Deriving the Verlet Method from Central Differences: A Comparative Approach

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1 Central Difference Approximation for Position and Velocity

The central difference method is a numerical scheme for approximating derivatives using discrete values at known time steps. In the context of Newtonian mechanics, we can use it to compute both position and velocity based on acceleration data.

1.1 Velocity Approximation from Central Differences

While the Verlet method updates position accurately, it does not directly provide velocity. However, we can approximate the velocity at time $t = n\Delta t$ using the central difference formula:

$$\boxed{\vec{v}_n \approx \frac{\vec{r}_{n+1} - \vec{r}_{n-1}}{2\Delta t}}$$

This gives a second-order accurate estimate of the velocity at the midpoint between t_{n+1} and t_{n-1}.

1.2 Summary of the Scheme

Given initial position $\vec{r}_0$, velocity $\vec{v}_0$, and acceleration $\vec{a}_0$, we can compute the next steps as follows:

1. **Initialisation (first step)**: use a forward difference to compute $\vec{r}_1$:

$$\vec{r}_1 = \vec{r}_0 + \vec{v}_0\Delta t + \frac{1}{2}\vec{a}_0\Delta t^2$$

2. **Loop:** for $n \geq 1$,

$$\vec{r}_{n+1} = 2\vec{r}_n - \vec{r}_{n-1} + \vec{a}_n\Delta t^2$$

3. **Velocity (optional)**: estimate as

$$\vec{v}_n = \frac{\vec{r}_{n+1} - \vec{r}_{n-1}}{2\Delta t}$$

1.3 Position Update from Central Differences

Let $\vec{r}_n$ be the position of the particle at time $t = n\Delta t$. The second derivative of position (i.e., acceleration) can be approximated using the central difference formula:

$$\vec{a}_n = \frac{\vec{r}_{n+1} - 2\vec{r}_n + \vec{r}_{n-1}}{\Delta t^2}$$

Solving for $\vec{r}_{n+1}$, we get the **Verlet position update**:

$$\boxed{\vec{r}_{n+1} = 2\vec{r}_n - \vec{r}_{n-1} + \vec{a}_n\Delta t^2}$$

This equation requires knowledge of positions at two previous steps (n and $n-1$), along with the acceleration at the current step.

2 Central Difference and Verlet Derivation

Consider a particle moving under Newton's second law:

$$\ddot{\vec{r}}(t) = \vec{a}(\vec{r}(t))$$

To derive the Verlet method, we use the Taylor expansions forward and backward in time:

$$\vec{r}(t+\Delta t) = \vec{r}(t) + \dot{\vec{r}}(t)\Delta t + \frac{1}{2}\ddot{\vec{r}}(t)\Delta t^2 + \frac{1}{6}\dddot{\vec{r}}(t)\Delta t^3 + \mathcal{O}(\Delta t^4)$$
$$\vec{r}(t-\Delta t) = \vec{r}(t) - \dot{\vec{r}}(t)\Delta t + \frac{1}{2}\ddot{\vec{r}}(t)\Delta t^2 - \frac{1}{6}\dddot{\vec{r}}(t)\Delta t^3 + \mathcal{O}(\Delta t^4)$$

Adding these:

$$\vec{r}(t+\Delta t) + \vec{r}(t-\Delta t) = 2\vec{r}(t) + \ddot{\vec{r}}(t)\Delta t^2 + \mathcal{O}(\Delta t^4)$$

Solving for $\vec{r}(t+\Delta t)$:

$$\boxed{\vec{r}(t+\Delta t) = 2\vec{r}(t) - \vec{r}(t-\Delta t) + \vec{a}(t)\Delta t^2}$$

This is the (position) Verlet algorithm, second-order accurate and time-reversible.

3 Error Analysis via Taylor Expansion

The local truncation error arises from neglecting $\mathcal{O}(\Delta t^4)$ terms. This makes Verlet a **second-order integrator**, with:

$$\text{Local error} = \mathcal{O}(\Delta t^4), \quad \text{Global error} = \mathcal{O}(\Delta t^2)$$

This is a significant improvement over Euler, which has global error $\mathcal{O}(\Delta t)$.

Moreover, the Verlet method is **symplectic**, meaning it preserves phase-space geometry, leading to better long-term energy stability.

4 Application to the Kepler Problem

The force is:

$$\vec{F} = -\frac{GMm}{r^3}\vec{r}, \quad \vec{a} = -\frac{GM}{r^3}\vec{r}$$

In Cartesian coordinates:

$$x_{n+1} = 2x_n - x_{n-1} + a_x(x_n, y_n)\Delta t^2$$
$$y_{n+1} = 2y_n - y_{n-1} + a_y(x_n, y_n)\Delta t^2$$

With:

$$a_x = -\frac{GMx_n}{(x_n^2 + y_n^2)^{3/2}}, \quad a_y = -\frac{GMy_n}{(x_n^2 + y_n^2)^{3/2}}$$

4.1 Comparison with Euler

Euler updates:

$$\vec{v}_{n+1} = \vec{v}_n + \vec{a}_n\Delta t, \qquad \vec{r}_{n+1} = \vec{r}_n + \vec{v}_n\Delta t$$

This introduces a secular drift in energy and angular momentum:

- Orbits spiral inward or outward.

- Energy is not conserved.

Verlet, on the other hand:

- Preserves orbital shape.
- Energy errors remain bounded and oscillatory.
- Captures elliptical/circular dynamics faithfully.

5 Example: Circular Orbit

Physical Context

Consider a particle of mass m orbiting a much heavier mass M (e.g., a planet orbiting a star) under the influence of gravity. For a perfectly circular orbit of radius r, the necessary condition is that the gravitational attraction provides the exact centripetal force required to keep the particle in circular motion.

Forces Involved

5.0.1 Gravitational Force

According to Newtons law of universal gravitation, the attractive force between the two masses is given by:

$$F_{\text{grav}} = \frac{GMm}{r^2}$$

where G is the gravitational constant.

Centripetal Force

The centripetal force required to keep a mass m moving at speed v in a circular path of radius r is:

$$F_{\text{centripetal}} = \frac{mv^2}{r}$$

5.1 Equating the Forces

For the orbit to remain circular (i.e., no radial acceleration toward or away from the centre), these two forces must balance:

$$F_{\text{grav}} = F_{\text{centripetal}} \quad \Rightarrow \quad \frac{GMm}{r^2} = \frac{mv^2}{r}$$

Cancel the mass m (which appears on both sides), and rearrange:

$$\frac{GM}{r^2} = \frac{v^2}{r} \quad \Rightarrow \quad v^2 = \frac{GM}{r}$$

Taking the square root:

$$\boxed{v = \sqrt{\frac{GM}{r}}}$$

Assume $r = r_0$, and circular motion implies:

$$v_0 = \sqrt{\frac{GM}{r_0}}, \quad \text{with } x_0 = r_0, \quad y_0 = 0$$

Initial conditions:

$$\vec{r}_0 = (r_0, 0),$$
$$\vec{v}_0 = \left(0, \sqrt{\frac{GM}{r_0}}\right)$$

To start Verlet, we compute $\vec{r}_1$ using Taylor expansion:

$$\vec{r}_1 = \vec{r}_0 + \vec{v}_0 \Delta t + \frac{1}{2}\vec{a}_0 \Delta t^2$$

Then iterate using:

$$\vec{r}_{n+1} = 2\vec{r}_n - \vec{r}_{n-1} + \vec{a}_n \Delta t^2$$

Simulation shows that the orbit remains circular, and total energy oscillates around a constant mean.