Newton-Cotes Integration Formulas

Chapra: Chapter-21

Newton-Cotes Integration Formulas

- The Newton-Cotes formulas are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate:

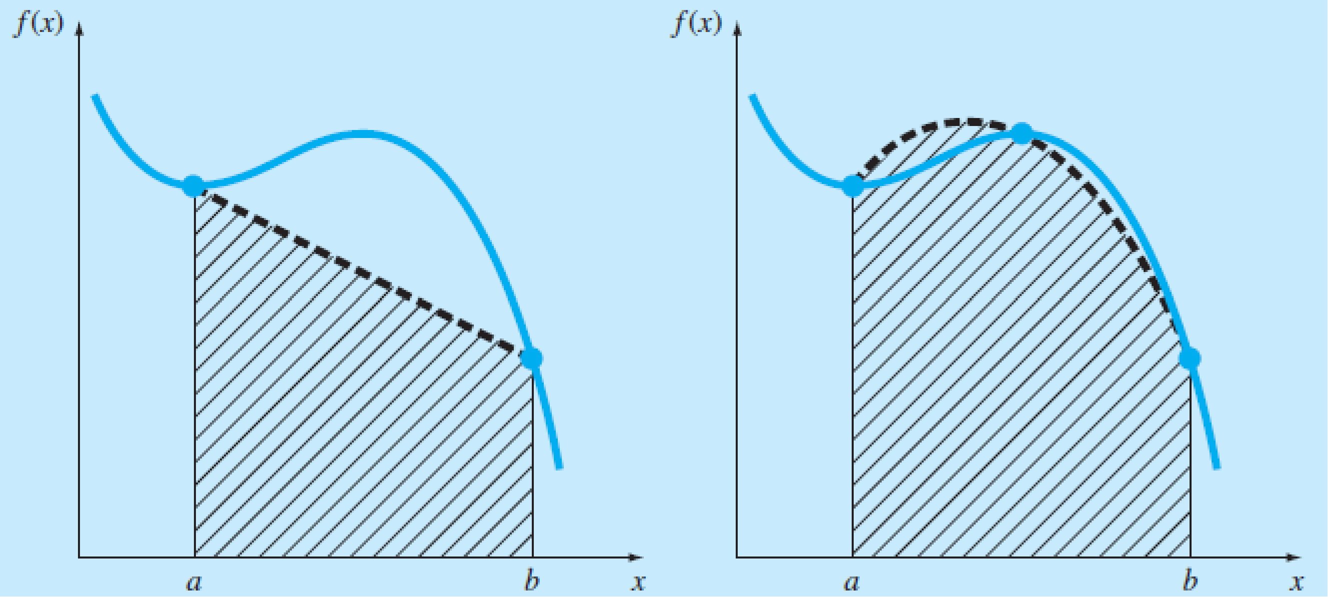
$$I = \int_a^b f(x)\,dx \cong \int_a^b f_n(x)\,dx$$

where $f_n(x)$ is a polynomial of the form

$$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n$$

Newton-Cotes Integration Formulas

- In the first figure, first-order polynomial (a straight line) is used as an approximation. In second figure, a parabola is employed for the same purpose.



The Trapezoidal Rule

- It corresponds to the case where the polynomial in equation of integration is first order:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_1(x)\,dx$$

- A straight line can be represented as

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

- The area under this straight line is an estimate of the integral of $f(x)$ between the limits a and b:

$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx$$

The Trapezoidal Rule

- The result of the integration is:

$$I = (b - a)\frac{f(a) + f(b)}{2}$$

- The formula for computing the area of a trapezoid is the height times the average of the bases.

$$I \cong \text{width} \times \text{average height}$$

The Trapezoidal Rule

- Use equation for trapezoidal rule to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. The exact value of the integral can be determined analytically to be 1.640533.

- The function values

$f(0) = 0.2$

$f(0.8) = 0.232$

can be substituted to yield $$I \cong 0.8\frac{0.2 + 0.232}{2} = 0.1728$$

which represents an error of $E_t = 1.640533 - 0.1728 = 1.467733$

which corresponds to a percent relative error of $\varepsilon_t = 89.5\%$.

The Multiple-Application Trapezoidal Rule

- One way to improve the accuracy of the trapezoidal rule is to divide the integration interval from a to b into a number of segments and apply the method to each segment.
- There are $n + 1$ equally spaced base points $(x_0, x_1, x_2, \ldots, x_n)$. There are n segments of equal width, $h = (b - a)/n$
- If a and b are designated as x_0 and x_n, respectively, the total integral can be represented as

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

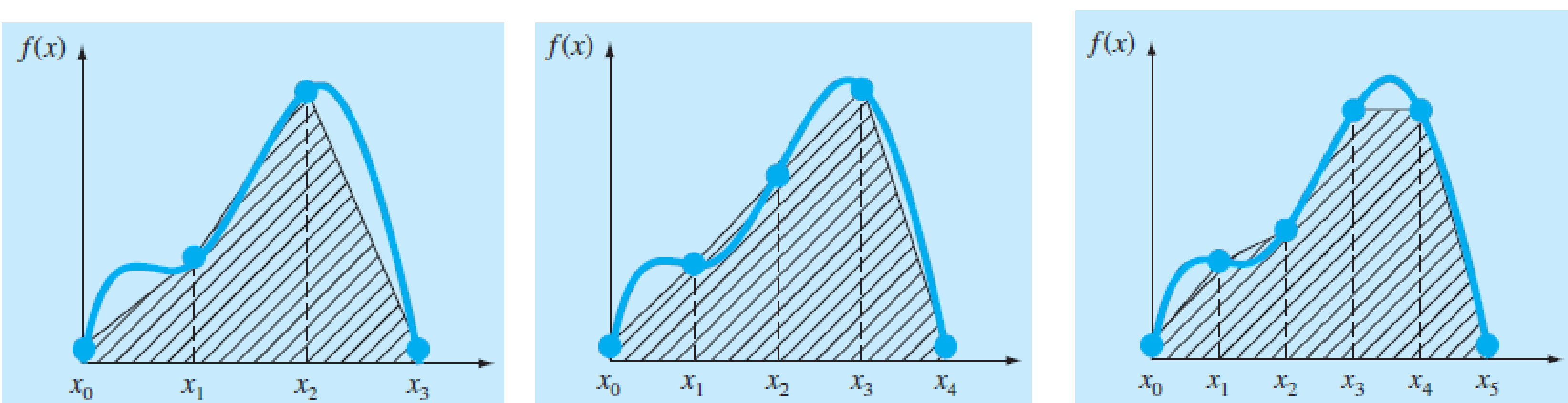
The Multiple-Application Trapezoidal Rule

- Substituting the trapezoidal rule for each integral yields

$$I = h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + \cdots + h\frac{f(x_{n-1}) + f(x_n)}{2}$$

- Or, grouping terms,

$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$



The Multiple-Application Trapezoidal Rule

- Use the two segment trapezoidal rule to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. The exact value of the integral can be determined analytically to be 1.640533.

- $n = 2$ ($h = 0.4$):

$f(0) = 0.2$ $\quad f(0.4) = 2.456$ $\quad f(0.8) = 0.232$

$$I = 0.8\frac{0.2 + 2(2.456) + 0.232}{4} = 1.0688$$

$E_t = 1.640533 - 1.0688 = 0.57173$ $\qquad \varepsilon_t = 34.9\%$

Simpson's 1/3 Rule

- Simpson's 1/3 rule results when a second-order interpolating polynomial is substituted into

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_2(x)\,dx$$

- If a and b are designated as x_0 and x_2 and $f_2(x)$ is represented by a second-order Lagrange polynomial, the integral becomes

$$I = \int_{x_0}^{x_2} \left[\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] dx$$

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- If a and b are designated as x_0 and x_2 and $f_2(x)$ is represented by a second-order Lagrange polynomial, the integral becomes

$$I = \int_{x_0}^{x_2} \left[\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] dx$$

Simpson's 1/3 Rule

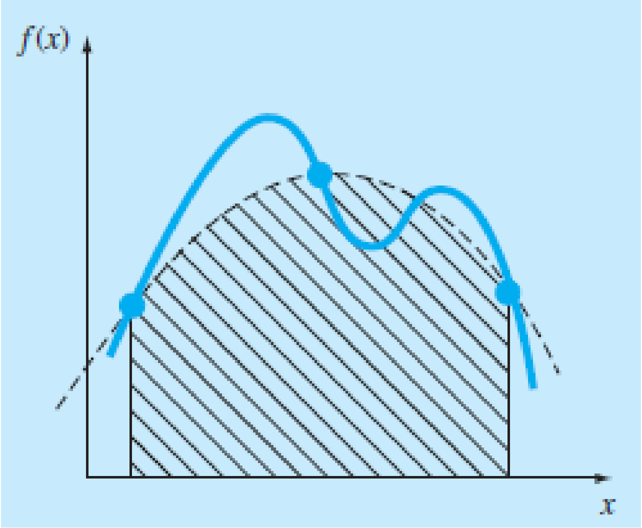
- After integration and algebraic manipulation, the following formula results:

$$I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

where, for this case, $h = (b - a)/2$.

- This equation is known as Simpson's 1/3 rule.
- It is the second Newton-Cotes closed integration formula.
- The label "1/3" stems from the fact that h is divided by 3 in above equation.

$$I \cong \underbrace{(b - a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{Average height}}$$



Simpson's 1/3 Rule

- Use Simpson's 1/3 rule to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. The exact value of the integral can be determined analytically to be 1.640533.

- $f(0) = 0.2$ $f(0.4) = 2.456$ $f(0.8) = 0.232$

$$I \cong 0.8 \frac{0.2 + 4(2.456) + 0.232}{6} = 1.367467$$

- which represents an exact error of

$$E_t = 1.640533 - 1.367467 = 0.2730667 \quad \varepsilon_t = 16.6\%$$

Multiple-Application Simpson’s 1/3 Rule

- Just as with the trapezoidal rule, Simpson’s rule can be improved by dividing the integration interval into a number of segments of equal width: $h = (b - a)/n$
- The total integral can be represented as

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

- Substituting Simpson’s 1/3 rule for the individual integral yields

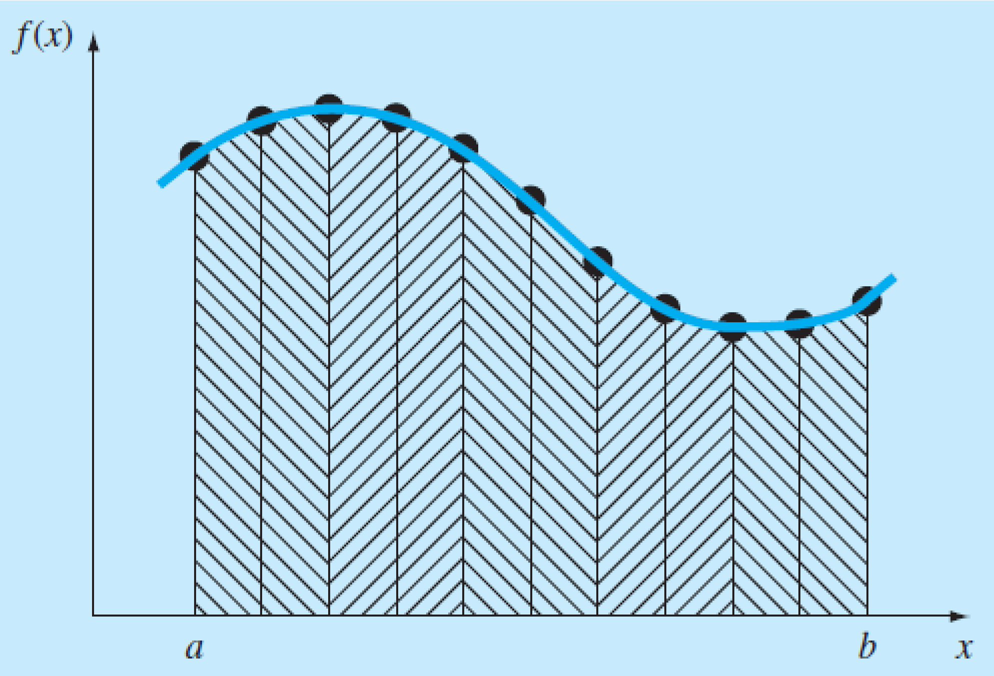
$$I \cong 2h\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + 2h\frac{f(x_2) + 4f(x_3) + f(x_4)}{6}$$

$$+ \cdots + 2h\frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}$$

Multiple-Application Simpson's 1/3 Rule

- Combining terms

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}}_{\text{Average height}}$$



Multiple-Application Simpson's 1/3 Rule

- Use Multiple-Application Simpson's 1/3 rule with $n = 4$ to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. The exact value of the integral can be determined analytically to be 1.640533.

- $n = 4$ ($h = 0.2$):

$f(0) = 0.2$ $\quad f(0.2) = 1.288$ $\quad f(0.4) = 2.456$ $\quad f(0.6) = 3.464$

$f(0.8) = 0.232$

$$I = 0.8\,\frac{0.2 + 4(1.288 + 3.464) + 2(2.456) + 0.232}{12} = 1.623467$$

$E_t = 1.640533 - 1.623467 = 0.017067 \quad \varepsilon_t = 1.04\%$

Simpson’s 3/8 Rule

- In a similar manner to the derivation of the trapezoidal and Simpson’s 1/3 rule, a third-order Lagrange polynomial can be fit to four points and integrated:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_3(x)\,dx$$

to yield

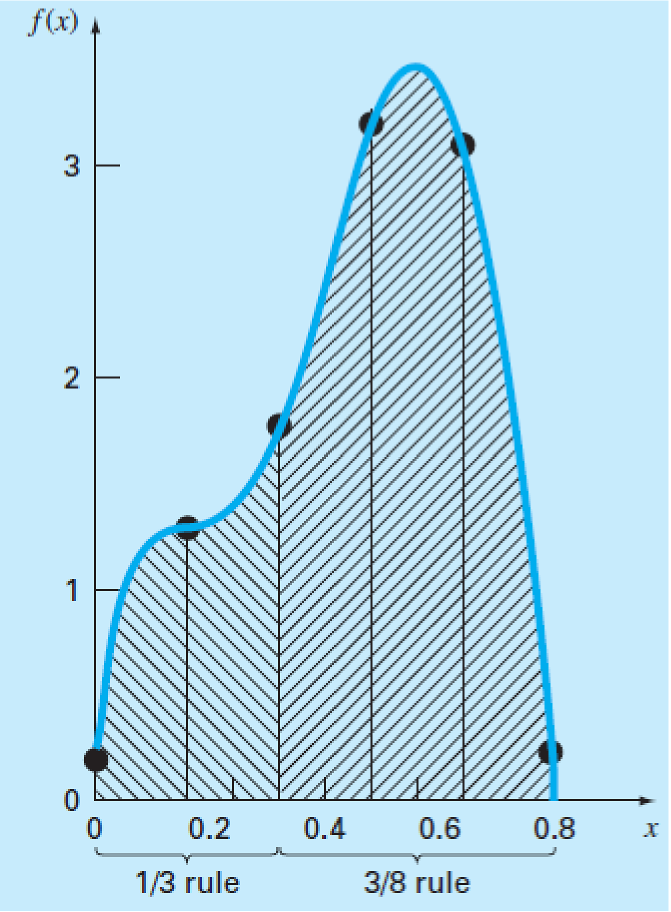
$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

where $h = (b - a)/3$.

Simpson's 3/8 Rule

- This equation is called Simpson's 3/8 rule because *h* is multiplied by 3/8.
- It is the third Newton-Cotes closed integration formula.
- The 3/8 rule can also be expressed as

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}}_{\text{Average height}}$$



Multiple-Application Simpson's 1/3 Rule

a) Use Simpson's 3/8 rule to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. The exact value of the integral can be determined analytically to be 1.640533.

b) Use it in conjunction with Simpson's 1/3 rule to integrate the same function for five segments.

- A single application of Simpson's 3/8 rule requires four equally spaced points:

  $f(0) = 0.2$ $\quad f(0.2667) = 1.432724$ $\quad f(0.5333) = 3.487177$

  $f(0.8) = 0.232$

Multiple-Application Simpson’s 1/3 Rule

$$I \cong 0.8 \frac{0.2 + 3(1.432724 + 3.487177) + 0.232}{8} = 1.519170$$

$E_t = 1.640533 – 1.519170 = 0.1213630 \quad \varepsilon_t = 7.4\%$

- The data needed for a five-segment application ($h = 0.16$) is

$f(0) = 0.2 \quad f(0.16) = 1.296919 \quad f(0.32) = 1.743393 \quad f(0.48) = 3.186015 \quad f(0.64) = 3.181929 \quad f(0.80) = 0.232$

The integral for the first two segments is obtained using Simpson’s 1/3 rule:

$$I \cong 0.32 \frac{0.2 + 4(1.296919) + 1.743393}{6} = 0.3803237$$

Multiple-Application Simpson's 1/3 Rule

For the last three segments, the 3/8 rule can be used to obtain

$$I \cong 0.48 \frac{1.743393 + 3(3.186015 + 3.181929) + 0.232}{8} = 1.264754$$

The total integral is computed by summing the two results:

$I =$ 0.3803237 + 1.264753 = 1.645077

E_t = 1.640533 – 1.645077 = –0.00454383 ε_t = –0.28%