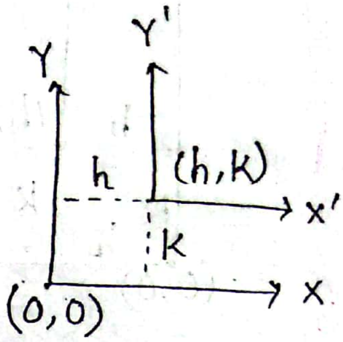
MAT104

1st October 2023

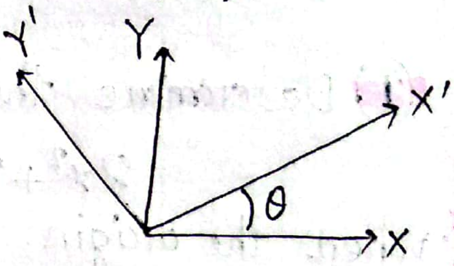
Change of Axes

Uses: graphics, robotics

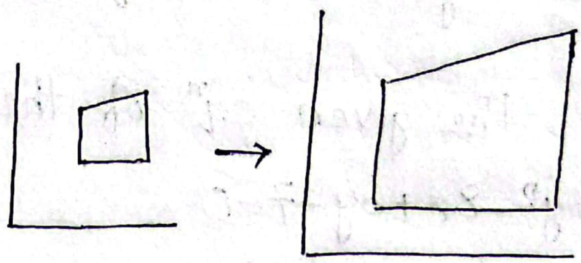


- Translation: origin কে shift করা
  (Shifting property)

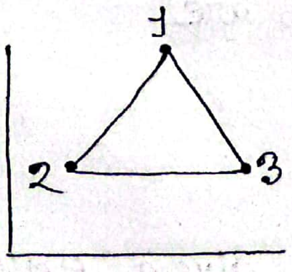
- Rotation: Origin কে fix রেখে অক্ষকে Rotate করলাম

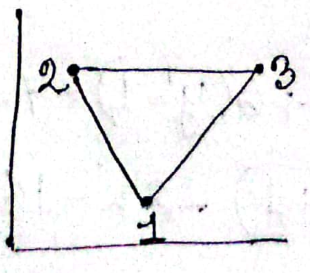


- Scaling:
  Resize করা

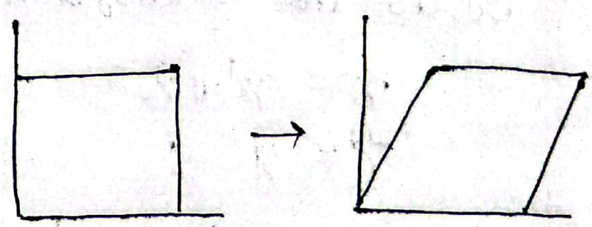


- Reflection:





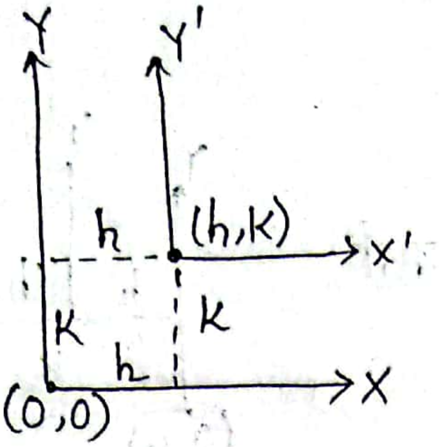
- Shear:



*** Translation of axes

*** Rotation of axes

Translation of axes :-



$x = x' + h$

$y = y' + h$

$(h,k) \rightarrow$ Shifting / Translating point

(1) Determine the equation of the curve :

$$2x^2+3y^2-8x+6y-7=0$$

when the origin is transferred to the point $(2,-1)$.

Answer: The given eq^n of the curve is

$$2x^2+3y^2-8x+6y-7=0 \quad \text{------} (1)$$

Origin is transferred to the point $(h,k)=(2,-1)$

So, as the transformed relation's are

$x = x'+2$

$y = y'-1$

Using the above tranformation given equation become,

$$2(x'+2)^2+3(y'-1)^2-8(x'+2)+6(y'-1)-7=0$$

$$\Rightarrow 2(x'^2+4x'+4)+3(y'^2-2y'+1)-8x'-16+6y'-6-7=0$$

$\Rightarrow 2x'^2 + 8x' + 8 + 3y'^2 - 6y' + 3 - 8x' - 16 + 6y' - 6 - 7 = 0$

$\Rightarrow 2x'^2 + 3y'^2 - 18 = 0$

$\Rightarrow 2x'^2 + 3y'^2 = 18.$

$\therefore \dfrac{x'^2}{9} + \dfrac{y'^2}{6} = 1$, which is the required equation.

② What does the equation : $x^2+y^2-4x-6y+6=0$ becomes when the origin is transferred to point (2, 3) and the direction of axis remain unchange

$x = x'+2$
$y = y'+3$

$(x'+2)^2 + (y'+3)^2 - 4(x'+2) - 6(y'+3) + 6 = 0$

$\Rightarrow (x'^2 + 4x' + 4) + y'^2 + 6y' + 9 - 4x' - 8 - 6y' - 18 + 6 = 0$

$\Rightarrow x'^2 + y'^2 - 7 = 0$

$\therefore x'^2 + y'^2 = 7$, which is the required equation.

③ Transform the equation $3x^2 + 14xy - 24y^2 - 22x + 110y - 121 = 0$

Shifting the origin to the point (-1, 2) and keeping the direction of axes fixed.

The given equation of the curve is

$$3x^2 + 14xy - 24y^2 - 22x + 110y - 121 = 0 \quad \text{-----} (1)$$

Origin is transferred to the point $(-1, 2)$

So, the transformed relations are,

$$x = x' - 1$$
$$y = y' + 2$$

Using the transformed relations in equation (1) we get,

$$3(x'-1)^2 + 14(x'-1)(y'+2) - 24(y'+2)^2 - 22(x'-1)$$
$$+ 110(y'+2) - 121 = 0$$

$$\Rightarrow 3(x'^2 - 2x' + (1)^2) + 14(x'y' + 2x' - y' - 2)$$
$$-24(y'^2 + 4y' + 4) - 22x' + 22 + 110y' + 220 - 121 = 0$$

$$\Rightarrow 3x'^2 - 6x' + 3 + 14x'y' + 28x' - 14y' - 28 - 24y'^2 - 96y'$$
$$-96 - 22x' + 22 + 110y' + 220 - 121 = 0$$

$$\Rightarrow 3x'^2 - 24y'^2 + 14x'y' = 0$$

$$\Rightarrow 3x'^2 - 24y'^2 + 14x'y' = 0$$

$\therefore$ This is the required equation.

(4) Transform to the parallel axis through the point $(3,5)$ the equation $x^2+y^2-6x-10y-2=0$

Ans: The given equation of the curve is,

$$x^2+y^2-6x-10y-2=0 \quad \text{------ (i)}$$

Origin is transferred to the point $(3,5)$, so the transferred relations are,

$$x = x'+3$$
$$y = y'+5$$

Using the transformed relations in the equation(i) we get,

$$(x'+3)^2+(y'+5)^2-6(x'+3)-10(y'+5)-2=0$$

$$\Rightarrow x'^2+6x'+9+y'^2+10y'+25-6x'-18-10y'-50-2=0$$

$$\Rightarrow x'^2+y'^2-36=0$$

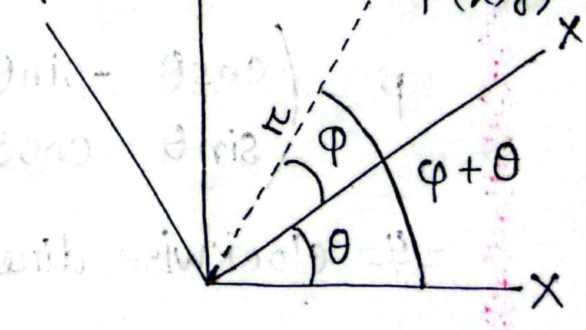
$\therefore x'^2+y'^2=36$, which is the required equation.

Rotation of axes:-

(x', y')

$x' = r\cos\varphi$

$y' = r\sin\varphi$



(x, y)

$x = r(\cos(\theta+\varphi))$

$y = r(\sin(\theta+\varphi))$

$$\begin{cases} \cos(a+b) = \cos a \cos b - \sin a \sin b \\ \sin(a+b) = \sin a \cos b + \cos a \sin b \end{cases}$$

$$\therefore x = r\cos(\theta+\varphi)$$
$$= r\cos\theta\cos\varphi - r\sin\theta\sin\varphi$$
$$= x'\cos\theta - y'\sin\theta$$

$$\therefore y = r\sin(\theta+\varphi)$$
$$= r\sin\theta\cos\varphi + r\sin\varphi\cos\theta$$
$$= x'\sin\theta + y'\cos\theta$$

$$\boxed{\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}}$$

Matrix Form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$X = PX'$$

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

\# clockwise direction $\theta = -\theta$

(1) Transform the equation $3x^2+5y^2-3=0$, to axes turned through 45°.

The given equation is $3x^2+5y^2-3=0$ ----(i)

Axes turned through $\theta = 45^\circ$, So as the tranformed relations are,

$$x = x'\cos 45^\circ - y'\sin 45^\circ = \frac{x'-y'}{\sqrt{2}}$$

$$y = x'\sin 45^\circ + y'\cos 45^\circ = \frac{x'+y'}{\sqrt{2}}$$

Using above transformation given equation (i) become,

$$3\left(\frac{x'-y'}{\sqrt{2}}\right)^2 + 5\left(\frac{x'+y'}{\sqrt{2}}\right)^2 - 3 = 0$$

$$\Rightarrow 3\left(\frac{x'^2-2x'y'+y'^2}{2}\right) + 5\left(\frac{x'^2+2x'y'+y'^2}{2}\right) - 3 = 0$$

$$\Rightarrow 3x'^2 - 6x'y' + 3y'^2 + 5x'^2 + 10x'y' + 5y'^2 - 6 = 0$$

$$\Rightarrow 8x'^2 + 8y'^2 + 4x'y' - 6 = 0$$

$$\Rightarrow 4x'^2 + 2x'y' + 4y'^2 - 3 = 0$$, this is the required equation.

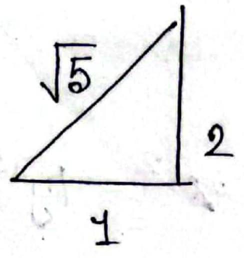
MAT 104

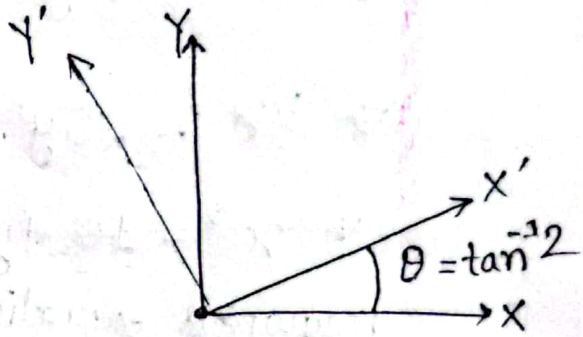
3rd October 2023

(2) If the axes be turned through an angle $\tan^{-1}2$, What does the equation $4xy - 3x^2 = a^2$ become?

$\theta = \tan^{-1}2$

$\Rightarrow \tan\theta = \frac{2}{1}$





$\therefore \cos\theta = \frac{1}{\sqrt{5}}$

$\therefore \sin\theta = \frac{2}{\sqrt{5}}$

We know, $x = x'\cos\theta - y'\sin\theta =$

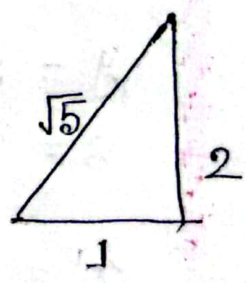
$y = x'\sin\theta + y'\cos\theta =$

In ~~x and y~~ XY cordinate, the given equation is

$$4xy - 3x^2 = a^2 \quad \text{-----(1)}$$

According the quention, $\theta = \tan^{-1}2$

$\therefore \tan\theta = 2$



$\therefore \sin\theta = \frac{2}{\sqrt{5}}, \ \cos\theta = \frac{1}{\sqrt{5}}$

$\therefore x = x'\cos\theta - y'\sin\theta = \frac{x' - 2y'}{\sqrt{5}}$

$\therefore y = x'\sin\theta + y'\cos\theta = \frac{2x' + y'}{\sqrt{5}}$

Thus,

(1) become, $4\left(\frac{x' - 2y'}{\sqrt{5}}\right)\left(\frac{2x' + y'}{\sqrt{5}}\right) - 3\left(\frac{x' - 2y'}{\sqrt{5}}\right)^2 = a^2$

$\Rightarrow 4\frac{(x' - 2y')(2x' + y')}{5} - 3\frac{(x' - 2y')^2}{5} = a^2$

$\Rightarrow 4(2x'^2 + x'y' - 4x'y' - 2y'^2) - 3(x'^2 - 4x'y' + 4y'^2) = 5a^2$

$\Rightarrow 8x'^2 + 4x'y' - 16x'y' - 8y'^2 - 3x'^2 + 12x'y' - 12y'^2$
$= 5a^2$

$\Rightarrow 5x'^2 - 20y'^2 = 5a^2$

$\cancel{\Rightarrow x'^2 - 10\,4y'^2 = k}$ $\therefore x'^2 - 4y'^2 = a^2$, is the required equation.

(3) Find the new co-ordinates of the point (2, 4) if the co-ordinate axis are rotated through an angle $\theta = 30°$.

We know, $x = \cancel{x' + 2}\, x'\cos 30° + y\sin 30°$

$\Rightarrow 4 \quad \cancel{y = y'} = \sqrt{3}x' - y' \;---\; (1)$

$y = x'\sin 30° + y'\cos 30°$
$8 = x' + \sqrt{3}y' \;----\; (2)$

Ans: $x' = \sqrt{3} + 2$, $y' = 2\sqrt{3} - 1$

From Equation (1), $4 = \sqrt{3}x' - y'$

$\Rightarrow y' = \sqrt{3}x' - 4$

putting value of y in eq (2)
$8 = x' + \sqrt{3}(\sqrt{3}x' - 4)$
$8 = x' + 3x' - 4\sqrt{3}$

$4x' = 8 + 4\sqrt{3}$

$\therefore x' = 2 + \sqrt{3}$

in eq(1), $4 = \sqrt{3}(2+\sqrt{3}) - y'$

$\Rightarrow 4 = 2\sqrt{3} + 3 - y'$

$\therefore y' = 2\sqrt{3} - 1$

(4) Let, an $X'Y'$ co-ordinate system be obtained by rotating an XY co-ordinate system through an angle of $45°$. To find an equation of the curve $3x^2 + y^2 = 6$ in the $X'Y'$ co-ordinate.

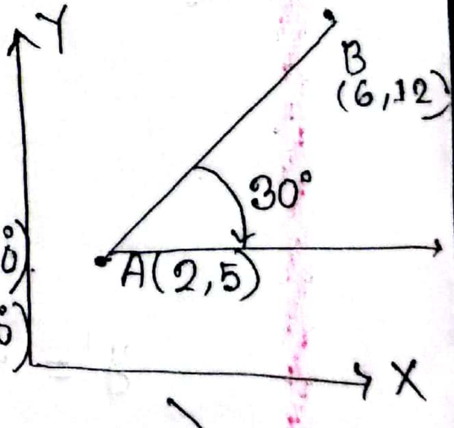
(5) Let an $X'Y'$ co-ordinate system be obtained by rotating an XY co-ordinate system through an angle of $45°$. To find an equation in XY co-ordinate of the curve $3x'^2 + y'^2 = 6$.

$x = x'\cos 45° - y'\sin 45°$

$\sqrt{2}x = x' - y'$ $\quad\quad \therefore x', y' = ?$

$\sqrt{2}y = x' + y'$

(6) Rotate line AB, whose end points are A(2,5), B(6,12). about origin through a 30° clockwise direction.



$AB \rightarrow XY \quad , \quad CD \rightarrow X'Y'$

$A = (2,5) \; , \; \theta = -30°$

$x = 2, y = 5 \quad \begin{cases} x = x'\cos(-30°) - y'\sin(-30°) \\ y = x'\sin(-30°) + y'\cos(-30°) \end{cases}$

$x' = ? \; , \; y' = ?$

(same for point B)

⑦ Rotate line CD, Whose end points are (3,4) and (12,15) about origin through a 45° anti-clockwise ~~cot~~ direction.

for C point, $x = x'\cos 45° - y'\sin 45°$

$$3 = x'\frac{\sqrt{2}}{2} - y\frac{\sqrt{2}}{2}$$

$$\Rightarrow 3\sqrt{2} = x' - y' \quad \text{---} \quad (i)$$

again, $4 = x'\frac{1}{\sqrt{2}} + y'\frac{1}{\sqrt{2}}$ $\therefore 4\sqrt{2} = x' + y' \text{ --- } (ii)$

$\therefore$ (ii)+(i) $\Rightarrow$

$$x' + y' = 4\sqrt{2}$$
$$x' - y' = 3\sqrt{2}$$

$$2x' = 7\sqrt{2}$$

$$\therefore x' = 7/\sqrt{2}$$

putting the value in eq(i),

$$\therefore y' = \frac{1}{\sqrt{2}}$$

Again, for D point,

$$x' = x\cos\theta + y\sin\theta$$
$$= 12\cos 45° + 15\sin 45°$$

$$\therefore x' = \frac{27}{\sqrt{2}}$$

$$y' = -x\sin\theta + y\cos\theta = \frac{3}{\sqrt{2}}$$

⑧ Determine the transform equation from $3x-2y+5=0$ when the origin is transferred to the point $(-2,-1)$ and the axis turned through an angle $45°$.

$(h,k) = (-2,-1)$

$x = x'+h = x'-2$
$y = y'+k = y'-1$

$\therefore\ 3(x'-2) - 2(y'-1)+5 = 0$
$\Rightarrow 3x'-6-2y'+2+5=0$
$\Rightarrow 3x'-2y'+1=0$

Removing suffixes, $3x-2y+1=0$ ---- ②

$x = x'\cos 45° - y'\sin 45° = \frac{x'-y'}{\sqrt{2}}$

$y = x'\sin 45° + y'\cos 45° = \frac{x'+y'}{\sqrt{2}}$

from eq ②, $3\left(\frac{x'-y'}{\sqrt{2}}\right) - 2\left(\frac{x'+y'}{\sqrt{2}}\right)+1=0$

$\Rightarrow 3x'-3y' - 2x'-2y'+\sqrt{2}=0$

$\Rightarrow x'-5y'+\sqrt{2}=0.$

*
another formula for this problem $= x = x'\cos\theta - y'\sin\theta + h$
$y = x'\sin\theta + y'\cos\theta + k$

H.W ⑨ Determine the transform equation $4x^2+xy-y^2-8x+2y+5=0$, when the origin is tranferred to the point $(-1,-2)$ and the axis turned through $45°$ angle