Using Laplace Convolution to do Integrals

November 20, 2015

Redoing what Andy did [2pt, same monomer]

$$I_{ii}=\int_0^N ds_2\int_0^N ds_1 G\left(\left|s_2-s_1\right|\right)$$

$$I_{ii}=2\int_0^N ds_2\int_0^{s_2} ds_1 G\left(\left|s_2-s_1\right|\right)$$

$$\mathcal{L}\left(I_{ii}\left(N\right)\right)\left(p\right)=2\int_0^\infty dNe^{-pN}\int_0^N ds_2\underbrace{\int_0^{s_2} ds_1 G\left(\left|s_2-s_1\right|\right)}_{f(s_2)}$$

$$\begin{aligned}\mathcal{L}\left\{f\right\}\mathcal{L}\left\{g\right\}=&\mathcal{L}\left(f*g\right)\\ =&\mathcal{L}\left(\int_0^t f\left(\tau\right)f\left(t-\tau\right)\right)\end{aligned}$$

$$\underbrace{\int_{N=0}^\infty e^{-pN}\int_{\tau=0}^N f\left(\tau\right)g\left(N-\tau\right)d\tau dN}_{\mathcal{L}(f*g)}=\underbrace{\int_{x_1=0}^\infty e^{-px_1}f\left(x_1\right)dx_1}_{F(p)}\underbrace{\int_{x_2=0}^\infty e^{-px_2}g\left(x_2\right)dx_2}_{G(p)}$$

See Arfken for proof.
Identify $g\left(p\right)=1$, $s_2\equiv\tau$, $x_1\to N$ and put a 2 out front.

$$2\int_{N=0}^\infty e^{-pN}\int_{s_2=0}^N f\left(s_2\right)ds_2 dN=2\int_{N=0}^\infty e^{-pN}f\left(N\right)dN\int_{x_2=0}^\infty e^{-px_2}dx_2$$

$$\mathcal{L}\left(I_{ii}\left(N\right)\right)\left(p\right)=2\int_{N=0}^\infty e^{-pN}f\left(N\right)dN\frac{1}{p}$$

[1]

$$\mathcal{L}\left(I_{ii}\left(N\right)\right)\left(p\right)=2\int_{N=0}^\infty e^{-pN}f\left(N\right)dN\frac{1}{p}$$

where $f\left(N\right)=\underbrace{\int_0^N ds_1 G\left(\left|N-s_1\right|\right)}_{f(N)}$

$$\mathcal{L}\left(I_{ii}\left(N\right)\right)\left(p\right)=2\int_{N=0}^\infty e^{-pN}\int_0^N ds_1 1\cdot G\left(\left|N-s_1\right|\right)dN\frac{1}{p}$$

Once again this is a convolution with $f\left(\tau\right)=1$, $\tau\to s_1$, $g\to G$:

$$\int_{N=0}^\infty e^{-pN}\int_{\tau=0}^N G\left(N-\tau\right)d\tau dN=\int_{x_1=0}^\infty e^{-px_1}dx_1\int_{x_2=0}^\infty e^{-px_2}G\left(x_2\right)dx_2$$

[1] $\int_{N=0}^\infty e^{-pN}f\left(N\right)dN$ is just the laplace transform of $f\left(N\right)$

$$\mathcal{L}\left(I_{ii}\left(N\right)\right)\left(p\right)=2\frac{-\mathcal{L}\left(f\left(N\right)\right)\left(p\right)}{p}$$

$$\mathcal{L}\left(I_{ii}\left(N\right)\right)\left(p\right)=2\int_{x_2=0}^{\infty}e^{-px_2}G\left(x_2\right)dx_2\frac{1}{p^2}$$

Identifying the Laplace transform and changing $x_2\to N$

$$\mathcal{L}\left\{I_{ii}\left(N\right)\right\}\left(p\right)=2\frac{\mathcal{L}\left\{G\left(N\right)\right\}\left(p\right)}{p^2}$$

Now we want to do an inverse Laplace transform by the residue theorem
[2]

In our case the residues are the residues described in the paper which each diverge signally [n=0] and the residue at zero for which n=1.

$$I_{ii}\left(N\right)=2\sum_{l=0}^{\infty}\lim_{p\to\epsilon_l}\left(p-\epsilon_l\right)e^{pN}\frac{G\left(p\right)}{p^2}+\lim_{p\to 0}\frac{\partial}{\partial p}\left(2e^{pN}G\left(p\right)\right)$$

Where we defined $G\left(p\right)\equiv\mathcal{L}\left\{G\left(N\right)\right\}$. Using L'Hopital's rule on the sum and product rule on the second term:

$$I_{ii}\left(N\right)=2\sum_{l=0}^{\infty}\frac{e^{\epsilon_l N}}{\epsilon_l^2}\frac{1}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)\Big|_{p=\epsilon_l}}+2\left.\frac{\partial}{\partial p}G\left(p\right)\right|_{p=0}+2NG\left(0\right)$$

How does the convergence compare to that of the original pole?

$$I_{ii}=2\int_0^N ds_2\int_0^{s_2}ds_1 G\left(\left|s_2-s_1\right|\right)$$

$$I_{ii}=2\int_0^N ds_2\int_0^{s_2}ds_1\sum_{l=0}^{\infty}\frac{e^{pN}}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)}$$

$$I_{ii}=2\int_0^N ds_2\int_0^{s_2}ds_1\sum_{l=0}^{\infty}Res_l\cdot e^{\epsilon_l N}$$

And these integrals can be done.

2 point different monomers?

$$I=\int_c^d ds_2\int_a^b ds_1 G\left(\left|s_2-s_1\right|\right)$$

where

$$a<b<c<d$$

[2]Cauchy's integral formula

$$\oint_\gamma\frac{u\left(z\right)}{\left(z-\epsilon\right)^{n+1}}dz=\frac{2\pi i}{n!}u^{(n)}\left(\epsilon\right)$$

where u is analytic. And the residue method for inverse Laplace transform:

$$f\left(x\right)=\frac{1}{2\pi i}\int_\gamma e^{sx}F\left(s\right)ds$$

Putting these together and renaming $z=s\to p$

$$f\left(x\right)=\frac{1}{\left(n-1\right)!}\lim_{p\to\epsilon}\frac{\partial^{n-1}}{\partial p^{n-1}}\left(\left(p-\epsilon\right)^n e^{px}F\left(p\right)\right)$$

The methods that don't seem to work

$$I=\int_0^{N_1} ds_2 \int_0^{N_2} ds_1 G\left(\left|D+s_2-s_1\right|\right)$$

$$\mathcal{L}\left\{I\left(N_1,N_2\right)\right\}\left(p_1,N_2\right)=\int_0^{\infty} dN e^{-pN_1}\int_0^{N_1} ds_2 \underbrace{\int_0^{N_2} ds_1 G\left(\left|D+s_2-s_1\right|\right)}_{f(s_2)}\cdot 1$$

Using the Laplace convolution theorem

$$\mathcal{L}\left\{I\left(N_1,N_2\right)\right\}\left(p_1,N_2\right)=\int_{s_2=0}^{\infty} e^{-ps_2} f\left(s_2\right) ds_2\cdot\frac{1}{p}$$

$$\mathcal{L}\left\{I\left(N_1,N_2\right)\right\}\left(p_1,N_2\right)=\int_{\boxed{s_2}=0}^{\infty} e^{-ps_2}\int_0^{\boxed{N_2}} ds_1 G\left(\left|\boxed{D+s_2}-s_1\right|\right) ds_2\cdot\frac{1}{p}$$

problem: in order to use the convolution we need the upper limit to be the integration variable

$$\underbrace{\int_{N=0}^{\infty} e^{-pN}\int_{\tau=0}^{N} f\left(\tau\right) g\left(N-\tau\right) d\tau dN}_{\mathcal{L}(f*g)}=\underbrace{\int_{x_1=0}^{\infty} e^{-px_1} f\left(x_1\right) dx_1}_{F(p)}\underbrace{\int_{x_2=0}^{\infty} e^{-px_2} g\left(x_2\right) dx_2}_{G(p)}$$

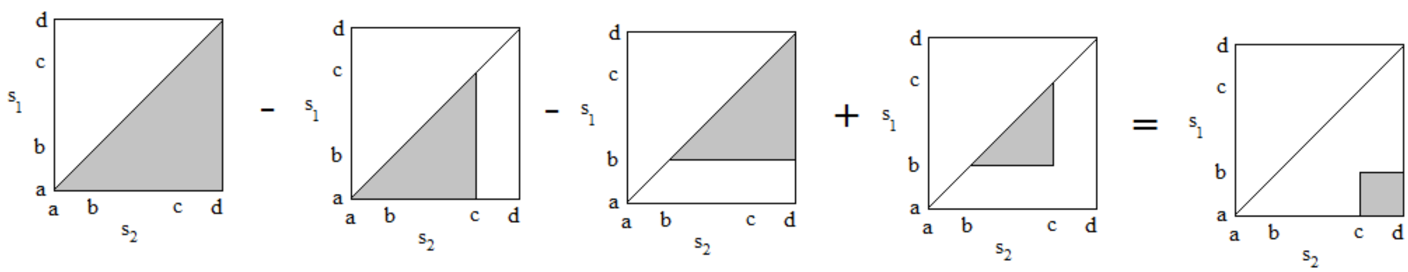
What happens if we take another Laplace transform $N_2\to p_2$?

$$\mathcal{L}^2\left\{I\left(N_1,N_2\right)\right\}\left(p_1,p_2\right)=\int_0^{\infty} dN_2 e^{-p_2N_2}\int_{s_2=0}^{\infty} e^{-p_1s_2}\int_0^{N_2} ds_1 G\left(\left|D+s_2-s_1\right|\right) ds_2\cdot\frac{1}{p_1}$$

$$\mathcal{L}^2\left\{I\left(N_1,N_2\right)\right\}\left(p_1,p_2\right)=\int_{s_2=0}^{\infty} e^{-p_1s_2}\underbrace{\int_0^{\infty} dN_2 e^{-p_2N_2}\int_0^{N_2} ds_1 G\left(\left|D+s_2-s_1\right|\right)}\, ds_2\cdot\frac{1}{p_1}$$

$$\mathcal{L}^2\left\{I\left(N_1,N_2\right)\right\}\left(p_1,p_2\right)=\int_{s_2=0}^{\infty} e^{-p_1s_2}\int_{s_1=0}^{\infty} e^{-ps_1} G\left(\left|D+s_2-s_1\right|\right) ds_1 ds_2\cdot\frac{1}{p_1p_2}$$

What works for 2 point



$$\int_a^d\int_a^d-\int_a^c\int_a^c-\int_b^d\int_b^d+\int_b^c\int_b^c=\int_c^d\int_a^b$$

As shown above the integral for a single triangle with bottom length N is:

$$T\left(N\right)=\sum_{l=0}^{\infty}\frac{e^{\epsilon_l N}}{\epsilon_l^2}\frac{1}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)\Big|_{p=\epsilon_l}}+\frac{\partial}{\partial p}G\left(p\right)\Big|_{p=0}+NG\left(0\right)$$

So the box $s_1\in(a,b)\wedge s_2\in(c,d)$ is:

$$I^{(2)}\left(a,b,c,d\right)=T\left(d-a\right)-T\left(c-a\right)-T\left(d-b\right)+T\left(c-b\right)$$

we see that the latter two terms cancel leaving

$$I^{(2)}\left(a,b,c,d\right)=\sum_{l=0}^{\infty}\frac{Z_l}{\epsilon_l^2}\frac{1}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)\Big|_{p=\epsilon_l}}$$

where

$$Z_l = e^{\epsilon_l(d-a)} - e^{\epsilon_l(c-a)} - e^{\epsilon_l(d-b)} + e^{\epsilon_l(c-b)}$$

$$Z_l = \left(e^{-\epsilon_l a} - e^{-\epsilon_l b}\right)\left(e^{\epsilon_l d} - e^{\epsilon_l c}\right)$$

$$Z_l = \left(e^{\epsilon_l(b-a)} - 1\right) e^{\epsilon_l(c-b)} \left(e^{\epsilon_l(d-c)} - 1\right)$$

How about higher orders?

Three points, all on same segment:

$$I_{iii}(N) = \int_0^N ds_2 \int_0^N ds_1 \int_0^N ds_0 \int du_0 \int du_1 \int du_2 G\left(\vec{q}_1, \vec{u}_1|\vec{u}_0, s_1 - s_0\right) G\left(\vec{q}_2, \vec{u}_2|\vec{u}_1, s_2 - s_1\right)$$

Lets use notation

$$G_{01}(N) \equiv G\left(\vec{q}_1, \vec{u}_1|\vec{u}_0, N\right)$$

$$\int d^3u \equiv \int du_0 \int du_1 \int du_2$$

$$I_{iii}(N) = 6\int_0^N ds_2 \int_0^{s_2} ds_1 \int_0^{s_1} ds_0 \int d^3u G_{01}(s_1 - s_0) G_{12}(s_2 - s_1)$$

$$\mathcal{L}\{I_{iii}(N)\}(p) = 6\int_{p=0}^{\infty} dp e^{-pN} \int_0^N ds_2 \underbrace{\int_0^{s_2} ds_1 \int_0^{s_1} ds_0 \int d^3u G_{01}(s_1 - s_0) G_{12}(s_2 - s_1) \cdot 1}_{f(s_2)}$$

the use the convolution theorem

$$\underbrace{\int_{N=0}^{\infty} e^{-pN} \int_{\tau=0}^{N} f(\tau) g(N-\tau) d\tau dN}_{\mathcal{L}(f*g)} = \underbrace{\int_{x_1=0}^{\infty} e^{-px_1} f(x_1) dx_1}_{F(p)} \underbrace{\int_{x_2=0}^{\infty} e^{-px_2} g(x_2) dx_2}_{G(p)}$$

$$\tau \to s_2; \quad g(N-\tau) \to 1$$

$$\mathcal{L}\{I_{iii}(N)\}(p) = 6\int_0^{\infty} dx_1 e^{-px_1} \overbrace{\int_0^{x_1} ds_1 \int_0^{s_1} ds_0 \int d^3u G_{01}(s_1 - s_0) G_{12}(x_1 - s_1)}^{f(x_1)} \frac{1}{p}$$

make the identification $x_1 \to N$ and also move u integrals to outside so they don't get in the way

$$\mathcal{L}\{I_{iii}(N)\}(p) = 6\int d^3u \int_0^{\infty} dN e^{-pN} \int_0^N ds_1 \underbrace{\int_0^{s_1} ds_0 G_{01}(s_1 - s_0)}_{f(s_1)} \underbrace{G_{12}(N - s_1)}_{g(N-s_1)} \frac{1}{p}$$

use convolution again $\tau \to s_1$

$$\mathcal{L}\{I_{iii}(N)\}(p) = 6\int d^3u \left(\int_{x_1=0}^{\infty} e^{-px_1} \int_0^{x_1} ds_0 G_{01}(x_1 - s_0) dx_1\right)\left(\int_{x_2=0}^{\infty} e^{-px_2} G_{12}(x_2) dx_2\right) \frac{1}{p}$$

rename $x_1 \to N$ for the first part and recognize the Laplace transform for the latter

$$\mathcal{L}\{I_{iii}(N)\}(p) = 6\int d^3u \left(\int_0^{\infty} e^{-pN} \int_0^N ds_0 1 \cdot G_{01}(N - s_0) dN\right) G_{12}(p)$$

using convolution again

$$\boxed{\mathcal{L}\{I_{iii}(N)\}(p) = 6\int d^3u G_{01}(p) G_{12}(p) \frac{1}{p^2}}$$

The inverse laplace transorm for this is in the case of WLC is worked out in a later section.

Attempt at three point with $j_1 = j_2 < j_3$

How do you construct

$$I(N) = 2\int d^3u \int_c^d ds_2 \int_a^b ds_1 \int_a^{s_1} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1)$$

out of

$$I(N) = 2\int d^3u \int_a^b ds_2 \int_a^{s_2} ds_1 \int_a^{s_1} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1)$$

$$\mathcal{L}\{I\}(p) = 2\int d^3u \int_0^\infty dN e^{-pN} \int_c^d ds_2 \int_a^b ds_1 \underbrace{\int_a^{s_1} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1)}_{f(s_1, s_2 - s_1)}$$

define $s_m, u_m(c,d), L_m(a,b)$ such that

$$\sum_{m=1}^{4} \eta_m \int_{L_m}^{u_m} \int_{L_m}^{u_m} = \int_a^d \int_a^d - \int_a^c \int_a^c - \int_b^d \int_b^d + \int_b^c \int_b^c$$

$$\sum_{m=1}^{4} \eta_m \int_{L_m}^{u_m} ds_2 \int_{L_m}^{u_m} ds_1 = \sum_{m=1}^{4} 2\eta_m \int_{L_m}^{u_m} ds_2 \int_{L_m}^{s_2} ds_1$$

$$I(N) = 2\int d^3u \sum_{m=1}^{4} 2\eta_m \int_{L_m}^{u_m} ds_2 \int_{L_m}^{s_2} ds_1 \int_0^{s_1} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1)$$

$$\begin{aligned} I(N) =& 2\int d^3u \sum_{m=1}^{4} 2\eta_m \int_{L_m}^{u_m} ds_2 \int_{L_m}^{s_2} ds_1 \int_{L_m}^{s_1} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1) \\ &+ 2\int d^3u \sum_{m=1}^{4} 2\eta_m \int_{L_m}^{u_m} ds_2 \int_{L_m}^{s_2} ds_1 \int_0^{L_m} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1) \end{aligned}$$

$$\begin{aligned} I(N) =& 2\int d^3u \sum_{m=1}^{4} 2\eta_m \int_{L_m}^{u_m} ds_2 \int_{L_m}^{s_2} ds_1 \int_{L_m}^{s_1} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1) \\ &+ 2\int d^3u \sum_{m=1}^{4} 2\eta_m \int_{L_m}^{u_m} ds_2 \int_{L_m}^{s_2} ds_1 \int_0^{L_m} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1) \end{aligned}$$

Attempt Three point with $j_0 = j_1 < j_2$ with 3 intigral done later

How do you construct

$$I(a,b,c,d) = 2\int d^3u \int_c^d ds_2 \int_a^b ds_1 \int_a^{s_1} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1)$$

sift to $a = 0$ and define $N = b - a$

$$I(c-a, d-a, N) = 2\int d^3u \int_{c-a}^{d-a} ds_2 \int_0^N ds_1 \int_0^{s_1} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1)$$

Laplace $N \to p$

$$\begin{aligned} \mathcal{L}\{I(c-a, d-a, N)\}(p) =& \int_0^\infty dN e^{-Np} 2\int d^3u \int_{c-a}^{d-a} ds_2 \int_0^N ds_1 \int_0^{s_1} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1) \\ =& 2\int d^3u \int_{c-a}^{d-a} ds_2 \int_0^\infty dN e^{-Np} \int_0^N ds_1 \underbrace{\int_0^{s_1} ds_0 G_{01}(s_1 - s_0) G_{12}(s_2 - s_1) \cdot 1}_{f(s_1)} \end{aligned}$$

we identify $s_1 \to \tau$, $1 =\to g(x)$,

$$\underbrace{\int_{N=0}^{\infty} e^{-pN} \int_{\tau=0}^{N} f(\tau)\, g(N-\tau)\, d\tau dN}_{\mathcal{L}(f*g)} = \underbrace{\int_{x_1=0}^{\infty} e^{-px_1} f(x_1)\, dx_1}_{F(p)} \underbrace{\int_{x_2=0}^{\infty} e^{-px_2} g(x_2)\, dx_2}_{G(p)}$$

$$\mathcal{L}\{I(c-a,d-a,N)\}(p) = 2\int d^3u \int_{c-a}^{d-a} ds_2 \int_0^{\infty} ds_1 e^{-ps_1} \overbrace{\int_0^{s_1} ds_0 G_{01}(s_1-s_0)\, G_{12}(s_2-s_1)} \frac{1}{p}$$

rename $s_1 \to N$ and $s_0 \to \tau$

$$\mathcal{L}\{I(c-a,d-a,N)\}(p) = 2\int d^3u \int_{c-a}^{d-a} ds_2 \int_0^{\infty} dN e^{-pN} \int_0^{N} d\tau G_{01}(N-\tau) \underbrace{G_{12}(s_2-N)}_{!?!?!?} \frac{1}{p}$$

I believe if I had done $j_0 < j_1 = j_2$ I would just have gotten $G(N-s_0)$ which would have been equally bad.

Confined intigrals

Using the convolution intigral

$$\underbrace{\int_{N=0}^{\infty} e^{-pN} \int_{\tau=0}^{N} f(\tau)\, g(N-\tau)\, d\tau dN}_{\mathcal{L}(f*g)} = \underbrace{\int_{x_1=0}^{\infty} e^{-px_1} f(x_1)\, dx_1}_{F(p)} \underbrace{\int_{x_2=0}^{\infty} e^{-px_2} g(x_2)\, dx_2}_{G(p)}$$

we try the forms

$$\int_0^{\infty} dN e^{-pN} \int_0^{N} d\tau f(\tau)\, h_a(N-\tau) = \int_{x_1=0}^{\infty} e^{-px_1} f(x_1)\, dx_1 \int_{x_2=0}^{\infty} e^{-px_2} h_a(x_1)\, dx_2$$

$$h_a(x) = \begin{cases} 0 & x < a \\ 1 & x > a \end{cases}$$

$$N - a > \tau$$

$$\boxed{\int_0^{\infty} dN e^{-pN} \int_0^{N-a} f(\tau) = \mathcal{L}\{f(\tau)\}(p)\, \frac{e^{-pa}}{p}}$$

also try the form

$$\int_0^{\infty} dN e^{-pN} \int_0^{N} d\tau f(\tau)\, h'_a(N-\tau) = \int_{x_1=0}^{\infty} e^{-px_1} f(x_1)\, dx_1 \int_{x_2=0}^{\infty} e^{-px_2} h'_a(x_1)\, dx_2$$

$$h'_a(x) = \begin{cases} 1 & x < a \\ 0 & x > a \end{cases}$$

$$N - a < \tau$$

$$\int_0^{\infty} dN e^{-pN} \int_{N-a}^{N} d\tau f(\tau) = \int_{x_1=0}^{\infty} e^{-px_1} f(x_1)\, dx_1 \int_0^{a} e^{-px_2} dx_2$$

$$\boxed{\int_0^{\infty} dN e^{-pN} \int_{N-a}^{N} d\tau f(\tau) = \mathcal{L}\{f(\tau)\}(p)\, \frac{1-e^{-pa}}{p}}$$

finally

$$\int_0^{\infty} dN e^{-pN} \int_0^{N} d\tau f(\tau)\, b_{a,b}(N-\tau) = \int_{x_1=0}^{\infty} e^{-px_1} f(x_1)\, dx_1 \int_{x_2=0}^{\infty} e^{-px_2} h'_a(x_1)\, dx_2$$

$$b_{a,b}(x) = \begin{cases} 0 & x < a \\ 1 & a \le x \le b \\ 0 & b < a \end{cases}$$

$$\boxed{\int_0^\infty dNe^{-pN}\int_{N-b}^{N-a} d\tau f(\tau) = \mathcal{L}\{f(\tau)\}(p)\left(\frac{e^{-ap}+e^{-bp}}{p}\right)}$$

Four point, all on one segment

$$I^{(4)}(N) = 12\int_0^N ds_3\int_0^{s_3} ds_2\int_0^{s_2} ds_1\int_0^{s_1} ds_0\int d^3uG_{01}(s_1-s_0)\,G_{12}(s_2-s_1)$$

Inverting Laplace WLC

The 2 point doesn't require a rotational part as the propagator is spyrically symmetric.

Rotation Part 3 point

We need to calculate:

$$G(0\to1\to2) = \frac{1}{(4\pi)^2}\int du_0\int du_1\int du_2\left(\sum_{l_0,l_f,m} Y_{l_f}^m\left(\Gamma_1^{-1}\vec{u}_1\right)Y_{l_0}^{*m}\left(\Gamma_1^{-1}\vec{u}_0\right)\mathcal{G}_{l_0}^{l_f,m}(K_1,p)\right)\cdot$$
$$\left(\sum_{l_0,l_f,m} Y_{l_f}^m\left(\Gamma_2^{-1}\vec{u}_2\right)Y_{l_0}^{*m}\left(\Gamma_2^{-1}\vec{u}_1\right)\mathcal{G}_{l_0}^{l_f,m}(K_2,p)\right)$$

From orthogonality $\int Y_l^m Y_{l'}^{m'} d\Omega = \frac{4\pi}{(2l+1)}\delta_{ll'}\delta_{mm'}$ and we note that $1 = \frac{1}{\sqrt{4\pi}}Y_0^0$

$$G(0\to1\to2) = \frac{1}{4\pi}\int du_1\left(\sum_l Y_l^{*0}\left(\Gamma_2^{-1}\vec{u}_1\right)Y_l^0\left(\Gamma_1^{-1}\vec{u}_1\right)\mathcal{G}_l^{0,0}(K_2,p)\,\mathcal{G}_0^{l,0}(K_1,p)\right)$$

And the Euler rotation $Y_l^m(\vec{u}_1) = \sum_{m'=-l}^{l} D_l^{m',m}(\alpha,\beta,\gamma)\,Y_l^{m'}(R(\alpha,\beta,\gamma)\,\vec{u}_1)$

$$G(0\to1\to2) = \frac{1}{4\pi}\int du_1\left(\sum_l Y_l^{*0}\left(\Gamma_2^{-1}\vec{u}_1\right)Y_l^0\left(\Gamma_1^{-1}\vec{u}_1\right)\mathcal{G}_l^{0,0}(K_2,p)\,\mathcal{G}_0^{l,0}(K_1,p)\right)$$

$$Y_l^m(\vec{u}_1) = \sum_{m'=-l}^{l} D_l^{m',m}(\alpha,\beta,\gamma)\,Y_l^{m'}(R(\alpha,\beta,\gamma)\,\vec{u}_1)$$

Because wee need only rotate $\vec{q}_1$ into $\vec{q}_2$ we have

$$Y_l^m(\vec{u}_1) = \sum_{m'=-l}^{l} D_l^{m',m}(0, acos(\vec{q}_1\cdot\vec{q}_2), 0)\,Y_l^{m'}(R(\alpha,\beta,\gamma)\,\vec{u}_1)$$

$$G(0\to1\to2) = \sum_l P_l^0(\vec{q}_1\cdot\vec{q}_2)\,\mathcal{G}_l^{0,0}(K_2,p)\,\mathcal{G}_0^{l,0}(K_1,p)$$

$$\mathcal{L}\{I_{iii}(N)\}(p) = 6\left(\sum_l P_l^0(\vec{q}_1\cdot\vec{q}_2)\,\mathcal{G}_l^{0,0}(K_2,p_2)\,\mathcal{G}_0^{l,0}(K_1,p_1)\right)\frac{1}{p^2}$$

a.k.a

$$\mathcal{L}\{I_{iii}(N)\}(p) = 6\left(\sum_\lambda P_\lambda^0(\vec{q}_1\cdot\vec{q}_2)\,\mathcal{G}_\lambda^{0,0}(K,p)\,\mathcal{G}_0^{\lambda,0}(K_1,p)\right)\frac{1}{p^2}$$

$$I_{iii}(N) = \frac{6}{(n-1)!}\lim_{p\to\epsilon}\frac{\partial^{n-1}}{\partial p^{n-1}}\left((p-\epsilon)^n e^{pN}\sum_l P_l^0(\vec{q}_1\cdot\vec{q}_2)\,\mathcal{G}_l^{0,0}(K_2,p)\,\mathcal{G}_0^{l,0}(K_1,p)\,\frac{1}{p^2}\right)$$

Apparently the eigenvalues ϵ_l don't depend on λ [written as l here]. Assuming that the $\vec{q}$'s have the same magnitude that means $K_1 = K_2$ and we have degenerate poles.

$$I_{iii}(N) = \frac{6}{1!}\lim_{p\to\epsilon}\frac{\partial}{\partial p}\left((p-\epsilon)^2 e^{pN}\sum_l P_l^0(\vec{q}_1\cdot\vec{q}_2)\,\mathcal{G}_l^{0,0}(K_2,p)\,\mathcal{G}_0^{l,0}(K_1,p)\frac{1}{p^2}\right)$$

Renaming $l\to\lambda$ so we can reuse l

$$I_{iii}(N)\frac{1}{6} = \lim_{p\to 0}\frac{\partial}{\partial p}\left(e^{pN}\sum_\lambda P_\lambda^0(\vec{q}_1\cdot\vec{q}_2)\,\mathcal{G}_\lambda^{0,0}(K,p)\,\mathcal{G}_0^{\lambda,0}(K,p)\right)$$
$$\sum_{l=0}^{\infty}\sum_\lambda P_\lambda^0(\vec{q}_1\cdot\vec{q}_2)\lim_{p\to\epsilon_l}\frac{\partial}{\partial p}\left((p-\epsilon_l)\,\mathcal{G}_\lambda^{0,0}(K,p)\,(p-\epsilon_l)\,\mathcal{G}_0^{\lambda,0}(K,p)\frac{e^{pN}}{p^2}\right)$$

$$I_{iii}(N)\frac{1}{6} = \sum_\lambda P_\lambda^0(\vec{q}_1\cdot\vec{q}_2)\left(N\mathcal{G}_\lambda^{0,0}(K_2,0)\,\mathcal{G}_0^{\lambda,0}(K_1,0) + \left(\frac{\partial}{\partial p}\mathcal{G}_\lambda^{0,0}(K,p)\right)_{p=0}\mathcal{G}_0^{\lambda,0}(K,0) + \mathcal{G}_\lambda^{0,0}(K_2,0)\left(\frac{\partial}{\partial p}\mathcal{G}_0^{\lambda,0}(K,p)\right)_{p=0}\right.$$
$$\left. + \sum_{l=0}^{\infty}\lim_{p\to\epsilon_l}\frac{\partial}{\partial p}\left(\underbrace{(p-\epsilon_l)\,\mathcal{G}_\lambda^{0,0}(K,p)}\,\underbrace{(p-\epsilon_l)\,\mathcal{G}_0^{\lambda,0}(K,p)}\frac{e^{pN}}{p^2}\right)\right)$$

Rotation Part 4 point

Done elsewhere/comming soon

Limit tricks

Consider the problem $\lim_{p\to\epsilon}\frac{\partial}{\partial p}\left((p-\epsilon)\,G(p)\right)$ where $G(p)$ blows up rougly as $(p-\epsilon)$. That is to say

$$\lim_{p\to\epsilon}\frac{(p-\epsilon)}{\left(\frac{1}{G(p)}\right)} = \lim_{p\to\epsilon}\frac{1}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)} = SFN$$

where SFN is some finite number. We approach this in the following way

$$\begin{aligned}
\lim_{p\to\epsilon}\frac{\partial}{\partial p}\left((p-\epsilon)\,G(p)\right) &= \lim_{p\to\epsilon}\frac{\partial}{\partial p}\left(\frac{(p-\epsilon)}{\left(\frac{1}{G(p)}\right)}\right)\\
&= \lim_{p\to\epsilon}\frac{\left(\frac{1}{G(p)}\right) - (p-\epsilon)\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)}{\left(\frac{1}{G(p)}\right)^2}\\
&= \lim_{p\to\epsilon}\frac{1-\frac{(p-\epsilon)}{\left(\frac{1}{G(p)}\right)}\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)}{\left(\frac{1}{G(p)}\right)} \sim \frac{0}{0}\\
&= \lim_{p\to\epsilon}\frac{-\frac{\partial}{\partial p}\left((p-\epsilon)\,G(p)\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)\right)}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)}\\
&= \lim_{p\to\epsilon}\frac{-(p-\epsilon)\,G(p)\frac{\partial^2}{\partial p^2}\left(\frac{1}{G(p)}\right) - \frac{\partial}{\partial p}\left((p-\epsilon)\,G(p)\right)\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)}\\
&= -\frac{\frac{\partial^2}{\partial p^2}\left(\frac{1}{G(p)}\right)_{p=\epsilon}}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)_{p=\epsilon}}\lim_{p\to\epsilon}(p-\epsilon)\,G(p) - \lim_{p\to\epsilon}\frac{\partial}{\partial p}\left((p-\epsilon)\,G(p)\right)
\end{aligned}$$

Identifying the rightmost term as negitive of the left and side

$$2\lim_{p\to\epsilon}\frac{\partial}{\partial p}\left((p-\epsilon)\,G(p)\right) = -\frac{\frac{\partial^2}{\partial p^2}\left(\frac{1}{G(p)}\right)_{p=\epsilon}}{\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)_{p=\epsilon}}\lim_{p\to\epsilon}(p-\epsilon)\,G(p)$$

We arrive at the handy equation[3].

$$\lim_{p\to\epsilon}\frac{\partial}{\partial p}\left(\left(p-\epsilon\right)G\left(p\right)\right)=-\frac{\frac{\partial^2}{\partial p^2}\left(\frac{1}{G(p)}\right)_{p=\epsilon}}{2\left(\frac{\partial}{\partial p}\left(\frac{1}{G(p)}\right)\right)^2_{p=\epsilon}}$$

Calculating derivatives of $\frac{1}{\mathcal{G}(p)}$

For the 2 point we only need to know $\mathcal{G}^{00}_0\left(p\right)=\frac{1}{j_0^{0(+)}}$. For 3 point we need $\mathcal{G}^0_{0,\lambda}$ and $\mathcal{G}^0_{\lambda_0,0}$.

$$\mathcal{G}^0_{\lambda_0,0}=\begin{cases}\ldots\\ ..\end{cases}$$

$$\mathcal{G}^0_{0,\lambda}=\begin{cases}w_0^{0(+)} & \lambda=0\\ w_0^{0(+)}\prod_{n=1}^{\lambda}iKa_n^0w_n^{0(+)} & \lambda>0\end{cases}$$

For four point we need $\mathcal{G}^\mu_{\lambda_0,\lambda}$.

$$\frac{1}{\mathcal{G}^\mu_{\lambda_0,\lambda}\left(p\right)}=\begin{cases}\left(a_\lambda^\mu K\right)^2w_{\lambda-1}^{\mu(-)}+P_\lambda+\left(a_{\lambda+1}^\mu K\right)^2w_{\lambda+1}^{\mu(+)} & \lambda_0=\lambda\\ \frac{\left(\left(a_\lambda^\mu K\right)^2w_{\lambda-1}^{\mu(-)}+P_\lambda+\left(a_{\lambda+1}^\mu K\right)^2w_{\lambda+1}^{\mu(+)}\right)\left(\prod_{n=1}^{\lambda-\lambda_0}w_{\lambda_0+n}^{\mu(+)}\right)^{-1}}{\prod_{n=1}^{\lambda-\lambda_0}iKa_{\lambda_0}^\mu} & \lambda_0<\lambda\\ \frac{\left(\left(a_\lambda^\mu K\right)^2w_{\lambda-1}^{\mu(-)}+P_\lambda+\left(a_{\lambda+1}^\mu K\right)^2w_{\lambda+1}^{\mu(+)}\right)\left(\prod_{n=1}^{\lambda_0-\lambda}w_{\lambda_0-n}^{\mu(-)}\right)^{-1}}{\prod_{n=1}^{\lambda_0-\lambda}iKa_{\lambda_0+1-n}^\mu} & \lambda_0>\lambda\end{cases}$$

Define $X^{(+)}=\left(\prod_{n=1}^{\lambda-\lambda_0}w_{\lambda_0+n}^{\mu(+)}\right)^{-1}$, $X^{(-)}=\left(\prod_{n=1}^{\lambda_0-\lambda}w_{\lambda_0-n}^{\mu(-)}\right)^{-1}$ and $X_1=\left(a_\lambda^\mu K\right)^2w_{\lambda-1}^{\mu(-)}+P_\lambda+\left(a_{\lambda+1}^\mu K\right)^2w_{\lambda+1}^{\mu(+)}$. Also define $c^{(+)}=\prod_{n=1}^{\lambda-\lambda_0}iKa_{\lambda_0}^\mu$ and $c^{(-)}=\prod_{n=1}^{\lambda_0-\lambda}iKa_{\lambda_0+1-n}^\mu$.

Nievely there will be a pole whenever $\frac{1}{\mathcal{G}^\mu_{\lambda_0,\lambda}(p)}=0$ which occures either when $X_1=0$ or $X_2=0$. However, when $X_2=0$ it turns out[4] that when $X_2=0$ that $W=\frac{1}{X_1}$ is also zero so there isn't really a pole. The poles that we need to calcuate the residues of are those that occure when $X_1=0$. At these poles we need to calculate

$$\begin{aligned}\frac{\partial}{\partial p}\left(\frac{1}{\mathcal{G}^\mu_{\lambda_0,\lambda}\left(p\right)}\right)=&\frac{\partial}{\partial p}\frac{X_1X^{(\pm)}}{c^{(\pm)}}\\ =&\frac{1}{c^{(\pm)}}\left(X_1\partial_pX^{(\pm)}+X^{(\pm)}\partial_pX_1\right)\\ =&\frac{1}{c^{(\pm)}}\left(X^{(\pm)}\partial_pX_1\right)\end{aligned}$$

$$\frac{\partial^2}{\partial p^2}\left(\frac{1}{\mathcal{G}^\mu_{\lambda_0,\lambda}\left(p\right)}\right)=\frac{1}{c^{(\pm)}}\left(X^{(\pm)}\partial_p^2X_1+\partial_pX^{(\pm)}\partial_pX_1\right)$$

This will require some tedious calcualtion. Define

$$j_\lambda^{\mu(\pm)}=\frac{1}{w_\lambda^{\mu(\pm)}}$$

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[3] This equation has been checked numerically.
[4] I verified this numerically and it is what Andy expected.