Realization, Internal Stability, and Controller Synthesis

Shih-Hao Tseng, (pronounced as "She-How Zen")

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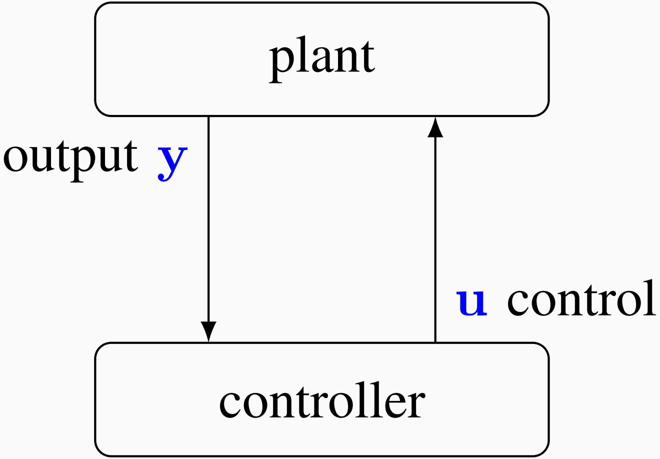
Synthesizing Internally Stabilizing Controllers



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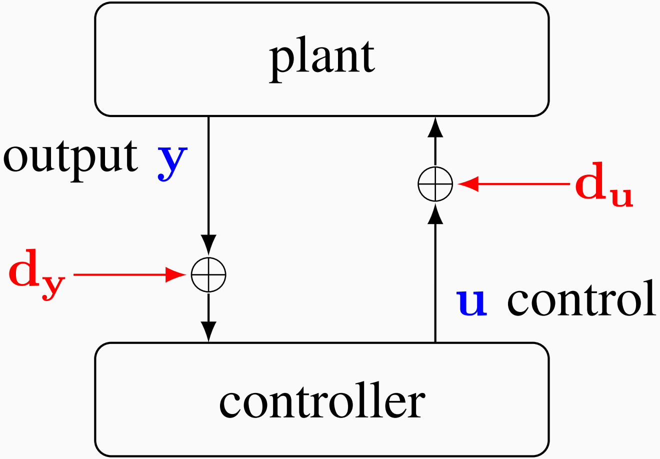

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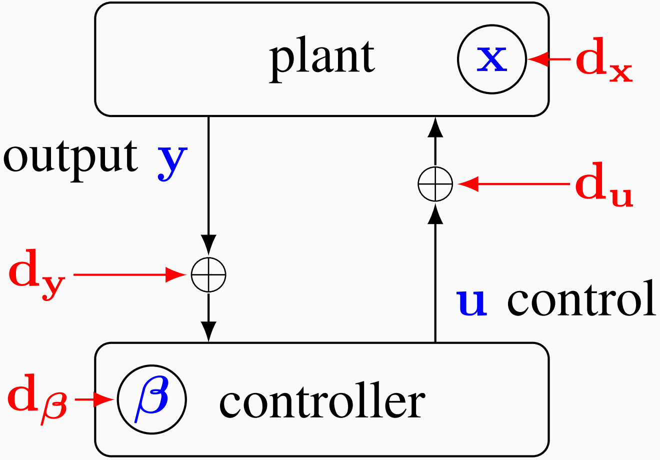

We Have Seen Multiple Controller Parameterizations

$$\begin{bmatrix} \mathbf{M}_l & -\mathbf{N}_l \\ -\mathbf{V}_l & \mathbf{U}_l \end{bmatrix} \begin{bmatrix} \mathbf{U}_r & \mathbf{N}_r \\ \mathbf{V}_r & \mathbf{M}_r \end{bmatrix} = I,$$

$$\mathbf{M}_{(\cdot)}, \mathbf{N}_{(\cdot)}, \mathbf{V}_{(\cdot)}, \mathbf{U}_{(\cdot)} \in \mathcal{RH}_\infty,$$

$$\mathbf{K} = (\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})(\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})^{-1}$$

(a) Youla Parameterization (Youla et al., 1976)

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$$\begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \mathbf{\Phi_x} \\ \mathbf{\Phi_u} \end{bmatrix} = I,$$

$$\mathbf{\Phi_x}, \mathbf{\Phi_u} \in z^{-1}\mathcal{RH}_\infty,$$

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(b) System Level Parameterization (Wang et al., 2017)

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$$\begin{bmatrix} I & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} = \begin{bmatrix} I & O \end{bmatrix},$$

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(c) Input-Output Parameterization (Furieri et al., 2019)

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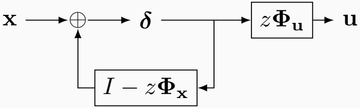
(d) Mixed Parameterization
(Zheng et al., 2019)

We Have Seen Multiple Realizations



(a) State-Feedback System Level Synthesis Realization
(Wang et al., 2017)

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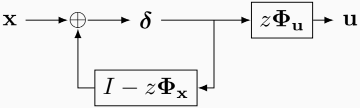

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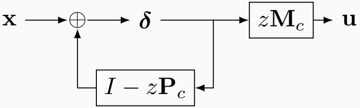

(b) Closed-Loop Design Separating (Li and Ho, 2020)

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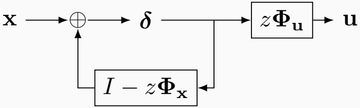

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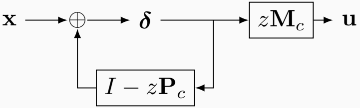

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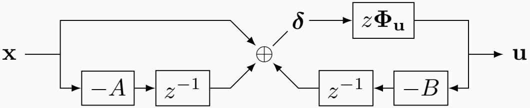

(c) Simpler Realization for Deployment
(Tseng and Anderson, 2020)

Are we done here?

How can we systematically find/understand different parameterizations and realizations?

Answer: Realization-Stability Lemma

- *All* existing parameterizations and realization results are special cases of the realization-stability lemma.

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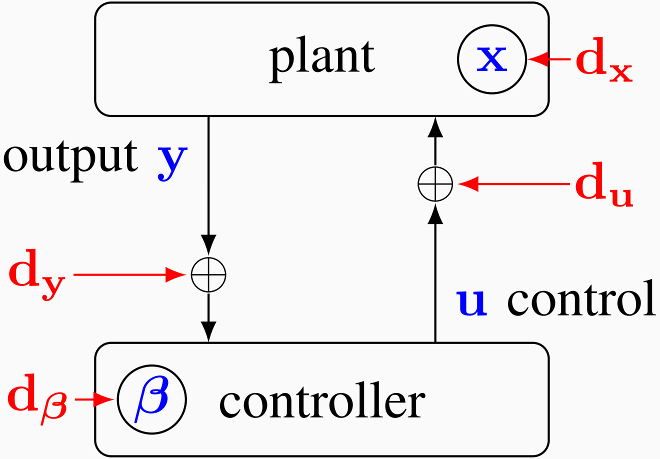
- The realization-stability lemma shows that equivalent systems can be derived from a transformation of external disturbances.
- The realization-stability lemma leads to the formulation of general controller synthesis problem.

How Can We Describe a Linear System?



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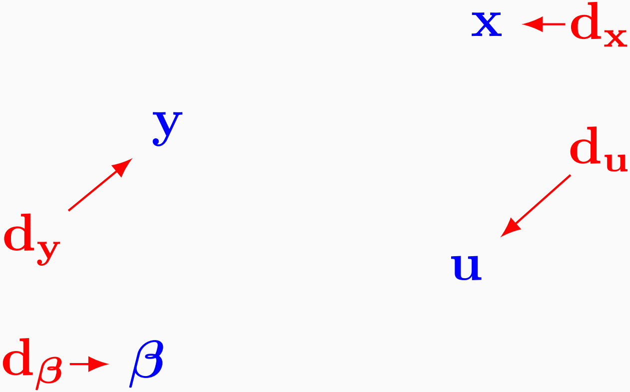

How Can We Describe a Linear System?

$\mathbf{x} \leftarrow \mathbf{d_x}$

$\mathbf{y}$

$\mathbf{u}$

$\boldsymbol{\beta}$

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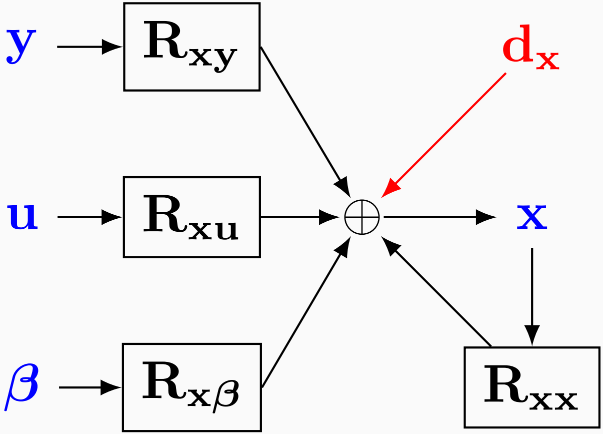


How Can We Describe a Linear System?

$$\mathbf{x} = \mathbf{R}_{\mathbf{x},:} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix} + \mathbf{d}_{\mathbf{x}}$$

How Can We Describe a Linear System?

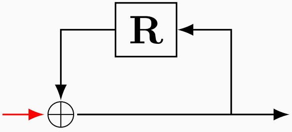
$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix} = \mathbf{R} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix} + \begin{bmatrix} \mathbf{d_x} \\ \mathbf{d_y} \\ \mathbf{d_u} \\ \mathbf{d}_{\boldsymbol{\beta}} \end{bmatrix}$$

How Can We Describe a Linear System?

$$\boldsymbol{\eta} = \mathbf{R} \quad \boldsymbol{\eta} \quad + \mathbf{d}$$

Realization and Internal Stability Matrices

$$\begin{bmatrix} \mathbf{d}_{\mathbf{x}} \\ \mathbf{d}_{\mathbf{y}} \\ \mathbf{d}_{\mathbf{u}} \\ \mathbf{d}_{\boldsymbol{\beta}} \end{bmatrix} = \mathbf{d}$$



$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix}$$

$$\boldsymbol{\eta} = \mathbf{R}\boldsymbol{\eta} + \mathbf{d}$$

closed-loop: realization matrix $\mathbf{R}$

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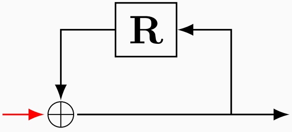

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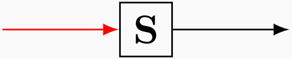


$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix}$$

$$\boldsymbol{\eta} = \mathbf{S}\mathbf{d}$$

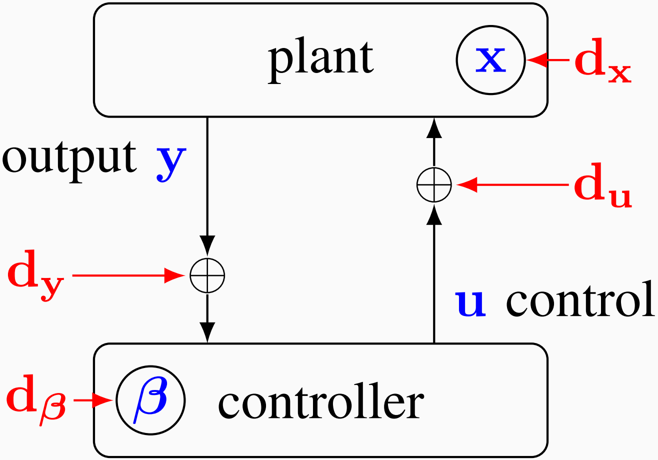
open-loop: internal stability matrix $\mathbf{S}$

Realization-Stability Lemma

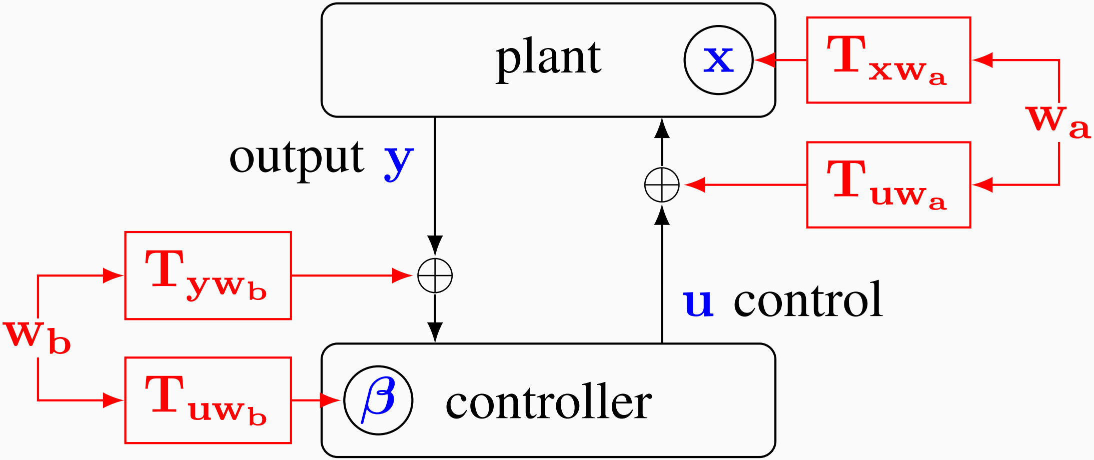
Let $\mathbf{R}$ be the realization matrix and $\mathbf{S}$ be the internal stability matrix, we have

$$(I - \mathbf{R})\mathbf{S} = \mathbf{S}(I - \mathbf{R}) = I.$$

Transformation and Equivalent Systems



Transformation and Equivalent Systems



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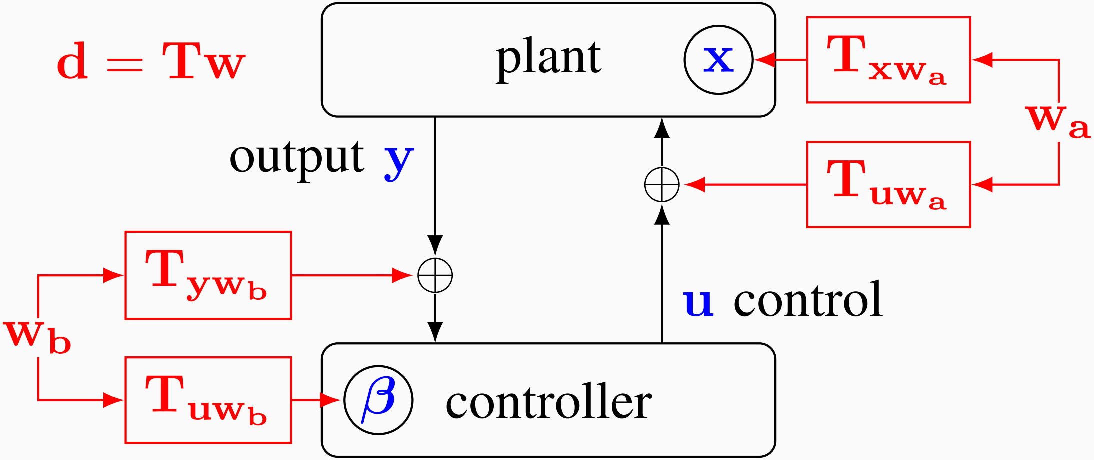

Transformation and Equivalent Systems

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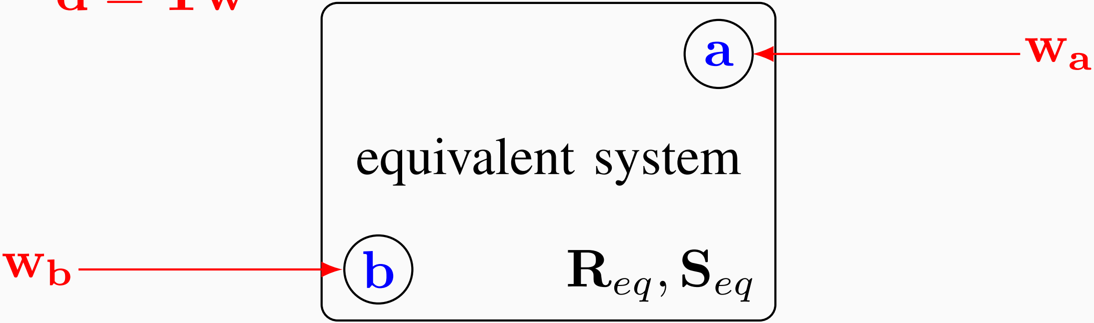

Transformation and Equivalent Systems

$$\mathbf{d} = \mathbf{T}\mathbf{w}$$

$$\mathbf{T}^{-1}(I - \mathbf{R}) = I - \mathbf{R}_{eq}$$

$$\mathbf{S}\mathbf{T} = \mathbf{S}_{eq}$$

General Controller Synthesis Problem

- Causality: $\mathbf{R_{ab}} \in \mathcal{R}_p$ for all $\mathbf{a} \neq \mathbf{b}$.
- Internal stability: $\mathbf{S} \in \mathcal{RH}_\infty$.

$$
\begin{aligned}
\min \quad & g(\mathbf{R}, \mathbf{S}) \\
\text{s.t.} \quad & (I - \mathbf{R})\mathbf{S} = \mathbf{S}(I - \mathbf{R}) = I \\
& \mathbf{R_{ab}} \in \mathcal{R}_p \qquad\qquad \forall \mathbf{a} \neq \mathbf{b} \\
& \mathbf{S} \in \mathcal{RH}_\infty \\
& (\mathbf{R}, \mathbf{S}) \in \mathcal{C}
\end{aligned}
$$

Unifying Existing Results

- Existing controller parameterizations are different ways of writing the general controller synthesis problem.

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- We can easily show the equivalence among parameterizations via transformation.
- Existing realization results study different equivalent systems

Example: System Level Parameterization (SLP)

$$x[t+1] = Ax[t] + Bu[t] + d_x[t]$$

$$u[t] = (K \star x)[t] + d_u[t]$$

Example: System Level Parameterization (SLP)

$$z\mathbf{x} = A\mathbf{x} + B\mathbf{u} + \mathbf{d_x}$$

$$\mathbf{u} = \mathbf{K}\mathbf{x} + \mathbf{d_u}$$

Example: System Level Parameterization (SLP)

$$\mathbf{x} = (A + (1 - z)I)\mathbf{x} + B\mathbf{u} + \mathbf{d_x}$$

$$\mathbf{u} = \mathbf{K}\mathbf{x} + \mathbf{d_u}$$

Example: System Level Parameterization (SLP)

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \underbrace{\begin{bmatrix} A + (1 - z)I & B \\ \mathbf{K} & O \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{d_x} \\ \mathbf{d_u} \end{bmatrix}$$

Example: System Level Parameterization (SLP)

$$\underbrace{\begin{bmatrix} zI - A & -B \\ -\mathbf{K} & I \end{bmatrix}}_{I - \mathbf{R}} \underbrace{\begin{bmatrix} \mathbf{S}_{\mathbf{xx}} & \mathbf{S}_{\mathbf{ux}} \\ \mathbf{S}_{\mathbf{ux}} & \mathbf{S}_{\mathbf{uu}} \end{bmatrix}}_{\mathbf{S}} = I$$

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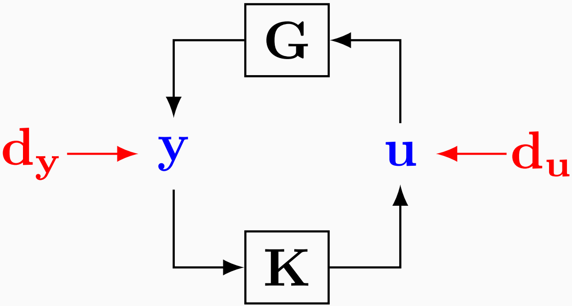
$$-\mathbf{K}\mathbf{\Phi_x} + \mathbf{\Phi_u} = O \;\Rightarrow\; \mathbf{K} = \mathbf{\Phi_u}\mathbf{\Phi_x}^{-1}$$

Example: Input-Output Parameterization (IOP)



Example: Input-Output Parameterization (IOP)

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \underbrace{\begin{bmatrix} O & \mathbf{G} \\ \mathbf{K} & O \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{d_y} \\ \mathbf{d_u} \end{bmatrix}$$

Example: Input-Output Parameterization (IOP)

$$\underbrace{\begin{bmatrix} I & -\mathbf{G} \\ -\mathbf{K} & I \end{bmatrix}}_{I-\mathbf{R}} \underbrace{\begin{bmatrix} \mathbf{S_{yy}} & \mathbf{S_{uy}} \\ \mathbf{S_{uy}} & \mathbf{S_{uu}} \end{bmatrix}}_{\mathbf{S}} = I$$

Example: Input-Output Parameterization (IOP)

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Equivalence of SLP and IOP via Transformation

$$\mathbf{T}^{-1}(I - \mathbf{R}_{SLP}) = I - \mathbf{R}_{IOP}$$

$$\mathbf{S}_{SLP}\mathbf{T} = \mathbf{S}_{IOP}$$

Equivalence of SLP and IOP via Transformation

$$\underbrace{\begin{bmatrix} (zI-A)^{-1} & O \\ O & I \end{bmatrix}}^{\mathbf{T}^{-1}} \overbrace{\begin{bmatrix} zI-A & -B \\ -\mathbf{K} & I \end{bmatrix}}^{I-\mathbf{R}_{SLP}} = \overbrace{\begin{bmatrix} I & -\mathbf{G} \\ -\mathbf{K} & I \end{bmatrix}}^{I-\mathbf{R}_{IOP}}$$

$$\underbrace{\begin{bmatrix} \mathbf{\Phi_x} & \mathbf{S_{ux}} \\ \mathbf{\Phi_u} & \mathbf{S_{uu}} \end{bmatrix}}_{\mathbf{S}_{SLP}} \underbrace{\begin{bmatrix} zI-A & O \\ O & I \end{bmatrix}}_{\mathbf{T}} = \underbrace{\begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix}}_{\mathbf{S}_{IOP}}$$

Conclusion and Future Directions

- Realization-Stability Lemma
  - unifies *all* existing parameterizations and realization results
  - introduces the transformation technique and the concept of equivalent systems
  - leads to the formulation of the general controller synthesis problem

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https://shih-hao-tseng.github.io/

Conclusion and Future Directions

- Realization-Stability Lemma
  - unifies *all* existing parameterizations and realization results
  - introduces the transformation technique and the concept of equivalent systems
  - leads to the formulation of the general controller synthesis problem
- Future directions:
  - better parameterizations/realizations
  - robust controller synthesis (arXiv: 2103.13650)

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