Domination of Queens on a Hexagonal Chess Board

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Abstract

For hexagonal chess boards, the domination numbers for their queen graphs are considered. We determine these numbers completely for any hexagonal chess board and develop a generic rule to determine these numbers.

Introduction

Hexagonal Chess is a modification of the traditional game of chess, played on a hexagonal board. A board is defined as an n-board if the board has n hexagons on the bottom row of the board. All boards used in the game are regular hexagonal boards. A regular hexagonal board has the same number of hexagons on each side of the board. Here is an example of a 5-board in Figure 1:

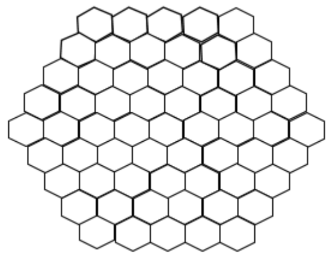

Figure 1

Traditional chess pieces are still used on a hexagonal chess board. However, due of the shape of each cell, possible moves of each piece must be redefined. While in this paper, we only considered the queen, similar algorithms can be developed for bishops, rooks and knights. The queen, in this version of the game, may move in both diagonals, horizontally and vertically. For example, in Figure 2, the black dot marks the placement of the queen and the red lines mark the spaces the queen threatens.

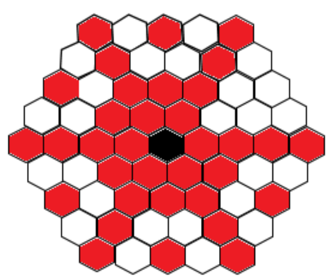

Figure 2

History

The study of domination of chess pieces began in 1860. In 1862, C. F. De Jaenish attempted to determine the minimum number of queens required to cover an n x n chess board. In 1892, W. W. Rouse Ball reported three basic types of problems that chess players studied during this time; Covering, Independent Covering and Independence. In 1958, Claude Berge introduced the domination number of a graph. In a graph G, a set $S \subseteq V(G)$ is a dominating set if every vertex not in S has a neighbor in S. The domination number $\gamma(G)$ is the minimum size of a dominating set in G.

Definitions

- **Domination Number:** Determine the minimum number of chess pieces of a given type that are necessary to dominate every square of a 'n' hexagonal chess board.
- **Independent Domination:** Determine the smallest number of mutually nonattacking chess pieces of a given type that are necessary to dominate every square of a 'n' hexagonal chess board.
- **Independence Number:** Determine the maximum number of chess pieces of a given type that can be placed on a 'n' hexagonal chess board such that no two pieces' attack each other.
- **Queen Graph:** On any graph, two vertices are said to be adjacent if they are joined by an edge. By definition, a given vertex is said to dominate itself and any adjacent vertices. A graph G is said to be dominated by a subset of vertices, say D, if any vertex in G is dominated by a vertex in D. Applying the above to the Queen's graph, a board is dominated by a set of queens if every square on the board is either occupied or attacked by a queen.

Problem

We develop a method to dominate an entire board with $\gamma(Q_n)$, the dominating number of queens, given a hexagonal chess board of any size. In addition to determining the dominating number, we also determine the independence number, $\alpha(Q_n)$, which refers to the maximum number of queens placed on the board such that no two queens attack one another.

Coordinate System

In order to do this, we defined a coordinate system. In this system, the x-axis is labeled on the top of the hexagon while the y axis is labeled down the side. Like the Cartesian coordinate system, y values change as you move up the graph. An example of this coordinate system is shown below in Figure 3:

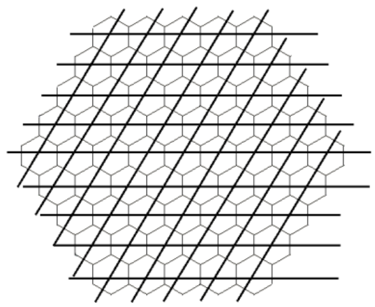

Figure 3

Lemmas and Theorem

Lemma 1: *Any queen on spot (a, b) threatens spot (b, a).*

Proof: Queen on (a, b) threatens at least all spots in the line y = a + b - x. Since (b, a) also satisfies this equation, we see that any queen which threatens a point (a, b) also threatens its symmetric point (b, a). Note that the points (a, b) and (b, a) are reflections across the line y = x, highlighted in green below.

If we set (x, y) = (a, b), we get a = a + b - b. Therefore, a queens on (a, b) always threatens the spot (b, a) Looking below in Figure 4 , we can also see that for any point (a, b) we choose, its symmetric point exists on the board.

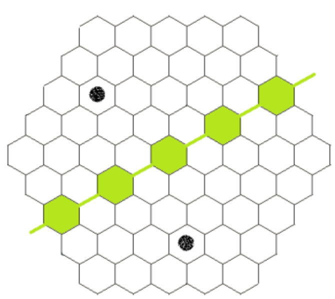

Figure 4

Theorem 1: Let T be the upper two squares, illustrated by the white region below. Then, the remainder of the graph (excluding T), can be covered by filling the base of the hexagon with queens (placing a queen on every spot in the line y= 2n-1). The red dots on the graph below in Figure 5 show a sample set up as described above that covers everything except the two squares at the top of the graph.

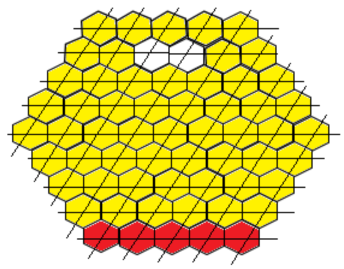

Figure 5

Queens on 3, 4, 5, 6, 8 Hexagonal Chess Boards

Based on the squares that a queen on a hexagonal board can dominate, we see that a 3 board can be dominated with one queen. In the board below in Figure 6, the black square represents the queen and the red squares show which squares the queen dominates.

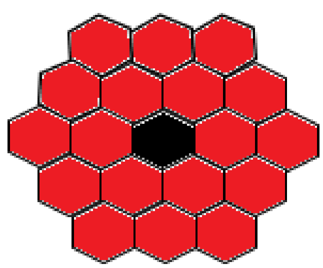

Figure 6

Before figuring out the domination number for the 5 board, we find the domination number for the 4 board. We notice that the 4 board is the 3 board, but with an additional outside ring, as shown below in Figure 7:

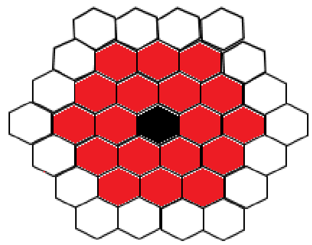

Figure 7

We try adding one more queen in an attempt to dominate all of the squares of the outside ring, as shown below in Figure 8.

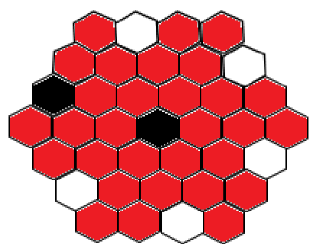

Figure 8

Since all of the squares are not covered so we add one more queen, as shown below in Figure 9.

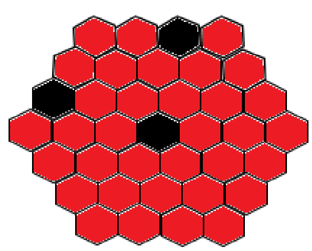

Figure 9

Since all of the squares are covered, the minimum number of queens required to dominate the 4 board is 3.

Figure 10 is an example of the two yellow squares that are not covered in the 5-hexagon game when queens are placed on the bottom row. The red spaces represent spaces currently dominated by the queens (black dots).

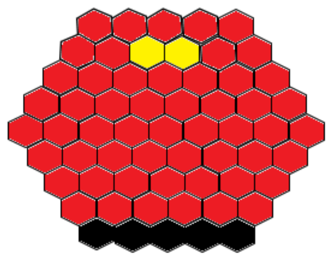

Figure 10

We notice that we need to add another queen in order to cover the two yellow squares, as shown below in Figure 11:

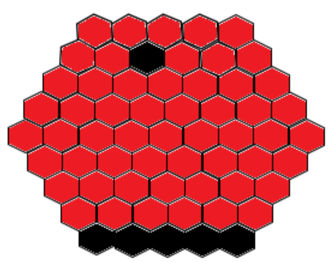

Figure 11

In the board above, 6 queens are needed to fully dominate the board. We investigate other placements of queens to see if we can minimize the number of queens to dominate the board. Below is an example of the domination of the board using 5 queens in Figure 12:

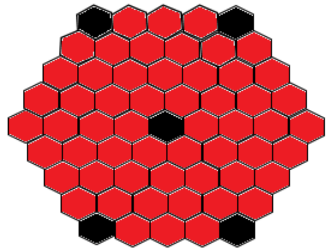

Figure 12

We now try dominating the board with 4 queens, as shown below in Figure 13:

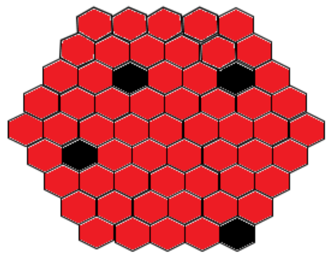

Figure 13

Thus, we know the minimum number of queens to dominate a 5 by 5 hexagonal board is at least 4. We now try dominating the board with 3 queens as shown in Figure 14.

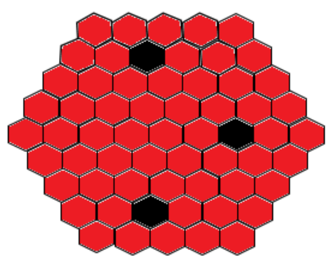

Figure 14

In order to prove that the minimum number of queens required to dominate the 5 by 5 board is 3, we must show that we cannot dominate the board with 2 queens in Figure 15.

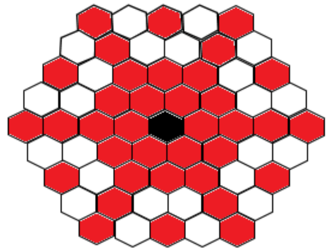

Figure 15

We see that there are 6 groups of 4 squares that are not dominated by the queen in the center of the board. We place another queen anywhere else on the board, as shown below in Figure 16:

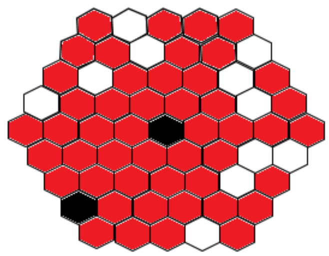

Figure 16

A single queen, placed on any square in the board, cannot cover all of the 6 groups of 4 squares. Therefore, 3 queens are needed to dominate a 5 board in Figure 16.

We now try the same technique with the 6 by 6 board. Figure 17 is an example of the four yellow squares that are not covered in the 6-hexagon game when queens are placed on the bottom row. The red spaces represent spaces currently dominated by the queens (black dots).

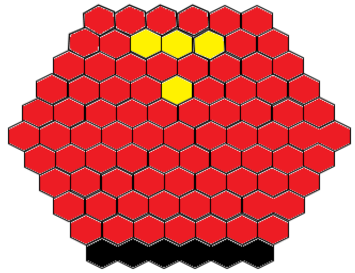

Figure 17

We notice that we need to add another queen in order to cover the four yellow squares, as shown below in Figure 18:

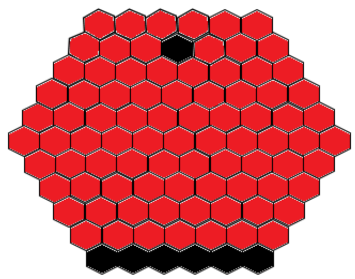

Figure 18

In the board above, 7 queens are needed to fully dominate the board. We investigate other placements of queens to see if we can minimize the number of queens to dominate the board. Figure 19 is an example of the domination of the board using 6 queens:

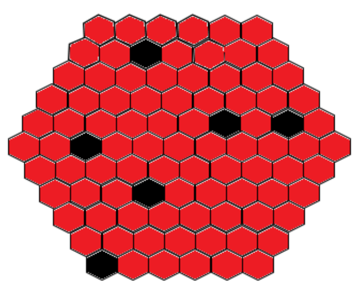

Figure 19

In the board above, 6 queens are needed to fully dominate the board. We investigate other placements of queens to see if we can minimize the number of queens to dominate the board. Figure 20 is an example of the domination of the board using 5 queens:

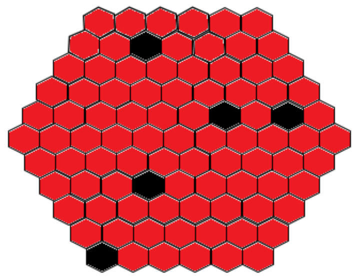

Figure 20

We now try dominating the board with 4 queens:

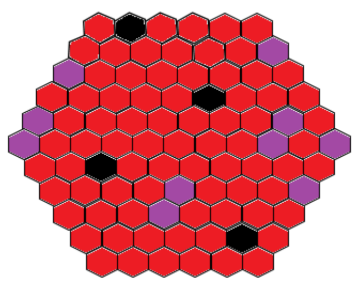

Figure 21

We see that there are 10 purple squares that are not covered by the 4 queens. Thus, 4 queens cannot fully dominate a 6 by 6 board. We can conclude that the minimum number of queens required to dominate the 6 by 6 board is 5.

We now try the same technique with the 8 by 8 board. The following board in Figure 22 shows the board dominated using 7 queens.

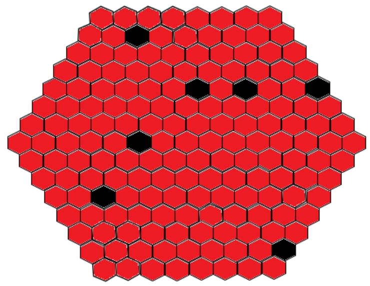

Figure 22

We now try dominating the board with 6 queens, as shown below in Figure 23:

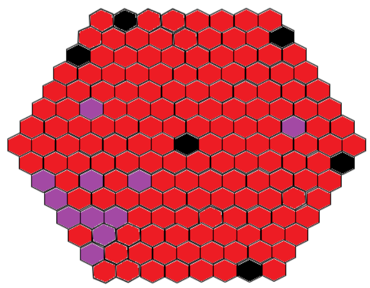

Figure 23

There are 12 squares that are not covered by the 6 queens. We can conclude that the minimum number of queens required to dominate the 8 board is 7.

n (number of hexagons in bottom row)	3	4	5	6	8
$\gamma(Q_n)$	1	3	3	5	7

From these 3 examples, we can come up with a formula for the domination number for any n board, where n is the number of hexagons on the bottom row of the board. If n is even, the dominating number is n – 1. If n is odd, the dominating number is n – 2.

While I was not able to determine a generic proof for the dominating number of any n board, I am able to prove that the dominating number for a board with a diagonal 4K + 3, where k is any positive integer, is 2K + 1.