MAT120

ASSIGNMENT 01

SUMMER 22

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Substitution methods

11) $\int \frac{\sqrt{x^2-1}}{x^4}\,dx$

Let, $x = \sec\theta$

$\Rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}(\sec\theta)$

$\Rightarrow \frac{dx}{d\theta} = \sec\theta\tan\theta$

$\Rightarrow dx = \sec\theta\tan\theta\,d\theta$

substituting the values in our integral,

$I = \int \frac{\sqrt{x^2-1}}{x^4}\,dx$

$= \int \frac{\sqrt{\sec^2\theta-1}}{\sec^4\theta} \times \sec\theta\tan\theta\,d\theta$

$= \int \frac{\sqrt{\tan^2\theta}}{\frac{1}{\cos^4\theta}} \times \frac{1}{\cos\theta} \times \frac{\sin\theta}{\cos\theta}\,d\theta$

$= \int \tan\theta \cdot \cos^4\theta \times \frac{\sin\theta}{\cos^2\theta}\,d\theta$

$$\Rightarrow \int \frac{\sin\theta}{\cos\theta} \times \cos^4\theta \times \frac{\sin\theta}{\cos^2\theta} \cdot d\theta$$

$$= \int \sin^2\theta \cdot \cos\theta \cdot d\theta$$

$$\therefore I = \int \sin^2\theta \cdot \cos\theta \cdot d\theta$$

Now assume,

$$u = \sin\theta$$

$$\Rightarrow \frac{du}{d\theta} = \frac{d}{d\theta}(\sin\theta)$$

$$\Rightarrow du = \cos\theta \, d\theta$$

substituting values we get,

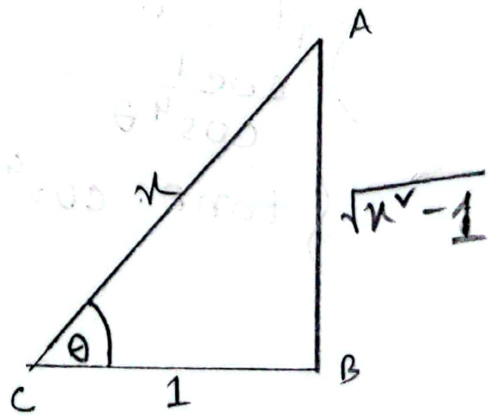
$$I = \int u^2 \cdot du$$

$$= \frac{u^3}{3} + C = \frac{\sin^3\theta}{3} + C$$

Again, we has assumed,

$$\sec\theta = \frac{x}{1} = \frac{hyp}{adj}$$

$$\therefore \sin\theta = \frac{\sqrt{x^2-1}}{x}$$



we get,

$$I = \left(\frac{\sqrt{x^2-1}}{x}\right)^3 \times \frac{1}{3} + C$$

$$= \frac{\left(\sqrt{x^2-1}\right)^3}{3x^3} + C$$

$$= \frac{(x^2-1)^{3/2}}{3x^3} + C$$

$\therefore$ The evaluated result of the integral,

$I_0 = \int \frac{\sqrt{x^2-1}}{x^4}\, dx$ is $I = \frac{(x^2-1)^{3/2}}{3x^3} + C$

Integration by parts

2. Evaluate : $\int (3t+5)\cos\left(\frac{t}{4}\right) dt$

Given integral,

$$I = \int (3t+5)\cos\left(\frac{t}{4}\right) dt$$

We know,

$$I = uv - \int v\, du$$

Now assume,

$$u = 3t+5$$

$$\Rightarrow \frac{du}{dt} = 3\frac{dt}{dt} + \frac{d5}{dt}$$

$$\Rightarrow du = 3\,dt$$

$$dv = \cos\left(\frac{t}{4}\right) dt$$

$$\Rightarrow \int dv = \int \cos\left(\frac{t}{4}\right) dt$$

$$\Rightarrow v = \frac{\sin\left(\frac{t}{4}\right)}{\frac{1}{4}} + C$$

$$\Rightarrow v = 4\sin\left(\frac{t}{4}\right) + C$$

Now,

$$I = (3t+5)\cdot 4\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)\cdot 3\,dt$$

$$= 4(3t+5)\sin\left(\frac{t}{4}\right) - 12\int \sin\left(\frac{t}{4}\right) dt$$

$$= 4(3t+5)\sin\left(\frac{t}{4}\right) + 12\frac{\cos\left(\frac{t}{4}\right)}{\frac{1}{4}} + C$$

$= 4(3t+5)\sin\left(\frac{t}{4}\right) + 48\cos\left(\frac{t}{4}\right) + C$

$\therefore$ Evaluated result of integral $I_0 = \int (3t+5)\cos\left(\frac{t}{4}\right) dt$ is $I = 4(3t+5)\sin\left(\frac{t}{4}\right) + 48\cos\left(\frac{t}{4}\right) + C$.

Rectangle method for finding areas

[4] Use defination with x_k^* as the i) right end point, ii) left end point and iii) mid point of each subinterval to find the area between the graph of $f(x) = x^3$ and the interval $[0, 5]$.

Solution: Here, interval $[a, b] = [0, 5]$

Width, $\Delta x = \frac{b-a}{n} = \frac{5}{n}$

1) [illegible]ht end point, $x_k^* = a + k\Delta x$

$= 0 + k \cdot \frac{5}{n}$

$= \frac{5k}{n}$

Now, our function, $f(x) = x^3$

$$\therefore f(x_k^*) = \left(\frac{5k}{n}\right)^3 = \frac{125k^3}{n^3}$$

$$\text{Area}, A = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{125k^3}{n^3} \cdot \frac{5}{n}$$

$$= \lim_{n\to+\infty} \frac{625}{n^4} \sum_{k=1}^{n} k^3$$

$$= 625 \lim_{n\to+\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3$$

$$= 625 \times \frac{1}{4}$$

$$= \frac{625}{4}$$

$\therefore$ Area using right end point, $A_1 = \frac{625}{4}$

ii) Left end point, $x_k^* = a + (k-1)\Delta x$

$$= 0 + (k-1) \cdot \frac{5}{n} \quad \left[\because \Delta x = \frac{5}{n}\right]$$

$$= \frac{5(k-1)}{n}$$

Now, $f(x) = x^3$

$$\Rightarrow f(x_k^*) = \left\{\frac{5(k-1)}{n}\right\}^3$$

$$= \frac{5^3(k-1)^3}{n^3} = \frac{125(k-1)^3}{n^3}$$

Now, Area, $A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*) \cdot \Delta x$

$$= \lim_{n \to +\infty} \sum_{k=1}^{n} \frac{125(k-1)^3}{n^3} \times \frac{5}{n}$$

$$= \lim_{n \to +\infty} \frac{625}{n^4} \sum_{k=1}^{n} (k-1)^3$$

$$= 625 \lim_{n \to +\infty} \frac{1}{n^4} \sum_{k=1}^{n} (k^3 - 3k^2 + 3k - 1)$$

$$= 625 \lim_{n \to +\infty} \left(\frac{1}{n^4} \sum_{k=1}^{n} k^3 - \frac{3}{n} \cdot \frac{1}{n^3} \sum_{k=1}^{n} k^2 + \frac{3}{n^2} \cdot \frac{1}{n^2} \sum_{k=1}^{n} k - \frac{1}{n^4} \cdot \sum_{k=1}^{n} 1 \right)$$

$$= 625\left(\lim_{n\to+\infty}\frac{1}{n^4}\sum_{k=1}^{n}k^3 - \lim_{n\to+\infty}\frac{3}{n}\cdot\frac{1}{n^3}\sum_{k=1}^{n}k^2 + \lim_{n\to+\infty}\frac{3}{n^2}\cdot\frac{1}{n^2}\sum_{k=1}^{n}k - \lim_{n\to+\infty}\frac{1}{n^4}\sum_{k=1}^{n}1\right)$$

$$= 625\left(\frac{1}{4} - \frac{3}{\infty}\cdot\frac{1}{3} + \frac{3}{(\infty)^2}\cdot\frac{1}{2} - \frac{1}{(\infty)^4}\cdot 1\right)$$

$$= 625\left(\frac{1}{4} - 0 + 0 - 0\right)$$

$$= \frac{625}{4}$$

$\therefore$ Areas with x_k^* as the left end point is $\frac{625}{4}$.

iii) Midpoint, $x_k = a + \left(k - \frac{1}{2}\right)\Delta x$

$$= 0 + \left(k - \frac{1}{2}\right) \times \frac{5}{n}$$

$$= \frac{5k}{n} - \frac{5}{2n}$$

$$= \frac{10k - 5}{2n}$$

Now, $f(x) = x^3$

$$\Rightarrow f(x_k^*) = \left(\frac{10k-5}{2n}\right)^3$$

$$= \frac{5^3(2k-1)^3}{8n^3}$$

Now,

$$\text{Area} = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*) \times \Delta x$$

$$= \lim_{n \to +\infty} \sum_{k=1}^{n} \frac{5^3}{8n^3}(2k-1)^3 \times \frac{5}{n}$$

$$= \frac{625}{8} \lim_{n \to +\infty} \sum_{k=1}^{n} \frac{(2k-1)^3}{n^4}$$

$$= \frac{625}{8} \lim_{n \to +\infty} \frac{1}{n^4} \sum_{k=1}^{n} 8k^3 - 3(2k^2)\cdot 1 + 3\cdot 2k - 1$$

$$= \frac{625}{8} \lim_{n \to +\infty} \frac{1}{n^4} \sum_{k=1}^{n} 8k^3 - 12k^2 + 6k - 1$$

$$= \frac{625}{8} \lim_{n \to +\infty} \left(\frac{8}{n^4} \sum_{k=1}^{n} k^3 - \frac{12}{n} \cdot \frac{1}{n^3} \sum_{k=1}^{n} k^2 + \frac{6}{n^2} \cdot \frac{1}{n^2} \sum_{k=1}^{n} k - \frac{1}{n^4} \sum_{k=1}^{n} 1 \right)$$

$$= \frac{625}{8}\left(\lim_{n\to+\infty} 8\cdot\frac{1}{n^4}\sum_{k=1}^{n} k^3 - \lim_{n\to+\infty}\frac{12}{n}\cdot\frac{1}{n^3}\sum_{k=1}^{n} k^2\right.$$

$$\left. + \lim_{n\to+\infty}\frac{6}{n^2}\cdot\frac{1}{n^2}\sum_{k=1}^{n} k - \lim_{n\to+\infty}\frac{1}{n^4}\sum_{k=1}^{n} 1\right)$$

$$= \frac{625}{8}\left(\frac{8}{4} - \frac{12}{\infty}\cdot\frac{1}{3} + \frac{6}{(\infty)^2}\cdot\frac{1}{2} - \frac{1}{(\infty)^4}\cdot 1\right)$$

$$= \frac{625}{8} \times \frac{8}{4}$$

$$= \frac{625}{4}$$

$\therefore$ Area with x_k^* as the mid point is $\frac{625}{4}$.