Numerical Methods in Classical Physics
PHYS3934 Assessment Questions

PHYS3934 (Advanced Students) – Exam Questions

1 Instructions

Provide written answers, including explanations where required, for all questions. If a question involves writing or modifying existing Python code, include your code in a documented notebook alongside your submission, with comments explaining your implementation. You are reminded that your work must be your own. For code, this means that all new parts required for a problem must be your own work.

2 A. Two-Body Problem

In lectures and labs, we studied the central force (Kepler) problem describing the motion of a planet or comet around a fixed massive object (the Sun). In this problem, we relax the assumption that the Sun is fixed and consider the full two-body problem, which models the dynamics of two masses interacting via mutual gravitational attraction.

Question A1: VelocityVerlet and Stability Analysis (10 points)

i. (2 points) Write the equations that describe the two-body motion under Newtonian gravity.

ii. (3 points) Implement the Velocity-Verlet method with an adaptive time step $\tau_n = \tau_0/(1+|\vec{a}_n|)$, using $\tau_0 = 0.05$.

iii. (3 points) Simulate the system for $a = 0.1$ and $a = 2$, using the same initial conditions for $\vec{r}_1$ and $\vec{v}_1$, with the centre of mass at the origin. Plot:

- orbital trajectories;
- τ_n versus time.
- phase space momentum-position

iv. (2 point) Estimate the maximum relative error in total energy for each case.

Question A2: Symplectic Euler Analysis and Implementation (10 points)

i. (2 points) Write a symplectic Euler integrator and explain how it differs from the standard Euler method in structure and purpose (hint: see notes).

ii. (3 points) Simulate the two-body problem using the symplectic Euler method for $a = 0.2$, 0.5, 1.0, and 2.0. Compare:

- orbital trajectories,
- energy drift over time,
- qualitative phase-space structure.

iii. (2 points) Perform a sensitivity analysis by introducing small perturbations in $\vec{v}_1$ and plotting the resulting orbits.
(hint: add a slight increment (e.g., $\delta = 10^{-3}$) to the y-component of the initial velocity vector of Body 1. This modified the initial velocity from $\vec{v}_1 = [0, 2 - \sqrt{3}]$ to$\vec{v}_1 = [0, 2 - \sqrt{3} + \delta]$.)

iv. (2 points) Assess how time step size affects the performance of symplectic Euler in long-term integration. Compare fixed and adaptive time-stepping.

v. (1 point) Compare angular momentum and energy conservation between the symplectic Euler and VelocityVerlet methods.

3 B. Heat Diffusion

Question B1: Advanced Stability Analysis of 2D Diffusion (10 points)

i. (3 points) Extend the FTCS scheme to include anisotropic diffusion:

$$\frac{\partial T}{\partial t} = \kappa_x \frac{\partial^2 T}{\partial x^2} + \kappa_y \frac{\partial^2 T}{\partial y^2}, \tag{1}$$

and derive the resulting numerical scheme.

ii. (3 points) Apply von Neumann analysis to the scheme and derive the stability condition for the general case $\kappa_x \neq \kappa_y$.

iii. (2 points) Discuss the implications of unequal diffusion rates in x and y on the stability and accuracy of the numerical solution.

iv. (2 points) Implement the extended FTCS scheme in two spatial dimensions and demonstrate it with a test case (e.g., an anisotropic Gaussian spike).