Functions of Complex Variable

Q : 1

(a) $f(z) = \frac{1}{z^2+1}$

Here, $z^2+1 \neq 0$

$\Rightarrow z^2 \neq -1$

$\Rightarrow z \neq \pm\sqrt{-1}$

$\Rightarrow z \neq \pm i \quad [i = \sqrt{-1}]$

$f(z) = \frac{1}{z^2+1}$, Domain $D(f)$ can be define on the whole finite plane except at the point when $z = \pm i$.

(b) $f(z) = \text{Arg}\left(\frac{1}{z}\right)$

We know, The principal value of $\arg(z)$, denoted by $\text{Arg}(z)$ is the unique value such that $(-\pi < \text{Arg}(z) < \pi)$

$\text{Arg}\left(\frac{1}{z}\right) = \tan^{-1}\left(\frac{\frac{-y}{x^2+y^2}}{\frac{x}{x^2+y^2}}\right)$

$= \tan^{-1}\left(\frac{-y}{x}\right)$

Here $x \neq 0$

$\therefore \text{Re}(z) \neq 0$

$\therefore$ Domain, $D(f)$ can be define on the whole plane except when $\text{Re}(z) = 0$.

$\frac{1}{z} = \frac{1}{x+iy}$

$= \frac{x-iy}{(x+iy)(x-iy)}$

$= \frac{x-iy}{x^2+y^2} \quad [i^2 = -1]$

$\therefore \frac{1}{z} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$

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(c) $f(z) = \dfrac{z}{z+\bar{z}}$

Here,

$\therefore z+\bar{z} \neq 0$

$\Rightarrow x+i\cancel{y}+x-i\cancel{y} \neq 0$

$\Rightarrow 2x \neq 0$

$\Rightarrow x \neq 0$

$\therefore Re(z) \neq 0$

$f(z) = \dfrac{z}{z+\bar{z}}$ can define on the whole plane except when $Re(z) = 0$.

$\therefore$ Domain $D(f)$ = All complex number except $x=0$ or $Re(z) = 0$.

(d) $f(z) = \dfrac{1}{1-|z|^2}$

Here, $1-|z|^2 \neq 0$

$\Rightarrow -|z|^2 \neq -1$

$\Rightarrow |z|^2 \neq 1$

$\Rightarrow |z| \neq 1$

$\therefore$ Domain $D(f)$ = All complex number except $|z| = 1$

$f(z) = \dfrac{1}{1-|z|^2}$ can be define on the whole plane except when $|z| = 1$

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Q : 2

$$f(z) = z^3 + z + 1$$

We know, $z = x + iy$

$$f(z) = z^3 + z + 1$$
$$= (x+iy)^3 + (x+iy) + 1$$
$$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3 + x + iy + 1$$
$$= x^3 + 3x^2 iy - 3xy^2 + (-iy^3) + x + iy + 1$$
$$= x^3 + 3x^2 iy - 3xy^2 - iy^3 + x + iy + 1$$
$$= x^3 - 3xy^2 + x + 1 + i(3x^2y - y^3 + y)$$

$f(z) = u(x,y) + i\,v(x,y)$ where

$$u = x^3 - 3xy^2 + x + 1$$
$$v = 3x^2y - y^3 + y$$

Q : 3 $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$ ----(i)

Given, $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

putting the values of x and in the (i) equation, we get,

$$= \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 - 2y + 2ix - 2i\left(\frac{z-\bar{z}}{2}\,\frac{z-\bar{z}}{2i}\right)$$

$$= \left(\frac{z+\bar{z}}{2}\right)^2 + \left(\frac{z-\bar{z}}{2}\right)^2 + i2(x+iy) + \left(\frac{z^2-\bar{z}^2}{2}\right)$$

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$$= \frac{z^2}{4} + \cancel{2\left(\frac{z}{2}\cdot\frac{\bar{z}}{2}\right)} + \frac{\bar{z}^2}{4} + \frac{z^2}{4} - \cancel{2\left(\frac{z}{2}\cdot\frac{\bar{z}}{2}\right)} + \frac{\bar{z}^2}{4} + 2iz - \frac{z^2}{2} + \frac{\bar{z}^2}{2}$$

$$= \frac{2z^2}{4} + \frac{2\bar{z}^2}{4} + 2iz - \frac{z^2}{2} + \frac{\bar{z}^2}{2}$$

$$= \cancel{\frac{z^2}{2}} + \frac{\bar{z}^2}{2} + 2iz - \cancel{\frac{z^2}{2}} + \frac{\bar{z}^2}{2}$$

$$= \frac{\bar{z}^2}{2} + \frac{\bar{z}^2}{2} + 2iz$$

$$= \bar{z}^2 + 2iz$$

$$\therefore f(z) = \bar{z}^2 + 2iz$$

Q:4: $f(z) = z + \frac{1}{z}$; $(z \neq 0)$

In polar form, $z = re^{i\theta} = r[\cos\theta + i\sin\theta]$

and, $\frac{1}{z} = \frac{1}{r}[e^{-i\theta}] = \frac{1}{r}[\cos\theta - i\sin\theta]$

$$f(z) = z + \frac{1}{z} = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= r\cos\theta + ir\sin\theta + \frac{1}{r}\cos\theta - \frac{1}{r}i\sin\theta$$

$$= \cos\theta\left(r + \frac{1}{r}\right) + i\sin\theta\left(r - \frac{1}{r}\right)$$

$$\therefore = u(r,\theta) + iv(r,\theta)$$

where, $u(r,\theta) = \left(r + \frac{1}{r}\right)\cos\theta$

$v(r,\theta) = i\left(r - \frac{1}{r}\right)\sin\theta$

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Limit and continuity

Q : 1

(a) $\lim_{z \to z_0} Re(z) = Re(z_0)$

Let $\varepsilon > 0$ be given, we have to find $\delta > 0$ such that, $0 < |z - z_0| < \delta$ then $|Re(z) - Re(z_0)| < \varepsilon$

$$\therefore |Re(z) - Re(z_0)| = |x - x_0|$$
$$= \sqrt{(x - x_0)^2}$$
$$= x - x_0$$
$$\leqslant |z - z_0| < \delta$$

$\therefore \delta = \varepsilon$

(b) $\lim_{z \to z_0} \overline{z} = \overline{z_0}$

Let $\varepsilon > 0$, we have to find $\delta > 0$ such that, $0 < |z - z_0| < \delta$, then $|\overline{z} - \overline{z}_0| < \varepsilon$

$$\therefore |\overline{z} - \overline{z}_0| = |\overline{z - z_0}|$$
$$= |z - z_0| < \delta$$

Hence $\delta = \varepsilon$

$\therefore \lim_{z \to z_0} \overline{z} = \overline{z_0}$ (proved)

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(c) $\lim_{z\to 0} \frac{\bar{z}^2}{z} = 0$

We have, $f(z) = \frac{\bar{z}^2}{z}$, Using definition,

Let $\varepsilon>0$, we have to find $\delta>0$ such that

$0<|z-0|<\delta$ then $|f(z)-0|<\varepsilon$

or, $0<|z|<\delta$, then $|f(z)|<\varepsilon$

$$\therefore |f(z)| = \left|\frac{\bar{z}^2}{z}\right| = \frac{|\bar{z}|^2}{|z|}$$

$$= \frac{|z|^2}{|z|} \qquad [|\bar{z}| = |z|]$$

$$= |z| < \delta = \varepsilon$$

$\therefore 0<|z|<\delta \Rightarrow f|(z)|<\epsilon$

Hence, $\delta = \varepsilon$

$\therefore \lim_{z\to 0} \frac{\bar{z}^2}{z} = 0$ (proved)

Q: 3

(a) $\lim_{z\to z_0} \frac{1}{z^n}$ $(z_0 \neq 0)$

$$= \frac{\lim_{z\to z_0} 1}{\lim_{z\to z_0} z^n}$$

$$= \frac{1}{z_0^n}$$

$\therefore \lim_{z\to z_0} \frac{1}{z^n} = \frac{1}{z_0^n}$ (Ans)

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(b) $\lim_{z \to i} \frac{iz^3 - 1}{z + i}$

$= \frac{i(i)^3 - 1}{i + i}$

$= \frac{i(-i) - 1}{2i} = \frac{-i^2 - 1}{2i}$

$= \frac{1 - 1}{2i}$ $[i^2 = 1]$

$= 0$

$\therefore \lim_{z \to i} \frac{iz^3 - 1}{z + i} = 0.$ (Ans)

(c) $\lim_{z \to z_0} \frac{P(z)}{Q(z)}$

$= \frac{\lim_{z \to z_0} P(z)}{\lim_{z \to z_0} Q(z)}$

$= \frac{P(z_0)}{Q(z_0)}$

$\therefore \lim_{z \to z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q(z_0)}$ (Ans)

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Q: 5

$$f(z) = \left(\frac{z}{\bar{z}}\right)^2$$

$\lim_{z \to 0} f(z)$ does not exist.

For, if it did exist, it could be found by letting the point z approach the origin in any manner. according to the question. $z = (x, 0)$ and $z = (x, x)$

when, $z = (x, 0)$ on the real axis,

$$f(z) = \left(\frac{x+i0}{x-i0}\right)^2 = \left(\frac{x}{x}\right)^2 = 1$$

and, when $z = (x, x)$ on the real axis and imaginary axis,

$$f(z) = \left(\frac{x+ix}{x-ix}\right)^2 = \left\{\frac{(x+ix)(x+ix)}{(x-ix)(x+ix)}\right\}^2$$

$$= \left(\frac{(x+ix)^2}{(x)^2-(ix)^2}\right)^2$$

$$= \left(\frac{x^2+2x(ix)+i^2x^2}{x^2-i^2x^2}\right)^2$$

$$= \left(\frac{x^2+2ix^2-x^2}{x^2+x^2}\right)^2$$

$$= (i)^2 = -1$$

Since limit is unique value, $\lim_{z \to 0} f(z) = \left(\frac{z}{\bar{z}}\right)^2$ does not exist.

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Q:10

(a) $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2}$

$= 4 \lim_{z\to\infty} \frac{z^2}{(z-1)^2}$

$= 4 \lim_{z\to\infty} \left(\frac{z}{z-1}\right)^2$

$= 4 \lim_{z\to\infty} z\left(\frac{z.1}{z\left(1-\frac{1}{z}\right)}\right)^2$

$= 4 \lim_{z\to\infty} \left(\frac{1}{1-\frac{1}{z}}\right)^2$

$= 4\left(\frac{\lim_{z\to\infty} 1}{\lim_{z\to\infty}\left(1-\frac{1}{z}\right)}\right)^2$

$= 4\left(\frac{1}{1-0}\right)^2 = 4$ (Showed)

(b) $\lim_{z\to 1} \frac{1}{(z-1)^3} = \lim_{z\to 1} 1 \cdot \frac{1}{(z-1)^3}$

$= \lim_{z\to 1} e^{\log\left(1.\frac{1}{(z-1)^3}\right)}$

$= e^{\left(\lim_{z\to 1} \log \frac{1}{z^3-3z^2+3z-1}\right)}$

$= e^{\infty}$

$\therefore \lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$ (Showed)

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(c) $\lim_{z\to\infty} \frac{z^2+1}{z-1}$

$= \lim_{z\to\infty} \frac{z\left(z+\frac{1}{z}\right)}{z\left(1-\frac{1}{z}\right)}$

$= \lim_{z\to\infty} \frac{z+\frac{1}{z}}{1-\frac{1}{z}}$

$= \frac{\lim_{z\to\infty} z+\frac{1}{z}}{\lim_{z\to\infty} 1-\frac{1}{z}}$

$= \frac{\infty+0}{1-0} = \infty$. (Showed)

<u>Q:11</u> $T(z) = \frac{az+b}{cz+d}$ $(ad-bc \neq 0)$

(a) $\lim_{z\to\infty} T(z) = \infty$, if $c=0$

$T(z) = \frac{az+b}{d}$ when $c=0$

$\therefore \lim_{z\to\infty} T(z) = \lim_{z\to\infty} \frac{az+b}{d}$

$= \frac{\lim_{z\to\infty} az+b}{\lim_{z\to\infty} d} = \frac{\infty}{d} = \infty$

$\therefore \lim_{z\to\infty} T(z) = \infty$ (Showed)

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(b) $\lim_{z\to\infty} T(z) = \frac{a}{c}$ and $\lim_{z\to -\frac{d}{c}} T(z) = \infty$ if $c \neq 0$.

$$\therefore \lim_{z\to\infty} T(z) = \lim_{z\to\infty} \frac{az+b}{cz+d}$$

$$= \lim_{z\to\infty} \frac{z\left(a+\frac{b}{z}\right)}{z\left(c+\frac{d}{z}\right)}$$

$$= \frac{\lim_{z\to\infty}\left(a+\frac{b}{z}\right)}{\lim_{z\to\infty}\left(c+\frac{d}{z}\right)}$$

$$= \frac{a+\frac{b}{\infty}}{c+\frac{d}{\infty}}$$

$$= \frac{a+0}{c+0} = \frac{a}{c} \qquad [c\neq 0]$$

$$\therefore \lim_{z\to\infty} T(z) = \frac{a}{c} \quad \text{when} \quad c\neq 0.$$

$$\therefore \lim_{z\to -\frac{d}{c}} T(z) = \lim_{z\to -\frac{d}{c}} \frac{az+b}{cz+d}$$

$$= \frac{\lim_{z\to -\frac{d}{c}} az+b}{\lim_{z\to -\frac{d}{c}} cz+d}$$

$$= \frac{a\left(-\frac{d}{c}\right)+b}{\cancel{c}\left(-\frac{d}{\cancel{c}}\right)+d}$$

$$= \frac{a(-d/c)+b}{-d+d} = \infty$$

$$\therefore \lim_{z\to -\frac{d}{c}} T(z) = \infty \quad \text{when} \quad c\neq 0.$$

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