Probability

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Probability

Probability is the **likeliness** of occurring any event(s).

Some related terms

Proby

Deterministic experiment Vs **Random experiment:**

An experiment whose outcome is predictable in advance is called deterministic experiment. Everyone conducting that experiment will get the same outcome.

An experiment whose outcome is not predictable with certainty in advance is called a random experiment. If a random experiment is performed then one of many possible outcomes will occur.

Some related terms

Deterministic experiment Vs **Random experiment:**

Example:

Deterministic experiment:

- Measuring linear distance from Dhaka to Chittagong.
- Measuring length of a scale.

Random experiment:

- Measuring weight of a person at different times.
- Tossing a coin.

Some related terms

Random

<u>Sample Space:</u>

The set of all possible outcomes of a random experiment is called sample space of that experiment and is denoted by S. Each individual outcome is called a sample point.

$S = \{ \}$

For example;

throwing a dice- S= {1, 2, 3, 4, 5, 6 }

Lifetime of a lightbulb- S= $\{x | 0 \le x < \infty\} = [0,\infty)$

Some related terms

- If the outcome of the experiment is the gender of a child, then

  S={G, B} ; Where outcome G means that the child is a girl and B that it is a boy.

- Consider an experiment that consists of rolling two balanced dice, one black and one red are thrown and number of dots on their upper faces are noted, also if b be the outcomes of the black die and r be the outcomes of the red die. If we let denote the outcome in which black dice has value b and red dice has value r, then the sample space of this experiment is:

$b \Downarrow$ \ $r \Rightarrow$	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1					
3					3,5	
4						
5	5,1					
6	6,1	6,2				6,6

Some related terms

Event:

Any subset E of a sample space S is an event.

For example;

Dice throw experiment- S= {1, 2, 3, 4, 5, 6 }

Say number 2 turned up in a throw. Then we will say, event E= {2} has occurred.

Some related terms

Mutually Exclusive events:

Two events are called mutually exclusive if both the events cannot occur simultaneously in a single trial. In other words, if one of those events occurs, the other event will not occur.

For example;

in a trial of coin toss experiment, event E_1= {Head} and event E_2={tail} will not occur simultaneously. So, E_1 and E_2 are mutually exclusive events.

On a day, Event E_1={Rain} & event E_2={Sunny} may occur simultaneously. These are not mutually exclusive events.

Some related terms

Collectively exhaustive events:

Collectively exhaustive events are those, which includes all possible outcomes.

For example;

In a coin tossing experiment events E1= {Head} and event E2={tail} are collectively exhaustive, because together they comprise the all the outcomes that are possible in a coin tossing experiment. There are no other possible outcomes of this experiment than these two.

Some related terms

Equally likely events:

The events of a random experiment are called equally likely if the chance of occurring those events are all equal.

For example;

In a coin tossing experiment, the events E1= {Head} and event E2={tail} are equally likely, because the chance of occurring E1 is as same as occurring E2.

On a day, Event E1={Rain} & event E2={No rain} may not be equally likely.

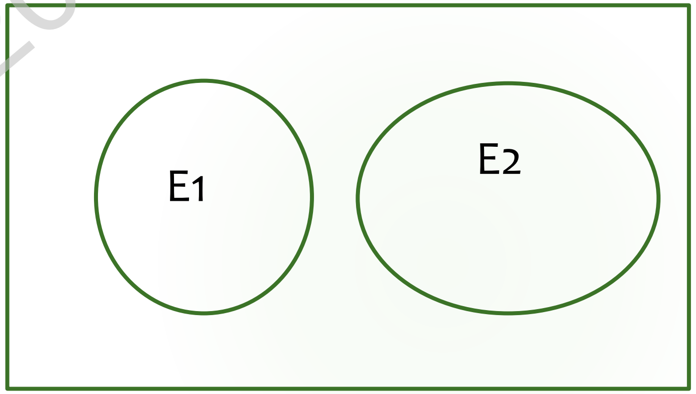
Some related terms

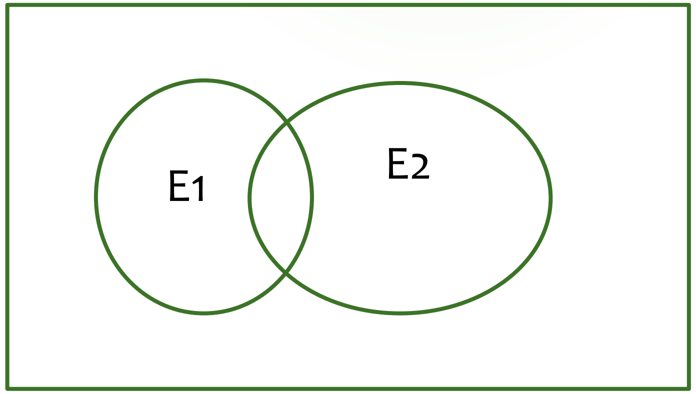
Disjoint events:

Two events are called disjoint, if they have no common elements between them.

Mutually exclusive events are disjoint events.

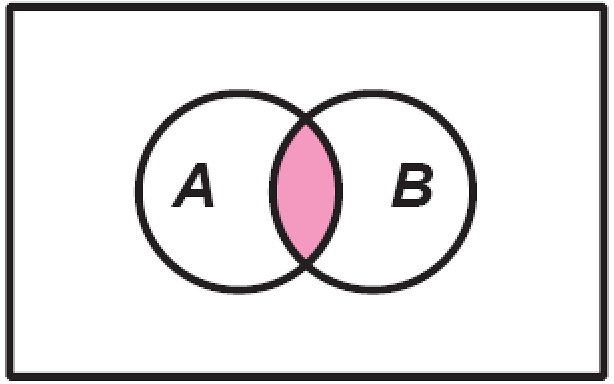
Disjoint events



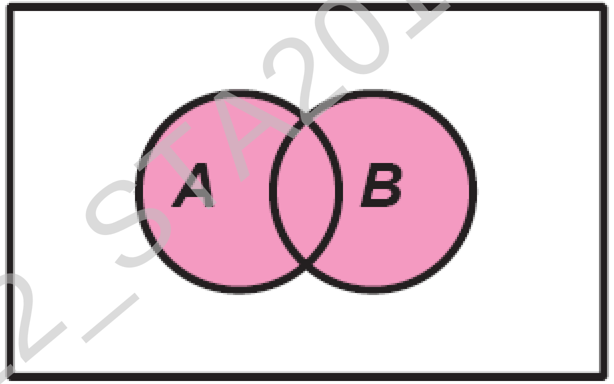


Joint events

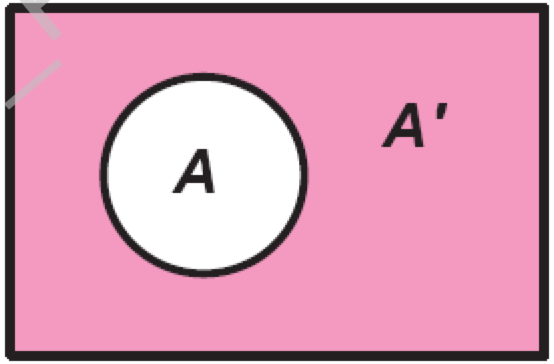
- UNION: The union of two regions defines an event that is either in A or in B or in both regions.
- INTERSECTION: The intersection of two regions defines an event *must* be in both A *and* B.
- COMPLEMENT: The complement A is everything in the event space that is not in A, i.e. A'.



intersection; A ∩ B



union; A ∪ B



not A; A'

Approaches of assigning probability

- At first we identify the sample space S of the random experiment.
- We then define our favorable event and assign probability to the event using one of the following 3 basic approaches-
  - Classical approach
  - Frequency approach
  - Subjective approach

Approaches of assigning probability

Classical approach:

(when the outcomes are equally likely, mutually exclusive and collectively exhaustive)

- If the sample space of a random experiment has a finite number (n_s) of outcomes
- n_E of these outcomes are favorable to an event E

Then, the probability of occurring event E, denoted by P(E) is-

$$P(E) = \frac{n_E}{n_S}$$

Approaches of assigning probability

Classical approach:

For example;

Dice throwing experiment-

S= {1, 2, 3, 4, 5, 6}

Consider two events, E1= {2} and E2= {2, 4, 6}

Here, n_{E_1}= 1 and n_{E_2}= 3. Also, n_S= 6

Therefore, probability of occurring event E1 is, $P(E_1) = \frac{n_{E_1}}{n_S} = \frac{1}{6}$

probability of occurring event E2 is, $P(E_2) = \frac{n_{E_2}}{n_S} = \frac{3}{6} = \frac{1}{2}$

Approaches of assigning probability

Frequency approach:

If an experiment is repeated n times under the same conditions and event E occurs f times out of n times, then

$$P(E) = \lim_{n \to \propto} \frac{f}{n}$$

That is, when n is very large, P(E) is very close to the relative frequency of event E.

Approaches of assigning probability

Frequency approach:

For example;

In a dice throwing experiment- S= {1, 2, 3, 4, 5, 6}

And our favorable event is E= {2}

Let, 2 occurred a total of 998 times out of total 6000 trials.

Therefore $P(E) = \lim_{n \to \propto} \frac{998}{6000} \approx \frac{1}{6}$

Approaches of assigning probability

Subjective approach:

Based on the judgement (personal experience, prior information and belief etc.), one can assign probability to an event E of a random experiment.

For example; on a day of summer someone made a statement on probability that rain will occur on that day is .70, based on his previous experience.

Axioms of probability

Valid probabilities will follow 3 axioms-

Axiom 1: (Axiom of positivizes) : $0 \leq P(E) \leq 1$

Axiom 2: (Axiom of certainty) : $P(S) = 1$

Axiom 3: (Axiom of additivity) : For a sequence of disjoint events $E_1, E_2, \ldots, E_n$-

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

Example 1

In a community of 400 people, 20 people has a particular disease. If a person is selected randomly from that community, what is the probability that he/ she does not has the disease?

Example 1

In a community of 400 people, 20 people has a particular disease. If a person is selected randomly from that community, what is the probability that he/ she does not has the disease?

Solution:

Let, D= the randomly selected person has the disease

$$here, P(D) = \frac{f}{n} = \frac{20}{400} = .05$$

$$\therefore P(D^c) = 1 - .05 = 0.95$$

So, the probability that he/ she does not have the disease is 0.95

Example 2

- A bag contains 4 white and 6 black balls. If one ball is drawn at random from the bag, what is the probability that it is

  i. Black, ii. White, iii. White or black and iv. Red.

Answer:

i) Let A be the event that the ball is black, then the number of outcomes favorable to A is 6. Hence

$$P(A) = \frac{m}{n} = \frac{6}{10};$$

$m =$ Favorable outcomes of an event A = Number of black balls

$n =$ Total number of outcomes of the experiment = Total number of balls

- ii. $P\ (B)=$
- iii. $P\ (C)=$
- iv. $P\ (D)=$

Exercise

Problem 1:

Two balanced dice, one black and one red are thrown and the number of dots on their upper faces are noted, let b be the outcomes of the black die and r be the outcomes of the red die. Now answer the following:

- List a sample space of the experiment.
- What is the probability of throwing a double?
- What is the probability that the sum is 5, that is $b+r=5$?
- What is the probability that the sum is even?
- What is the probability that $r \leq 2$ or $b \leq 3$?
- What is the probability that the number on the red die is at least 4 greater than the number on the black dice.

Exercise

Problem 2:

A card is drawn from a pack of 52 cards. Find the probability that it is a) a red card b) a spade c) a queen d) not a spade e) a king or queen.

Problem 3:

The following table gives a distribution of weekly wages of 4000 employees of a firm.

Wages in Tk.	Below 500	500-750	750-1000	1000-1250	1250-1500	1500-1750	1750 and above
No. of workers	36	472	1912	800	568	140	72

An individual is selected random. What is the probability that his wage are

i) under Tk.750 ii) above Tk. 1250 iii) between Tk. 750 and 1250.

Exercise

- A balanced coin is tossed until head appears, it is tossed maximum 4 times.

a) construct the sample space of the experiment

b) Is the sample space simple?

c) Calculate the probability that head appears after first throw.

d) What is the probability that head does not appear at all?

Exercise

Problem:

Let S={1,2,3,4,5,6}, A={1,3,5}, B={4,6} and C={1,4} find

- $A \cap B$
- $B \cup C$
- $A \cup (B \cap C)$
- $(A \cup B)^C$

Probability with Sampling & without Sampling

Sampling with replacement:

- If the elements of a sample are drawn randomly one by one and after each draw the element is returned to the population then the drawing is said to be done with replacement and the process of having the sample is called random sampling with replacement.

Sampling without replacement:

- If the elements of a sample are drawn randomly one by one and after each draw the element is not returned to the population then the drawing is said to be done without replacement and the process of having the sample is called random sampling without replacement.

Exercise

(ii) $\left(\frac{2}{7}\times\frac{1}{6}\right)+\left(\frac{3}{7}\times\frac{2}{6}\right)+\left(\frac{2}{7}\times\frac{1}{6}\right)$

Problem 5:

A box contains seven balls – two red, three blue and two yellow. Consider an experiment that consists of drawing a ball from the box.

- What is the probability that the first ball drawn is yellow?
- What is the probability that the same colored ball is drawn twice without replacement?
- What is the probability that the same colored ball is drawn twice with replacement?

R 2, B 3, Y 2, 7

(i) $\frac{2}{7}$

Exercise

Problem 6:

A jar consists of 21 sweets. 12 are green and 9 are blue. William picked two sweets at random.

a) Draw a tree diagram to represent the experiment.

b) Find the probability that

i) both sweets are blue.

ii) One sweet is blue and one sweet is green.

c) William randomly took a third sweet. Find the probability that:

i) All three sweets are green?

ii) At least one of the sweet is blue?

Probability Laws:

Addition Law

- **For disjoint events A and B-**

The probability that, either event A or event B will occur is,

$$P(A \cup B) = P(A) + P(B)$$

- **For disjoint events A, B, C, … , and Z-**

The probability that, either event A or event B or event C or … or event Z will occur is,

$$P(A \cup B \cup C \cup \cdots \cup Z) = P(A) + P(B) + P(C) + \ \ldots + P(Z)$$

Probability Laws:

Addition Law

- **For joint events A and B-**

The probability that, either event A or event B or both will occur is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **For joint events A, B, and C**

The probability that, either event A or event B or event C or any two of them or all will occur is,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Probability Laws:

Addition Law

Example 2:

In a company, 60% of the employees have motorcycle, 40% has private car and 20% has both.

If an employee is selected randomly from that company, then

a. What is the probability that the employee has either motorcycle or private car?

b. What is the probability that the employee has neither motorcycle nor private car?

Probability Laws:

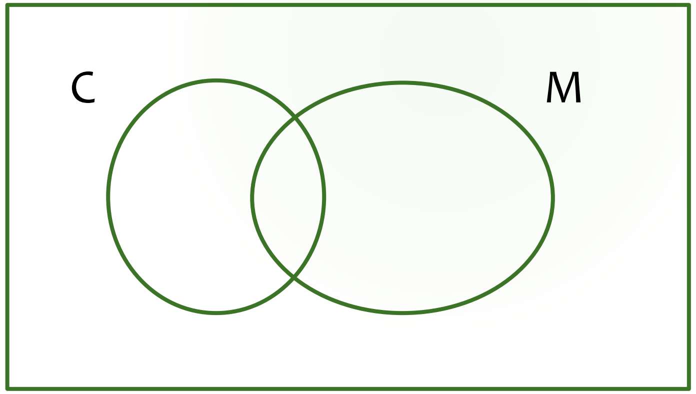
Addition Law

Solution:

Let,

M= the randomly selected employee has motorcycle

C= the randomly selected employee has car



Here, $P(M) = \frac{60}{100} = 0.6,\ P(C) = \frac{40}{100} = 0.4$

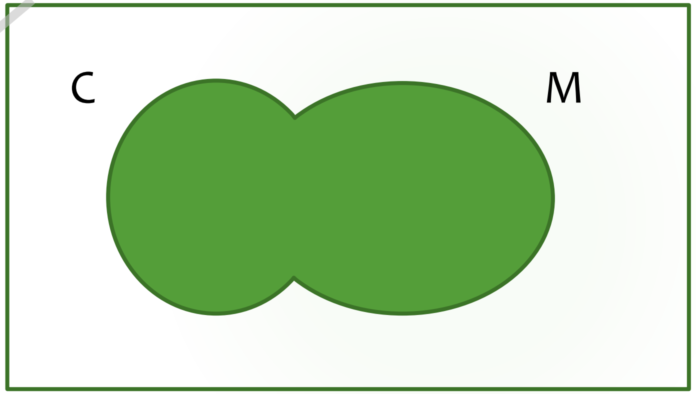
$$P(M \cap C) = \frac{20}{100} = 0.2$$

Probability Laws:

Addition Law

Solution(contd.)**:**

a. probability that the person has either motorcycle or private car is-
$P(M \cup C) = P(M) + P(C) - P(M \cap C)$
$= 0.6 + 0.4 - 0.2 = 0.8$



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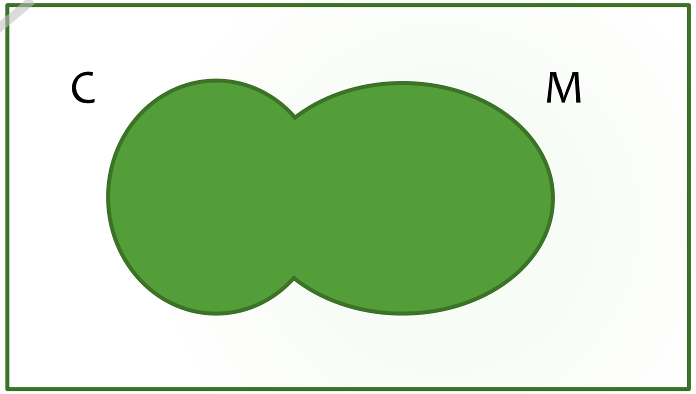
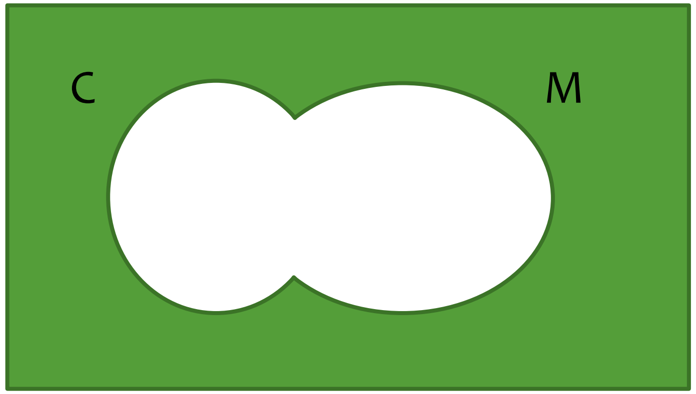


b. probability that the person has neither motorcycle nor private car is-
$P\left((M \cup C)^c\right) = 1 - P(M \cup C)$
$= 1 - 0.8 = 0.2$



Probability Laws:

Conditional Probability:

The conditional probability of an event E, given that another event F has already happened is,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \qquad for\ P(F) > 0$$

Probability Laws:

Example 3:

In a company, 60% of the employees have motorcycle, 40% has private car and 20% has both.

If an employee is selected randomly from that company, then

a. What is the probability that the employee has a car?

b. If it is known that the employee has a motorcycle, then what is the probability that the employee also has a car?

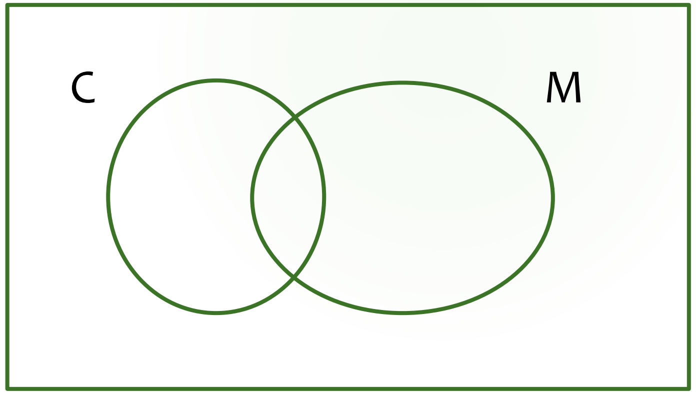
Probability Laws:

Solution:

Let,

M= the randomly selected employee has motorcycle

C= the randomly selected employee has car



Here, $P(M) = \frac{60}{100} = 0.6,\ P(C) = \frac{40}{100} = 0.4$

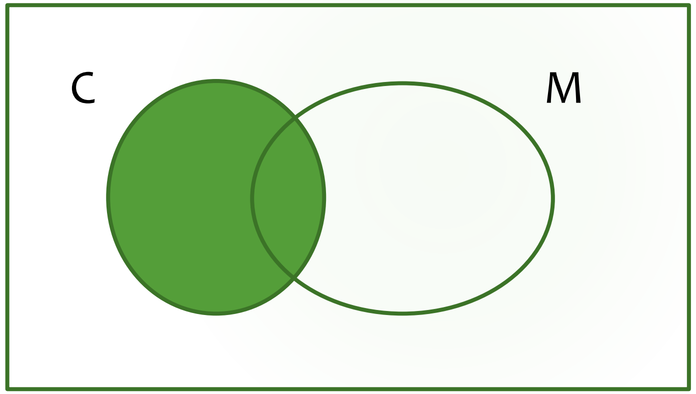
$$P(M \cap C) = \frac{20}{100} = 0.2$$

Probability Laws:

Solution(contd.)**:**

a. probability that the employee has a car is-

$$P(C) = 0.4$$

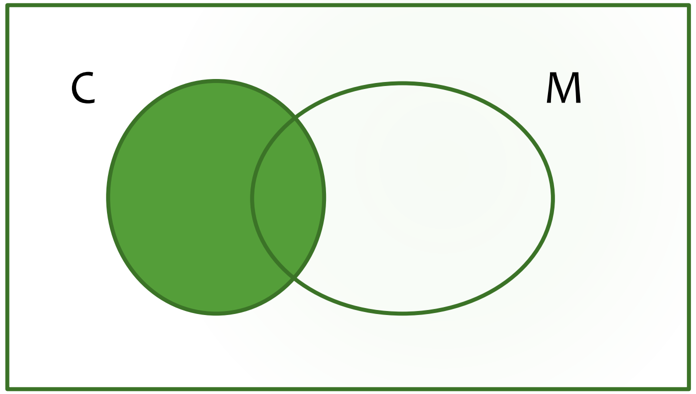


Probability Laws:

Solution(contd.):

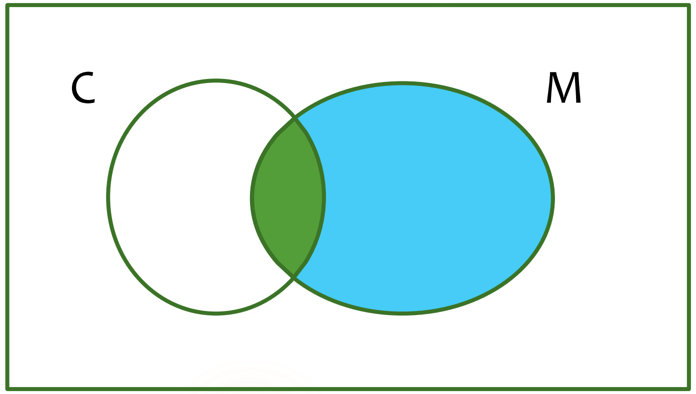
a. probability that the employee has a car is-

$$P(C) = 0.4$$



b. If it is known that the employee has a motorcycle, then what is the probability that the employee also has a car is,

$$P(C|M) = \frac{P(M \cap C)}{P(M)} = \frac{0.2}{0.6} = \frac{1}{3} = 0.33$$



Probability Laws:

Multiplication Law

- **For two dependent events E and F-**

The probability that, both event E and event F will occur simultaneously is,

$$P(E \cap F) = P(E|F)\,P(F)$$

Here, occurrence of event E depends on occurrence of event F.

- **For two independent events E and F-**

The probability that, both event E and event F will occur simultaneously is,

$$P(E \cap F) = P(E)\,P(F)$$

Probability Laws:

Multiplication Law

Example 4:

In rainy season, it rains 70% of the days in Bangladesh. When it rains, 80% times it makes thunderstorms. What is the probability that, in a particular day of rainy season, it will rain and it will thunderstorm?

Solution:

Let,

R= it will rain on that particular day

T= it will thunderstorm on that particular day

Probability Laws:

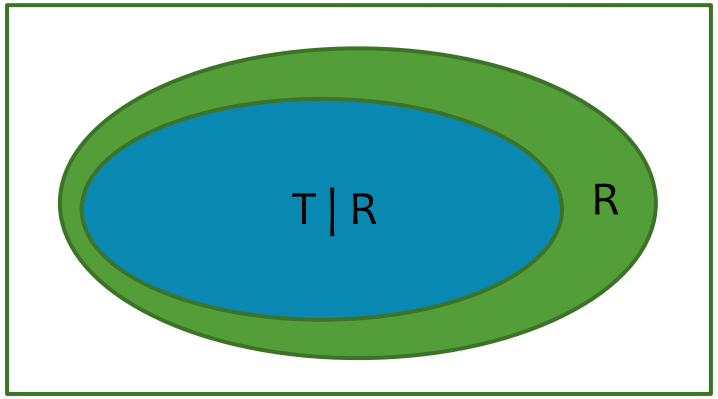
Multiplication Law

Solution(contd.)**:**

Here, given that, $P(R) = \frac{70}{100} = 0.7$ and $P(T|R) = \frac{80}{100} = 0.8$

Therefore, the probability that, on that particular day of rainy season, it will rain and it will thunderstorm is-

$$P(R \cap T) = P(T|R)P(R)$$
$$= 0.8 * 0.7 = 0.56$$



Probability Laws:

Multiplication Law

Example 5:

Mr. Fahad and Mr. Khan has to tour abroad for their business frequently. Mr. Fahad tours 65% of the times in a year at abroad and Mr. Khan tours 50% of the times in a year at abroad. What is the probability that, on January 01, 2016, both Mr. Fahad and Mr. Khan will be at abroad?

Solution:

Let,

F= Mr. Fahad will be at abroad

K= Mr. Khan will be at abroad

Probability Laws:

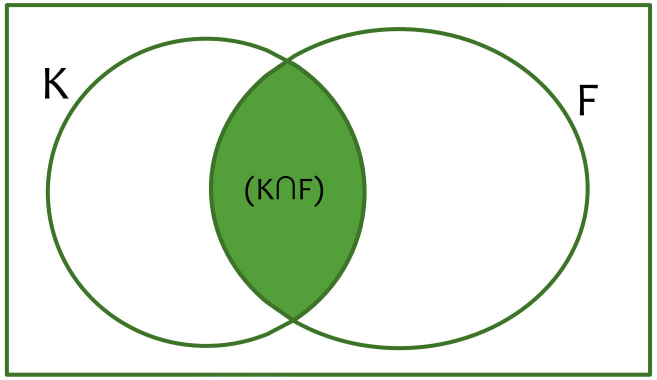
Multiplication Law

Solution(contd.)**:**

Here, given that, $P(F) = \frac{65}{100} = 0.65$ and $P(K) = \frac{50}{100} = 0.5$

Therefore, the probability that, on January 01, 2016, both Mr. Fahad and Mr. Khan will be at abroad is-

$$P(K \cap F) = P(K)P(F)$$
$$= 0.5 * 0.65 = 0.325$$



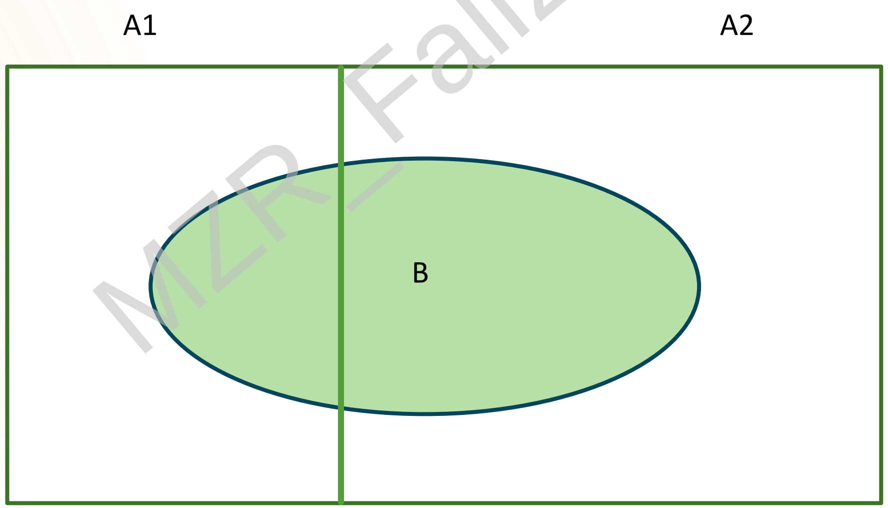
Bayes' Theorem

Let, events A1 and A2 form partition of S. Let B be an event with P(B)>0. Then,

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(A_1)\, P(B|A_1)}{P(A_1)\, P(B|A_1) + P(A_2)\, P(B|A_2)}$$

Note:

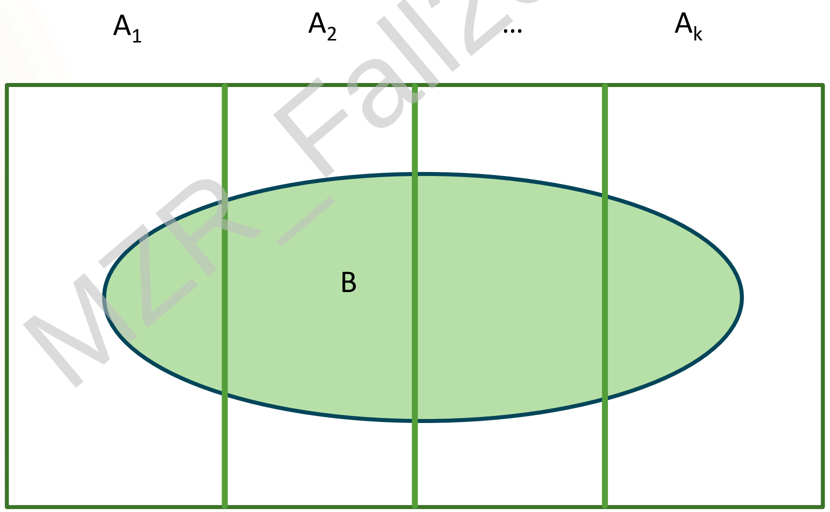
$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) = P(A_1)\, P(B|A_1) + P(A_2)\, P(B|A_2)$$



Bayes' Theorem (more general):

Let, events $A_1, A_2, \ldots, A_k$ form partition of S. Let B be an event with P(B)>0. Then, for an event A_j (j= 1, 2,…, k),

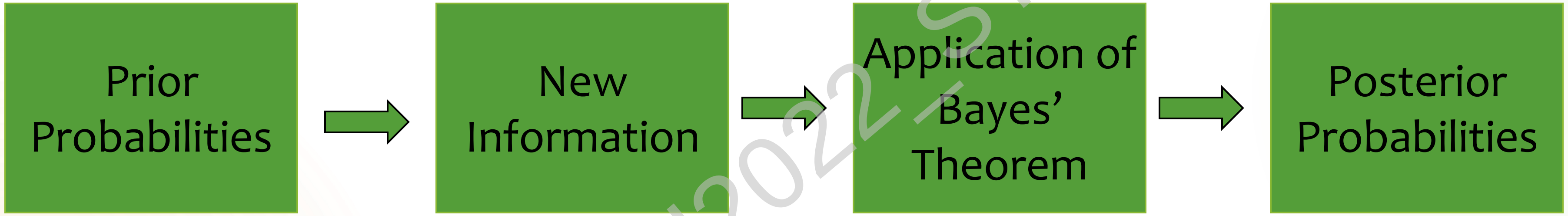
$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(A_j)\, P(B|A_j)}{\sum_{i=1}^{k} P(A_i)\, P(B|A_i)}$$



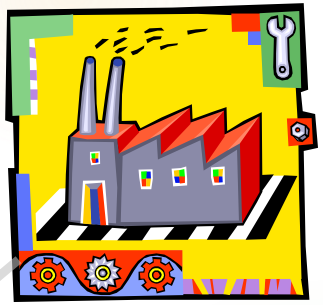
Bayes' Theorem

Suppose we have estimated **prior** probabilities for events we are concerned with, and then obtain new information. We would like to a sound method to computed *revised* or **posterior** probabilities. Bayes' theorem gives us a way to do this.

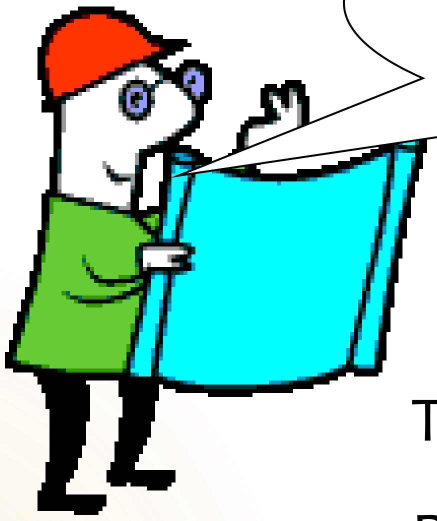
Probability Revision using Bayes' Theorem



Application of Bayes' Theorem



- Consider a manufacturing firm that receives shipment of parts from two suppliers.
- Let A_1 denote the event that a part is received from supplier 1; A_2 is the event the part is received from supplier 2



We get 65 percent of our parts from supplier 1 and 35 percent from supplier 2.

Thus:

$P(A_1) = .65$ and $P(A_2) = .35$

Quality levels differ between suppliers

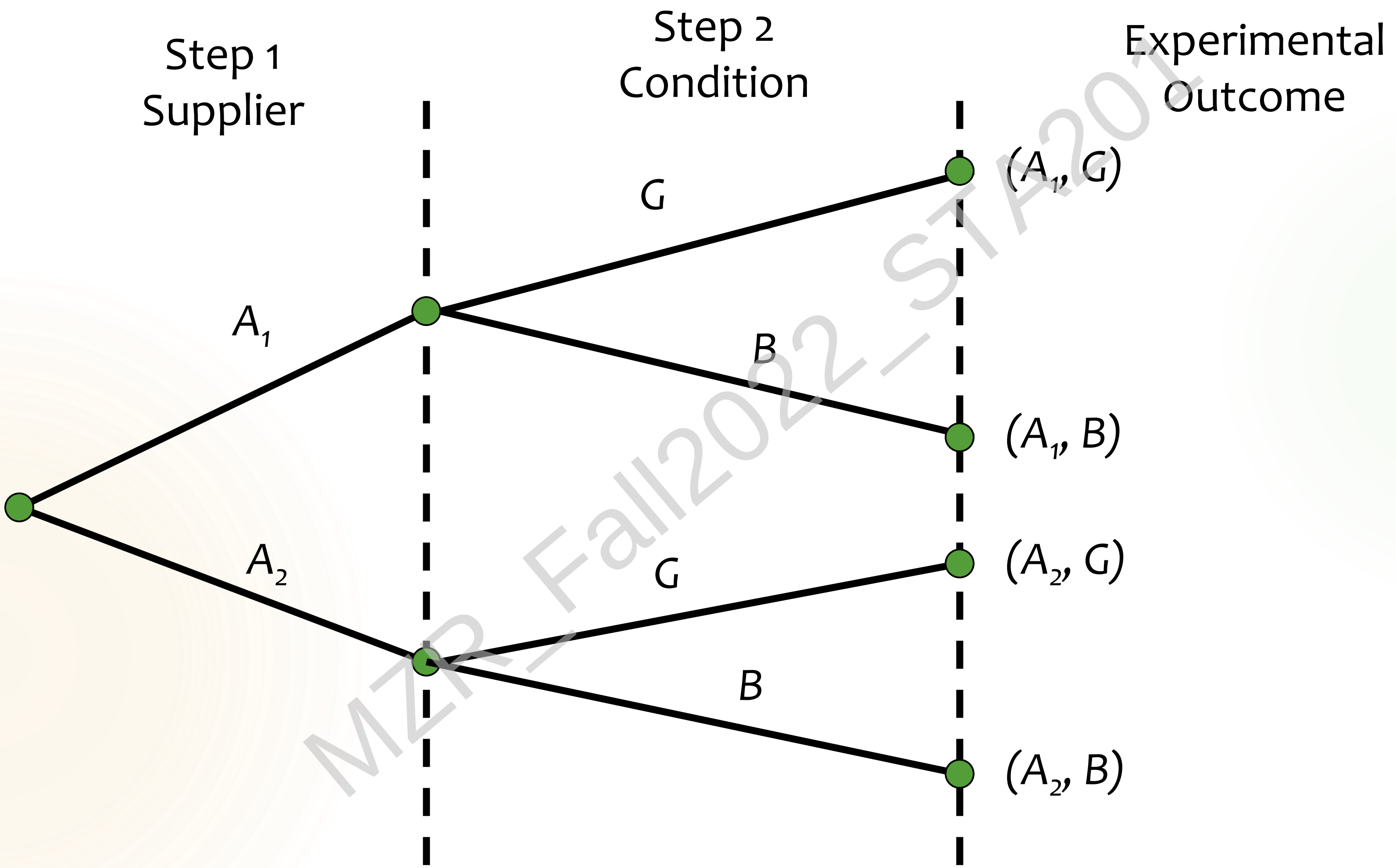
	Percentage Good Parts	Percentage Bad Parts
Supplier 1	98	2
Supplier 2	95	5

Let G denote that a part is good and B denote the event that a part is bad. Thus we have the following conditional probabilities:

$$P(G \mid A_1) = .98 \text{ and } P(B \mid A_2) = .02$$

$$P(G \mid A_2) = .95 \text{ and } P(B \mid A_2) = .05$$

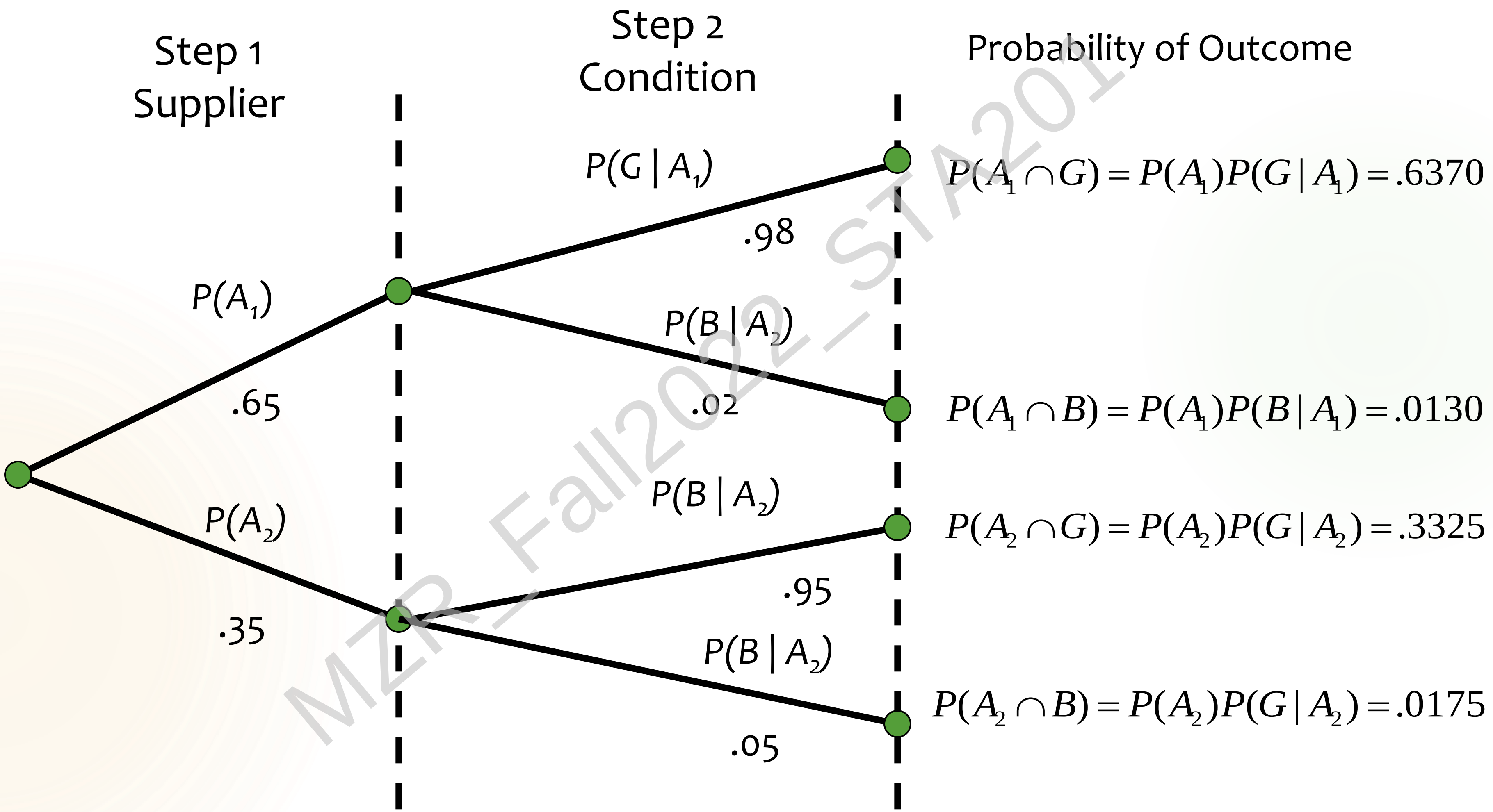
Tree Diagram for Two-Supplier Example



Each of the experimental outcomes is the intersection of 2 events. For example, the probability of selecting a part from supplier 1 that is good is given by:

$$P(A_1, G) = P(A_1 \cap G) = P(A_1)P(G \mid A_1)$$

Probability Tree for Two-Supplier Example



A bad part broke one of our machines—so we're through for the day. What is the probability the part came from suppler 1?

We know from the law of conditional probability that:

$$P(A_1 | B) = \frac{P(A_1 \cap B)}{P(B)} \qquad (4.14)$$

Observe from the probability tree that:

$$P(A_1 \cap B) = P(A_1)P(B | A_1) \qquad (4.15)$$

The probability of selecting a bad part is found by adding together the probability of selecting a bad part from supplier 1 and the probability of selecting bad part from supplier 2.

That is:

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B)$$

$$= P(A_1)P(B \mid A_1) + P(A_2)P(B / A_2) \qquad (4.16)$$

Bayes' Theorem for 2 events

By substituting equations (4.15) and (4.16) into (4.14), and writing a similar result for $P(B \mid A_2)$, we obtain Bayes' theorem for the 2 event case:

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

Do the Math

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

$$= \frac{(.65)(.02)}{(.65)(.02) + (.35)(.05)} = \frac{.0130}{.0305} = .4262$$

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

$$= \frac{(.35)(.05)}{(.65)(.02) + (.35)(.05)} = \frac{.0175}{.0305} = .5738$$

Bayes' Theorem

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \ldots + P(A_n)P(B \mid A_n)}$$

Tabular Approach to Bayes' Theorem—2-Supplier Problem

(1) Events A_i	(2) Prior Probabilities $P(A_i)$	(3) Conditional Probabilities $P(B \mid A_1)$	(4) Joint Probabilities $P(A_i \cap B)$	(5) Posterior Probabilities $P(A_i \mid B)$
A_1	.65	.02	.0130	.0130/.0305 =.4262
A_2	.35	.05	.0175	.0175/.0305 =.5738
	1.00		P(B)=.0305	**1.0000**

Using Excel to Compute Posterior Probabilities

Events	Prior Probabilities	Conditional Probabilities	Joint Probabilities	Posterior Probabilities			
A_1	0.65	0.02	0.013	0.426229508			
A_2	0.35	0.05	0.0175	0.573770492			
			0.0305	1.0000			

Problem

Problem: A consulting firm submitted a bid for a large consulting contract. The firm's management felt id had a 50-50 change of landing the project. However, the agency to which the bid was submitted subsequently asked for additional information. Past experience indicates that that for 75% of successful bids and 40% of unsuccessful bids the agency asked for additional information.

a. What is the prior probability of the bid being successful (that is, prior to the request for additional information).
b. What is the conditional probability of a request for additional information given that the bid will be ultimately successful.
c. Compute the posterior probability that the bid will be successful given a request for additional information.

Solution

Let S_1 denote the event of successfully obtaining the project.

S_2 is the event of not obtaining the project.

B is the event of being asked for additional information about a bid.

a. $P(S_1) = .5$

b. $P(B \mid S_1) = .75$

c. Use Bayes' theorem to compute the posterior probability that a request for information indicates a successful bid.

$$P(S_1 \mid B) = \frac{P(S_1 \cap B)}{P(S_1 \cap B) + P(S_2 \cap B)} = \frac{(.5)(.75)}{(.5)(.75) + (.5)(.4)}$$

$$= \frac{.375}{.575} \cong .652$$

Bayes' Theorem

Example 6:

60% of the students in a class are male. 5% of the males and 10% of the females are in the photography club. If a student is randomly selected from the class.

a. What is the probability that the student is in photography club?
b. If the randomly selected student is in the photography club, what is the chance that the student is male?

Bayes' Theorem

<u>Solution:</u>

Let,

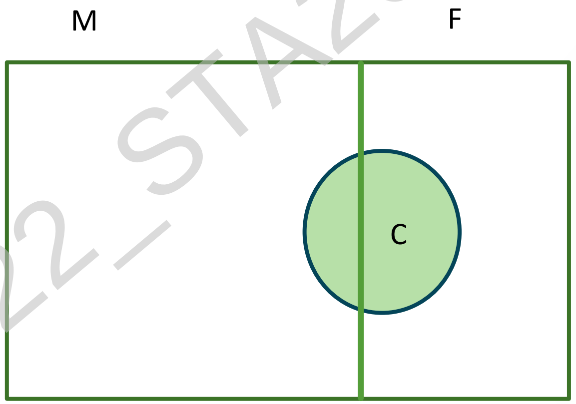
C= The student is in photography club

M= The student is male

F= The student is female

P(M)= 0.6 P(F)=0.4

P(C|M)= 0.05 P(C|F)= 0.10



a. $P(C) = P(M)\,P(C|M) + P(F)\,P(C|F)$

$$= 0.6 \times 0.05 + 0.4 \times 0.10$$

$$= 0.03 + 0.04 = 0.07$$

b. $P(M|C) = \frac{P(M)\,P(C|M)}{P(C)} = \frac{0.6 \times 0.05}{0.07} = \frac{0.03}{0.07} = 0.43$

Probability using contingency table

Example 7:

Below given a contingency table for Smoking status and Cancer status.

Smoking Status/ cancer Status	Cancer	Healthy	Total
Smoker	7860	1530	**9390**
Non-smoker	5390	11580	**16970**
Total	**13250**	**13110**	**26360**

1. What is the probability that a randomly selected person is a smoker
2. What is the probability that a randomly selected person has cancer?
3. What is the probability that a randomly selected person is both smoker and has cancer?
4. If a person is smoker, what is the probability that he also has cancer?

Probability using contingency table

Solution:

Let,

S= The person is smoker, N= The person is non-smoker

C= The person has cancer, H= The person is healthy

1. The probability that a randomly selected person is a smoker

$$P(S) = \frac{9390}{26360} = 0.356$$

Probability using contingency table

Solution (contd.)**:**

2. The probability that a randomly selected person has cancer

$$P(C) = \frac{13250}{26360} = 0.503$$

3. The probability that a randomly selected person is both smoker and has cancer

$$P(S \cap C) = \frac{7860}{26360} = 0.298 = P(C|S)\,P(S) = \frac{7860}{9390} \times \frac{9390}{26360}$$

Probability using contingency table

Solution **(contd.):**

4. If a person is smoker, what is the probability that he also has cancer

$$P(C|S) = \frac{7860}{9390} = 0.837 = \frac{P(S \cap C)}{P(S)} = \frac{\frac{7860}{26360}}{\frac{9390}{26360}}$$