Sphere

The sphere is a geometrical three-dimensional solid having a curved surface.

Example: Basketball, Marbles, World

circle: • $(x_0, y_0), r$

circle $(x-x_0)^2+(y-y_0)^2=r^2$

sphere: • $(x_0, y_0, z_0), r$

sphere $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$

• Volume $= \frac{4}{3}\pi r^3$

• Surface Area $= 4\pi r^2$

• Half sphere $\longrightarrow$ Hemisphere

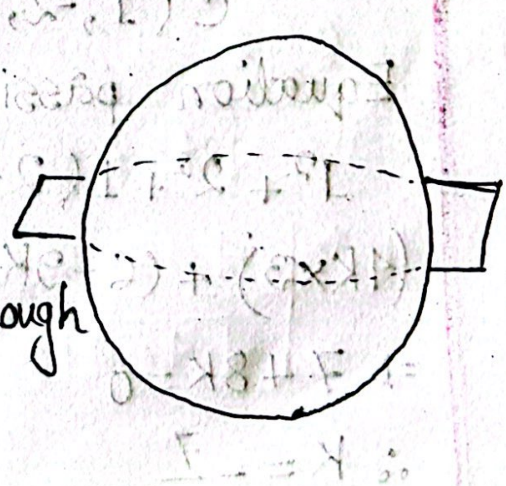
$$\begin{cases} \text{circle: } x^2+y^2+2gx+2fy+c=0 \\ \text{Sphere: } x^2+y^2+z^2+2gx+2fy+2hz+c=0 \end{cases}$$

• center $(-g, -f, -h)$

• $r = \sqrt{g^2+f^2+h^2-c}$

\# plane cut sphere $\rightarrow$ circle

$S=0$, $P=0$



Equation of sphere passing through the circle, $S+KP=0$

$S=0$, $S'=0$

$$S+K(S-S')=0$$

$$x^2+y^2+z^2+2gx+2fy+2hz+c_1=0$$
$$x^2+y^2+z^2+2g'x+2f'y+2h'z+c_2=0$$

$ax+by+cz+d=0$

Find the equation of the sphere through the circle $x^2+y^2+2x+3y+6=0$, $x-2y+4z-9=0$ and the center of the sphere, $x^2+y^2+z^2-2x+4y-6z+5=0$

Ans: The equation of the sphere passing through the circle,

$$S+KP=0$$

$$\Rightarrow x^2+y^2+2x+3y+6+K(x-2y+4z-9)=0$$

$$\Rightarrow x^2+y^2+x(2+K)+(3-2K)y+4Kz+(6-9K)=0 \quad \text{---(i)}$$

Given sphere,

$$x^2+y^2+z^2-2x+4y-6z+5=0$$
$$C(1,-2,3)$$

Equation passing through the center $(1,2,-3)$

$$1^2+2^2+1(2+K)+(3-2K)(-2)+\cancel{4(-2)-6(3)+5=0}$$
$$(4K\times3)+(6-9K)=0$$

$$\Rightarrow 7+8K=0$$

$$\therefore K=-\frac{7}{8}$$

(i), $x^2+y^2+x\left(2-\frac{7}{8}\right)+\left(3-2\left(-\frac{7}{8}\right)\right)y+4\left(-\frac{7}{8}\right)z+\left(6-9\left(-\frac{7}{8}\right)\right)=0$

$$\Rightarrow x^2+y^2+\frac{9}{8}x+\frac{19}{4}y-\frac{7}{2}z-\frac{111}{8}=0$$

$\Rightarrow \cancel{x^2+y^2}\ 8x^2+8y^2+9x+38y-28z-111=0$, which is the required sphere.

Find the equation of the sphere having its center on the plane $4x-5y-z=3$ and passing through the circle

$$x^2+y^2+z^2-2x-3y+4z+8=0 \text{ and}$$
$$x^2+y^2+z^2+4x+5y-6z+2=0$$

Solve: The equation of the sphere passing through the circle is, $S+K(S-S')=0$

$\Rightarrow x^2+y^2-2x-3y+4z+8+K(-6x-8y+10z+6)=0$

$\Rightarrow x^2+y^2+z^2+x(-2-6k)+(-3-8k)y+(4+10k)z+(8+6k)=0$

Comparing eqn (i) with,

$$x^2+y^2+z^2+2gx+2fy+2hz+c=0$$

$2g=-2-6k \qquad \Rightarrow g=-(1+3k)$

$2f=-3-8k \qquad \Rightarrow f=-\frac{(3+8k)}{2}$

$2h=4+10k \qquad \Rightarrow h=2+5k$

The center, $C(-g, -f, -h) = C\left(1+3k, \frac{3+8k}{2}, -(2+5k)\right)$ lies on the plane $4x-5y-z=3$

$4(1+3k) - 5\left(\frac{3+8k}{2}\right) + 2+5k = 3$

$\therefore k = -\frac{3}{2}$ ⇒ K এর মান (i) নং এ বসালে (ans)

Question: Find the value of a for which the plane $x+y+z=a\sqrt{3}$ touches the ~~plane~~ sphere $x^2+y^2+z^2-2x-2y-2z-6=0$

$\Rightarrow \left|\frac{\sqrt{3}.\sqrt{3} - a\sqrt{3}}{\sqrt{3}}\right| = 3$

$\Rightarrow |\sqrt{3} - a| = 3$

$\therefore \sqrt{3} - a = \pm 3$

$(+)\ a = \sqrt{3} - 3$, $(-)\ a = \sqrt{3} + 3$

\# Find the two tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6y-6z+5=0$ which are parallel to the plane $2x+2y-z=0$

$ax+by=c$

$ax+by+k=0$

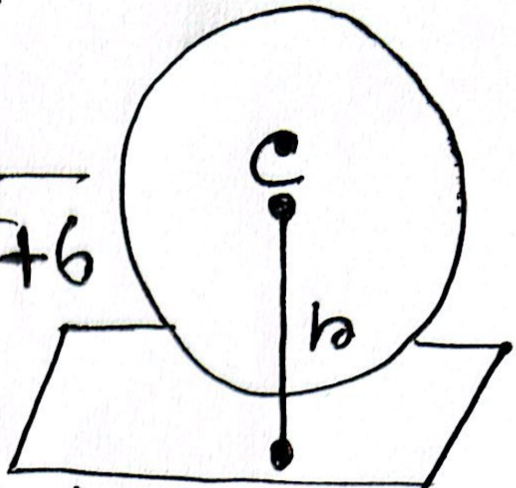
Solve: tangent plane: $2x+2y-z+k=0$ - - - (i)

$C(2, -1, 3)$

$r = \sqrt{4+1+9-5} = 3$

$\Rightarrow \left|\frac{4-2-3+k}{\sqrt{4+4+1}}\right| = 3$

$C(1,1,1)$



$r = \sqrt{(-1)^2 + (-1)^2 + (-1)^2 + 6}$

$= 3$

By the condition of tangency

$$\left| \frac{1+1+1-a\sqrt{3}}{\sqrt{1^2+1^2+1^2}} \right| = 3$$

$$\Rightarrow \left| \frac{3-a\sqrt{3}}{\sqrt{3}} \right| = 3$$

$$\Rightarrow \left| \frac{\sqrt{3}.\sqrt{3}-a\sqrt{3}}{\sqrt{3}} \right| = 3$$

$$\Rightarrow |\sqrt{3}-a| = 3$$

$$\Rightarrow \sqrt{3}-a = \pm 3$$

$\cancel{a = \sqrt{3} \pm 3}$ $(+)\ a = \sqrt{3}-3$

$(-)\ a = \sqrt{3}+3$

$$\Rightarrow \left|\frac{-1+K}{3}\right| = 3$$

$$\Rightarrow \frac{K-1}{3} = \pm 3$$

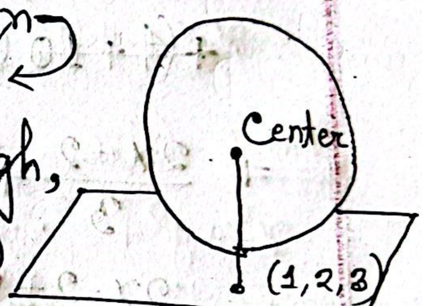
$$\Rightarrow (+)\quad K = 10 \qquad (-)\quad K = -8$$

Find the equation of the tangent plane to the sphere $3(x^2+y^2+z^2)-2x-3y-4z-22=0$ at the point $(1,2,3)$

$\Rightarrow$ বিন্দু দিয়ে এমন plane এর eqn

Solve: $\Rightarrow a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

The eqn of plane passing through,



$$a(x-1)+b(y-2)+c(z-3)=0 \quad \text{---- (i)}$$

$$3(x^2+y^2+z^2)-2x-3y-4z-22=0$$

$$\Rightarrow x^2+y^2+z^2-\frac{2}{3}x-y-\frac{4}{3}z-\frac{22}{3}=0$$

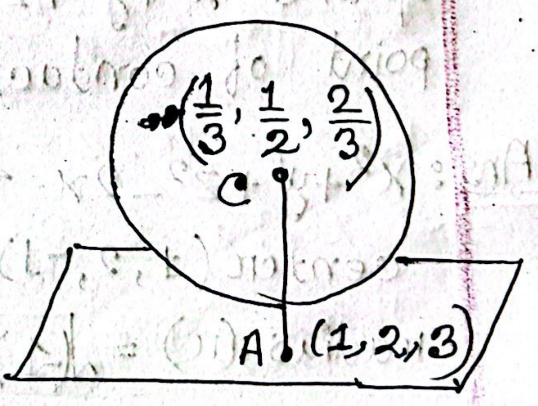
$(x_2-x_1, y_2-y_1, z_2-z_1)$

Comparing with $\rightarrow$ $x^2+y^2+z^2+2gx+2fy+2hz+c=0$

$$\Rightarrow 2g=-\frac{2}{3}\ ;\ g=-\frac{1}{3}$$

$$\Rightarrow 2f=-1\ ,\ f=-\frac{1}{2}$$

$$\Rightarrow 2h=-\frac{4}{3}\ ,\ h=-\frac{2}{3}$$



$$C(-g,-f,-h)=C\left(\frac{1}{3},\frac{1}{2},\frac{2}{3}\right)$$

d.r of CA, d.r $\left(1-\frac{1}{3}, 2-\frac{1}{2}, 3-\frac{2}{3}\right)$

d.r $\left(\frac{2}{3}, \frac{3}{2}, \frac{7}{3}\right)$

(a, b, c)

(i) $\Rightarrow a(x-1)+b(y-2)+c(z-3)=0$.

$\Rightarrow \frac{2}{3}(x-1)+\frac{3}{2}(y-2)+\frac{7}{3}(z-3)=0$

~~(i) $\Rightarrow x^2+y^2+z^2+\left(-2-6\left(-\frac{3}{2}\right)\right)x+\left(-3-8\left(-\frac{3}{2}\right)\right)y$~~

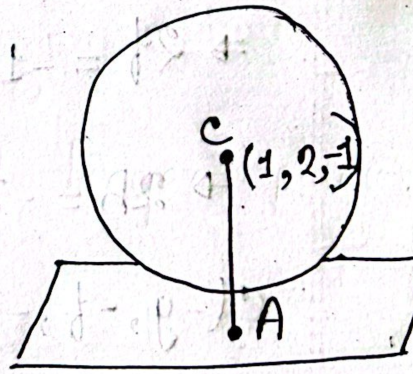
~~$+\left(4+10\left(-\frac{3}{2}\right)\right)z+8+6\left(-3/2\right)=0$~~

$\Rightarrow \frac{2x-2}{3}+\frac{3y-6}{2}+\frac{7z-21}{3}=0$

$\Rightarrow \frac{2x-2+9y-18+7z-21}{6}=0$

$\Rightarrow 2x+9y+7z-41=0$

Show that, the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z=3$ and find the point of contact.



Ans: $x^2+y^2+z^2-2x-4y+2z=0$

center $(1, 2, -1)$

radius, $(r)=\sqrt{(-1)^2+(-2)^2+1^2-(-3)}$

$r=3$

Given plane, $2x-2y+z+12=0$ ------ (i)

perpendicular distance from $(1,2,-1)$ on (i)

$$\left|\frac{2-4-1+12}{\sqrt{2^2+(-2)^2+1^2}}\right| = \left|\frac{9}{3}\right| = 3$$

Since, perpendicular distance = radius, so, equation (i) touches the given sphere.

Direction of ratio of CA is $(2,-2,1)$

$$d.c\left(\frac{2}{\sqrt{4+4+1}}, \frac{-2}{\sqrt{4+4+1}}, \frac{1}{\sqrt{4+4+1}}\right) = \left(\frac{2}{\sqrt{9}}, \frac{-2}{\sqrt{9}}, \frac{1}{\sqrt{9}}\right)$$

$$d.c\left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$$

The line CA passes through $(1,2,-1)$

$$\frac{x-1}{\frac{2}{3}} = \frac{y-2}{\frac{-2}{3}} = \frac{z+1}{\frac{1}{3}}$$

সূত্র: (x_1, y_1, z_1) দিয়ে যায়, $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

$$\Rightarrow \frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{3} = s \text{ (let)}$$

$$\Rightarrow x = 2s+1, \quad y = -2s+2, \quad z = s-1$$

The point x, y, z lies on equation (1),

$$2(2s+1) - 2(-2s+2) + (s-1) + 12 = 0$$

$$\Rightarrow 4s+2+4s-4+s-1+12 = 0$$

$$\Rightarrow 9s+9 = 0$$

$$\Rightarrow 9s = -9$$

$$\Rightarrow s = -1$$

$$x = 2(-1)+1 = -1$$

$$y = -2(-1)+2 = 4$$

$$z = -1-1 = -2$$

∴ The touching contact $(-1, 4, -2)$

Find the equation of the sphere which touches the sphere $x^2+y^2+z^2-x+3y+2z-3=0$ at the point $(1, 1, -1)$ and passes through the origin.

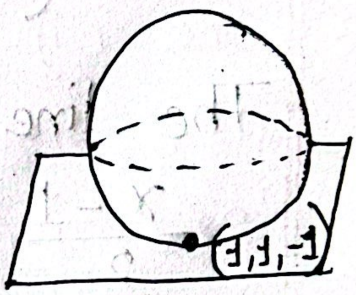
Formula: $x^2+y^2+z^2+2gx+2fy+2hz+c$

tangent plane,

$xx_1+yy_1+zz_1+g(x+x_1)+f(y+y_1)+h(z+z_1)+c=0$

Ans: Given sphere,

$x^2+y^2+z^2-x+3y+2z-3=0$



~~equation of~~ center $\left(\frac{1}{2}, \frac{-3}{2}, -1\right)$

The equation of tangent plane passes through $(1, 1, -1)$,

$$x.1+y.1+z.(-1)+\left(-\frac{1}{2}\right)(x+1)+\left(\frac{3}{2}\right)(y+1)+1(z-1)-3=0$$

$$\Rightarrow x+y-z-\frac{x-1}{2}+\frac{3y+3}{2}+z-1-3=0$$

$$\Rightarrow 2x+2y-\cancel{2z}-x-1+3y+3+\cancel{2z}-8=0$$

$\Rightarrow x+5y-6=0$, this is equation of tangent plane.

The required sphere,

$S+KP=0$

$$x^2+y^2+z^2-x+3y+2z-3+K(x+5y-6)=0$$

$$x^2+y^2+z^2+x(K-1)+y(5K+3)+2z-(3+6K)=0 \quad \ldots\ldots (i)$$

which passes through $(0,0,0)$

$$0+0+0+0+0+0-(3+6K)=0$$

$$\Rightarrow -3-6K=0$$

$$\Rightarrow K=-\frac{1}{2}$$

$$(i) \Rightarrow x^2+y^2+z^2+x\left(-\frac{1}{2}-1\right)+y\left(5\left(-\frac{1}{2}\right)+3\right)+2z-\left(3+6\left(-\frac{1}{2}\right)\right)$$

$$\Rightarrow x^2+y^2+z^2-\frac{3x}{2}+\frac{y}{2}+2z=0$$

$$\Rightarrow 2x^2+2y^2+2z^2-3x+y+4z=0$$