CSE106 Discrete Mathematics
Credit Hours: 3 + 0 = 3
Prerequisite: CSE103 Structured Programming

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Text book:

Kenneth H. Rosen, *Discrete Mathematics and Its Applications with Combinatorics and Graph Theory*, 7th Edition, Tata McGraw-Hill Publishing Company Limited, New Delhi, 2015.

Midterm II

Link:
https://drive.google.com/file/d/1OyGwIkoHmmybFivJz8PonUl6BnOEIE0r/view?usp=sharing

Chapter 2

Basic Structures: Sets, Functions, Sequences, and Sums

2.1 Sets

Definition 1: A *set* is an unordered collection of objects.

Definition 2:
The objects in a set are called the *elements*, or *members*, of the set.
We write $a \in A$ to denote that a is an element of the set A.
The notation $a \notin A$ denotes that a is not a member of the set A.

A set is described by listing all the members of the set.

Example 1: The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

Example 2: The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$.

Example 3: $\{a, 2, \text{Fred}, \text{New Jersey}\}$ is the set containing the four elements a, 2, Fred, and New Jersey.

Example 4: The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$.

Another way of describing a set is to use **set builder notation**. We characterize all those elements in the set by stating the property or properties they must have to be members.

The set O of all odd positive integers less that 10 can be written as
$O = \{x \mid x$ is an odd positive integer less than $10\}$

These sets play an important role in discrete mathematics:
$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of i**ntegers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**

Definition 3: Two sets are *equal* if and only if they have the same elements. That is, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

Example 6: The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal.

Example: The sets $\{1, 3, 5\}$ and $\{3, 5, 1, 3, 5\}$ are equal.

Sets can be represented graphically using Venn diagrams.
In Venn diagrams the **universal set** *U*, which contains all the objects under consideration, is represented by a rectangle.
Inside this rectangle, circles are used to represent sets.
Sometimes points are used to represent the particular elements of the set.

Example 7: Draw a Venn diagram that represents *V*, the set of vowels in the English alphabet.

Solution:
Universal set *U* is the set of the 26 letters of the English alphabet.

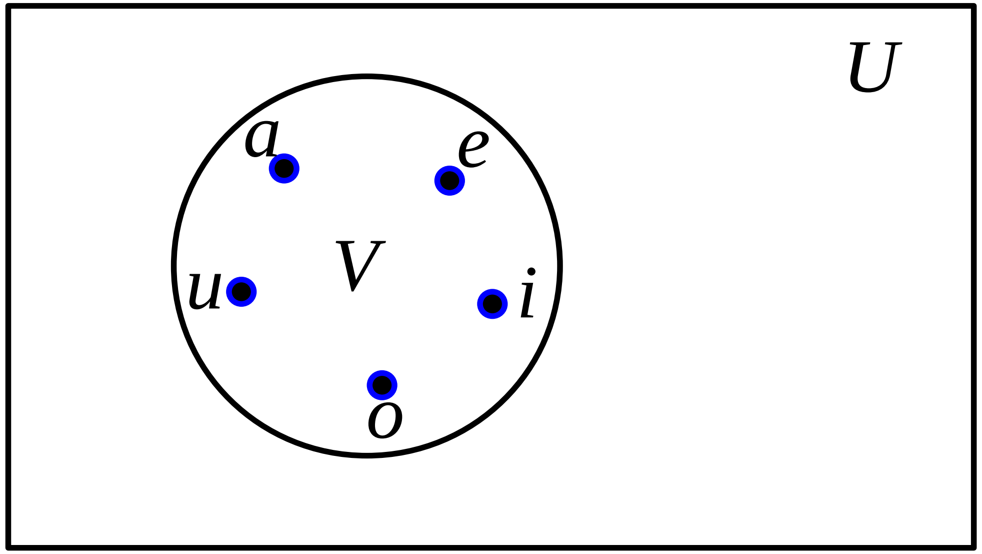


Figure 1 Venn Diagram for the Set of Vowels.

There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by $\emptyset$. The empty set can also be denoted by { }.

A set with one element is called a **singleton set**.

Definition 4: The set A is said to be a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.
We see that $A \subseteq B$ if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.

Example 8:
The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10.
The set of rational numbers is a subset of the set of real numbers.

Theorem 1: For any set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

When we wish to emphasize that a set A is a subset of the set B but $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B.

$A \subset B$ if and only if $\forall x(x \in A \to x \in B) \wedge \exists x(x \in B \wedge x \notin A)$ is true.

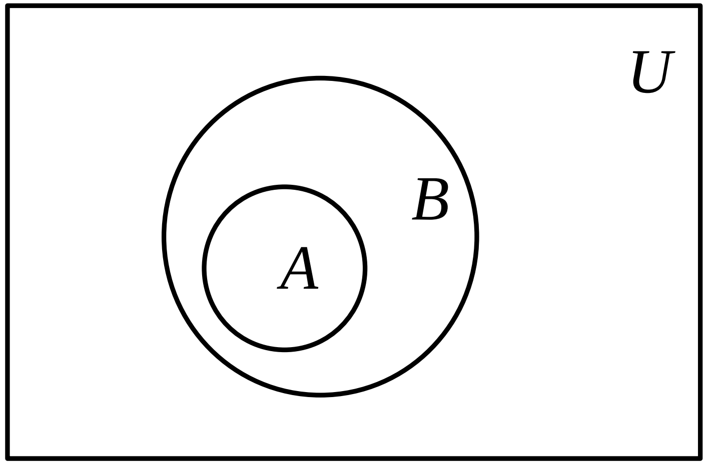


Figure 2 Venn Diagram Showing that A is a subset of B.

Sets may have other sets as member.

$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
$B = \{x \mid x$ is a subset of the set $\{a, b\}\}$
These two sets are equal, that is, $A = B$.

Note that $\{a\} \in A$, but $a \notin A$.

Definition 5: Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

Example 9: Let A be the set of odd positive integers less than 10, that is, A = {1, 3, 5, 7, 9}. Then $|A| = 5$.

Example 10: Let S be the set of letters in the English alphabet. Then $|S| = 26$.

Example 11: Since the null set has no elements, $|\emptyset| = 0$.

Definition 6: A set is said to be *infinite* if it is not finite.

Example 12: The set of positive integers is infinite.

The Power Set

Definition 7: Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $P(S)$.

Example 13: What is the power set of the set $\{0, 1, 2\}$?

Solution:
$P(\{0, 1, 2\})$
$= \{\varnothing, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

Example 14: What is the power set of the empty set? What is the power set of the set $\{\varnothing\}$?

Solution:
$P(\varnothing) = \{\varnothing\}$.
$P(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$

If a set has n elements, then its power set has 2^n elements.

Cartesian Products

Definition 8: The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, …, and a_n as its nth element.

We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal.
In other words, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$.

2-tuples are called **ordered pairs**.

The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

(a, b) and (b, a) are not equal unless $a = b$.

Definition 9: Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
Hence, $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

Example 16: What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution:
$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B.
The elements of R are ordered pairs, where the first element belongs to A and the second to B.

For example,
$R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$
is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$.

The Cartesian product $A \times B$ and $B \times A$ are not equal, unless $A = \varnothing$ or $B = \varnothing$ (so that $A \times B = \varnothing$) or unless $A = B$.

Example 17: Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where $A = \{1, 2\}$ and $B = \{a, b, c\}$.

Solution:
$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1). (c, 2)\}$
$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$
Therefore, $B \times A \neq A \times B$.

Example 18: What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Solution:

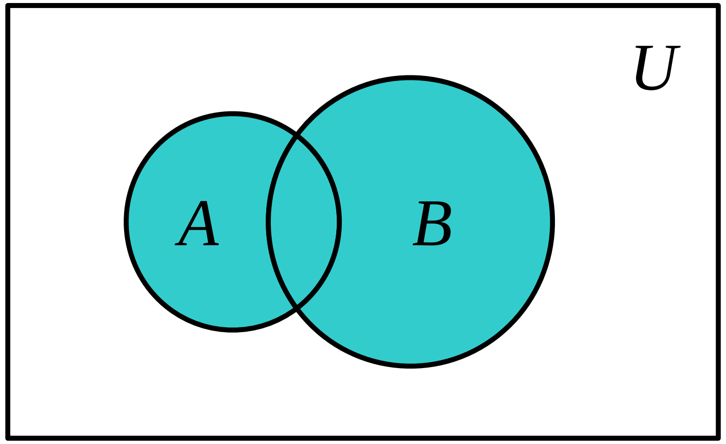
$A \times B$
$= \{0, 1\} \times \{1, 2\}$
$= \{(0, 1), (0, 2), (1, 1), (1, 2)\}$

$(A \times B) \times C$
$= \{(0, 1), (0, 2), (1, 1), (1, 2)\} \times \{0, 1, 2\}$
$= \{(0, 1, 0), (0, 1, 1), (0, 1, 2),$
$(0, 2, 0), (0, 2, 1), (0, 2, 2),$
$(1, 1, 0), (1, 1, 1), (1, 1, 2),$
$(1, 2, 0), (1, 2, 1), (1, 2, 2)\}$

2.2 Set Operations

Definition 1: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are in A or in B, or in both.

$A \cup B = \{x \mid x \in A \vee x \in B\}$.



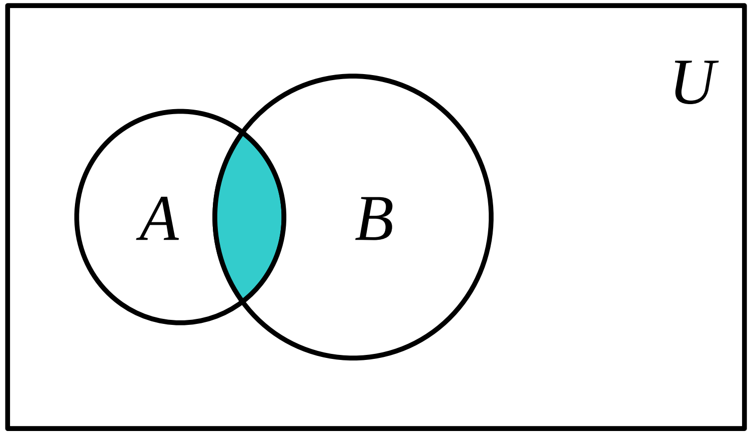
$A \cup B$ is shaded

Figure 1 Venn Diagram Representing the Union of A and B.

Example 1: $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$

Definition 2: Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set that contains those elements in both A and B.

$A \cap B = \{x \mid x \in A \land x \in B\}$.



$A \cap B$ is shaded

Figure 2 Venn Diagram Representing the Intersection of A and B.

Example 3: $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$

Definition 3: Two sets are called *disjoint* if their intersection is the empty set.

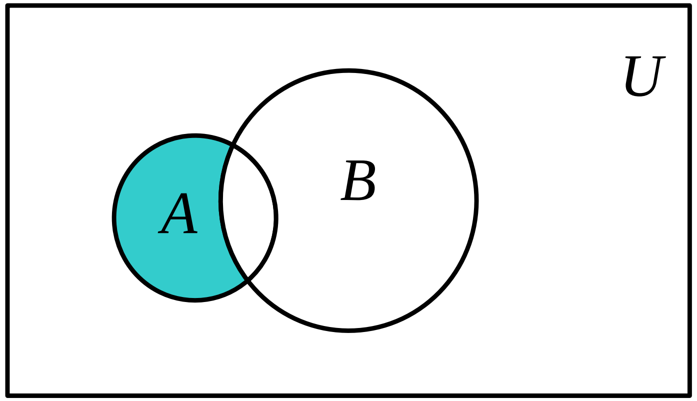
Example 5: Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Since $A \cap B = \varnothing$, A and B are disjoint.

Definition 4: Let A and B be sets. The difference of A and B, denoted by $A-B$, is the set containing those elements that are in A but not in B.

$$A-B = \{x \mid x \in A \wedge x \notin B\}$$

The difference of A and B is also called the *complement of B with respect to A.*

$$A-B = A \cap \bar{B}$$



$A - B$ is shaded.

Figure 3 Venn Diagram for the Difference of A and B.

Example 6:

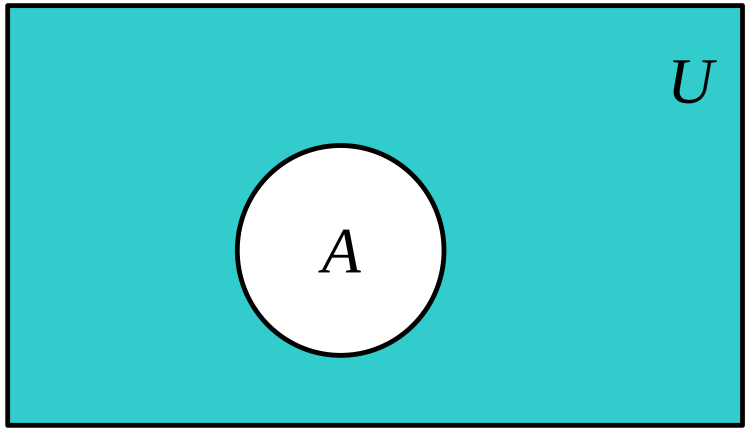
$\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$

$\{1, 2, 3\} - \{1, 3, 5\} = \{2\}$

Definition 5: Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U.

$\bar{A} = \{x \mid x \notin A\}$

In other words, the complement of the set A is $U–A$.



$\bar{A}$ is shaded.

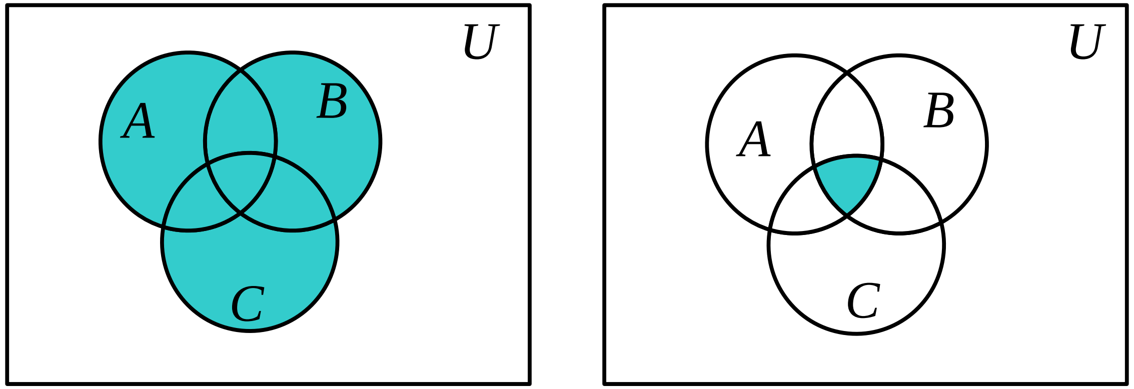
Figure 4 Venn Diagram for the Complement of the Set A.

Example 8: Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet).
Then $\bar{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$

Example 9: Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers).
Then $\bar{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Generalized Unions and Intersections

$A \cup B \cup C$ contains those elements that are in at least one of the sets A, B, and C.
$A \cap B \cap C$ contains those elements that are in all of sets A, B, and C.



(a) $A \cup B \cup C$ is shaded. (b) $A \cap B \cap C$ is shaded.

Figure 5 The Union and Intersection of A, B, and C.

Example 15:
Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

Solution:
$A \cup B = \{0, 2, 4, 6, 8\} \cup \{0, 1, 2, 3, 4\} = \{0, 1, 2, 3, 4, 6, 8\}$
$(A \cup B) \cup C = \{0, 1, 2, 3, 4, 6, 8\} \cup \{0, 3, 6, 9\} = \{0, 1, 2, 3, 4, 6, 8, 9\}$

$A \cap B = \{0, 2, 4, 6, 8\} \cap \{0, 1, 2, 3, 4\} = \{0, 2, 4\}$
$(A \cap B) \cap C = \{0, 2, 4\} \cap \{0, 3, 6, 9\} = \{0\}$

Computer Representation of Sets

Assume that the universal set U is finite.
First, specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$.
Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Example 18: Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order. What bit strings represent the subset A of all odd integers in U, the subset B of all even integers in U, and the subset C of integers not exceeding 5 in U?

Solution:
$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$A = \{1, 3, 5, 7, 9\}$
$B = \{2, 4, 6, 8, 10\}$
$C = \{1, 2, 3, 4, 5\}$

Bit strings
$U = 11\ 1111\ 1111$
$A = 10\ 1010\ 1010$
$B = 01\ 0101\ 0101$
$C = 11\ 1110\ 0000$

Example 19: The universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The given set is $A = \{1, 3, 5, 7, 9\}$. What is the bit string for $\bar{A}$?

Solution:

U = 11 1111 1111

A = 10 1010 1010

$\bar{A}$ = 01 0101 0101 (Bit-wise complement)

Example 20:
The universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
Two sets are $A = \{1, 2, 3, 4, 5\}$ and $B = (1, 3, 5, 7, 9\}$.
Use bit strings to find $A \cup B$, $A \cap B$, and $A - B$.

Solution:
U = 11 1111 1111

Union:
A = 11 1110 0000
B = 10 1010 1010

$A \cup B$ = 11 1110 1010 (bit-wise OR)
The corresponding set is $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$

Intersection:
A = 11 1110 0000
B = 10 1010 1010

$A \cap B$ = 10 1010 0000 (bit-wise AND)
The corresponding set is $A \cap B = \{1, 3, 5\}$

Example 20 (contd):

The universal set is U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.
Two sets are A = {1, 2, 3, 4, 5} and B = (1, 3, 5, 7, 9}.
Use bit strings to find $A \cup B$, $A \cap B$, and $A - B$.

Solution:
U = 11 1111 1111

Difference:
A = 11 1110 0000
B = 10 1010 1010
$\bar{B}$ = 01 0101 0101

A = 11 1110 0000
$\bar{B}$ = 01 0101 0101

$A - B = A \cap \bar{B}$ = 01 0100 0000 (Bit-wise AND)
The corresponding set is $A - B$ = {2, 4}

2.3 Functions

Definition 1:

Let A and B be sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A.
We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
If f is a function from A to B, we write $f : A \to B$.

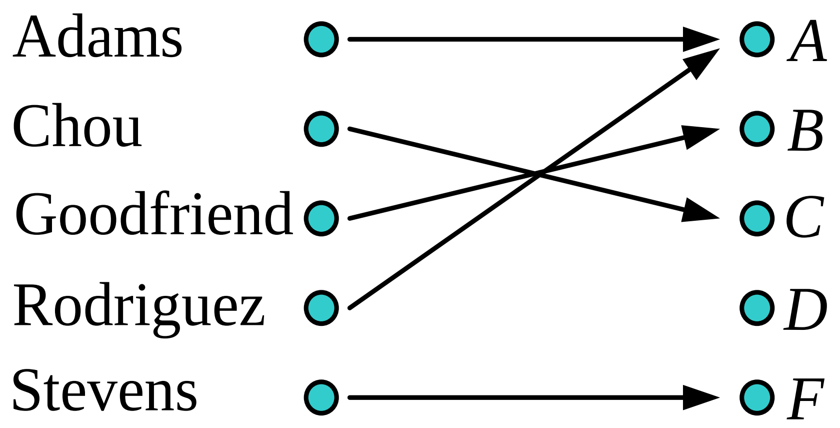


Figure 1 Assignment of Grades in a Discrete Mathematics Class.

Definition 2:

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f.
If $f(a) = b$, we say that b is the *image* of a and a is a *pre-image* of b.
The *range* of f is the set of all images of elements of A.
Also, if f is a function from A to B, we say that f maps A to B.

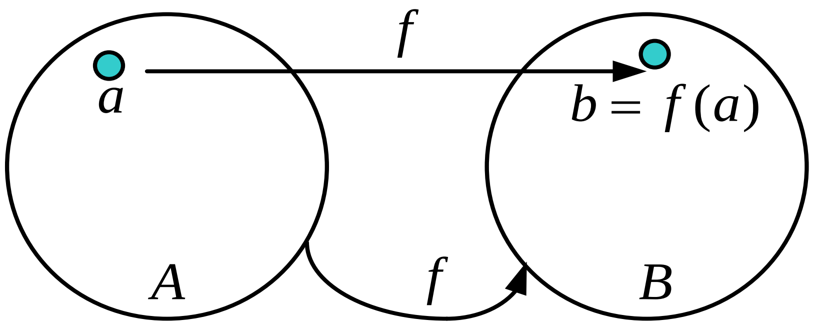
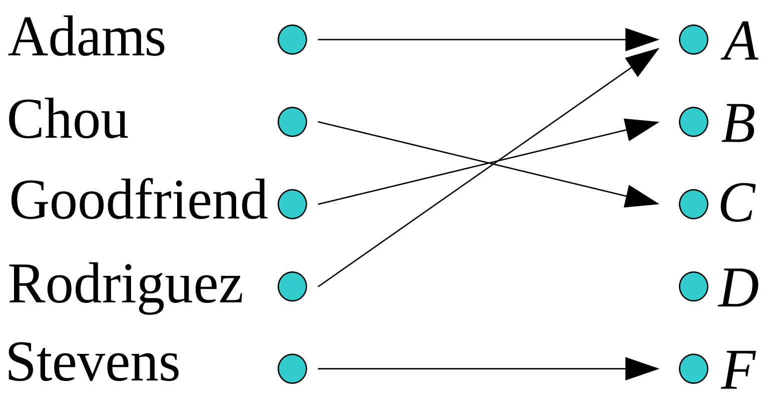


Figure 2: The function f maps A to B.

Example 1: What are the domain, codomain, and range of the function that assigns grades to students in discrete mathematics class described in the following figure?



Solution:
Let G be the function that assigns a grade to a student in discrete mathematics class.

$G(\text{Adams}) = A$.

The domain of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}

The codomain of G is the set $\{A, B, C, D, F\}$

The range of G is the set $\{A, B, C, F\}$.

Example 2: Let R be the relation consisting of ordered pairs
(Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22),
where each pair consists of a graduate student and the age of this student. What is the function that this relation determines?

Solution:
This relation defines the function *f*, where
f(Abdul) = 22
f(Brenda) = 24
f(Carla) = 21
f(Desire) = 22
f(Eddie) = 24
f(Felicia) = 22.

Here the domain is the set
{Abdul, Brenda, Carla, Desire, Eddie, Felicia}.

The codomain is the set of positive integers.

The range is the set {21, 22, 24}.

Example 4: Let $f : \mathbf{Z} \to \mathbf{Z}$ assign the square of an integer to an integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \ldots\}$.

Definition 3: Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined by

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$,

$(f_1 f_2)(x) = f_1(x)\, f_2(x)$.

Example 6: Let f_1 and f_2 be functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution:

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$

$(f_1 f_2)(x) = f_1(x) f_2(x) = x^2\,(x - x^2) = x^3 - x^4$

One-to-One and Onto Functions

Definition 5: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

Example 8: Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Solution:
The function f is one-to-one, since f takes on different values at the four elements of its domain.

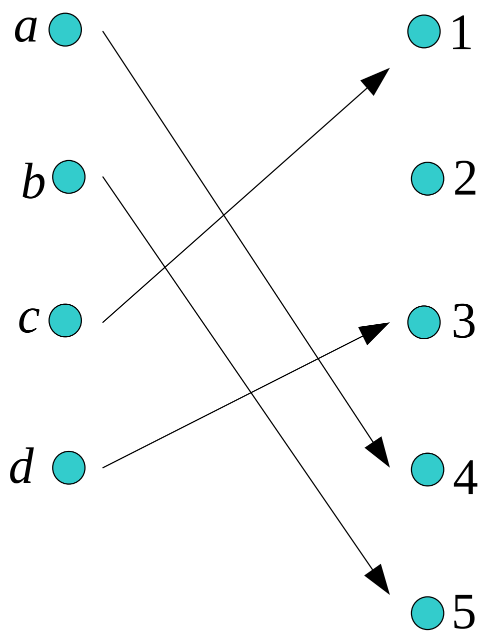


Figure 3 A one-to-one Function.

Example 9: Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution:
The function $f(x) = x^2$ is not one-to-one, since $f(1) = f(-1) = 1$, but $1 \neq -1$.

Example 10: Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

Solution:
The function $f(x) = x + 1$ is one-to-one, since $x + 1 \neq y + 1$ when $x \neq y$.

Definition 7: A function f from A to B is called *onto*, or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

Example 11: Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution:
Since all three elements of the codomain are images of elements in the domain, f is onto.

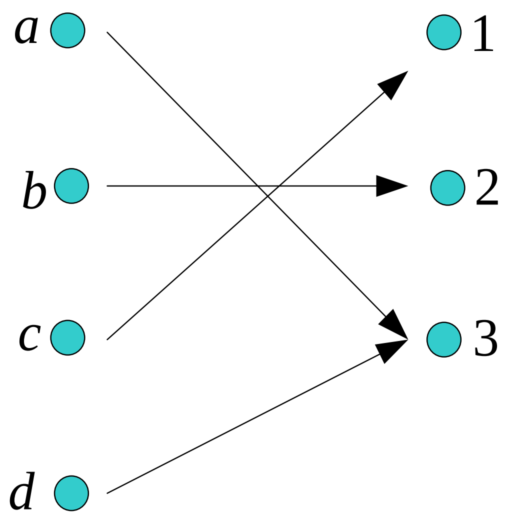


Figure 4 An Onto Function.

Example 12: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution:
The function f is not onto since there is no integer x with $x^2 = -1$, for example.

Example 13: Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

Solution:
This function is onto, since for every integer y from the codomain, there is an integer x in the domain such that $f(x) = y = x + 1$.

Definition 8: The function f is a *one-to-one correspondence*, or a *bijective*, if it is both one-to-one and onto.

Example 14: Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijective function?

Solution:
$f(a) = 4$
$f(b) = 2$
$f(c) = 1$
$f(d) = 3$

Each element of the domain maps to unique element of the codomain. So, the function f is one-to-one.
All elements of the codomain are mapped. So, the function f is onto.
As the function f is both one-to-one and onto function, it is a bijective function.

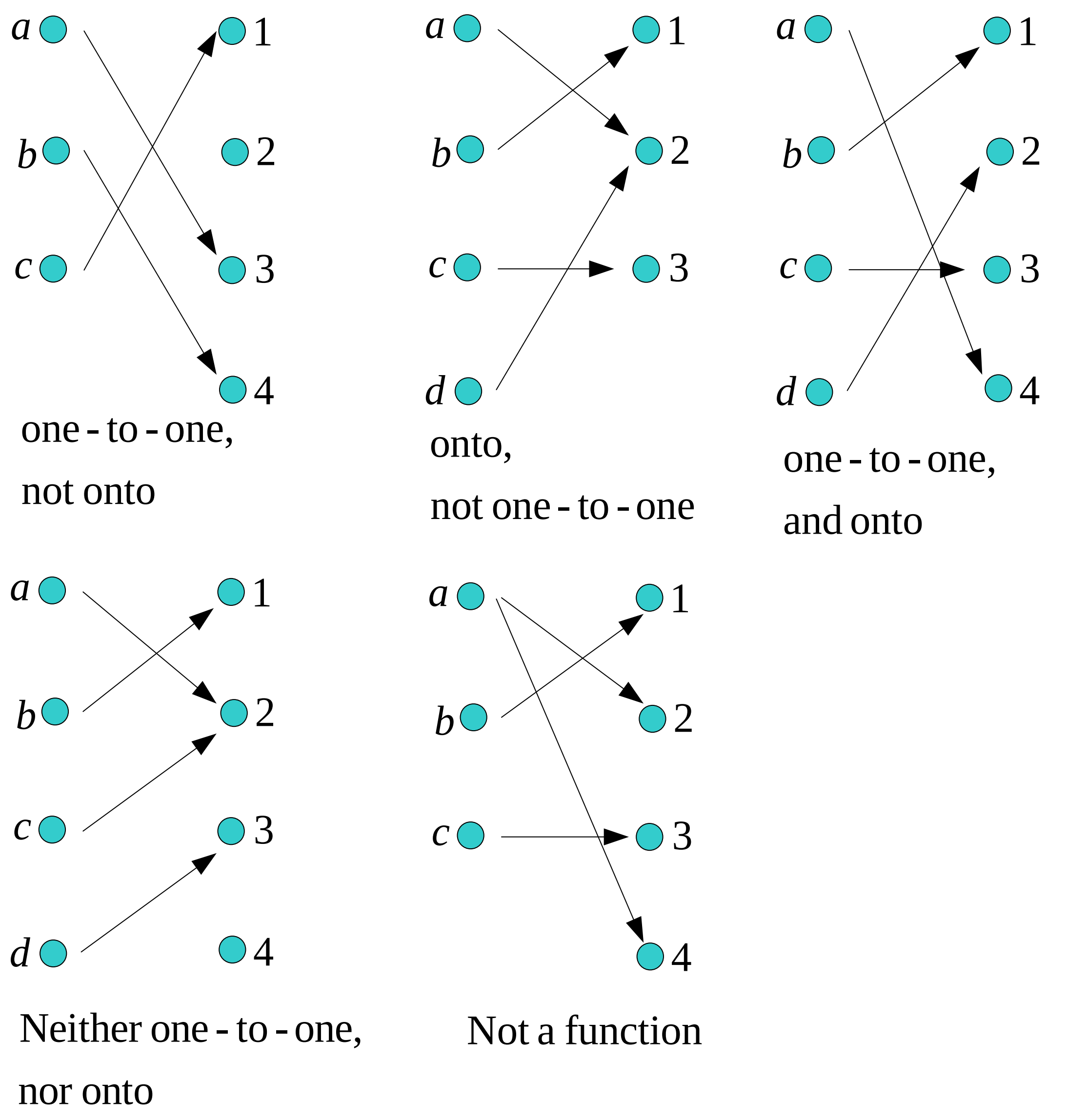


Figure 5 Examples of Different Types of Correspondance.

Inverse Functions and Compositions of Functions

Definition 9: Let f be a bijective function from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

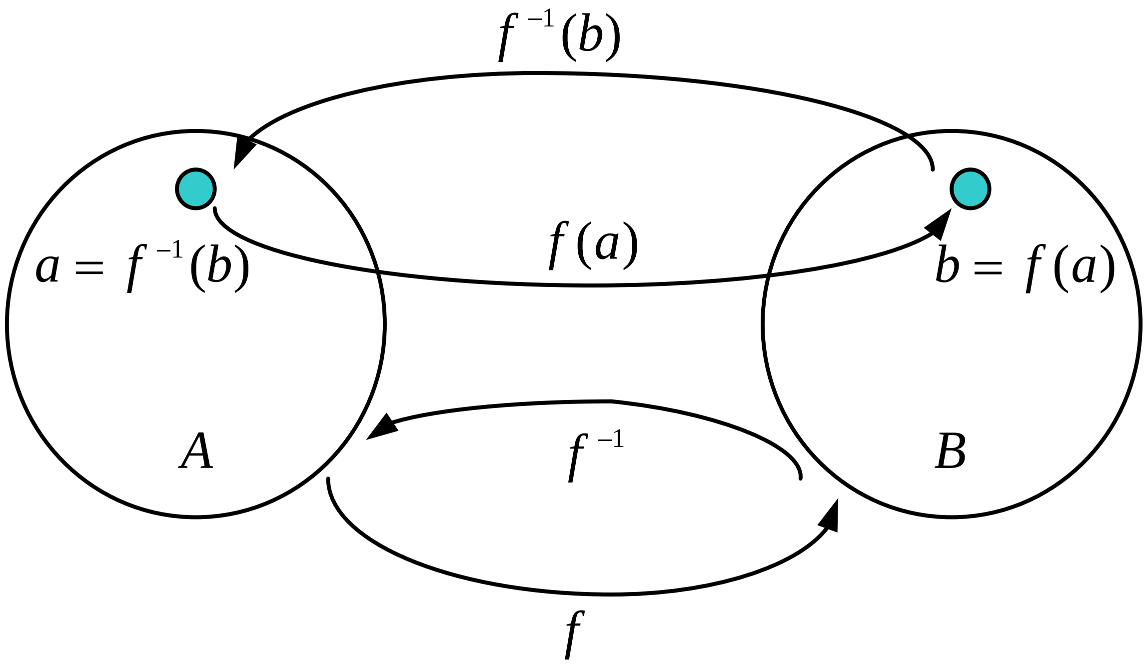


Figure 6 The Function f^{-1} is the Inverse of Function f.

- If a function f is not bijective, we cannot define an inverse function of f.
- A bijective function is called **invertible** because we can define an inverse of this function.
- A function is **not invertible** if it is not bijective function, because the inverse of such function does not exist.

Example 16: Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution:
$f(a) = 2$
$f(b) = 3$
$f(c) = 1$

The function f is one-to-one and onto, so it is a bijective function and is invertible.

The inverse function f^{-1} is
$f^{-1}(1) = c$
$f^{-1}(2) = a$
$f^{-1}(3) = b$.

Example 17: Let f: $\mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution:
The function f is a one-to-one function.
The function f is an onto function.

So, the function f is a bijective function and is invertible.

Suppose that y is the image of x, so that $y = x + 1$.
Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbf{Z}$ that is sent to y by f.
Consequently, $f^{-1}(y) = y - 1$.

Example 18: Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ with $f(x) = x^2$. Is f invertible?

Solution:
Since $f(-2) = f(2) = 4$, f is not one-to-one. So, f is not a bijective function. Hence, f is not invertible.

Example 19: Let f be the function from the set of all nonnegative real numbers to the set of all nonnegative real numbers with $f(x) = x^2$. Show that f is invertible.

Solution:
If $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 – y^2 = (x + y)(x – y) = 0$.
This means that $x + y = 0$ or $x – y = 0$, so $x = –y$ or $x = y$. Because both x and y are nonnegative, we must have $x = y$. so, $f(x) = x^2$ is one-to-one.

Furthermore, $f(x) = x^2$ is onto when the codomain is the set of all nonnegative real numbers, because each nonnegative real number has a square root. That is, if y is a nonnegative real number, there exists a nonnegative real number x such that $x = \sqrt{y}$, which means that $x^2 = y$.

Because the function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one and onto, it is a bijective function and is invertible.

Definition 10: Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.

To find $(f \circ g)(a)$ we first apply the function g to a to obtain $g(a)$ and then we apply the function f to the result of $g(a)$ to obtain $(f \circ g)(a) = f(g(a))$.

The composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.

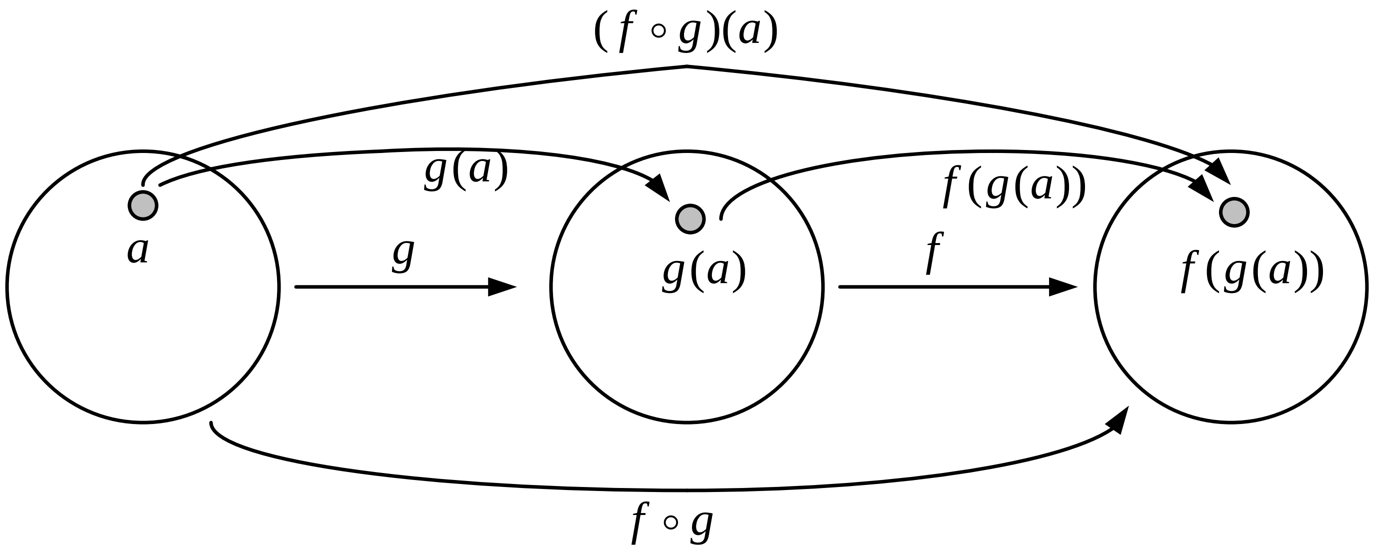


Figure 7 The Composition of the Functions f and g.

Example 20: Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Solution:

$g(a) = b$
$g(b) = c$
$g(c) = a$

$f(a) = 3$
$f(b) = 2$
$f(c) = 1$

The composition $f \circ g$ is defined by
$(f \circ g)(a) = f(g(a)) = f(b) = 2$,
$(f \circ g)(b) = f(g(b)) = f(c) = 1$,
$(f \circ g)(c) = f(g(c)) = f(a) = 3$.

$g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Example 21: Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution:
$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$
$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Chapter 4
Induction and Recursion

4.3 Recursive Definitions and Structural Induction

Recursively Defined Functions

We use two steps to define a function with the set of nonnegative integers as its domain:

BASIS STEP: Specify the value of the function at zero.
RECURSIVE STEP: Derive a rule for finding its value at an integer from its values at smaller integers.

Such a definition is called a **recursive** or **inductive** definition.

Example 1: Suppose that f is defined recursively by
$f(0) = 3$
$f(n) = 2f(n-1) + 3$
Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

Solution:
$f(0) = 3$
$f(1) = 2f(0) + 3 = 2 \cdot 3 + 3 = 9$
$f(2) = 2f(1) + 3 = 2 \cdot 9 + 3 = 21$
$f(3) = 2f(2) + 3 = 2 \cdot 21 + 3 = 45$
$f(4) = 2f(3) + 3 = 2 \cdot 45 + 3 = 93$

Example 2: Give a recursive definition of the factorial function $F(n) = n!$.

Solution:

$F(0) = 1$

$F(n) = n \cdot F(n - 1)$ for $n = 1, 2, 3, \ldots$

$$
\begin{aligned}
F(5) &= 5 \cdot F(4) \\
&= 5 \cdot 4 \cdot F(3) \\
&= 5 \cdot 4 \cdot 3 \cdot F(2) \\
&= 5 \cdot 4 \cdot 3 \cdot 2 \cdot F(1) \\
&= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot F(0) \\
&= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1 \\
&= 120
\end{aligned}
$$

Example 3: Give a recursive definition of a^n, where a is a nonzero real number and n is a nonnegative integer.

Solution:

$a^0 = 1$

$a^n = a \cdot a^{n-1}$ for $n = 1, 2, 3 \ldots$

Example 5: The *Fibonacci numbers*, f_0, f_1, f_2, …, are defined by the equations

$f_0 = 0$

$f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$

Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

Solution:

$f_0 = 0$

$f_1 = 1$

$f_2 = f_1 + f_0 = 1 + 0 = 1$

$f_3 = f_2 + f_1 = 1 + 1 = 2$

$f_4 = f_3 + f_2 = 2 + 1 = 3$

$f_5 = f_4 + f_3 = 3 + 2 = 5$

$f_6 = f_5 + f_4 = 5 + 3 = 8$

Chapter 7
Relations

7.1 Relations and Their properties

Defintion 1: Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

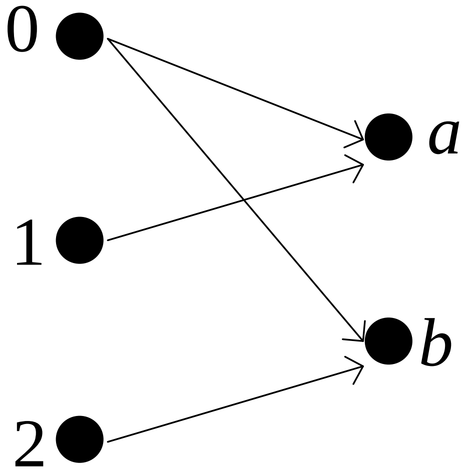
A binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

We use the notation aRb to denote that $(a, b) \in R$.

When (a, b) belongs to R, a is said to be **related** to b by R.

Example 3: Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$.
Then $R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B.
This means that $0Ra$.

Relations can be represented graphically, as shown in Figure 1, using arrows to represent ordered pairs. Another way to represent this relation is to use a table, which is also done in Figure 1.



R	a	b
0	X	X
1	X	
2		X

Figure 1. Displaying the Ordered Pairs in the Relation R from Example 3.

Functions as Relations

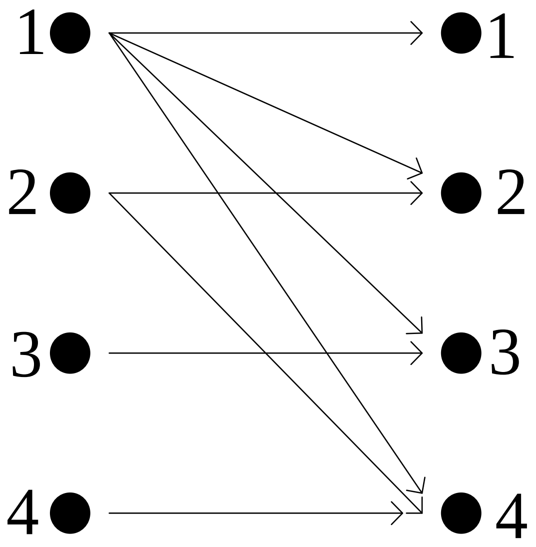
A function f from a set A to a set B assigns exactly one element of B to each element of A. The graph of f is the set of ordered pairs (a, b) such that $b = f(a)$. Since the graph of f is a subset of $A \times B$, it is a relation from A to B. Moreover, the graph of a function has the property that every element of A is the first element of exactly one ordered pair of the graph.

Relations on a Set

Definition 2: A *relation on* the set A is a relation from A to A.

Example 4: Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution:
$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$



	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Example 5: Consider these relations on the set of integers:

$R_1 = \{(a, b) \mid a \leq b\}$
$R_2 = \{(a, b) \mid a > b\}$
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$
$R_4 = \{(a, b) \mid a = b\}$
$R_5 = \{(a, b) \mid a = b + 1\}$
$R_6 = \{(a, b) \mid a + b \leq 3\}$

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, –1), and (2, 2)?

Solution:
(1, 1) is in R_1, R_3, R_4, R_6
(1, 2) is in R_1, R_6
(2, 1) is in R_2, R_5, R_6
(1, –1) is in R_2, R_3, R_6
(2, 2) is in R_1, R_3, R_4

Properties of Relations on sets

Definition 3: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.
The relation R on the set A is reflexive if $\forall a \in A\ ((a, a) \in R)$.
A relation on A is reflexive if every element of A is related to itself.

Example 7: Consider the following relations on {1, 2, 3, 4}:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$
$R_2 = \{(1, 1), (1, 2), (2, 1)\}$
$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$
$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$
$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
$R_6 = \{(3, 4)\}$

Which of these relations are reflexive?

Solution:
The following relations are reflexive:
$R_3 = \{\mathbf{(1, 1)}, (1, 2), (1, 4), (2, 1), \mathbf{(2, 2)}, \mathbf{(3, 3)}, (4, 1), \mathbf{(4, 4)}\}$
$R_5 = \{\mathbf{(1, 1)}, (1, 2), (1, 3), (1, 4), \mathbf{(2, 2)}, (2, 3), (2, 4), \mathbf{(3, 3)}, (3, 4), \mathbf{(4, 4)}\}$

Example 8: Which of the following relations on integers are reflexive?

$R_1 = \{(a, b) \mid a \le b\}$
$R_2 = \{(a, b) \mid a > b\}$
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$
$R_4 = \{(a, b) \mid a = b\}$
$R_5 = \{(a, b) \mid a = b + 1\}$
$R_6 = \{(a, b) \mid a + b \le 3\}$

Solution:
The following relations are reflexive:

$R_1 = \{(a, b) \mid a \le b\}$ (since $a \le a$ for every integer a)
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$ (since $a = a$ for every integer a)
$R_4 = \{(a, b) \mid a = b\}$ (since $a = a$ for every integer a)

Example 9: Is the "divides" relation on the set of positive integers reflexive?

Solution:
Since $a \mid a$ whenever a is a positive integer, the "divides" relation is reflexive.

Definition 4: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.
The relation R on the set A is symmetric if
$\forall a \in A \ \forall b \in A \ ((a, b) \in R \to (b, a) \in R)$.

Example 10: Which of the following relations on {1, 2, 3, 4}are symmetric?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$
$R_2 = \{(1, 1), (1, 2), (2, 1)\}$
$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$
$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$
$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
$R_6 = \{(3, 4)\}$

Solution:
The following relations are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does:

$R_2 = \{(1, 1), \mathbf{(1, 2)}, \mathbf{(2, 1)}\}$
$R_3 = \{(1, 1), \mathbf{(1, 2)}, \mathbf{(1, 4)}, \mathbf{(2, 1)}, (2, 2), (3, 3), \mathbf{(4, 1)}, (4, 4)\}$

Example 11: Which of the following relations on integers are symmetric?

$R_1 = \{(a, b) \mid a \leq b\}$
$R_2 = \{(a, b) \mid a > b\}$
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$
$R_4 = \{(a, b) \mid a = b\}$
$R_5 = \{(a, b) \mid a = b + 1\}$
$R_6 = \{(a, b) \mid a + b \leq 3\}$

Solution:
The following relations are symmetric:

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$ [If $a = b$ then $b = a$. If $a = -b$ then $b = -a$]
$R_4 = \{(a, b) \mid a = b\}$ [If $a = b$ then $b = a$.]
$R_6 = \{(a, b) \mid a + b \leq 3\}$ [If $a + b \leq 3$ then $b + a \leq 3$]

Example 12: Is the “divides” relation on the set of positive integers symmetric?

Solution:
This relation is not symmetric, since $1 \mid 2$, but $2 \nmid 1$.

Definition 5: A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.
The relation R on a set A is transitive if we have
$\forall a \in A\ \forall b \in A\ \forall c \in A\ (((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$

Example 13: Which of the following relations on $\{1, 2, 3, 4\}$ are transitive?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$
$R_2 = \{(1, 1), (1, 2), (2, 1)\}$
$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$
$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$
$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$
$R_6 = \{(3, 4)\}$

Solution:
The following relation is transitive:

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$
[(3, 2) and (2, 1) $\rightarrow$ (3, 1)
(4, 2) and (2, 1) $\rightarrow$ (4, 1)
(4, 3) and (3, 1) $\rightarrow$ (4, 1)
(4, 3) and (3, 2) $\rightarrow$ (4, 2)]

Example 13 (contd.):

The following relation is transitive:

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

[(1, 1) and (1, 2) $\rightarrow$ (1, 2)
(1, 1) and (1, 3) $\rightarrow$ (1, 3)
(1, 1) and (1, 4) $\rightarrow$ (1, 4)
(1, 2) and (2, 2) $\rightarrow$ (1, 2)
(1, 2) and (2, 3) $\rightarrow$ (1, 3)
(1, 2) and (2, 4) $\rightarrow$ (1, 4)
(1, 3) and (3, 3) $\rightarrow$ (1, 3)
(1, 3) and (3, 4) $\rightarrow$ (1, 4)
(1, 4) and (4, 4) $\rightarrow$ (1, 4)
(2, 2) and (2, 3) $\rightarrow$ (2, 3)
(2, 2) and (2, 4) $\rightarrow$(2, 4)
(2, 3) and (3, 3) $\rightarrow$(2, 3)
(2, 3) and (3, 4) $\rightarrow$(2, 4)
(2, 4) and (4, 4) $\rightarrow$(2, 4)
(3, 3) and (3, 4) $\rightarrow$(3, 4)
(3, 4) and (4, 4) $\rightarrow$ (3, 4)]

Example 13 (contd.):

The following relation is transitive:

$R_6 = \{(3, 4)\}$

[There is no pair like (4, *c*) which would imply the existence of the pair (3, *c*)]

The following relations are not transitive:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$

[There are pairs (3, 4) and (4, 1) but not (3, 1)]

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$

[There are pairs (2, 1) and (1, 2) but not (2, 2)]

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$

[There are pairs (4, 1) and (1, 2) but not (4, 2)]

Example 14: Which of the following relations on integers are transitive?

$R_1 = \{(a, b) \mid a \le b\}$
$R_2 = \{(a, b) \mid a > b\}$
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$
$R_4 = \{(a, b) \mid a = b\}$
$R_5 = \{(a, b) \mid a = b + 1\}$
$R_6 = \{(a, b) \mid a + b \le 3\}$

Solution:
The following relations are transitive:

$R_1 = \{(a, b) \mid a \le b\}$ $[(a \le b) \text{ and } (b \le c) \rightarrow (a \le c)]$
$R_2 = \{(a, b) \mid a > b\}$ $[(a > b) \text{ and } (b > c) \rightarrow (a > c)]$
$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$ $[(a = \pm b) \text{ and } (b = \pm c) \rightarrow (a = \pm c)]$
$R_4 = \{(a, b) \mid a = b\}$ $[(a = b) \text{ and } (b = c) \rightarrow (a = c)]$

The following relations are not transitive:

$R_5 = \{(a, b) \mid a = b + 1\}$ $[(2, 1), (1, 0) \in R_5 \text{ but } (2, 0) \notin R_5)]$
$R_6 = \{(a, b) \mid a + b \le 3\}$ $[(2, 1), (1, 2) \in R_6 \text{ but } (2, 2) \notin R_6)]$

Example 15: Is the “divides” relation on the set of positive integers transitive?

Solution:
If a divides b, then $b = ak$, where k is a positive integer
If b divides c, then $c = bl$, where l is a positive integer
Then $c = bl = akl = a(kl)$, where kl is a positive integer. Therefore, a divides c
Hence, the “divides” relation is transitive.

Chapter 5

Counting

5.1 The Basics of Counting

<u>Basic Counting principles</u>

THE PRODUCT RULE
Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are n_1n_2 ways to do the procedure.

Example 1: A new company with just two employees, Sahin and Parvin, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution:
An office can be assigned to Sahin in 12 ways.
After assigning an office to Sahin, an office can be assigned to Parvin in 11 different ways.
Therefore, by the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

Example 2: The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution:
A letter can be assigned in 26 different ways
An integer can be assigned in 100 different ways
The different ways that a chair can be labeled is $26 \cdot 100 = 2600$
Therefore, the largest number of chairs that can be labeled differently is 2600.

Example 4: How many different bit strings are there of length seven?

Solution:
Each of the seven bits can be chosen in two ways
Therefore, there are a total of $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$ different bit strings of length seven.

Example 9: What is the value of k after the following code has been executed?

$k := 0$
for $i_1 := 1$ **to** n_1
  for $i_2 := 1$ **to** n_2
    .
    .
    .
    for $i_m := 1$ **to** n_m
      $k := k + 1$

Solution:
The initial value of k is zero
Each time the nested loop is traversed, 1 is added to k
The nested loop is traversed $n_1 n_2 \ldots n_m$ times
Therefore, the final value of k is $n_1 n_2 \ldots n_m$

THE SUM RULE

If a first task can be done in n_1 ways and second task in n_2 ways, and if these tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do one of these tasks.

Example 11: Suppose that either a member of the mathematics faculty or a student who is mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors?

Solution:
Choosing a member of the mathematics faculty can be done in 37 ways.
Choosing mathematics major can be done in 83 ways.
There are 37 + 83 = 120 possible ways to pick this representative.

Example 12: A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. How many possible projects are there to choose from?

Solution:
The student can choose a project from the first list in 23 ways, from the second list in 15 ways, and from the third list in 19 ways.
Hence, there are 23 + 15 + 19 = 57 projects to choose from.

Example 13: What is the value of k after the following code has been executed?

$k := 0$
for $i_1 := 1$ **to** n_1
    $k := k + 1$
for $i_2 := 1$ **to** n_2
    $k := k + 1$
    .
    .
    .
for $i_m := 1$ **to** n_m
    $k := k + 1$

Solution:
The initial value of k is zero.
This block of code is made up of m different loops. Each time a loop is traversed, 1 is added to k.
The final value of k is $n_1 + n_2 + \ldots + n_m$.

More Complex Counting problems

Example 14: In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variables names are there in this version of BASIC?

Solution:
Let V be the number of different variable names in this version of BASIC.

Let V_1 be the number of these that are one character long.
$V_1 = 26$

Let V_2 be the number of these that are two characters long
$V_2 = 26 \cdot 36 - 5 = 931$

$V = V_1 + V_2 = 26 + 931 = 957$

Example 15: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution:
Let P be the total number of possible passwords, and P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively.

The number of strings of six characters is 36^6
The number of strings with no digits is 26^6
$\therefore\ P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560$

Similarly,
$P_7 = 36^7 - 26^7 = 78{,}367{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920$

$P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880$

$P = P_6 + P_7 + P_8 = 1{,}867{,}866{,}560 + 70{,}332{,}353{,}920 + 2{,}612{,}282{,}842{,}880 = 2{,}684{,}483{,}063{,}360$

The Inclusion-Exclusion Principle

When two tasks can be done at the same time, we cannot use the sum rule to count the number of ways to do one of the two tasks. Adding the number of ways to do each task leads to an overcount, since the ways to do both tasks are counted twice. To correctly count the number of ways to do one of the two tasks, we add the number of ways to do each of the two tasks and then subtract the number of ways to do both tasks. This technique is called the **principle of inclusion-exclusion**.

Example 17: How many bit strings of length eight either starts with a 1 bit or end with the two bits 00?

Solution:
Constructing a bit string of length eight beginning with a 1 bit can be done in $2^7 = 128$ ways.

Constructing a bit string of length eight ending with the two bits 00 can be done in $2^6 = 64$ ways.

Constructing a bit string of length eight that begins with a 1 and ends with 00 can be done in $2^5 = 32$ ways.

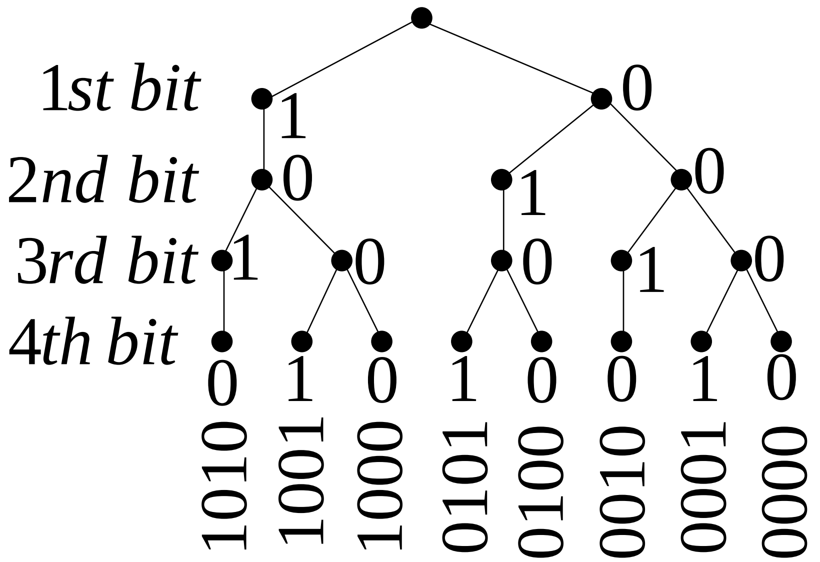
Consequently, the number of bit strings of length eight that begin with a 1 or end with a 00 equals
$128 + 64 – 32 = 160$.

Tree Diagrams

Counting problems can be solved using **tree diagrams**. To use trees in counting, we use a branch to represent each possible choice. We represent the possible outcomes by the leaves.

Example 19: How many bit string of length four do not have two consecutive 1s?

Solution:



The tree diagram displays all bit strings of length four without two consecutive 1s. We see that there are eight bit strings of length four without two consecutive 1s.

5.2 The Pigeonhole Principle

Theorem 1: THE PIGEONHOLE PRINCIPLE
If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is **at least one box** containing two or more (**at least two**) of the objects.

Proof.
Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, since there are at least $k + 1$ objects.

Example 1: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example 2: In any group of 27 English words, there must be at least two that begin with the same letter, since there are 26 letters in the English alphabet.

Example 3: How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100?

Solution:
There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

The Generalized Pigeonhole principle

Theorem 2: THE GENERALIZED PIGEONHOLE PRINCIPLE
If N objects are placed into k boxes, then there is at least one box containing at least $r = \lceil N/k \rceil$ objects.

Proof:
Suppose that none of the boxes contain more than $\lceil N/k \rceil - 1$ objects. Then, the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N$$

This is a contradiction, since there are a total of N objects.

A common type of problem asks for the minimum number of objects N so that at least r of these objects must be in one of k boxes when these objects are distributed among the boxes.

When we have N objects, the generalized pigeonhole principle tells us that there must be at least r objects in one of the boxes as long as

$\lceil N/k \rceil \geq r$
$N/k + 1 > r$
$N/k > (r - 1)$
$N > k(r - 1)$
$N = k(r - 1) + 1$ is the smallest integer satisfying the condition

Example 5: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example 6: What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Solution:
Let
N = minimum number of students to satisfy the condition
k = 5 = total number of grades
r = 6 = minimum number of students receiving the same grade

$N = k(r - 1) + 1 = 5\,(6 - 1) + 1 = 26$