Truncation Errors and Taylor Series

Chapra: Chapter-4

Truncation Error

- *Truncation errors* are those that result from using an approximation in place of an exact mathematical procedure.
- *Example:*

$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

- *How much error is introduced with this approximation?*
- We can use *Taylor's Series* to estimate truncation error.

Taylor Series

n^{th} order approximation

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \ldots$$
$$+ \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n + R_n$$

$(x_{i+1} - x_i) = h$ *step size* (define first)

$$R_n = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} h^{(n+1)} \qquad x_i \le \varepsilon \le x_{i+1}$$

Remainder term, R_n, accounts for all terms from $(n+1)$ to infinity.

Taylor Series Approximation of a Polynomial

Problem Statement: Use zero- through fourth-order Taylor series expansions to approximate the function

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

from $x_i = 0$ with h = 1. That is, predict the function's value at $x_{i+1} = 1$.

Solution: $f(0) = 1.2$ and $f(1) = 0.2$. Thus the true value that we are trying to predict is 0.2.

The Taylor series approximation with $n = 0$ is $f(x_{i+1}) \cong 1.2$

Truncation error : $E_t = 0.2 - 1.2 = -1.0$ at $x = 1$.

For $n = 1$, the first derivative at $x = 0$;

$f'(0) = -0.4(0.0)^3 - 0.45(0.0)^2 - 1.0(0.0) - 0.25 = -0.25$

Taylor Series Approximation of a Polynomial

The first-order approximation is $f(x_{i+1}) \cong 1.2 - 0.25h$

which can be used to compute $f(1) = 0.95$.

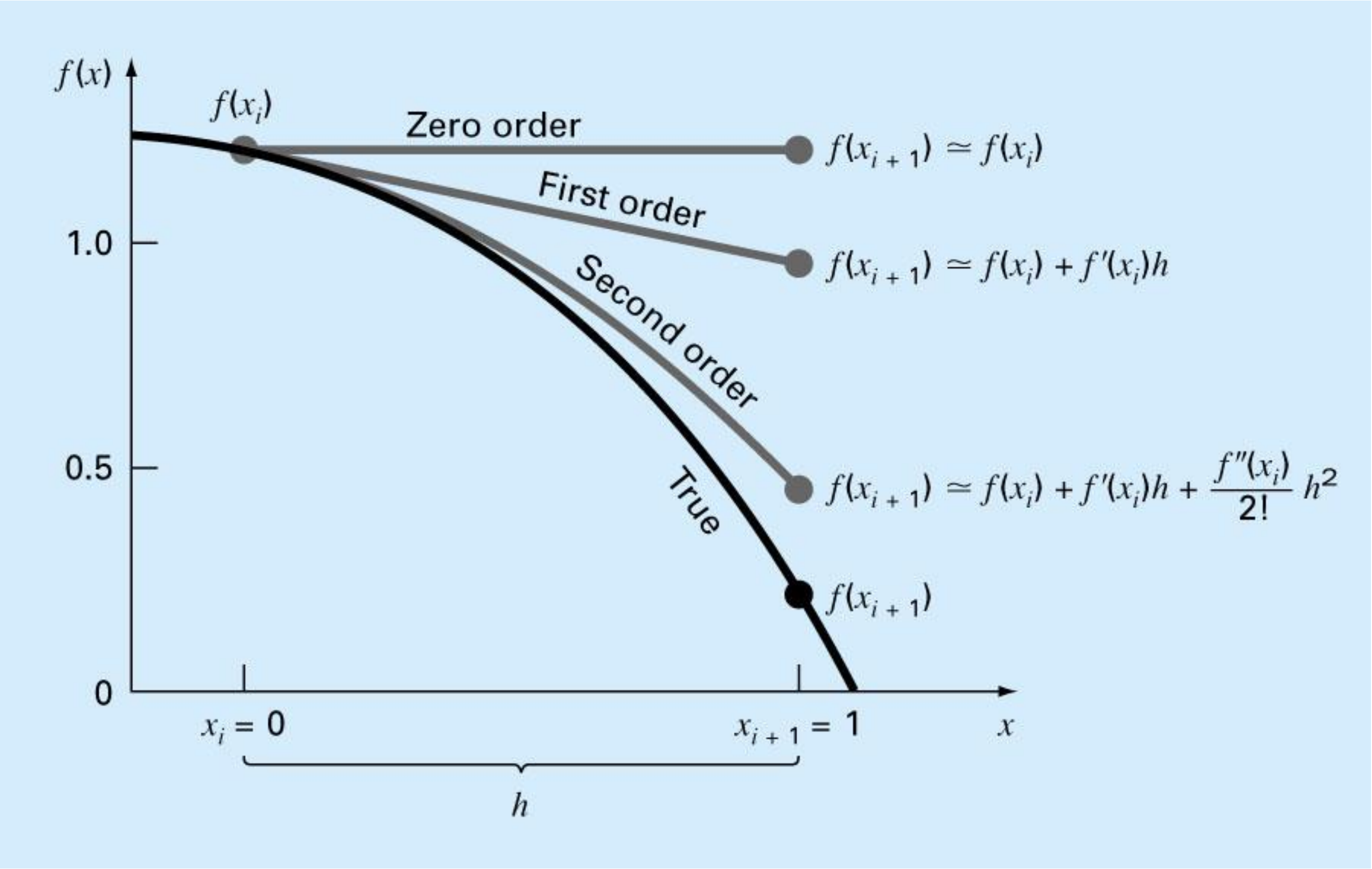
Truncation error : $E_t = 0.2 - 0.95 = -0.75$

Additional terms would improve the approximation even more.

The remainder

$$R_4 = \frac{f^{(5)}(\varepsilon)}{5!} h^5 = 0$$

because the fifth derivative of a fourth-order polynomial is zero.



$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

The approximation of *f(x)* at *x*=1 by zero-order, first-order and second-order Taylor series expansion

Insight

- Each additional term contributes to the approximation
- n^{th}-order Taylor Series gives exact value of n^{th}-order polynomial
- Inclusion of a few terms gives an approximation that is good enough for practical purpose.
- The Remainder:
  - ε is not exactly known.
  - Need to determine $f^{n+1}(x_{i+1})$, which require the determination of the $(n+1)$th derivative of $f(x)$. If we know $f(x)$ then we do not need to use Taylor's series!
  - Yet, $R_n = O(h^{n+1})$ gives insight into error e.g., if error is $O(h)$ then halving step size will halve the error. If error is $O(h^2)$ then halving the step size will quarter the error, and so on.

Effect of Step Size

Estimate

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c}{m} v(t_i) \right] (t_{i+1} - t_i)$$

Actual

$$v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

m=	68.1
c=	12.5
9=	9.81
Δt=	2

t	Actual	Estimate	Error%
0	0	0	
2	16.42172	19.62	19.47591
4	27.79763	32.03736	15.25213
6	35.67812	39.89621	11.82263
8	41.13722	44.87003	9.074043
10	44.91893	48.01792	6.899074
12	47.53865	50.01019	5.199019
14	49.35343	51.27109	3.885579
16	50.61058	52.06911	2.881848

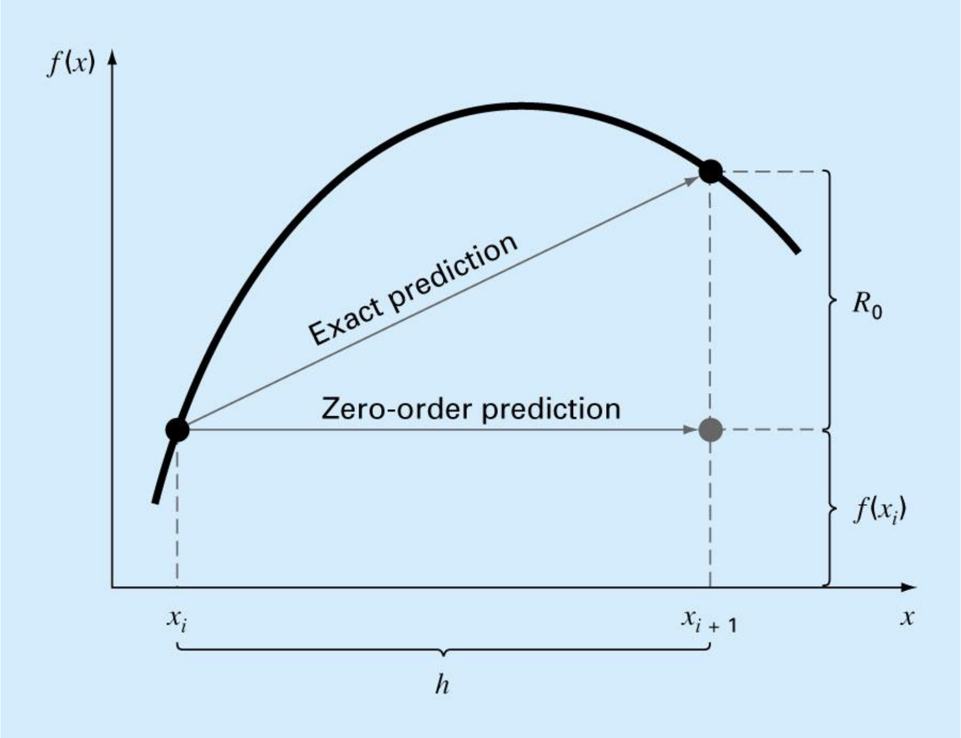
t	Actual	Estimate	Error%
0	0	0	
1	8.962318	9.81	9.458288
2	16.42172	17.81934	8.510793
3	22.63024	24.35854	7.637128
4	27.79763	29.69744	6.834436
5	32.09849	34.05637	6.099607
6	35.67812	37.6152	5.429315
7	38.65748	40.52079	4.820066
8	41.13722	42.89306	4.268249
9	43.20112	44.82988	3.770176
10	44.91893	46.4112	3.322138
11	46.34867	47.70225	2.920443
12	47.53865	48.75633	2.561458
13	48.52908	49.61693	2.241647
14	49.35343	50.31957	1.957596
15	50.03953	50.89323	1.706043
16	50.61058	51.36159	1.483897

Insight : R_n

$$f(x_{i+1}) \cong f(x_i)$$

$$R_0 = f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!} + \cdots\cdots$$

$$R_0 \cong f'(x_i)h$$

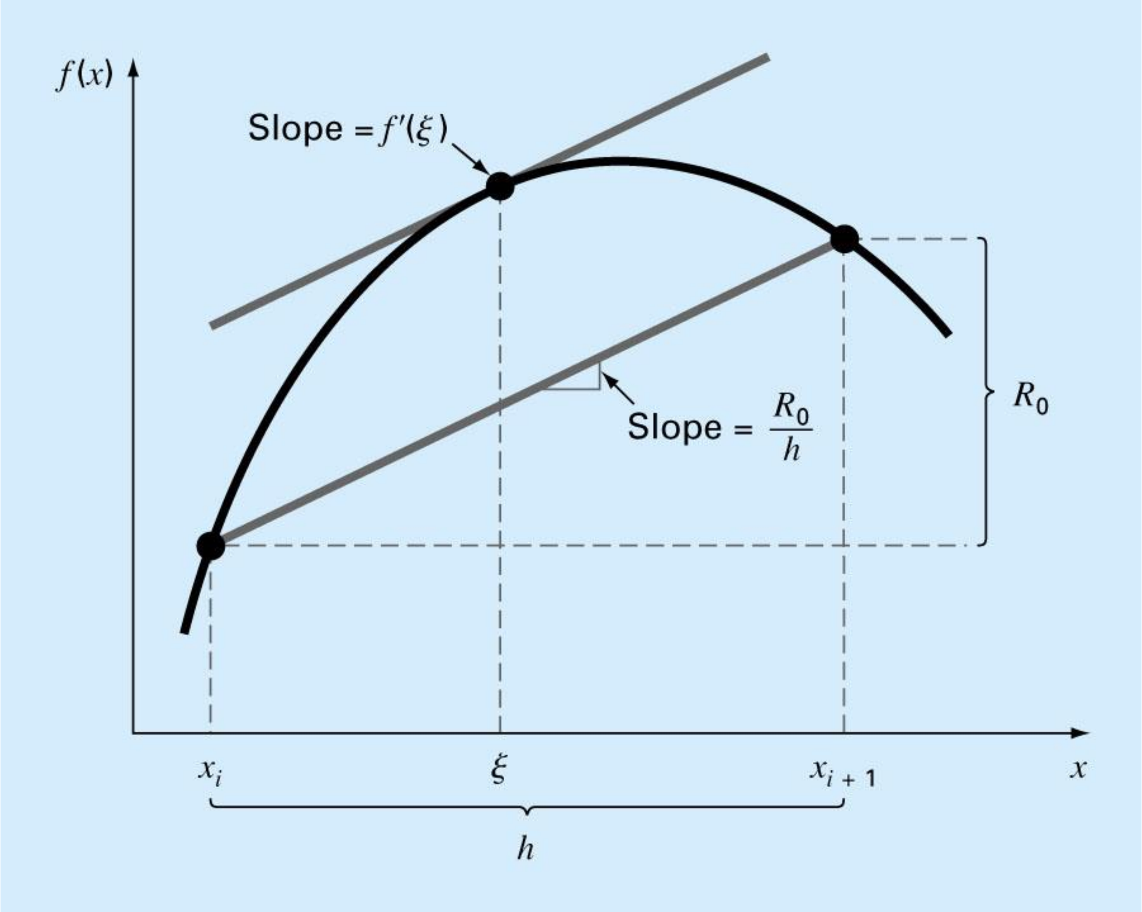


Insight : R_n

$$R_0 \cong f'(x_i)h$$

$$R_0 = f'(\varepsilon)h$$

$$R_1 = \frac{f''(\varepsilon)}{2!}h^2$$



Using the Taylor Series to Estimate Truncation Errors

- We have to understand the application of Taylor series expansion to the numerical methods.
- In case of falling parachutist, we needed to predict the velocity, $v(t)$ as a function of time.
- $v(t)$ can be expanded in Taylor series as:

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + \frac{v''(t_i)}{2!}(t_{i+1} - t_i)^2 + \cdots + R_n$$

- Now let us truncate the series after first derivative term:

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + R_1$$

Using the Taylor Series to Estimate Truncation Errors

$$v'(t_i) = \underbrace{\frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}}_{\substack{\text{First-order} \\ \text{approximation}}} - \underbrace{\frac{R_1}{t_{i+1} - t_i}}_{\substack{\text{Truncation} \\ \text{error}}}$$

- The truncation error associated with the approximation of the derivative:

$$\frac{R_1}{t_{i+1} - t_i} = \frac{v''(\xi)}{2!}(t_{i+1} - t_i) \quad \text{or} \quad \frac{R_1}{t_{i+1} - t_i} = O(t_{i+1} - t_i)$$

- The error of the derivative approximation should be proportional to the step size.

How to get derivatives?

- We will be given value of unknown $f(x)$ for some value of x

- We can estimate $f^n(a)$, i.e. the n^{th} order derivative of $f(x)$ at $x=a$ numerically without knowing $f(x)$