Gauss Elimination

Chapra: Chapter-9

Equation Solving

- Linear Algebra
  - Iterative Methods
    - Jacobi’s
    - Gauss-Seidel
  - Direct Methods
    - Gauss Elimination
    - Gauss-Jordan
    - Choleski’s

System of Linear Equations

- A set of *n* equations and *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\begin{matrix} \vdots & & \vdots \end{matrix}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

Gauss Elimination

- One of the most popular techniques for solving simultaneous linear equations of the form.
- Consists of 2 steps

  1. Forward Elimination of Unknowns.
  2. Back Substitution

$$[A][X]=[B]$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Forward Elimination of Unknowns

- It is designed to reduce the set of equations to an upper triangular system.
- The initial step will be to eliminate the first unknown, x_1, from the second through the nth equations.
- Multiply the first equation by a_{21}/a_{11}.

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \frac{a_{21}}{a_{11}}a_{13}x_3 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

- Subtract it from the second equation.

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

- Or $$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

Forward Elimination of Unknowns

- The procedure is then repeated for the remaining equations. For instance, the first equation can be multiplied by a_{31}/a_{11} and the result subtracted from the third.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \quad \ldots.(1)$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2 \quad \ldots.(2)$$

$$a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n = b'_3 \quad \ldots.(3)$$

- Multiply eq. (2) by a'_{32}/a'_{22} and subtract the result from eq.(3)

$$a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3$$

- After (n – 1)th iteration; $a_{nn}^{n-1}x_n = b_n^{n-1}$

Back Substitution

- The last equation can be solved for x_n.

$$x_n = \frac{b_n^{n-1}}{a_{nn}^{n-1}}$$

- This result can be back-substituted into the (n – l)th equation to solve for x_{n-1} .
- The procedure which is repeated to evaluate the remaining x's, can be represented by the following formula;

$$x_i = \frac{b_i^{i-1} - \sum_{j=i+1}^{n} a_{ij}^{i-1} x_j}{a_{ii}^{i-1}} \qquad for\ \ i = n-1, n-2, \ldots, 1$$

Example

- Use Gauss Elimination to solve:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \qquad \ldots.(1)$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \qquad \ldots.(2)$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \qquad \ldots.(3)$$

Carry six significant figures during computation.

- Multiply (1) by 0.1/3 and subtract the result from (2)
- Multiply (1) by 0.3/3 and subtract the result from (3)

$$3x_1 \quad -0.1x_2 \quad -0.2x_3 = 7.85 \qquad \ldots.(4)$$
$$7.00333x_2 - 0.293333x_3 = -19.5617 \qquad \ldots.(5)$$
$$-0.190000x_2 + 10.0200x_3 = 70.6150 \qquad \ldots.(6)$$

- Multiply (5) by – 0.19/7.00333 and subtract the result from (6)

Example

- The upper triangular form is:

$$\begin{aligned} 3x_1 \quad -0.1x_2 \quad -0.2x_3 &= 7.85 \\ 7.00333x_2 - 0.293333x_3 &= -19.5617 \\ 10.0120x_3 &= 70.0843 \end{aligned}$$

- Now x_3 can be found;

$$x_3 = \frac{70.0843}{10.0120} = 7.0000$$

$$x_2 = \frac{-19.5617 + 0.293333(7.0000)}{7.00333} = -2.50000$$

$$x_1 = \frac{7.85 + 0.1(-2.50000) + 0.2(7.0000)}{3} = 3.00000$$

Problem

- The upward velocity of a rocket is given at three different times.

Time, t (s)	**Velocity, v (m/s)**
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 \ , \qquad 5 \le t \le 12.$$

Find the velocity at t=6, 9 and 11 seconds.

Pitfalls of Elimination Methods

- Divide by zero:

$$
\begin{aligned}
2x_2 + 3x_3 &= 8 \\
4x_1 + 6x_2 + 7x_3 &= -3 \\
2x_1 + x_2 + 6x_3 &= 5
\end{aligned}
$$

- Round-Off errors:
  - As we use decimals of some significant figures instead of fraction, the round-off error can occur when large number of equations are to be solved.

Techniques for Improving Solutions

- Partial Pivoting:
  - Gaussian Elimination with partial pivoting applies row switching to normal Gaussian Elimination.

    How?
  - At the beginning of the k-th step of forward elimination, find the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \ldots\ldots\ldots\ldots\ldots, |a_{nk}|$$

  - If the maximum of the values is $|a_{pk}|$ in the p-th row, $k \le p \le n$,
  - then switch rows p and k.

Example

- Use Gauss elimination to solve

$$0.0003x_1 + 3.0000x_2 = 2.0001$$
$$1.0000x_1 + 1.0000x_2 = 1.0000$$

The exact solution is $x_1 = 1/3$ and $x_2 = 2/3$

- Multiply the first equation by 1/(0.0003) yields

$$x_1 + 10{,}000x_2 = 6667$$
$$-9999x_2 = -6666$$
$$x_2 = 2/3$$

$$x_1 = \frac{2.0001 - 3(2/3)}{0.0003}$$

Example

Significant Figures	x_2	x_1	Absolute Value of Percent Relative Error for x_1
3	0.667	−3.33	1099
4	0.6667	0.0000	100
5	0.66667	0.30000	10
6	0.666667	0.330000	1
7	0.6666667	0.3330000	0.1

The equations are solved in reverse order, the row with the larger pivot element is normalized.

$$1.0000x_1 + 1.0000x_2 = 1.0000$$
$$0.0003x_1 + 3.0000x_2 = 2.0001$$

Elimination and substitution yield $x_2 = 2/3$

$$x_1 = \frac{1 - (2/3)}{1}$$

Example

Significant Figures	x_2	x_1	Absolute Value of Percent Relative Error for x_1
3	0.667	0.333	0.1
4	0.6667	0.3333	0.01
5	0.66667	0.33333	0.001
6	0.666667	0.333333	0.0001
7	0.6666667	0.3333333	0.00001

Thus, a pivot strategy is much more satisfactory.