Lecture Notes: Symplectic Euler Method

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Introduction

The Symplectic Euler method (also known as semi-implicit Euler) is a numerical integration scheme used particularly in physics simulations where long-term energy conservation is important. Unlike the standard (explicit) Euler method, the Symplectic Euler is symplectic — it preserves the geometric structure of Hamiltonian systems, making it well suited for problems in classical mechanics.

Equations of Motion

Consider Newton's second law:

$$\frac{d\vec{x}}{dt} = \vec{v}, \quad \frac{d\vec{v}}{dt} = \vec{a}(\vec{x})$$

The standard (explicit) Euler method updates the position and velocity as:

$$\vec{x}_{n+1} = \vec{x}_n + \tau\vec{v}_n, \quad \vec{v}_{n+1} = \vec{v}_n + \tau\vec{a}(\vec{x}_n)$$

This method is not symplectic, and it tends to drift in energy over time.

Symplectic Euler Formulation

There are two common variants of the Symplectic Euler method. One of the most used versions updates the velocity before the position:

$$\vec{v}_{n+1} = \vec{v}_n + \tau\vec{a}(\vec{x}_n)$$

$$\vec{x}_{n+1} = \vec{x}_n + \tau\vec{v}_{n+1}$$

This version is called **velocity-first symplectic Euler**.

Alternatively, one can use the **position-first** variant:

$$\vec{x}_{n+1} = \vec{x}_n + \tau\vec{v}_n$$

$$\vec{v}_{n+1} = \vec{v}_n + \tau\vec{a}(\vec{x}_{n+1})$$

Advantages

- **Symplecticity**: Preserves the phase-space volume (Liouvilles Theorem).
- **Energy Behaviour**: Unlike explicit Euler, total energy remains bounded over long times (may oscillate but doesn't drift).
- **Stability**: Better suited for conservative systems like planetary motion or oscillators.

Comparison with Other Methods

- **Explicit Euler**: Simple but unstable for long-term simulation of oscillatory systems.
- **Runge-Kutta 4**: High accuracy but not symplectic. May exhibit long-term energy drift.
- **Verlet (Leapfrog)**: Also symplectic, but more accurate than Symplectic Euler; often preferred for molecular dynamics or celestial mechanics.

Python Example

```
# Symplectic Euler for 1D harmonic oscillator
x, v = 1.0, 0.0
tau = 0.1
for _ in range(100):
    v += -tau * x
    x += tau * v
```

Conclusion

The Symplectic Euler method offers a simple and powerful approach to simulate conservative systems. While not as accurate as higher-order methods for short-term predictions, its qualitative behaviour over long times is often superior.