

Topic 3 Notes

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3 Line integrals and Cauchy's theorem

3.1 Introduction

The basic theme here is that complex line integrals will mirror much of what we've seen for multivariable calculus line integrals. But, just like working with $e^{i\theta}$ is easier than working with sine and cosine, complex line integrals are easier to work with than their multivariable analogs. At the same time they will give deep insight into the workings of these integrals.

To define complex line integrals, we will need the following ingredients:

- The complex plane: $z = x + iy$
- The complex differential $dz = dx + idy$
- A curve in the complex plane: $\gamma(t) = x(t) + iy(t)$, defined for $a \leq t \leq b$.
- A complex function: $f(z) = u(x, y) + iv(x, y)$

3.2 Complex line integrals

Line integrals are also called **path or contour** integrals. Given the ingredients we define the **complex line integral** $\int_{\gamma} f(z) dz$ by

$$\int_{\gamma} f(z) dz := \int_a^b f(\gamma(t))\gamma'(t) dt. \quad (1a)$$

You should note that this notation looks just like integrals of a real variable. We don't need the vectors and dot products of line integrals in \mathbf{R}^2 . Also, make sure you understand that the product $f(\gamma(t))\gamma'(t)$ is just a product of complex numbers.

An alternative notation uses $dz = dx + idy$ to write

$$\int_{\gamma} f(z) dz = \int_{\gamma} (u + iv)(dx + idy) \quad (1b)$$

Let's check that Equations 1a and 1b are the same. Equation 1b is really a multivariable calculus expression, so thinking of $\gamma(t)$ as $(x(t), y(t))$ it becomes

$$\int_{\gamma} f(z) dz = \int_a^b [u(x(t), y(t)) + iv(x(t), y(t))] (x'(t) + iy'(t)) dt$$

But,

$$u(x(t), y(t)) + iv(x(t), y(t)) = f(\gamma(t))$$

and

$$x'(t) + iy'(t) = \gamma'(t)$$

so the right hand side of this equation is

$$\int_a^b f(\gamma(t))\gamma'(t) dt.$$

That is, it is exactly the same as the expression in Equation 1a.

Example 3.1. Compute $\int_{\gamma} z^2 dz$ along the straight line from 0 to $1 + i$.

Solution: We parametrize the curve as $\gamma(t) = t(1 + i)$ with $0 \leq t \leq 1$. So $\gamma'(t) = 1 + i$. The line integral is

$$\int z^2 dz = \int_0^1 t^2(1 + i)^2(1 + i) dt = \frac{2i(1 + i)}{3}.$$

Example 3.2. Compute $\int_{\gamma} \bar{z} dz$ along the straight line from 0 to $1 + i$.

Solution: We can use the same parametrization as in the previous example. So,

$$\int_{\gamma} \bar{z} dz = \int_0^1 t(1 - i)(1 + i) dt = 1.$$

Example 3.3. Compute $\int_{\gamma} z^2 dz$ along the unit circle.

Solution: We parametrize the unit circle by $\gamma(\theta) = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. We have $\gamma'(\theta) = ie^{i\theta}$. So, the integral becomes

$$\int_{\gamma} z^2 dz = \int_0^{2\pi} e^{2i\theta} ie^{i\theta} d\theta = \int_0^{2\pi} ie^{3i\theta} d\theta = \frac{e^{3i\theta}}{3} \Big|_0^{2\pi} = 0.$$

Example 3.4. Compute $\int \bar{z} dz$ along the unit circle.

Solution: Parametrize C : $\gamma(t) = e^{it}$, with $0 \leq t \leq 2\pi$. So, $\gamma'(t) = ie^{it}$. Putting this into the integral gives

$$\int_C \bar{z} dz = \int_0^{2\pi} \overline{e^{it}} i e^{it} dt = \int_0^{2\pi} i dt = \boxed{2\pi i}.$$

3.3 Fundamental theorem for complex line integrals

This is exactly analogous to the fundamental theorem of calculus.

Theorem 3.5. (Fundamental theorem of complex line integrals) If $f(z)$ is a complex analytic function on an open region A and γ is a curve in A from z_0 to z_1 then

$$\int_{\gamma} f'(z) dz = f(z_1) - f(z_0).$$

Proof. This is an application of the chain rule. We have

$$\frac{df(\gamma(t))}{dt} = f'(\gamma(t)) \gamma'(t).$$

So

$$\int_{\gamma} f'(z) dz = \int_a^b f'(\gamma(t)) \gamma'(t) dt = \int_a^b \frac{df(\gamma(t))}{dt} dt = f(\gamma(t)) \Big|_a^b = f(z_1) - f(z_0).$$

Another equivalent way to state the fundamental theorem is: if f has an antiderivative F , i.e. $F' = f$ then

$$\int_{\gamma} f(z) dz = F(z_1) - F(z_0).$$

Example 3.6. Redo $\int_{\gamma} z^2 dz$, with γ the straight line from 0 to $1 + i$.

Solution: We can check by inspection that z^2 has an antiderivative $F(z) = z^3/3$. Therefore the fundamental theorem implies

$$\int_{\gamma} z^2 dz = \frac{z^3}{3} \Big|_0^{1+i} = \frac{(1+i)^3}{3} = \frac{2i(1+i)}{3}.$$

Example 3.7. Redo $\int_{\gamma} z^2 dz$, with γ the unit circle.

Solution: Again, since z^2 had antiderivative $z^3/3$ we can evaluate the integral by plugging the endpoints of γ into the $z^3/3$. Since the endpoints are the same the resulting difference will be 0!

3.4 Path independence

We say the integral $\int_{\gamma} f(z) dz$ is path independent if it has the same value for any two paths with the same endpoints. More precisely, if $f(z)$ is defined on a region A then $\int_{\gamma} f(z) dz$ is path independent in A , if it has the same value for any two paths in A with the same endpoints.

The following theorem follows directly from the fundamental theorem. The proof uses the same argument as Example 3.7.

Theorem 3.8. If $f(z)$ has an antiderivative in an open region A , then the path integral $\int_{\gamma} f(z) dz$ is path independent for all paths in A .

Proof. Since $f(z)$ has an antiderivative $F(z)$, the fundamental theorem tells us that the integral only depends on the endpoints of γ , i.e.

$$\int_{\gamma} f(z) dz = F(z_1) - F(z_0)$$

where z_0 and z_1 are the beginning and end point of γ .

An alternative way to express path independence uses closed paths.

Theorem 3.9. The following two things are equivalent.

1. The integral $\int_{\gamma} f(z) dz$ is path independent.
2. The integral $\int_{\gamma} f(z) dz$ around any closed path is 0.

Proof. This is essentially identical to the equivalent multivariable proof. We have to show two things:

- (i) Path independence implies the line integral around any closed path is 0.
- (ii) If the line integral around all closed paths is 0 then we have path independence.

To see (i), assume path independence and consider the closed path C shown in figure (i) below. Since the starting point z_0 is the same as the endpoint z_1 the line integral $\int_C f(z) dz$ must have the same value as the line integral over the curve consisting of the single point z_0 . Since that is clearly 0 we must have the integral over C is 0.

To see (ii), assume $\int_C f(z) dz = 0$ for any closed curve. Consider the two curves C_1 and C_2 shown in figure (ii). Both start at z_0 and end at z_1 . By the assumption that integrals over closed paths are 0 we have $\int_{C_1-C_2} f(z) dz = 0$. So,

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz.$$

That is, any two paths from z_0 to z_1 have the same line integral. This shows that the line integrals are path independent.

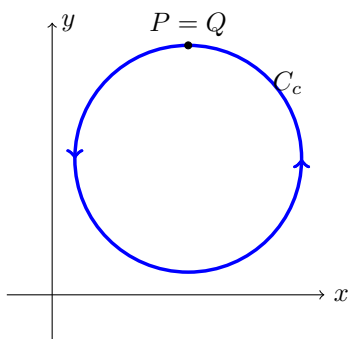


Figure (i)

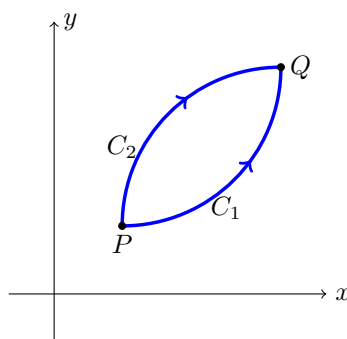


Figure (ii)

3.5 Examples

Example 3.10. Why can't we compute $\int_{\gamma} \bar{z} dz$ using the fundamental theorem.

Solution: Because \bar{z} doesn't have an antiderivative. We can also see this by noting that if \bar{z} had an antiderivative, then its integral around the unit circle would have to be 0. But, we saw in Example 3.4 that this is not the case.

Example 3.11. Compute $\int_{\gamma} \frac{1}{z} dz$ over each of the following contours

- (i) The line from 1 to $1 + i$.

- (ii) The circle of radius 1 around $z = 3$.
- (iii) The unit circle.

Solution: For parts (i) and (ii) there is no problem using the antiderivative $\log(z)$ because these curves are contained in a simply connected region that doesn't contain the origin.

(i)

$$\int_{\gamma} \frac{1}{z} dz = \log(1+i) - \log(1) = \log(\sqrt{2}) + i\frac{\pi}{4}.$$

(ii) Since the beginning and end points are the same, we get

$$\int_{\gamma} \frac{1}{z} dz = 0$$

(iii) We parametrize the unit circle by $\gamma(\theta) = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$. We compute $\gamma'(\theta) = ie^{i\theta}$. So the integral becomes

$$\int_{\gamma} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = \int_0^{2\pi} i d\theta = 2\pi i.$$

Notice that we could use $\log(z)$ if we were careful to let the argument increase by 2π as it went around the origin once.

Example 3.12. Compute $\int_{\gamma} \frac{1}{z^2} dz$, where γ is the unit circle in two ways.

- (i) Using the fundamental theorem.
- (ii) Directly from the definition.

Solution: (i) Let $f(z) = -1/z$. Since $f'(z) = 1/z^2$, the fundamental theorem says

$$\int_{\gamma} \frac{1}{z^2} dz = \int_{\gamma} f'(z) dz = f(\text{endpoint}) - f(\text{start point}) = 0.$$

It equals 0 because the start and endpoints are the same.

(ii) As usual, we parametrize the unit circle as $\gamma(\theta) = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$. So, $\gamma'(\theta) = ie^{i\theta}$ and the integral becomes

$$\int_{\gamma} \frac{1}{z^2} dz = \int_0^{2\pi} \frac{1}{e^{2i\theta}} ie^{i\theta} d\theta = \int_0^{2\pi} ie^{-i\theta} d\theta = -e^{-i\theta} \Big|_0^{2\pi} = 0.$$

3.6 Cauchy's theorem

Cauchy's theorem is analogous to Green's theorem for curl free vector fields.

Theorem 3.13. (Cauchy's theorem) Suppose A is a simply connected region, $f(z)$ is analytic on A and C is a simple closed curve in A . Then the following three things hold:

(i) $\int_C f(z) dz = 0$

(i') We can drop the requirement that C is simple in part (i).

(ii) Integrals of f on paths within A are path independent. That is, two paths with the same endpoints integrate to the same value.

(iii) f has an antiderivative in A .

Proof. We will prove (i) using Green's theorem – we could give a proof that didn't rely on Green's, but it would be quite similar in flavor to the proof of Green's theorem.

Let R be the region inside the curve. And write $f = u + iv$. Now we write out the integral as follows

$$\int_C f(z) dz = \int_C (u + iv)(dx + idy) = \int_C (u dx - v dy) + i(v dx + u dy).$$

Let's apply Green's theorem to the real and imaginary pieces separately. First the real piece:

$$\int_C u dx - v dy = \int_R (-v_x - u_y) dx dy = 0.$$

We get 0 because the Cauchy-Riemann equations say $u_y = -v_x$, so $-v_x - u_y = 0$.

Likewise for the imaginary piece:

$$\int_C v dx + u dy = \int_R (u_x - v_y) dx dy = 0.$$

We get 0 because the Cauchy-Riemann equations say $u_x = v_y$, so $u_x - v_y = 0$.

To see part (i') you should draw a few curves that intersect themselves and convince yourself that they can be broken into a sum of simple closed curves. Thus, (i') follows from (i).¹

Part (ii) follows from (i) and Theorem 3.9.

To see (iii), pick a base point $z_0 \in A$ and let

$$F(z) = \int_{z_0}^z f(w) dw.$$

Here the integral is over any path in A connecting z_0 to z . By part (ii), $F(z)$ is well defined. If we can show that $F'(z) = f(z)$ then we'll be done. Doing this amounts to managing the notation to apply the fundamental theorem of calculus and the Cauchy-Riemann equations. So, let's write

$$f(z) = u(x, y) + iv(x, y), \quad F(z) = U(x, y) + iV(x, y).$$

Then we can write

$$\frac{\partial f}{\partial x} = u_x + iv_x, \text{ etc.}$$

We can formulate the Cauchy-Riemann equations for $F(z)$ as

$$F'(z) = \frac{\partial F}{\partial x} = \frac{1}{i} \frac{\partial F}{\partial y} \tag{2a}$$

i.e.

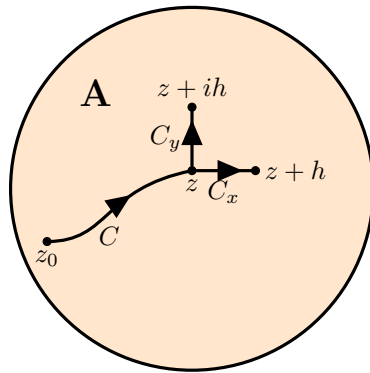
$$F'(z) = U_x + iV_x = \frac{1}{i}(U_y + iV_y) = V_y - iU_y. \tag{2b}$$

¹In order to truly prove part (i') we would need a more technically precise definition of simply connected so we could say that all closed curves within A can be continuously deformed to each other.

For reference, we note that using the path $\gamma(t) = x(t) + iy(t)$, with $\gamma(0) = z_0$ and $\gamma(b) = z$ we have

$$\begin{aligned} F(z) &= \int_{z_0}^z f(w) dw = \int_{z_0}^z (u(x, y) + iv(x, y))(dx + idy) \\ &= \int_0^b (u(x(t), y(t)) + iv(x(t), y(t)))(x'(t) + iy'(t)) dt. \end{aligned} \quad (3)$$

Our goal now is to prove that the Cauchy-Riemann equations given in Equation 3 hold for $F(z)$. The figure below shows an arbitrary path from z_0 to z , which can be used to compute $F(z)$. To compute the partials of F we'll need the straight lines that continue C to $z + h$ or $z + ih$.



Paths for proof of Cauchy's theorem

To prepare the rest of the argument we remind you that the fundamental theorem of calculus implies

$$\lim_{h \rightarrow 0} \frac{\int_0^h g(t) dt}{h} = g(0). \quad (4)$$

(That is, the derivative of the integral is the original function.)

First we'll look at $\frac{\partial F}{\partial x}$. So, fix $z = x + iy$. Looking at the paths in the figure above we have

$$F(z + h) - F(z) = \int_{C+C_x} f(w) dw - \int_C f(w) dw = \int_{C_x} f(w) dw.$$

The curve C_x is parametrized by $\gamma(t) = x + t + iy$, with $0 \leq t \leq h$. So,

$$\begin{aligned} \frac{\partial F}{\partial x} &= \lim_{h \rightarrow 0} \frac{F(z + h) - F(z)}{h} = \lim_{h \rightarrow 0} \frac{\int_{C_x} f(w) dw}{h} \\ &= \lim_{h \rightarrow 0} \frac{\int_0^h u(x + t, y) + iv(x + t, y) dt}{h} \\ &= u(x, y) + iv(x, y) \\ &= f(z). \end{aligned} \quad (5)$$

The second to last equality follows from Equation 4.

Similarly, we get (remember: $w = z + it$, so $dw = i dt$)

$$\begin{aligned}
 \frac{1}{i} \frac{\partial F}{\partial y} &= \lim_{h \rightarrow 0} \frac{F(z + ih) - F(z)}{ih} = \lim_{h \rightarrow 0} \frac{\int_{C_y} f(w) dw}{ih} \\
 &= \lim_{h \rightarrow 0} \frac{\int_0^h u(x, y + t) + iv(x, y + t) i dt}{ih} \\
 &= u(x, y) + iv(x, y) \\
 &= f(z).
 \end{aligned} \tag{6}$$

Together Equations 5 and 6 show

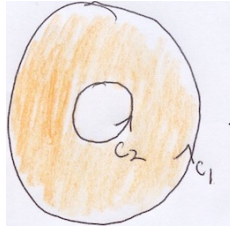
$$f(z) = \frac{\partial F}{\partial x} = \frac{1}{i} \frac{\partial F}{\partial y}$$

By Equation 2a we have shown that F is analytic and $F' = f$. \square

3.7 Extensions of Cauchy's theorem

Cauchy's theorem requires that the function $f(z)$ be analytic on a simply connected region. In cases where it is not, we can extend it in a useful way.

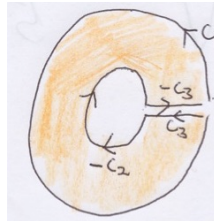
Suppose R is the region between the two simple closed curves C_1 and C_2 . Note, both C_1 and C_2 are oriented in a counterclockwise direction.



Theorem 3.14. (Extended Cauchy's theorem) If $f(z)$ is analytic on R then

$$\int_{C_1 - C_2} f(z) dz = 0.$$

Proof. The proof is based on the following figure. We 'cut' both C_1 and C_2 and connect them by two copies of C_3 , one in each direction. (In the figure we have drawn the two copies of C_3 as separate curves, in reality they are the same curve traversed in opposite directions.)



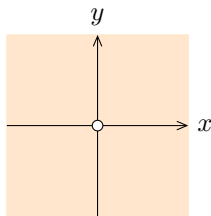
With C_3 acting as a cut, the region enclosed by $C_1 + C_3 - C_2 - C_3$ is simply connected, so Cauchy's Theorem 3.13 applies. We get

$$\int_{C_1+C_3-C_2-C_3} f(z) dz = 0$$

The contributions of C_3 and $-C_3$ cancel, which leaves $\int_{C_1-C_2} f(z) dz = 0$. QED

Note. This clearly implies $\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$.

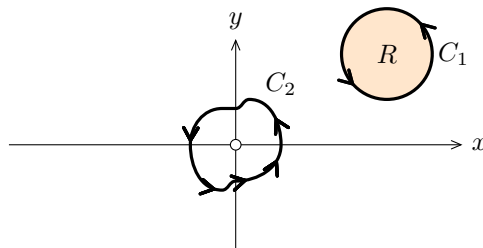
Example 3.15. Let $f(z) = 1/z$. $f(z)$ is defined and analytic on the punctured plane.



Punctured plane: $\mathbf{C} - \{0\}$

Question: What values can $\int_C f(z) dz$ take for C a simple closed curve (positively oriented) in the plane?

Solution: We have two cases (i) C_1 not around 0, and (ii) C_2 around 0



Case (i): Cauchy's theorem applies directly because the interior does not contain the problem point at the origin. Thus,

$$\int_{C_1} f(z) dz = 0.$$

Case (ii): we will show that

$$\int_{C_2} f(z) dz = 2\pi i.$$

Let C_3 be a small circle of radius a centered at 0 and entirely inside C_2 .

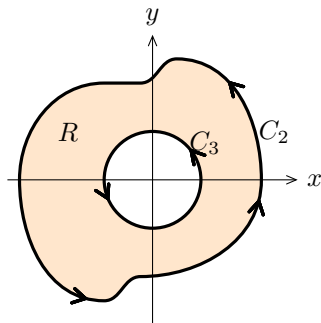


Figure for part (ii)

By the extended Cauchy theorem we have

$$\int_{C_2} f(z) dz = \int_{C_3} f(z) dz = \int_0^{2\pi} i dt = 2\pi i.$$

Here, the line integral for C_3 was computed directly using the usual parametrization of a circle.

Answer to the question: The only possible values are 0 and $2\pi i$.

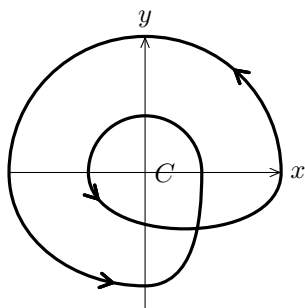
We can extend this answer in the following way:

If C is not simple, then the possible values of

$$\int_C f(z) dz$$

are $2\pi ni$, where n is the number of times C goes (counterclockwise) around the origin 0.

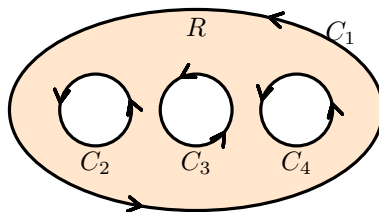
Definition. n is called the *winding number* of C around 0. n also equals the number of times C crosses the positive x -axis, counting +1 for crossing from below and -1 for crossing from above.



A curve with winding number 2 around the origin.

Example 3.16. A further extension: using the same trick of cutting the region by curves to make it simply connected we can show that if f is analytic in the region R shown below then

$$\int_{C_1 - C_2 - C_3 - C_4} f(z) dz = 0.$$



That is, $C_1 - C_2 - C_3 - C_4$ is the boundary of the region R .

Orientation. It is important to get the orientation of the curves correct. One way to do this is to make sure that the region R is always to the left as you traverse the curve. In the above example. The region is to the right as you traverse C_2 , C_3 or C_4 in the direction indicated. This is why we put a minus sign on each when describing the boundary.
