CS166 WI24: Homework 5 (Due Friday March 1 11:59pm)

※ Problem 1: n-qubit Hadamard

In lecture, we encountered the following identity for the n-qubit Hadamard.

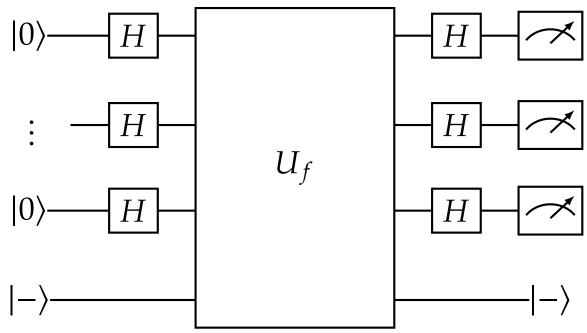
Proposition 1.1 (n-qubit Hadamard)**.** Let $x = x_1 x_2 \cdots x_n$ be the binary expansion of x. In other words, x_i is the i-th bit of x when x is written in binary. Then, we have the following identity:

$$
\begin{aligned}
H^{\otimes n} |x\rangle &= H|x_1\rangle \otimes H|x_2\rangle \otimes \cdots \otimes H|x_n\rangle && (1)\\
&= \frac{(|0\rangle + (-1)^{x_1}|1\rangle)}{\sqrt{2}} \otimes \cdots \otimes \frac{(|0\rangle + (-1)^{x_n}|1\rangle)}{\sqrt{2}} && (2)\\
&= \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle && (3)
\end{aligned}
$$

where $x \cdot y$ is the bit wise dot product of x and y (i.e., $x \cdot y = x_1 y_1 + \cdots + x_n y_n$).

1. Consider the case where $n = 3$, and $x = 101$.
   (a) Write down equations (1), and (2) for this case explicitly.
   (b) Distribute the tensor product you have from equation (2) and verify that each coefficient matches equation (3).
2. Consider the case where $n = 4$, and $x = 0000$.
   (a) What is equation (2) in this instance?
   (b) What is the probability that we measure 0010 if we measure the four qubits in the standard basis?
   (c) More generally, what can we say about the distribution of outputs when we measure this state in the standard basis?
3. Prove the proposition.

※ Problem 2: Deutsch-Josza Explicit Example



Let f be a function that takes 2 bits as inputs and outputs a single bit. The function takes the two bits it received, adds them all together, and outputs the answer mod 2.

1. Is this function constant or balanced?
2. What is $U_f\,|110\rangle\,|-\rangle$?
3. What is the state of the algorithm after the U_f gate is applied?
4. What is the state of the algorithm after the second layer of H gates? Don't use summation notation, and explicitly write out the coefficients.
5. What are the possible measurement results and their corresponding probabilities?

※ Problem 3: Simon's problem example

The function $f : \{0,1\}^3 \to \{0,1\}^2$ is defined as follows:

$$f(000) = 11 f(100) = 01 \tag{4}$$
$$f(001) = 01 f(101) = 11 \tag{5}$$
$$f(010) = 00 f(110) = 10 \tag{6}$$
$$f(011) = 10 f(111) = 00 \tag{7}$$
$$\tag{8}$$

1. The inputs are paired so that for $x \neq x'$, $f(x) = f(x')$ if and only if $x = x' \oplus a$ for some fixed a. Can you figure out a by inspection?

2. In this question, you will simulate Simon's algorithm for the function f. The unitary operator U_f maps $|x\rangle |y\rangle$ to $|x\rangle |y \oplus f(x)\rangle$, where x is a 3-bit string and y is a 2-bit string. The operator $\oplus$ is bit-wise addition, mod 2. Write down the state of the algorithm after each step.

   - Start with $|000\rangle |00\rangle$.
   - Apply $H^{\otimes 3} \otimes I^{\otimes 2}$.
   - Apply U_f.
   - Measure the last two qubits.
   - Drop the last two qubits and apply $H^{\otimes 3}$.
   - Measure the first three qubits.

3. Verify that a string x is a possible outcome of the last measurement if and only if $x \cdot a = 0 \bmod 2$.