A General Approach to Robust Controller Analysis and Synthesis

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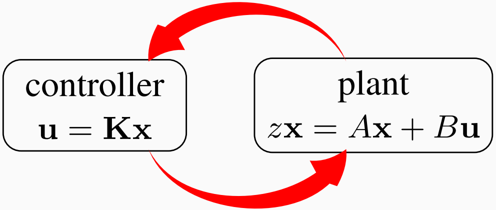
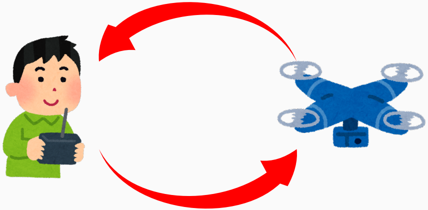
Perturbation/Uncertainty and Robust Controller Synthesis

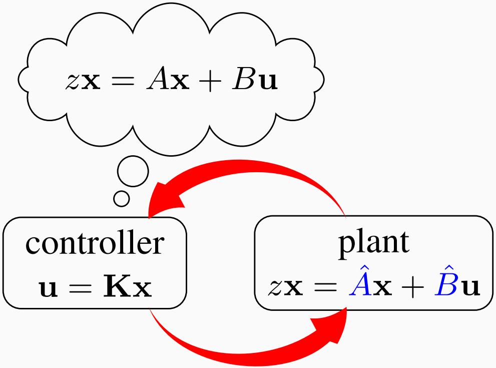


Ideal model



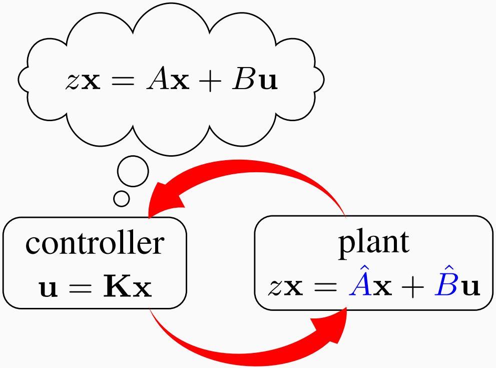
Real system

Sources of Uncertainty/Perturbation: Plant Uncertainty/Perturbation



How to synthesize a controller that can stabilize a perturbed system?

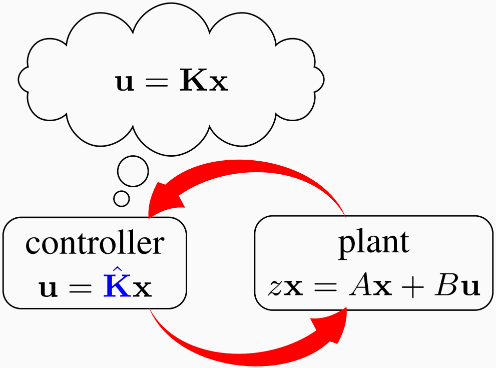
Sources of Uncertainty/Perturbation: Plant Uncertainty/Perturbation



- **μ-synthesis** (Doyle, 1982) (Doyle, 1985) (Zhou and Doyle, 1998)
- **Robust primal-dual Youla parameterization** (Niemann and Stoustrup, 2002)
- **Robust input-output parameterization (IOP)** (Zheng et al., 2020)

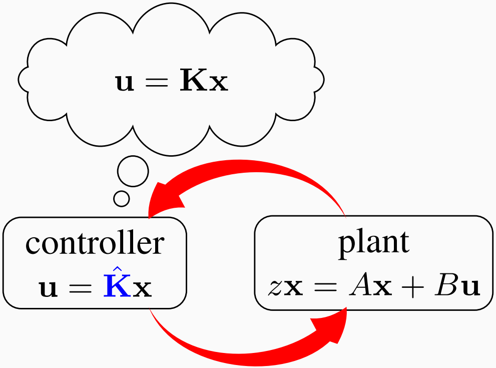
How to synthesize a controller that can stabilize a perturbed system?

Sources of Uncertainty/Perturbation: Controller Resolution/Inaccuracy



How to ensure a perturbed controller realization can still stabilize the original system?

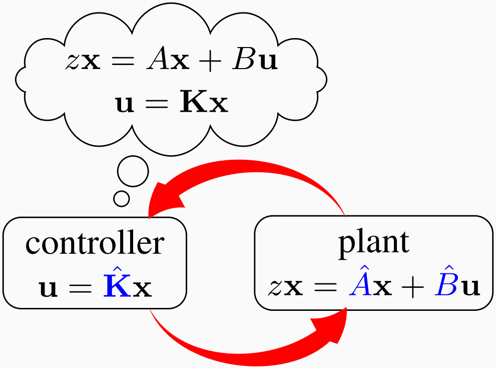
Sources of Uncertainty/Perturbation: Controller Resolution/Inaccuracy



- **Robust system level synthesis (SLS)** (Matni, Wang, and Anderson, 2017) (Boczar, Matni, and Recht, 2018) (Anderson et al., 2019)

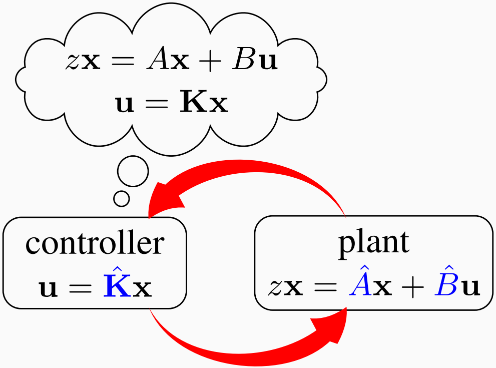
How to ensure a perturbed controller realization can still stabilize the original system?

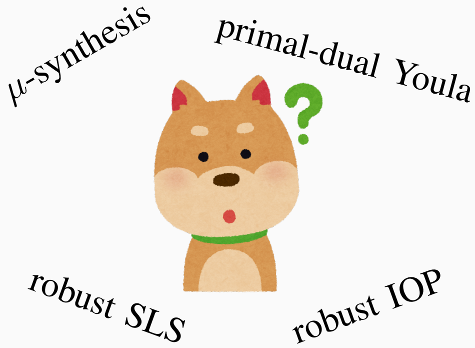
Issues



1. Both plant and controller can be subject to uncertainty

Issues





1. Both plant and controller can be subject to uncertainty
2. Which method to use?

Unified Approach: Robust Stability Conditions for General Systems

- We provide a unified approach through the *robust stability conditions for general systems*.

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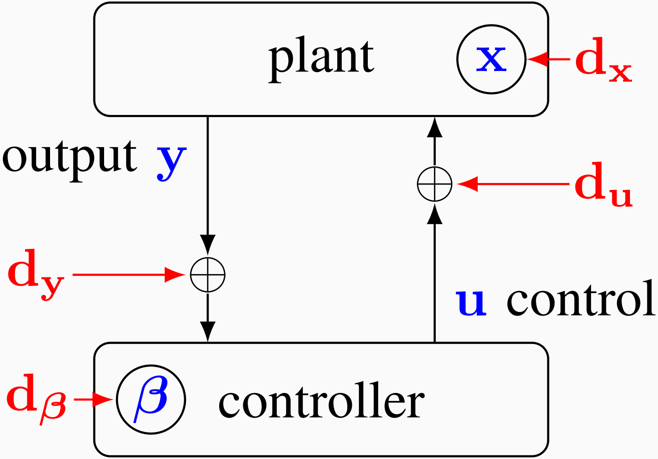
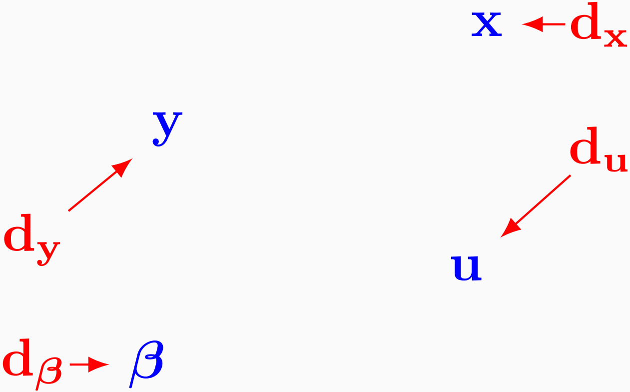
Unified Approach: Robust Stability Conditions for General Systems

- We provide a unified approach through the *robust stability conditions for general systems*.
- The condition is derived from the *realization* abstraction, which investigates both the plant and controller together as a closed-loop system.
- Existing results can be derived from the condition, and we can also derive new results accordingly. $\Rightarrow$ An effective analysis approach

Describing a Linear System by its Realization



Describing a Linear System by its Realization



Describing a Linear System by its Realization

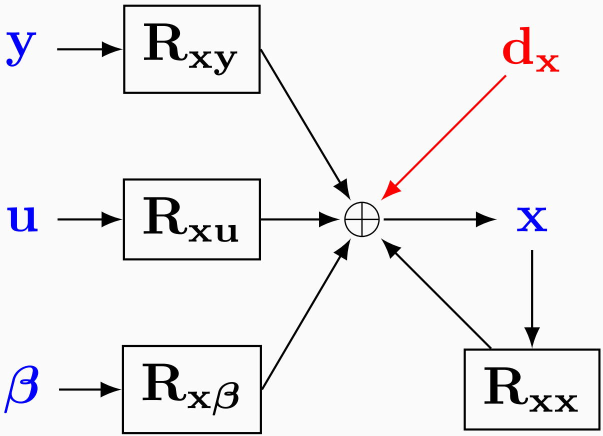
$$\mathbf{x} \leftarrow \mathbf{d}_{\mathbf{x}}$$

$$\mathbf{y}$$

$$\mathbf{u}$$

$$\boldsymbol{\beta}$$

Describing a Linear System by its Realization



Describing a Linear System by its Realization

$$\mathbf{x} = \mathbf{R}_{\mathbf{x},:} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix} + \mathbf{d}_{\mathbf{x}}$$

Describing a Linear System by its Realization

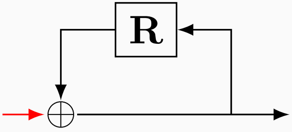
$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix} = \mathbf{R} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix} + \begin{bmatrix} \mathbf{d}_{\mathbf{x}} \\ \mathbf{d}_{\mathbf{y}} \\ \mathbf{d}_{\mathbf{u}} \\ \mathbf{d}_{\boldsymbol{\beta}} \end{bmatrix}$$

Describing a Linear System by its Realization

$$\boldsymbol{\eta} = \mathbf{R} \quad \boldsymbol{\eta} + \mathbf{d}$$

Realization and Internal Stability Matrices

$$\begin{bmatrix} \mathbf{d_x} \\ \mathbf{d_y} \\ \mathbf{d_u} \\ \mathbf{d}_{\boldsymbol{\beta}} \end{bmatrix} = \mathbf{d}$$



$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix}$$

$$\boldsymbol{\eta} = \mathbf{R}\boldsymbol{\eta} + \mathbf{d}$$

closed-loop: realization matrix $\mathbf{R}$

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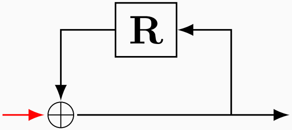

$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix}$$

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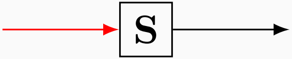
closed-loop: realization matrix $\mathbf{R}$

$$\begin{bmatrix} \mathbf{d_x} \\ \mathbf{d_y} \\ \mathbf{d_u} \\ \mathbf{d}_{\boldsymbol{\beta}} \end{bmatrix} = \mathbf{d}$$



$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix}$$

$$\boldsymbol{\eta} = \mathbf{S}\mathbf{d}$$

open-loop: internal stability matrix $\mathbf{S}$

Realization-Stability Lemma

Let $\mathbf{R}$ be the realization matrix and $\mathbf{S}$ be the internal stability matrix, we have

$$(I - \mathbf{R})\mathbf{S} = \mathbf{S}(I - \mathbf{R}) = I.$$

Realization-Stability Lemma

Let $\mathbf{R}$ be the realization matrix and $\mathbf{S}$ be the internal stability matrix, we have

$$(I - \mathbf{R})\mathbf{S} = \mathbf{S}(I - \mathbf{R}) = I$$

and the system is stable iff

$$\mathbf{S} \in \mathcal{RH}_\infty.$$

Robust Stability Condition

Consider a system perturbed according to some uncertain parameter $\boldsymbol{\Delta} \in \mathcal{D}$, i.e., with realization $\mathbf{R}(\boldsymbol{\Delta})$. By the realization-stability lemma:

$$(I - \mathbf{R}(\boldsymbol{\Delta}))\mathbf{S}(\boldsymbol{\Delta}) = \mathbf{S}(\boldsymbol{\Delta})(I - \mathbf{R}(\boldsymbol{\Delta})) = I$$

and the perturbed system is robustly stable iff

$$\mathbf{S}(\boldsymbol{\Delta}) \in \mathcal{RH}_\infty \qquad \forall \boldsymbol{\Delta} \in \mathcal{D}.$$

General Robust Controller Synthesis Problem

- Causality: $\mathbf{R}(\mathbf{\Delta})_{\mathbf{ab}} \in \mathcal{R}_p$ for all $\mathbf{a} \neq \mathbf{b}$.
- Robust internal stability: $\mathbf{S}(\mathbf{\Delta}) \in \mathcal{RH}_\infty$, $\forall \mathbf{\Delta} \in \mathcal{D}$.

$$
\begin{aligned}
\min \quad & g(\mathbf{R}(\mathbf{\Delta}), \mathbf{S}(\mathbf{\Delta}), \mathcal{D}) & \\
\text{s.t.} \quad & (I - \mathbf{R}(\mathbf{\Delta}))\mathbf{S}(\mathbf{\Delta}) = \mathbf{S}(\mathbf{\Delta})(I - \mathbf{R}(\mathbf{\Delta})) = I & \forall \mathbf{\Delta} \in \mathcal{D} \\
& \mathbf{R}(\mathbf{\Delta})_{\mathbf{ab}} \in \mathcal{R}_p & \forall \mathbf{\Delta} \in \mathcal{D}, \mathbf{a} \neq \mathbf{b} \\
& \mathbf{S}(\mathbf{\Delta}) \in \mathcal{RH}_\infty & \forall \mathbf{\Delta} \in \mathcal{D} \\
& (\mathbf{R}(\mathbf{\Delta}), \mathbf{S}(\mathbf{\Delta})) \in \mathcal{C} & \forall \mathbf{\Delta} \in \mathcal{D}
\end{aligned}
$$

Additive Perturbation

Suppose

$$\mathbf{R}(\boldsymbol{\Delta}) = \hat{\mathbf{R}} + \boldsymbol{\Delta}.$$

The corresponding stability matrix is given by

$$\mathbf{S}(\boldsymbol{\Delta}) = \hat{\mathbf{S}}(I - \boldsymbol{\Delta}\hat{\mathbf{S}})^{-1} = (I - \hat{\mathbf{S}}\boldsymbol{\Delta})^{-1}\hat{\mathbf{S}},$$

where $\hat{\mathbf{S}}$ is the nominal stability matrix satisfying the realization-stability lemma

$$(I - \hat{\mathbf{R}})\hat{\mathbf{S}} = \hat{\mathbf{S}}(I - \hat{\mathbf{R}}) = I.$$

General Robust Controller Synthesis Problem Under Additive Perturbation

- Causality: $\mathbf{R}(\mathbf{\Delta})_{\mathbf{ab}} = (\hat{\mathbf{R}} + \mathbf{\Delta})_{\mathbf{ab}} \in \mathcal{R}_p$ for all $\mathbf{a} \neq \mathbf{b}$.
- Robust internal stability: $\mathbf{S}(\mathbf{\Delta}) = \hat{\mathbf{S}}(I - \mathbf{\Delta}\hat{\mathbf{S}})^{-1} \in \mathcal{RH}_\infty$, $\forall \mathbf{\Delta} \in \mathcal{D}$.

$$
\begin{aligned}
\min \quad & g(\hat{\mathbf{R}}, \hat{\mathbf{S}}, \mathcal{D}) & \\
\text{s.t.} \quad & (I - \hat{\mathbf{R}})\hat{\mathbf{S}} = \hat{\mathbf{S}}(I - \hat{\mathbf{R}}) = I & \\
& \mathbf{R}(\mathbf{\Delta})_{\mathbf{ab}} = (\hat{\mathbf{R}} + \mathbf{\Delta})_{\mathbf{ab}} \in \mathcal{R}_p & \forall \mathbf{\Delta} \in \mathcal{D}, \mathbf{a} \neq \mathbf{b} \\
& \mathbf{S}(\mathbf{\Delta}) = \hat{\mathbf{S}}(I - \mathbf{\Delta}\hat{\mathbf{S}})^{-1} \in \mathcal{RH}_\infty & \forall \mathbf{\Delta} \in \mathcal{D} \\
& (\mathbf{R}(\mathbf{\Delta}), \mathbf{S}(\mathbf{\Delta})) \in \mathcal{C} & \forall \mathbf{\Delta} \in \mathcal{D}
\end{aligned}
$$

Deriving Existing and New Results Using Robust Stability Condition

- The robust stability condition provides a unified and simpler way to reproduce existing results – they are all special cases of the robust stability condition.

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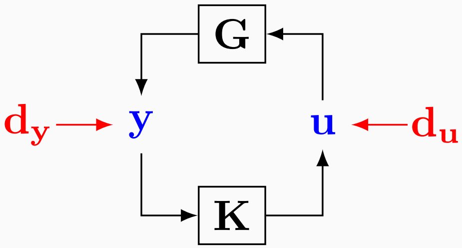
- The robust stability condition provides a unified and simpler way to reproduce existing results – they are all special cases of the robust stability condition.
- Such a unified approach also allows one to easily extend existing results to different settings and derive new robust results.

Example: Robust Input-Output Parameterization (IOP)



Example: Robust Input-Output Parameterization (IOP)

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \underbrace{\begin{bmatrix} O & \mathbf{G} \\ \mathbf{K} & O \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{d_y} \\ \mathbf{d_u} \end{bmatrix}$$

Example: Robust Input-Output Parameterization (IOP)

$$\underbrace{\begin{bmatrix} I & -\mathbf{G} \\ -\mathbf{K} & I \end{bmatrix}}_{I-\mathbf{R}} \underbrace{\begin{bmatrix} \mathbf{S_{yy}} & \mathbf{S_{uy}} \\ \mathbf{S_{uy}} & \mathbf{S_{uu}} \end{bmatrix}}_{\mathbf{S}} = I$$

Example: Robust Input-Output Parameterization (IOP)

$$\underbrace{\begin{bmatrix} I & -\mathbf{G} \\ -\mathbf{K} & I \end{bmatrix}}_{I-\mathbf{R}} \underbrace{\begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix}}_{\mathbf{S}} = I$$

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$$\underbrace{\begin{bmatrix} I & -\mathbf{G} \\ -\mathbf{K} & I \end{bmatrix}}_{I-\mathbf{R}} \underbrace{\begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix}}_{\mathbf{S} \in \mathcal{RH}_\infty} = I$$

Example: Robust Input-Output Parameterization (IOP)

Let $\mathcal{K}_\epsilon$ be

$$\mathcal{K}_\epsilon = \{\mathbf{K} : \mathbf{K} \text{ internally stabilizes } \mathbf{G}(\mathbf{\Delta_G}), \forall \mathbf{\Delta_G} \in \mathcal{D}_\epsilon\}$$

where

$$\mathbf{G}(\mathbf{\Delta_G}) = \hat{\mathbf{G}} + \mathbf{\Delta_G} \quad \text{and} \quad \mathcal{D}_\epsilon = \{\mathbf{\Delta_G} : \|\mathbf{\Delta_G}\|_\infty < \epsilon\}\,.$$

Example: Robust Input-Output Parameterization (IOP)

$$\begin{bmatrix} I & -\hat{\mathbf{G}} \\ -\hat{\mathbf{K}} & I \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Y}} & \hat{\mathbf{W}} \\ \hat{\mathbf{U}} & \hat{\mathbf{Z}} \end{bmatrix} = I$$

$$\hat{\mathbf{G}} \Rightarrow \hat{\mathbf{G}} + \boldsymbol{\Delta}_{\mathbf{G}}$$

$\hat{\mathbf{K}}$ still stablizing $\forall \boldsymbol{\Delta}_{\mathbf{G}} \in \mathcal{D}_\epsilon$?

Example: Robust Input-Output Parameterization (IOP)

$$\mathbf{R}(\boldsymbol{\Delta}_{\mathbf{G}}) = \underbrace{\begin{bmatrix} O & \hat{\mathbf{G}} \\ \hat{\mathbf{K}} & O \end{bmatrix}}_{\hat{\mathbf{R}}} + \underbrace{\begin{bmatrix} O & \boldsymbol{\Delta}_{\mathbf{G}} \\ O & O \end{bmatrix}}_{\boldsymbol{\Delta}}$$

Example: Robust Input-Output Parameterization (IOP)

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$$\Rightarrow \mathbf{S}(\boldsymbol{\Delta}) = \hat{\mathbf{S}}(I - \boldsymbol{\Delta}\hat{\mathbf{S}})^{-1} \in \mathcal{RH}_\infty$$

Example: Robust Input-Output Parameterization (IOP)

$$\hat{\mathbf{S}} \in \mathcal{RH}_\infty \Rightarrow$$

$$\left(I - \begin{bmatrix} O & \mathbf{\Delta}_{\mathbf{G}} \\ O & O \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Y}} & \hat{\mathbf{W}} \\ \hat{\mathbf{U}} & \hat{\mathbf{Z}} \end{bmatrix} \right)^{-1} \in \mathcal{RH}_\infty$$

$$\Rightarrow \mathbf{S}(\mathbf{\Delta}) = \hat{\mathbf{S}}(I - \mathbf{\Delta}\hat{\mathbf{S}})^{-1} \in \mathcal{RH}_\infty$$

Example: Robust Input-Output Parameterization (IOP)

$$\left(I - \begin{bmatrix} O & \mathbf{\Delta_G} \\ O & O \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Y}} & \hat{\mathbf{W}} \\ \hat{\mathbf{U}} & \hat{\mathbf{Z}} \end{bmatrix}\right)^{-1} \in \mathcal{RH}_\infty$$

$$\Leftrightarrow (I - \mathbf{\Delta_G}\hat{\mathbf{U}})^{-1} \in \mathcal{RH}_\infty$$

Example: Robust Input-Output Parameterization (IOP)

$$
\left(I - \begin{bmatrix} O & \mathbf{\Delta_G} \\ O & O \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Y}} & \hat{\mathbf{W}} \\ \hat{\mathbf{U}} & \hat{\mathbf{Z}} \end{bmatrix} \right)^{-1} \in \mathcal{RH}_\infty
$$

$$
\Leftrightarrow (I - \mathbf{\Delta_G} \hat{\mathbf{U}})^{-1} \in \mathcal{RH}_\infty
$$

$$
\Leftrightarrow \left\| \hat{\mathbf{U}} \right\|_\infty \leq \epsilon^{-1}
$$

Example: Robust Input-Output Parameterization (IOP)

$$\begin{bmatrix} I & -\hat{\mathbf{G}} \\ -\hat{\mathbf{K}} & I \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Y}} & \hat{\mathbf{W}} \\ \hat{\mathbf{U}} & \hat{\mathbf{Z}} \end{bmatrix} = I$$

$$\hat{\mathbf{S}} \in \mathcal{RH}_\infty$$

$$\left\| \hat{\mathbf{U}} \right\|_\infty \leq \epsilon^{-1}$$

Example: Robust Output-Feedback System Level Synthesis (SLS)

$$
\begin{aligned}
z\mathbf{x} &= A\mathbf{x} + B\mathbf{u} + \mathbf{d_x} \\
\mathbf{u} &= \hat{\mathbf{K}}\mathbf{y} + \mathbf{d_u} \\
\mathbf{y} &= C\mathbf{x} + D\mathbf{u} + \mathbf{d_y}
\end{aligned}
$$

Example: Robust Output-Feedback System Level Synthesis (SLS)

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \\ \mathbf{y} \end{bmatrix} = \underbrace{\begin{bmatrix} A + I - zI & B & O \\ O & O & \hat{\mathbf{K}} \\ C & D & O \end{bmatrix}}_{\hat{\mathbf{R}}} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} \mathbf{d}_{\mathbf{x}} \\ \mathbf{d}_{\mathbf{u}} \\ \mathbf{d}_{\mathbf{y}} \end{bmatrix}$$

Example: Robust Output-Feedback System Level Synthesis (SLS)

$$\underbrace{\begin{bmatrix} zI - A & -B & O \\ O & I & -\hat{\mathbf{K}} \\ -C & -D & I \end{bmatrix}}_{I - \hat{\mathbf{R}}} \underbrace{\begin{bmatrix} \hat{\mathbf{\Phi}}_{\mathbf{xx}} & \hat{\mathbf{S}}_{\mathbf{xu}} & \hat{\mathbf{\Phi}}_{\mathbf{xy}} \\ \hat{\mathbf{\Phi}}_{\mathbf{ux}} & \hat{\mathbf{S}}_{\mathbf{uu}} & \hat{\mathbf{\Phi}}_{\mathbf{uy}} \\ \hat{\mathbf{S}}_{\mathbf{yx}} & \hat{\mathbf{S}}_{\mathbf{yu}} & \hat{\mathbf{S}}_{\mathbf{yy}} \end{bmatrix}}_{\hat{\mathbf{S}}} = I$$

Example: Robust Output-Feedback System Level Synthesis (SLS)

$$\underbrace{\begin{bmatrix} zI - A & -B & O \\ O & I & -\hat{\mathbf{K}} \\ -C & -D & I \end{bmatrix}}_{I - \hat{\mathbf{R}}} \underbrace{\begin{bmatrix} \hat{\mathbf{\Phi}}_{\mathbf{xx}} & \hat{\mathbf{S}}_{\mathbf{xu}} & \hat{\mathbf{\Phi}}_{\mathbf{xy}} \\ \hat{\mathbf{\Phi}}_{\mathbf{ux}} & \hat{\mathbf{S}}_{\mathbf{uu}} & \hat{\mathbf{\Phi}}_{\mathbf{uy}} \\ \hat{\mathbf{S}}_{\mathbf{yx}} & \hat{\mathbf{S}}_{\mathbf{yu}} & \hat{\mathbf{S}}_{\mathbf{yy}} \end{bmatrix}}_{\hat{\mathbf{S}} \in \mathcal{RH}_\infty} = I$$

Example: Robust Output-Feedback System Level Synthesis (SLS)

Let $\mathcal{K}_\epsilon$ be

$$\mathcal{K}_\epsilon = \{\mathbf{K} : \mathbf{K} \text{ internally stabilizes perturbed plant}, \forall \mathbf{\Delta} \in \mathcal{D}_\epsilon\}$$

where

$$\begin{bmatrix} A(\mathbf{\Delta}) & B(\mathbf{\Delta}) \\ C(\mathbf{\Delta}) & D(\mathbf{\Delta}) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} \mathbf{\Delta}_A & \mathbf{\Delta}_B \\ \mathbf{\Delta}_C & \mathbf{\Delta}_D \end{bmatrix}$$

and

$$\mathcal{D}_\epsilon = \left\{ \mathbf{\Delta} = \begin{bmatrix} \mathbf{\Delta}_A & \mathbf{\Delta}_B & O \\ O & O & O \\ \mathbf{\Delta}_C & \mathbf{\Delta}_D & O \end{bmatrix} : \|\mathbf{\Delta}\|_\infty < \epsilon \right\}.$$

Example: Robust Output-Feedback System Level Synthesis (SLS)

$$\begin{bmatrix} zI - A & -B & O \\ O & I & -\hat{\mathbf{K}} \\ -C & -D & I \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_{\mathbf{xx}} & \hat{\mathbf{S}}_{\mathbf{xu}} & \hat{\mathbf{\Phi}}_{\mathbf{xy}} \\ \hat{\mathbf{\Phi}}_{\mathbf{ux}} & \hat{\mathbf{S}}_{\mathbf{uu}} & \hat{\mathbf{\Phi}}_{\mathbf{uy}} \\ \hat{\mathbf{S}}_{\mathbf{yx}} & \hat{\mathbf{S}}_{\mathbf{yu}} & \hat{\mathbf{S}}_{\mathbf{yy}} \end{bmatrix} = I$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} \mathbf{\Delta}_A & \mathbf{\Delta}_B \\ \mathbf{\Delta}_C & \mathbf{\Delta}_D \end{bmatrix}$$

$$\hat{\mathbf{K}} \text{ still stablizing } \forall \begin{bmatrix} \mathbf{\Delta}_A & \mathbf{\Delta}_B \\ \mathbf{\Delta}_C & \mathbf{\Delta}_D \end{bmatrix} \in \mathcal{D}_\epsilon?$$

Example: Robust Output-Feedback System Level Synthesis (SLS)

$$\mathbf{R}(\boldsymbol{\Delta}) = \underbrace{\begin{bmatrix} A + I - zI & B & O \\ O & O & \hat{\mathbf{K}} \\ C & D & O \end{bmatrix}}_{\hat{\mathbf{R}}} + \underbrace{\begin{bmatrix} \boldsymbol{\Delta}_A & \boldsymbol{\Delta}_B & O \\ O & O & O \\ \boldsymbol{\Delta}_C & \boldsymbol{\Delta}_D & O \end{bmatrix}}_{\boldsymbol{\Delta}}$$

Example: Robust Output-Feedback System Level Synthesis (SLS)

$$\mathbf{R}(\boldsymbol{\Delta}) = \underbrace{\begin{bmatrix} A + I - zI & B & O \\ O & O & \hat{\mathbf{K}} \\ C & D & O \end{bmatrix}}_{\hat{\mathbf{R}}} + \underbrace{\begin{bmatrix} \boldsymbol{\Delta}_A & \boldsymbol{\Delta}_B & O \\ O & O & O \\ \boldsymbol{\Delta}_C & \boldsymbol{\Delta}_D & O \end{bmatrix}}_{\boldsymbol{\Delta}}$$

$$\Rightarrow \mathbf{S}(\boldsymbol{\Delta}) = \hat{\mathbf{S}}(I - \boldsymbol{\Delta}\hat{\mathbf{S}})^{-1} \in \mathcal{RH}_\infty$$

Example: Robust Output-Feedback System Level Synthesis (SLS)

$$\hat{\mathbf{S}} \in \mathcal{RH}_\infty \Rightarrow (I - \mathbf{\Delta}\hat{\mathbf{S}})^{-1} \in \mathcal{RH}_\infty \Leftrightarrow$$

$$\left(I - \begin{bmatrix} \mathbf{\Delta}_A & \mathbf{\Delta}_B \\ \mathbf{\Delta}_C & \mathbf{\Delta}_D \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_{\mathbf{xx}} & \hat{\mathbf{\Phi}}_{\mathbf{xy}} \\ \hat{\mathbf{\Phi}}_{\mathbf{ux}} & \hat{\mathbf{\Phi}}_{\mathbf{uy}} \end{bmatrix} \right)^{-1} \in \mathcal{RH}_\infty$$

$$\Rightarrow \mathbf{S}(\mathbf{\Delta}) = \hat{\mathbf{S}}(I - \mathbf{\Delta}\hat{\mathbf{S}})^{-1} \in \mathcal{RH}_\infty$$

Example: Robust Output-Feedback System Level Synthesis (SLS)

$$\left(I - \begin{bmatrix} \mathbf{\Delta}_A & \mathbf{\Delta}_B \\ \mathbf{\Delta}_C & \mathbf{\Delta}_D \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_{\mathbf{xx}} & \hat{\mathbf{\Phi}}_{\mathbf{xy}} \\ \hat{\mathbf{\Phi}}_{\mathbf{ux}} & \hat{\mathbf{\Phi}}_{\mathbf{uy}} \end{bmatrix}\right)^{-1} \in \mathcal{RH}_\infty$$

$$\Leftrightarrow \left\| \begin{bmatrix} \hat{\mathbf{\Phi}}_{\mathbf{xx}} & \hat{\mathbf{\Phi}}_{\mathbf{xy}} \\ \hat{\mathbf{\Phi}}_{\mathbf{ux}} & \hat{\mathbf{\Phi}}_{\mathbf{uy}} \end{bmatrix} \right\|_\infty \leq \epsilon^{-1}$$

Example: Robust Output-Feedback System Level Synthesis (SLS)

$$\begin{bmatrix} zI - A & -B & O \\ O & I & -\hat{\mathbf{K}} \\ -C & -D & I \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_{\mathbf{xx}} & \hat{\mathbf{S}}_{\mathbf{xu}} & \hat{\mathbf{\Phi}}_{\mathbf{xy}} \\ \hat{\mathbf{\Phi}}_{\mathbf{ux}} & \hat{\mathbf{S}}_{\mathbf{uu}} & \hat{\mathbf{\Phi}}_{\mathbf{uy}} \\ \hat{\mathbf{S}}_{\mathbf{yx}} & \hat{\mathbf{S}}_{\mathbf{yu}} & \hat{\mathbf{S}}_{\mathbf{yy}} \end{bmatrix} = I$$

$$\hat{\mathbf{S}} \in \mathcal{RH}_\infty, \qquad \left\| \begin{bmatrix} \hat{\mathbf{\Phi}}_{\mathbf{xx}} & \hat{\mathbf{\Phi}}_{\mathbf{xy}} \\ \hat{\mathbf{\Phi}}_{\mathbf{ux}} & \hat{\mathbf{\Phi}}_{\mathbf{uy}} \end{bmatrix} \right\|_\infty \leq \epsilon^{-1}$$

Conclusion

- Robust stability condition – an analysis approach that
  - unifies several existing robust results
  - allows easy derivation of new results
  - leads to the formulation of the general robust controller synthesis problem

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Conclusion

- Robust stability condition – an analysis approach that
  - unifies several existing robust results
  - allows easy derivation of new results
  - leads to the formulation of the general robust controller synthesis problem
- Can we unify all robust controller synthesis results?

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