LU Decomposition and Matrix Inversion

Chapra: Chapter-10

LU Decomposition

- Gauss elimination requires a bulk of computational efforts.
- To solve the linear algebraic equations, [A]{X} = {B}, LU decomposition methods separate the time-consuming elimination of the matrix[A] from the manipulations of the right-hand side {B}.
- Thus, once [A] has been "decomposed," multiple right-hand-side vectors can be evaluated in an efficient manner.

LU Decomposition

- Rearranging the equations, [A]{X} – {B} = 0. …..(1)
- This could be expressed as an upper triangular system

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

- Rearranging the above equations, [U]{X} – {D} = 0. …..(2)
- Assume a lower diagonal matrix with 1's on the diagonal

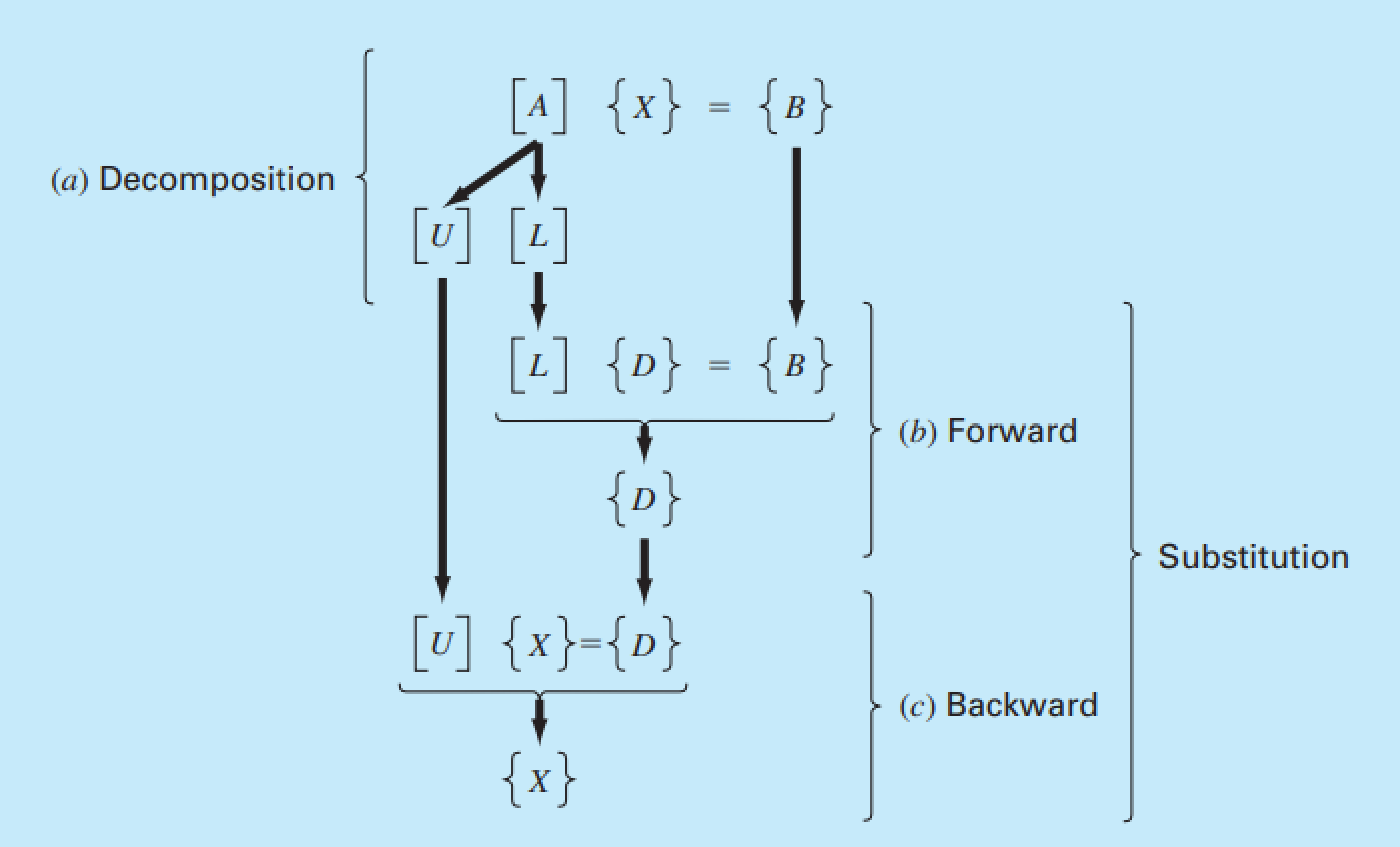
$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

[L]{[U]{X} – {D}} = [A]{X} – {B} …..(3)

LU Decomposition

- [L][U] = [A] and [L]{D} = {B}
- A two-step strategy
  - *LU decomposition step*. [A] is factored or “decomposed” into lower [L] and upper [U] triangular matrices.
  - *Substitution step*. [L] and [U] are used to determine a solution {X} for a right-hand-side {B}. This step itself consists of two steps.
    - First, generation of an intermediate vector {D} by forward substitution.
    - The result is substituted into (2), which can be solved by back substitution for {X}.

LU Decomposition



LU Decomposition Version of Gauss Elimination

- The forward elimination step is intended to reduce the original coefficient matrix [A] to the form

$$[U] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

- The first step in Gauss elimination is to multiply row 1 by the factor, $f_{21} = a_{21}/a_{11}$.
- Similarly, row 1 is multiplied by $f_{31} = a_{31}/a_{11}$.
- The final step is to multiply the modified second row by $f_{32} = a'_{32}/a'_{22}$.

LU Decomposition Version of Gauss Elimination

- The [A] matrix can therefore be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ f_{21} & a'_{22} & a'_{23} \\ f_{31} & f_{32} & a''_{33} \end{bmatrix}$$

- [A] = [L][U]

$$[U] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \qquad [L] = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}$$

Example

- Derive an LU decomposition based on the Gauss elimination performed for

$$\begin{aligned}3x_1 - 0.1x_2 - 0.2x_3 &= 7.85\\0.1x_1 + 7x_2 - 0.3x_3 &= -19.3\\0.3x_1 - 0.2x_2 + 10x_3 &= 71.4\end{aligned}$$

- Matrix A =

$$[A] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}$$

- Matrix U =

$$[U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

Example

- $f_{21} = 0.1/3 = 0.03333333$ $f_{31} = 0.3/3 = 0.1000000$
  $f_{32} = -0.19/7.00333 = -0.0271300$
- The lower triangular matrix is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix}$$

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 0.0333333 & 1 & 0 \\ 0.100000 & -0.0271300 & 1 \end{bmatrix} \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

$$[L][U] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.0999999 & 7 & -0.3 \\ 0.3 & -0.2 & 9.99996 \end{bmatrix}$$

Matrix Inverse

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$$

- $[A]^{-1}$ is the inverse matrix of [A].
- Then, $[A]\,[A]^{-1} = [I]$ or $[A]^{-1}[A] = [I]$

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}$$

- Let $Z = [A]^{-1}$
- AZ = I
- LUZ = I
- LY = I
- UZ = Y

$$LY_1 = I_1 \Rightarrow \begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}y_{11}\\ y_{21}\\ y_{31}\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} \Rightarrow \begin{bmatrix}y_{11}\\ y_{21}\\ y_{31}\end{bmatrix}=\begin{bmatrix}1\\ -2.56\\ 3.2\end{bmatrix}$$

$$LY_2 = I_2 \Rightarrow \begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}y_{12}\\ y_{22}\\ y_{32}\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix} \Rightarrow \begin{bmatrix}y_{12}\\ y_{22}\\ y_{32}\end{bmatrix}=\begin{bmatrix}0\\ 1\\ -3.5\end{bmatrix}$$

$$LY_3 = I_3 \Rightarrow \begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}y_{13}\\ y_{23}\\ y_{33}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \Rightarrow \begin{bmatrix}y_{13}\\ y_{23}\\ y_{33}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

Matrix Inverse

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$$

- Z = [A]$^{-1}$
- AZ = I
- LUZ = I
- LY = I
- UZ = Y

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}$$

$$UZ_1=Y_1 \Rightarrow \begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}\begin{bmatrix}z_{11}\\ z_{21}\\ z_{31}\end{bmatrix}=\begin{bmatrix}1\\ -2.56\\ 3.2\end{bmatrix} \Rightarrow \begin{bmatrix}z_{11}\\ z_{21}\\ z_{31}\end{bmatrix}=\begin{bmatrix}0.04762\\ -0.9524\\ 4.571\end{bmatrix}$$

$$UZ_2=Y_2 \Rightarrow \begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}\begin{bmatrix}z_{12}\\ z_{22}\\ z_{32}\end{bmatrix}=\begin{bmatrix}0\\ 1\\ -3.5\end{bmatrix} \Rightarrow \begin{bmatrix}z_{12}\\ z_{22}\\ z_{32}\end{bmatrix}=\begin{bmatrix}-0.08333\\ 1.4167\\ -5\end{bmatrix}$$

$$UZ_3=Y_3 \Rightarrow \begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}\begin{bmatrix}z_{13}\\ z_{23}\\ z_{33}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \Rightarrow \begin{bmatrix}z_{13}\\ z_{23}\\ z_{33}\end{bmatrix}=\begin{bmatrix}0.0357\\ -0.4643\\ 1.4286\end{bmatrix}$$

Gauss-Seidel

Chapra: Chapter-11

Gauss-Seidel

- The most commonly used iterative method.
- Assume that we are given a set of n equations:

$$[A]\{X\} = \{B\}$$

- Suppose that for conciseness we limit ourselves to a 3 X 3 set of equations.
- If the diagonal elements are all nonzero, the first equation can be solved for x_1, the second for x_2, and the third for x_3 to yield

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Gauss-Seidel

- Choose guesses for the x's. Assume that they are all zero.
- These zeros can be substituted into (1), which can be used to calculate a new value for $x_1 = b_1/a_{11}$.
- Then, we substitute this new value of x_1 along with the previous guess of zero for x_3 into (2) to compute a new value for x_2.
- The process is repeated for (3) to calculate a new estimate for x_3.
- We return to the first equation and repeat the entire procedure until our solution converges closely enough to the true values.

$$|\varepsilon_{a,i}| = \left|\frac{x_i^j - x_i^{j-1}}{x_i^j}\right| 100\% < \varepsilon_s$$

Example

- Use the Gauss-Seidel method to obtain the solution of the same system used for

$$
\begin{aligned}
3x_1 - 0.1x_2 - 0.2x_3 &= 7.85 \\
0.1x_1 + 7x_2 - 0.3x_3 &= -19.3 \\
0.3x_1 - 0.2x_2 + 10x_3 &= 71.4
\end{aligned}
$$

Recall that the true solution is $x_1 = 3$, $x_2 = -2.5$, and $x_3 = 7$.

- First, solve each of the equations for its unknown on the diagonal.

$$x_1 = \frac{7.85 + 0.1x_2 + 0.2x_3}{3} \qquad \ldots(1)$$

$$x_2 = \frac{-19.3 - 0.1x_1 + 0.3x_3}{7} \qquad \ldots(2)$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10} \qquad \ldots(3)$$

Example

- By assuming that x_2 and x_3 are zero, (1) can be used to compute

$$x_1 = \frac{7.85 + 0 + 0}{3} = 2.616667$$

- This value, along with the assumed value of $x_3 = 0$, can be substituted into (2) to calculate

$$x_2 = \frac{-19.3 - 0.1(2.616667) + 0}{7} = -2.794524$$

- The first iteration is completed by substituting the calculated values for x_1 and x_2 into (3) to yield

$$x_3 = \frac{71.4 - 0.3(2.616667) + 0.2(-2.794524)}{10} = 7.005610$$

Example

- For the second iteration, the same process is repeated to compute

$$x_1 = \frac{7.85 + 0.1(-2.794524) + 0.2(7.005610)}{3} = 2.990557 \qquad |\varepsilon_t| = 0.31\%$$

$$x_2 = \frac{-19.3 - 0.1(2.990557) + 0.3(7.005610)}{7} = -2.499625 \qquad |\varepsilon_t| = 0.015\%$$

$$x_3 = \frac{71.4 - 0.3(2.990557) + 0.2(-2.499625)}{10} = 7.000291 \qquad |\varepsilon_t| = 0.0042\%$$

- The method is, therefore, converging on the true solution. Additional iterations could be applied to improve the answers.
- However, in an actual problem, we would not know the true answer a priori.