- **Course Title : Differential Equations and Special Functions**
- **Course Code: MAT102**
- **Section-3 // Lecture-10 (March 20, 2023)**

Today's Lecture Topics:

- **Ordinary Differential Equations**

- **Linear differential equations (second or higher order) with constant coefficients: The general form of non-homogeneous equations and its solution procedures**

- **Course Instructor: Dr. Akter Hossain, Assistant Professor of MPS Department, EWU, Dhaka, BD**

- **Linear differential equations (second or higher order) with constant coefficients**

- **The general form of non-homogeneous equations is as follows**

$$\boldsymbol{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = F(x)} \text{ or } \mathbf{X} ------- -(1)$$

or, $\boldsymbol{f(D)y = F(x)\ or\ X}$ $------------------- -(2)$

where $\boldsymbol{f(D)} = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n$ **where D** $\equiv \frac{d}{dx}$

- **Inverse operator: The particular integral,** $\boldsymbol{y_p}$ **of the differential equation having the form** $\boldsymbol{f(D)y = X}$ **is given by**

$$\boldsymbol{\frac{1}{f(D)} f(D) y_p = \frac{1}{f(D)} X} \text{ or } \boldsymbol{y_p = \frac{1}{f(D)} X}$$

- **Linear differential equations (second or higher order) with constant coefficients**
- Inverse operator: The particular integral, y_p of the differential equation having the form, $f(D)y = X$ is given by

$$\frac{1}{f(D)} f(D) y_p = \frac{1}{f(D)} X$$

or $y_p = \frac{1}{f(D)} X$ ---------------------- (3)

- **Working rule for finding y_p when X is the form of e^{ax}**

- **Formulae I : $y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ provided that $f(a) \neq 0$**

- Linear differential equations (second or higher order) with constant coefficients
- Working rule I for finding $y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ provided that $f(a) \neq 0$
- Example:1 Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{-x} + 3e^{2x}$.

➢ Solution: Given the differential equation is

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{-x} + 3e^{2x} \quad ---------------- -(1)$$

Let $y = e^{mx}$ be a trial solution of the homogeneous equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0 \quad -------------------- -(2).$$

Then the auxiliary equation of (2) is $m^2 - 6m + 25 = 0$

$$\Rightarrow m^2 - 2.m.3 + 9 + 16 = 0 \Rightarrow (m-3)^2 = -16$$

- **Linear differential equations (second or higher order) with constant coefficients**
- **Working rule I for finding $y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ provided that $f(a) \neq 0$**
- **Example:1 Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{-x} + 3e^{2x}$.**

➢ **Solution:**

Then the auxiliary equation of (2) is $m^2 - 6m + 25 = 0$

$$\Rightarrow m^2 - 2.m.3 + 9 + 16 = 0$$

$$\Rightarrow (m-3)^2 = -16$$

$$\Rightarrow (m-3)^2 = 16i^2 \Rightarrow (m-3)^2 = (4i)^2$$

$$\therefore m = 3 \pm 4i$$

Therefore, the complementary solution (y_c) of (1) is

$$y_c = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) \text{ ------------------ } (3)$$

- **Linear differential equations (second or higher order) with constant coefficients**
- **Working rule I for finding $y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ provided that $f(a) \neq 0$**
- **Example:1 Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{-x} + 3e^{2x}$.**

➢ **Solution: Now the particular integral is**

$$y_p = \frac{1}{D^2-6D+25}\left(e^{-x} + 3e^{2x}\right)$$

$$= \frac{1}{D^2-6D+25} e^{-x} + \frac{1}{D^2-6D+25} 3e^{2x}$$

$$= \frac{1}{(-1)^2-6(-1)+25} e^{-x} + \frac{1}{(2)^2-6(2)+25} 3e^{2x}$$

$$= \frac{e^{-x}}{1+6+25} + \frac{3e^{2x}}{4-12+25} = \frac{1}{32} e^{-x} + \frac{3}{17} e^{2x}$$

Therefore, the general solution is

$$y = y_c + y_p = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + \frac{1}{32} e^{-x} + \frac{3}{17} e^{2x} \text{ (Ans.)}$$

- **Linear differential equations (second or higher order) with constant coefficients**
- **Non-homogeneous equation**
- **Inverse operator: The particular integral, y_p of the differential equation having the form, $f(D)y = X$ is given by**

$$\frac{1}{f(D)} f(D) y_p = \frac{1}{f(D)} X$$

$$\text{or} \quad y_p = \frac{1}{f(D)} X \text{ --------------------- } (3)$$

- **If $f(a) = 0$ then $f(D)$ must possess a factor of the type $(D-a)^r$. Under this condition, we have to use the formulae II.**

- **Formulae II: If $y_p = \frac{1}{f(D)} X = \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$ where r = 1,2,3…**

- **Linear differential equations (second or higher order) with constant coefficients**

> **Non-homogeneous differential equation**

- **Formulae II: If** $y_p = \frac{1}{f(D)}X = \frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}$ **where r = 1,2,3…**

- **Example: 1 Find** $y_p = \frac{1}{D^3-D^2-D+1}e^x$

> **Solution: Here** $f(D) = D^3 - D^2 - D + 1$

$$\therefore f(1) = 1^3 - 1^2 - 1 + 1 = 0$$

Therefore, we have to factorize $f(D)$ *and* **let us do it. Therefore**

$$f(D) = D^2(D-1) - 1(D-1) = \left(D^2 - 1\right)(D-1) = (D-1)^2(D+1)$$

Thus, we can write, $y_p = \frac{1}{(D-1)^2(D+1)}e^x$

$$= \frac{1}{(D-1)^2}\left[\frac{1}{(D+1)}e^x\right] = \frac{1}{(D-1)^2}\frac{1}{2}e^x = \frac{1}{2}\frac{1}{(D-1)^2}e^x = \frac{1}{2}\frac{x^2}{2!} = \frac{1}{4}x^2$$

- **Linear differential equations (second or higher order) with constant coefficients**
- **Non-homogeneous equation**
- **Inverse operator: The particular integral, y_p of the differential equation having the form, $f(D)y = X$ is given by**

$$\frac{1}{f(D)} f(D) y_p = \frac{1}{f(D)} X$$

$$\text{or} \quad y_p = \frac{1}{f(D)} X \text{ ---------------------------- } (3)$$

- **If $f(a) = 0$ then $f(D)$ must possess a factor of the type $(D-a)^r$. Under this condition, we have to use the formulae II.**

- **Formulae II: If $y_p = \frac{1}{f(D)} X = \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$ where r = 1,2,3…**

- **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation**

- **Formulae II: If** $y_p = \frac{1}{f(D)} X = \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$ **where r = 1,2,3…**

- **Example: 1 Find** $y_p = \frac{1}{D^3-D^2-D+1} e^x$

➢ **Solution: Here** $f(D) = D^3 - D^2 - D + 1$

$$\therefore f(1) = 1^3 - 1^2 - 1 + 1 = 0$$

Therefore, we have to factorize $f(D)$ *and* **let us do it. Therefore**

$$f(D) = D^2 (D-1) - 1(D-1) = \left(D^2 - 1\right)(D-1) = (D-1)^2(D+1)$$

Thus, we can write, $y_p = \frac{1}{(D-1)^2(D+1)} e^x$

$$= \frac{1}{(D-1)^2} \left[\frac{1}{(D+1)} e^x\right] = \frac{1}{(D-1)^2} \frac{1}{2} e^x = \frac{1}{2} \frac{1}{(D-1)^2} e^x = \frac{1}{2} \frac{x^2}{2!} = \frac{1}{4} x^2$$

- Linear differential equations (second or higher order) with constant coefficients

➢ Non-homogeneous differential equation

- Formulae II: If $y_p = \frac{1}{f(D)}X = \frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}$ where r = 1,2,3…
- Example: Solve $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = e^{2x}\cosh x$
- Solution: Let $y = e^{mx}$ be the solution of the homogeneous equation $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = 0$

Therefore, the auxiliary equation of homogeneous equation is

$$m^3 - 5m^2 + 7m - 3 = 0$$

$$\Rightarrow m^3 - m^2 - 4m^2 + 4m + 3m - 3 = 0$$

$$\Rightarrow m^2(m-1) - 4m(m-1) + 3(m-1) = 0$$

$$\Rightarrow (m-1)\left(m^2 - 4m + 3\right) = 0 \Rightarrow (m-1)\left(m^2 - 3m - m + 3\right) = 0$$

➢ For Hyperbolic function, please see this link: https://en.wikipedia.org/wiki/Hyperbolic_functions

- **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation**

- **Formulae II: If** $y_p = \frac{1}{f(D)} X = \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$ **where r = 1,2,3…**
- **Example: Solve** $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = e^{2x} \cosh x$

- **Solution:** $\Rightarrow (m-1)\left(m^2 - 3m - m + 3\right) = 0$

$$\Rightarrow (m-1)\,[m(m-1) - 1(m-3)] = 0$$

$$\Rightarrow (m-1)^2 (m-3) = 0$$

$$\therefore m = 1, 1 \text{ and } 3$$

Therefore, the complementary solution of the given differential equation is $y_c = (c_1 + c_2 x)e^x + c_3 e^{3x}$

- Linear differential equations (second or higher order) with constant coefficients

➢ Non-homogeneous differential equation

- Formulae II: If $y_p = \frac{1}{f(D)} X = \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$ where r = 1,2,3…

- Example: Solve $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = e^{2x} coshx$

- Solution: Therefore the particular integral of the given differential equation can be written as

$$y_p = \frac{1}{D^3 - 5D^2 + 7D - 3} e^{2x} coshx = \frac{1}{(D-1)^2(D-3)} e^{2x} coshx$$

$$= \frac{1}{(D-1)^2(D-3)} e^{2x} \left[\frac{e^x + e^{-x}}{2}\right]$$

$$= \frac{1}{(D-1)^2(D-3)} \frac{1}{2} \left(e^{3x} + e^x\right)$$

$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2}}$$

- **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation**

- **Formulae II: If** $y_p = \frac{1}{f(D)} X = \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$ **where r = 1,2,3…**

- **Example: Solve** $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = e^{2x}coshx$

- **Solution: Therefore the particular integral of the given differential equation can be written as**

$$y_p = \frac{1}{D^3-5D^2+7D-3} e^{2x}coshx = \frac{1}{2}\frac{1}{(D-1)^2(D-3)} e^{3x} + \frac{1}{2}\frac{1}{(D-1)^2(D-3)} e^{x}$$

$$= \frac{1}{2}\frac{1}{(D-3)}\left[\frac{1}{(D-1)^2} e^{3x}\right] + \frac{1}{2}\frac{1}{(D-1)^2}\left[\frac{1}{(D-3)} e^{x}\right] = \frac{1}{2}\frac{1}{(D-3)}\frac{1}{(2)^2} e^{3x} - \frac{1}{2}\frac{1}{(D-1)^2}\frac{1}{2} e^{x}$$

$$= \frac{1}{8}\frac{1}{(D-3)} e^{3x} - \frac{1}{4}\frac{1}{(D-1)^2} e^{x} = \frac{1}{8}\frac{x}{1!} e^{3x} - \frac{1}{4}\frac{x^2}{2!} e^{x}$$

- **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation**

- **Formulae II: If** $y_p = \frac{1}{f(D)} X = \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$ **where r = 1,2,3…**
- **Example: Solve** $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = e^{2x}\cosh x$

- **Solution:**

Therefore the general solution of the given differential equation is

$$y = y_c + y_p = (c_1 + c_2 x)e^x + c_3 e^{3x} + \frac{x}{8} e^{3x} - \frac{x^2}{8} e^x$$

- **Linear differential equations (second or higher order) with constant coefficients**
  - **Non-homogeneous differential equation:** $f(D)y = X$ **where** X **is** e^{ax}

Exercise-13

Ex. 1. *Solve the following differential equations :*

(*a*) $(D^2 - 3D + 2)\, y = e^{3x}$.

(*b*) $(4D^2 + 12D + 9)\, y = 144\, e^{-3x}$.

(*c*) $D^2 (D + 1)^2 (D^2 + D + 1)^2\, y = e^x$

Answers: (a) $y = c_1 e^x + c_2 e^{2x} + (1/2)\, e^{3x}$.

(b) $y = (c_1 + c_2 x)\, e^{-3x/2} + 16\, e^{-3x}$.

(c) $y = c_1 + c_2 x + (c_3 + c_4 x)\, e^{-x}$

$$+ e^{-x/2} [(c_5 + c_6 x) \cos (\sqrt{3}x/2)] + [(c_7 + c_8 x) \sin (\sqrt{3}x/2)] + (1/36)\, e^x.$$

$9e^{-2x}$

Ex 2 *Solve the following differential equations :*

(*a*) $(D + 2)(D - 1)^3\, y = e^x$.

(*b*) $(D - 1)^2 (D^2 + 1)^2\, y = e^x$

Ans.(a) $y = c_1 e^{-2x} + (c_2 + c_3 x + c_4 x^2)\, e^x + (1/18)\, x^3 e^x$.

(*b*) $y = (c_1 + c_2 x)\, e^x + (c_3 + c_4 x) \cos x + (c_5 + c_6 x) \sin x + (x^2/8)\, e^x$.

- **Linear differential equations (second or higher order) with constant coefficients**

  - **Non-homogeneous differential equation:** $f(D)y = X$ where X is x^m or a polynomial of degree m, m being positive integer.

Exercise-14

Ex. 1. *Solve* $(D^4 - D^2)\, y = 2.$

Ans. $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} - x^2$

Ex. 2. *Solve* $(D^4 - a^4)\, y = x^4.$

Ans. $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax - (1/a^4)\,(x^4 + 24/a^4).$

Ex. 3 *Solve* $(D^2 + 2D + 2)\, y = x^2.$

Ans. $y = e^{-x}\,(c_1 \cos x + c_2 \sin x) + (x^2 - 2x + 1)/2.$

Ex. 4 *Solve* $(D^3 - D^2 - D - 2)\, y = x.$

Ans. $y = c_1 e^{2x} + e^{-x/2}\,\{c_3 \cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)\} - (1/8) \times (4x - x^2).$

- **Linear differential equations (second or higher order) with constant coefficients** HW (It is a must event)

- **Non-homogeneous differential equation**

- **Suppose $y_p = \frac{1}{f(D)} X$ where X is the form of $\cos ax$ or $\sin ax$**

- **Working rule for finding y_p when X is the form of $\cos ax$ or $\sin ax$**

- **Formulae III : We have to express $f(D)$ as the function of D^2, say $\phi(D^2)$ and then replace D^2 by $-a^2$.**

  If $\phi(-a^2) \neq 0$, then we have to use following result.

- $If\ y_p = \frac{1}{f(D)} \cos ax = \frac{1}{\phi(D^2)} \cos ax$ **then** $y_p = \frac{1}{\phi(-a^2)} \cos ax$

- **Example: Find $y_p = \frac{1}{D^4+D^2+1} \cos 2x$** (Only particular integral is calculated)

- Solution: Given $y_p = \frac{1}{D^4+D^2+1} \cos 2x = \frac{1}{(D^2)^2+D^2+1} \cos 2x = \frac{1}{(-2^2)^2-2^2+1} \cos 2x$

$$= \frac{1}{(-2^2)^2-2^2+1} \cos 2x = \frac{1}{13} \cos 2x$$

- **Linear differential equations (second or higher order) with constant coefficients**

HW (It is a must event)

- **Non-homogeneous differential equation**
- **Suppose $y_p = \frac{1}{f(D)}X$ where X is the form of $\cos ax$ or $\sin ax$**
- **Working rule for finding y_p when X is the form of $\cos ax$ or $\sin ax$**
- **Formulae III : We have to express $f(D)$ as the function of D^2, say $\phi\left(D^2\right)$ and then replace D^2 by $-a^2$.**

  If $\phi\left(-a^2\right) \neq 0$, then we have to use following result.
- $If\ y_p = \frac{1}{f(D)}\cos ax = \frac{1}{\phi\left(D^2\right)}\cos ax$ then $y_p = \frac{1}{\phi\left(-a^2\right)}\cos ax$
- **Case: 1: Sometimes, we cannot form $\phi\left(D^2\right)$. Then we will try to get $\phi\left(D^2, D\right)$ that is a function of $D\ and\ D^2$.**

In such cases, we have to use the procedures as given below:

$$y_p = \frac{1}{D^2-2D+1}\cos 3x = \frac{1}{-3^2-2D+1}\cos 3x = \frac{1}{-8-2D}\cos 3x$$

- **Linear differential equations (second or higher order) with constant coefficients**

HW (It is a must event)

- **Non-homogeneous differential equation**

- **Case: 1: Sometimes, we cannot form $\phi\left(D^2\right)$. Then we will try to get$\phi\left(D^2, D\right)$ that is a function of $D\ and\ D^2$.**

In such case, we have to use the following procedures:

$$y_p = \frac{1}{-8-2D} \cos 3x = \frac{1}{-2} \frac{1}{D+4} \cos 3x$$

$$= \frac{1}{-2} \frac{D-4}{D^2-4^2} \cos 3x$$

$$= -\frac{1}{2} \ \frac{D-4}{-3^2-4^2} \cos 3x$$

$$= \frac{1}{50} (D-4) \cos 3x$$

➢ ☞ Instructions:

The following slides are given to all of you in advance for your convenience, and please check and try to understand the given slides (lecture martials) before joining the next class.

If you do so, then it would be easier for you to understand the next class. Besides, you will feel comfortable in asking relevant questions in the classroom, if any!

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**

- **Case: 1: Sometimes, we cannot form $\phi\left(D^2\right)$. Then we will try to get $\phi\left(D^2, D\right)$ that is a function of $D \text{ and } D^2$.**

In such case, we have to use the following procedures:

$$y_p = \frac{1}{50}(D-4)\cos 3x = \frac{1}{50}\left(-3\sin 3x - 4\cos 3x\right) \text{ (Ans.)}$$

- **Case 2: If $\phi\left(-a^2\right) = 0$, then we will use the following formulae $\frac{1}{f(D)}e^{ax} = e^{ax}\frac{1}{f(D+a)}e^{0x}$ and we have to follow the procedure as presented in the next slide.**

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**
- **Case 2: Let** $y_p = \frac{1}{D^2+a^2} \sin ax$

$$= \text{Imaginary Part of } \left(\frac{1}{D^2+a^2} \cos ax + i \frac{1}{D^2+a^2} \sin ax\right)$$

$$= \text{Imaginary Part of } \frac{1}{D^2+a^2} (\cos ax + i \sin ax$$

$$\therefore\ y_p = \text{Imaginary Part of } \frac{1}{D^2+a^2} e^{iax} ------ -(1)$$

But $\frac{1}{D^2+a^2} e^{iax} = e^{iax} \frac{1}{(D+ia)^2+a^2} e^{0x}$

$$= e^{iax} \frac{1}{D^2+2Dai} e^{0x} = e^{iax} \frac{1}{D(D+2ai)} e^{0x}$$

$$= e^{iax} \frac{1}{D} \frac{1}{(D+2ai)} e^{0x}$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**
- **Case 2: If $\phi\left(-a^2\right)=0$, then** $\frac{1}{f(D)}e^{ax}=e^{ax}\frac{1}{f(D+a)}e^{0x}$

Now, $\frac{1}{D^2+a^2}\,e^{iax}=e^{iax}\,\frac{1}{D}\,\frac{1}{(D+2ai)}\,e^{0x}=\;e^{iax}\,\frac{1}{D}\,\frac{1}{(0+2ai)}\,e^{0x}$

$$=\;e^{iax}\frac{1}{D}\,\frac{1}{2ai}\,1=\;e^{iax}\,\frac{x}{2ai}$$

$$=\frac{x}{2ai}\;e^{iax}=\frac{x}{2ai}\;(\cos ax+i\sin ax)$$

$$=\frac{x}{2ai}\cos ax+\frac{x}{2a}\sin ax$$

$$=-\frac{xi}{2a}\cos ax+\frac{x}{2a}\sin ax$$

Therefore, imaginary part of $\frac{1}{D^2+a^2}\;e^{iax}=-\frac{x}{2a}\cos ax$

Hence, we get from (1), $y_p=\;\frac{1}{D^2+a^2}\;\sin ax=\;-\frac{x}{2a}\cos ax$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation**

➢ **Formulae IV:** $(a)\ y_p = \frac{1}{D^2+a^2}\ sin\ ax = -\frac{x}{2a}\cos \mathrm{ax}$

$$(b)\ y_p = \frac{1}{D^2+a^2}\ cos\ ax = \frac{x}{2a}\sin \mathrm{ax}$$

Example: Solve $\frac{d^2y}{dx^2} + 4y = 8\ cos\ 2x$

Solution: Let $y = e^{mx}$ **the trail solution of** $\frac{d^2y}{dx^2} + 4y = 0$**. Therefore, the corresponding auxiliary equation is** $m^2 + 4 = 0\ \therefore m = \pm 2i$

Therefore, the complementary solution of given equation is

$$y_c = c_1 cos 2x + c_2 \sin 2x ------------------(1)$$

- **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation**

Example: Solve $\frac{d^2y}{dx^2} + 4y = 8\cos 2x$

Solution: The particular integral, $y_p = \frac{1}{D^2+4} 8\cos 2x$

$$\Rightarrow y_p = 8\frac{1}{D^2+4}\cos 2x$$

$$\left[We\ know\ that, y_p = \frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin \mathrm{ax}\right]$$

$$\therefore\ y_p = 8\frac{x}{2.\,2}\sin 2x = 2x\sin 2x$$

Therefore, the general solution of given equation is

$$y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + 2\mathrm{x}\sin 2\mathrm{x}\ (\mathrm{Ans.})$$

■ **Linear differential equations (second or higher order) with constant coefficients**

➢ **Non-homogeneous differential equation**

Exercise-15

Solve the following differential equations :

1. $(D^2+9)\,y = \cos 2x + \sin 2x.$ **Ans.** $y = c_1 \cos 3x + c_2 \sin 3x + (1/5)(\cos 2x + \sin 2x)$

2. $(D^3+D^2-D-1)\,y = \cos 2x.$ **Ans.** $y = c_1 e^x + (c_2 + c_3 x)\,e^{-x} - (1/25)(2 \sin 2x + \cos 2x)$

3. $(D^2-5D+6)\,y = \sin 3x.$ **Ans.** $y = c_1 e^{2x} + c_2 e^{3x} + (1/78)(5 \cos 3x - \sin 3x)$

4. $(D^2+D+1)\,y = \sin 2x.$ **Ans.** $y = e^{-x/2}\left\{c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2)\right\} - (1/13)(2\cos 2x + 3\sin 2x)$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**

- **Suppose $y_p = \frac{1}{f(D)} X$ where X is the form of $e^{ax} V$ where V is any function of x.**

- **Working rule for finding y_p when X is the form of of $e^{ax} V$**

- **Formulae V : $y_p = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$ where $\frac{1}{f(D+a)} V$ can be calculated by the methods which have already discussed in the previous section.**

- **Example: Solve : $\left(D^2 + 3D + 2\right)y = e^{2x} sinx$**

Solution: Let $y = e^{mx}$ the trail solution of $\left(D^2 + 3D + 2\right)y = 0$. Then the corresponding auxiliary is $m^2 + 3m + 2 = 0$

$$\Rightarrow m^2 + 2m + \mathrm{m} + 2 = 0$$

$$\Rightarrow m(m + 2) + 1(m + 2) = 0 \Rightarrow (m + 2)\ (m + 1) = 0\ \therefore m = -1, -2$$

- **Linear differential equations (second or higher order) with constant coefficients**

- **Non-homogeneous differential equation**
- **Example: Solve :** $\left(D^2+3D+2\right)y = e^{2x}sinx$

Solution: Since $m=-1,-2,$ the complementary solution of the given equation is

$$y_c = C_1e^{-x} + C_2e^{-2x}$$

Now, the particular integral, $y_p = \frac{1}{D^2+3D+2}e^{2x}sinx$

$$= e^{2x}\frac{1}{(D+2)^2+3(D+2)+2}sinx$$

$$= e^{2x}\frac{1}{D^2+4D+4+3D+6+2}sinx$$

$$= e^{2x}\frac{1}{-1^2+4D+4+3D+6+2}sinx$$

$$= e^{2x}\frac{1}{11+7D}sinx$$

■ Linear differential equations (second or higher order) with constant coefficients

➢ **Non-homogeneous differential equation**

➢ **Example: Solve :** $\left(D^2+3D+2\right)y = e^{2x} sinx$

Solution: Now, the particular integral, $y_p = e^{2x}\ \frac{1}{11+7D}\ sinx$

$$= e^{2x}\ \frac{11-7D}{(11+7D)(11-7D)}\ sinx$$

$$= e^{2x}\ \frac{11-7D}{(121-49D^2)}\ sinx$$

$$= e^{2x}\ \frac{11-7D}{(121+49)}\ sinx$$

$$= \frac{e^{2x}}{170}\ (11-7D)\ sinx$$

$$= \frac{e^{2x}}{170}\ (11sinx-7cosx)$$

Therefore, the general solution is $y = y_c + y_p$

$$= C_1e^{-x} + C_2e^{-2x} + \frac{e^{2x}}{170}\ (11sinx-7cosx)$$

- Linear differential equations (second or higher order) with constant coefficients
- Non-homogeneous differential equation

Exercise-16

Solve *the following differential equations:*

1. $(D^2 - 2D + 4)\,y = e^x \cos x$ **Ans.** $y = e^x\,[c_1 \cos(x\sqrt{3}) + c_2 \sin(x\sqrt{3})] + (1/2)\,e^x \cos x$
2. $(D+1)^3\,y = x^2 e^{-x}$ **Ans.** $y = (c_1 + c_2 x + c_3 x^2)\,e^{-x} + (x^5/60) \times e^{-x}$
3. $(D^2 - 2D + 5)\,y = e^{2x} \sin x$ **Ans.** $y = e^x\,(c_1 \cos 2x + c_2 \sin 2x) - (1/10)\,e^{2x}(\cos x - 2 \sin x)$
4. $(D^2 - 1)\,y = e^x \cos x.$ **Ans.** $y = c_1 e^x + c_2 e^{-x} + (1/5)\,e^x\,(2 \sin x - \cos x)$

No need to solve.

Thank you for your attendance and attention