p = e^{2x} \frac{1}{11+7D} sinx$

$$= e^{2x} \frac{11-7D}{(11+7D)(11-7D)} sinx$$

$$= e^{2x} \frac{11-7D}{(121-49D^2)} sinx$$

$$= e^{2x} \frac{11-7D}{(121+49)} sinx$$

$$= \frac{e^{2x}}{170} (11 - 7D) sinx$$

$$= \frac{e^{2x}}{170} (11sinx - 7cosx)$$

Therefore, the general solution is $y = y_c + y_p$

$$= C_1e^{-x} + C_2e^{-2x} + \frac{e^{2x}}{170} (11sinx - 7cosx)$$

- Linear differential equations (second or higher order) with constant coefficients
- Non-homogeneous differential equation

Exercise-16

Solve *the following differential equations:*

1. $(D^2 - 2D + 4)\, y = e^x \cos x$ **Ans.** $y = e^x\,[c_1 \cos (x\sqrt{3}\,) + c_2 \sin (x\sqrt{3}\,)] + (1/2)\, e^x \cos x$

2. $(D + 1)^3\, y = x^2 e^{-x}$ **Ans.** $y = (c_1 + c_2 x + c_3 x^2)\, e^{-x} + (x^5/60) \times e^{-x}$

3. $(D^2 - 2D + 5)\, y = e^{2x} \sin x$ **Ans.** $y = e^x\,(c_1 \cos 2x + c_2 \sin 2x) - (1/10)\, e^{2x} (\cos x - 2 \sin x)$

4. $(D^2 - 1)\, y = e^x \cos x.$ **Ans.** $y = c_1 e^x + c_2 e^{-x} + (1/5)\, e^x (2 \sin x - \cos x)$

No need to solve.

- Linear differential equations (second or higher order) with constant coefficients

- Non-homogeneous differential equation: $f(D)y = X \; [X = \phi(x) = F(x)]$

- Method of Variation of Parameters for finding solution of $f(D)y = X$ (for Second order linear differential equation) where $D \equiv \frac{d}{dx}$

- Working rule for solving $f(D)y = X$ by the Method of Variation of Parameters

Step 1: Reduce the general form of non-homogeneous linear differential equation with constant coefficients: $f(D)y = X$ to 2nd order differential equation and re-write it as $y_2 + Py_1 + Qy = R$ ———————————— —(1) (where P and Q are constant and R is a function of x) in which the coefficient of y_2 must be unity.

- **Linear differential equations (second or higher order) with constant coefficients**

- **Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters**

Step 2. Consider $$y_2+Py_1+Qy=0 \qquad \text{... (2)}$$
which is obtained by taking $R=0$ in (1). Solve (2) by methods **discussed in the previous lectures**
Let the general solution of (2) *i.e.*, C.F. of (1) be
$$\text{C.F.} = C_1u + C_2v,\ C_1,\ C_2 \text{ being arbitrary constants} \qquad \text{... (3)}$$
Step 3. General solution of (1) is $$y = \text{C.F.} + \text{P.I.} \qquad \text{... (4)}$$
where $$\text{C.F.} = C_1u + C_2v,\ C_1,\ C_2 \text{ being arbitrary constants} \qquad \text{... (5)}$$
and $$\text{P.I.} = u\,f(x) + v\,g(x) \qquad \text{... (6)}$$
where $$f(x) = -\int \frac{vR}{W}\,dx \qquad \text{and} \qquad g(x) = \int \frac{uR}{W}\,dx, \qquad \text{... (7)}$$
where $$W = \text{Wronskian of } u \text{ and } v = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = uv_1 - u_1v \qquad \text{... (8)}$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Working rule for solving: $y_2+Py_1 + Qy = R$ *by the* Method of Variation of Parameters**

Example 1. *Apply the method of variation of parameters to solve*

$(i)\ y_2 + n^2y = sec\ nx$;

$(ii)\ y_2 + y = sec\ x$

$(iii)\ y_2 + 4y = sec\ 2x$;

$(iv)\ y_2 + 9y = sec\ 3x$

Solution: (i) Given $y_2 + n^2y = \sec nx$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\,y = R$, we have $R = \sec nx$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving:** $y_2 + Py_1 + Qy = R$ *by the* **Method of Variation of Parameters**

Solution $(i)\ y_2 + n^2 y = sec\ nx$

Consider $y_2 + n^2 y = 0$ or $(D^2 + n^2)\, y = 0$, where $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $m^2 + n^2 = 0$ so that $m = \pm\, in$.

C.F. of (1) $= C_1 \cos nx + C_2 \sin nx$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos nx$, $v = \sin nx$ Also, here $R = \sec nx$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos nx & \sin nx \\ -n \sin nx & n \cos nx \end{vmatrix} = n \neq 0 \qquad \text{... (5)}$$

Then, P.I. of (1) $= u\, f(x) + v\, g(x)$ ... (6)

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving: $y_2 + Py_1 + Qy = R$ *by the* Method of Variation of Parameters**

Solution $(i)\ y_2 + n^2 y = sec\ nx$

$$u = \cos nx,\quad v = \sin nx,\quad R = \sec nx$$

Then, P.I. of $(1) = u\, f(x) + v\, g(x)$ ... (6)

where $f(x) = -\int \frac{vR}{W} dx = -\int \frac{\sin nx \sec nx}{n} dx = \frac{1}{n^2} \log \cos nx$, by (4) and (5)

$g(x) = \int \frac{uR}{W} dx = \int \frac{\cos nx \quad \sec nx}{n} dx = \frac{x}{n}$, *by* (4) and (5)

$\therefore$ P.I. of $(1) = (\cos nx) \times (1/n^2) \log \cos nx + (\sin nx) \times (x/n)$, by (6)

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $y = C_1 \cos nx + C_2 \sin nx + (1/n^2) \times \cos nx \log \cos nx + (x/n) \times \sin nx$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving:** $y_2+Py_1 + Qy = R$ *by the* **Method of Variation of Parameters**

Solution: (ii) $y_2 + y = sec\ x$

(iii) $y_2 + 4y = sec\ 2x$

(iv) $y_2 + 9y = sec\ 3x$

The general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

$y = C_1 \cos nx + C_2 \sin nx + (1/n^2) \times \cos nx \log \cos nx + (x/n) \times \sin nx$

$\boxed{(i)\ y_2 + n^2 y = sec\ nx}$

(*ii*) Compare it with part (*i*). Here $n = 1$. Now do as in part (*i*).

The required solution is $y = C_1 \cos x + C_2 \sin x + \cos x \log \cos x + x \sin x$.

(*iii*) Proceed as in part (*i*). Note that here $n = 2$.

Ans. $y = C_1 \cos 2x + C_2 \sin 2x + (1/4) \times \cos 2x \log \cos 2x + (x/2) \times \sin 2x$

(*iv*) Proceed as in part (*i*). Note that here $n = 3$.

Ans. $y = C_1 \cos 3x + C_2 \sin 3x + (1/9) \times \cos 3x \log \cos 3x + (x/3) \times \sin 3x$

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Working rule for solving: $y_2 + Py_1 + Qy = R$ *by the* Method of Variation of Parameters**

Example 2. *Apply the method of variation of parameters to solve*

(i) $y_2 + a^2 y = cosec\ ax$

(ii) $y_2 + y = cosec\ x$

(iii) $y_2 + 9y = cosec\ 3x$

Solution: (*i*) Given $\quad y_2 + a^2 y = \operatorname{cosec} ax \quad \ldots (1)$

Comparing (1) with $y_2 + Py_1 + Q\,y = R$, we have $\quad R = \operatorname{cosec} ax$

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Working rule for solving:** $y_2 + Py_1 + Qy = R$ *by the* **Method of Variation of Parameters**

Solution: $(i)\ y_2 + a^2 y = cosec\ ax$

Consider $\quad y_2 + a^2 y = 0 \quad$ or $\quad (D^2 + a^2)\,y = 0, \quad D \equiv d/dx \quad$ ... (2)

Auxiliary equation of (2) is $\quad \mathbf{m}^2 + a^2 = 0 \quad$ so that $\quad \mathbf{m} = \pm\, ai$

$\therefore$ C.F. of (1) $= C_1 \cos ax + C_2 \sin ax$, C_1 and C_2 being arbitrary constants ... (3)

Let $\quad u = \cos ax, \quad v = \sin ax.\quad$ Also, here $\quad R = \operatorname{cosec} ax \quad$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a \neq 0 \qquad \text{... (5)}$$

Then, $\quad$ P.I. of (1) $= u\, f(x) + v\, g(x)$, ... (6)

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters**

Solution: $(i)\ y_2+a^2y=cosec\ ax$

where
$$f(x)=-\int\frac{vR}{W}dx=-\int\frac{\sin ax\ \mathrm{cosec}\ ax}{a}dx=-\frac{x}{a},\ \text{by (4) and (5)}$$

$$\boxed{\begin{array}{c}u=\cos ax\\ v=\sin ax.\\ R=\mathrm{cosec}\ ax\end{array}}$$

$$g(x)=\int\frac{uR}{W}dx=\int\frac{\cos ax\ \mathrm{cosec}\ ax}{a}dx=(1/a^2)\times\log\sin ax,\ \text{by (4) and (5)}$$

$\therefore$ P.I. of (1) $=(\cos ax)\times(-x/a)+(\sin ax)\times(1/a^2)\times\log\sin ax$, by (6)

$$\boxed{\text{P.I. of (1)}=u\,f(x)+v\,g(x).}$$

Hence the general solution of (1) is $\quad y=\text{C.F.}+\text{P.I.}$

i.e., $\quad y=C_1\cos ax+C_2\sin ax-(x/a)\times\cos ax+(1/a^2)\times\sin ax\log\sin ax$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters**

$$\boxed{(i)\ y_2 + a^2 y = cosec\ ax}$$

The general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $y = C_1 \cos ax + C_2 \sin ax - (x/a) \times \cos ax + (1/a^2) \times \sin ax \log \sin ax$

Solution: $(ii)\ y_2 + y = cosec\ x$

$(iii)\ y_2 + 9y = cosec\ 3x$

(ii) Proceed as in part (i). Note that here $a = 1$.

Ans. $y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \log \sin x$

(iii) Proceed as in part (i). Note that have $a = 3$

Ans. $y = C_1 \cos 3x + C_2 \sin 3x - (x/3) \times \cos 3x + (1/9) \times \sin 3x \log \sin 3x$

- **Linear differential equations (second or higher order) with constant coefficients**

➢ **Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters**

Example:4 *Solve* $d^2y/dx^2 - 2(dy/dx) = e^x \sin x$ *by using the method of variation of parameters.*

Solution: Given $(D^2 - 2D)\,y = e^x \sin x,$ where $D \equiv d/dx$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\,y = R,$ here $R = e^x \sin x$

Consider $(D^2 - 2D)\,y = 0$ ... (2)

Auxiliary equation of (2) is $\mathbf{m}^2 - 2 = 0$ so that $\mathbf{m} = 0, 2.$

C.F. of (1) $= C_1 + C_2\,e^{2x}$, C_1 and C_2 being arbitrary constants. ... (3)

Let $u = 1$ and $v = e^{2x}.$ Also, here $R = e^x \sin x$ ... (4)

Here
$$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x} \neq 0 \qquad \text{... (5)}$$

Then, P.I. of (1) $= u\,f(x) + v\,g(x),$ ... (6)

Linear differential equations (second or higher order) with constant coefficients: Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters

Example:4 *Solve $d^2y/dx^2 - 2(dy/dx) = e^x \sin x$ by using the method of variation of parameters.*

Solution: where

$$\boxed{u = 1 \quad v = e^{2x}. \quad R = e^x \sin x}$$

$$f(x) = -\int \frac{vR}{W}dx = -\int \frac{e^{2x}e^x \sin x}{2e^{2x}} = -\frac{1}{2}\int e^x \sin x\, dx, \text{ by (4) and (5)}$$

$$= -\frac{1}{2}\frac{e^x}{1^2+1^2}(\sin x - \cos x), \text{as} \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)$$

$$= -(1/4) \times e^x (\sin x - \cos x)$$

and

$$g(x) = \int \frac{uR}{W}dx = \int \frac{e^x \sin x}{2e^{2x}}dx = \frac{1}{2}\int e^{-x} \sin dx, \text{by (4) and (5)}$$

$$= \frac{1}{2}\frac{e^{-x}}{(-1)^2+1^2}\{(-1)\sin x - \cos x\} = -\frac{e^{-x}}{4}(\sin x + \cos x)$$

$$\boxed{\text{P.I. of (1)} = u\, f(x) + v\, g(x).}$$

$\therefore$ P.I. of (1) $= -(1/4) \times e^x(\sin x - \cos x) + e^{2x} \times (-1/4) \times e^{-x}(\sin x + \cos x)$, by (6)

$= -(1/4) \times e^x \{(\sin x - \cos x) + (\sin x + \cos x)\} = -(1/2) \times e^x \sin x$

Hence the required general solution is $y = \text{C.F} + \text{P.I.}$

i.e., $y = C_1 + C_2 e^{2x} - (1/2) \times e^x \sin x$, C_1, C_2 being arbitrary constants.

■ **Linear differential equations (second or higher order) with constant coefficients: Working rule for solving: $y_2+Py_1+Qy=R$ *by the* Method of Variation of Parameters**

Exercise-18

1. *Apply the method of variation of parameters to solve the equations:*

(i) $y_2-2y_1+y=e^x$ **Ans.** $y=(C_1+C_2x)\,e^x+(x^2/2)\times e^x$

(ii) $y_2-2y_1+y=(1/x^3)\,e^x$ **Ans.** $y=(c_1+c_2x)\,e^x-(1/2x)\times e^x$

(iii) $y_2+a^2y=\cos ax$ **Ans.** $y=C_1\cos ax+C_2\sin ax+(x/2a)\times\sin ax$

(iv) $y_2+4y=\sin x$ **Ans.** $y=C_1\cos 2x+C_2\sin 2x+(1/3)\times\sin x$

No need to solve

■ Linear differential equations (second or higher order) with constant coefficients

- ➢ **Non-homogeneous differential equation**: $f(D)y = X = \phi(x) = F(x)$
- ➢ **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$ and $D \equiv \frac{d}{dx}$**

SIMPLE FORM OF THE METHOD

The *method of undetermined coefficients* is applicable only if $\phi(x)$ and *all* of its derivatives can be written in terms of **a** *finite* set of linearly independent functions, which we denote by $\{y_1(x), y_2(x), \ldots, y_n(x)\}$. The method is initiated by assuming a particular solution of the form

$$y_p(x) = A_1 y_1(x) + A_2 y_2(x) + \cdots + A_n y_n(x)$$

where $A_1, A_2, \ldots, A_n$ denote arbitrary multiplicative constants. These arbitrary constants are then evaluated by substituting the proposed solution into the given differential equation and equating the coefficients of like terms.

■ Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation**: $\boldsymbol{f(D)y = X = \phi(x) = F(x)}$
- **Method of undetermined coefficient for finding particular integral of $\boldsymbol{f(D)y = X = \phi(x) = F(x)}$ and $\boldsymbol{D \equiv \frac{d}{dx}}$**

Case 1. $\phi(x) = p_n(x)$, an nth-degree polynomial in x. Assume a solution of the form

$$y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0 \tag{11.1}$$

where A_j ($j = 0, 1, 2, \ldots, n$) is a constant to be determined.

Case 2. $\phi(x) = ke^{\alpha x}$ where k and α are known constants. Assume a solution of the form

$$y_p = Ae^{\alpha x} \tag{11.2}$$

where A is a constant to be determined.

■ **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**: $f(D)y = X = \phi(x) = F(x)$
- **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

Case 3. $\phi(x) = k_1 \sin \beta x + k_2 \cos \beta x$ **where** k_1, k_2, **and** β **are known constants.** Assume a solution of the form

$$y_p = A \sin \beta x + B \cos \beta x \tag{11.3}$$

where A and B are constants to be determined.

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Non-homogeneous differential equation**: $\boldsymbol{f(D)y = X = \phi(x) = F(x)}$

➢ **Method of undetermined coefficient for finding particular integral of $\boldsymbol{f(D)y = X = \phi(x) = F(x)}$ and $\boldsymbol{D \equiv \frac{d}{dx}}$**

GENERALIZATIONS

If $\phi(x)$ is the product of terms considered in Cases 1 through 3, take y_p to be the product of the corresponding assumed solutions and algebraically combine arbitrary constants where possible. In particular, if $\phi(x) = e^{\alpha x}p_n(x)$ is the product of a polynomial with an exponential, assume

$$y_p = e^{\alpha x}(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0) \tag{11.4}$$

where A_j is as in Case 1. If, instead, $\phi(x) = e^{\alpha x}p_n(x) \sin \beta x$ is the product of a polynomial, exponential, and sine term, or if $\phi(x) = e^{\alpha x}p_n(x) \cos \beta x$ is the product of a polynomial, exponential, and cosine term, then assume

$$y_p = e^{\alpha x} \sin \beta x \,(A_n x^n + \cdots + A_1 x + A_0) + e^{\alpha x} \cos \beta x \,(B_n x^n + \cdots + B_1 x + B_0) \tag{11.5}$$

where A_j and B_j $(j = 0, 1, \ldots, n)$ are constants which still must be determined.

If $\phi(x)$ is the sum (or difference) of terms already considered, then we take y_p to be the sum (or difference) of the corresponding assumed solutions and algebraically combine arbitrary constants where possible.

■ **Linear differential equations (second or higher order) with constant coefficients**

- ➢ **Non-homogeneous differential equation:** $\boldsymbol{f(D)y = X = \phi(x) = F(x)}$
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MODIFICATIONS

If any term of the assumed solution, disregarding multiplicative constants, is also a term of y_h (the homogeneous solution), then the assumed solution must be modified by multiplying it by x^m, where m is the smallest positive integer such that the product of x^m with the assumed solution has no terms in common with y_h.

LIMITATIONS OF THE METHOD

In general, if $\phi(x)$ is not one of the types of functions considered above, or if the differential equation *does not have constant coefficients*

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- **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(x) = F(x)$ and $D \equiv \frac{d}{dx}$**

Case 1. **$\phi(x) = p_n(x)$, an nth-degree polynomial in x.**

Assume a solution of the form $y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$ *(11.1)*

Example:1 Solve $y'' - y' - 2y = 4x^2$.

Solution:

$y_h = c_1 e^{-x} + c_2 e^{2x}$. Here $\phi(x) = 4x^2$, a second-degree polynomial. Using *(11.1)*, we assume that

$$y_p = A_2 x^2 + A_1 x + A_0 \qquad (1)$$

Thus, $y_p' = 2A_2 x + A_1$ and $y_p'' = 2A_2$. Substituting these results into the differential equation, we have

$$2A_2 - (2A_2 x + A_1) - 2(A_2 x^2 + A_1 x + A_0) = 4x^2$$

■ Linear differential equations (second or higher order) with constant coefficients

- ➢ **Non-homogeneous differential equation:** $f(D)y = X = \phi(x) = F(x)$
- ➢ **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

Example:1 Solve $y'' - y' - 2y = 4x^2$.

Solution:

or, equivalently,

$$(-2A_2)x^2 + (-2A_2 - 2A_1)x + (2A_2 - A_1 - 2A_0) = 4x^2 + (0)x + 0$$

Equating the coefficients of like powers of x, we obtain

$$-2A_2 = 4 \qquad -2A_2 - 2A_1 = 0 \qquad 2A_2 - A_1 - 2A_0 = 0$$

Solving this system, we find that $A_2 = -2$, $A_1 = 2$, and $A_0 = -3$. Hence (*1*) becomes

$$y_p = -2x^2 + 2x - 3$$

and the general solution is

$$y = y_h + y_p = c_1e^{-x} + c_2e^{2x} - 2x^2 + 2x - 3$$

➢ ☞ Instructions:

The following slides are given to all of you in advance for your convenience, and please check and try to understand the given slides (lecture martials) before joining the next class.

If you do so, then it would be easier for you to understand the next class. Besides, you will feel comfortable in asking relevant questions in the classroom, if any!

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation:** $f(D)y = X = \phi(x) = F(x)$
- **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

Example 2: Solve $y'' - y' - 2y = e^{3x}$.

Case 2. $\phi(x) = ke^{\alpha x}$ **where** k **and** α **are known constants.**

Assume a solution of the form $y_p = Ae^{\alpha x}$ (11.2)

Solution:

$y_h = c_1 e^{-x} + c_2 e^{2x}$. Here $\phi(x)$ has the form displayed in Case 2 with $k = 1$ and $\alpha = 3$. Using (11.2), we assume that

$$y_p = Ae^{3x} \tag{1}$$

Thus, $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substituting these results into the differential equation, we have

$$9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} = e^{3x} \quad \text{or} \quad 4Ae^{3x} = e^{3x}$$

It follows that $4A = 1$, or $A = \frac{1}{4}$, so that (1) becomes $y_p = \frac{1}{4}e^{3x}$. The general solution then is

$$y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{4}e^{3x}$$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation:** $f(D)y = X = \phi(x) = F(x)$

➢ **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \dfrac{d}{dx}$

Case 3. $\phi(x) = k_1 \sin \beta x + k_2 \cos \beta x$ **where** k_1, k_2, **and** β **are known constants.**

Assume a solution of the form $y_p = A \sin \beta x + B \cos \beta x$ (11.3)

Example 3: Solve $y'' - y' - 2y = \sin 2x$.

Solution:

$y_h = c_1 e^{-x} + c_2 e^{2x}$. Here $\phi(x)$ has the form displayed in Case 3 with $k_1 = 1$, $k_2 = 0$, and $\beta = 2$. Using (11.3), we assume that

$$y_p = A \sin 2x + B \cos 2x \tag{1}$$

Thus, $y_p' = 2A \cos 2x - 2B \sin 2x$ and $y_p'' = -4A \sin 2x - 4B \cos 2x$. Substituting these results into the differential equation, we have

$$(-4A \sin 2x - 4B \cos 2x) - (2A \cos 2x - 2B \sin 2x) - 2(A \sin 2x + B \cos 2x) = \sin 2x$$

or, equivalently,

$$(-6A + 2B) \sin 2x + (-6B - 2A) \cos 2x = (1) \sin 2x + (0) \cos 2x$$

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Non-homogeneous differential equation**: $\boldsymbol{f(D)y = X = \phi(x) = F(x)}$

➢ **Method of undetermined coefficient for finding particular integral of $\boldsymbol{f(D)y = X = \phi(x) = F(x)}$ and $\boldsymbol{D \equiv \frac{d}{dx}}$**

Example 3: Solve $y'' - y' - 2y = \sin 2x$.

Solution:

Equating coefficients of like terms, we obtain

$$-6A + 2B = 1 \qquad -2A - 6B = 0$$

Solving this system, we find that $A = -3/20$ and $B = 1/20$. Then from (*1*),

$$y_p = -\frac{3}{20}\sin 2x + \frac{1}{20}\cos 2x$$

and the general solution is

$$y = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{20}\sin 2x + \frac{1}{20}\cos 2x$$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation:** $f(D)y = X = \phi(t) = F(t)$

➢ **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(t) = F(t)$ **and** $D \equiv \frac{d}{dt}$

Case 1. $\phi(x) = p_n(x)$, **an** n**th-degree polynomial in** x.

Assume a solution of the form $y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$ (11.1)

Example: 5 Solve $\ddot{y} - 6\dot{y} + 25y = 50t^3 - 36t^2 - 63t + 18$.

Solution:

$$y_h = c_1 e^{3t}\cos 4t + c_2 e^{3t}\sin 4t$$

Here $\phi(t)$ is a third-degree polynomial in t. Using (11.1) with t replacing x, we assume that

$$y_p = A_3 t^3 + A_2 t^2 + A_1 t + A_0 \qquad (1)$$

Consequently,

$$\dot{y}_p = 3A_3 t^2 + 2A_2 t + A_1$$

and

$$\ddot{y}_p = 6A_3 t + 2A_2$$

Substituting these results into the differential equation, we obtain

$$(6A_3 t + 2A_2) - 6(3A_3 t^2 + 2A_2 t + A_1) + 25(A_3 t^3 + A_2 t^2 + A_1 t + A_0) = 50t^3 - 36t^2 - 63t + 18$$

Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation**: $f(D)y = X = \phi(t) = F(t)$
- **Method of undetermined coefficient for finding particular integral of $f(D)y = X = \phi(t) = F(t)$ and $D \equiv \frac{d}{dt}$**

Example: 5 Solve $\ddot{y} - 6\dot{y} + 25y = 50t^3 - 36t^2 - 63t + 18.$

Solution:

or, equivalently,

$$(25A_3)t^3 + (-18A_3 + 25A_2)t^2 + (6A_3 - 12A_2 + 25A_1) + (2A_2 - 6A_1 + 25A_0) = 50t^3 - 36t^2 - 63t + 18$$

Equating coefficients of like powers of t, we have

$$25A_3 = 50; \quad -18A_3 + 25A_2 = -36; \quad 6A_3 - 12A_2 + 25A_1 = -63; \quad 2A_2 - 6A_1 + 25A_0 = 18$$

Solving these four algebraic equations simultaneously, we obtain $A_3 = 2$, $A_2 = 0$, $A_1 = -3$, and $A_0 = 0$, so that (*1*) becomes

$$y_p = 2t^3 - 3t$$

The general solution is

$$y = y_h + y_p = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t + 2t^3 - 3t$$

Linear differential equations (second or higher order) with constant coefficients

- **Non-homogeneous differential equation**: $f(D)y = X = \phi(x) = F(x)$
- **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

$$\boxed{y_p = e^{\alpha x}(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0) \qquad (11.4)}$$

Example: 6 Solve $y''' - 6y'' + 11y' - 6y = 2xe^{-x}$.

Solution: $y_h = c_1e^x + c_2e^{2x} + c_3e^{3x}$. Here $\phi(x) = e^{\alpha x}p_n(x)$, where $\alpha = -1$ and $p_n(x) = 2x$, a first-degree polynomial. Using Eq. (*11.4*), we assume that $y_p = e^{-x}(A_1x + A_0)$, or

$$y_p = A_1xe^{-x} + A_0e^{-x} \qquad (1)$$

Thus,

$$y_p' = -A_1xe^{-x} + A_1e^{-x} - A_0e^{-x}$$

$$y_p'' = A_1xe^{-x} - 2A_1e^{-x} + A_0e^{-x}$$

$$y_p''' = -A_1xe^{-x} + 3A_1e^{-x} - A_0e^{-x}$$

Substituting these results into the differential equation and simplifying, we obtain

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation:** $f(D)y = X = \phi(x) = F(x)$

➢ **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

Example: 6 Solve $y''' - 6y'' + 11y' - 6y = 2xe^{-x}$.

Solution:

$$-24A_1xe^{-x} + (26A_1 - 24A_0)e^{-x} = 2xe^{-x} + (0)e^{-x}$$

Equating coefficients of like terms, we have

$$-24A_1 = 2 \quad 26A_1 - 24A_0 = 0$$

from which $A_1 = -1/12$ and $A_0 = -13/144$.

Equation (*1*) becomes

$$y_p = -\frac{1}{12}xe^{-x} - \frac{13}{144}e^{-x}$$

and the general solution is

$$y = c_1e^{x} + c_2e^{2x} + c_3e^{3x} - \frac{1}{12}xe^{-x} - \frac{13}{144}e^{-x}$$

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Non-homogeneous differential equation:** $f(D)y = X = \phi(x) = F(x)$

➢ **Method of undetermined coefficient for finding particular integral of** $f(D)y = X = \phi(x) = F(x)$ **and** $D \equiv \frac{d}{dx}$

$$\boxed{y_p = e^{\alpha x}(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0) \qquad (11.4)}$$

Example 7: Solve $y' - 5y = x^2 e^x - xe^{5x}$.

Solution:

$y_h = c_1 e^{5x}$. Here $\phi(x) = x^2 e^x - xe^{5x}$, which is the difference of two terms, each in manageable form. For $x^2 e^x$ we would assume a solution of the form

$$e^x(A_2 x^2 + A_1 x + A_0) \qquad (1)$$

For xe^{5x} we would try initially a solution of the form

$$e^{5x}(B_1 x + B_0) = B_1 xe^{5x} + B_0 e^{5x}$$

But this supposed solution would have, disregarding multiplicative constants, the term e^{5x} in common with y_h. We are led, therefore, to the modified expression

$$xe^{5x}(B_1 x + B_0) = e^{5x}(B_1 x^2 + B_0 x) \qquad (2)$$

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Example 7: Solve $y' - 5y = x^2e^x - xe^{5x}$.

Solution:

We now take y_p to be the sum of (1) and (2):

$$y_p = e^x(A_2x^2 + A_1x + A_0) + e^{5x}(B_1x^2 + B_0x) \tag{3}$$

Substituting (3) into the differential equation and simplifying, we obtain

$$e^x[(-4A_2)x^2 + (2A_2 - 4A_1)x + (A_1 - 4A_0)] + e^{5x}[(2B_1)x + B_0]$$
$$= e^x[(1)x^2 + (0)x + (0)] + e^{5x}[(-1)x + (0)]$$

Equating coefficients of like terms, we have

$$-4A_2 = 1 \quad 2A_2 - 4A_1 = 0 \quad A_1 - 4A_0 = 0 \quad 2B_1 = -1 \quad B_0 = 0$$

from which $A_2 = -\frac{1}{4} \quad A_1 = -\frac{1}{8} \quad A_0 = -\frac{1}{32} \quad B_1 = -\frac{1}{2} \quad B_0 = 0$

Equation (3) then gives $y_p = e^x\left(-\frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32}\right) - \frac{1}{2}x^2e^{5x}$

and the general solution is $y = c_1e^{5x} + e^x\left(-\frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32}\right) - \frac{1}{2}x^2e^{5x}$

- **Linear differential equations (second or higher order) with constant coefficients**
- **Non-homogeneous differential equation: $f(D)y = X = \phi(x) = F(x)$**
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Exercise-17

Solve, using the method of undetermined coefficients :

1. $y'' + y = \sin x$ **Ans.** $y = c_1 \cos x + c_2 \sin x - (x/2) \cos x$

1. $(D^2 + 2D + 1)\, y = x^2 - \cos x$ **Ans.** $y = (c_1 + c_2 x)\, e^{-x} + x^2 - 4x + 6 - (1/6) \sin x\, e^{-2x}$

3. $(D^2 - D - 2)\, y = 4x^2$ **Ans.** $y = c_1\, e^{2x} + c_2\, e^{-x} - 3 + 2x - 2x^2$

4. $(D^2 - 1)\, y = e^x \sin 2x.$ **Ans.** $y = c_1\, e^x + c_2\, e^{-x} - e^x (\sin 2x + \cos 2x)/8$

Thank you for your attendance and attention