Discrete Mathematics

Relations

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Cartesian product (review)

Let $A=\{a_1, a_2, ..a_k\}$ and $B=\{b_1,b_2,..b_m\}$.

The Cartesian product A x B is defined by a set of pairs
$\{(a_1\ b_1), (a_1, b_2), \ldots\ (a_1, b_m), \ldots, (a_k,b_m)\}$.

Cartesian product defines a product set, or a set of all ordered arrangements of elements in sets in the Cartesian product.

Binary relation

Definition: Let A and B be two sets. A **binary relation from A to B** is a subset of a Cartesian product A x B.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$.
- We use the notation **a R b to denote (a,b) $\in$ R** and **a $\not R$ b to denote (a,b) $\notin$ R**. If **a R b**, we say a is related to b by R.

Example: Let A={a,b,c} and B={1,2,3}.

- Is R={(a,1),(b,2),(c,2)} a relation from A to B? **Yes.**
- Is Q={(1,a),(2,b)} a relation from A to B? **No.**
- Is P={(a,a),(b,c),(b,a)} a relation from A to A? **Yes**

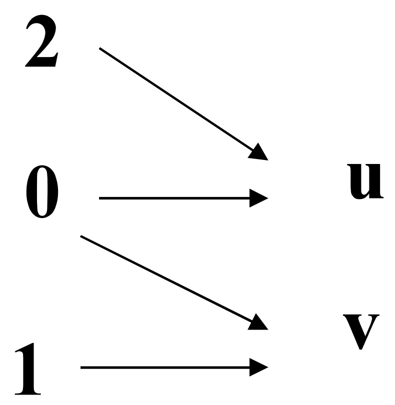
Representing binary relations

- We can graphically represent a binary relation R as follows:
  - if **a R b** then draw an arrow from a to b.

$$\mathbf{a} \rightarrow \mathbf{b}$$

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- Note: $R \subseteq A \times B$.
- **Graph:**



Representing binary relations

- We can represent a binary relation R by a **table** showing (marking) the ordered pairs of R.

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- **Table:**

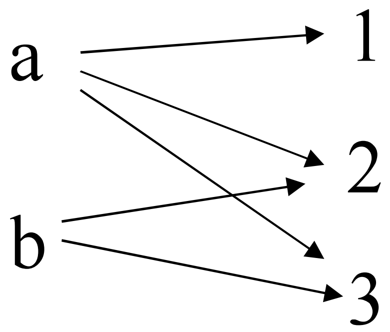
R	u	v
0	x	x
1		x
2	x	

or

R	u	v
0	1	1
1	0	1
2	1	0

Relations and functions

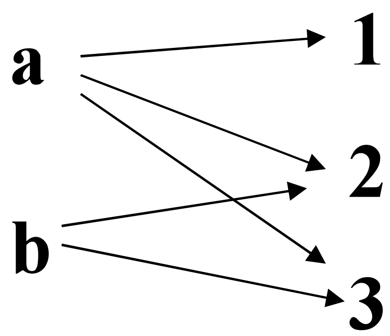
- Relations represent **one to many relationships** between elements in A and B.
- **Example:**



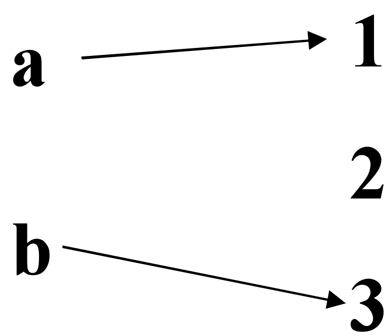
- What is the difference between a **relation and a function from A to B**?

Relations and functions

- Relations represent **one to many relationships** between elements in A and B.
- **Example:**



- What is the difference between a **relation and a function from A to B**? A function defined on sets A,B $A \rightarrow B$ assigns to each element in the domain set A exactly one element from B. So it is **a special relation.**



Relation on the set

<u>Definition:</u> **A relation on the set A** is a relation from A to itself.

Example 1:

- Let A = {1,2,3,4} and R_{div} = {(a,b)| a divides b}
- What does R_{div} consist of?
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
-

R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Relation on the set

Example:

- Let A = {1,2,3,4}.
- Define a $R_{\neq}$ b if and only if a ≠ b.

$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

-

R	1	2	3	4
1		x	x	x
2	x		x	x
3	x	x		x
4	x	x	x	

Binary relations

- **Theorem:** The number of binary relations on a set A, where $|A| = n$ is:

$$2^{n^2}$$

- **Proof:**

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- If $|A| = n$ then the cardinality of the Cartesian product $|A \times A| = n^2$.
- R is a binary relation on A if $R \subseteq A \times A$ (that is, R is a subset of A x A).
- The number of subsets of a set with k elements : 2^k
- The number of subsets of A x A is : $2^{|AxA|} = 2^{n^2}$

Properties of relations

Definition (reflexive relation) : A relation R on a set A is called **reflexive** if $(a,a) \in R$ for every element $a \in A$.

Example 1:

- Assume relation $R_{div} = \{(a\ b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- **Is R_{div} reflexive?**
- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Answer:** Yes. $(1,1), (2,2), (3,3)$, and $(4,4) \in A$.

Reflexive relation

Reflexive relation

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

$$MR_{div} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

- **A relation R is reflexive** if and only if MR has 1 in every position on its main diagonal.

Properties of relations

Definition (reflexive relation) : A relation R on a set A is called **reflexive** if $(a,a) \in R$ for every element $a \in A$.

Example 2:

- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} reflexive?**
- **No.** It is not reflexive since $(1,1) \notin R_{fun}$.

Properties of relations

Definition (irreflexive relation): A relation R on a set A is called **irreflexive** if **(a,a) $\notin$ R for every a $\in$ A.**

Example 1:

- Assume relation $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a $\neq$ b.
- **Is $R_{\neq}$ irreflexive?**
- **$R_{\neq}$ = …**

Properties of relations

Definition (irreflexive relation): A relation R on a set A is called **irreflexive** if **(a,a) $\notin$ R for every a $\in$ A.**

Example 1:

- Assume relation $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a $\neq$ b.
- **Is $R_{\neq}$ irreflexive?**
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**
- **Answer:** Yes. Because (1,1),(2,2),(3,3) and (4,4) $\notin$ $R_{\neq}$

Irreflexive relation

Irreflexive relation

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if a $\neq$ b.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

- **A relation R is irreflexive** if and only if MR has 0 in every position on its main diagonal.

Properties of relations

Definition (irreflexive relation): A relation R on a set A is called **irreflexive** if **(a,a) $\notin$ R for every a $\in$ A.**

Example 2:

- R_{fun} on A = {1,2,3,4} defined as:
    - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} irreflexive?**
- **Answer: No**. Because (2,2) and (3,3) $\in$ R_{fun}

Properties of relations

Definition (symmetric relation): A relation R on a set A is called **symmetric** if

$$\forall\, a, b \in A \quad (a,b) \in R \rightarrow (b,a) \in R.$$

Example 1:

- $R_{div} = \{(a\ b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- **Is R_{div} symmetric?**
- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Answer: No.** It is not symmetric since $(1,2) \in R$ but $(2,1) \notin R$.

Properties of relations

Definition (symmetric relation): A relation R on a set A is called **symmetric** if

$$\forall\, a, b \in A \quad (a,b) \in R \rightarrow (b,a) \in R.$$

Example 2:

- $R_{\neq}$ on A={1,2,3,4}, such that $\mathbf{a}\ \mathbf{R_{\neq}}\ \mathbf{b}$ if and only if $a \neq b$.
- **Is $R_{\neq}$ symmetric ?**
- $\mathbf{R_{\neq}=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}}$
- **Answer: Yes. If** $(a,b) \in R_{\neq} \rightarrow (b,a) \in R_{\neq}$

Symmetric relation

Symmetric relation:

- $R_{\neq}$ on A={1,2,3,4}, such that **a $R_{\neq}$ b** if and only if $a \neq b$.
- **$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}**

$$MR = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

- **A relation R is symmetric** if and only if $m_{ij} = m_{ji}$ for all i,j.

Properties of relations

Definition (symmetric relation): A relation R on a set A is called **symmetric** if

$$\forall\, a, b \in A \quad (a,b) \in R \rightarrow (b,a) \in R.$$

Example 3:

- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} symmetric?**
- **Answer: No.** For $(1,2) \in R_{fun}$ there is no $(2,1) \in R_{fun}$

Anti-symmetric relation

Definition (anti-symmetric relation): A relation on a set A is called **anti-symmetric** if

- $[(a,b) \in R \text{ and } (b,a) \in R] \rightarrow a = b$ where $a, b \in A$.

Example 3:

- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} anti-symmetric?**
- **Answer: Yes.** It is anti-symmetric

Anti-symmetric relation

Antisymmetric relation

- relation $R_{fun} = \{(1,2),(2,2),(3,3)\}$

$$MR_{fun} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

- A relation is **antisymmetric** if and only if $m_{ij} = 1 \rightarrow m_{ji} = 0$ for $i \neq j$.

Transitive relation

Definition (transitive relation): A relation R on a set A is called **transitive** if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 1:**
- $R_{div} = \{(a\ b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is R_{div} transitive?**
- **Answer:**

Transitive relation

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- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 1:**
- $R_{div} = \{(a\ b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$
- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$
- **Is R_{div} transitive?**
- **Answer: Yes.**

Transitive relation

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- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 2**:
- $R_{\neq}$ on $A=\{1,2,3,4\}$, such that $\mathbf{a\ R_{\neq}\ b}$ if and only if $a \neq b$.
- $\mathbf{R_{\neq}=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}}$
- **Is $R_{\neq}$ transitive ?**
- **Answer:**

Transitive relation

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- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 2**:
- $R_{\neq}$ on $A=\{1,2,3,4\}$, such that **a $R_{\neq}$ b** if and only if $a \neq b$.
- $\mathbf{R_{\neq}=\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}}$
- **Is $R_{\neq}$ transitive?**
- **Answer: No.** It is not transitive since $(1,2) \in R$ and $(2,1) \in R$ but $(1,1)$ is not an element of R.

Transitive relations

Definition (transitive relation): A relation R on a set A is called **transitive** if

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- **Example** 3:
- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} transitive?**
- **Answer:**

Transitive relations

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- **Example** 3:
- Relation R_{fun} on $A = \{1,2,3,4\}$ defined as:
  - $R_{fun} = \{(1,2),(2,2),(3,3)\}$.
- **Is R_{fun} transitive?**
- **Answer: Yes.** It is transitive.

Transitive relations

Exercise: Let $A = \{1,2,3\}$ and R be the relation on A whose matrix is

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that R is transitive. (**Hint**: Check if $(\mathbf{M}_R)^2_{\odot} = \mathbf{M}_R$)

Equivalence relation

Definition: A relation R on a set A is called an **equivalence relation** if it is **reflexive, symmetric and transitive.**

***Refer the book for more details*