Laplace Inversion

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Contents

1 2 point, same monomer [Redoing what Andy did]

$$I_{ii} = \int_0^N ds_2 \int_0^N ds_1 G\left(|s_2 - s_1|\right)$$

$$I_{ii} = 2\int_0^N ds_2 \int_0^{s_2} ds_1 G\left(|s_2 - s_1|\right)$$

$$\mathcal{L}\left(I_{ii}\left(N\right)\right)\left(p\right) = 2\int_0^\infty dN e^{-pN} \int_0^N ds_2 \underbrace{\int_0^{s_2} ds_1 G\left(|s_2 - s_1|\right)}_{f(s_2)}$$

$$\begin{aligned}\mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\} =& \mathcal{L}\left(f * g\right)\\ =& \mathcal{L}\left(\int_0^t f\left(\tau\right) f\left(t-\tau\right)\right)\end{aligned}$$

$$\underbrace{\int_{N=0}^\infty e^{-pN} \int_{\tau=0}^N f\left(\tau\right) g\left(N-\tau\right) d\tau dN}_{\mathcal{L}(f*g)} = \underbrace{\int_{x_1=0}^\infty e^{-px_1} f\left(x_1\right) dx_1}_{F(p)} \underbrace{\int_{x_2=0}^\infty e^{-px_2} g\left(x_2\right) dx_2}_{G(p)} \quad (1)$$

See Arfken for proof.
Identify $g\left(p\right) = 1$, $s_2 \equiv \tau$, $x_1 \rightarrow N$ and put a 2 out front.

$$2\int_{N=0}^{\infty} e^{-pN}\int_{s_2=0}^{N} f(s_2)\,ds_2 dN = 2\int_{N=0}^{\infty} e^{-pN} f(N)\,dN \int_{x_2=0}^{\infty} e^{-px_2}dx_2$$

$$\mathcal{L}(I_{ii}(N))(p) = 2\int_{N=0}^{\infty} e^{-pN} f(N)\,dN\frac{1}{p}$$

Note [1].

where $f(N) = \underbrace{\int_0^N ds_1 G(|N-s_1|)}_{f(N)}$

$$\mathcal{L}(I_{ii}(N))(p) = 2\int_{N=0}^{\infty} e^{-pN}\int_0^N ds_1 1\cdot G(|N-s_1|)\,dN\frac{1}{p}$$

Once again this is a convolution with $f(\tau)=1$, $\tau\to s_1$, $g\to G$:

$$\int_{N=0}^{\infty} e^{-pN}\int_{\tau=0}^{N} G(N-\tau)\,d\tau dN = \int_{x_1=0}^{\infty} e^{-px_1}dx_1\int_{x_2=0}^{\infty} e^{-px_2}G(x_2)\,dx_2$$

$$\mathcal{L}(I_{ii}(N))(p) = 2\int_{x_2=0}^{\infty} e^{-px_2}G(x_2)\,dx_2\frac{1}{p^2}$$

Identifying the Laplace transform and changing $x_2\to N$

$$\mathcal{L}\{I_{ii}(N)\}(p) = 2\frac{\mathcal{L}\{G(N)\}(p)}{p^2}$$

Now we want to do an inverse Laplace transform by the residue theorem[2]

In our case the residues are the residues described in the paper which each diverge signally [n=1] and the residue at zero for which n=2.

$$I_{ii}(N) = 2\sum_{l=0}^{\infty}\lim_{p\to\epsilon_l}(p-\epsilon_l)\,e^{pN}\frac{G(p)}{p^2} + \lim_{p\to 0}\frac{\partial}{\partial p}\left(2e^{pN}G(p)\right)$$

Where we defined $G(p)\equiv\mathcal{L}\{G(N)\}$. Using L'Hopital's rule on the sum and product rule on the second term:

$$\boxed{I_{ii}(N) = 2\sum_{l=0}^{\infty}\frac{e^{\epsilon_l N}}{\epsilon_l^2}\frac{1}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)\Big|_{p=\epsilon_l}} + 2\left.\frac{\partial}{\partial p}G(p)\right|_{p=0} + 2NG(0)}$$

1.1 How does the convergence compare to that of the original pole?

$$I_{ii} = 2\int_0^N ds_2\int_0^{s_2} ds_1 G(|s_2-s_1|)$$

$$I_{ii} = 2\int_0^N ds_2\int_0^{s_2} ds_1\sum_{l=0}^{\infty}\frac{e^{p(s_2-s_1)}}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)}$$

[1] $\int_{N=0}^{\infty} e^{-pN} f(N)\,dN$ is just the laplace transform of $f(N)$

$$\mathcal{L}(I_{ii}(N))(p) = 2\frac{-\mathcal{L}(f(N))(p)}{p}$$

[2] Cauchy's integral formula

$$\oint_\gamma \frac{u(z)}{(z-\epsilon)^n}dz = \frac{2\pi i}{(n-1)!}u^{(n-1)}(\epsilon)$$

where u is analytic. And the residue method for inverse Laplace transform:

$$f(x) = \frac{1}{2\pi i}\int_\gamma e^{sx}F(s)\,ds$$

Putting these together and renaming $z=s\to p$

$$f(x) = \sum_{\epsilon}\frac{1}{(n-1)!}\lim_{p\to\epsilon}\frac{\partial^{n-1}}{\partial p^{n-1}}\left((p-\epsilon)^n e^{px}F(p)\right)$$

Defining $Res_l = \left(\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)\Big|_{p=\epsilon_l}\right)^{-1}$ we have[3]

$$
\begin{aligned}
I_{ii} &= 2\int_0^N ds_2 \int_0^{s_2} ds_1 \sum_{l=0}^{\infty} Res_l e^{\epsilon_l (s_2 - s_1)} \\
&= 2\sum_{l=0}^{\infty} Res_l \int_0^N ds_2 e^{\epsilon_l s_2} \int_0^{s_2} ds_1 e^{-\epsilon_l s_1} \\
&= 2\sum_{l=0}^{\infty} Res_l \int_0^N ds_2 e^{\epsilon_l s_2} \left(\frac{e^{-\epsilon_l s_2}}{-\epsilon_l} - \frac{1}{-\epsilon_l}\right) \\
&= 2\frac{1}{\epsilon_l}\sum_{l=0}^{\infty} Res_l \int_0^N ds_2 \left(e^{\epsilon_l s_2} - 1\right) \\
&= 2\frac{1}{\epsilon_l}\sum_{l=0}^{\infty} Res_l \left(\frac{e^{\epsilon_l N}}{\epsilon_l} - N - \frac{1}{\epsilon_l}\right) \\
&= 2\sum_{l=0}^{\infty} Res_l \left(\frac{e^{\epsilon_l N}}{\epsilon_l^2} - \frac{N}{\epsilon_l} - \frac{1}{\epsilon_l^2}\right)
\end{aligned}
$$

comparing this with above we have the two identities

$$\sum_{l=0}^{\infty} Res_l \frac{-1}{\epsilon_l} = G(0) \tag{2}$$

and

$$\sum_{l=0}^{\infty} Res_l \frac{-1}{\epsilon_l^2} = \frac{\partial}{\partial p} G(p)\Big|_{p=0} \tag{3}$$

2 Alturnitive methode finding identities

In general suppose we have a function and it's laplace transform

$$\mathcal{G}(p) = \int_0^{\infty} G(N)\, e^{-pN} dN$$

with the inverse transform

$$G(N) = \sum_l Res_l e^{\epsilon_l N}$$

where

$$Res = \lim_{p\to\epsilon_l} (p - \epsilon_l)\, G(p)$$

where I assume only simple poles.
By substitution

[3] We define the quntity

$$
\begin{aligned}
Res_l &\equiv \lim_{p\to\epsilon} (\epsilon - p)\, G(p) \\
&= \left(\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)\Big|_{p=\epsilon_l}\right)^{-1}
\end{aligned}
$$

even though for the complex path intigral

$$\int dN e^{pN} \frac{G(p)}{p^2}$$

you would usually refer to the "residue" as

$$\lim_{p\to pole} \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial p^{n-1}} (p - pole)^n\, e^{pN} \frac{G(p)}{p^2}$$

$$\mathcal{G}(p) = \int_0^\infty \sum_l Res_l e^{(\epsilon_l - p)N} dN$$

We can do the intigral if $p > \epsilon$

$$\begin{aligned}\mathcal{G}(p) &= \sum_l Res_l \left.\frac{e^{(\epsilon_l - p)N}}{(\epsilon_l - p)}\right|_{N=0}^{N=\infty} \\ &= \sum_l Res_l \frac{-1}{(\epsilon_l - p)} \quad \text{if} \quad p > \{\epsilon_l\}\end{aligned}$$

I belive you can analytically continue $\mathcal{G}$ so that

$$\mathcal{G}(p) = \sum_l Res_l \frac{-1}{(\epsilon_l - p)} \quad \text{if} \quad p \neq \{\epsilon_l\}$$

Which gives us 2 and 8. Taking the derivative with respect to p we have

$$\mathcal{G}'(p) = \sum_l Res_l \frac{-1}{(\epsilon_l - p)^2} \quad \text{if} \quad p \neq \{\epsilon_l\}$$

which includes the identity 2 when you let $p = 0$. While we are at it, we may as well also state that

$$\frac{\partial^n}{\partial p^n} \mathcal{G}(p) = \sum_l Res_l \frac{-(n-1)!}{(\epsilon_l - p)^n} \quad \text{if} \quad p \neq \{\epsilon_l\}$$

3 Three Point Same Monomer

3.1 Notation

We have the propagator

$$\mathcal{G}(p) = \mathcal{L}_{s \to p} \{G(\Delta s)\}$$

The both G and $\mathcal{G}$ also depends on number of arguments $\{K, \lambda_0, \lambda, \mu\}$ which don't matter for the present discussion. I will refer to them with a single indicy.

$$\mathcal{G}_1(p) \equiv \mathcal{G}_{\lambda_1}^{\lambda_{0_1}, \mu_1}(K_1, p)$$

We have an expression for $\mathcal{G}$ in Shafig's paper. Because G has only first order poles so it is determined by the inverse Laplace transform

$$G_1(\Delta s) \equiv \sum_{l=0}^{\infty} Res_1(\epsilon_{1,l}) e^{\epsilon_{1,l} \Delta s} \tag{4}$$

$$Res_1(\epsilon_l) = \frac{1}{\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{G}_1(p)}\right)\Big|_{p=\epsilon_l}}$$

The function $\mathcal{G}_1(p)$ can be evaluated at it's own l^{th} pole $\mathcal{G}_1(\epsilon_l(1))$ or at the pole of some other function $\mathcal{G}_1(\epsilon_l(2))$ which may [or may not coincide with the pole of interest. I will sometimes abbreviate

$$\epsilon_1 \equiv \epsilon_{1,l_1}$$

Using this notation the two identities 2 and 3 become

$$\sum_{l=0}^{\infty} Res_l \frac{-1}{\epsilon_l^2} = \left.\frac{\partial}{\partial p} \mathcal{G}(p)\right|_{p=0}$$

and

$$\sum_{l=0}^{\infty} Res_l \frac{-1}{\epsilon_l} = \mathcal{G}(0)$$

3.2 Laplace Convolution Trick

We have shown that

$$I_{1,2}^{(3)}(N)=\int_{0}^{N}ds_2\int_{0}^{s_2}ds_1\int_{0}^{s_1}ds_0 G_1(s_1-s_0)\,G_2(s_2-s_1) \tag{5}$$

which, after plugging in equation 4 is

$$I_{1,2}^{(3)}(N)=\sum_{l_1,l_2=0}^{\infty}Res_1(\epsilon_{1,l_2})\,Res_2(\epsilon_{2,l_2})\int_{0}^{N}ds_2\int_{0}^{s_2}ds_1\int_{0}^{s_1}ds_0 e^{\epsilon_{1,l_1}(s_1-s_0)}e^{\epsilon_{2,l_2}(s_2-s_1)}$$

is equivalent to

$$I_{1,2}^{(3)}(N)=\sum_{\epsilon=\epsilon_{1,l},\epsilon_{2,l},0}\frac{1}{(n-1)!}\lim_{p\to\epsilon}\frac{\partial^{n-1}}{\partial p^{n-1}}\left((p-\epsilon)^n\frac{e^{pN}}{p^2}\mathcal{G}_2(p)\,\mathcal{G}_1(p)\right) \tag{6}$$

Proof:

$$\mathcal{L}\left\{I_{1,2}^{(3)}(N)\right\}(p)=\int_{N=0}^{\infty}dNe^{-pN}\int_{0}^{N}ds_2\underbrace{\int_{0}^{s_2}ds_1\int_{0}^{s_1}ds_0 G_1(s_1-s_0)\,G_2(s_2-s_1)}_{f(s_2)}\cdot 1$$

the use the convolution theorem

$$\underbrace{\int_{N=0}^{\infty}e^{-pN}\int_{\tau=0}^{N}f(\tau)\,g(N-\tau)\,d\tau dN}_{\mathcal{L}(f*g)}=\underbrace{\int_{x_1=0}^{\infty}e^{-px_1}f(x_1)\,dx_1}_{F(p)}\underbrace{\int_{x_2=0}^{\infty}e^{-px_2}g(x_2)\,dx_2}_{G(p)}$$

$$\tau\to s_2;\quad g(N-\tau)\to 1$$

$$\mathcal{L}\left\{I_{1,2}^{(3)}(N)\right\}(p)=\int_{0}^{\infty}dx_1e^{-px_1}\overbrace{\int_{0}^{x_1}ds_1\int_{0}^{s_1}ds_0 G_1(s_1-s_0)\,G_2(x_1-s_1)}^{f(x_1)}\frac{1}{p}$$

make the identification $x_1\to N$ and also move u integrals to outside so they don't get in the way

$$\mathcal{L}\left\{I_{1,2}^{(3)}(N)\right\}(p)=\int_{0}^{\infty}dNe^{-pN}\int_{0}^{N}ds_1\underbrace{\int_{0}^{s_1}ds_0 G_1(s_1-s_0)}_{f(s_1)}\underbrace{G_2(N-s_1)}_{g(N-s_1)}\frac{1}{p}$$

use convolution again $\tau\to s_1$

$$\mathcal{L}\left\{I_{1,2}^{(3)}(N)\right\}(p)=\left(\int_{x_1=0}^{\infty}e^{-px_1}\int_{0}^{x_1}ds_0 G_1(x_1-s_0)\,dx_1\right)\left(\int_{x_2=0}^{\infty}e^{-px_2}G_2(x_2)\,dx_2\right)\frac{1}{p}$$

rename $x_1\to N$ for the first part and recognize the Laplace transform for the latter

$$\mathcal{L}\left\{I_{1,2}^{(3)}(N)\right\}(p)=\left(\int_{0}^{\infty}e^{-pN}\int_{0}^{N}ds_0 1\cdot G_1(N-s_0)\,dN\right)\mathcal{G}_2(p)$$

using convolution again

$$\mathcal{L}\left\{I_{1,2}^{(3)}(N)\right\}(p)=\mathcal{G}_1(p)\,\mathcal{G}_2(p)\,\frac{1}{p^2}$$

inverting the Laplace transform completes the proof.

3.3 By integration

Inserting the inverse Laplace transform 4 into equation 5 we have

$$\begin{aligned}
I_{1,2}^{(3)}(N) &= \sum_{l_1,l_2=0}^{\infty} Res_1(\epsilon_1)\, Res_2(\epsilon_2) \int_0^N ds_2 \int_0^{s_2} ds_1 \int_0^{s_1} ds_0 e^{\epsilon_1(s_1-s_0)} e^{\epsilon_2(s_2-s_1)} \\
&= \ldots \int_0^N ds_2 \int_0^{s_2} ds_1 \int_0^{s_1} ds_0 e^{-\epsilon_1 s_0} e^{(\epsilon_1-\epsilon_2)s_1} e^{\epsilon_2 s_2} \\
&= \ldots \frac{1}{\epsilon_1} \int_0^N ds_2 \int_0^{s_2} ds_1 \left(e^{\epsilon_1 s_1} - 1\right) e^{(\epsilon_1-\epsilon_2)s_1} e^{\epsilon_2 s_2} \\
&= \ldots \\
&= \sum_{l_1,l_2=0}^{\infty} Res_1(\epsilon_1)\, Res_2(\epsilon_2) \frac{\epsilon_1^2 \mathrm{expl}(2, N\epsilon_2) - \epsilon_2^2 \mathrm{expl}(2, N\epsilon_1)}{-\epsilon_1^2\epsilon_2^2(\epsilon_1-\epsilon_2)}
\end{aligned}$$

The integrals in this step are straight forward. I did them with the computer and verified they numerically. Here I have defined the function

$$\begin{aligned}
\mathrm{expl}(j, x) &\equiv \sum_{n=j}^{\infty} \frac{x^n}{n!} \qquad (7) \\
&= e^x - 1 - x - \ldots \frac{x^{n-1}}{(n-1)!}
\end{aligned}$$

Using the expl function we are able to evaluate the fraction accurately but the fraction does not go to zero as the $\epsilon \to -\infty$ so all orders of l are needed. This is bad. For the purpose of comparing this to the single Laplace version I will expand ...

$$\begin{aligned}
I_{1,2}^{(3)}(N) &= \sum_{l_1,l_2=0}^{\infty} Res_1(\epsilon_1)\, Res_2(\epsilon_2) \frac{\epsilon_1^2\left(e^{N\epsilon_2} - 1 - N\epsilon_2\right) - \epsilon_2^2\left(e^{N\epsilon_1} - 1 - N\epsilon_1\right)}{-\epsilon_1^2\epsilon_2^2(\epsilon_1-\epsilon_2)} \\
&= \sum_{l_1,l_2=0}^{\infty} Res_1(\epsilon_1)\, Res_2(\epsilon_2) \left[\frac{e^{N\epsilon_1}\epsilon_1^{-2} - e^{N\epsilon_2}\epsilon_2^{-2}}{\epsilon_1-\epsilon_2} + \frac{1}{\epsilon_2\epsilon_1^2} + \frac{1}{\epsilon_1\epsilon_2^2} + N\frac{1}{\epsilon_1\epsilon_2}\right]
\end{aligned}$$

Using the identities from the two point correlations [equations 2 and 3] $\sum_{l=0}^{\infty} Res_l \frac{-1}{\epsilon_l^2} = \frac{\partial}{\partial p}\mathcal{G}(p)\Big|_{p=0}$ and $\sum_{l=0}^{\infty} Res_l \frac{-1}{\epsilon_l} = \mathcal{G}(0)$ we have ...

$$I_{1,2}^{(3)}(N) = \left(\sum_{l_1,l_2=0}^{\infty} Res_1(\epsilon_1)\, Res_2(\epsilon_2) \frac{e^{N\epsilon_1}\epsilon_1^{-2} - e^{N\epsilon_2}\epsilon_2^{-2}}{\epsilon_1-\epsilon_2}\right) + N\mathcal{G}_2(0)\,\mathcal{G}_1(0) + \mathcal{G}_2'(0)\,\mathcal{G}_1(0) + \mathcal{G}_2(0)\,\mathcal{G}_1'(0)$$

3.4 Inverting the single Laplace

From equation 6

$$I_{1,2}^{(3)}(N) = \sum_{residues} \frac{1}{(n-1)!} \lim_{p\to\epsilon} \frac{\partial^{n-1}}{\partial p^{n-1}} \left((p-\epsilon)^n \frac{e^{pN}}{p^2}\mathcal{G}_2(p)\,\mathcal{G}_1(p)\right)$$

The only second order pole is the one at zero. The others are single. The pole at zero gives us

$$\lim_{p\to 0}\frac{\partial}{\partial p}\left(e^{pN}\mathcal{G}_2(p)\,\mathcal{G}_1(p)\right) = N\mathcal{G}_2(0)\,\mathcal{G}_1(0) + \mathcal{G}_2'(0)\,\mathcal{G}_1(0) + \mathcal{G}_2(0)\,\mathcal{G}_1'(0)$$

where $\mathcal{G}_2'(p) = \frac{\partial}{\partial p}\mathcal{G}_2(p)$.
Assuming the poles do not coincide, the terms with the poles in $\mathcal{G}$ are

$$I_{1,2}^{(3)}(N) = \sum_{l} \lim_{p\to\epsilon_{1,l}} \left(\frac{e^{pN}}{p^2}\mathcal{G}_2(p)\,(p-\epsilon_{1,l})\,\mathcal{G}_1(p)\right) + \ldots_{1\leftrightarrow 2} + \text{zero term}$$

because there is a pole in $\mathcal{G}_1$ at $\epsilon_{1,l}$ by assumption we use L'Hospital's rule

$$I_{1,2}^{(3)}(N)=\sum_l\left(\frac{e^{\epsilon_{1,l}N}}{\epsilon_{1,l}^2}\mathcal{G}_2(\epsilon_{1,l})\frac{1}{\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{G}_1(p)}\right)\Big|_{p=\epsilon_l}}\right)+\ldots_{1\leftrightarrow 2}+\text{zero term}$$

$$I_{1,2}^{(3)}(N)=\sum_l\left(\frac{e^{\epsilon_{1,l}N}}{\epsilon_{1,l}^2}\mathcal{G}_2(\epsilon_{1,l})\,Res_1(\epsilon_{1,l})\right)+\ldots_{1\leftrightarrow 2}+N\mathcal{G}_2(0)\,\mathcal{G}_1(0)+\mathcal{G}_2'(0)\,\mathcal{G}_1(0)+\mathcal{G}_2(0)\,\mathcal{G}_1'(0)$$

3.5 Three point identity

Comparing the to results for $I_{1,2}^{(3)}(N)$ via the two methods we have

$$\sum_{l_1,l_2=0}^{\infty}Res_1(\epsilon_1)\,Res_2(\epsilon_2)\,\frac{e^{N\epsilon_1}\epsilon_1^{-2}-e^{N\epsilon_2}\epsilon_2^{-2}}{\epsilon_1-\epsilon_2}=\sum_{l_1}\left(\frac{e^{\epsilon_1 N}}{\epsilon_1^2}Res_1(\epsilon_1)\,\mathcal{G}_2(\epsilon_1)\right)+\sum_{l_2}\left(\frac{e^{\epsilon_2 N}}{\epsilon_2^2}Res_2(\epsilon_2)\,\mathcal{G}_1(\epsilon_2)\right)$$

$$\sum_{l_1}\left(\frac{e^{\epsilon_1 N}}{\epsilon_1^2}Res_1(\epsilon_1)\sum_{l_2}\frac{Res_2(\epsilon_2)}{\epsilon_1-\epsilon_2}\right)+\sum_{l_2=0}\left(\frac{e^{\epsilon_2 N}}{\epsilon_2^2}Res_2(\epsilon_2)\sum_{l_1}\frac{Res_1(\epsilon_1)}{\epsilon_2-\epsilon_1}\right)=\sum_{l_1}\left(\frac{e^{\epsilon_1 N}}{\epsilon_1^2}Res_1(\epsilon_1)\,\mathcal{G}_2(\epsilon_1)\right)+\sum_{l_2}\left(\frac{e^{\epsilon_2 N}}{\epsilon_2^2}Res_2(\epsilon_2)\,\mathcal{G}_1(\epsilon_2)\right)$$

it would appear that[4]

$$\sum_{l_2}\frac{Res_2(\epsilon_{2,l_2})}{\epsilon_1-\epsilon_{2,l_2}}=\mathcal{G}_2(\epsilon_1)\tag{8}$$

Note: There may be some requirement like $\epsilon_1>\{\epsilon_2\}$ for this to be true.

3.6 If the poles coincide

If $\epsilon_{1,l_1}=\epsilon_{2,l_2}\equiv\epsilon$ for some l_1 and l_2 we have that the particular term involving the second order pole would be

$$term_{\epsilon_{1,l_1}=\epsilon_{2,l_2}}=\lim_{p\to\epsilon}\frac{\partial}{\partial p}\left((p-\epsilon)^2\,\frac{e^{pN}}{p^2}\mathcal{G}_2(p)\,\mathcal{G}_1(p)\right)$$

$$term_{\epsilon_1=\epsilon_2}=\lim_{p\to\epsilon}\frac{\partial}{\partial p}\left(\frac{e^{pN}}{p^2}(p-\epsilon)\,\mathcal{G}_2(p)\,(p-\epsilon)\,\mathcal{G}_1(p)\right)$$

Doing the derivative term by term

$$term_{\epsilon_1=\epsilon_2}=\left(N\frac{e^{\epsilon N}}{\epsilon^2}-2\frac{e^{\epsilon N}}{\epsilon^3}\right)Res_2(\epsilon)\,Res_1(\epsilon)+\frac{e^{\epsilon N}}{\epsilon^2}\left(\frac{-\frac{\partial^2}{\partial p^2}\left(\frac{1}{\mathcal{G}_2(p)}\right)_{p=\epsilon}}{2\left(\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{G}_2(p)}\right)\right)^2_{p=\epsilon}}\right)Res_1(\epsilon)+\frac{e^{\epsilon N}}{\epsilon^2}Res_2(\epsilon)\left(\frac{-\frac{\partial^2}{\partial p^2}\left(\frac{1}{\mathcal{G}_1(p)}\right)_{p=\epsilon}}{2\left(\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{G}_1(p)}\right)\right)^2_{p=\epsilon}}\right)$$

where I used the result of a limit trick, see equation 9.

$$term_{\epsilon_1=\epsilon_2}=(N\epsilon-2)\,\frac{e^{\epsilon N}}{\epsilon^3}Res_2(\epsilon)\,Res_1(\epsilon)+\frac{e^{\epsilon N}}{\epsilon^2}\left(\frac{-\frac{\partial^2}{\partial p^2}\left(\frac{1}{\mathcal{G}_2(p)}\right)_{p=\epsilon}}{2Res_2(\epsilon)^2}\right)Res_1(\epsilon)+\frac{e^{\epsilon N}}{\epsilon^2}Res_2(\epsilon)\left(\frac{-\frac{\partial^2}{\partial p^2}\left(\frac{1}{\mathcal{G}_1(p)}\right)_{p=\epsilon}}{2Res_1(\epsilon)^2}\right)$$

If one does the integrals directly

$$\begin{aligned}term_{\epsilon_1=\epsilon_2}&=Res_2(\epsilon)\,Res_1(\epsilon)\int_0^N ds_3\int_0^{s_3}ds_2\int_0^{s_2}ds_1 e^{\epsilon(s_3-s_1)}\\&=Res_2(\epsilon)\,Res_1(\epsilon)\,\frac{-2\mathrm{expl}(3,N\epsilon)+N\epsilon\mathrm{expl}(2,N\epsilon)}{\epsilon^3}\end{aligned}$$

this is basically straight forward integration, and has been verified numerically. Expanding using the definition of expl [see 7] we have

$$term_{\epsilon_1=\epsilon_2}=Res_2(\epsilon)\,Res_1(\epsilon)\left((N\epsilon-2)\,\frac{e^{\epsilon N}}{\epsilon^3}+N\frac{1}{\epsilon^2}+\frac{2}{\epsilon^3}\right)$$

[4] We can make and argument based on the above being true for all N. Laplace transforms are unique so there no way to make an exponential out of a bunch of other exponentials with different fall offs.

4 Three point on different segments

Next we consider the case where the first two points are on the same block and the other one is on the other block. In the diblock case this is in S_{AAB}. Once again we will make the assumption that none of the poles coincide. We would like to evaluate the intigral

$$I^{(3)}_{\underset{AAB}{1,2}}(N)=\sum_{l_1,l_2=0}^{\infty}Res_1(\epsilon_1)\,Res_2(\epsilon_2)\int_{Nf_A}^{N}ds_2\int_0^{Nf_A}ds_1\int_0^{s_1}ds_0e^{\epsilon_1(s_1-s_0)}e^{\epsilon_2(s_2-s_1)}$$

Each of these intigrals is easily doable and, after some rearangement, we have

$$I^{(3)}_{\underset{AAB}{1,2}}(N)=\sum_{l_1,l_2=0}^{\infty}Res_1(\epsilon_1)\,Res_2(\epsilon_2)\,[int]$$

$$[int]=\frac{e^{Nf_A\epsilon_1+Nf_B\epsilon_2}}{\epsilon_1\epsilon_2(\epsilon_1-\epsilon_2)}+\frac{1}{\epsilon_1}\frac{e^{Nf_B\epsilon_2}}{\epsilon_2^2}-\frac{e^{Nf_A\epsilon_1}}{\epsilon_1^2\epsilon_2}-\frac{1}{\epsilon_1\epsilon_2^2}+\frac{1}{\epsilon_1-\epsilon_2}\frac{e^{Nf_A\epsilon_2}-e^{N\epsilon_2}}{\epsilon_2^2}+\frac{e^{Nf_A\epsilon_1}}{(\epsilon_2-\epsilon_1)\,\epsilon_1^2}$$

Now we can use the rules:

$$\mathcal{G}(p)=\sum_l Res_l\frac{-1}{(\epsilon_l-p)}\quad\text{if}\quad p\neq\{\epsilon_l\}$$

$$\mathcal{G}'(p)=\sum_l Res_l\frac{-1}{(\epsilon_l-p)^2}\quad\text{if}\quad p\neq\{\epsilon_l\}$$

we have

$$\begin{aligned}I^{(3)}_{\underset{AAB}{1,2}}(N)=&\sum_{l_1,l_2=0}^{\infty}Res_1(\epsilon_1)\,Res_2(\epsilon_2)\frac{e^{Nf_A\epsilon_1+Nf_B\epsilon_2}}{\epsilon_1\epsilon_2(\epsilon_1-\epsilon_2)}\\&+(-\mathcal{G}_1(0))\sum_{l_2=0}^{\infty}Res_2(\epsilon_2)\frac{e^{Nf_B\epsilon_2}}{\epsilon_2^2}\\&-(-\mathcal{G}'_2(0))\sum_{l_1=0}^{\infty}Res_1(\epsilon_1)\frac{e^{Nf_A\epsilon_1}}{\epsilon_1^2}\\&-(-\mathcal{G}_1(0))(-\mathcal{G}'_2(0))\\&+\sum_{l_2=0}^{\infty}(-\mathcal{G}_1(\epsilon_2))\,Res_2(\epsilon_2)\frac{e^{Nf_A\epsilon_2}-e^{N\epsilon_2}}{\epsilon_2^2}\\&+\sum_{l_1=0}^{\infty}(-\mathcal{G}_2(\epsilon_1))\,Res_1(\epsilon_1)\frac{e^{Nf_A\epsilon_1}}{\epsilon_1^2}\end{aligned}$$

In case you were wondering the case of S_{ABB} is quite similar with

$$\int_{Nf_A}^{N}\int_{Nf_A}^{s_3}\int_0^{Nf_A}e^{\epsilon_1(s_1-s_0)}e^{\epsilon_2(s_2-s_1)}=$$

$$\frac{e^{Nf_A\epsilon_1+Nf_B\epsilon_2}}{\epsilon_1\epsilon_2(\epsilon_2-\epsilon_1)}+\frac{1}{\epsilon_2}\frac{e^{Nf_A\epsilon_1}}{\epsilon_1^2}-\frac{1}{\epsilon_1}\frac{e^{Nf_B\epsilon_2}}{\epsilon_2^2}-\frac{1}{\epsilon_1^2\epsilon_2}+\frac{e^{Nf_B\epsilon_1}-e^{N\epsilon_1}}{\epsilon_1^2(\epsilon_2-\epsilon_1)}+\frac{e^{Nf_B\epsilon_2}}{\epsilon_2^2(\epsilon_1-\epsilon_2)}$$

where we see that this is the same as above with $\epsilon_1\leftrightarrow\epsilon_2$.

5 Four points all on one segment

5.1 By sigle inverselaplace tranform

$$I^{(4)}_{1,2,3}(N)=\int_0^N ds_3\int_0^{s_3}ds_2\int_0^{s_2}ds_1\int_0^{s_1}ds_0G_1(s_1-s_0)\,G_2(s_2-s_1)\,G_3(s_3-s_2)$$

$$\mathcal{L}\left(I^{(4)}_{1,2,3}(N)\right)(p)=\int_{N=0}^{\infty}e^{-pN}\int_0^N ds_3\underbrace{\int_0^{s_3}ds_2\int_0^{s_2}ds_1\int_0^{s_1}ds_0G_1(s_1-s_0)\,G_2(s_2-s_1)\,G_3(s_3-s_2)}_{f(s_3)}$$

using convolution

$$\breve{I}^{(4)}(p)=\frac{1}{p}\int_0^\infty ds_3e^{-ps_3}\int_0^{s_3}ds_2\underbrace{\int_0^{s_2}ds_1\int_0^{s_1}ds_0G_1(s_1-s_0)\,G_2(s_2-s_1)}_{f(s_2)}\underbrace{G_3(s_3-s_2)}_{g(s_3-s_2)}$$

using convolution again

$$\breve{I}^{(4)}(p)=\frac{\mathcal{G}_3(p)}{p}\int_0^\infty ds_3e^{-ps_2}\int_0^{s_2}ds_1\underbrace{\int_0^{s_1}ds_0G_1(s_1-s_0)}_{f(s_1)}\underbrace{G_2(s_2-s_1)}_{f(s_2-s_1)}$$

and again

$$\breve{I}^{(4)}(p)=\frac{\mathcal{G}_3(p)\,\mathcal{G}_2(p)}{p}\int_0^\infty ds_1e^{-ps_1}\int_0^{s_1}ds_0G_1(s_1-s_0)$$

and again

$$\breve{I}^{(4)}_{1,2,3}(p)=\frac{\mathcal{G}_3(p)\,\mathcal{G}_2(p)\,\mathcal{G}_1(p)}{p^2}$$

5.2 Inverting 4-point single laplace

Assuming that no pole coinside we have

$$I^{(4)}_{1,2,3}(N)=\sum_{residues}\frac{1}{(n-1)!}\lim_{p\to\epsilon}\frac{\partial^{n-1}}{\partial p^{n-1}}\left((p-\epsilon)^n\frac{e^{pN}}{p^2}\mathcal{G}_3(p)\,\mathcal{G}_2(p)\,\mathcal{G}_1(p)\right)$$

The only second order pole is the one at zero. The others are single. The pole at zero gives us

$$\lim_{p\to0}\frac{\partial}{\partial p}\left(e^{pN}\mathcal{G}_3(p)\,\mathcal{G}_2(p)\,\mathcal{G}_1(p)\right)=N\mathcal{G}_3(0)\,\mathcal{G}_2(0)\,\mathcal{G}_1(0)+\mathcal{G}_3'(0)\,\mathcal{G}_2(0)\,\mathcal{G}_1(0)+\mathcal{G}_3(0)\,\mathcal{G}_2'(0)\,\mathcal{G}_1(0)+\mathcal{G}_3(0)\,\mathcal{G}_2(0)\,\mathcal{G}_1'(0)$$

Assuming that none of the poles coinside and using the notation $R_1=Res_1$ and $\mathcal{G}_1(0)$ shorted to $\mathcal{G}_1$

$$\begin{aligned}\underset{AAAA}{I^{(4)}_{1,2,3}}(N)=&\sum_{l_1}\mathcal{G}_2(\epsilon_1)\,\mathcal{G}_3(\epsilon_1)\,R_1\frac{e^{\epsilon_1Nf_A}}{\epsilon_1^2}+\ldots_{1\leftrightarrow2}+\ldots_{1\leftrightarrow3}\\&+Nf_A\mathcal{G}_3\mathcal{G}_2\mathcal{G}_1+\mathcal{G}_3'\mathcal{G}_2\mathcal{G}_1+\mathcal{G}_3\mathcal{G}_2'\mathcal{G}_1+\mathcal{G}_3\mathcal{G}_2\mathcal{G}_1'\end{aligned}$$

5.3 4-point AAAB

In a similar manner to three point we have

$$\underset{AAAB}{I^{(4)}_{1,2,3}}(N)=\sum_{l_1,l_2,l_3}R_1(\epsilon_1)\,R_2(\epsilon_2)\,R_3(\epsilon_3)\int_{Nf_A}^{N}ds_3\int_0^{Nf_A}ds_2\int_0^{s_2}ds_1\int_0^{s_1}ds_0e^{\epsilon_1(s_1-s_0)}e^{\epsilon_2(s_2-s_1)}e^{\epsilon_1(s_3-s_2)}$$

where I have used $R_1\equiv Res_1$. Also in the section only G means $\mathcal{G}$, I'm just too lasy to type "mathcal" all the time and G_1 is short for $G_1(0)$. By doing the intigrals, algebra, and the identities above I found that

$$\begin{aligned}\underset{AAAB}{I^{(4)}_{1,2,3}}(N)=&-G_1G_2G_3'-G_1G_2\sum_{l_3}R_3\frac{e^{Nf_B\epsilon_3}}{\epsilon_3^2}+\sum_{l_3}G_1(\epsilon_3)\,G_2(\epsilon_3)\,R_3\frac{e^{N\epsilon_3}-e^{Nf_A\epsilon_3}}{\epsilon_3^2}\\&+\sum_{l_2}(G_3-G_3(\epsilon_2))\,G_1(\epsilon_2)\,R_2\frac{e^{Nf_A\epsilon_2}}{\epsilon_2^2}+\sum_{l_1}(G_3-G_3(\epsilon_1))\,G_2(\epsilon_1)\,R_1\frac{e^{Nf_A\epsilon_1}}{\epsilon_1^2}\\&+\sum_{l_1,l_3}G_2(\epsilon_1)\,R_1R_3\frac{e^{Nf_A\epsilon_1+Nf_B\epsilon_3}}{\epsilon_1\epsilon_3(\epsilon_1-\epsilon_3)}+\sum_{l_2,l_3}G_1(\epsilon_2)\,R_2R_3\frac{e^{Nf_A\epsilon_2+Nf_B\epsilon_3}}{\epsilon_2\epsilon_3(\epsilon_2-\epsilon_3)}\end{aligned}$$

5.4 4-point AABB

$$I^{(4)}_{\substack{1,2,3\\AABB}}(N) = \sum_{l_1,l_2,l_3} R_1(\epsilon_1) R_2(\epsilon_2) R_3(\epsilon_3) \int_{Nf_A}^{N} ds_3 \int_{Nf_A}^{s_4} ds_2 \int_0^{Nf_A} ds_1 \int_0^{s_1} ds_0 e^{\epsilon_1(s_1-s_0)} e^{\epsilon_2(s_2-s_1)} e^{\epsilon_1(s_3-s_2)}$$

once again using the notation above

$$\begin{aligned}
I^{(4)}_{\substack{1,2,3\\AABB}}(N) =& -G_1G_3G_2' + G_3\sum_{l_1}(G_2 - G_2(\epsilon_1)) R_1 \frac{e^{Nf_A\epsilon_1}}{\epsilon_1^2} + G_1\sum_{l_3}(G_2 - G_2(\epsilon_3)) R_3 \frac{e^{Nf_B\epsilon_3}}{\epsilon_3^2} \\
& - \sum_{l_2} G_3G_1(\epsilon_2) R_2 \frac{e^{Nf_A\epsilon_2}}{\epsilon_2^2} - \sum_{l_2} G_1G_3(\epsilon_2) R_2 \frac{e^{Nf_B\epsilon_2}}{\epsilon_2} + \sum_{l_2} G_1(\epsilon_2) G_3(\epsilon_2) R_2 \frac{e^{N\epsilon_2}}{\epsilon_2^2} \\
& + \sum_{l_1,l_2} G_3(\epsilon_2) R_1R_2 \frac{e^{Nf_B\epsilon_2+Nf_A\epsilon_1}}{\epsilon_1\epsilon_2(\epsilon_1-\epsilon_2)} + \sum_{l_2,l_3} G_1(\epsilon_2) \frac{e^{Nf_B\epsilon_3+Nf_A\epsilon_2}}{(\epsilon_3-\epsilon_2)\epsilon_2\epsilon_3} \\
& + \sum_{l_1,l_3} (G_2(\epsilon_1) - G_2(\epsilon_3)) R_1R_3 \frac{e^{Nf_B\epsilon_3+Nf_A\epsilon_1}}{(\epsilon_3-\epsilon_1)\epsilon_1\epsilon_3}
\end{aligned}$$

6 Limit tricks

Consider the problem $\lim_{p\to\epsilon} \frac{\partial}{\partial p}((p-\epsilon)G(p))$ where $G(p)$ blows up rougly as $(p-\epsilon)$. That is to say

$$\lim_{p\to\epsilon} \frac{(p-\epsilon)}{\left(\frac{1}{G(p)}\right)} = \lim_{p\to\epsilon} \frac{1}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)} = SFN$$

where SFN is some finite number. We approach this in the following way

$$\begin{aligned}
\lim_{p\to\epsilon} \frac{\partial}{\partial p}((p-\epsilon)G(p)) &= \lim_{p\to\epsilon} \frac{\partial}{\partial p}\left(\frac{(p-\epsilon)}{\left(\frac{1}{G(p)}\right)}\right) \\
&= \lim_{p\to\epsilon} \frac{\left(\frac{1}{G(p)}\right) - (p-\epsilon)\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)}{\left(\frac{1}{G(p)}\right)^2} \\
&= \lim_{p\to\epsilon} \frac{1 - \frac{(p-\epsilon)}{\left(\frac{1}{G(p)}\right)}\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)}{\left(\frac{1}{G(p)}\right)} \sim \frac{0}{0} \\
&= \lim_{p\to\epsilon} \frac{-\frac{\partial}{\partial p}\left((p-\epsilon)G(p)\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)\right)}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)} \\
&= \lim_{p\to\epsilon} \frac{-(p-\epsilon)G(p)\frac{\partial^2}{\partial p^2}\left(\frac{1}{G(p)}\right) - \frac{\partial}{\partial p}((p-\epsilon)G(p))\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)} \\
&= -\frac{\frac{\partial^2}{\partial p^2}\left(\frac{1}{G(p)}\right)_{p=\epsilon}}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)_{p=\epsilon}} \lim_{p\to\epsilon}(p-\epsilon)G(p) - \lim_{p\to\epsilon}\frac{\partial}{\partial p}((p-\epsilon)G(p))
\end{aligned}$$

Identifying the rightmost term as negitive of the left and side

$$2\lim_{p\to\epsilon} \frac{\partial}{\partial p}((p-\epsilon)G(p)) = -\frac{\frac{\partial^2}{\partial p^2}\left(\frac{1}{G(p)}\right)_{p=\epsilon}}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)_{p=\epsilon}} \lim_{p\to\epsilon}(p-\epsilon)G(p)$$

We arive at the handy equation[5].

$$\lim_{p\to\epsilon} \frac{\partial}{\partial p}((p-\epsilon)G(p)) = -\frac{\frac{\partial^2}{\partial p^2}\left(\frac{1}{G(p)}\right)_{p=\epsilon}}{2\left(\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)\right)^2_{p=\epsilon}} \tag{9}$$

[5] This equation has been checked numerically.

6.0.1 Second Derivative

Given that

$$\lim_{p\to\epsilon} \frac{(p-\epsilon)}{\left(\frac{1}{G(p)}\right)} = \lim_{p\to\epsilon} \frac{1}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)} = SFN$$

It is a certified, testified, verified, golden-approved fact that

$$\lim_{p\to\epsilon} \frac{\partial^2}{\partial p^2}\left(\left(p-\epsilon\right)G\left(p\right)\right) = \left.\frac{\left(\frac{\partial^2}{\partial p^2}\frac{1}{G(p)}\right)^2}{2\left(\frac{\partial}{\partial p}\frac{1}{G(p)}\right)^3} - \frac{\frac{\partial^3}{\partial p^3}\frac{1}{G(p)}}{3\left(\frac{\partial}{\partial p}\frac{1}{G(p)}\right)^2}\right|_{p=\epsilon} \tag{10}$$

Proof: see mathcad document.