MAT 216

Linear Algebra & Fourier Analysis

Example Problems PART A

Contents:

- ***Odd Even Functions***
- ***Fourier Series***

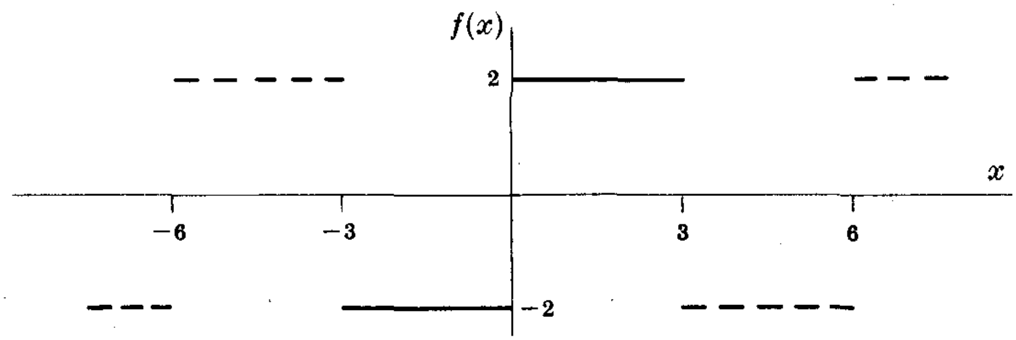
Reading Module:

Schaum's Outline Series Theory problems of Fourier Analysis – Murray R. Spiegel

Example 1:

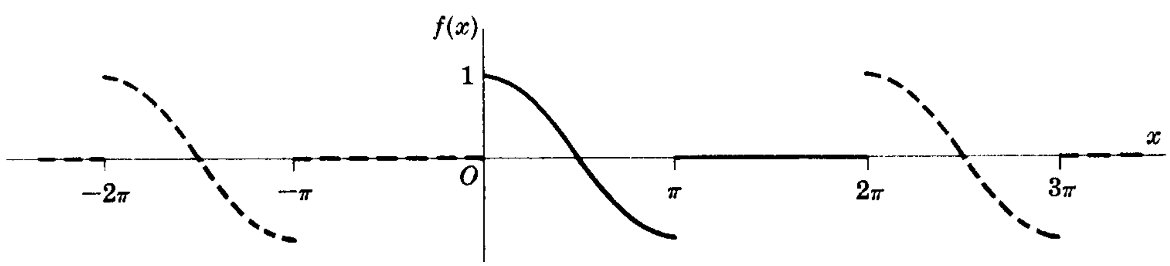
Classify each of the following functions according as they are even, odd, or neither even nor odd.

$$(a)\quad f(x) = \begin{cases} 2 & 0 < x < 3 \\ -2 & -3 < x < 0 \end{cases} \qquad \text{Period} = 6$$



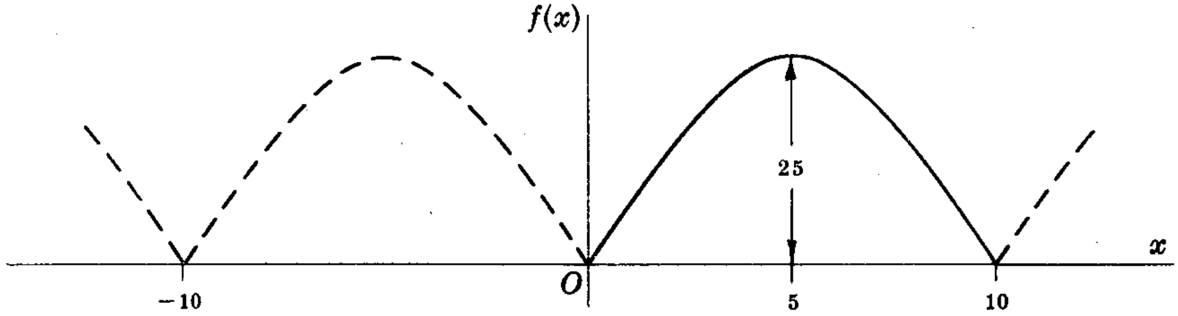
The graph of the function is seen to be odd.

$$(b)\quad f(x) = \begin{cases} \cos x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases} \qquad \text{Period} = 2\pi$$



The graph of the function is neither odd nor even.

(c) $f(x) = x(10 - x), \quad 0 < x < 10, \quad$ Period $= 10.$



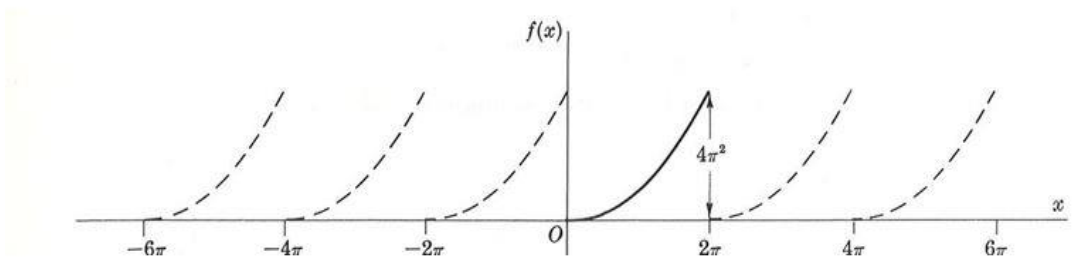
The graph of the function is seen to be even.

Example 2:

Note: the response from $-\pi$ to π is exactly the same as from 0 to 2π
so integrating over either is the same…..and the later is easier

Expand $f(x) = x^2, \ 0 < x < 2\pi$, in a Fourier series if the period is 2π.

The graph of $f(x)$ with period 2π is shown



$$Period = 2L = 2\pi \quad \therefore L = \pi$$

We know

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi}{L}(x) + \sum_{n=1}^{\infty} b_n sin\frac{n\pi}{L}(x)$$

Now we will evaluate the coefficients a_0, a_n, b_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}(x)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x^2 cos\frac{n\pi}{\pi}(x)dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x^2\cos(nx)\,dx$$

$$\boldsymbol{let\ x^2 = u\ \ and\ \cos(nx) = v, applying} \int (\boldsymbol{uv})\boldsymbol{dx} = \boldsymbol{u}\int \boldsymbol{v\,dx} - \int \{\frac{\boldsymbol{d}}{\boldsymbol{dx}}(\boldsymbol{u})\int (\boldsymbol{v})\boldsymbol{dx}\}\boldsymbol{dx}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin(nx)}{n}\right) - \int 2x\left(\frac{\sin(nx)}{n}\right)dx\right]$$

$$let\ u = 2x, and\ v = \frac{\sin(nx)}{n}$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin(nx)}{n} - \left\{2x\left(-\frac{\cos(nx)}{n^2}\right) - \int 2\left(-\frac{\cos(nx)}{n^2}\right)dx\right\}\right]$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin(nx)}{n} + 2x\frac{\cos(nx)}{n^2} - \frac{2}{n^2}\int \cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin(nx)}{n} + 2x\frac{\cos(nx)}{n^2} - 2\frac{\sin(nx)}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[4\pi^2\frac{\sin(2\pi n)}{n} + 4\pi\frac{\cos(2\pi n)}{n^2} - 2\frac{\sin(2\pi n)}{n^3} - 0^2\frac{\sin(0)}{n} - 2(0)\frac{\cos(0)}{n^2} + 2\frac{\sin(0)}{n^3}\right]$$

Note:

- $\boldsymbol{\sin(n\pi) = 0, \forall n}$
- $\boldsymbol{\cos(n\pi) = \begin{cases} +1, & for\ \text{"n" even} \\ -1, & for\ \text{"n"}\ odd \end{cases}}$
- **In this case $\cos(2\pi n) = +1$, $\because 2\times(\pi n)$ is an even number as we know 2 multiplied by any number is an even number.**

$$= \frac{1}{\pi}\left[0 + 4\pi\frac{1}{n^2} - 0 - 0 - 0 + 0\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right] = \frac{4}{n^2} \qquad while\ n \neq 0\ \because n\ is\ in\ the\ denominator$$

$$a_n: [n = 0] \xrightarrow{\text{yields}} a_0 = \frac{1}{L}\int_{-L}^{L} x^2 cos\frac{n\pi}{L}(x)dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x^2 cos\frac{0.\pi}{\pi}(x)dx;\ since\ n = 0$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} x^2 \cos(0)\ dx,$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} x^2 (1) dx$$

$$= \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{2\pi} = \frac{8\pi^2}{3}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) sin \frac{n\pi}{L}(x) dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} x^2 \sin(nx)\, dx$$

$$let\ u = x^2, \qquad v = \sin(nx)$$

$$= \frac{1}{\pi} \left[x^2 \left(-\frac{\cos(nx)}{n} \right) - \int 2x \left(-\frac{\cos(nx)}{n} \right) dx \right]$$

$$= \frac{1}{\pi} \left[x^2 \left(-\frac{\cos(nx)}{n} \right) + \int 2x \left(\frac{\cos(nx)}{n} \right) dx \right]$$

$$let\ u = 2x, \qquad v = \frac{\cos(nx)}{n}$$

$$= \frac{1}{\pi} \left[-x^2 \frac{\cos(nx)}{n} + \left\{ 2x \left(\frac{\sin(nx)}{n^2} \right) - \int 2 \frac{\sin(nx)}{n^2} dx \right\} \right]$$

$$= \frac{1}{\pi} \left[-x^2 \frac{\cos(nx)}{n} + 2x \frac{\sin(nx)}{n^2} - 2 \left(-\frac{\cos(nx)}{n^3} \right) \right]$$

$$= \frac{1}{\pi} \left[-x^2 \frac{\cos(nx)}{n} + 2x \frac{\sin(nx)}{n^2} + 2 \frac{\cos(nx)}{n^3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-4\pi^2 \frac{\cos(2\pi n)}{n} + 4\pi \frac{\sin(2\pi n)}{n^2} + 2 \frac{\cos(2\pi n)}{n^3} - \left\{ -0^2 \frac{\cos(0)}{n} + 2(0) \frac{\sin(0)}{n^2} + 2 \frac{\cos(0)}{n^3} \right\} \right]$$

$$= \frac{1}{\pi} \left[-4\pi^2 \frac{1}{n} + 4\pi(0) + 2 \frac{1}{n^3} - \left\{ -0 + 0 + 2 \frac{1}{n^3} \right\} \right]$$

$$= \frac{1}{\pi} \left[\frac{-4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3} \right] = \frac{1}{\pi} \left[-\frac{4\pi^2}{n} \right] = -\frac{4\pi}{n}$$

Since

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi}{L}(x) + \sum_{n=1}^{\infty} b_n sin\frac{n\pi}{L}(x)$$

$$= \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2} cos\frac{n\pi}{\pi}(x) + \sum_{n=1}^{\infty} \left(\frac{-4\pi}{n}\right) sin\frac{n\pi}{\pi}(x)$$

$$= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} 4\left(\frac{1}{n^2}\cos(nx) - \frac{\pi}{n}\sin(nx)\right).$$
