

A model for the ∞ -category of stratified spaces

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Abstract

Let P be a poset. In this note we define a combinatorial simplicial model structure on the category of simplicial sets over the nerve of P whose underlying ∞ -category is the ∞ -category of P -stratified spaces considered in [4].

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Overview

Let P be a poset. In recent work with Clark Barwick and Saul Glasman [4], we introduced an ∞ -category \mathbf{Str}_P of P -stratified spaces [4, Definition 2.1.1]. The ∞ -category \mathbf{Str}_P is the full subcategory of the overcategory $\mathbf{Cat}_{\infty/P}$ spanned by those ∞ -categories C over P such that the structure morphism $C \rightarrow P$ is a conservative functor, or, equivalently, such that each fiber of the structure morphism $C \rightarrow P$ is an ∞ -groupoid (i.e., a *space*). Since the ∞ -category \mathbf{Str}_P of P -stratified spaces is presentable, \mathbf{Str}_P can be presented as the underlying ∞ -category of *some* combinatorial simplicial model category [HTT,

Proposition A.3.7.6]. The purpose of this note is to make the work of [4] more accessible to readers more familiar with model-categorical techniques by providing an explicit presentation of the ∞ -category \mathbf{Str}_P as the underlying ∞ -category of a combinatorial simplicial model structure on the category $s\mathbf{Set}_{/P}$ of simplicial sets over (the nerve of) P .

It is not difficult to define a model structure on $s\mathbf{Set}_{/P}$ whose fibrant objects are ∞ -categories with a conservative functor to P : we take the left Bousfield localization of the Joyal model structure inherited on the overcategory $s\mathbf{Set}_{/P}$ that inverts all simplicial homotopies $X \times \Delta^1 \rightarrow Y$ respecting the stratifications of X and Y by P . We call the resulting model structure on $s\mathbf{Set}_{/P}$ the *Joyal–Kan model structure* (see § 1 for a precise definition). The Joyal model structure is *not* simplicial: the simplicial set of maps between two ∞ -categories is an ∞ -category, but not generally an ∞ -groupoid. On the other hand, if C and D are ∞ -categories with conservative functors to P , then every natural transformation between functors $C \rightarrow D$ over P is necessarily invertible, and one might hope that the Joyal–Kan model structure actually *is* simplicial. We prove exactly this; the following two theorems are the main results of this note.

A Theorem. *Let P be a poset.*

- (A.1) *There exists a left proper combinatorial model structure on the overcategory $s\mathbf{Set}_{/P}$ called the Joyal–Kan model structure with cofibrations monomorphisms and fibrant objects the ∞ -categories C over P such that the structure morphism $C \rightarrow P$ is a conservative functor (Propositions 1.5 and 1.10).*
- (A.2) *If $f : X \rightarrow Y$ a morphism in $s\mathbf{Set}_{/P}$ and all of the fibers of X and Y over points of P are ∞ -groupoids (e.g., X and Y are fibrant objects), then f is an equivalence in the Joyal–Kan model structure if and only if f is an equivalence in the Joyal model structure (Proposition 3.12).*
- (A.3) *Weak equivalences in the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ are stable under filtered colimits (Proposition 3.17).*
- (A.4) *The Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ is simplicial (Theorem 3.18).*

B Theorem (Corollary 3.19). *Let P be a poset. Then the underlying ∞ -category of the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ is the ∞ -category \mathbf{Str}_P of P -stratified spaces of [4, Definition 2.1.1]. That is, we have a canonical equivalence of ∞ -categories*

$$N_\Delta(s\mathbf{Set}_{/P}^\circ) \simeq \mathbf{Str}_P,$$

where N_Δ denotes the simplicial nerve and $s\mathbf{Set}_{/P}^\circ$ is the fibrant simplicial category given by the full subcategory of $s\mathbf{Set}_{/P}$ spanned by the fibrant objects in the Joyal–Kan model structure.

In § 1 we define and develop the basics of the Joyal–Kan model structure. Sections 2 and 3 build up the machinery to prove Theorems A and B. In § 4 we explain how equivalences in the ∞ -category \mathbf{Str}_P are detected on the level of stata and links, and what this means for P -stratified topological spaces. This provides a variant of a result of David Miller [16, Theorem 6.10; 17, Theorem 6.3].

Relation to other work

Recently, a number of approaches have been developed to study the homotopy theory of stratified spaces. First, in his thesis André Henriques conjectured that one should be able to define a model structure on P -stratified simplicial sets [11, §4.7]. He later wrote a note [12] defining a model structure on P -stratified simplicial sets and proving that it is Quillen equivalent to the projective model structure (with respect to the Kan model structure on $s\mathbf{Set}$) on the category $\mathrm{Fun}(\mathrm{sd}(P)^{op}, s\mathbf{Set})$ of simplicial presheaves on the subdivision of P (cf. [4, §4]).

Sylvain Douteau [6] also recently defined a model structure on $s\mathbf{Set}_{/P}$ to study stratified homotopy theory. He also explores this model structure extensively in his thesis [7]. Douteau’s model structure coincides with Henriques’ model structure on $s\mathbf{Set}_{/P}$: this is an immediate consequence of the fact that the cofibrations and fibrant objects in the two model structures coincide [14, Proposition E.1.10; 20, Theorem 15.3.1]. The Joyal–Kan model structure is a left Bousfield localization of the Henriques–Douteau model structure, and is generally not equivalent to the Henriques–Douteau model structure (cf. [7, Proposition 9.3.6]).

After introducing more terminology, we will discuss the relationship between our work and the topological approaches to stratified homotopy theory of David Ayala, John Francis, and Nick Rozenblyum (Remark 1.21) as well as Stephen Nand-Lal and Jon Woolf (Remark 1.22).

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o Terminology & notations

o.1. We use the language and tools of higher category theory, particularly in the model of *quasicategories*, as defined by Boardman–Vogt and developed by Joyal and Lurie [HTT; HA].

- We write $s\mathbf{Set}$ for the category of simplicial sets and $\mathrm{Map}: s\mathbf{Set}^{op} \times s\mathbf{Set} \rightarrow s\mathbf{Set}$ for the internal-Hom in simplicial sets.
- To avoid confusion, we call weak equivalences in the Joyal model structure on $s\mathbf{Set}$ *Joyal equivalences* and we call weak equivalences in the Kan model structure on $s\mathbf{Set}$ *Kan equivalences*.
- We write $s\mathbf{Set}^{Joyal}$ for the model category of simplicial sets in the Joyal model structure.
- The *underlying quasicategory* of a simplicial model category \mathbf{A} is the simplicial nerve $N_\Delta(\mathbf{A}^\circ)$ of the full subcategory $\mathbf{A}^\circ \subset \mathbf{A}$ spanned by the fibrant-cofibrant objects (which forms a fibrant simplicial category).

► If C is an ordinary category, we simply write $C \in s\mathbf{Set}$ for its nerve.

0.2 Definition. Let P be a poset. The category of *P -stratified simplicial sets* is the over-category $s\mathbf{Set}_{/P}$ of simplicial sets over (the nerve of) P .

Given a P -stratified simplicial set $f : X \rightarrow P$ and point $p \in P$, we write $X_p := f^{-1}(p)$ for the p^{th} *stratum* of X .

1 The Joyal–Kan model structure

In this section we define a *Joyal–Kan* model structure on simplicial sets stratified over a poset P by taking the left Bousfield localization of the Joyal model structure that inverts those simplicial homotopies $X \times \Delta^1 \rightarrow Y$ over P respecting stratifications.

1.1 Notation. Let P be a poset. Write E_P for the set of morphisms in $s\mathbf{Set}_{/P}$ consisting of the endpoint inclusions $\Delta^{\{0\}}, \Delta^{\{1\}} \subset \Delta^1$ over P for which the stratification $f : \Delta^1 \rightarrow P$ is constant.

1.2 Definition. Let P be a poset. The *Joyal–Kan model structure* on $s\mathbf{Set}_{/P}$ is the $s\mathbf{Set}^{\text{Joyal}}$ -enriched left Bousfield localization of the Joyal model structure on $s\mathbf{Set}_{/P}$ with respect to the set E_P .

We now proceed to verify that the Joyal–Kan model structure exists as well as explore its basic properties. First we fix some notation.

1.3 Notation. Let P be a poset. Write $\times_P - : s\mathbf{Set}_{/P} \times s\mathbf{Set} \rightarrow s\mathbf{Set}_{/P}$ for the standard tensoring of $s\mathbf{Set}_{/P}$ over $s\mathbf{Set}$, defined on objects by sending an object $X \in s\mathbf{Set}_{/P}$ and a simplicial set $K \in s\mathbf{Set}$ to the product $X \times_P K := X \times K$ in $s\mathbf{Set}$, where the structure morphism $X \times_P K \rightarrow P$ is given by the composite

$$X \times K \xrightarrow{\text{pr}_1} X \longrightarrow P.$$

The assignment on morphisms is the obvious one. When unambiguous, we write \times rather than \times_P , leaving the poset P implicit.

We write $\text{Map}_{/P} : s\mathbf{Set}_{/P}^{\text{op}} \times s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}$ for the standard simplicial enrichment, whose assignment on objects is given by

$$\text{Map}_{/P}(X, Y) := s\mathbf{Set}_{/P}(X \times_P \Delta^*, Y),$$

and the assignment on morphisms is the obvious one.

1.4 Remark. Let P be a poset, $i : X \rightarrow Y$ a morphism in $s\mathbf{Set}_{/P}$, and $j : A \rightarrow B$ a morphism of simplicial sets. Then on underlying simplicial sets, the pushout-tensor

$$i \hat{\times} j : (X \rtimes B) \sqcup^{X \rtimes A} (Y \rtimes A) \rightarrow Y \rtimes B$$

is simply the pushout-product

$$i \hat{\times} j : (X \times B) \sqcup^{X \times A} (Y \times A) \rightarrow Y \times B$$

in $s\mathbf{Set}$. Since the pushout-product of monomorphisms in $s\mathbf{Set}$ is a monomorphism and the forgetful functor $s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}$ detects monomorphisms, if i and j are monomorphisms, then $i \hat{\times} j$ is a monomorphism.

Since the Joyal model structure on $s\mathbf{Set}_{/P}$ is $s\mathbf{Set}^{\text{Joyal}}$ -enriched, a direct application of [2, Theorems 4.7 & 4.46] shows that the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ exists and satisfies the expected properties which we summarize in **Proposition 1.5**.

1.5 Proposition. *Let P be a poset. The Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ exists and satisfies the following properties.*

- (1.5.1) *The Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ is combinatorial.*
- (1.5.2) *The Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ is $s\mathbf{Set}^{\text{Joyal}}$ -enriched.*
- (1.5.3) *The cofibrations of in the Joyal–Kan model structure are precisely the monomorphisms of simplicial sets; in particular, the Joyal–Kan model structure is left proper.*
- (1.5.4) *The fibrant objects in the Joyal–Kan model structure are precisely the fibrant objects in the Joyal model structure on $s\mathbf{Set}_{/P}$ that are also E_P -local.*
- (1.5.5) *The weak equivalences in the Joyal–Kan model structure are the E_P -local weak equivalences.*

1.6 Remark. When $P = *$ is the terminal poset, the Joyal–Kan model structure on $s\mathbf{Set} = s\mathbf{Set}_{/*}$ coincides with the Kan model structure.

Fibrant objects in the Joyal–Kan model structure

We now identify the fibrant objects in the Joyal–Kan model structure.

1.7 Recollection. By [HTT, Corollary 2.4.6.5] if C is a quasicategory, then a morphism of simplicial sets $f: X \rightarrow C$ is a fibration in the Joyal model structure on $s\mathbf{Set}$ if and only if the following conditions are satisfied:

- (1.7.1) The morphism f is an inner fibration.
- (1.7.2) For every equivalence $e: c \simeq c'$ in C and object $\tilde{c} \in X$ such that $f(\tilde{c}) = c$, there exists an equivalence $\tilde{e}: \tilde{c} \simeq \tilde{c}'$ in X such that $f(\tilde{e}) = e$.

A morphism of simplicial sets satisfying (1.7.1) and (1.7.2) is called an **isofibration**. (See also [5, §2].)

We make use of the following obvious fact.

1.8 Lemma. *Let C be a quasicategory whose equivalences are precisely the degenerate edges (e.g., a poset). Then a morphism of simplicial sets $f: X \rightarrow C$ is an isofibration if and only if f is an inner fibration.*

1.9 Proposition. *Let P be a poset. An object X of $s\mathbf{Set}_{/P}$ is fibrant in the Joyal–Kan model structure if and only if the structure morphism $X \rightarrow P$ is an inner fibration and for every $p \in P$ the stratum X_p is a Kan complex.*

Proof. Since the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ is the left Bousfield localization of the Joyal model structure on $s\mathbf{Set}_{/P}$ with respect to E_P , the fibrant objects in the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ are the fibrant objects in the Joyal model structure on $s\mathbf{Set}_{/P}$ that are also E_P -local. An object $X \in s\mathbf{Set}_{/P}$ is fibrant in the Joyal model structure if and only if the structure morphism $X \rightarrow P$ is an isofibration, or, equivalently the structure morphism $X \rightarrow P$ is an inner fibration ([Lemma 1.8](#)).

Now we analyze the E_P -locality condition. A Joyal-fibrant object $X \in s\mathbf{Set}_{/P}$ is E_P -local if and only if for every 1-simplex $\sigma: \Delta^1 \rightarrow P$ such that $\sigma(0) = \sigma(1)$, evaluation morphisms

$$\mathrm{ev}_i: \mathrm{Map}_{/P}(\Delta^1, X) \rightarrow \mathrm{Map}_{/P}(\Delta^{[i]}, X)$$

for $i = 0, 1$ are isomorphisms in the homotopy category of $s\mathbf{Set}^{Joy}$. Let $p \in P$ be such that $\sigma(0) = \sigma(1) = p$. Then

$$\mathrm{Map}_{/P}(\Delta^1, X) \cong \mathrm{Map}(\Delta^1, X_p)$$

and

$$\mathrm{Map}_{/P}(\Delta^{[i]}, X) \cong \mathrm{Map}(\Delta^{[i]}, X_p),$$

for $i = 0, 1$. Under these identifications, the evaluation morphisms

$$\mathrm{ev}_i: \mathrm{Map}_{/P}(\Delta^1, X) \rightarrow \mathrm{Map}_{/P}(\Delta^{[i]}, X)$$

are identified with the evaluation morphisms

$$\mathrm{ev}_i: \mathrm{Map}(\Delta^1, X_p) \rightarrow \mathrm{Map}(\Delta^{[i]}, X_p) \cong X_p,$$

for $i = 0, 1$. Since the strata of X are quasicategories, X is E_P -local if and only if for every $p \in P$, the evaluation morphisms

$$\mathrm{ev}_i: \mathrm{Map}(\Delta^1, X_p) \rightarrow \mathrm{Map}(\Delta^{[i]}, X_p) \cong X_p,$$

for $i = 0, 1$, are Joyal equivalences. To conclude, recall that for a quasicategory C , the evaluation morphisms $\mathrm{ev}_0, \mathrm{ev}_1: \mathrm{Map}(\Delta^1, C) \rightarrow C$ are Joyal equivalences if and only if C is a Kan complex. \square

Combining [Proposition 1.9](#) with [\[HTT, Proposition 2.3.1.5\]](#) we deduce:

1.10 Proposition. *Let P be a poset, X a simplicial set, and $f: X \rightarrow P$ a morphism of simplicial sets. The following are equivalent:*

- (1.10.1) *The object $f: X \rightarrow P$ of $s\mathbf{Set}_{/P}$ is fibrant in the Joyal–Kan model structure.*
- (1.10.2) *The morphism $f: X \rightarrow P$ is an inner fibration with all fibers Kan complexes.*
- (1.10.3) *The simplicial set X is a quasicategory and all of the fibers of f are Kan complexes.*
- (1.10.4) *The simplicial set X is a quasicategory and f is a conservative functor between quasicategories.*

A number of facts are now immediate.

1.11 Corollary. *Let P be a poset. The set of equivalences between two fibrant objects in the Joyal–Kan model structure on $\mathbf{sSet}_{/P}$ is the set of Joyal equivalences over P , i.e., fully fully faithful and essentially surjective functors over P .*

1.12 Corollary. *Let P be a poset. A morphism in $\mathbf{sSet}_{/P}$ between fibrant objects in the Joyal–Kan model structure is conservative functor.*

Proof. Note that if a composite functor gf is conservative and g is conservative, then f is conservative. \square

Relation to P -stratified topological spaces

Now we examine when the P -stratified simplicial set associated to a P -stratified topological space is fibrant.

1.13 Recollection. A poset P has a natural topology, the *Alexandroff topology*, in which a subset $U \subset P$ is open if and only if $x \in U$ and $y \geq x$ implies that $y \in U$. We write $A(P)$ for the set P equipped with the Alexandroff topology.

1.14. To fix a convenient category of topological spaces, we write \mathbf{Top} for the category of *numerically generated* topological spaces (also called Δ -generated or I -generated topological spaces) [8; 9; 10, §3; 21]. For the present work, the category of numerically generated topological spaces is preferable to the more standard category of compactly generated weakly Hausdorff topological spaces [15, Chapter 5] because any poset in the Alexandroff topology numerically generated, whereas a poset is weakly Hausdorff if and only if it is discrete. (This point is not important for the present section, but is relevant in §4.)

1.15 Recollection ([HA, §A.6]). The category of P -stratified topological spaces is the overcategory

$$\mathbf{Top}_{/P} := \mathbf{Top}_{/A(P)}.$$

There is a natural stratification $\pi_P : |P| \rightarrow A(P)$ of the geometric realization of (the nerve of) P by the Alexandroff space $A(P)$. Post-composition with π_P defines a left adjoint functor $|-|_P : \mathbf{sSet}_{/P} \rightarrow \mathbf{Top}_{/P}$ with right adjoint $\mathrm{Sing}^P : \mathbf{Top}_{/P} \rightarrow \mathbf{sSet}_{/P}$ computed by the pullback of simplicial sets

$$\mathrm{Sing}^P(T) := P \times_{\mathrm{Sing}(A(P))} \mathrm{Sing}(T),$$

where the morphism $P \rightarrow \mathrm{Sing}(A(P))$ is adjoint to π_P .

1.16. Let $f : T \rightarrow A(P)$ be a P -stratified topological space. Then for each $p \in P$, the stratum $\mathrm{Sing}^P(T)_p$ is isomorphic to the Kan complex $\mathrm{Sing}(f^{-1}(p))$.

1.17 Corollary. *Let P be a poset. If $T \in \mathbf{Top}_{/P}$ is a P -stratified topological space, then the simplicial set $\mathrm{Sing}^P(T)$ is a fibrant object in the Joyal–Kan model structure on $\mathbf{sSet}_{/P}$ if and only if $\mathrm{Sing}^P(T)$ is a quasicategory.*

Proof. Combine Proposition 1.10 and (1.16). \square

Lurie proves [HA, Theorem A.6.4] that if $T \in \mathbf{Top}_{/P}$ is *conically stratified*¹, then the simplicial set $\mathrm{Sing}^P(T)$ is a quasicategory. Hence we deduce:

1.18 Corollary. *Let P be a poset. If $T \in \mathbf{Top}_{/P}$ is a conically stratified topological space, then the simplicial set $\mathrm{Sing}^P(T)$ is a fibrant object in the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$.*

Not all stratified topological spaces — even those arising as geometric realizations of quasicategories — are conically stratified.

1.19 Example. Stratify the quasicategory Λ_0^2 over $[1]$ via the map sending 0 and 1 to 0 and 2 to 1. The $[1]$ -stratified topological space $|\Lambda_0^2|_{[1]}$ is not conically stratified. Moreover, the $[1]$ -stratified simplicial set $\mathrm{Sing}^{[1]}|\Lambda_0^2|_{[1]}$ is not a quasicategory.

1.20 Warning. Example 1.19 shows that unlike the Kan model structure (i.e., the Joyal–Kan model structure where $P = *$), if P is a non-discrete poset, the functor $\mathrm{Sing}^P|-|_P$ is *not* a fibrant replacement for the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$.

1.21 Remark (on the work of Ayala–Francis–Rozenblyum). Ayala, Francis, and Rozenblyum introduced a theory of *conically smooth stratified spaces* [1, §1]. Their results [1, Lemma 3.3.9 & Theorem 4.2.8] show that the assignment $X \mapsto \mathrm{Sing}^P(X)$ defines a fully faithful functor from their ∞ -category of P -stratified spaces (obtained by localizing the category of conically smooth P -stratified spaces at the stratified homotopy equivalences [1, Theorem 2.4.5]) to the ∞ -category \mathbf{Str}_P introduced in [4, Definition 2.1.1]. The latter ∞ -category is presented by the Joyal–Kan model structure as a result of our main theorem (Theorem B).

1.22 Remark (on the work of Nand-Lal–Woelf). In his thesis, Nand-Lal extensively studied the homotopical foundations of stratified topological spaces via simplicial methods [18, Chapter 8]. Nand-Lal proved that the Joyal (equivalently, Joyal–Kan) model structure transfers along the functor Sing^P to define $s\mathbf{Set}^{\mathrm{Joy}}$ -enriched model structure on the full subcategory of $\mathbf{Top}_{/P}$ spanned by those objects T such that $\mathrm{Sing}^P(T)$ is a quasicategory [18, Theorem 8.2.3.2].²

Example 1.19 shows that the stratified geometric realization does not land in Nand-Lal’s subcategory, and a naïve comparison between the Joyal–Kan model structure and Nand-Lal’s model structure will not work. One would like to extend Nand-Lal’s model structure to all of $\mathbf{Top}_{/P}$; the many difficulties in doing so are surveyed in [18, §8.4].

Nand-Lal’s thesis is the first step in forthcoming work of Nand-Lal and Woelf to define a homotopy theory for a large class of stratified topological spaces without requiring detailed topological assumptions [19].

2 Stratified horn inclusions

In this section we identify horn inclusions in $s\mathbf{Set}_{/P}$ that are Joyal–Kan equivalences. We will use these horn inclusions in our proof that the Joyal–Kan model structure is simplicial (see §3).

¹See [HA, Definitions A.5.3 & A.5.5] for the definition of a conically stratified topological space.

²Nand-Lal uses a different convenient category of topological spaces, so more precisely he transfers the Joyal model structure along Sing^P to a subcategory his convenient category of topological spaces.

2.1 Proposition. *Let P be a poset. A horn inclusion $i: \Lambda_k^n \hookrightarrow \Delta^n$ over P stratified by a morphism $f: \Delta^n \rightarrow P$ is a Joyal–Kan equivalence in $\mathbf{sSet}_{/P}$ if and only if one of the following conditions holds:*

(2.1.1) $0 < k < n$.

(2.1.2) $k = 0$ and $f(0) = f(1)$.

(2.1.3) $k = n$ and $f(n-1) = f(n)$.

Proof. First we show that the class horn inclusions (2.1.1)–(2.1.3) are Joyal–Kan equivalences. It is clear that inner horn inclusions $\Lambda_k^n \hookrightarrow \Delta^n$ are weak equivalences in the Joyal–Kan model structure on $\mathbf{sSet}_{/P}$ as they are already weak equivalences in the Joyal model structure. If $n = 1$, then the endpoint inclusions $\Lambda_0^1, \Lambda_1^1 \hookrightarrow \Delta^1$ where $f(0) = f(1)$ are Joyal–Kan equivalences by the definition of the Joyal–Kan model structure.

Now we tackle the case of higher outer horns. We treat the case of left horns $\Lambda_0^n \hookrightarrow \Delta^n$ where the stratification $f: \Delta^n \rightarrow P$ has the property that $f(0) = f(1)$ (i.e., the class specified by (2.1.2)); the case of right horns is dual. We prove the claim by induction on n .

For the base case where $n = 2$, write D_0^2 for the (nerve of the) preorder given by $0 \leq 1 \leq 2$ along with $0 \geq 1$, and stratify D_0^2 by the unique extension of f to D_0^2 . All stratifications will be induced by f via inclusions into D_0^2 . We prove the claim by showing that the inclusions $\Lambda_0^2, \Delta^2 \hookrightarrow D_0^2$ are Joyal–Kan equivalences and conclude by the 2-of-3 property. Write E for the walking isomorphism category $0 \cong 1$ and consider the cube

$$(2.2) \quad \begin{array}{ccccc} & \Delta^{\{0\}} & \xrightarrow{\quad} & \Delta^{\{0,2\}} & \\ & \parallel \downarrow \wr & & \parallel \downarrow \wr & \\ \Delta^{\{0\}} & \xrightarrow{\quad} & \Delta^{\{0,2\}} & & \\ \wr \downarrow & & \downarrow & \wr & \downarrow \\ & \Delta^{\{0,1\}} & \xrightarrow{\quad} & \Lambda_0^2 & \\ & \wr \downarrow & \wr & \downarrow & \\ E & \xrightarrow{\quad} & L_0^2 & & \end{array},$$

where the front face is a pushout defining the simplicial set L_0^2 and the back face is a pushout square. Since $f(0) = f(1)$, the inclusion $\Delta^{\{0\}} \hookrightarrow \Delta^{\{0,1\}}$ is a trivial Joyal–Kan cofibration; the fact that the back face of (2.2) is a pushout shows that the inclusion $\Delta^{\{0,2\}} \hookrightarrow \Lambda_0^2$ is a trivial Joyal–Kan cofibration. Since the inclusion $\Delta^{\{0\}} \hookrightarrow E$ is a trivial Joyal cofibration and the front face of (2.2) is a pushout, the inclusion $\Delta^{\{0,2\}} \hookrightarrow L_0^2$ is a trivial Joyal cofibration. By the 2-of-3 property, the induced map on pushouts $\Lambda_0^2 \hookrightarrow L_0^2$ is a trivial Joyal–Kan cofibration. Similarly, the inclusion

$$\Lambda_1^2 \hookrightarrow L_1^2 := E \cup^{\Delta^{\{1\}}} \Delta^{\{1,2\}}$$

is a trivial Joyal–Kan cofibration. The inclusions $L_0^2, L_1^2 \hookrightarrow D_0^2$ are inner anodyne, so in particular the composite inclusion

$$\Lambda_0^2 \hookrightarrow L_0^2 \hookrightarrow D_0^2$$

is a trivial Joyal–Kan cofibration. Finally, to see that the inclusion $\Delta^2 \hookrightarrow D_0^2$ is a Joyal–Kan equivalence note that we have a commutative square

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{\sim} & \Delta^2 \\ \downarrow \wr & & \downarrow \\ L_1^2 & \xrightarrow{\sim} & D_0^2, \end{array}$$

where the horizontal morphisms are inner anodyne and the inclusion $\Lambda_1^2 \hookrightarrow L_1^2$ is a Joyal–Kan equivalence. This concludes the base case.

Now we prove the induction step with $n \geq 3$ and $\Lambda_0^n \hookrightarrow \Delta^n$ an outer horn inclusion over P where the stratification $f: \Delta^n \rightarrow P$ has the property that $f(0) = f(1)$. Write

$$\Lambda_{\{0,2\}}^n := \bigcup_{j \in [n] \setminus \{0,2\}} \Delta^{\{0, \dots, j-1, j+1, \dots, n\}} \subset \Delta^n$$

and note that by [3, Lemma 12.13] the inclusion $\Lambda_{\{0,2\}}^n \hookrightarrow \Delta^n$ is inner anodyne. Since we have a factorization of the inclusion $\Lambda_{\{0,2\}}^n \hookrightarrow \Delta^n$ as a composite

$$\Lambda_{\{0,2\}}^n \hookrightarrow \Lambda_0^n \hookrightarrow \Delta^n,$$

the claim is equivalent to showing that the inclusion $\Lambda_{\{0,2\}}^n \hookrightarrow \Lambda_0^n$ is a Joyal–Kan equivalence in $\mathbf{sSet}_{/P}$. To see this, note that we have a pushout square in $\mathbf{sSet}_{/P}$

$$(2.3) \quad \begin{array}{ccc} \Lambda_0^{\{0,1,3,\dots,n\}} & \xrightarrow{\sim} & \Delta^{\{0,1,3,\dots,n\}} \\ \downarrow & & \downarrow \\ \Lambda_{\{0,2\}}^n & \xrightarrow{\sim} & \Lambda_0^n, \end{array}$$

where the inclusion $\Lambda_0^{\{0,1,3,\dots,n\}} \hookrightarrow \Delta^{\{0,1,3,\dots,n\}}$ is a trivial Joyal–Kan cofibration by the induction hypothesis.

Now we prove the horn inclusions given by the classes (2.1.1)–(2.1.3) are the only horn inclusions over P that are trivial Joyal–Kan cofibrations. Equivalently, if $i: \Lambda_k^n \hookrightarrow \Delta^n$ is an outer horn and either $k = 0$ and $f(0) \neq f(1)$, or $k = n$ and $f(n-1) \neq f(n)$, then i is not a Joyal–Kan equivalence. We treat the case that $k = 0$; the case that $k = n$ is dual. The cases where $n = 1$ and $n = 2$ require slightly different (but easier) arguments than when $n \geq 3$, so we tackle those first.

When $n = 1$, we need to show that the endpoint inclusion $\Delta^{\{0\}} \hookrightarrow \Delta^1$ is not a Joyal–Kan fibration, where the stratification $f: \Delta^1 \rightarrow P$ is a monomorphism. In this case, by Proposition 1.10 both $\Delta^{\{0\}}$ and Δ^1 are fibrant in the Joyal–Kan model structure, so by Corollary 1.11 we just need to check that the inclusion $i: \Delta^{\{0\}} \hookrightarrow \Delta^1$ is not a Joyal equivalence, which is clear.

For $n = 2$, note that the simplicial set Λ_0^2 is a 1-category. Since $f(0) \neq f(1)$, we have $f(0) \neq f(2)$, so the functor $f: \Lambda_0^2 \rightarrow P$ is conservative; applying Proposition 1.10 shows that Λ_0^2 is fibrant in the Joyal–Kan model structure. To see that the inclusion $\Lambda_0^2 \hookrightarrow \Delta^2$

is not a trivial Joyal–Kan cofibration, note that the lifting problem

$$\begin{array}{ccc} \Lambda_0^2 & \xlongequal{\quad} & \Lambda_0^2 \\ \downarrow & \nearrow & \downarrow f \\ \Delta^2 & \xrightarrow{f} & P \end{array}$$

does not admit a solution because the inclusion of simplicial sets $\Lambda_0^2 \hookrightarrow \Delta^2$ does not admit a retraction.

To prove the claim for $n \geq 3$, one can easily construct a 1-category $C_{0,f}^n$ along with a natural inclusion $\phi_f: \Lambda_0^n \hookrightarrow C_{0,f}^n$ that does not extend to Δ^n as follows: adjoin a new morphism $a: 1 \rightarrow n$ to Δ^n so that a and the unique morphism $1 \rightarrow n$ are equalized by the unique morphism $0 \rightarrow 1$, then formally adjoin inverses to all morphisms $i \rightarrow j$ such that $f(i) = f(j)$. The inclusion $\phi_f: \Lambda_0^n \hookrightarrow C_{0,f}^n$ is not the standard one, but one with the property that the edge $\Delta^{\{1,n\}}$ is sent to the morphism a . Thus ϕ_f does not extend to Δ^n . The morphism $f|_{\Lambda_0^n}$ extends to a stratification $\bar{f}: C_{0,f}^n \rightarrow P$ that makes $C_{0,f}^n$ a fibrant object in the Joyal–Kan model structure, and the inclusion $\Lambda_0^n \hookrightarrow \Delta^n$ is not a trivial Joyal–Kan cofibration in $s\mathbf{Set}_P$ since the lifting problem

$$\begin{array}{ccc} \Lambda_0^n & \xhookrightarrow{\phi_f} & C_{0,f}^n \\ \downarrow & \nearrow & \downarrow \bar{f} \\ \Delta^n & \xrightarrow{f} & P \end{array}$$

does not admit a solution. □

2.4 Notation. Let P be a poset. Write $J_P \subset \mathbf{Mor}(s\mathbf{Set}_P)$ for the set of all horn inclusions $i: \Lambda_k^n \hookrightarrow \Delta^n$ over P that are Joyal–Kan equivalences.

We can use the set J_P to identify fibrations between fibrant objects of the Joyal–Kan model structure on $s\mathbf{Set}_P$. First we record a convenient fact.

2.5 Lemma. *Let $f: X \rightarrow Y$ be a conservative functor between quasicategories. The following are equivalent:*

- (2.5.1) *For every equivalence $e: y \simeq y'$ in Y and object $\tilde{y} \in X$ such that $f(\tilde{y}) = y$, there exists an equivalence $\tilde{e}: \tilde{y} \simeq \tilde{y}'$ in X such that $f(\tilde{e}) = e$.*
- (2.5.2) *For every equivalence $e: y \simeq y'$ in Y and object $\tilde{y} \in X$ such that $f(\tilde{y}) = y$, there exists a morphism $\tilde{e}: \tilde{y} \rightarrow \tilde{y}'$ in X such that $f(\tilde{e}) = e$.*

2.6 Proposition. *Let P be a poset and $f: X \rightarrow Y$ a morphism in $s\mathbf{Set}_P$ between fibrant objects in the Joyal–Kan model structure. Then the following are equivalent:*

- (2.6.1) *The morphism f is a Joyal–Kan fibration.*
- (2.6.2) *The morphism f is a Joyal fibration, equivalently, an isofibration.*

- (2.6.3) *The morphism f satisfies the right lifting property with respect to J_P .*
- (2.6.4) *The morphism f is an inner fibration and the restriction of f to each stratum is a Kan fibration.*
- (2.6.5) *The morphism f is an inner fibration and satisfies the right lifting property with respect to E_P .*

Proof. The equivalence of (2.6.1) and (2.6.2) is immediate from the fact that the Joyal–Kan model structure is a left Bousfield localization of the Joyal model structure.

Now we show that (2.6.2) implies (2.6.3). Assume that f is an isofibration. Since f is an isofibration, f is an inner fibration, hence lifts against inner horns in J_P . Now consider the lifting problem

$$(2.7) \quad \begin{array}{ccc} \Delta^{\{i\}} & \xrightarrow{h} & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^1 & \xrightarrow{h'} & Y \end{array}$$

where the inclusion $\Delta^{\{i\}} \hookrightarrow \Delta^1$ is in J_P . Since Y is fibrant in the Joyal–Kan model structure, the edge $h'(\Delta^1)$ is an equivalence in Y . **Lemma 2.5** (and its dual) now shows that the lifting problem (2.7) admits a solution. Finally, if $n \geq 2$ and $k = 0$ or $k = n$, then given a lifting problem

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{h} & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \xrightarrow{h'} & Y \end{array}$$

where the horn inclusion $\Lambda_k^n \hookrightarrow \Delta^n$ is in J_P , since X and Y are fibrant in the Joyal–Kan model structure:

- If $k = 0$, then $h(\Delta^{\{0,1\}})$ and $h'(\Delta^{\{0,1\}})$ are equivalences.
- If $k = n$, then $h(\Delta^{\{n-1,n\}})$ and $h'(\Delta^{\{n-1,n\}})$ are equivalences.

In either case, the desired lift exists because f is an inner fibration and the outer horn is “special” [13, Theorem 2.2; 20, p. 236].

The fact that (2.6.3) implies (2.6.4) is obvious from the identification of J_P (**Proposition 2.1**).

The fact that (2.6.4) implies (2.6.5) is obvious from the definition of E_P and the fact that the restriction of f to each stratum is a Kan fibration.

Now we show that (2.6.5) implies (2.6.2). Assume that f is an inner fibration and satisfies the right lifting property with respect to E_P . Since f is conservative (**Corollary 1.12**) and the equivalences in Y lie in individual strata, **Lemma 2.5** combined with the fact that f satisfies the right lifting property with respect to E_P show that f is an isofibration. \square

2.8 Corollary. *Let P be a poset and X an object of $s\mathbf{Set}_{J_P}$. Then X is fibrant in the Joyal–Kan model structure if and only if the stratification $X \rightarrow P$ satisfies the right lifting property with respect to J_P .*

3 Simpliciality of the Joyal–Kan model structure

Unlike the Kan model structure on $s\mathbf{Set}$, the Joyal model structure is *not* simplicial. As a result, it does not follow formally from the definition that the Joyal–Kan model structure on $s\mathbf{Set}_P$ is simplicial. This section is dedicated to showing that, nevertheless, the Joyal–Kan model structure is simplicial ([Theorem 3.18](#)).

3.1. By appealing to [[HTT](#), Proposition A.3.1.7], we can prove that the Joyal–Kan model structure is simplicial by proving the following three claims:

(3.1.1) The collection of weak equivalences in the Joyal–Kan model structure on $s\mathbf{Set}_P$ is stable under filtered colimits.

(3.1.2) Given a monomorphism of simplicial sets $j: A \rightarrowtail B$ and a Joyal–Kan cofibration $i: X \rightarrowtail Y$ in $s\mathbf{Set}_P$, the pushout-tensor

$$i \hat{\rtimes} j: (X \rtimes B) \sqcup^{X \rtimes A} (Y \rtimes A) \rightarrow Y \rtimes B$$

is a Joyal–Kan cofibration.

(3.1.3) For every $n \geq 0$ and every object $X \in s\mathbf{Set}_P$, the natural map

$$X \rtimes \Delta^n \rightarrow X \rtimes \Delta^0 \cong X$$

is a Joyal–Kan equivalence.

Note that (3.1.2) follows from [Remark 1.4](#) and the fact that cofibrations in the Joyal–Kan model structure are monomorphisms of simplicial sets ([Proposition 1.5](#)). However, items (3.1.1) and (3.1.3) are non-obvious and occupy the rest of the section.

We first concern ourselves with (3.1.3). Since the natural map

$$X \rtimes \Delta^n \rightarrow X \rtimes \Delta^0 \cong X$$

admits a section $X \cong X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^n$, it suffices to show that this section is a trivial Joyal–Kan cofibration. In fact, we prove a more precise claim.

3.2 **Notation.** Let P be a poset.

- Write $I_P \subset J_P$ for the *inner* horn inclusions in J_P .
- Write $L_P \subset J_P$ for those horn inclusions $\Lambda_k^n \hookrightarrow \Delta^n$ in J_P where $n \geq 1$ and $0 \leq k < n$, i.e., the *left* horn inclusions in J_P .

Note that [Proposition 2.1](#) gives complete characterizations of I_P and L_P .

3.3 **Proposition.** *Let P be a poset and $n \geq 0$ an integer. For any object $X \in s\mathbf{Set}_P$, the inclusion*

$$X \cong X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^n$$

is in the weakly saturated class generated by L_P , in particular, a trivial Joyal–Kan cofibration in $s\mathbf{Set}_P$.

The next proposition (and its proof) is a stratified variant of [HTT, Propositions 2.1.2.6 & 3.1.1.5] which we use to prove [Proposition 3.3](#).

3.4 Proposition. *Let P be a poset. Consider the following classes of morphisms in $s\mathbf{Set}_P$:*

(3.4.1) *All inclusions*

$$(\partial\Delta^m \rtimes \Delta^1) \sqcup^{\partial\Delta^m \rtimes \Delta^{\{0\}}} (\Delta^m \rtimes \Delta^{\{0\}}) \hookrightarrow \Delta^m \rtimes \Delta^1,$$

where $m \geq 0$ and $\Delta^m \in s\mathbf{Set}_P$ is any m -simplex over P .

(3.4.2) *All inclusions*

$$(A \rtimes \Delta^1) \sqcup^{A \rtimes \Delta^{\{0\}}} (B \rtimes \Delta^{\{0\}}) \hookrightarrow B \rtimes \Delta^1,$$

where $A \hookrightarrow B$ is any monomorphism in $s\mathbf{Set}_P$.

The classes (3.4.1) and (3.4.2) generate the same weakly saturated class of morphisms in $s\mathbf{Set}_P$. Moreover, this weakly saturated class of morphisms generated by (3.4.1) or (3.4.2) is contained in the weakly saturated class of morphisms generated by L_P .

Proof. Since the inclusions $\partial\Delta^m \hookrightarrow \Delta^m$ in $s\mathbf{Set}_P$ generate the monomorphisms in $s\mathbf{Set}_P$, to see that each of the morphisms specified in (3.4.2) is contained in the weakly saturated class generated by (3.4.1), it suffices to work simplex-by-simplex with the inclusion $A \hookrightarrow B$. The converse is obvious since the class specified by (3.4.1) is contained in the class specified by (3.4.2).

To complete the proof, we show that for each P -stratified m -simplex $\Delta^m \in s\mathbf{Set}_P$, the inclusion

$$(3.5) \quad (\partial\Delta^m \rtimes \Delta^1) \sqcup^{\partial\Delta^m \rtimes \Delta^{\{0\}}} (\Delta^m \rtimes \Delta^{\{0\}}) \hookrightarrow \Delta^m \rtimes \Delta^1$$

belongs to the weakly saturated class generated by L_P . The proof of this is *verbatim* the same as the proof of [HTT, Proposition 2.1.2.6], which writes the inclusion (3.5) as a composite of pushouts of horn inclusions, all of which are in L_P . \square

3.6 Corollary. *Let P be a poset. For any P -stratified simplicial set $X \in s\mathbf{Set}_P$, the inclusion $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^1$ is in the weakly saturated class generated by L_P .*

Proof. In (3.4.2) set $A = \emptyset$ and $B = X$. \square

3.7 Notation. Let $n \geq 0$ be an integer. Write $\mathbf{Spn}^n \subset \Delta^n$ for the *spine* of Δ^n , defined by

$$\mathbf{Spn}^n := \Delta^{\{0,1\}} \cup^{\Delta^{\{1\}}} \dots \cup^{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}.$$

Now we use [Corollary 3.6](#) and the fact that the spine inclusion $\mathbf{Spn}^n \hookrightarrow \Delta^n$ is inner anodyne to address [Proposition 3.3](#).

3.8 Lemma. *Let P be a poset and $n \geq 0$ an integer. For any P -stratified simplicial set $X \in s\mathbf{Set}_P$, the inclusion $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \mathbf{Spn}^n$ is in the weakly saturated class generated by L_P .*

Proof. Noting that $\mathrm{Spn}^1 = \Delta^1$, factor the inclusion $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \mathrm{Spn}^n$ as a composite

$$X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^1 \hookrightarrow X \rtimes \mathrm{Spn}^2 \hookrightarrow \dots \hookrightarrow X \rtimes \mathrm{Spn}^n .$$

The inclusion $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^1$ is in the weakly saturated class generated by L_P ([Corollary 3.6](#)), so it suffices to show that for $1 \leq k \leq n-1$, the inclusion $X \rtimes \mathrm{Spn}^k \hookrightarrow X \rtimes \mathrm{Spn}^{k+1}$ is in the weakly saturated class generated by L_P . To see this, note that the inclusion $X \rtimes \mathrm{Spn}^k \hookrightarrow X \rtimes \mathrm{Spn}^{k+1}$ is given by the pushout

$$\begin{array}{ccc} X \rtimes \Delta^{\{k\}} & \hookrightarrow & X \rtimes \Delta^{\{k,k+1\}} \\ \downarrow & \lrcorner & \downarrow \\ X \rtimes \mathrm{Spn}^k & \hookrightarrow & X \rtimes \mathrm{Spn}^{k+1} , \end{array}$$

and by [Corollary 3.6](#) the inclusion $X \rtimes \Delta^{\{k\}} \hookrightarrow X \rtimes \Delta^{\{k,k+1\}}$ is in the weakly saturated class generated by L_P . \square

Proof of [Proposition 3.3](#). The inclusion $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^n$ factors as a composite

$$X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \mathrm{Spn}^n \hookrightarrow X \rtimes \Delta^n .$$

To conclude, first note that by [Lemma 3.8](#) the inclusion $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \mathrm{Spn}^n$ is in the weakly saturated class generated by L_P . Second, since the inclusion $\mathrm{Spn}^n \hookrightarrow \Delta^n$ is inner anodyne, the inclusion

$$X \times \mathrm{Spn}^n = X \rtimes \mathrm{Spn}^n \hookrightarrow X \rtimes \Delta^n = X \times \Delta^n$$

is inner anodyne [[HTT](#), Corollary 2.3.2.4], hence in the weakly saturated class generated by L_P . \square

Stability of weak equivalences under filtered colimits

In this subsection we explain how to fibrantly replace simplicial sets over P whose strata are Kan complexes without changing their strata, and use this to deduce that Joyal–Kan equivalences between such objects are Joyal equivalences ([Proposition 3.12](#)). We leverage this to show that Joyal–Kan equivalences are stable under filtered colimits ([Proposition 3.17](#)). We conclude by proving that the Joyal–Kan model structure is simplicial ([Theorem 3.18](#)), and deduce that the Joyal–Kan model structure presents the ∞ -category Str_P of P -stratified spaces ([Corollary 3.19](#)).

3.9 Notation. Let P be a poset. Write $I_P^{nv} \subset I_P$ for the inner horn inclusions $\Delta_k^n \hookrightarrow \Delta^n$ over P that are *not vertical* in the sense that the stratification $\Delta^n \rightarrow P$ is not a constant map.

3.10 Lemma. *Let $f: X \rightarrow Y$ be a morphism in $s\mathbf{Set}_{/P}$. Then there exists a commutative square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & & \downarrow j \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y}, \end{array}$$

in $s\mathbf{Set}_{/P}$ where:

- (3.10.1) *The morphisms i and j are I_P^{nv} -cell maps.*
- (3.10.2) *The morphism \tilde{f} satisfies the right lifting property with respect to I_P^{nv} .*
- (3.10.3) *The morphisms i and j restrict to isomorphism on strata, i.e., for all $p \in P$ the morphisms i and j restrict to isomorphisms of simplicial sets $i: X_p \xrightarrow{\sim} \tilde{X}_p$ and $j: Y_p \xrightarrow{\sim} \tilde{Y}_p$.*
- (3.10.4) *If, in addition, all of the strata of X and Y are quasicategories, then \tilde{X} and \tilde{Y} can be chosen to be quasicategories.*

In particular, if all of the strata of X and Y are Kan complexes, then \tilde{X} and \tilde{Y} can be chosen to be fibrant in the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$.

Proof. Since the morphisms in I_P^{nv} all have small domains, we can apply the small object argument to construct a square

$$(3.11) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & & \downarrow j \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y}, \end{array}$$

where i and j are I_P^{nv} -cell maps and \tilde{f} has the right lifting property with respect to I_P^{nv} , which proves (3.10.1) and (3.10.2).

To prove (3.10.3) we examine the constructions of \tilde{X} and \tilde{Y} via the small object argument. Both morphisms i and j are obtained by a transfinite composite of pushouts of inner horn inclusions $\Lambda_k^n \hookrightarrow \Delta^n$ in I_P^{nv} . Hence to prove (3.10.3) it suffices to show that given an object $A \in s\mathbf{Set}_{/P}$ and a morphism $f: \Lambda_k^n \rightarrow A$, where $\Lambda_k^n \in s\mathbf{Set}_{/P}$ is the domain of a morphism $g: \Lambda_k^n \hookrightarrow \Delta^n$ in I_P^{nv} , the morphism \bar{g} in the pushout square

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{f} & A \\ g \downarrow & & \downarrow \bar{g} \\ \Delta^n & \xrightarrow{\tilde{f}} & A' \end{array}$$

induces an isomorphism (of simplicial sets) on strata. To see this, let $\sigma: \Delta^n \rightarrow P$ denote the stratification of the target of g . Since $g \in I_P^{nv}$, the stratification σ is not a constant map. We claim that for all $p \in P$ and $m \geq 0$, the m -simplices of $A_p \subset A'_p$ and A'_p coincide. If $p \notin \sigma(\Delta^n)$ or $m < n - 1$, this is obvious. Let us consider the remaining cases.

- If $m = n - 1$, then note that the only additional $(n - 1)$ -simplex adjoined to A in the pushout defining A' is the image of the face $\Delta^{\{0, \dots, k-1, k+1, \dots, n\}} \subset \Delta^n$. Since the horn $\Lambda_k^n \subset \Delta^n$ is an inner horn, both vertices $\Delta^{\{0\}}$ and $\Delta^{\{n\}}$ are contained in $\Delta^{\{0, \dots, k-1, k+1, \dots, n\}}$. Since the stratification $\sigma: \Delta^n \rightarrow P$ is not constant, the image of $\Delta^{\{0, \dots, k-1, k+1, \dots, n\}}$ in A' intersects more than one stratum. Hence for each $p \in P$, the $(n - 1)$ -simplices of the strata A_p and A'_p coincide.
- If $m = n$, then note that the only additional nondegenerate n -simplex adjoined to A in the pushout defining A' is the unique nondegenerate n -simplex of Δ^n . Since the stratification $\sigma: \Delta^n \rightarrow P$ is non-constant, the image of this top-dimensional simplex under \tilde{f} intersects more than one stratum. Similarly, note that since the image of the face $\Delta^{\{0, \dots, k-1, k+1, \dots, n\}} \subset \Delta^n$ in A' intersects more than one stratum (by the previous point), all of its degeneracies intersect more than one stratum. But the image of Δ^n and images of the degeneracies of $\Delta^{\{0, \dots, k-1, k+1, \dots, n\}}$ under \tilde{f} are the only n -simplices adjoined to A in the pushout defining A' . Hence for each $p \in P$, the n -simplices of the strata A_p and A'_p coincide.
- If $m > n$, then the claim follows from the fact that the ℓ -simplices of A_p and A'_p coincide for all $\ell \leq n$ and the n -skeletality of Δ^n .

Now we prove (3.10.4); assume that the strata of X and Y are quasicategories. To see that \tilde{Y} is a quasicategory, note that by the construction of the factorization (3.11) via the small object argument, \tilde{Y} is given by factoring the unique morphism $Y \rightarrow P$ to the final object as a composite

$$Y \xrightarrow{j} \tilde{Y} \xrightarrow{h} P$$

of the I_p^{nv} -cell map j followed by a morphism h with the right lifting property with respect to I_p^{nv} . To show that \tilde{Y} is a quasicategory, we prove that h is an inner fibration. By the definition of I_p^{nv} , the morphism h lifts against all inner horns $\Lambda_k^n \hookrightarrow \Delta^n$ over P where the stratification of Δ^n is not constant. Thus to check that $h: \tilde{Y} \rightarrow P$ is an inner fibration, it suffices to check that for every inner horn $\Lambda_k^n \hookrightarrow \Delta^n$ over P where the stratification $\sigma: \Delta^n \rightarrow P$ is constant at a vertex $p \in P$, every lifting problem

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & \tilde{Y} \\ \downarrow & \nearrow & \downarrow h \\ \Delta^n & \xrightarrow{\sigma} & P \end{array}$$

admits a solution. The desired lift exists by (3.10.3) because the stratum $\tilde{Y}_p \cong Y_p$ is a quasicategory by assumption.

We conclude that \tilde{X} is a quasicategory by showing that the stratification $\tilde{X} \rightarrow P$ is an inner fibration. First, note that since $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ and $h: \tilde{Y} \rightarrow P$ have the right lifting property with respect to I_p^{nv} , so does the stratification $h\tilde{f}: \tilde{X} \rightarrow P$. Hence to show that the stratification $\tilde{X} \rightarrow P$ is an inner fibration, it suffices to show that $\tilde{X} \rightarrow P$ lifts against inner horns $\Lambda_k^n \hookrightarrow \Delta^n$ over P where the stratification $\sigma: \Delta^n \rightarrow P$ is constant. Again, the desired lift exists by (3.10.3) because the strata of \tilde{X} are quasicategories. \square

3.12 Proposition. *Let P be a poset and $f: X \rightarrow Y$ a morphism in $s\mathbf{Set}_{/P}$. If the strata of X and Y are all Kan complexes, then f is a Joyal–Kan equivalence if and only if f is a Joyal equivalence.*

Proof. Since the Joyal–Kan equivalences between fibrant objects of the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ are precisely the Joyal equivalences (Corollary 1.11), by 2-of-3 it suffices to show that there exists a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & & \downarrow j \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y}, \end{array}$$

in $s\mathbf{Set}_{/P}$, where \tilde{X} and \tilde{Y} are Joyal–Kan fibrant objects, and $i: X \simeq \tilde{X}$ and $j: Y \simeq \tilde{Y}$ are Joyal equivalences. This follows from Lemma 3.10 since I_P^{nv} -cell maps are, in particular, Joyal equivalences. \square

3.13. Since Joyal equivalences are stable under filtered colimits [HTT, Theorem 2.2.5.1 & p. 90], to show that Joyal–Kan equivalences are stable under filtered colimits, Proposition 3.12 reduces us to constructing a functor $F: s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}_{/P}$ that lands in strata-wise Kan complexes, admits a natural weak equivalence $\mathrm{id}_{s\mathbf{Set}_{/P}} \simeq F$, and preserves filtered colimits. We accomplish this by applying Kan’s Ex^∞ functor vertically to each stratum.

3.14 Construction. Let P be a poset. Define a functor $\mathrm{VEx}_P^\infty: s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}_{/P}$ by the assignment

$$X \mapsto X \sqcup^{\mathrm{Obj}(P) \times_P X} \mathrm{Ex}^\infty(\mathrm{Obj}(P) \times_P X) \cong X \sqcup^{\coprod_{p \in P} X_p} \left(\coprod_{p \in P} \mathrm{Ex}^\infty(X_p) \right),$$

where the pushout is taken in $s\mathbf{Set}_{/P}$, and the two stratifications $\coprod_{p \in P} \mathrm{Ex}^\infty(X_p) \rightarrow P$ and $\coprod_{p \in P} X_p \rightarrow P$ are induced by the constant maps $\mathrm{Ex}^\infty(X_p) \rightarrow P$ and $X_p \rightarrow P$ at $p \in P$.

We claim that the natural inclusion

$$X \hookrightarrow \mathrm{VEx}_P^\infty(X)$$

is a trivial Joyal–Kan cofibration. To see this, observe that for each $p \in P$, the inclusion $X_p \hookrightarrow \mathrm{Ex}^\infty(X_p)$ is a trivial Kan cofibration, so in the weakly saturated class generated by the horn inclusions $\Lambda_k^n \hookrightarrow \Delta^n$ in $s\mathbf{Set}$, where $n \geq 0$ and $0 \leq k \leq n$. Thus, stratifying X_p and $\mathrm{Ex}^\infty(X_p)$ via the constant maps at $p \in P$, by Proposition 2.1 the inclusion $X_p \hookrightarrow \mathrm{Ex}^\infty(X_p)$ is a trivial Joyal–Kan cofibration in $s\mathbf{Set}_{/P}$. To conclude, note that by definition the inclusion $X \hookrightarrow \mathrm{VEx}_P^\infty(X)$ is a pushout of the trivial Joyal–Kan cofibration

$$\coprod_{p \in P} X_p \hookrightarrow \coprod_{p \in P} \mathrm{Ex}^\infty(X_p).$$

In particular, note that by Proposition 3.12 a morphism $f: X \rightarrow Y$ in $s\mathbf{Set}_{/P}$ is a Joyal–Kan equivalence if and only if

$$\mathrm{VEx}_P^\infty(f): \mathrm{VEx}_P^\infty(X) \rightarrow \mathrm{VEx}_P^\infty(Y)$$

is a Joyal equivalence.

3.15 Warning. The functor VEx_P^∞ in general does *not* preserve quasicategories over P and is not a fibrant replacement for the Joyal–Kan model structure unless P is discrete.

3.16 Lemma. *Let P be a poset. Then the functor $\mathrm{VEx}_P^\infty : s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}_{/P}$ preserves filtered colimits.*

Proof. First, since filtered colimits commute with finite limits in $s\mathbf{Set}$, the functor

$$\mathrm{Obj}(P) \times_P - : s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}_{/P}$$

preserves filtered colimits. Second, since Kan’s Ex^∞ functor preserves filtered colimits, the functor $s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}_{/P}$ given by the assignment

$$X \mapsto \mathrm{Ex}^\infty(\mathrm{Obj}(P) \times_P X) \cong \coprod_{p \in P} \mathrm{Ex}^\infty(X_p)$$

preserves filtered colimits. The claim is now clear from the definition of VEx_P^∞ . \square

Combining our observation (3.13) with Lemma 3.16 we deduce:

3.17 Proposition. *For any poset P , Joyal–Kan equivalences in $s\mathbf{Set}_{/P}$ are stable under filtered colimits.*

Synthesizing the material of this section shows that the Joyal–Kan model structure is simplicial.

3.18 Theorem. *For any poset P , the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ is simplicial.*

Proof. Since $s\mathbf{Set}_{/P}$ is tensored and cotensored over $s\mathbf{Set}$ and every object is cofibrant in the Joyal–Kan model structure (Proposition 1.5), by [HTT, Proposition A.3.1.7] it suffices to show the three conditions (3.1.1)–(3.1.3).

First, (3.1.1) is the content of Proposition 3.17. Second, as noted in (3.1), (3.1.2) follows from Remark 1.4 and the fact that cofibrations in the Joyal–Kan model structure are monomorphisms of simplicial sets. Finally, (3.1.3) follows from Proposition 3.3. \square

We immediately deduce the following.

3.19 Corollary. *Let P be a poset. Then the underlying quasicategory of the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ is the quasicategory \mathbf{Str}_P of P -stratified spaces of [4, Definition 2.1.1].*

4 Checking equivalences on strata and links

In work with Barwick and Glasman we introduced a Segal space model for P -stratified spaces called *spatial décollages over P* [4, Definition 4.1.3] and proved that the ∞ -category of spatial décollages over P is equivalent to the ∞ -category \mathbf{Str}_P [4, Theorem 4.2.4]. Implicit in this equivalence of ∞ -categories is the fact that equivalences in \mathbf{Str}_P are detected on strata and *links*. This provides an ∞ -categorical variant of a result of Miller [16, Theorem 6.10; 17, Theorem 6.3]. In this section we give a direct proof of this fact and explain what this means for P -stratified topological spaces (Corollary 4.9).

4.1 Notation. We write \mathbf{Spc} for the ∞ -category of spaces (i.e., ∞ -groupoids), which may be defined as the underlying ∞ -category of the Kan model structure on $s\mathbf{Set}$.

4.2. Let P be a poset and $p \in P$. Observe that the functor $\mathbf{Str}_p \rightarrow \mathbf{Spc}$ given by taking the p^{th} stratum is corepresented by the P -stratified space $\{p\} \hookrightarrow P$.

4.3. Let P be a poset and $p < q$ elements of P . We regard $\{p < q\}$ as an object of \mathbf{Str}_p via the inclusion $\{p < q\} \hookrightarrow P$.

4.4 Definition. Let P be a poset, X an object of \mathbf{Str}_p , and $p, q \in P$ with $p < q$. The *link* from p to q in X is the ∞ -groupoid

$$\text{Link}_{p<q}(X) := \text{Map}_{\mathbf{Str}_p}(\{p < q\}, X) = \text{Map}_{/P}(\{p < q\}, X) .$$

Evaluation at p and q define source and target maps

$$\text{Link}_{p<q}(X) \rightarrow X_p \quad \text{and} \quad \text{Link}_{p<q}(X) \rightarrow X_q .$$

4.5 Proposition. Let P be a poset and let $f : X \rightarrow Y$ be a morphism \mathbf{Str}_p . Then f is an equivalence if and only if the following conditions are satisfied:

(4.5.1) For each $p \in P$, the induced map on strata $f_p : X_p \rightarrow Y_p$ is an equivalence in \mathbf{Spc} .

(4.5.2) For all $p, q \in P$ with $p < q$, the induced map on links

$$f_* : \text{Link}_{p<q}(X) \rightarrow \text{Link}_{p<q}(Y)$$

is an equivalence in \mathbf{Spc} .

Proof. Since taking strata and links define functors $\mathbf{Str}_p \rightarrow \mathbf{Spc}$, if $f : X \rightarrow Y$ is an equivalence in \mathbf{Str}_p then f induces equivalences on strata in links.

For the converse, first note that since \mathbf{Str}_p is a full subcategory of $\mathbf{Cat}_{\infty, /P}$, the morphism f is an equivalence if and only if f is fully faithful and essentially surjective. Note that since the strata of X and Y are ∞ -groupoids, the statement (4.5.1) is equivalent to the statement:

(4.5.1') The functor f is essentially surjective and for each $p \in P$ and objects $x, x' \in X_p$, the natural map

$$\text{Map}_X(x, x') = \text{Map}_{X_p}(x, x') \rightarrow \text{Map}_{Y_p}(f(x), f(x')) = \text{Map}_Y(f(x), f(x'))$$

is an equivalence in the ∞ -category of spaces.

We claim that (4.5.1) and (4.5.2) together imply the statement:

(4.5.2') For all $p, q \in P$ with $p < q$, and for every pair of objects $x_p \in X_p$ and $x_q \in X_q$, the induced map $\text{Map}_X(x_p, x_q) \rightarrow \text{Map}_Y(f(x_p), f(x_q))$ is an equivalence in \mathbf{Spc} .

This will complete the proof since (4.5.1') and (4.5.2') together say that f is fully faithful and essentially surjective. To see that (4.5.1) and (4.5.2) imply (4.5.2'), consider the cube

$$(4.6) \quad \begin{array}{ccccc} & \text{Map}_X(x_p, x_q) & \xrightarrow{\quad} & \text{Link}_{p<q}(X) & \\ & \swarrow & \downarrow \lrcorner & \swarrow f_* & \downarrow \\ \text{Map}_Y(f(x_p), f(x_q)) & \xrightarrow{\quad} & \text{Link}_{p<q}(Y) & & \\ \downarrow \lrcorner & & \downarrow & \swarrow & \downarrow \\ & \{(x_p, x_q)\} & \xrightarrow{\quad} & X_p \times X_q & \\ & \swarrow & \downarrow & \swarrow f_p \times f_q & \\ \{(f(x_p), f(x_q))\} & \xrightarrow{\quad} & Y_p \times Y_q & & \end{array},$$

where the vertical morphisms are given by projection onto source and target. If

$$f_* : \text{Link}_{p<q}(X) \rightarrow \text{Link}_{p<q}(Y)$$

is an equivalence for all $p < q$ in P , then because the front and back faces of (4.6) are pull-back squares in \mathbf{Spc} and the three diagonal morphisms $\{(x_p, x_q)\} \rightarrow \{(f(x_p), f(x_q))\}$, $f_p \times f_q$, and f_* are equivalences in \mathbf{Spc} , the natural morphism

$$\text{Map}_X(x_p, x_q) \rightarrow \text{Map}_Y(f(x_p), f(x_q))$$

is an equivalence. This verifies (4.5.2'). \square

4.7 Definition. Let P be a poset, T a P -stratified topological space (Recollection 1.15), and $p, q \in P$ with $p < q$. The (*topological*) *link* from p to q in T is the topological space

$$\text{TLink}_{p<q}(T) := \text{Map}_{\mathbf{Top}_{/P}}(|\{p < q\}|_P, T),$$

where $\text{Map}_{\mathbf{Top}_{/P}}$ denotes the topological space of maps in $\mathbf{Top}_{/P}$.

4.8 Remark. Let T be a P -stratified topological space. Then we have an isomorphism of simplicial sets

$$\text{Sing}(\text{TLink}_{p<q}(T)) \cong \text{Map}_{/P}(\{p < q\}, \text{Sing}^P(T)).$$

This follows easily by transposing across a few adjunctions:

$$\begin{aligned} \text{Sing}(\text{TLink}_{p<q}(T))_n &= \mathbf{Top}(|\Delta^n|, \text{Map}_{\mathbf{Top}_{/P}}(|\{p < q\}|_P, T)) \\ &\cong \mathbf{Top}_{/P}(|\{p < q\}|_P \rtimes |\Delta^n|, T) \\ &\cong \mathbf{Top}_{/P}(|\{p < q\} \rtimes \Delta^n|_P, T) \\ &\cong s\mathbf{Set}_{/P}(\{p < q\} \rtimes \Delta^n, \text{Sing}^P(T)) \\ &= \text{Map}_{/P}(\{p < q\}, \text{Sing}^P(T))_n \end{aligned}$$

In particular, if $\text{Sing}^P(T)$ is a quasicategory (so that $\text{Sing}^P(T)$ defines an object of \mathbf{Str}_P), then we have a canonical equivalence

$$\text{Sing}(\text{TLink}_{p<q}(T)) \simeq \text{Link}_{p<q}(\text{Sing}^P(T)).$$

Since the Joyal–Kan model structure on $s\mathbf{Set}_{/P}$ presents the ∞ -category \mathbf{Str}_P (Corollary 3.19), we deduce the following.

4.9 Corollary. *Let P be a poset and let $f : S \rightarrow T$ be a morphism in $\mathbf{Top}_{/P}$. If $\mathrm{Sing}^P(S)$ and $\mathrm{Sing}^P(T)$ are quasicategories, then the morphism $\mathrm{Sing}^P(f)$ is an equivalence in when regarded as a morphism in \mathbf{Str}_P if and only if the following conditions are satisfied:*

(4.9.1) *For each $p \in P$, the induced map on strata $f_p : S_p \rightarrow T_p$ is a weak homotopy equivalence.*

(4.9.2) *For all $p, q \in P$ with $p < q$, the induced map on topological links*

$$f_* : \mathrm{TLink}_{p < q}(S) \rightarrow \mathrm{TLink}_{p < q}(T)$$

is a weak homotopy equivalence.

Proof. Combine (1.16), Remark 4.8, and Proposition 4.5 with the fact that a map g of topological spaces is a weak homotopy equivalence if and only if $\mathrm{Sing}(g)$ is an equivalence when regarded as a morphism in the ∞ -category \mathbf{Spc} of spaces. \square

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