Lec: 02
MAT: 120

Techniques of Integration by substitution

U-substitution:

(i) $\int \frac{x^2+1}{\sqrt{x^3+3x}}\,dx$ (ii) $\int \cos^3 x\,dx$ (iii) $\int x\sqrt{x^2+1}\,dx$

(iv) $\int \frac{1}{\sqrt{x}} \sin\sqrt{x}\,dx$ (v) $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx$

(vi) $\int \sin^2 x \cos x\,dx$. (vii) $\int \frac{dx}{(\frac{1}{3}x - 1)}$

Summation Formulas:

(a) $\sum_{k=1}^{n} k$

(b) $\sum_{k=1}^{n} k^2$

(c) $\sum_{k=1}^{n} k^3$

A definition of Area: Suppose that the function f is continuous and nonnegative on the interval $[a, b]$, and let R denote the region bounded below by the x-axis, bounded on the sides by the vertical lines $x=a$ & $x=b$, and bounded above by the curve $y=f(x)$

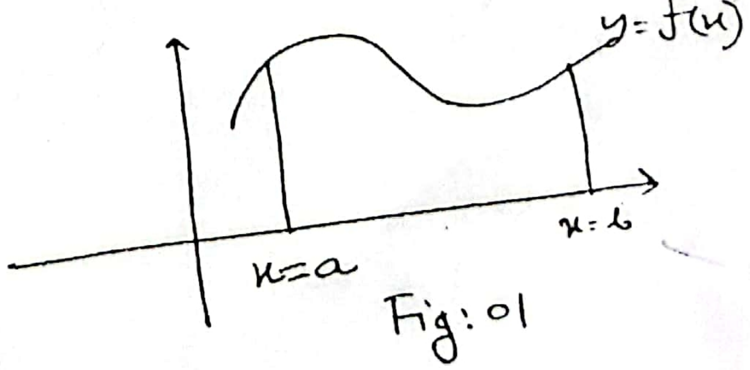


Fig: 01

• Divide the interval $[a,b]$ into n equal subinterval by inserting $n-1$ equally spaced points between a & b and denote those points by $x_1, x_2, \dots, x_{n-1}$ (Figure: 02)

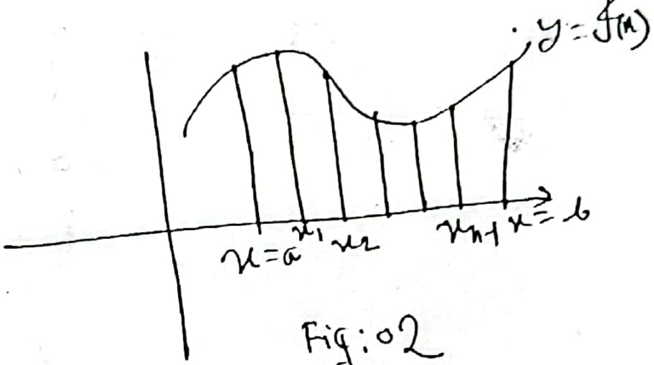
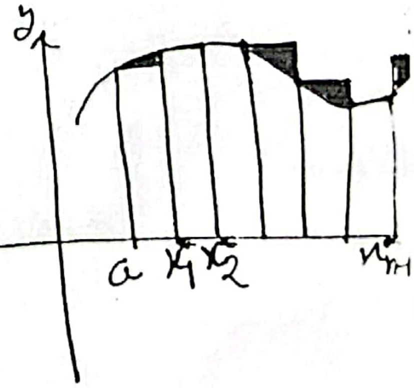


Fig: 02

• Each of these subintervals has width $(b-a)/n$, which is customarily denoted by

$$\Delta x = \frac{b-a}{n}.$$

• Over each subinterval construct a rectangle whose height is the value of f at an arbitrarily selected point in the subint. (Figure: 03)



denote the points selected in the subintervals, then the rectangles will have heights $f(x_1^*), f(x_2^*), \dots$
$\dots f(x_n^*)$ and area

$$f(x_1^*)\Delta x,\ f(x_2^*)\Delta x, \dots, f(x_n^*)\Delta x.$$

The left endpoint, right endpoint, and midpoint cho[illegible]
$x_1^*, x_2^*, \ldots, x_n^*$ are given by

Left endpoint: $x_k^* = x_{k-1} = a + (k-1)\Delta x$

Right endpoint: $x_k^* = x_k = a + k\Delta x$

Midpoint: $x_k^* = \frac{1}{2}(x_{k-1} + x_k) = a + \left(k - \frac{1}{2}\right)\Delta x$.

Theorem:

(a) $\lim\limits_{n \to +\infty} \frac{1}{n} \sum\limits_{k=1}^{n} 1 = 1$.

(b) $\lim\limits_{n \to +\infty} \frac{1}{n^2} \sum\limits_{k=1}^{n} k = \frac{1}{2}$;

(c) $\lim\limits_{n \to +\infty} \frac{1}{n^3} \sum\limits_{k=1}^{n} k^2 = \frac{1}{3}$.

(d) $\lim\limits_{n \to +\infty} \frac{1}{n^4} \sum\limits_{k=1}^{n} k^3 = \frac{1}{4}$.

Lesson 02

Example: Use definition (*) with x_i^* as the right endpoint of each subinterval to find the area between the graph of $f(x) = x^2$ and the interval $[0,1]$.

Solution:

The length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

Now for the right endpoint

$$x_k^* = a + k\Delta x = 0 + k\frac{1}{n} = \frac{k}{n}$$

Thus,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} (x_k^*)^2 \Delta x$$

$$= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{1}{n^3}\left[\frac{(n^2+n)(2n+1)}{6}\right]$$

$$= \frac{1}{n^3}\left[\frac{2n^3+n^2+2n^2+n}{6}\right]$$

$$= \frac{1}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right]$$

$$= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

$$\therefore \sum_{k=1}^{n} f(x_k^*)\, \Delta x = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

By the definition:

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n \to \infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right)$$

$$= \lim_{n \to \infty} \frac{1}{3} + \lim_{n \to \infty} \frac{1}{2n} + \lim_{n \to +\infty} \frac{1}{6n^2}$$

$$= \frac{1}{3}$$

Example: Use the definition (*) with x_k^* as the midpoint of each subinterval to find the area under the parabola $y = f(x) = 9 - x^2$ and over the interval $[0, 3]$