# General Form of Ordinary Differential Equations Suitable for Numerical Solution

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## Introduction

Ordinary Differential Equations (ODEs) are equations involving one or more functions and their derivatives. Many physical and engineering problems can be described using ODEs. Often, analytical solutions are not possible, and we must resort to numerical techniques. This document introduces the general form of ODEs suitable for numerical solution and derives the analytical solutions of first and second-order ODEs where possible.

## 1 General Form of an ODE

An ODE can be expressed in the general form as:

$$F\left(t, y, y', y'', \ldots, y^{(n)}\right) = 0, \tag{1}$$

where $t$ is the independent variable, $y(t)$ is the dependent function, and $y^{(n)}$ denotes the $n$-th derivative with respect to $t$.

For numerical purposes, we often convert higher-order ODEs into a system of first-order equations.

## 2 First-Order ODE

The standard form of a first-order ODE is:

$$\frac{dy}{dt} = f(t, y), \tag{2}$$

with an initial condition:

$$y(t_0) = y_0. \tag{3}$$

### Analytical Solution (If Separable)

Suppose the equation is separable:

$$\frac{dy}{dt} = g(t)h(y). \tag{4}$$

Then we separate variables:

$$\frac{1}{h(y)}dy = g(t)dt. \tag{5}$$

Integrating both sides:

$$\int \frac{1}{h(y)}dy = \int g(t)dt + C, \tag{6}$$

where $C$ is an integration constant determined from the initial condition.

# 3 Second-Order ODE

The general second-order ODE has the form:

$$\frac{d^2y}{dt^2} = f(t, y, y'). \tag{7}$$

To solve numerically, we define:

$$y_1 = y, \quad y_2 = y' = \frac{dy}{dt}, \tag{8}$$

Then:

$$\frac{dy_1}{dt} = y_2, \tag{9}$$
$$\frac{dy_2}{dt} = f(t, y_1, y_2). \tag{10}$$

Thus, the second-order ODE is transformed into a system of two first-order ODEs.

### Analytical Solution (Linear and Homogeneous)

Consider:

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = 0. \tag{11}$$

This is a second-order linear homogeneous ODE with constant coefficients.

We assume a solution of the form $y(t) = e^{rt}$, and substitute:

$$r^2 e^{rt} + are^{rt} + be^{rt} = 0. \tag{12}$$

Factoring out $e^{rt} \neq 0$:

$$r^2 + ar + b = 0. \tag{13}$$

The roots $r_1$, $r_2$ of the characteristic equation determine the solution:

- **Distinct real roots:** $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
- **Repeated root:** $y(t) = (C_1 + C_2 t)e^{rt}$
- **Complex roots:** $y(t) = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$, where $r = \alpha \pm i\beta$

Constants $C_1$, $C_2$ are determined by initial conditions.

# 4 General Form of a System of ODEs

A system of ODEs consists of $N$ coupled first-order equations. It can be expressed as:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t), \tag{14}$$

where:

- $t$ is the independent variable (often time).
- $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T$ is a vector of $N$ dependent variables.
- $\mathbf{f}(\mathbf{x}, t) = [f_1(\mathbf{x}, t), f_2(\mathbf{x}, t), \ldots, f_N(\mathbf{x}, t)]^T$ is a vector function known as the right-hand side (RHS).

### Initial Value Problem (IVP)

Given the initial condition:

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{15}$$