Random Variable & Mathematical Expectation

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Random Variable

A random variable is a variable that takes on **numerical values** as a result of a random experiment or measurement; associates a numerical value with each possible outcome.

The differences between variable and random variable are-

- Random variable always takes **numerical values**
- There is a **probability associated** with each possible values

Random variable is denoted by capital letters such as X, Y, Z etc.

And the possible outcomes are denoted by small letters such as x, y, z etc.

Random Variable

Example 1:

A coin is tossed. It has two possible outcomes- Head and Tail.

Consider a variable, X= outcome of a coin toss= $\begin{cases} H, & if\,Head\ appears \\ T, & if\ Tail\ appears \end{cases}$

Here, S= {H, T}.

But, these are not numerical values.

Random Variable

Example 1(contd.):

Consider a variable, X= Number of heads obtained in a trial

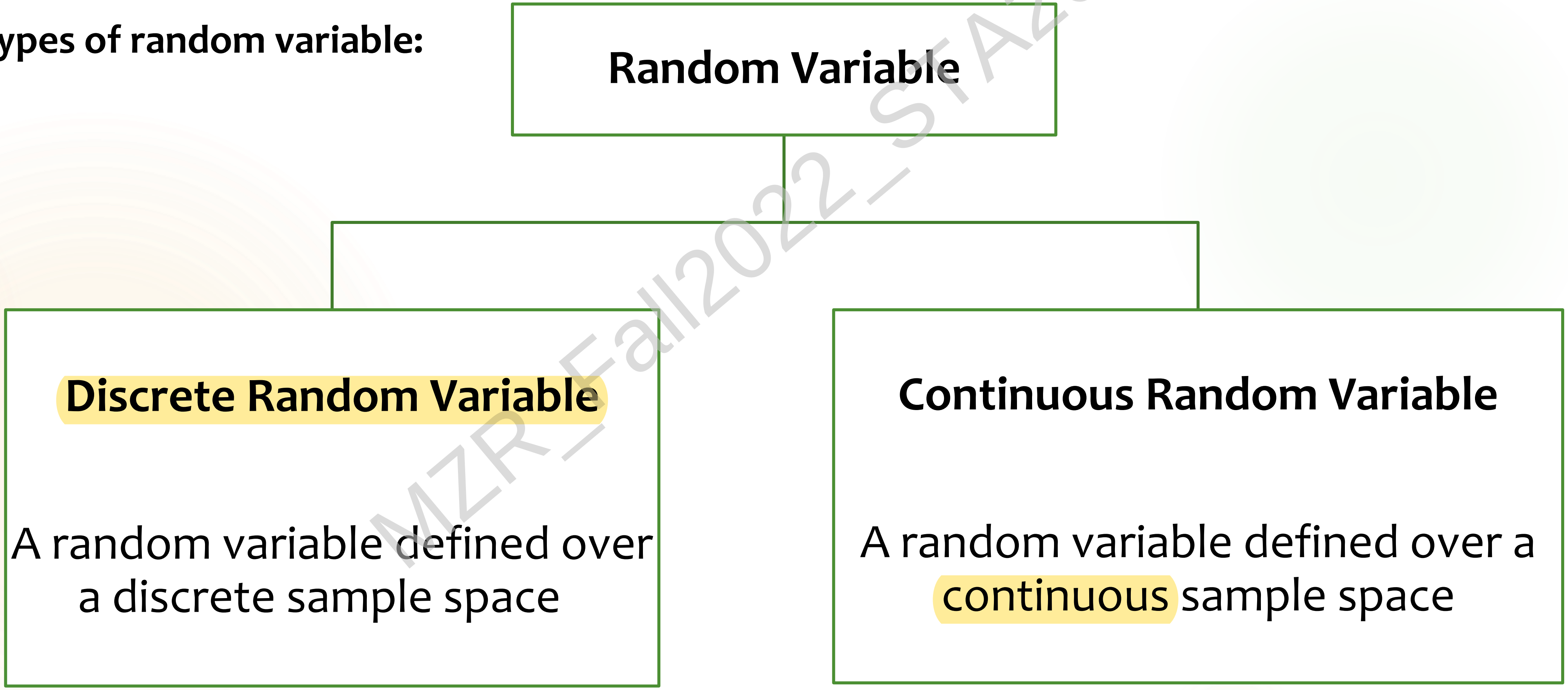
Then, $X = \begin{cases} 1, & if\ Head\ appears \\ 0, & if\ Tail\ appears \end{cases}$

For a fair coin, we can write, P(X=1) = ½ and P(X=0) = ½

So, X is a random variable.

Random Variable

Types of random variable:



Random Variable

Examples:

Discrete Random Variable:

1. X= Number of correct answers in a 100-MCQ test= 0, 1, 2, …, 100
2. X= Number of cars passing a toll both in a day= 0, 1, 2, …, ∞
3. X= Number of balls required to take the first wicket = 1, 2, 3, …, ∞
4. X=The number of telephone calls received in a telephone booth during one day=1,2,..

Continuous Random Variable:

1. X= Weight of a person. $0<X<\infty$
2. X= Monthly Profit. $-\infty<X<\infty$
3. X=Temperature recorded by the meteorological office. $0<X<\infty$

Probability Distributions

Distribution of the probabilities among the different values of a random variable.

Discrete probability distribution- probability distribution of a discrete random variable

Continuous probability distribution- probability distribution of a continuous random variable

Probability Distributions

Examples:

Discrete probability distribution-

- Tossing a coin 2 times.

X= Number of Heads appeared

S= {HH, HT, TH, TT}

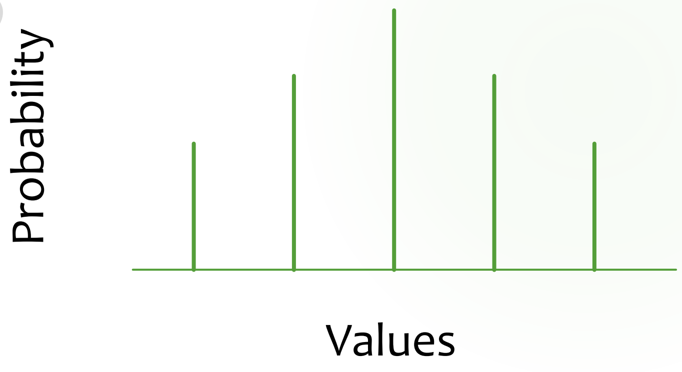
x	0	1	2
P(x)	¼	2/4	¼

Probability Distributions

Different types of probability distributions:

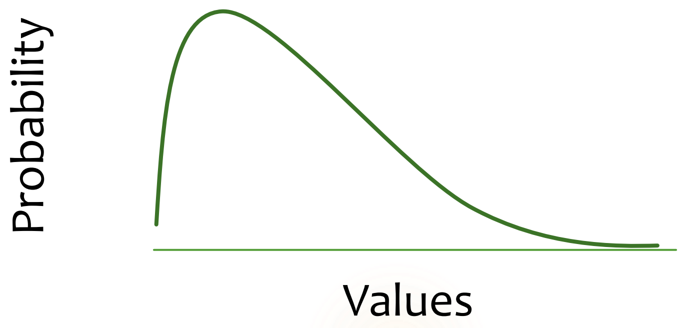
Discrete probability distribution-

1. Bernoulli Distribution
2. *Binomial Distribution*
3. Poisson Distribution etc.



Continuous probability distribution-

1. Uniform Distribution
2. *Normal Distribution*
3. Exponential Distribution
4. t-distribution etc.



PMF and PDF

Probability Mass Function (pmf)- the probability distribution function of a discrete random variable X is called a pmf and is denoted by p(x)

Properties of probability function:

If p(x) is probability function of a discrete random variable X, then p(x) satisfies the following two properties:

- $0 \leq P(x) \leq 1$, For each possible value of X,
- $\sum P(x_i)=1$

Probability Density Function (pdf)- the probability distribution function of a continuous random variable X is called a pdf and is denoted by f(x)

If f(x) is probability function of a discrete random variable X, then f(x) satisfies the following two properties:

1. $f(x) = 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P[a \leq x \leq b] = \int_{a}^{b} f(x)dx$

PMF

Example:

Let X be a random variable with probability function defined as follows

Values of X : x	- 2	0	4	11
$f(x)$	1/10	2/10	4/10	3/10

Find:

i. $P[-2 \le x < 4]$ ii. $P[x > 0]$ iii. $P[x \le 4]$

Answer:

i. $P[-2 \le x < 4] = P[X = -2] + P[X = 0] = \ldots\ \ldots\ \ldots\ \ldots =$

ii. $P[x > 0] = \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots = \ldots\ \ldots\ \ldots\ \ldots =$

iii. $P[x \le 4] = \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots = \ldots\ \ldots\ \ldots\ \ldots =$

PMF

Problem:

A random variable X has the following probability function:

X Values of :x	0	1	2	3	4	5	6	7	8
$f(x)$	a	3a	5a	7a	9a	11a	13a	15a	17a

i. Determine the value of a.

ii. Find $P[x<3], P[x \geq 3]$ and $P[0<x<5]$

Problem:

A coin is tossed three times in which the probability of head is twice as the probability of tail. If the number of heads is a random variable, find the probability function of the random variable. Also find

a. $P[x \geq 1]$ **b.** $P[x=2]$ **c.** $P[x \leq 1]$

Mathematical Expectations

- For a discrete random variable X with pmf p(x), the mathematical expectation of X is-

$$\mu = E(X) = \sum_{x} x\, p(x)$$

- For a continuous random variable X with pdf f(x), the mathematical expectation of X is-

$$\mu = E(X) = \int_{x} x\, f(x)$$

Mathematical expectation is also known as population mean or expected value.

Mathematical Expectations

$$E(X^2) = \begin{cases} \sum_x x^2\, p(x) & , if\ x\ is\ a\ discrete\ r.v. \\ \int_x x^2\, f(x) & , if\ x\ is\ a\ continuous\ r.v. \end{cases}$$

Variance:

$$\sigma^2 = Var(X) = E[X - E(X)]^2 = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2$$

Standard deviation: $\sigma = \sqrt{Var(X)}$

Properties of Mathematical Expectations

Let, c is a constant number

X and Y are two independent random variables

1. $E(c) = c$
2. $E(c\,X) = c\,E(x)$
3. $E(X + c) = E(x) + c$
4. $E(X+Y) = E(X) + E(Y)$
5. $E(X-Y) = E(X) - E(Y)$
6. $E(XY) = E(X) \,.\, E(Y)$

1. $Var(c) = 0$
2. $Var(c\,X) = c^2\,Var(x)$
3. $Var(X + c) = Var(x)$
4. $Var(X+Y) = Var(X) + Var(Y)$
5. $Var(X-Y) = Var(X) + Var(Y)$

Mathematical Expectation

Example 2-

A company estimates the net profit on a new product, it is launching, to be Rs. 3 million during first year, if it is 'successful', Rs. 1 million if it is 'moderately successful', and a loss of Rs. 1 million if it is 'unsuccessful'.

The company assigns the following probabilities to first year prospects for the product-

Successful: 0.25, Moderately successful: 0.40, and Unsuccessful: 0.35

What are the **expected value** and **standard deviation** of the first year net profit for the product? Also, find the expected value of net profit if there is a fixed cost of Rs. 0.2 million, whatever the success status is.

Mathematical Expectation

Solution-

Let,

X= Net profit on the new product in the 1st year (Rs. Million)

Given that,

x	3	1	-1
P(x)	0.25	0.4	0.35

Expected net profit, $E(X) = \sum x\, p(x) = (3 * 0.25) + (1 * 0.4) + (-1 * 0.35)$

$$= 0.8\ million$$

Mathematical Expectation

Solution (contd.)-

$$E(X^2) = \sum x^2\, p(x) = (3^2 * 0.25) + (1^2 * 0.4) + \left((-1)^2 * 0.35\right)$$
$$= (9 * 0.25) + (1 * 0.4) + (1 * 0.35) = 3$$

$$Var(X) = E(X^2) - [E(X)]^2 = 3 - 0.8^2 = 2.36$$
$$\therefore SD(X) = \sqrt{Var(X)} = \sqrt{2.36} = 1.54\ million$$

If there is a fixed cost of Rs. 0.2 million, then expected net profit-

$$E(X - 0.2) = E(X) - 0.2 = 0.8 - 0.2 = 0.6\ million$$

Mathematical Expectation

Sometimes $E(X)$ is called as mathematical expectation of X or expected value of X or mean of the distribution.

Problem:

Find the mean of a random variable having probability function defined as follows:

Values of $X : x$	- 2	0	4	11
$f(x)$	1/10	2/10	4/10	3/10