SET - 8

MAT110

SUMMER 21

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SEC : 04 (FAB)

SET : 08

Ans to the qr no 1

Given,

$$f(x) = x^4 - 12x^3$$

Now, $f'(x) = 4x^3 - 36x^2$

if $f'(x) = 0$

$\Rightarrow 4x^3 - 36x^2 = 0$

$\Rightarrow 4x^3 = 36x^2$

$\Rightarrow$

Now, $f''(x) = 12x^2 - 72x$

At $x = 0$,

$f''(0) = 0$

~~for~~ since $f''(x) = 0$, this is an inconclusive point.

Again at $x = 9$,

$f''(9) = 12 \times (9)^2 - 72 \times 9$

$= 12 \times 81 - 72 \times 9$

$= 324$

since $f''(x) > 0$ at $x = 9$, this function has a relative maximum at $x = 9$ which is,

$f(9) = 9^4 - 12(9)^3 = -2187$

Ans to the or no 2

Given, $f(x) = \dfrac{1}{x+2}$

Now,

$$f'(x) = -\frac{1}{(x+2)^2}$$

$$f''(x) = -\frac{(-2)}{(x+2)^3} = \frac{2}{(x+2)^3}$$

$$f'''(x) = \frac{2(-3)}{(x+2)^4} = \frac{-6}{(x+2)^4}$$

About $x = 3$,

$$f(3) = \frac{1}{3+2} = 5$$

$$f'(3) = -\frac{1}{(3+2)^2} = -\frac{1}{25}$$

$$f''(x) = \frac{2}{(3+2)^3} = \frac{2}{125}$$

$$f'''(x) = \frac{-6}{(3+2)^4} = \frac{-6}{625}$$

Taylor series about $x=3$ is,

$$P_3(3) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + \cdots$$

$$\Rightarrow P_3(3) = \frac{1}{5} - \frac{1}{25}(x-3) + \frac{1}{125}(x-3)^2 - \frac{1}{625}(x-3)^3 + \cdots$$

$$\Rightarrow P_3(3) = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^{n+1} \cdot (-1)^n \cdot (x-3)^n$$

$$\Rightarrow P_3(3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{(5)^{n+1}}$$

$\therefore$ The taifor series : $\frac{1}{5} - \frac{1}{25}(x-3) + \frac{1}{125}(x-3)^2 - \frac{1}{625}(x-3)^3 + \cdots$

sigma notation of the series : $\sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{(5)^{n+1}}$

Ans to the qr no 3

Given, $T = x^2y - xy^3 + 2$

$x = r\cos\theta$

$y = r\sin\theta$

Now, $\dfrac{\partial T}{\partial x} = 2xy - y^3$

$\dfrac{\partial T}{\partial y} = x^2 - 3xy^2$

$\dfrac{\partial x}{\partial \theta} = -r\sin\theta$

$\dfrac{\partial y}{\partial \theta} = r\cos\theta$

$\dfrac{\partial x}{\partial r} = \cos\theta$

$\dfrac{\partial \cancel{x}y}{\partial r} = \sin\theta$

Now,

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= (2xy - y^3)\cos\theta + (x^2 - 3xy^2)\sin\theta$$

$$= (2r^2\cos\theta\sin\theta - r^3\sin^3\theta)\cos\theta +$$

$$\{(r^2\cos^2\theta - 3r^3\cos\theta\sin^2\theta)\sin\theta\}$$

$$= 2r^2\cos^2\theta\sin\theta - r^3\sin^3\theta\cos\theta +$$

$$r^2\cos^2\theta\sin\theta - 3r^3\cos\theta\sin^3\theta$$

$$3r^2\cos^2\theta\sin\theta - 4r^3\sin^3\theta\cos\theta$$

$$=$$

And, $\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial \theta}$

$$= (2xy - y^3)(-r\sin\theta) + (x^2 - 3xy^2) \cdot r\cos\theta$$

$$= (2r^2\cos\theta\sin\theta - r^3\sin^3\theta)(-r\sin\theta) + (r^2\cos^2\theta$$

$$- 3r^3\cos\theta \cdot \sin^2\theta)\, r\cos\theta$$

$$= -2r^3\cos\theta\sin^2\theta + r^4\sin^4\theta + r^3\cos^3\theta - 3r^4\cos^2\theta\sin^2\theta$$

$$= r^4\sin^4\theta + r^3\cos^3\theta - 2r^3\cos\theta\sin^2\theta - 3r^4\cos^2\theta\sin^2\theta$$

Ans to the qr no 4

Given,

$$f(x,y) = x^2 + xy + y^2 - 3x$$

Now, $f_x = 2x + y - 3$

$f_y = x + 2y$

if $f_x = 0$

$\Rightarrow 2x + y - 3 = 0$

$\Rightarrow 2x + y = 3 \quad \cdots (i)$

And $f_y = 0$

$\Rightarrow x + 2y = 0$

$\Rightarrow x = -2y \quad \cdots (ii)$

Substituting value of x in equation (i),

$2(-2y) + y = 3$

$\Rightarrow -4y + y = 3$

$\Rightarrow -3y = 3$

$\Rightarrow y = -1$

Substituting value of y in equation (11),

$x = -2(-1) = 2$

$\therefore$ critical point $(x, y) = (2, -1)$

Now, $f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = 1$

Since $f_{xy} = 1 < f_{xx} \cdot f_{yy} = 4$

And $f_{xx} = f_{yy} = 2 > 0$,

There is a relative minima at point $(2, -1)$ which is,

$f(2, -1) = 4 - 2 + 1 - 6 = -3$

Ans to the qr no 5

Given, $f(x,y) = \tan^{-1}(x+2y)$

Now, $f_x = \frac{1}{1+(x+2y)^2}$

$f_y = \frac{2}{1+(x+2y)^2}$

$f_{xx} = \frac{-1}{\{1+(x+2y)^2\}^2} \cdot 2(x+2y)$

$f_{yy} = \frac{-2}{\{1+(x+2y)^2\}^2} \cdot 2(x+2y) \cdot 2$

$f_{xy} = \frac{-1}{\{1+(x+2y)^2\}^2} \cdot 2(x+2y) \cdot 2$

since we have to evaluate this at the point $(1,0)$

$f(1,0) = \tan^{-1}(1) = \frac{\pi}{4}$

$f_x(1,0) = \frac{1}{1+1} = \frac{1}{2}$

$f_y(1,0) = \frac{2}{1+1} = 1$

$f_{xx}(1,0) = \frac{-1}{(1+1)^2} \cdot 2 = -\frac{2}{4} = -\frac{1}{2}$

$f_{yy}(1,0) = \frac{-8}{(1+1)^2} = -2$

$f_{xy}(1,0) = \frac{-4}{4} = -1$

Now, to calculate the second degree polynomial $Q(x,y)$ we shall first get the first degree polynomial $L(x,y)$.

$\therefore L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$

$\Rightarrow L(x,y) = \frac{\pi}{4} + \frac{1}{2}(x-1) + y$

Now, the seond degree polynomial,

$Q(x,y) = L(x,y) + \frac{f_{xx}(1,0)}{2!}(x-1)^2 + f_{xy}(1,0)(x-1)(y-0) + \frac{f_{yy}(1,0)}{2!}(y-0)^2$

$$\Rightarrow Q(x,y) = \frac{\pi}{4} + \frac{1}{2}(x-1) + y + -\frac{1}{2} \times \frac{1}{2}(x-1)^2$$

$$+ (-1)(x-1)y + \frac{-2}{2} y^2$$

$$\Rightarrow Q(x,y) = \frac{\pi}{4} + \frac{1}{2}(x-1) + y - \frac{1}{4}(x-1)^2 - (x-1)y - y^2$$

$\therefore$ The second degree polynomial of the given function is,

$$Q(x,y) = \frac{\pi}{4} + \frac{1}{2}(x-1) + y - \frac{1}{4}(x-1)^2 - (x-1)y - y^2$$

Ans to the qr no 6

Given, $\vec{F} = e^{x}\hat{i} + \ln(xy)\hat{j} + e^{xyz}\hat{k}$

Divergence, $\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(e^{x}\hat{i} + \ln(xy)\hat{j} + e^{xyz}\hat{k}\right)$

$\Rightarrow \vec{\nabla}\cdot\vec{F} = e^{x} + \frac{1}{xy}\cdot\frac{\partial}{\partial y}(xy) + e^{xyz}\cdot\frac{\partial}{\partial z}(xyz)$

$\Rightarrow \vec{\nabla}\cdot\vec{F} = e^{x} + \frac{x}{xy} + e^{xyz}\cdot xy$

$\Rightarrow \vec{\nabla}\cdot\vec{F} = e^{x} + \frac{1}{y} + xye^{xyz}$

curl $\vec{F}$, $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x} & \ln(xy) & e^{xyz} \end{vmatrix}$

$\Rightarrow \vec{\nabla}\times\vec{F} = \hat{i}\left\{\frac{\partial}{\partial y}(e^{xyz}) - \frac{\partial}{\partial z}\ln(xy)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(e^{xyz}) - \frac{\partial}{\partial z}e^{x}\right\}$
$+ \hat{k}\left\{\frac{\partial}{\partial x}\ln(xy) - \frac{\partial}{\partial y}(e^{x})\right\}$

$\Rightarrow \vec{\nabla}\times\vec{F} = \left\{e^{xyz}\cdot\frac{\partial}{\partial y}(xyz) - 0\right\}\hat{i} - \hat{j}\left\{e^{xyz}\cdot\frac{\partial}{\partial x}(xyz) - 0\right\}$
$+ \left\{\frac{1}{xy}\frac{\partial}{\partial x}(xy) - 0\right\}\hat{k}$

$$\Rightarrow \vec{\nabla} \times \vec{F} = xz\, e^{xyz}\, \hat{i} \cancel{+} - yz\, e^{xyz}\, \hat{j} + \frac{1}{x}\, \hat{k}$$

$$\therefore \text{Div } \vec{F} = e^{x} + \frac{1}{y} + xy\, e^{xyz}$$

$$\text{Curl } \vec{F} = \cancel{xye}\; xz\, e^{xyz}\, \hat{i} - yz\, e^{xyz}\, \hat{j} + \frac{1}{x}\, \hat{k}$$

Ans to the qr no 7

Given,

$$-x^2+4y^2-2x-16y+11=0$$

$$\Rightarrow 4y^2-16y-x^2-2x+11=0$$

$$\Rightarrow 4y^2-16y+16-(x^2+2x+1)+1-16+11=0$$

$$\Rightarrow (2y-4)^2-(x+1)^2=4$$

$$\Rightarrow \frac{4(y-2)^2}{4}-\frac{(x+1)^2}{4}=1$$

$$\Rightarrow \frac{(y-2)^2}{1^2}-\frac{(x+1)^2}{2^2}=1 \Rightarrow \frac{(y-2)^2}{1^2}-\frac{x-(-1)^2}{2^2}=1$$

comparing this equation with the standard form of the equation of the hyperbola $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ we get,

$a=1$, $b=2$

(a) center : $(h, k) = (-1, 2)$

(b) vertices :

The vertices of this paracpola would vary on the y axis.

$\therefore$ vertices, $(h, k \pm a) = (-1, 2 \pm 1)$

$\therefore$ vertices are $(-1, 3)$, $(-1, 1)$

(c) Foci :

To find the foci we need to find the ~~ed~~ eccentricity, $e = \sqrt{1 + b^2/a^2}$

$= \sqrt{1 + 2^2} = \sqrt{5}$

Foci, $F = (h, k \pm ae)$

$= (-1, 2 \pm 1\sqrt{5})$

$\therefore$ Foci are at $(-1, 2+\sqrt{5})$ and $(-1, 2-\sqrt{5})$