Cauchy-Riemann Equation

Q:1

(a) $f(z) = \bar{z}$

$$= x - iy$$

$$= u(x,y) + iv(x,y)$$

$u(x,y) = x \quad ; \quad v(x,y) = -y$

$U_x = 1 \qquad U_y = 0$

$V_x = 0 \qquad V_y = -1$

Here, $U_x \neq V_y$

Since, the first cauchy-riemann equation is not satisfied, $f(z) = \bar{z}$ does not satisfy necessary condition Therefore, $f'(z)$ does not exist at any point z.

(b) $f(z) = z - \bar{z}$

$$= x + iy - (x - iy)$$

$$= x + iy - x + iy$$

$$= 2iy \quad = iv(x,y)$$

$u(x,y) = 0 \quad ; \quad V(x,y) = 2y$

$U_x = 0 \qquad U_y = 0$

$V_x = 0 \qquad V_y = 2$

Since, the cauchy-riemann equation does not satisfy the conditions, $U_x \neq V_y$, . Therefore $f'(x)$ does not exist at any point z.

(c) $f(z) = 2x + ixy^2$

$$f(z) = 2x + ixy^2$$
$$= u(x,y) + iv(x,y)$$

$u(x,y) = 2x$; $v(x,y) = xy^2$

$U_x = 2$ $\quad$ $U_y = 0$

$V_x = y^2$ $\quad$ $V_y = 2y$

Here, $U_x \neq V_y$

$-V_x \neq U_y$

Since, Cauchy Riemann equations are not satisfied, Therefore $f'(z)$ does not exist at any point z.

(d) $f(z) = e^x e^{-iy}$

$$f(z) = e^x \cos(y) - ie^x \sin(y)$$
$$= u(x,y) + iv(x,y)$$

$u(x,y) = e^x \cos(y)$; $v(x,y) = -e^x \sin(y)$

$U_x = e^x \cos(y)$, $U_y = -e^x \sin(y)$

$V_x = -e^x \sin(y)$, $V_y = -e^x \cos(y)$

Here, $U_x \neq V_y$

$U_y \neq -V_x$

Since, Cauchy Riemann equations are not satisfied. Therefore $f'(z)$ does not exist at any point z.

Q : 2

(a) $f(z) = iz + 2$

$$f(z) = i(x+iy) + 2$$
$$= ix - y + 2$$
$$= 2 - y + ix \quad = u(x,y) + iv(x,y)$$

where, $u(x,y) = 2-y \quad ; \quad v(x,y) = x$

$$U_x = 0 \qquad V_x = 1$$
$$U_y = -1 \qquad V_y = 0$$

$U_x = V_y = 0$ and $U_y = -V_x = -1$

Since, Cauchy Riemann equation is satisfied.
$f(z)$ is differentiable everywhere in the complex plane.

$$\therefore f'(z) = \frac{d}{dz}(iz+2) = i$$

$$\therefore f''(z) = \frac{d}{dz}(i) = 0$$

(b) $f(z) = e^{-x}e^{-iy}$

$$f(z) = e^{-x}(\cos y - i\sin y)$$
$$= e^{-x}\cos y - ie^{-x}\sin y \quad = u(x,y) + iv(x,y)$$

$u(x,y) = e^{-x}\cos y$

$v(x,y) = -e^{-x}\sin y$

$$U_x = -e^{-x}\cos y \qquad U_y = -e^{-x}\sin y$$
$$V_x = e^{-x}\sin y \qquad V_y = -e^{-x}\cos y$$

Since, $U_x = V_y = -e^{-x}\cos y$ and $U_y = -V_x = -e^{-x}\sin y$

Cauchy Riemann equation is satisfied. $f(z)$ is differentiable everywhere in the complex plane.

$$\therefore f'(z) = \frac{d}{dz} e^{-x} e^{-iy}$$
$$= \frac{d}{dz} e^{-(x+iy)}$$
$$= \frac{d}{dz} e^{-z}$$
$$= -e^{-z}$$
$$\therefore f''(z) = \frac{d}{dz}(-e^{-z})$$
$$= e^{-z} = f(z)$$

(c) $f(z) = z^3$

$$f(z) = z^3 = (x+iy)^3$$
$$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$$
$$\therefore f(z) = x^3 + 3ix^2y - 3xy^2 - iy^3$$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$
$$= u(x,y) + iv(x,y)$$

Here, $u(x,y) = x^3 - 3xy^2$

$v(x,y) = 3x^2y - y^3$

$U_x = 3x^2 - 3y^2$ $\qquad$ $U_y = -6xy$

$V_x = 6xy$ $\qquad$ $V_y = 3x^2 - 3y^2$

$U_x = V_y = 3x^2 - 3y^2$ and $U_y = -V_x = -6xy$

$\therefore$ Cauchy Riemann equation satisfied. $f(z)$ is differentiable everywhere in the complex plane.

$$\therefore f'(z) = \frac{d}{dz} z^3$$

$$= 3z^2$$

$$\therefore f''(z) = \frac{d}{dz}(3z^2)$$

$$= 6z$$

(d) $f(z) = \cos x \cosh y - i \sin x \sinh y$

$f(z) = u(x,y) + iv(x,y)$

where, $u(x,y) = \cos x \cosh y$

$v(x,y) = -\sin x \sinh y$

$u_x = -\sin x \cosh y$	$u_y = \cos x \sinh y$
$v_x = -\cos x \sinh y$	$v_y = -\sin x \cosh y$

$u_x = v_y = -\sin x \cosh y$ and $-v_x = u_y = \cos x \sinh y$
Cauchy Riemann equation is satisfied, $f(z)$ is differentiable everywhere in the complex plane.

$$\therefore f'(z) = u_x + iv_y = -\sin x \cosh y - i \cos x \sinh y$$

$$= -(\sin x \cosh y + i \cos x \sinh y)$$

$$\therefore f''(z) = \frac{d}{dx}(-\sin x \cosh y) - \frac{d}{dx} i(\cos x \sinh y)$$

$$= -\cos x \cosh y + i \sin x \sinh y$$

$$= -f(z)$$

Q : 3

(a) $f(z) = \frac{1}{z}$

$$f(z) = \frac{1}{z} = \frac{1}{x+iy}$$

$$= \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$\therefore f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$u(x,y) = \frac{x}{x^2+y^2}$$

$$v(x,y) = -\frac{y}{x^2+y^2}$$

$$U_x = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad U_y = \frac{-2xy}{(x^2+y^2)^2}$$

$$V_x = \frac{2xy}{(x^2+y^2)^2} \qquad V_y = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Since Cauchy Riemann equations are satisfied because $U_x = V_y$ and $U_y = -V_x$, $f(z) = \frac{1}{z}$ is differentiable everywhere except $z=0$

$$\therefore f'(z) = \frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2}$$

Therefore $f'(z)$ exists everywhere except at $z=0$ and it's value is $f'(z) = -\frac{1}{z^2}$