Computational Physics: Analytic and Numerical Solution of the Heat Spike Problem

Prepared by Marco Fronzi

April 28, 2025

Introduction

In this document, we derive the analytic solution for a one-dimensional heat spike problem, and then we present a numerical solution using the Forward Time Centered Space (FTCS) difference scheme with Dirichlet boundary conditions.

1 Analytic Solution of the Heat Spike

The one-dimensional diffusion (heat) equation is given by:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where $T(x,t)$ is the temperature, κ is the thermal diffusivity, x is position, and t is time.

We consider the initial condition:

$$T(x,0) = \delta(x - x_0) \tag{2}$$

where δ is the Dirac delta function centered at $x = x_0$.

The solution to the diffusion equation with a delta function initial condition in an infinite domain is a Gaussian:

$$T(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{(x - x_0)^2}{4\kappa t}\right) \tag{3}$$

This Gaussian spreads out over time, with width increasing as $\sqrt{t}$.

If the domain is bounded (e.g., $x \in [0,1]$) with Dirichlet boundary conditions $T(0,t) = T(1,t) = 0$, the solution must be modified by the method of images or Fourier series to respect the boundaries.

2 Numerical Solution Using FTCS Scheme

We discretize the space and time:

$$x_i = i\Delta x, \quad i = 0, 1, \ldots, L \tag{4}$$

$$t_n = n\Delta t, \quad n = 0, 1, 2, \ldots \tag{5}$$

where Δx and Δt are the spatial and temporal steps, respectively.

The discrete approximation of the diffusion equation using the Forward Time Centered Space (FTCS) method is:

$$T_i^{n+1} = T_i^n + \frac{\kappa \Delta t}{(\Delta x)^2} \left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right) \tag{6}$$

Defining the diffusion number:

$$s = \frac{\kappa \Delta t}{(\Delta x)^2} \tag{7}$$

The update rule becomes:

$$T_i^{n+1} = T_i^n + s\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right) \tag{8}$$

2.1 Boundary Conditions

We apply Dirichlet boundary conditions:

$$T_0^n = 0, \quad T_L^n = 0 \quad \forall n \tag{9}$$

which means the temperature is fixed at zero at both ends of the domain for all times.

2.2 Initial Condition

The initial heat spike is approximated by setting:

$$T_i^0 = \begin{cases} \frac{1}{\Delta x} & \text{if } x_i \approx x_0 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

which mimics the Dirac delta function.

3 Step-by-Step Implementation of the Numerical Solution

1. **Define parameters:** Set the number of spatial points L, the domain size, the thermal diffusivity κ, the spatial step Δx, the time step Δt, and the total number of time steps.

2. **Initialize the temperature array:** Set T_i^0 according to the initial spike condition.

3. **Set boundary conditions:** Ensure that $T_0^n = 0$ and $T_L^n = 0$ for all time steps.

4. **Compute the diffusion number s:** Calculate $s = \kappa \Delta t/(\Delta x)^2$. Ensure $s \leq 0.5$ for stability.

5. **Time marching:**

   (a) For each time step n:

   - For each interior point $i = 1$ to $L - 1$:

   $$T_i^{n+1} = T_i^n + s\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)$$

- Update all points simultaneously or use a temporary array to avoid overwriting values.

6. **Visualize results:** Plot $T(x, t)$ at different times to observe the diffusion of the heat spike.

4 Summary

- The analytic solution for an infinite domain is a Gaussian centered at x_0 with width increasing as $\sqrt{t}$.
- The FTCS numerical scheme approximates the solution by updating the temperature profile using nearest neighbors.
- Dirichlet boundary conditions ensure that the temperature is zero at the domain boundaries at all times.
- Stability of the FTCS scheme requires $s \leq \frac{1}{2}$.