Interpolation

Chapra: Chapter-18

Newton's Divided-difference Interpolating Polynomials

- Estimate *ln* 2 using linear interpolation. First, perform the computation by interpolating between *ln* 1 = 0 and *ln* 6 = 1.791759. Then, repeat the procedure, but use a smaller interval from *ln* 1 to *ln* 4 (1.386294). Note that the true value of *ln* 2 is 0.6931472.
- We use

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

and a linear interpolation for *ln*(2) from $x_0 = 1$ to $x_1 = 6$ to give

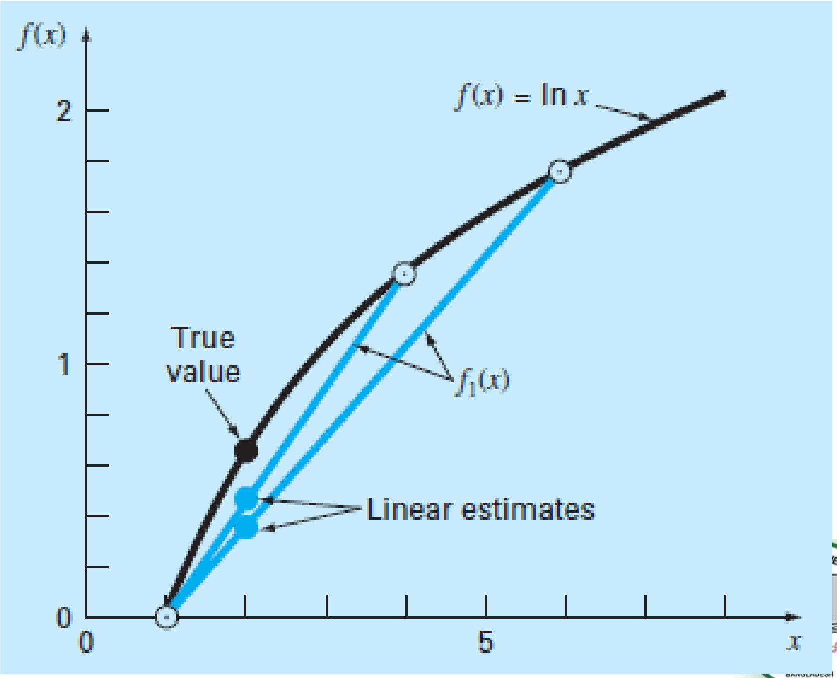
$$f_1(2) = 0 + \frac{1.791759 - 0}{6 - 1}(2 - 1) = 0.3583519$$

Newton's Divided-difference Interpolating Polynomials

which represents an error of $\varepsilon_t = 48.3\%$. Using the smaller interval from $x_0 = 1$ to $x_1 = 4$ yields

$$f_1(2) = 0 + \frac{1.386294 - 0}{4 - 1}(2 - 1) = 0.4620981$$

Thus, using the shorter interval reduces the percent relative error to $\varepsilon_t = 33.3\%$.



Newton's Divided-difference Interpolating Polynomials

If three data points are available, this can be accomplished with a second-order polynomial (also called a quadratic polynomial or a parabola). A particularly convenient form for this purpose is

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Or, $f_2(x) = b_0 + b_1x - b_1x_0 + b_2x^2 + b_2x_0x_1 - b_2xx_0 - b_2xx_1$

Or, collecting terms, $f_2(x) = a_0 + a_1x + a_2x^2$

Where,

$$a_0 = b_0 - b_1x_0 + b_2x_0x_1$$

$$a_1 = b_1 - b_2x_0 - b_2x_1$$

$$a_2 = b_2$$

Newton's Divided-difference Interpolating Polynomials

Putting $x = x_0$; $\quad b_0 = f(x_0)$

Putting $x = x_1$; $\quad b_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

Putting $x = x_2$;

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Newton’s Divided-difference Interpolating Polynomials

- Fit a second-order polynomial to the three points used in previous example.

$$x_0 = 1 \quad f(x_0) = 0$$
$$x_1 = 4 \quad f(x_1) = 1.386294$$
$$x_2 = 6 \quad f(x_2) = 1.791759$$

Use the polynomial to evaluate *ln* 2.

- $b_0 = 0$, $\quad b_1 = \dfrac{1.386294 - 0}{4 - 1} = 0.4620981$,

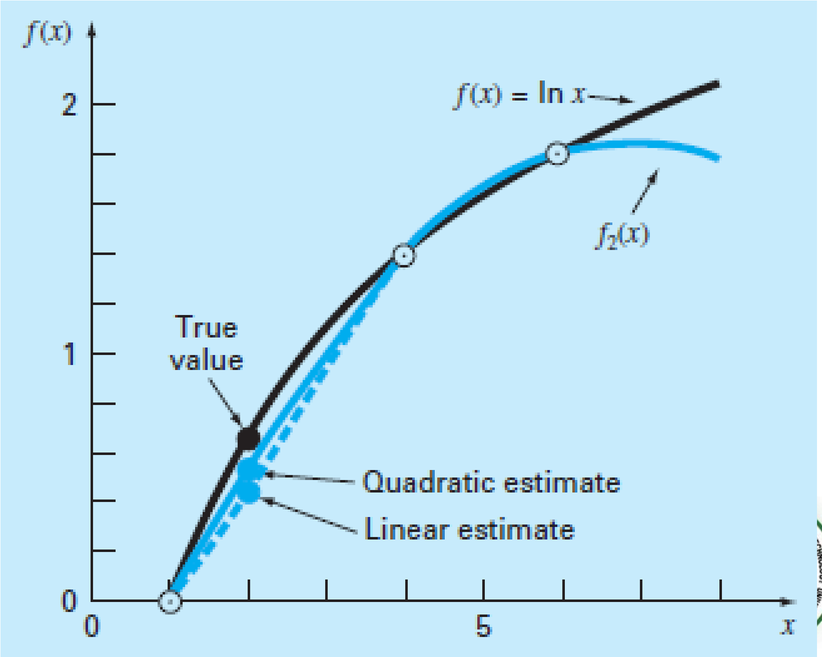
$$b_2 = \frac{\dfrac{1.791759 - 1.386294}{6 - 4} - 0.4620981}{6 - 1} = -0.0518731$$

Newton’s Divided-difference Interpolating Polynomials

- Substituting these values the quadratic formula becomes

$$f_2(x) = 0 + 0.4620981(x - 1) - 0.0518731(x - 1)(x - 4)$$

- which represents a relative error of $\varepsilon_t = 18.4\%$.



General Form of Newton's Interpolating Polynomials

- The preceding analysis can be generalized to fit an *n*th-order polynomial to $n + 1$ data points. The *n*th-order polynomial is

$$f_n(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

- We use the following equations to evaluate the coefficients:

$$
\begin{aligned}
b_0 &= f(x_0) \\
b_1 &= f[x_1, x_0] \\
b_2 &= f[x_2, x_1, x_0] \\
&\;\vdots \\
b_n &= f[x_n, x_{n-1}, \ldots, x_1, x_0]
\end{aligned}
$$

General Form of Newton's Interpolating Polynomials

- The finite divided differences are:

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

$$f[x_n, x_{n-1}, \ldots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_0]}{x_n - x_0}$$

General Form of Newton's Interpolating Polynomials

- In previous example, data points at $x_0 = 1$, $x_1 = 4$, and $x_2 = 6$ were used to estimate *ln* 2 with a parabola. Now, adding a fourth point [$x_3 = 5$; $f(x_3) = 1.609438$], estimate *ln* 2 with a third-order Newton's interpolating polynomial.
- The third-order polynomial with $n = 3$, is

$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

The first divided differences for the problem are

$$f[x_1, x_0] = \frac{1.386294 - 0}{4 - 1} = 0.4620981$$

$$f[x_2, x_1] = \frac{1.791759 - 1.386294}{6 - 4} = 0.2027326$$

$$f[x_3, x_2] = \frac{1.609438 - 1.791759}{5 - 6} = 0.1823216$$

General Form of Newton's Interpolating Polynomials

The second divided differences are

$$f[x_2, x_1, x_0] = \frac{0.2027326 - 0.4620981}{6 - 1} = -0.05187311$$

$$f[x_3, x_2, x_1] = \frac{0.1823216 - 0.2027326}{5 - 4} = -0.02041100$$

The third divided difference is

$$f[x_3, x_2, x_1, x_0] = \frac{-0.02041100 - (-0.05187311)}{5 - 1} = 0.007865529$$

The equation becomes

$$f_3(x) = 0 + 0.4620981(x - 1) - 0.05187311(x - 1)(x - 4) + 0.007865529(x - 1)(x - 4)(x - 6)$$

which represents a relative error of $\varepsilon_t = 9.3\%$.

Lagrange Interpolating Polynomials

The Lagrange interpolating polynomial is simply a reformulation of the Newton polynomial that avoids the computation of divided differences. It can be represented concisely a

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

Where,

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{n} \frac{x - x_j}{x_i - x_j}$$

Lagrange Interpolating Polynomials

The linear version (n = 1) is

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

The second-order version is

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

Lagrange Interpolating Polynomials

- Use a Lagrange interpolating polynomial of the first and second order to evaluate *ln* 2 on the basis of the data:

$$x_0 = 1 \quad f(x_0) = 0$$
$$x_1 = 4 \quad f(x_1) = 1.386294$$
$$x_2 = 6 \quad f(x_2) = 1.791759$$

- The first-order polynomial

$$f_1(2) = \frac{2-4}{1-4}0 + \frac{2-1}{4-1}1.386294 = 0.4620981$$

- The second-order polynomial

$$f_2(2) = \frac{(2-4)(2-6)}{(1-4)(1-6)}0 + \frac{(2-1)(2-6)}{(4-1)(4-6)}1.386294 + \frac{(2-1)(2-4)}{(6-1)(6-4)}1.791760 = 0.5658444$$

Coefficients of an Interpolating Polynomial

- Newton and the Lagrange polynomials do not provide a convenient polynomial of the conventional form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

- We want to compute the coefficients of the parabola

$$f(x) = a_0 + a_1 x + a_2 x^2$$

- Three data points are required: $[x_0, f(x_0)]$, $[x_1, f(x_1)]$, and $[x_2, f(x_2)]$. Each can be substituted into the equation to give

$$f(x_0) = a_0 + a_1 x_0 + a_2 x_0^2$$
$$f(x_1) = a_0 + a_1 x_1 + a_2 x_1^2$$
$$f(x_2) = a_0 + a_1 x_2 + a_2 x_2^2$$

- These equations can be solved by an elimination method.