PHY109 Engineering Physics I

Chapter 5 (Electricity)

Part-2 (Electricity-II)

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5.4 Capacitors and Dielectrics

Figure 5.19 shows two nearby conductors, which are permitted to be of any shape, carrying equal and opposite charges. Such an arrangement is called a *capacitor*, the conductors being called *plates*.

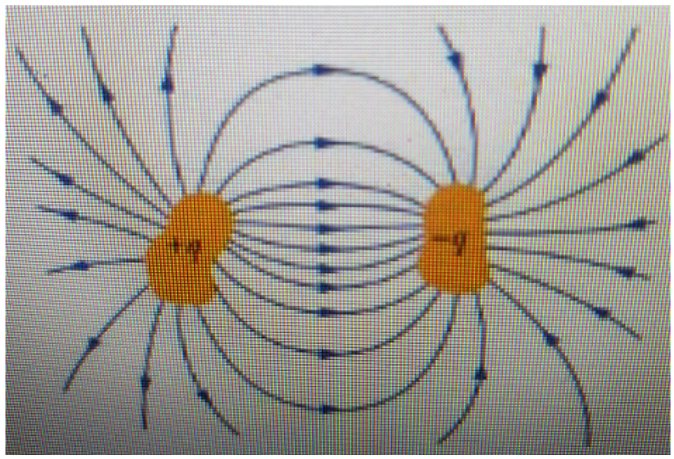

Fig. 5.19 Two isolated conductors carrying equal and opposite charges form a capacitor.

The equal and opposite charges might be established by connecting the plates momentarily to opposite poles of a battery. The capacitance C of a capacitor is defined by the following equation:

$$C = \frac{q}{V}. \tag{5.38}$$

In Eq. (5.38), V is the potential difference between the two plates and q is the magnitude of the charge on either plate.

A Parallel-Plate Capacitor

Figure 5.20 shows a parallel-plate capacitor formed with two parallel conducting plates of area A separated by a distance d in air. If we connect each plate to the terminal of a battery, a charge $+q$ will appear on one plate and a charge $-q$ on the other. If d is small compared with the plate dimensions, the electric field strength **E** between the plates will be uniform, which means that the lines of force will be parallel and evenly spaced.

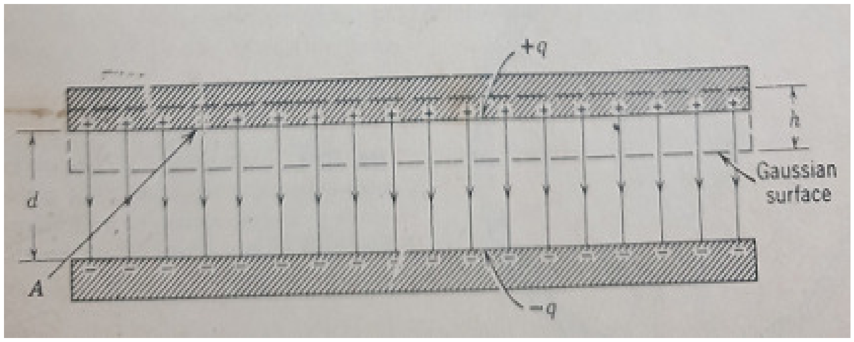

Fig. 5.20 A parallel-plate capacitor with plates of area A separated by a distance d. The dashed line represents a Gaussian surface whose height is h and whose top and bottom caps are the same shape and size as the capacitor plates.

We can calculate the capacitance of this device using the Gauss's law. Figure 5.20 shows a Gaussian surface in the form of a rectangular box of height h closed by plane caps of area A that are the shape and size of the capacitor plates. The flux of **E** is zero for the part of the Gaussian surface that lies inside the top capacitor plate because the electric field inside a conductor carrying a static charge is zero. The flux of **E** through the wall of the Gaussian surface is zero because, to the extent that the fringing of the lines of force can be neglected, **E** lies in the wall.

This leaves only the face of the Gaussian surface that lies between the plates. Here **E** is constant and the flux Φ_E is simply EA. Then the Gauss's law gives

$$\varepsilon_0 \Phi_E = q$$

or

$$\varepsilon_0 \oiint_S \mathbf{E} \cdot d\mathbf{S} = \varepsilon_0 EA = q. \tag{5.39}$$

The work required to carry a test charge q_0 from one plate to the other can be expressed either $q_0 V$ or as the product of a force $q_0 E$ times a distance d or $q_0 Ed$. These expressions must be equal: $q_0 V = q_0 Ed$, which gives $V = Ed$. More formally, this relation is written as

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = -\int E\, dl \cos 180^o = Ed, \tag{5.40}$$

where V is the potential difference between the plates.

If we substitute Eqs. (5.39) and (5.40) into the relation $C = q/V$, we obtain

$$C = \frac{q}{V} = \frac{\varepsilon_0 EA}{Ed} = \frac{\varepsilon_0 A}{d}. \tag{5.41}$$

The above Eq. (5.41) holds only for capacitors of the parallel-plate type; different formulas hold for capacitors of different geometry.

A Cylindrical Capacitor

A cylindrical capacitor consists of two coaxial cylinders as in Fig. 5.21 of radiuses a and b and length l in air. We would like to find the capacitance of this capacitor. We assume that the capacitor is very long (that is, that $l >> b$) so that fringing of the lines of force at the ends can be ignored for the purpose of calculating the capacitance.

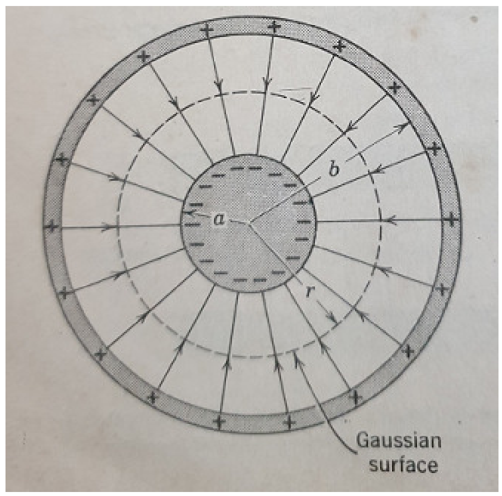

Fig. 5.21 A cross-section of a cylindrical capacitor. The dashed circle is a cross-section of a cylindrical Gaussian surface of radius r and length l.

As a Gaussian surface we construct a coaxial cylinder of radius r and length l, closed by plane caps. Gauss's law

$$\varepsilon_0 \Phi_E = q$$

or

$$\varepsilon_0 \oint\!\!\oint_S \mathbf{E} \cdot d\mathbf{S} = q,$$

gives

$$\varepsilon_0 E(2\pi r)(l) = q,$$

the flux being entirely through the cylindrical surface and not through the end caps. Solving for E yields

$$E = \frac{q}{2\pi\varepsilon_0 rl}.$$

The potential difference between the plates is given by

$$V = -\int_a^b \mathbf{E}\cdot d\mathbf{r} = \int_a^b E dr = \int_a^b \frac{q}{2\pi\varepsilon_0 l}\frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 l}\ln\left(\frac{b}{a}\right).$$

Thus, the capacitance of the cylindrical capacitor is

$$C = \frac{q}{V} = \frac{2\pi\varepsilon_0 l}{\ln(b/a)}. \tag{5.42}$$

Like the relation for the parallel-plate capacitor, Eq. (5.41), this relation also depends only on the geometrical factors.

Parallel Combination of Capacitors

Figure 5.22 shows three capacitors connected in parallel. We would like to find an expression for the equivalent capacitance for this connection. Equivalent means that if the parallel combination and the single capacitor were each in a box with wires a and b connected to terminals, it would not be possible to distinguish the two by electrical measurements external to the box.

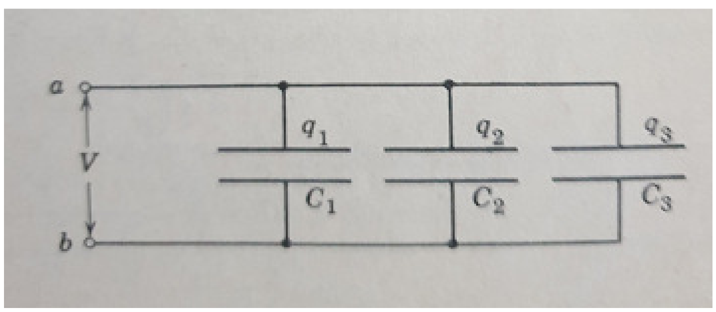

Fig. 5.22 Three capacitors in parallel.

The potential difference across each capacitor in Fig. 5.22 will be the same. This follows because all of the upper plates are connected together at terminal a, whereas all of the lower plates are connected together at terminal b. Applying the relation $q = CV$ to each capacitor yields

$$q_1 = C_1 V, \qquad q_2 = C_2 V, \qquad \text{and} \qquad q_3 = C_3 V.$$

The total charge q on the combination is

$$q = q_1 + q_2 + q_3 = (C_1 + C_2 + C_3)V.$$

The equivalent capacitance C is

$$C = \frac{q}{V} = C_1 + C_2 + C_3 . \tag{5.43}$$

This result can easily be extended to any number of parallel-connected capacitors.

Series Combination of Capacitors

Figure 5.23 shows three capacitors connected in series. We would like to find an expression for the equivalent capacitance for this connection.

For capacitors connected as shown in Fig. 5.23, the magnitude q of the charge on each plate must be the same. This is true because the net charge on the part of the circuit enclosed by the dashed line in the figure must be zero; that is, the charge present on these plates initially is zero and connecting a battery between a and b will only produce a charge separation, the net charge on these plates still being zero.

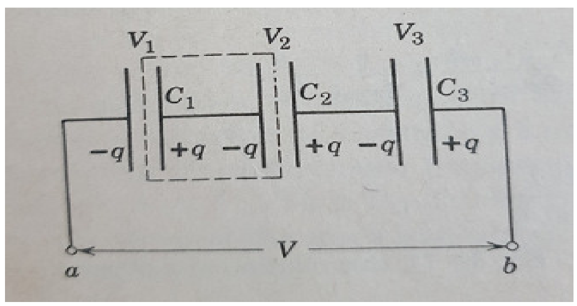

Fig. 5.23 Three capacitors in series.

Applying the relation $q = CV$ to each capacitor yields

$$V_1 = \frac{q}{C_1}, \qquad V_2 = \frac{q}{C_2}, \qquad \text{and} \qquad V_3 = \frac{q}{C_3}.$$

The potential difference for the series combination is

$$V = V_1 + V_2 + V_3 = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right).$$

The equivalent capacitance C is

$$C = \frac{q}{V} = \frac{1}{\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)}$$

or

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}. \tag{5.44}$$

The equivalent series capacitance is always less than the smallest capacitance in the chain. It is to be mentioned here also that the result can easily be extended to any number of parallel-connected capacitors.

A Parallel-Plate Capacitor with a Dielectric

The expression for the capacitance of a parallel-plate capacitor is $C = \varepsilon_0 A / d$, which holds only for a parallel-plate capacitor with its plates in a vacuum. Michael Faraday, in 1837, first investigated the effect of filling the space between the plates with a dielectric.

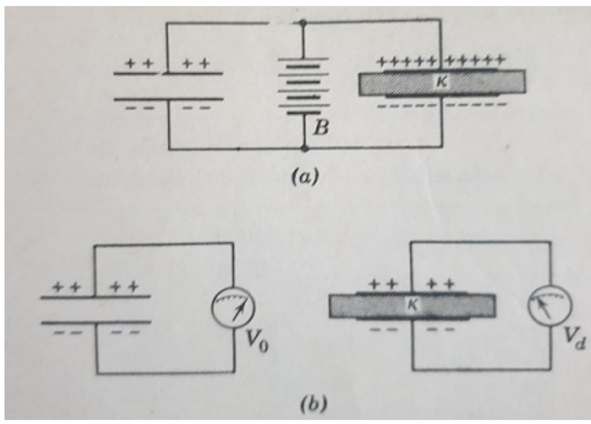

Fig. 5.24 (a) Battery B supplies the same potential difference to each capacitor; the one on the right has a higher charge. (b) Both capacitors carry the same charge; the one on the right has the lower potential difference.

Figure 5.24*a* shows two identical capacitors, in one of which there is a dielectric, the other containing only air at normal pressure. When both capacitors were charged to the same potential difference, Faraday found by experiment that the charge on the one containing the dielectric was greater than that on the other.

Since q is larger for the same V, if a dielectric is present, it follows from the relation $C = q / V$ that *the capacitance increases if a dielectric is placed between the plates*.

The *dielectric constant* κ of a dielectric material is defined as the ratio of the capacitance with the dielectric to that without the dielectric. Thus, the capacitance of the parallel-plate capacitor is modified to

$$C = \frac{\kappa \varepsilon_0 A}{d}. \tag{5.45}$$

Energy Storage in an Electric Field

We know that all charge configurations have a certain *electric potential energy U*, equal to the work W (which may be positive or negative) that must be done to assemble them from their individual components, originally assumed to be infinitely far apart and at rest. For a simple example, work must be done to separate two equal and opposite charges. This energy is stored in the system and can be recovered if the charges are allowed to come together again. Similarly, a charged capacitor has stored in it an electrical potential energy U equal to the work W required to charge it. This energy can be recovered if the capacitor is allowed to discharge. We can visualize the work of charging by imagining that an external agent pulls electrons from the positive plate and pushes them onto the negative plate, thus bringing about the charge separation; normally the work of charging is done by a battery, at the expense of its stored of chemical energy.

Let us suppose that at a time t a charge $q'(t)$ has been transferred from one plate to the other. The potential difference $V(t)$ between the plates at that moment will be $q'(t)/C$, where C is the capacitance of the capacitor. If an extra increment of charge dq' is transferred, the small amount of additional work needed will be $dW = V\,dq' = \left(q'/C\right)dq'$.

If this process is continued until a total charge q has been transferred, the total work will be found from

$$W = \int dW = \int_0^q \left(\frac{q'}{C}\right) dq' = \frac{1}{2}\frac{q^2}{C}. \tag{5.46}$$

By using the relation $q = CV$, we can rewrite Eq. (5.46) as

$$W(=U) = \frac{1}{2}CV^2. \tag{5.47}$$

It is to be mentioned here that the energy stored in a capacitor resides in the electric field.

In a parallel-plate capacitor, the electric field can be assumed to be same for all points between the plates. Thus, the *energy density u*, which is the stored energy per unit volume, should also be uniform:

$$u = \frac{Energy}{Volume} = \frac{U}{Ad} = \frac{CV^2/2}{Ad},$$

where the volume of the parallel plate capacitor is Ad.

We know the expression for the capacitance of the parallel plate capacitor is

$$C = \kappa\varepsilon_0 A / d \ .$$

Thus,

$$u = \frac{\frac{1}{2}CV^2}{Ad} = \frac{\frac{1}{2}\kappa\varepsilon_0 A / d}{Ad}(Ed)^2 \ ,$$

from which we obtain

$$u = \frac{1}{2}\kappa\varepsilon_0 E^2 \ . \tag{5.48}$$

5.5 Electromotive Force and Circuits

A Simple Electric Circuit

Let us consider the following circuit, where a charge dq passes through any cross section of the circuit in time dt. In particular, this charge enters the source of *emf* (electromotive force) E at its low-potential end and leaves at its high-potential end. The source must do an amount of work dW on the positive charge carriers to force them to go to the point of higher potential. The *emf* E of the source is defined from the following expression:

$$E = \frac{dW}{dq}. \tag{5.49}$$

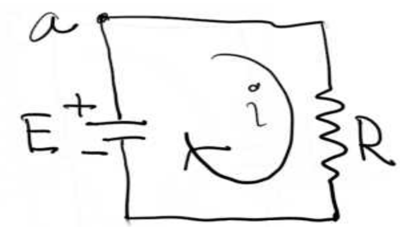

Fig. 5.25: A simple electric circuit.

The unit of *emf* is the *joule/coul* which is nothing but the *volt*. We may then say that a battery has an *emf* of 1 volt if it maintains a difference in potential of 1 volt between its terminals.

If a source of *emf* does work on a charge carrier, energy must be transferred within the source. In a battery, for example, chemical energy is transferred into electrical energy.

Now let us calculate the current in the simple circuit. In a time dt an amount of energy given by $i^2 R\, dt$ will appear in the resistor of Fig. 5.25 as Joule heat. During this same time a charge dq (idt) will have moved through the *seat* of *emf*, and the source will have done work on this charge given by

$$dW = E\,dq = Ei\,dt\ .$$

From the conservation of energy principle, the work done by the source must be equal to the Joule heat:

$$Ei\ dt = i^2 R\, dt\ .$$

Solving for i, we obtain

$$i = \frac{E}{R}. \tag{5.50}$$

Laws for Solving Circuit Equations

Kirchhof's Voltage Law (KVL)

If we start at any point in the circuit of Fig. 5.25, in imagination, go around the circuit in either direction, adding up algebraically the changes in potential that we encounter, we must arrive at the same potential when we return to our starting point. In other words,

"*The algebraic sum of the changes in potential encountered in making a complete loop of the circuit must be zero*".

This is known as *Kirchhoff's voltage law*; for brevity we call it the *loop theorem*.

In Fig. 5.25, let us start at point a, whose potential is V_a, and traverse the circuit clockwise. In going through the resistor, there is *a* change in potential of $-iR$. The minus sign shows that the top of the resistor is higher in potential than the bottom, which must be true, because positive charge carriers move of their own accord from high to low potential. As we traverse the battery from bottom to top, there is an increase of potential $+E$ because the battery does positive work on the charge carriers, that is, it moves them from a point of low potential to one of high potential. Adding the algebraic sum of the changes in potential to the initial potential V_a must yield the identical value V_a:

$$V_a - iR + E = V_a.$$

We write this as

$$-iR + E = 0,$$

which is independent of the value of V_a and which asserts explicitly that the algebraic sum of the potential changes for a complete circuit or loop traversal is zero.

To prepare for the study of more complex circuits, let us examine the rules for finding potential differences.

- If a resistor is traversed in the direction of the current, the change in potential is $-iR$, in the opposite direction it is $+ iR$.
- If a source of *emf* is traversed in the direction of the *emf*, the change in potential is $+E$; in the opposite direction it is $-E$.

Kirchhof's Current Law

Figure 5.26 shows a circuit containing two loops. For simplicity we neglect the internal resistance of the batteries. There are two junctions, *b* and *d*, and three branches connecting these junctions. The branches are the left branch *bad*, the right branch *bcd*, and the central branch *bd*. If the emfs and the resistances are given, what are the currents in the various branches?

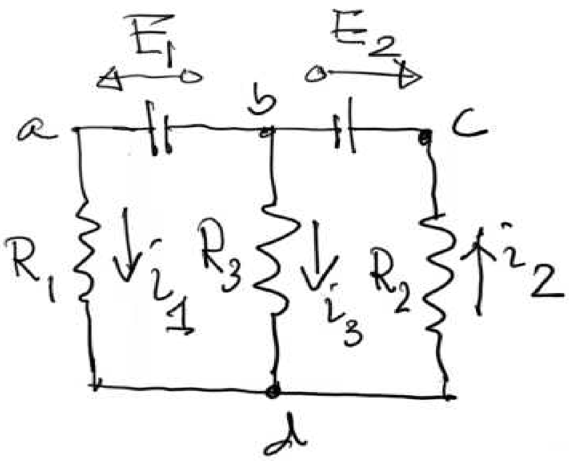

Fig. 5.26 A multiloop circuit.

The three currents i_1, i_2, and i_3 carry charge either toward junction *d* or away from it. Charge does not accumulate at junction *d*, nor does it drain away from this junction because the circuit is in a steady-state condition. Thus charge must be removed from the junction by the currents at the same rate that it is brought into it. If we arbitrarily call a current approaching the junction positive and the one leaving the junction negative, then

$$i_1 + i_3 - i_2 = 0.$$

This equation suggests a general principle for the solution of multiloop circuits:

"*At any junction the algebraic sum of the currents must be zero*".

This *junction theorem* is also known as *Kirchhoff's current law*.

We note that it is simply a statement of the conservation of charge.

Thus, our basic tools for solving circuits are (*a*) the Conservation of Energy and (*b*) the Conservation of Charge. In multiloop circuits there is more than one loop, and the current in general will not be the same in all parts of any given loop.

If we traverse the left loop of Fig. 5.26 in a counterclockwise direction, the loop theorem gives

$$E_1 - i_1 R_1 + i_3 R_3 = 0. \tag{5.51}$$

The right loop gives

$$-i_3 R_3 - i_2 R_2 - E_2 = 0. \tag{5.52}$$

The junction theorem at point d is

$$i_1 + i_3 - i_2 = 0. \tag{5.53}$$

Solving these three equations for the unknowns i_1, i_2, and i_3, we can obtain the solutions as

$$i_1 = \frac{E_1(R_2 + R_3) - E_2 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1}, \tag{5.54}$$

$$i_2 = \frac{E_1 R_3 - E_2(R_1 + R_3)}{R_1 R_2 + R_2 R_3 + R_3 R_1}, \tag{5.55}$$

and

$$i_3 = \frac{-E_1 R_2 - E_2 R_1}{R_1 R_2 + R_2 R_3 + R_3 R_1}. \tag{5.56}$$

A Simple RC Circuit

In the preceding sections we have dealt with circuits in which the circuit elements were resistors and in which the currents did not vary with time. Here, we introduce the capacitor as a circuit element, which will lead us to the concept of time-varying currents.

Charging the Capacitor

In Fig. 5. 27, let us on the switch *S* that is let the switch be thrown to position *a*. We can now ask what current is set up in the single-loop circuit shown.

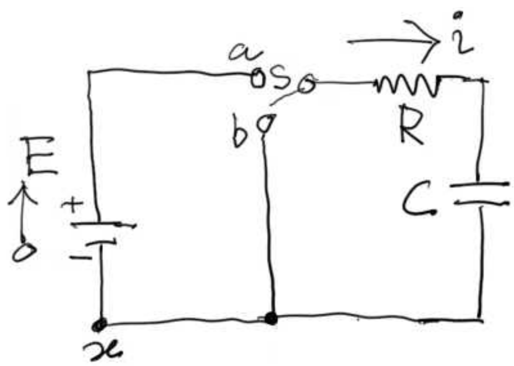

Fig. 5.27 An RC circuit.

Let us apply the conservation of energy principle here. In time dt a charge dq (= idt) moves through the circuit. The work done by the source of the emf which is $dW = E\,dq$ must be equal to the energy that appears as Joule heat in the resistor R during the time dt which is $i^2R\,dt$ plus the increase in the amount of energy that is stored in the capacitor which is $dU = d(q^2/2C)$. In equation form, we have

$$E\,dq = i^2R\,dt + d\left(\frac{q^2}{2C}\right)$$

which gives

$$E\,dq = i^2R\,dt + \frac{q}{C}dq.$$

Dividing by dt, the above equation yields

$$E\frac{dq}{dt} = i^2R + \frac{q}{C}\frac{dq}{dt}.$$

But $dq/dt = i$, so that this equation becomes

$$Ei = i^2 R + \frac{q}{C} i,$$

which gives

$$E = iR + \frac{q}{C}. \tag{5.57}$$

It is to be mentioned here that the above equation can also be derived by using the loop theorem.

We would also like to write down Eq. (5.57) in the following form

$$E = R\frac{dq}{dt} + \frac{q}{C}. \tag{5.58}$$

Our task now is to find the function $q(t)$ that satisfies the first-order differential equation, Eq. (5.58). It is very easy to solve this differential equation and the solution is

$$q(t) = CE\left(1 - e^{-t/RC}\right). \tag{5.59}$$

Differentiating Eq. (5.59) gives the current i in the circuit:

$$i(t) = \frac{dq}{dt} = \frac{E}{R} e^{-t/RC}. \tag{5.60}$$

Figures 5.28 shows a plot of Eq. (5.59) for a particular case. About the circuit current, we observe the following:

(a) at $t = 0$, $q = 0$ and $i = E/R$, and
(b) as $t \to \infty$, $q \to CE$ and $i \to 0$:

that is, the current is initially E/R and finally zero; the charge on the capacitor plates is initially zero and finally CE.

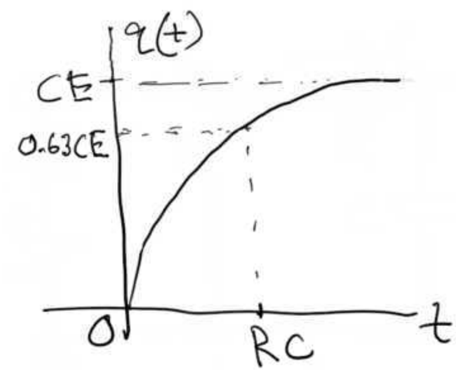

Fig. 5.28 Charging of the capacitor with respect to time t.

The quantity RC appears in Eqs. (5.59) and (5.60) has the dimension of time and is called the *capacitive time constant* (t_{RC}) of the circuit. It is the time at which the charge on the capacitor has increased to within a factor of $(1-e^{-1}) \equiv 0.63\ (63\%)$ of its equilibrium value. To show this, let us put $t = RC$ in Eq. (5.59) to obtain

$$q = CE(1-e^{-1}) = 0.63CE\,,$$

where CE is the equilibrium charge on the capacitor corresponding to $t \to \infty$.

Let us assume now that the switch S in Fig. 5.27 has been in position a for a sufficient time $t >> RC$. The capacitor is then fully charged for all practical purposes.

Discharging the Capacitor

After the capacitor in Fig. 5.27 has been fully charged, the switch S is then thrown to position b. How do the charge of the capacitor and the current vary with time?

With the switch S closed on b, there is no *emf* in the circuit and Eq. (5.57) for the circuit, with E = 0, becomes

$$iR + \frac{q}{C} = 0. \qquad (5.61)$$

Putting $i = dq/dt$ allows us to write the above equation as

$$R\frac{dq}{dt} + \frac{q}{C} = 0. \qquad (5.62)$$

The solution of this equation is

$$q(t) = q_0 e^{-t/RC}\,, \qquad (5.63)$$

q_0 being the initial charge on the capacitor. The capacitive time constant $t_{RC} = RC$ appears in the above expression for the capacitor discharge as well.

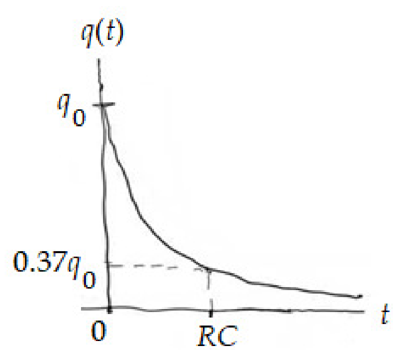

Fig. 5.29 Discharging of the capacitor with respect to time t.

We observe that at a time equal to the capacitive time constant $t = RC$, the capacitor charge is reduced to $q_0 e^{-1} = 0.37 q_0$, which is 37% of the initial charge q_0. This is shown in Fig. 5.29.

The current during discharge follows from differentiating Eq. (5.63)

$$i = \frac{dq}{dt} = -\frac{q_0}{RC} e^{-t/RC}. \tag{5.64}$$

The negative sign in Eq. (5.64) shows that the current is in the direction opposite to that shown in Fig. 5.27. This is as it should be, since the capacitor is discharging rather than charging. Since the equilibrium charge in the capacitor is $q_0 = CE$, we may write Eq. (5.64) as

$$i = -\frac{E}{R} e^{-t/RC}, \tag{5.65}$$

In which E/R appears as the initial current at $t = 0$, we can call it i_0. This is reasonable because the initial potential difference for the fully charged capacitor is E.

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N.B. This Chapter is prepared for the students mainly based on Physics Part II by David Halliday and Robert Resnick Forty-Second Wiley Eastern Reprint, July 1992