Computational Physics: Von Neumann Analysis and Lax Method

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Introduction

The Von Neumann analysis and the Lax method are two fundamental tools in the numerical analysis of partial differential equations (PDEs).

Von Neumann analysis is a method used to assess the stability of numerical methods for solving linear PDEs. It involves analyzing the frequency components of a numerical solution to determine whether perturbations grow or decay over time. Essentially, the numerical solution is expressed as a sum of sinusoidal modes, and the behavior of each mode is studied as the computation progresses. If all modes remain bounded (or decay), the method is stable.

The Lax method (or Lax scheme) is a numerical technique used to solve hyperbolic equations, such as the wave equation or linear advection equations. The Lax method slightly modifies the central difference method by introducing an "averaging" step that improves stability. In the Lax scheme, the value of the function at a point is updated using an average of neighboring points and a finite difference for the spatial term, ensuring stability under certain conditions.

In essence, the Von Neumann analysis helps you understand whether a numerical method is stable, while the Lax method is a concrete example of a stable numerical scheme for hyperbolic equations.

Von Neumann Stability Analysis

We consider a one-dimensional linear partial differential equation and study the stability of a finite difference scheme using Von Neumann analysis.

1. Governing Equation

The linear advection equation is given by:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

where $c \in \mathbb{R}$ is a constant advection velocity.

2. Discretisation (FTBS Scheme)

Using the Forward-Time Backward-Space (FTBS) scheme, the discrete update rule is:

$$u_j^{n+1} = u_j^n - \lambda(u_j^n - u_{j-1}^n),$$

where $\lambda = \frac{c\Delta t}{\Delta x}$ is the Courant number.

3. Fourier Mode Assumption

We assume a solution of the form:

$$u_j^n = \xi^n e^{ikj\Delta x},$$

where:

- ξ is the amplification factor,
- k is the wavenumber,
- Δx is the spatial grid spacing.

4. Substitution into the Scheme

Left-hand side:

$$u_j^{n+1} = \xi^{n+1} e^{ikj\Delta x}$$

Right-hand side:

$$\begin{aligned} u_j^n - \lambda(u_j^n - u_{j-1}^n) &= \xi^n e^{ikj\Delta x} - \lambda\left(\xi^n e^{ikj\Delta x} - \xi^n e^{ik(j-1)\Delta x}\right) \\ &= \xi^n e^{ikj\Delta x}\left[1 - \lambda(1 - e^{-ik\Delta x})\right] \end{aligned}$$

Equating both sides and simplifying:

$$\xi^{n+1} e^{ikj\Delta x} = \xi^n e^{ikj\Delta x}\left[1 - \lambda(1 - e^{-ik\Delta x})\right] \Rightarrow \xi = 1 - \lambda(1 - e^{-ik\Delta x})$$

5. Modulus of the Amplification Factor

Using Euler's formula:

$$e^{-ik\Delta x} = \cos(k\Delta x) - i\sin(k\Delta x),$$

we write:

$$\xi = 1 - \lambda(1 - \cos(k\Delta x) + i\sin(k\Delta x)) = 1 - \lambda + \lambda\cos(k\Delta x) - i\lambda\sin(k\Delta x)$$

The modulus squared is:

$$\begin{aligned} |\xi|^2 &= (1 - \lambda + \lambda\cos(k\Delta x))^2 + (\lambda\sin(k\Delta x))^2 \\ &= (1-\lambda)^2 + 2\lambda(1-\lambda)\cos(k\Delta x) + \lambda^2\cos^2(k\Delta x) + \lambda^2\sin^2(k\Delta x) \\ &= (1-\lambda)^2 + 2\lambda(1-\lambda)\cos(k\Delta x) + \lambda^2\underbrace{(\cos^2(k\Delta x) + \sin^2(k\Delta x))}_{=1} \\ &= (1-\lambda)^2 + 2\lambda(1-\lambda)\cos(k\Delta x) + \lambda^2 \\ &= 1 - 2\lambda + 2\lambda^2 + 2\lambda(1-\lambda)\cos(k\Delta x) \end{aligned}$$

6. Stability Condition

We require that $|\xi| \leq 1$ for all k. The worst-case scenario is when $\cos(k\Delta x) = -1$, which gives:

$$|\xi|^2 = 1 - 2\lambda + 2\lambda^2 - 2\lambda(1-\lambda) = 1 - 4\lambda + 4\lambda^2$$

For stability:

$$|\xi|^2 \leq 1 \Rightarrow 1 - 4\lambda + 4\lambda^2 \leq 1 \Rightarrow 4\lambda(1-\lambda) \geq 0$$

This inequality holds if and only if:

$$0 \leq \lambda \leq 1$$

7. Conclusion

The FTBS scheme for the advection equation is **stable** under the condition:

$$\boxed{0 \leq \lambda \leq 1}$$

This result is derived using the Von Neumann analysis by requiring the amplification factor ξ to satisfy $|\xi| \leq 1$ for all wavenumbers k.

Lax Method:

The Lax method for the advection equation $u_t + cu_x = 0$ is given by:

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{c\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n),$$

affirming the independence of the stability on the CourantFriedrichsLewy (CFL) condition:

$$\frac{c\Delta t}{\Delta x} \leq 1.$$

This ensures that the method remains stable under the appropriate time-step and spatial discretization.

1. Finite Difference Update as a Matrix Equation

Consider a linear finite difference scheme written in matrix form:

$$\mathbf{u}^{n+1} = \mathbf{A}\mathbf{u}^n,$$

where:

- $\mathbf{u}^n$ is the solution vector at time step n,
- $\mathbf{A}$ is the time-propagator matrix, which can be written as $\mathbf{I} + \mathbf{D}$.

2. Stability and the Spectral Radius

The stability of the scheme is determined by the spectral radius of $\mathbf{A}$. That is, the scheme is stable if:

$$|\mu| \leq 1 \quad \text{for all eigenvalues } \mu \text{ of } \mathbf{A}.$$

3. FTBS Scheme and the Propagator Matrix

For the forward-time, backward-space (FTBS) scheme applied to the advection equation:

$$u_j^{n+1} = (1-\lambda)u_j^n + \lambda u_{j-1}^n,$$

the matrix $\mathbf{A}$ has the form:

$$\mathbf{A} = \begin{bmatrix} 1-\lambda & 0 & \cdots & 0 & \lambda \\ \lambda & 1-\lambda & 0 & \cdots & 0 \\ 0 & \lambda & 1-\lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda & 1-\lambda \end{bmatrix}$$

This assumes periodic boundary conditions, making $\mathbf{A}$ a circulant matrix.

4. Eigenvalues via Fourier Modes

Because the scheme is translation invariant, the eigenvectors of $\mathbf{A}$ are discrete Fourier modes:

$$v_j = e^{ikj\Delta x}$$

Applying $\mathbf{A}$ to v_j gives:

$$\mathbf{A}v_j = \left[(1-\lambda) + \lambda e^{-ik\Delta x}\right] v_j = \mu(k)v_j$$

Therefore, the eigenvalues of $\mathbf{A}$ are:

$$\boxed{\mu(k) = 1 - \lambda(1 - e^{-ik\Delta x})}$$

5. Connection to Von Neumann Analysis

In Von Neumann analysis, we write the solution as:

$$u_j^n = \xi^n e^{ikj\Delta x}$$

By substituting this into the finite difference scheme, we found:

$$\xi = 1 - \lambda(1 - e^{-ik\Delta x})$$

Thus, the amplification factor ξ is the same as the eigenvalue $\mu(k)$. The Courant number λ directly controls both the magnitude and phase of these eigenvalues.

6. Final Statement

- The eigenvalues of the time-propagator matrix $\mathbf{A}$ are:

$$\boxed{\mu(k) = \xi(k) = 1 - \lambda(1 - e^{-ik\Delta x})}$$

- The scheme is stable if:

$$\boxed{|\mu(k)| = |\xi(k)| \leq 1 \quad \forall k}$$

- The Courant number λ controls the spectral radius of $\mathbf{A}$ and hence determines the stability.