Bracketing Methods

Chapra: Chapter-5

Roots of Equations

- Roots of Equations
  - Bracketing Methods
    - Bisection method
    - False Position Method
  - Open Methods
    - Simple fixed point iteration
    - Newton Raphson
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  - System of Nonlinear Equations
  - Roots of polynomials
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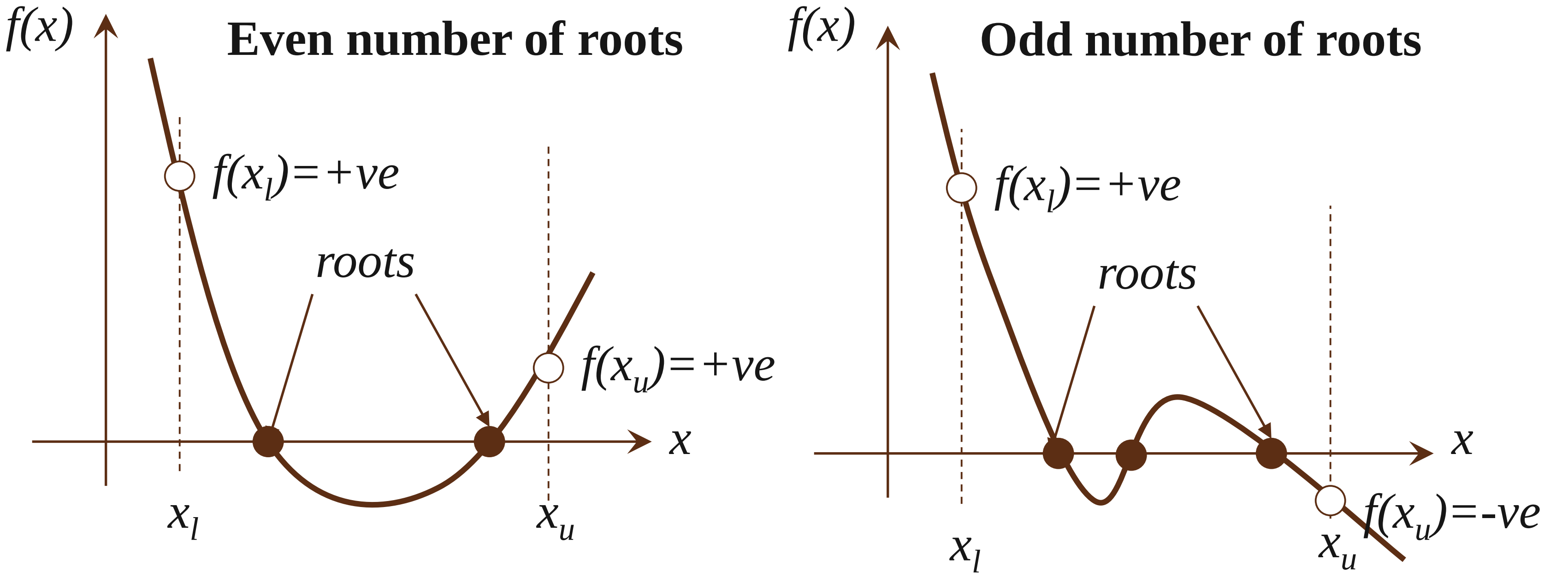
Bracketing Methods

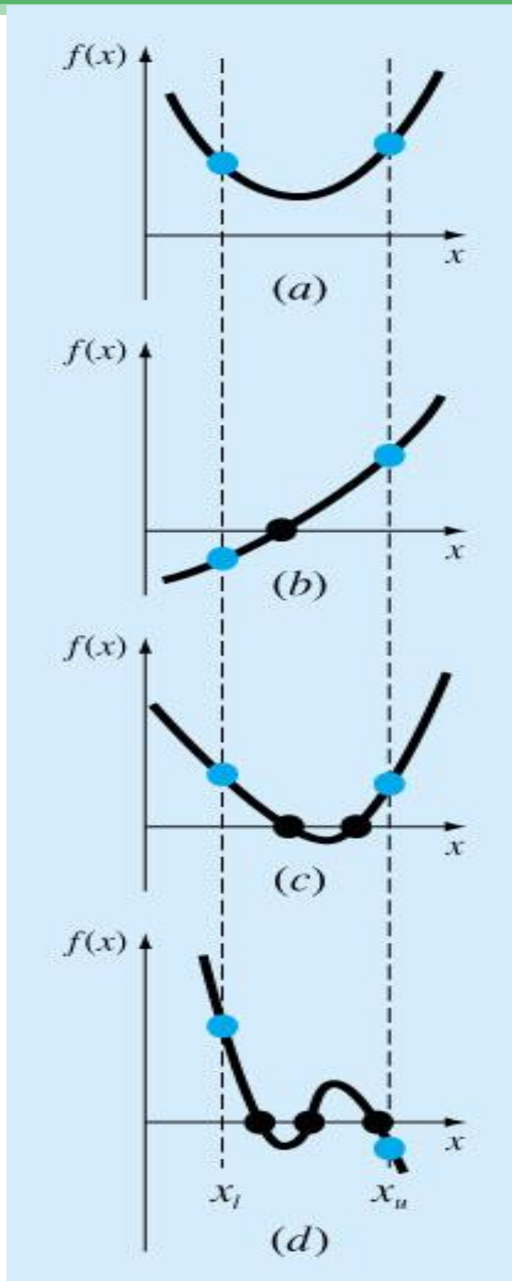


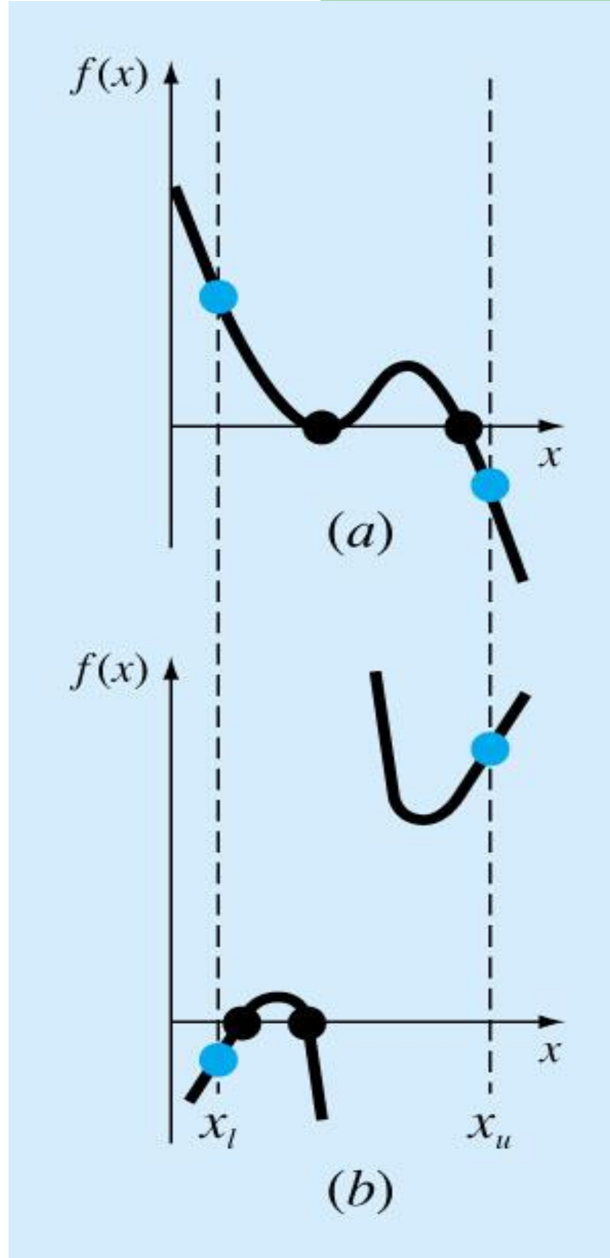
Typically changes sign in the vicinity of a root

Bracketing Methods

- Two initial guesses (x_l and x_u) are required for the root which *bracket* the root (s).
- If one root of a **real** and **continuous** function, $f(x)=0$, is bounded by values x_l , x_u then $f(x_l).f(x_u) < 0$.

(The function changes sign on opposite sides of the root)





Special Cases

Bisection Method

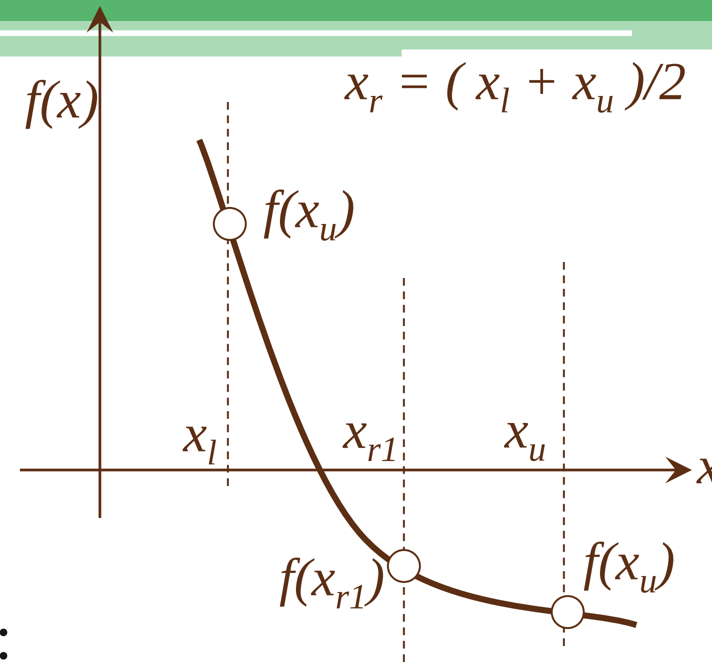
- Generally, if $f(x)$ is real and continuous in the interval x_l to x_u *and* $f(x_l).f(x_u)<0$, then there is at least one real root between x_l and x_u to this function.
- The interval at which the function changes sign is located. Then the interval is divided in half with the root lies in the midpoint of the subinterval. This process is repeated to obtained refined estimates.

Step 1: Choose lower x_l and upper x_u guesses for the root such that:

$$f(x_l).f(x_u)<0$$

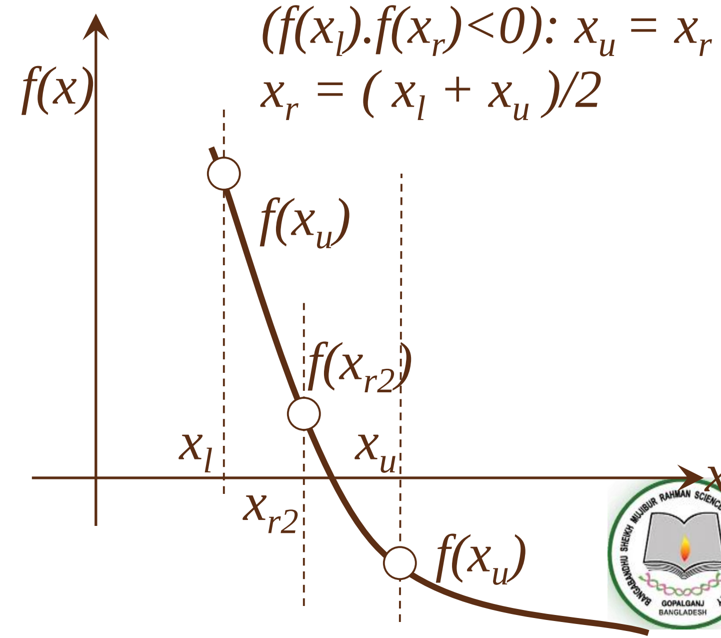
Step 2: The root estimate is:

$$x_r = (\, x_l + x_u \,)/2$$



Step 3: Subdivide the interval according to:

- If $(f(x_l).f(x_r)<0)$ the root lies in the lower subinterval; $x_u = x_r$ and go to step 2.
- If $(f(x_l).f(x_r)>0)$ the root lies in the upper subinterval; $x_l = x_r$ and go to step 2.
- If $(f(x_l).f(x_r)=0)$ the root is x_r and stop



Bisection Method – Termination Criteria

True relaive Error :

$$\varepsilon_t = \left| \frac{X_{true} - X_{approximate}}{X_{true}} \right| \times 100\%$$

Approximate relative Error :

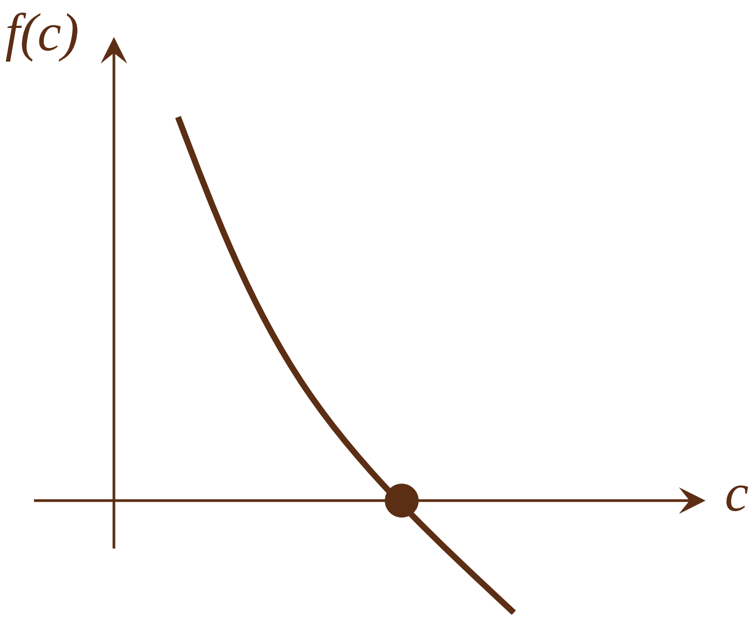
$$\varepsilon_a = \left| \frac{X_r^n - X_r^{n-1}}{X_r^n} \right| \times 100\%$$

$$\varepsilon_a = \left| \frac{X_u - X_l}{X_u + X_l} \right| \times 100\% \text{ (Bisection)}$$

- For the Bisection Method $e_a > e_t$
- The computation is terminated when e_a becomes less than a certain criterion ($e_a < e_s$)

Bisection Method: Example

- The parachutist velocity is $v = \dfrac{mg}{c}\left(1 - e^{-\frac{c}{m}t}\right)$
- What is the drag coefficient c needed to reach a velocity of 40 m/s if m = 68.1 kg, t = 10 s, g= 9.8 m/s^2



$$f(c) = \frac{mg}{c}\left(1 - e^{-\frac{c}{m}t}\right) - v$$

$$f(c) = \frac{667.38}{c}\left(1 - e^{-0.146843c}\right) - 40$$

1. **Assume** x_l =12 and x_u=16
   $f(x_l)$=6.067 and $f(x_u)$=-2.269

2. **The root**: $x_r=(x_l+x_u)/2= 14$

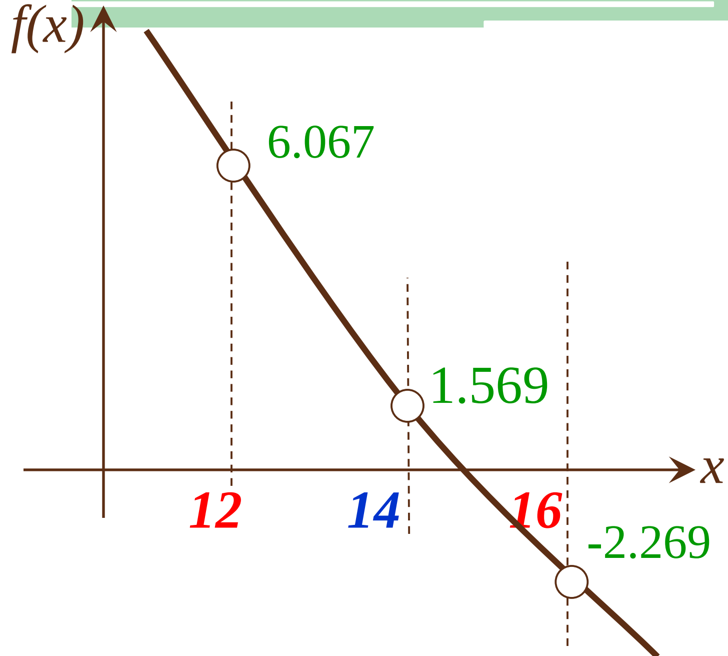
3. ***Check*** *f(12).f(14) = 6.067•1.569=9.517 >0;*
   the root lies between 14 and 16.

4. **Set** x_l = 14 and x_u=16, thus the new root
   $x_r=(14+ 16)/2= 15$

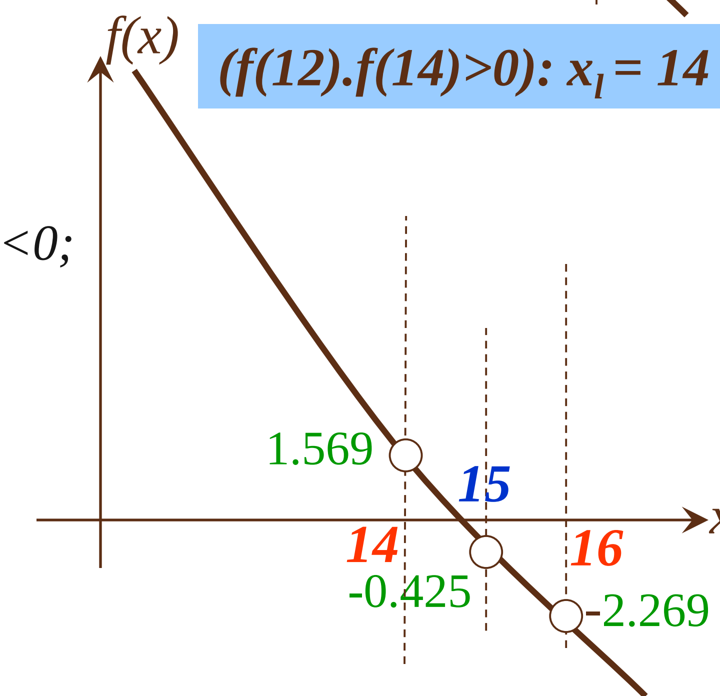
5. ***Check*** *f(14).f(15) = 1.569•-0.425= -0.666 <0;*
   the root lies between 14 and 15.

6. **Set** x_l = 14 and x_u=15, thus the new root
   $x_r=(14+ 15)/2= 14.5$

and so on…...



(f(12).f(14)>0): x_l = 14

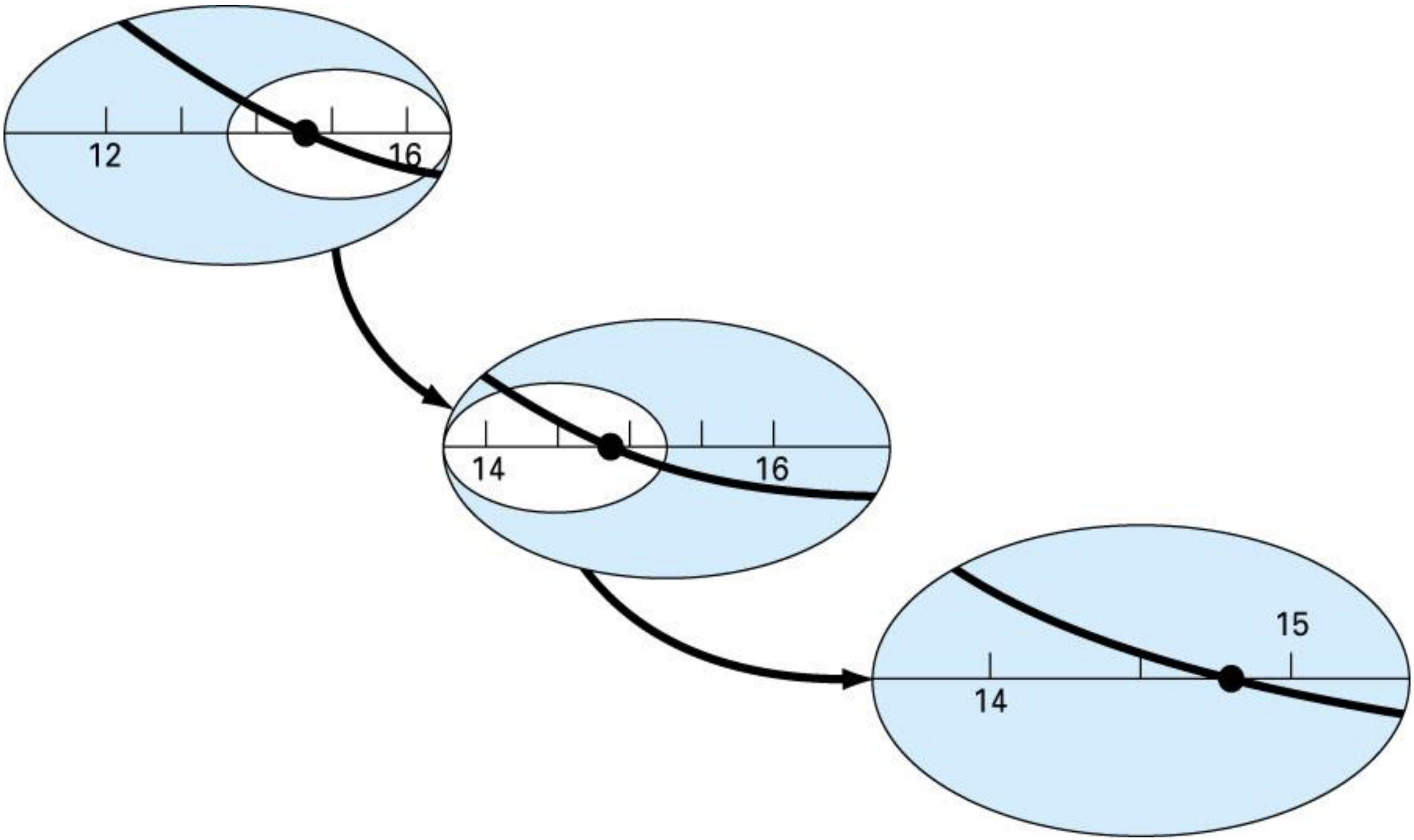


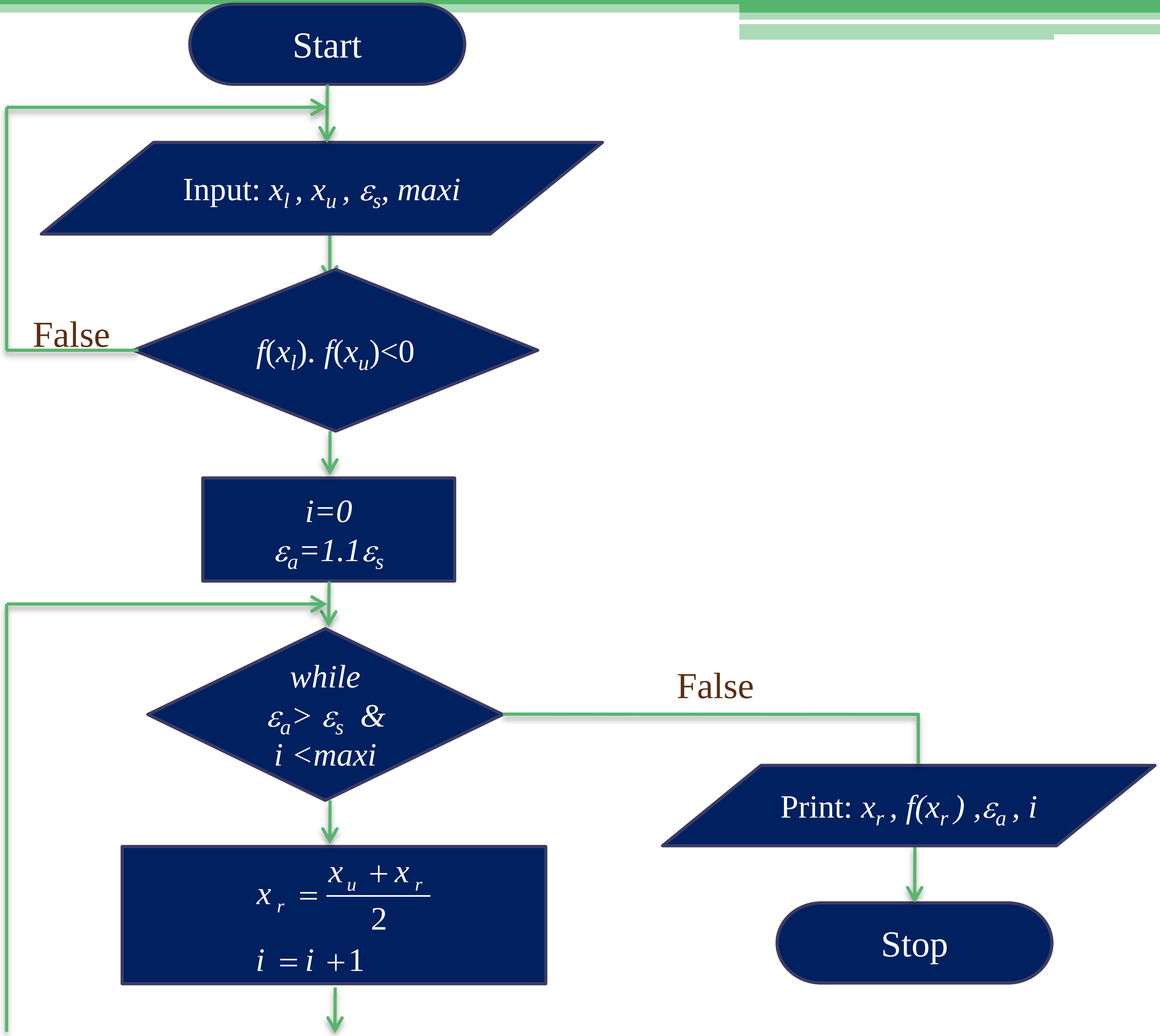
Bisection Method: Example

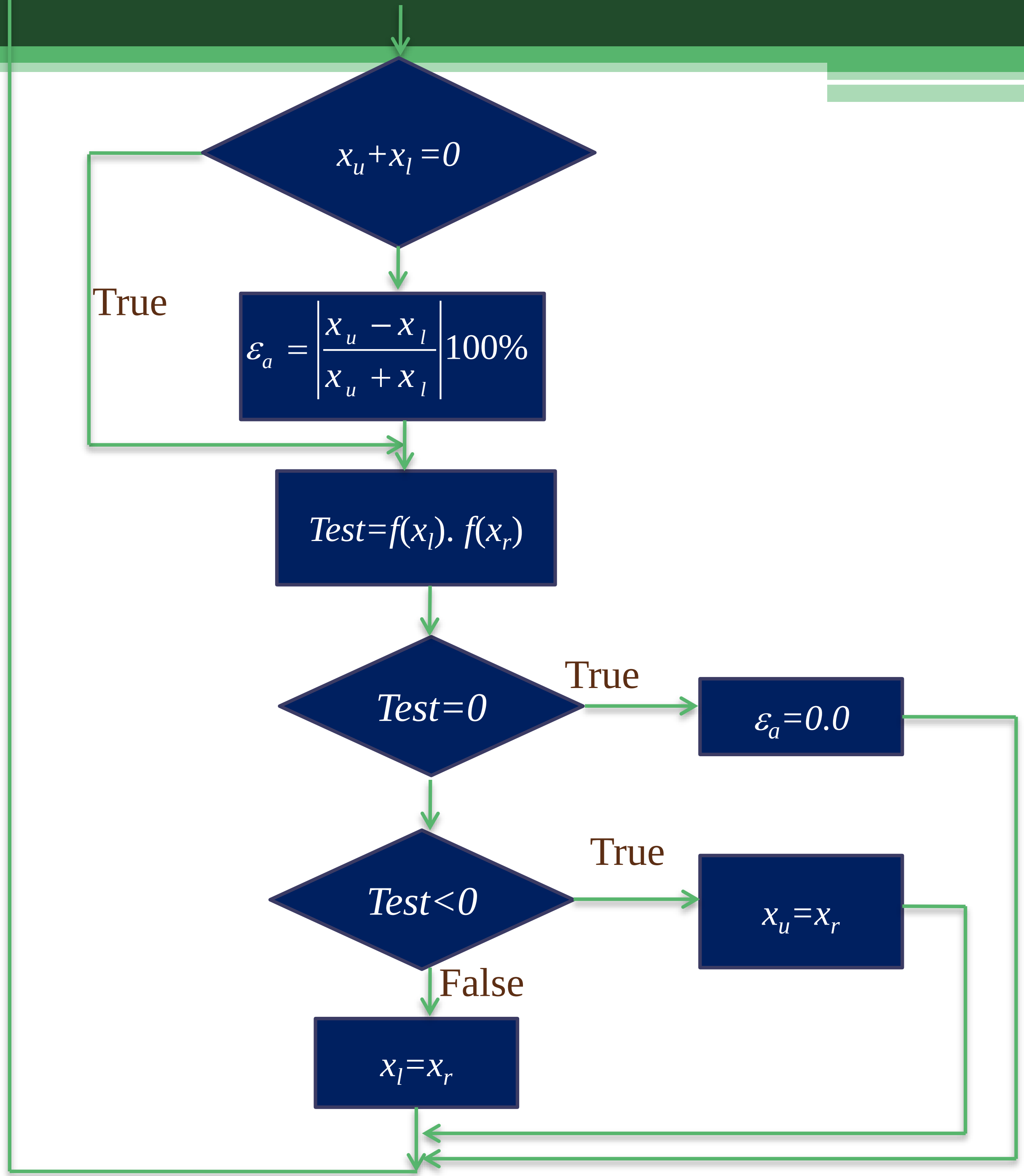
- In the previous example, if the stopping criterion is $e_s = 0.5\%$; what is the root?

Iter.	X_l	X_u	X_r	$e_a\%$	$e_t\%$
1	*12*	*16*	*14*	*5.279*	*--*
2	*14*	*16*	*15*	*6.667*	*1.487*
3	*14*	*15*	*14.5*	*3.448*	*1.896*
4	*14.5*	*15*	*14.75*	*1.695*	*1.204*
5	*14.75*	*15*	*14.875*	*0.84*	*0.641*
6	*14.74*	*14.875*	*14.813*	*0.422*	*0.291*

Bisection Method




Start
Input: x_l , x_u , ε_s, $maxi$
False
$f(x_l). f(x_u)<0$
$i=0$
$\varepsilon_a=1.1\varepsilon_s$
while
$\varepsilon_a> \varepsilon_s$ &
$i <maxi$
False
Print: x_r , $f(x_r)$,ε_a , i
$x_r = \frac{x_u + x_r}{2}$
$i = i + 1$
Stop

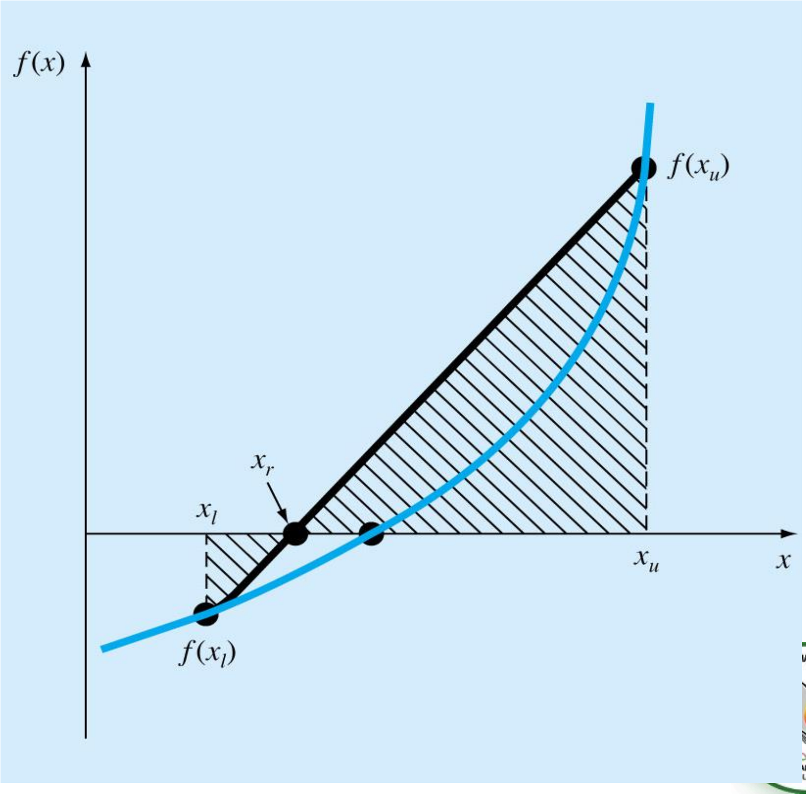


$x_u+x_l=0$
True
$\varepsilon_a = \left|\frac{x_u - x_l}{x_u + x_l}\right| 100\%$
$Test=f(x_l).\ f(x_r)$
$Test=0$
True
$\varepsilon_a=0.0$
$Test<0$
True
$x_u=x_r$
False
$x_l=x_r$

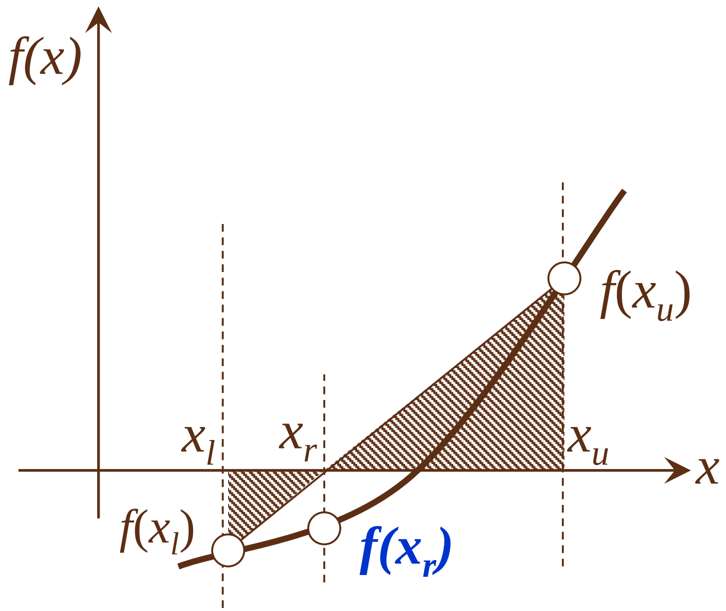
False-Position Method

- The bisection method divides the interval x_l to x_u in half not accounting for the magnitudes of $f(x_l)$ and $f(x_u)$. For example if $f(x_l)$ is closer to zero than $f(x_u)$, then it is more likely that the root will be closer to $f(x_l)$.

- False position method is an alternative approach where $f(x_l)$ and $f(x_u)$ are joined by a straight line; the intersection of which with the x-axis represents an improved estimate of the root.



False-Position Method: Procedure



$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

False-Position Method: Procedure

Step 1: Choose lower x_l and upper x_u guesses for the root such that: $f(x_l).f(x_u)<0$

Step 2: The root estimate is:

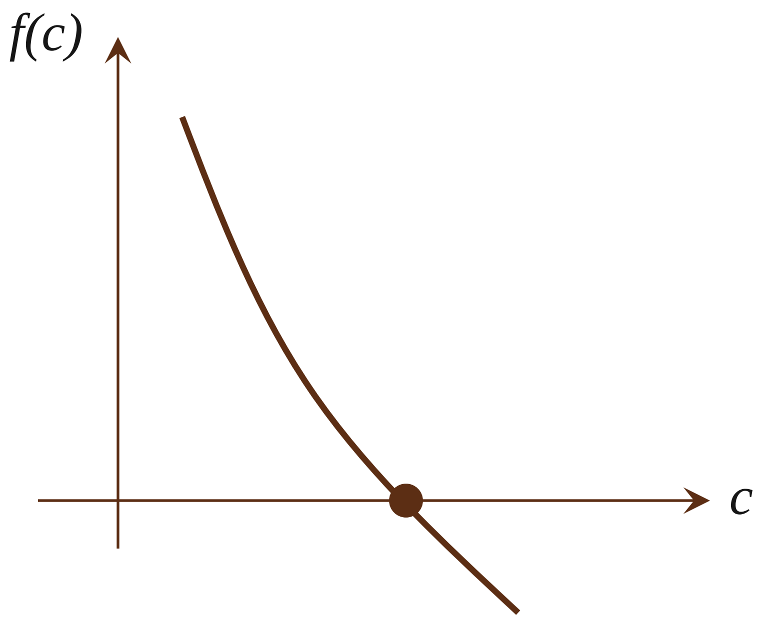
$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

Step 3: Subdivide the interval according to:

- If $(f(x_l).f(x_r)<0)$ the root lies in the lower subinterval; $x_u = x_r$ and go to step 2.
- If $(f(x_l).f(x_r)>0)$ the root lies in the upper subinterval; $x_l = x_r$ and go to step 2.
- If $(f(x_l).f(x_r)=0)$ the root is x_r and stop

False-Position Method: Example

- The parachutist velocity is $$v = \frac{mg}{c}(1 - e^{-\frac{c}{m}t})$$
- What is the drag coefficient c needed to reach a velocity of 40 m/s if m =68.1 kg, t =10 s, g= 9.8 m/s^2

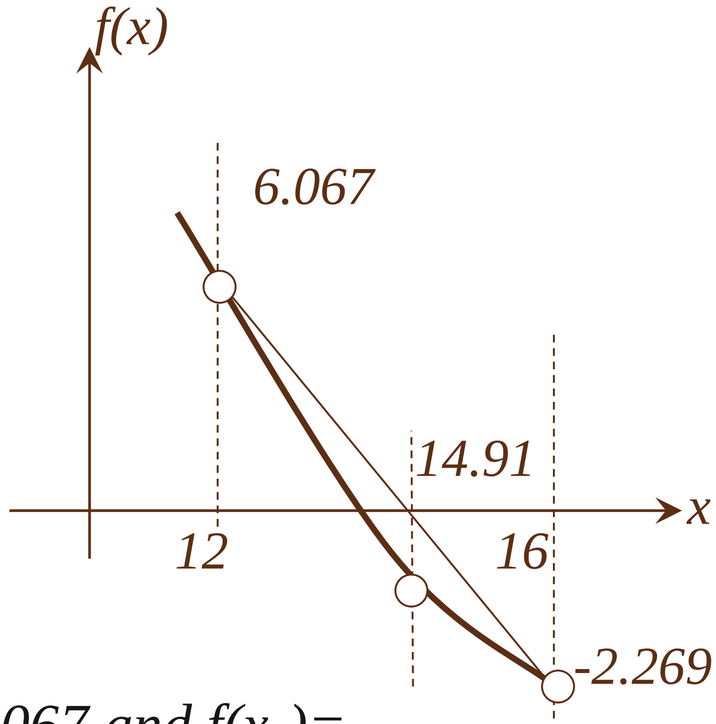


$$f(c) = \frac{mg}{c}(1 - e^{-\frac{c}{m}t}) - v$$

$$f(c) = \frac{667.38}{c}(1 - e^{-0.146843c}) - 40$$

1. Assume x_l = 12 and x_u=16

   $f(x_l)$= 6.067 *and* $f(x_u)$= -2.269

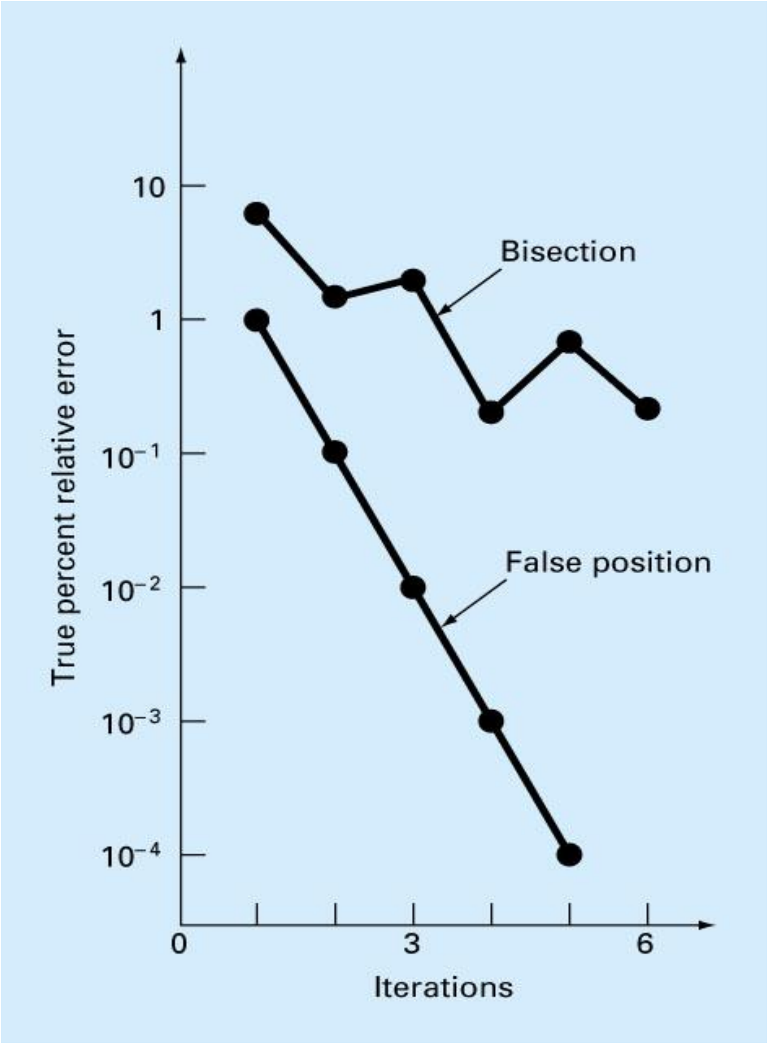


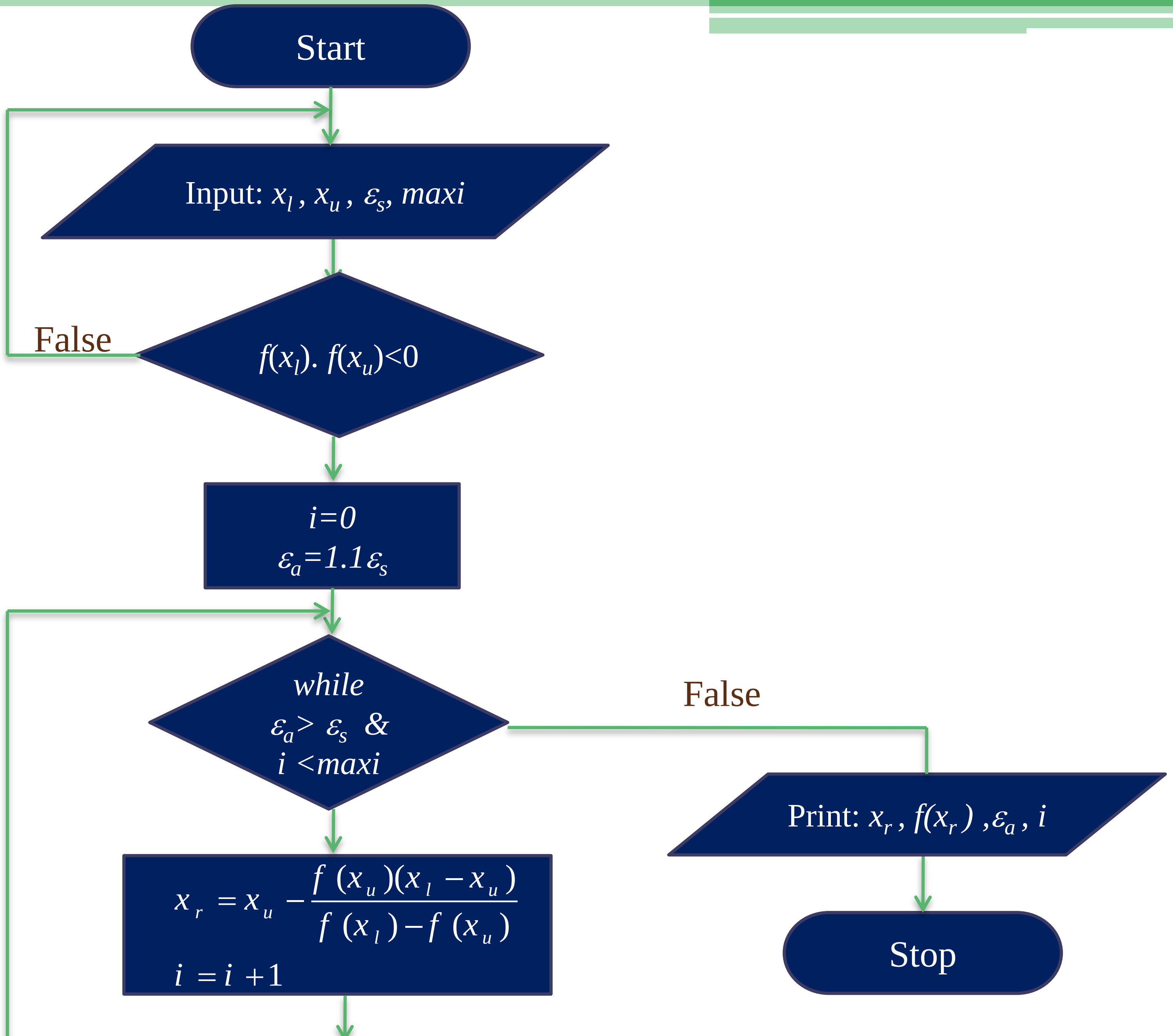
2. The root: x_r=*14.9113*

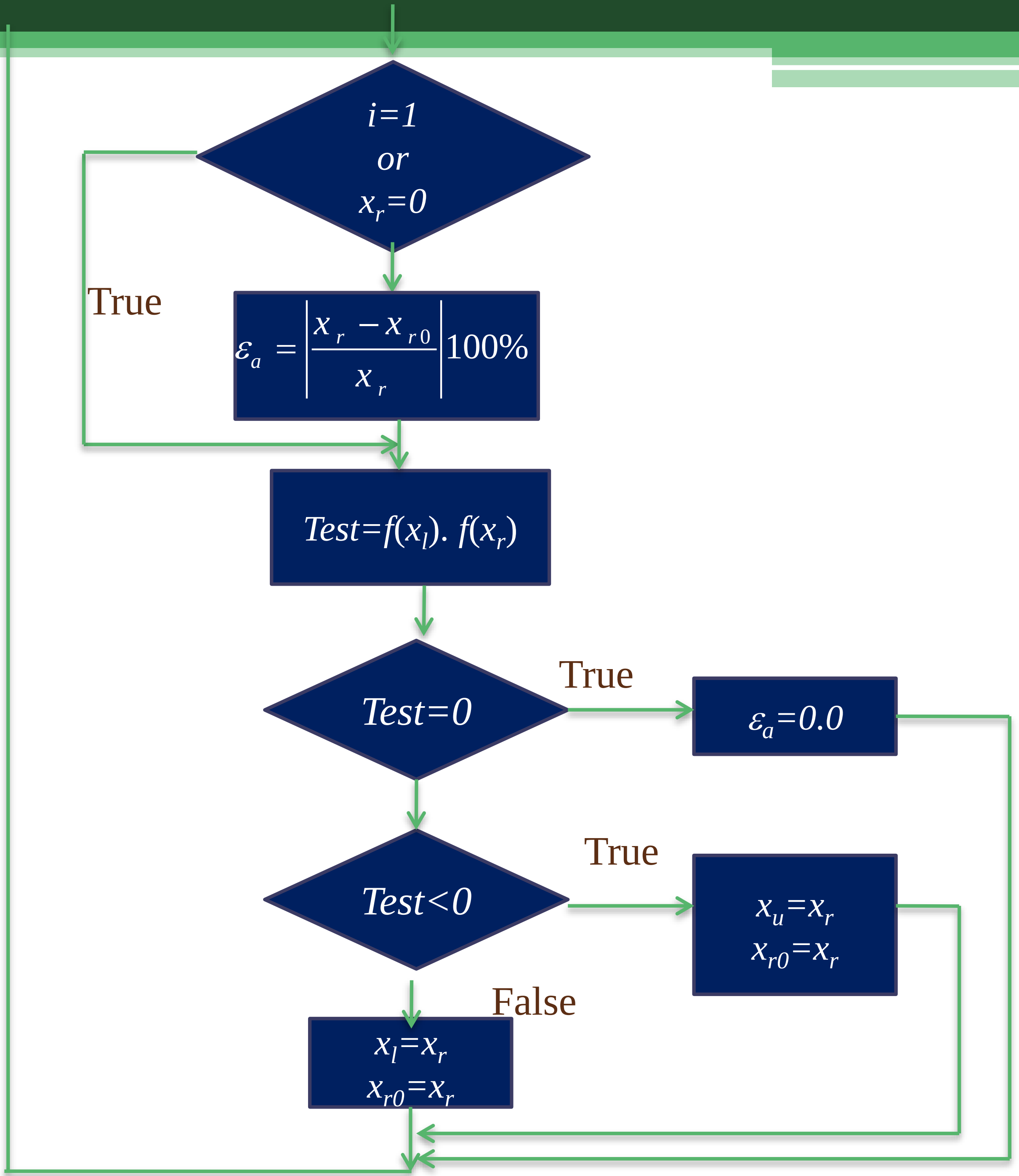
   $f(12) \cdot f(14.9113) = -1.5426 < 0;$

3. The root lies between 12 and 14.9113.
4. Assume x_l = 12 and x_u=14.9113, $f(x_l)=6.067$ *and* $f(x_u)$=-*0.2543*
5. The new root x_r= *14.7942*
6. This has an approximate error of 0.79%

False-Position Method: Example




Start
Input: x_l , x_u , ε_s, $maxi$
False
$f(x_l).\ f(x_u)<0$
$i=0$
$\varepsilon_a=1.1\varepsilon_s$
while
$\varepsilon_a > \varepsilon_s$ &
$i < maxi$
False
$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$
$i = i + 1$
Print: x_r , $f(x_r)$, ε_a , i
Stop



$i=1$
or
$x_r=0$
True
$\varepsilon_a = \left|\frac{x_r - x_{r0}}{x_r}\right| 100\%$
$Test=f(x_l).\ f(x_r)$
$Test=0$
True
$\varepsilon_a=0.0$
$Test<0$
True
$x_u=x_r$
$x_{r0}=x_r$
False
$x_l=x_r$
$x_{r0}=x_r$

Pitfalls of the False-Position Method

A case where Bisection is preferable to False-position

Problem Statement. Use bisection and false position to locate the root of

$$f(x) = x^{10} - 1$$

between $x = 0$ and 1.3.

Solution. Using bisection, the results can be summarized as

Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	0	1.3	0.65	100.0	35
2	0.65	1.3	0.975	33.3	2.5
3	0.975	1.3	1.1375	14.3	13.8
4	0.975	1.1375	1.05625	7.7	5.6
5	0.975	1.05625	1.015625	4.0	1.6

Thus, after five iterations, the true error is reduced to less than 2 percent. For false position, a very different outcome is obtained:

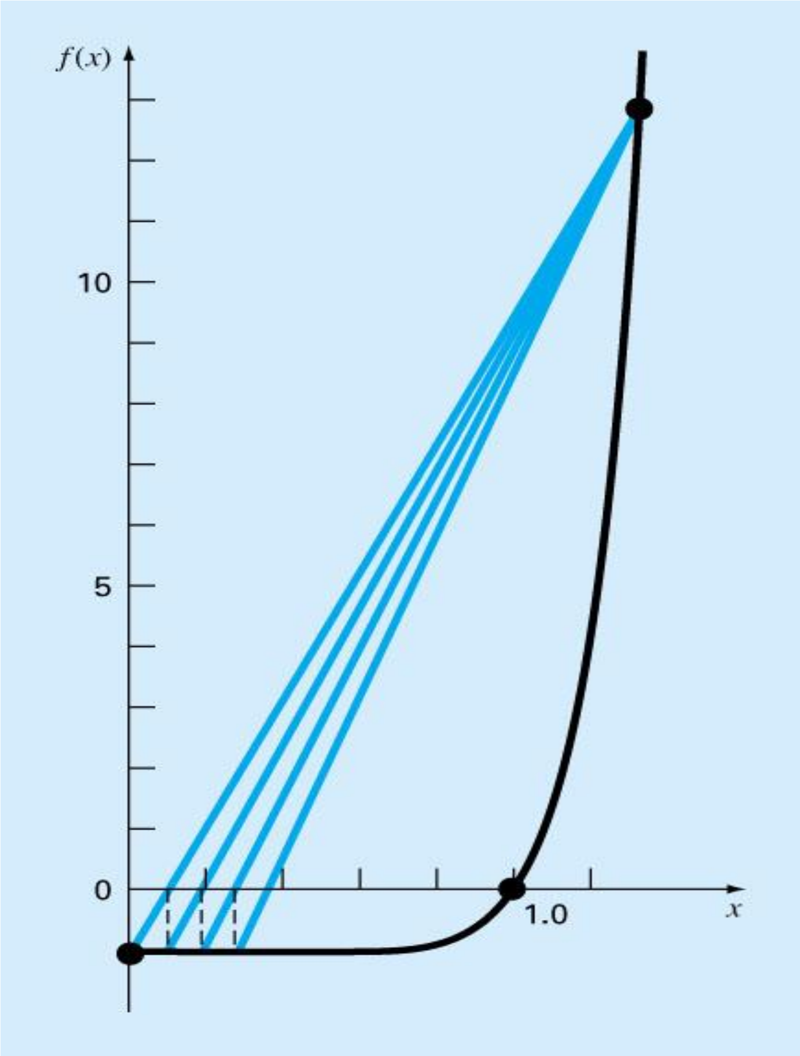
Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	0	1.3	0.09430		90.6
2	0.09430	1.3	0.18176	48.1	81.8
3	0.18176	1.3	0.26287	30.9	73.7
4	0.26287	1.3	0.33811	22.3	66.2
5	0.33811	1.3	0.40788	17.1	59.2

Pitfalls of the False-Position Method

Plot of $f(x) = x^{10} - 1$,

illustrating slow convergence

of the false-position method.



Pitfalls of the False-Position Method

- Although a method such as false position is often superior to bisection, there are some cases (when function has significant curvature) that violate this general conclusion.
- In such cases, the approximate error might be misleading and the results should always be checked by substituting the root estimate into the original equation and determining whether the result is close to zero.
- Major weakness of the false-position method: its one sidedness. That is, as iterations are proceeding, one of the bracketing points will tend to stay fixed which leads to poor convergence.

Modified False-Position

One way to mitigate the "one-sided" nature of false position is to make the algorithm detect when one of the bounds is stuck. If this occurs, the function value at the stagnant bound is divided in half. This is thought to fasten the convergence.