- **Course Title : Differential Equations and Special Functions**
- **Course Code: MAT102**
- **Section-3**
- **Lecture-8 (March 15, 2023)**

Today's Lecture Topics:

- **Ordinary Differential Equations**

- **Linear differential equations (second or higher order) with constant coefficients**
- **Linear differential equations (second or higher order) with constant coefficients: The general form of non-homogeneous equations**

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■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Homogeneous equations**

➢ **Example 2: Solve** $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \ or \ (D^2 + 5D + 6)y = 0 \ \ where$

D **stands for** d/dx **and** D^2 **stands for** d^2/dx^2

Solution: Given differential equation is

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \ or \ (D^2 + 5D + 6)y = 0 \ \ --------- -(1)$$

Let $y = e^{mx}$ **be a trial solution of the equation** (1)**. Then the auxiliary equation of the equation** (1) **is**

$$m^2 + 5m + 6 = 0 \Rightarrow m^2 + 3m + 2m + 6 = 0 \Rightarrow (m + 3)(m + 2) = 0$$

$$\therefore m = -2, -3$$ (**Real and distinct roots**)

Therefore, the general solution of (1) is $y = c_1e^{-2x} + c_2e^{-3x} \ (Ans.)$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Homogeneous equations**

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny=0 \text{ ---------------- }(2)$$

$$\Downarrow$$

Auxiliary equation: $a_0m^n+a_1m^{n-1}+\cdots+a_{n-1}m+a_n=0 \text{ ---- }(3)$

<u>Case 2</u>: Repeated real roots

Suppose the roots of (3) are the n repeated real numbers $m, m, \ldots, m$

Then the general solution of (2) is

$$y=\left(c_1+c_2x+c_3x^2+\cdots+c_nx^{n-1}\right)e^{mx}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

■ Linear differential equations (second or higher order) with constant coefficients

➢ Homogeneous equations

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \text{ ------------------- } (2)$$

$$\Downarrow$$

Auxiliary equation: $a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \text{ ---- } (3)$

<u>Case 2 (i)</u>: Certain number of repeated real roots and the rest of the roots are distinct

Now if there are the k repeated real roots *m*, *m*, ... , *m* and the rest of the roots are distinct then the general solution of eq. (2) is

$$y = \left(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}\right) e^{mx} + c_{k+1} e^{m_{k+1} x} + c_{k+2} e^{m_{k+2} x} + \cdots + c_n e^{m_n x}$$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Homogeneous equations**

➢ **Example 1: Solve** $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0$

$$\text{or } \left(D^3 - 4D^2 - 3D + 18\right)y = 0$$

Solution: Given differential equation is

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0 --------------(1)$$

Let $y = e^{mx}$ **be a trial solution of the equation (1). Then, the auxiliary equation of (1) is** $m^3 - 4m^2 - 3m + 18 = 0$

$\Rightarrow m^3 - 3m^2 - m^2 + 3m - 6m + 18 = 0$

$\Rightarrow m^2(m-3) - m(m-3) - 6(m-3) = 0 \Rightarrow (m-3)\left(m^2 - m - 6\right) = 0$

$\Rightarrow (m-3)(m-3)(m+2) = 0 \therefore m = 3, 3, -2$ **(Repeated real roots)**

Therefore, the general solution of (1) **is** $y = (c_1 + c_2x)e^{3x} + c_3e^{-2x}$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Homogeneous equations**
- **Example 2: Solve** $\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 16\frac{dy}{dx} - 12y = 0$

$$\text{or } \left(D^3 - 7D^2 + 16D - 12\right)y = 0$$

Solution: Given differential equation is

$$\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 16\frac{dy}{dx} - 12y = 0 ---------------(1)$$

Let $y = e^{mx}$ be a trial solution of the equation (1). Then, the auxiliary equation of (1) is $m^3 - 7m^2 + 16m - 12 = 0$

$$\Rightarrow m^3 - 2m^2 - 5m^2 + 10m + 6m - 12 = 0$$

$$\Rightarrow m^2(m-2) - 5m(m-2) + 6(m-2) = 0 \Rightarrow (m-2)\left(m^2 - 5m + 6\right) = 0$$

$$\Rightarrow (m-2)(m-2)(m-3) = 0 \therefore m = 2, 2, 3$$ **(Repeated real roots)**

Therefore, the general solution of (1) is $y = (c_1 + c_2x)e^{2x} + c_3e^{3x}$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Homogeneous equations**

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 ------------- -(2)$$

Auxiliary equation: $a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0$ $--- -(3)$

Case 3: Conjugate complex roots

Suppose the auxiliary equation has two complex conjugate roots $a + ib$ and $a - ib$ which are non-repeated.

So, the general solution can be written as

$$y = k_1 e^{(a+ib)x} + k_2 e^{(a-ib)x} = k_1 e^{ax+ibx} + k_2 e^{ax-ibx}$$
$$= k_1 e^{ax}(\cos b\,x + i \sin b\,x) + k_2 e^{ax}(\cos b\,x - i \sin b\,x)$$
$$= e^{ax}[(k_1 + k_2) \cos b\,x + i(k_1 - k_2) \sin b\,x] \therefore y = e^{ax}(c_1 \cos b\,x + c_2 \sin b\,x)$$

where $c_1 = k_1 + k_2$ and $c_2 = i(k_1 - k_2)$ are new arbitrary constants.

- **Linear differential equations (second or higher order) with constant coefficients**

- **Homogeneous equations**
- **Example 1: Solve** $\frac{d^2y}{dx^2} + 4y = 0 \ or \ (D^2 + 4)y = 0$

Solution: Given differential equation is

$$\frac{d^2y}{dx^2} + 4y = 0 \ or \ (D^2 + 4)y = 0 \text{ ------------- } (1)$$

Let $y = e^{mx}$ **be a trial solution of the equation (1). Then the auxiliary equation of the equation (1) is**

$$m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m^2 = 4i^2$$

$$\therefore m = \pm 2i = 0 \pm 2i \ \ (i.e., 0 + 2i, 0 - 2i)$$

Therefore, the general solution of (1) is

$$y = e^{0x}(c_1 \cos 2x + c_2 \sin 2x) = c_1 \cos 2x + c_2 \sin 2x \ (Ans.)$$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Homogeneous equations**

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny=0 ----------------(2)$$

Auxiliary equation of (2): $a_0m^n+a_1m^{n-1}+\cdots+a_{n-1}m+a_n=0 \;--(3)$

Case 4: Conjugate complex roots (repeated)

(i) Suppose the roots of auxiliary equation (3) are $a\pm ib$ (occur twice).

So, the general solution of (2) can be written as

$y=e^{ax}[(c_1+c_2x)\cos b\,x+(c_3+c_4x)\sin bx]$ **containing four arbitrary constants.**

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Homogeneous equations**

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \text{ --------------- } (2)$$

Auxiliary equation of (2): $a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \ \text{--}(3)$

Case 4: Conjugate complex roots (repeated)

(ii) Suppose the roots of auxiliary equation (3) are $a \pm ib$ (occur thrice).

So, the general solution of (2) can be written as
$y = e^{ax}[(c_1 + c_2 x + c_3 x^2) \cos b\,x + (c_4 + c_5 x + c_6 x^2) \sin bx]$ **containing six arbitrary constants.**

- **Linear differential equations (second or higher order) with constant coefficients**

- **Homogeneous equations**

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny=0\text{ ---------------- }(2)$$

Auxiliary equation of (2): $a_0m^n+a_1m^{n-1}+\cdots+a_{n-1}m+a_n=0\ --(3)$

Case 4: Conjugate complex roots (repeated)

(iii) Suppose the roots of auxiliary equation (3) are $a\pm ib$ (occur k times)

So, the general solution of (2) can be written as

$y=e^{ax}\left[\left(c_1+c_2x+c_3x^2+\cdots+c_kx^{k-1}\right)\cos bx+\left(c_{k+1}+c_{k+2}x+c_{k+3}x^2+\cdots+c_{2k}x^{k-1}\right)\sin bx\right]$ **where** $c_1, c_2, \ldots\ldots, c_{2k}$ **are arbitrary constants.**

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Homogeneous equations**

➢ **Example 1: Solve** $[(D^2+1)^3(D^2+D+1)^2]y=0$

Solution: Given differential equation is

$$[(D^2+1)^3(D^2+D+1)^2]y=0 \text{ ------------- } -(1)$$

Let $y=e^{mx}$ **be a trial solution of the equation (1). Then the auxiliary equation of the equation (1) is**

$$(m^2+1)^3(m^2+m+1)^2=0$$

$$\Rightarrow m=0\pm i \text{ (thrice) and}$$

$$-\frac{1}{2}\pm i\frac{\sqrt{3}}{2} \text{ (twice)}$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Homogeneous equations**

- **Example 1: Solve** $[\left(D^2+1\right)^3\left(D^2+D+1\right)^2]y=0$

Solution: Given differential equation is

$$\left(D^2+1\right)^3\left(D^2+D+1\right)^2 y=0 \text{ ------------- } (1)$$

The roots of auxiliary equation of the equation (1) are

$$m=0\pm i \text{ (thrice) and } -\frac{1}{2}\pm i\frac{\sqrt{3}}{2} \text{ (twice)}$$

Therefore, the general solution of (1) is

$$y = e^{0x}\left[\left(c_1+c_2x+c_3x^2\right)cosx+\left(c_4+c_5x+c_6x^2\right)sinx\right] + e^{-\frac{x}{2}}\left[(c_7+c_8x)cos\frac{\sqrt{3}}{2}x+(c_9+c_{10}x)sin\frac{\sqrt{3}}{2}x\right](Ans.)$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Homogeneous equations**
- **Example 2: Solve** $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0, y(0) = -3, y'(0) = -1$.

Solution: Given differential equation is

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0 \quad ------------------ -(1)$$

Let $y = e^{mx}$ **be the trail solution of** (1)**. Then the auxiliary equation of (1) is** $m^2 - 6m + 25 = 0$

$$\Rightarrow m^2 - 2.m.3 + 9 + 16 = 0 \Rightarrow (m-3)^2 = -16$$

$$\Rightarrow (m-3)^2 = 16i^2$$

$$\Rightarrow (m-3)^2 = (4i)^2$$

$$\Rightarrow m - 3 = \pm 4i$$

$$\therefore m = 3 \pm 4i$$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Homogeneous equations**

➢ **Example 2: Solve** $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0, y(0) = -3, y'(0) = -1.$

Solution: Given differential equation is

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0 \;-------------------- -(1)$$

The roots of auxiliary equation of (1) is $m = 3 \pm 4i$. **Therefore the general solution of (1) is**

$$y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) \;-------------------- -(2)$$

$$\therefore y'$$

$$= 3e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + e^{3x}(-4c_1 \sin 4x + 4c_2 \cos 4x) - -(3)$$

When $x = 0, y = -3$, **we can write from (2) is as follows**

$$-3 = e^0(c_1 \cos 0 + c_2 \sin 0) \therefore c_1 = -3$$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Homogeneous equations**

➢ **Example 2: Solve** $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0, y(0) = -3, y'(0) = -1.$

Solution:

$$\therefore y' = 3e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + e^{3x}(-4c_1 \sin 4x + 4c_2 \cos 4x) - -(3)$$

Again, $x = 0, y' = -1$, **we can write from (3) is as follows**

$$-1 = 3e^0(c_1 \cos 0 + c_2 \sin 0) + e^0(-4c_1 \sin 0 + 4c_2 \cos 0)$$

$$\Rightarrow -1 = 3c_1 + 4c_2 \Rightarrow -1 = -9 + 4c_2(\text{ as } c_1 = -3) \Rightarrow 4c_2 = 8 \therefore c_2 = 2$$

Therefore, the general solution of the given differential equation is

$$y = e^{3x}(-3\cos 4x + 2\sin 4x) \text{ (Ans.)}$$

■ Linear differential equations (second or higher order) with constant coefficients

HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS → **Even numbered problems**

Solve the following differential equations :

1. (*a*) $(D^3+6D^2+11D+6)y=0$ **Ans.** $y=c_1e^{-x}+c_2e^{-2x}+c_3e^{-3x}$

(*b*) $d^2y/dx^2+2(dy/dx)+5y=0$ **Ans.** $y=e^{-x}(c_1\cos 4x+c_2\sin 4x)$

(*c*) $d^3y/dx^3-6(d^2y/dx^2)+9(dy/dx)=0$ **Ans.** $y=c_1+(c_2+xc_3)e^{3x}$

2. $(D^3+6D^2+12D+8)y=0.$ **Ans.** $y=(c_1+c_2x+c_3x^2)e^{-2x}$

3. $(d^2y/dx^2)+2p(dy/dx)+(p^2+q^2)y=0.$ **Ans.** $y=e^{-px}(c_1\cos qx+c_2\sin qx)$

4. $(D^4-2D^3+5D^2-8D+4)y=0.$ **Ans.** $y=(c_1+c_2x)e^x+c_3\cos 2x+c_4\sin 2x$

5. $(D^4+D^2+1)y=0.$

Ans. $y=e^{x/2}[c_1\cos(x\sqrt{3}/2)+c_2\sin(x\sqrt{3}/2)]+e^{-x/2}[c_3\cos(x\sqrt{3}/2)+c_4\sin(x\sqrt{3}/2)]$

6. $(D^4+4D^3-5D^2-36D-36)y=0.$ **Ans.** $y=c_1e^{-3x}+c_2e^{3x}+(c_3+c_4x)e^{-2x}$

7. $(D^4-7D^3+18D^2-20D+8)y=0.$ **Ans.** $y=c_1e^x+(c_2+c_3x+c_4x^2)e^{2x}$

8. $(D^2\pm w^2)y=0,\ w\neq 0.$ **Ans.** $y=c_1\cos wx+c_2\sin wx+c_3e^{wx}+c_4e^{-wx}$

9. $\{D^3+D^2(2\sqrt{3}-1)+D(3-2\sqrt{3})-3\}y=0$ **Ans.** $y=c_1e^x+(c_2+c_3x)e^{-x\sqrt{3}}$

10. (*a*) $(D^5-13D^3+26D^2+82D+104)y=0$

Ans. (*a*) $y=c_1e^{-4x}+e^{-x}(c_2\cos x+c_3\sin x)+e^{3x}(c_4\cos 2x+c_5\sin 2x)$

➢ ☞ Instructions:

The following slides are given to all of you in advance for your convenience, and please check and try to understand the given slides (lecture martials) before joining the next class.

If you do so, then it would be easier for you to understand the next class. Besides, you will feel comfortable in asking relevant questions in the classroom, if any!

- **Linear differential equations (second or higher order) with constant coefficients**

- **The general form of non-homogeneous equations is as follows**

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = F(x) \text{ or } X \text{ ------- } -(1)$$

or, $f(D)y = F(x)$ *or* X ---------------------- $-(2)$

where $f(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$ where $D \equiv \frac{d}{dx}$

- **Inverse operator: The particular integral, y_p of the differential equation having the form $f(D)y = X$ is given by**

$$\frac{1}{f(D)} f(D) y_p = \frac{1}{f(D)} X \text{ or } y_p = \frac{1}{f(D)} X$$

- **Linear differential equations (second or higher order) with constant coefficients**
- **Inverse operator: The particular integral, y_p of the differential equation having the form, $f(D)y = X$ is given by**

$$\frac{1}{f(D)} f(D) y_p = \frac{1}{f(D)} X$$

or $y_p = \frac{1}{f(D)} X$ ---------------------- (3)

- **Working rule for finding y_p when X is the form of e^{ax}**

- **Formulae I : $y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ provided that $f(a) \neq 0$**

■ **Linear differential equations (second or higher order) with constant coefficients**

■ **Working rule I for finding** $y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ **provided that** $f(a) \neq 0$

■ **Example:1 Solve** $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{-x} + 3e^{2x}$.

➢ **Solution: Given the differential equation is**

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{-x} + 3e^{2x} \text{ ------------------ } -(1)$$

Let $y = e^{mx}$ be a trial solution of the homogeneous equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0 \text{ ------------------------ } -(2).$$

Then the auxiliary equation of (2) is $m^2 - 6m + 25 = 0$

$$\Rightarrow m^2 - 2.m.3 + 9 + 16 = 0 \Rightarrow (m-3)^2 = -16$$

- **Linear differential equations (second or higher order) with constant coefficients**
- **Working rule I for finding** $y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ **provided that** $f(a) \neq 0$
- **Example:1 Solve** $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{-x} + 3e^{2x}$.

➢ **Solution:**

Then the auxiliary equation of (2) is $m^2 - 6m + 25 = 0$

$$\Rightarrow m^2 - 2.m.3 + 9 + 16 = 0$$
$$\Rightarrow (m-3)^2 = -16$$
$$\Rightarrow (m-3)^2 = 16i^2 \Rightarrow (m-3)^2 = (4i)^2$$
$$\therefore m = 3 \pm 4i$$

Therefore, the complementary solution (y_c) of (1) is

$$y_c = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) \text{ ----------------- } (3)$$

- **Linear differential equations (second or higher order) with constant coefficients**
- **Working rule I for finding** $y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ **provided that** $f(a) \neq 0$
- **Example:1 Solve** $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{-x} + 3e^{2x}$.

➢ **Solution: Now the particular integral is**

$$y_p = \frac{1}{D^2-6D+25}\left(e^{-x} + 3e^{2x}\right)$$

$$= \frac{1}{D^2-6D+25} e^{-x} + \frac{1}{D^2-6D+25} 3e^{2x}$$

$$= \frac{1}{(-1)^2-6(-1)+25} e^{-x} + \frac{1}{(2)^2-6(2)+25} 3e^{2x}$$

$$= \frac{e^{-x}}{1+6+25} + \frac{3e^{2x}}{4-12+25} = \frac{1}{32} e^{-x} + \frac{3}{17} e^{2x}$$

Therefore, the general solution is

$$y = y_c + y_p = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + \frac{1}{32} e^{-x} + \frac{3}{17} e^{2x} \text{ (Ans.)}$$

- **Linear differential equations (second or higher order) with constant coefficients**

HW

- **Working rule II for finding particular integral ,** $y_p = \frac{1}{f(D)} X$

where X **is** x^m **or a polynomial of degree** m**,** m **being positive integer.**

Short method of finding P.I. when $X = x^m$, m being a positive integer.

Working rule for evaluating $\{1/f(D)\}\, x^m$.

Step I. Bring out the lowest degree term from $f(D)$ so that the remaining factor in the denominator is of the form $[1 + \phi(D)]^n$ or $[1 - \phi(D)]^n$, n being a positive integer.

Step II. We take $[1 + \phi(D)]^n$ or $[1 - \phi(D)]^n$ in the numerator so that it takes the form $[1 + \phi(D)]^{-n}$ or $[1 - \phi(D)]^{-n}$.

Step III. We expand $[1 \pm \phi(D)]^{-n}$ by the binomial theorem, namely

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

In particular the following binomial expansions should be remembered.

$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$; $\quad (1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$; $\quad (1 - x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

In any case, the expansion is to be carried upto D^m, since $D^{m+1} x^m = 0$, $D^{m+2} x^m = 0$, and all the higher differential coefficients of x^m vanish.

- **Linear differential equations (second or higher order) with constant coefficients**

HW

- **Working rule II for finding particular integral , $y_p = \frac{1}{f(D)}X$**

where X is x^m or a polynomial of degree m, m being positive integer.

- **Example:2 Solve** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6$

➢ **Solution: The given differential equation is**

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6 \quad -------------(1)$$

Let $y = e^{mx}$ be a trial solution of the homogeneous equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 0 \quad -------------------(2)$$

Then the auxiliary equation of (2) is $m^2 + 4m - 2 = 0$

$$\Rightarrow m^2 + 2.m.2 + 4 - 6 = 0 \Rightarrow (m+2)^2 = 6 \Rightarrow m + 2 = \pm\sqrt{6}$$

$$\therefore m = -2 \pm \sqrt{6}$$

- **Linear differential equations (second or higher order) with constant coefficients**

HW

- **Working rule II for finding particular integral,** $y_p = \frac{1}{f(D)}X$

  where X is x^m or a polynomial of degree m, m being positive integer.

- **Example:2 Solve** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6$

➢ **Solution: The given differential equation is**

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6 \quad \text{-----------} (1)$$

Therefore, the complementary solution of (1) is

$$y_c = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{-(2-\sqrt{6})x}$$

when the roots of auxiliary equation of

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y =$**0 are** $-2+\sqrt{6}$ **and** $-2-\sqrt{6}$**.**

- **Linear differential equations (second or higher order) with constant coefficients**

HW

- **Working rule II for finding particular integral , $y_p = \frac{1}{f(D)}X$**

  where X is x^m or a polynomial of degree m, m being positive integer.

- **Example:2 Solve** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6$

➢ **Solution:**

Now the particular integral is $y_p = \frac{1}{D^2+4D-2}\left(2x^2 - 3x + 6\right)$

$$= \frac{1}{-2\left(1 - 2D - \frac{1}{2}D^2\right)}\left(2x^2 - 3x + 6\right)$$

$$= -\frac{1}{2}\left(1 - 2D - \frac{1}{2}D^2\right)^{-1}\left(2x^2 - 3x + 6\right)$$

$$= -\frac{1}{2}\left\{1 - \left(2D + \frac{1}{2}D^2\right)\right\}^{-1}\left(2x^2 - 3x + 6\right)$$

- **Linear differential equations (second or higher order) with constant coefficients**

HW

- **Working rule II for finding particular integral , $y_p = \frac{1}{f(D)} X$**

where X is x^m or a polynomial of degree m, m being positive integer.

- **Example:2 Solve** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6$

➢ **Solution:**

Now the particular integral is

$$y_p = -\frac{1}{2}\left\{1 + \left(2D + \frac{1}{2}D^2\right) + \left(2D + \frac{1}{2}D^2\right)^2 + \cdots\right\}(2x^2 - 3x + 6)$$

$$\left[\text{We know that } (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\right]$$

$$= -\frac{1}{2}\left(1 + 2D + \frac{1}{2}D^2 + 4D^2\right)(2x^2 - 3x + 6)$$

$$= -\frac{1}{2}\left(1 + 2D + \frac{9}{2}D^2\right)(2x^2 - 3x + 6) \text{ -------- }(2)$$

- **Linear differential equations (second or higher order) with constant coefficients**

HW

- **Working rule II for finding particular integral ,** $y_p = \frac{1}{f(D)} X$ **where** X **is** x^m **or a polynomial of degree** m**,** m **being positive integer.**

- **Example:2 Solve** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = 2x^2 - 3x + 6$

➢ **Solution: Now the particular integral is**

$$y_p = -\frac{1}{2}\left(2x^2 - 3x + 6 + 8x - 6 + 18\right)$$

$$\left[\text{by using } Eq.(2): -\frac{1}{2}\left(1 + 2D + \frac{9}{2}D^2\right)\left(2x^2 - 3x + 6\right)\right]$$

$$= -\frac{1}{2}\left(2x^2 + 5x + 18\right) = -x^2 - \frac{5}{2}x - 9$$

Therefore, the general solution is $y = y_c + y_p$

$$= c_1 e^{(-2+\sqrt{6})x} + c_2 e^{-(2-\sqrt{6})x} - x^2 - \frac{5}{2}x - 9 \text{ (Ans.)}$$

- **Linear differential equations (second or higher order) with constant coefficients**

HW

- **Non-homogeneous equation**
- Inverse operator: The particular integral, y_p of the differential equation having the form, $f(D)y = X$ is given by

$$\frac{1}{f(D)} f(D) y_p = \frac{1}{f(D)} X$$

or $$y_p = \frac{1}{f(D)} X \text{ ------------------------ } (3)$$

- **If $f(a) = 0$ then $f(D)$ must possess a factor of the type $(D-a)^r$. Under this condition, we have to use the formulae II.**

- **Formulae II: If $y_p = \frac{1}{f(D)} X = \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$ where r = 1,2,3…**

■ **Linear differential equations (second or higher order) with constant coefficients**

HW

➢ **Non-homogeneous differential equation**

■ **Formulae II: If** $y_p = \frac{1}{f(D)}X = \frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}$ **where r = 1,2,3…**

■ **Example: 1 Find** $y_p = \frac{1}{D^3-D^2-D+1}e^x$

➢ **Solution: Here** $f(D) = D^3 - D^2 - D + 1$

$$\therefore f(1) = 1^3 - 1^2 - 1 + 1 = 0$$

Therefore, we have to factorize $f(D)$ *and* **let us do it. Therefore**

$f(D) = D^2(\mathrm{D}-1) - 1(\mathrm{D}-1) = \left(D^2 - 1\right)(\mathrm{D}-1) = (D-1)^2(D+1)$

Thus, we can write, $y_p = \frac{1}{(D-1)^2(D+1)}e^x$

$$= \frac{1}{(D-1)^2}\left[\frac{1}{(D+1)}e^x\right] = \frac{1}{(D-1)^2}\frac{1}{2}e^x = \frac{1}{2}\frac{1}{(D-1)^2}e^x = \frac{1}{2}\frac{x^2}{2!} = \frac{1}{4}x^2$$

Thank you for your attendance and attention