Chapter 11

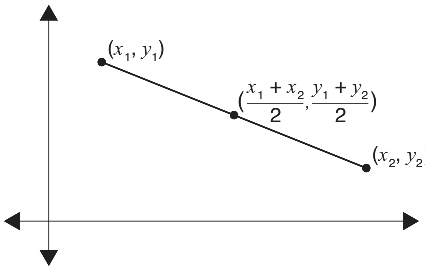
Conic Sections

11.1 Midpoint and Distance Formulas

The *midpoint* of a line segment is the point equidistant from the endpoints of the line segment. If you know the coordinates of the endpoints of a line segment, you can find the coordinates of the midpoint by averaging the coordinates.

Midpoint Formula

The midpoint of a line segment with endpoints (x_1,y_1) and (x_2,y_2) is the point $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$.



Example: Find the coordinates of the midpoint of line segment *AB* if *A*(–4, 3) and *B*(6, –1).

Solution: Use the midpoint formula.

$A(-4, 3), B(6, -1)$

$$M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

$$M=\left(\frac{-4+6}{2},\frac{3+(-1)}{2}\right)=(1, 1)$$

The midpoint of $\overline{AB}$ is (1, 1).

If you know the midpoint and one endpoint of a segment, you can use the midpoint formula to calculate the other endpoint.

Example: The midpoint of line segment *CD* is (4, –2) and *C*(2, 3). What are the coordinates of *D*?

Solution: Write equations for the *x* and *y* coordinates of *D* using the midpoint formula and the coordinates of the midpoint. Solve for *x* and *y*.

$M = (4, -2), C(2, 3)$

$$M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

$$(4,-2)=\left(\frac{2+x_2}{2},\frac{3+y_2}{2}\right)$$

$4=\dfrac{2+x_2}{2}$	$-2=\dfrac{3+y_2}{2}$	Write equations for x_2 and y_2 from the midpoint formula.
$8=2+x_2$	$-4=3+y_2$	Multiply both sides of each equation by 2.
$6=x_2$	$-7=y_2$	Solve.

The coordinates of D are (6, –7).

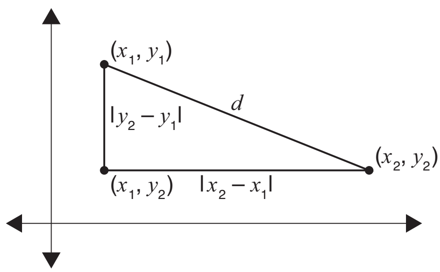
To find the *distance* between two points, you can use the *distance formula*, which is derived from the Pythagorean Theorem. Recall that the Pythagorean Theorem says that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse, or $c^2 = a^2 + b^2$.

$$c^2 = a^2 + b^2$$

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Distance Formula

The distance between two points (x_1,y_1) and (x_2,y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example: Find the length of line segment *AB* if *A*(–4, 3) and *B*(6, –1).

Solution: Use the distance formula.

A(–4, 3), *B*(6, –1)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ Distance formula.

$$d = \sqrt{(6-(-4))^2 + (-1-3)^2}$$ Substitute known values.

$$d = \sqrt{10^2 + (-4)^2} \quad \text{Simplify.}$$

$$d = \sqrt{116} \approx 10.77 \quad \text{Simplify.}$$

The length of $\overline{AB}$ is approximately 10.77 units.

Practice Exercises

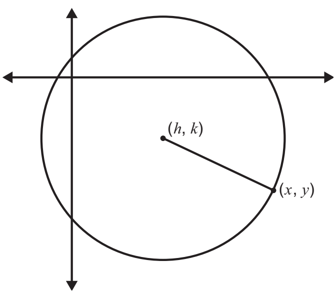
11.1 Find the coordinates of the midpoint of line segment *AB* if *A*(1, 7) and *B*(–5, 3).

11.2 The midpoint of line segment *CD* is (–3, 1) and endpoint *C* is (2, 5). What are the coordinates of *D*?

11.3 Find the length of line segment *AB* if *A*(–2, –5) and *B*(8, –4).

11.2 Circles

A *circle* is the set of points in a plane that are equidistant from a single point, called the *center* of the circle. The *radius* of the circle is a line segment whose endpoints are the center of the circle and any point on the circle.



Since a circle is defined by a distance from a point, you can use the distance formula to write the equation of a circle with center (*h*, *k*).

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d \quad \text{Distance formula.}$$

$$\sqrt{(x-h)^2 + (y-k)^2} = r \quad \text{Substitute values.}$$

$$(x-h)^2 + (y-k)^2 = r^2 \quad \text{Square both sides.}$$

Standard Form of the Equation of a Circle

Center at (h, k) and radius r	**Center (0, 0) and radius r**
$(x-h)^2 + (y-k)^2 = r^2$	$x^2 + y^2 = r^2$

You can think of the graph of a circle with center (h, k) as a transformation of the circle with center (0, 0). Every point on the circle is translated h units horizontally and k units vertically.

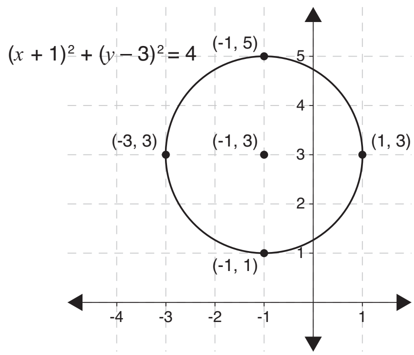
Example: Graph $(x+1)^2+(y-3)^2=4$. Is the circle a function?

Solution: Identify the center and the radius of the circle. Graph the center and sketch the circle using the radius.

$(x-h)^2+(y-k)^2=r^2$	Standard equation of a circle.
$(x+1)^2+(y-3)^2=4$	Given equation.
$(x-(-1))^2+(y-3)^2=2^2$	Transform to standard form.

Center (–1, 3), radius 2

The equation of a circle is not a function. You can see in the graph that there are two y values for most x values. The circle fails the vertical line test for a function.



Note that the center of a circle is a reference point and not part of the circle. The only points on the circle are the points that are r units away from the center. That includes the points marked with dots as well as the infinitely many points that lie on the graph of the circle.

Example: Write the equation of a circle with center (6, –5) and radius $\sqrt{7}$.

Solution: Substitute h, k, and r in the standard form of the equation of a circle.

$h = 6$, $k = -5$, $r = \sqrt{7}$

$(x-h)^2+(y-k)^2=r^2$	Standard equation of a circle.
$(x-6)^2+(y-(-5))^2=\sqrt{7}^2$	Substitute h, k, and r.
$(x-6)^2+(y+5)^2=7$	Simplify.

The *diameter* of a circle is a line segment that passes through the center of a circle and has endpoints on the circle. You can write the equation for a circle if you know the endpoints of the diameter by using the distance and midpoint formulas.

Example: Write the equation of a circle if the endpoints of a diameter are (2, 5) and (–6, 3).

Solution: Use the midpoint formula to find the center of the circle. Use the distance formula to find the radius. Substitute *h*, *k,* and *r* in the standard form of the equation of a circle.

$M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$	Midpoint formula.
$(h,k) = \left(\dfrac{2 + -6}{2}, \dfrac{5+3}{2}\right)$	Substitute endpoints of diameter.
$(h, k) = (-2, 4)$	Simplify.

The center of the circle is (–2, 4).

Find the distance between the center and another point on the circle.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance formula.
$r = \sqrt{(-2-2)^2 + (4-5)^2}$	Substitute center (–2, 4) and point (2, 5).
$r = \sqrt{16+1}$	Simplify.
$r = \sqrt{17}$	Simplify.

The radius of the circle is $\sqrt{17}$.

Use the center and the radius to write the equation of the circle.

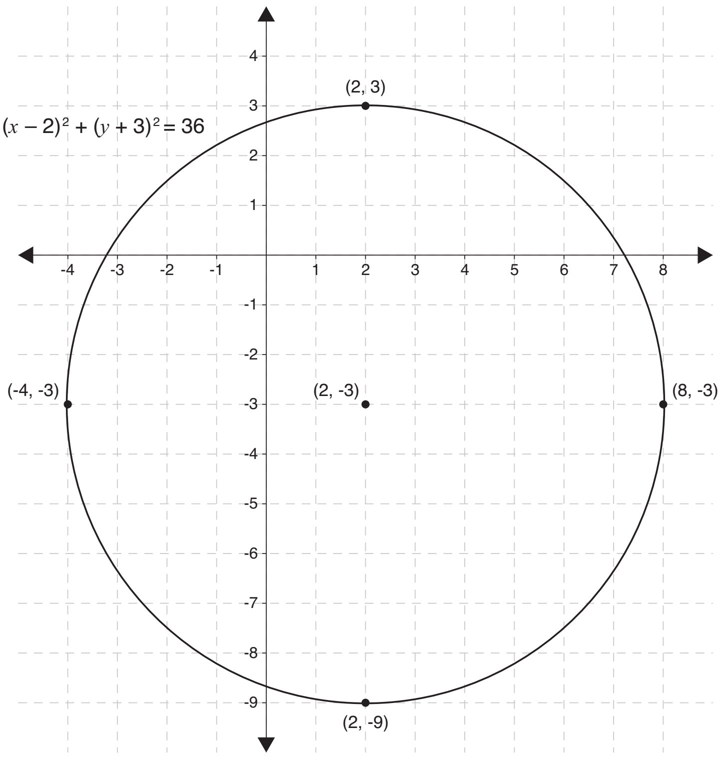
$(x-h)^2 + (y-k)^2 = r^2$	Standard equation of a circle.
$(x-(-2))^2 + (y-4)^2 = \sqrt{17}^2$	Substitute *h*, *k,* and *r*.
$(x+2)^2 + (y-4)^2 = 17$	Simplify.

If you are given the equation of a circle that is not in standard form, you may have to complete the square in order to transform the equation to standard form. To review completing the square, visit Lesson 5.3.

Example: Graph the circle $x^2 + y^2 - 4x + 6y = 23$.

Solution: Transform the equation into the standard form for the equation of a circle by completing the square for *x* and *y*. Identify the center and the radius and sketch the graph.

$x^2 + y^2 - 4x + 6y = 23$	
$(x^2 - 4x) + (y^2 + 6y) = 23$	Regroup.
$(x^2 - 4x + 4) + (y^2 + 6y + 9) = 23 + 13$	$\left(\frac{-4}{2}\right)^2 = 4; \left(\frac{6}{2}\right)^2 = 9$. Add 13 to both sides.
$(x-2)^2 + (y+3)^2 = 36$	Factor the perfect square trinomials.
$(x-2)^2 + (y+3)^2 = 6^2$	Write in standard form.
Center (2, –3), radius 6	Identify the center and radius.



Practice Exercises

11.4 Graph $(x+5)^2+(y+2)^2=1$.

11.5 Write the equation of a circle with center (2, 1) and radius 9.

11.6 Write the equation of a circle if the endpoints of the diameter are (–1, –4) and (3, –8).

11.7 Write the equation of the circle $x^2+y^2+12x+18y=-106$ in standard form.

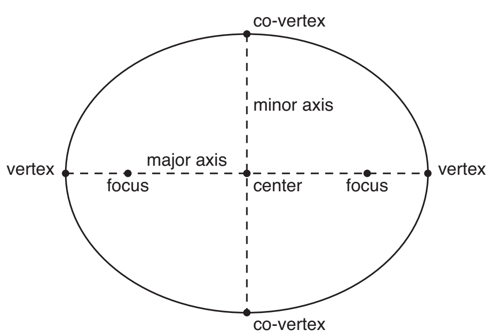
11.3 Ellipses

An ellipse looks like a circle that has been flattened. An *ellipse* is the set of all points in the plane such that the sum of the distances from two fixed points is constant. Each fixed point is called a *focus*, or together, the *foci*.

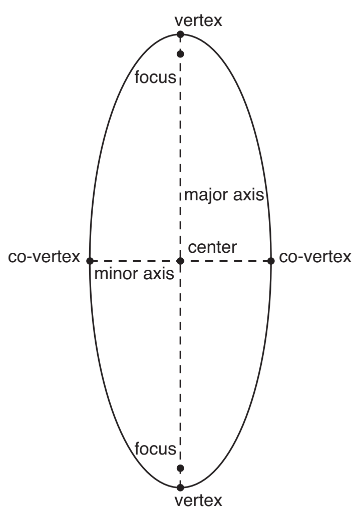
An ellipse has two axes of symmetry: the longer one called the *major axis* and the shorter one called the *minor axis*. The axes are perpendicular and intersect at the midpoint of the foci, or the *center* of the ellipse. The foci lie on the major axis.

An ellipse has four vertices. Each vertex is the point of intersection of the ellipse and an axis of symmetry. Some call only the intersections of the major axis and the ellipse *vertices* and call the intersection of the minor axis and the ellipse *minor vertices* or *co-vertices*.

Ellipse with horizontal major axis



Ellipse with vertical major axis



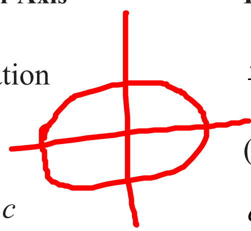
There are two forms of the standard equation of an ellipse centered at (0, 0), depending on whether the major axis is horizontal or vertical.

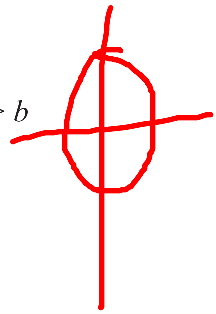
When an ellipse is written in standard form, you can identify the direction of the major axis by identifying which variable has the larger denominator. If x^2 has a larger denominator, the major axis will be the x-axis or parallel to the x-axis. If y^2 has a larger denominator, the major axis will be the y-axis or parallel to the y-axis.

If $a = b$, then the major and minor axes are equal, and the ellipse is a circle.

Standard Form of the Equation of an Ellipse Centered at (0, 0)

Major Axis	**Horizontal**	**Vertical**
Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,\ a > b$
Foci	$(-c, 0), (c, 0)$	$(0, -c), (0, c)$
a, b, c	$c^2 = a^2 - b^2$	$c^2 = a^2 - b^2$
Length of major axis	$2a$	$2a$
Length of minor axis	$2b$	$2b$
Vertices	$(-a, 0), (a, 0)$	$(0, -a), (0, a)$
Co-vertices	$(0, -b), (0, b)$	$(-b, 0), (b, 0)$

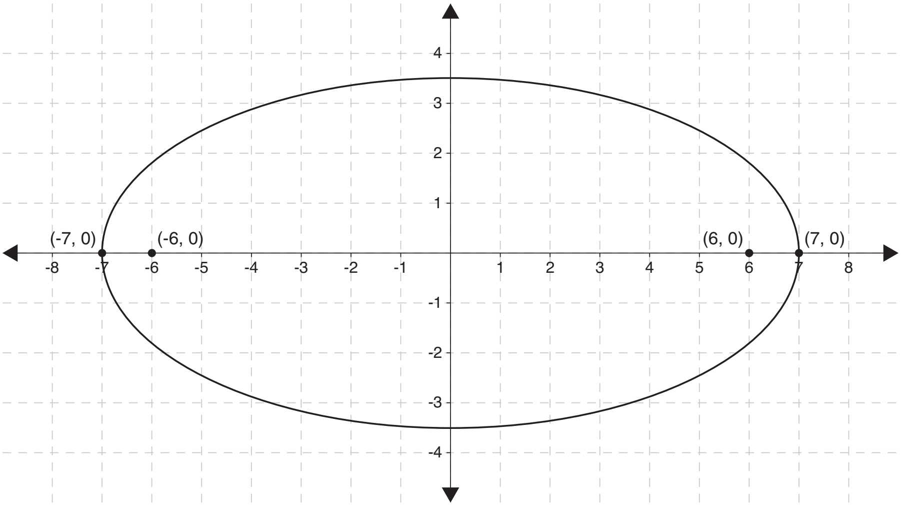




You can write the equation of an ellipse when you are given the coordinates of the vertices and the foci.

Example: Write the equation of the ellipse shown on the next page.

Solution: The ellipse has a horizontal major axis. Find the length of the major axis and divide by 2 to find a. Calculate a^2. Identify c from the foci, and use $c^2 = a^2 - b^2$ to find b^2. Substitute a^2 and b^2 into the standard form of the equation of an ellipse with a horizontal major axis.



The coordinates of the vertices that lie on the major axis are (–7, 0) and (7, 0), so the length of the major axis is 14.

$2a = 14$	Length of the major axis
$a = 7$	Divide both sides of the equation by 2.
$a^2 = 49$	Calculate a^2.

The foci are (–6, 0) and (6, 0), so c is 6.

$c^2 = a^2 - b^2$	Relationship among a, b, and c.
$6^2 = 49 - b^2$	Substitute $c = 6$ and $a^2 = 49$.
$36 = 49 - b^2$	Simplify.
$-13 = -b^2$	Subtract 49 from both sides of the equation.
$13 = b^2$	Multiply both sides of the equation by –1.
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Standard equation of ellipse with a horizontal major axis.
$\frac{x^2}{49} + \frac{y^2}{13} = 1$	Substitute $a^2 = 49$ and $b^2 = 13$.

You can write the equation of an ellipse centered at the origin if you know the lengths of both axes.

Example: If the length of the horizontal axis of an ellipse is 20 and the length of the vertical axis is 16, write the equation of the ellipse centered at the origin in standard form and graph the ellipse.

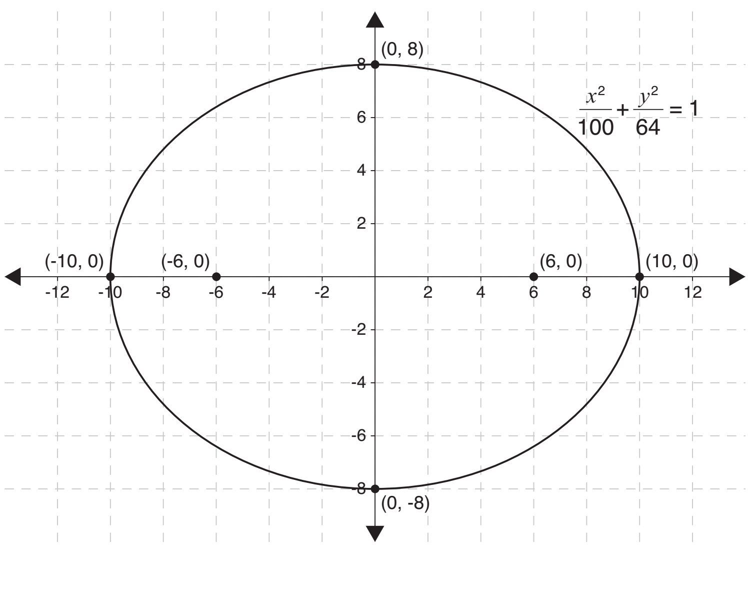
Solution: Since the horizontal axis is longer, it is the major axis. The length of the major axis is $2a$, and the length of the minor axis is $2b$. Set each expression equal to the length of each axis and find the value of a and b. Substitute a and b into $c^2 = a^2 - b^2$ and solve to find the value of c. Substitute a and b in the standard form of the equation of an ellipse with a horizontal major axis. Identify and graph the foci, vertices, and co-vertices. Sketch a graph of the ellipse.

$$2a = 20 \qquad 2b = 16 \qquad c^2 = a^2 - b^2$$
$$a = 10 \qquad b = 8 \qquad c^2 = 10^2 - 8^2$$
$$c^2 = 36$$
$$c = 6$$

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ Standard equation of ellipse with a horizontal major axis.

$\dfrac{x^2}{10^2} + \dfrac{y^2}{8^2} = 1$ Substitute $a = 10$ and $b = 8$.

$\dfrac{x^2}{100} + \dfrac{y^2}{64} = 1$ Simplify.



Foci: $(-c, 0), (c, 0) \rightarrow (-6, 0), (6, 0)$

Vertices: $(-a, 0), (a, 0) \rightarrow (-10, 0), (10, 0)$

Co-Vertices: $(0, -b), (0, b) \rightarrow (0, -8), (0, 8)$

Notice that in standard form, the equation for an ellipse is equal to 1. You may have to transform an equation into standard form.

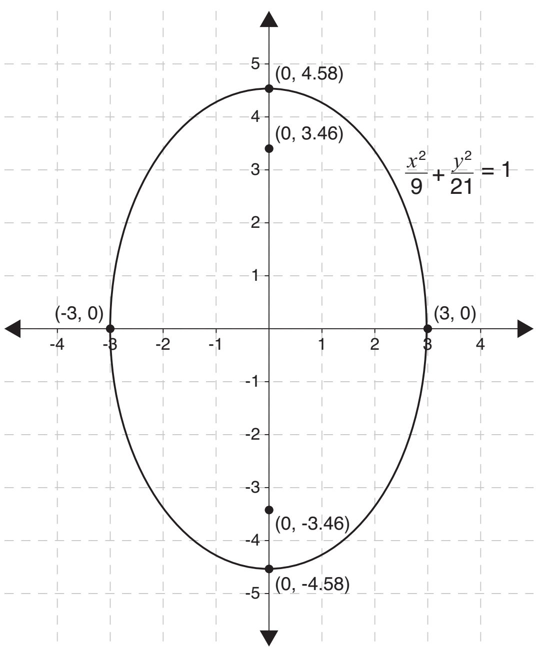
Example: Write the equation $7x^2 + 3y^2 = 63$ in standard form and sketch a graph of the ellipse.

Solution: Divide both sides of the equation by 63. Identify a^2 and b^2 and solve to find a and b. Use the equation $c^2 = a^2 - b^2$ to find c. Plot the foci and vertices and sketch the graph.

$7x^2 + 3y^2 = 63$

$\dfrac{x^2}{9} + \dfrac{y^2}{21} = 1$ Divide both sides of the equation by 63.

Since the denominator of y^2 is larger, the ellipse has a vertical major axis and the standard equation is $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$.



$a^2 = 21$ $\quad b^2 = 9$ $\quad c^2 = a^2 - b^2$

$a = \sqrt{21}$ $\quad b = 3$ $\quad c^2 = 21 - 9$

$a \approx 4.58$ $\quad c^2 = 12$

$c \approx 3.46$

Foci: $(0, -c), (0, c) \rightarrow (0, -3.46), (0, 3.46)$

Vertices: $(0, -a), (0, a) \rightarrow (0, -4.58), (0, 4.58)$

Co-Vertices: $(-b, 0), (b, 0) \rightarrow (-3, 0), (3, 0)$

The graph of an ellipse can be translated so that the center is (h, k). Just as you have seen with other translated graphs, $x - h$ and $y - k$ replace x and y.

Standard Form of the Equation of an Ellipse Centered at (*h*, *k*)

Major Axis	**Horizontal**	**Vertical**
Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,\ a > b$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1,\ a > b$
Foci	$(h - c, k), (h + c, k)$	$(h, k - c), (h, k + c)$
a, b, c	$c^2 = a^2 - b^2$	$c^2 = a^2 - b^2$
Length of major axis	$2a$	$2a$
Length of minor axis	$2b$	$2b$
Vertices	$(h - a, k), (h + a, k)$	$(h, k - a), (h, k + a)$
Co-vertices	$(h, k - b), (h, k + b)$	$(h - b, k), (h + b, k)$

If the equation of an ellipse is not in standard form, you may have to complete the square in order to transform the equation to standard form.

Example: Write the equation $2x^2 + 8y^2 - 20x + 48y = -106$ in standard form and sketch a graph of the ellipse.

Solution: Transform the equation into standard form by completing the square for *x* and *y,* and then divide both sides of the equation by a value that will make the equation equal to 1. Identify a^2 and b^2 and solve to find a and b. Use the equation $c^2 = a^2 - b^2$ to find c. Plot the foci and vertices and sketch the graph.

$$2x^2 + 8y^2 - 20x + 48y = -106$$

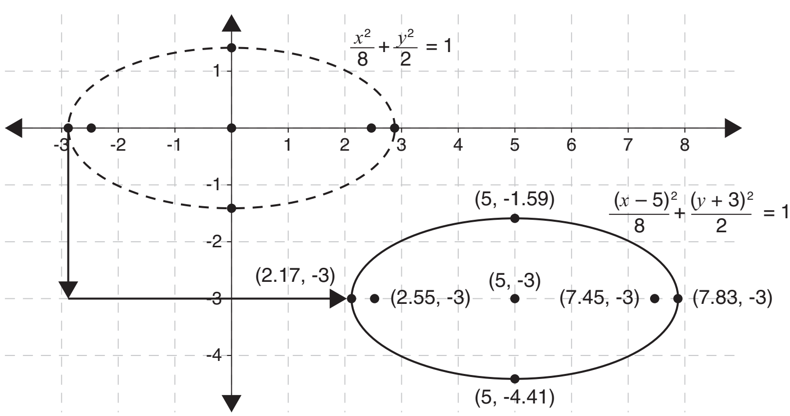
$$2(x^2 - 10x) + 8(y^2 + 6y) = -106 \qquad \text{Regroup and factor.}$$

$$2(x^2 - 10x + 25) + 8(y^2 + 6y + 9) = -106 + 122 \qquad 2\left(\frac{10}{2}\right)^2 = 50,\ 8\left(\frac{6}{2}\right)^2 = 72$$

$$2(x-5)^2 + 8(y+3)^2 = 16 \qquad \text{Factor.}$$

$$\frac{(x-5)^2}{8} + \frac{(y+3)^2}{2} = 1 \qquad \text{Divide both sides by 16.}$$

Since the denominator of x^2 is larger, the ellipse has a horizontal major axis and the standard equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. The center of the ellipse is (5, –3).



$a^2 = 8$ $\quad b^2 = 2$ $\quad c^2 = a^2 - b^2$

$a = \sqrt{8}$ $\quad b^2 = 2$ $\quad c^2 = 8 - 2$

$a \approx 2.83$ $\quad b \approx 1.41$ $\quad c^2 = 6$

$c \approx 2.45$

Foci: $(h - c, k), (h + c, k)$

$(5 - 2.45, -3), (5 + 2.45, -3)$

$(2.55, -3), (7.45, -3)$

Vertices: $(h - a, k), (h + a, k)$

$(5 - 2.83, -3), (5 + 2.83, -3)$

$(2.17, -3), (7.83, -3)$

Co-Vertices: $(h, k - b), (h, k + b)$

$(5, -3 - 1.41), (5, -3 + 1.41)$

$(5, -4.41), (5, -1.59)$

Notice in the previous example that you could also graph the ellipse by translating the graph of $\frac{x^2}{8} + \frac{y^2}{2} = 1$ down 3 and to the right 5.

You can also write the equation of an ellipse if you know the coordinates of the vertices and the co-vertices.

Example: Write the equation of the ellipse in standard form if the vertices of the ellipse are (2, 19) and (2, –7) and the co-vertices are (–3, 6), and (7, 6).

Solution: Since the vertices have the same x coordinate, the major axis is vertical. Find the center of the ellipse by finding the midpoint of the major or minor axis. Find the length of the major and minor axes. The length of the major axis is $2a$, and the length of the minor axis is $2b$. Set each expression equal to the length of each axis and find the value of a and b. Substitute a and b in the standard form of the equation of an ellipse with a vertical major axis.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$ Midpoint formula.

$$M = \left(\frac{2+2}{2}, \frac{19+(-7)}{2}\right)$$ Substitute the coordinates of the vertices.

$M = (2, 6)$

The center of the ellipse is (2, 6). Or, $h = 2$ and $k = 6$.

Length of the vertical axis = 19 – (–7) = 26.

Length of the horizontal axis = 7 – (–3) = 10.

$2a = 26$ $\quad$ $2b = 10$

$a = 13$ $\quad$ $b = 5$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$ Standard equation of ellipse with vertical major axis.

$$\frac{(x-2)^2}{5^2} + \frac{(y-6)^2}{13^2} = 1$$ Substitute known values.

$$\frac{(x-2)^2}{25} + \frac{(y-6)^2}{169} = 1$$ Simplify.

Practice Exercises

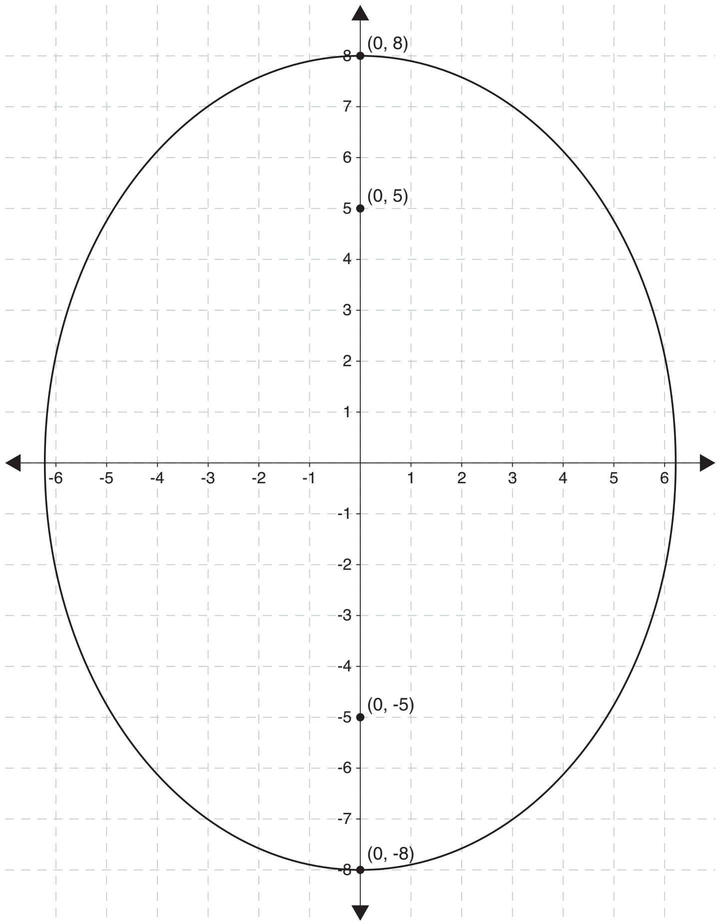
11.8 Write the equation of the ellipse on the next page.

11.9 If the length of the horizontal axis is 8 and the length of the vertical axis is 10, what is the equation of the ellipse centered at the origin?

11.10 Write the equation $x^2 + 4y^2 = 8$ in standard form and sketch the graph of the ellipse.

11.11 Write the equation $25x^2 + 4y^2 + 50x - 32y = 11$ in standard form and sketch the graph of the ellipse.

11.12 Write the equation of the ellipse with vertices (–3, 1), (1, 1) and co-vertices (–1, 2) and (–1, 0).

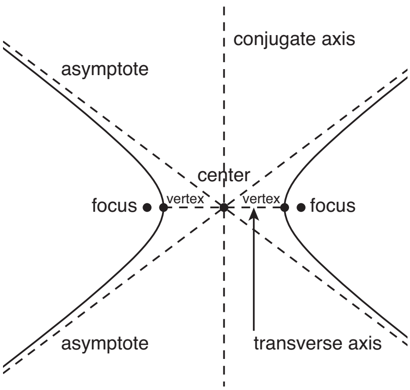


11.4 Hyperbolas

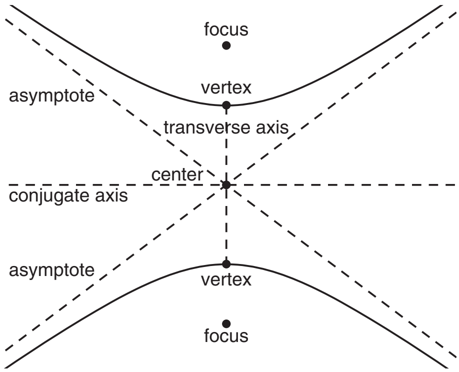
Hyperbolas share much in common with ellipses. While an ellipse is the set of all points in the plane such that the *sum* of the distances from two fixed points is constant, a *hyperbola* is the set of all points in the plane such that the *difference* of the distances from two points is constant. The two fixed points are called foci.

A hyperbola has two branches and two axes. The *transverse axis* connects the two vertices of the hyperbola. The *conjugate axis* is perpendicular to the transverse axis and intersects it at the midpoint of the two vertices, or the *center* of the hyperbola. As the branches move away from each center, they approach, but do not cross, one of two asymptotes.

Hyperbola with horizontal transverse axis



Hyperbola with vertical transverse axis



There are two forms of the standard equation of a hyperbola centered at (0, 0), depending on whether the transverse axis is horizontal or vertical.

Standard Form of the Equation of a Hyperbola Centered at (0, 0)

Transverse Axis	Horizontal	Vertical
Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Foci	$(-c, 0), (c, 0)$	$(0, -c), (0, c)$
a, b, c	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
Vertices	$(-a, 0), (a, 0)$	$(0, -a), (0, a)$
Asymptotes	$y=\frac{b}{a}x, y=-\frac{b}{a}x$	$y=\frac{a}{b}x, y=-\frac{a}{b}x$

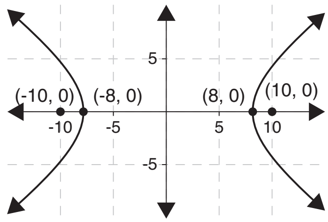
Notice the many similarities between the equations and properties of an ellipse centered at (0, 0) and a hyperbola centered at (0, 0). Do note, however, the different relationship between a, b, and c. For an ellipse, $c^2 = a^2 - b^2$ and for a hyperbola, $c^2 = a^2 + b^2$.

> When a hyperbola is written in standard form, you can identify the direction of the transverse axis by identifying which variable is not being subtracted. If x^2 is not being subtracted, the transverse axis will be the x-axis or parallel to the x-axis. If y^2 is not being subtracted, the transverse axis will be the y-axis or parallel to the y-axis.

You can write the equation of a hyperbola if you know the coordinates of the vertices and the foci.

Example: Write the equation of the hyperbola.

Solution: The hyperbola has a horizontal transverse axis and is centered at the origin. The coordinates of the foci are (–10, 0), and (10, 0), so $c = 10$. The coordinates of the vertices are (–8, 0) and (8, 0), so $a = 8$. Use the equation $c^2 = a^2 + b^2$ to find the value of b^2.



Write the equation of the hyperbola using the standard equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$c^2 = a^2 + b^2$	Relationship among *a, b*, and *c*.
$10^2 = 8^2 + b^2$	Substitute $c = 10$ and $a = 8$.
$100 = 64 + b^2$	Simplify.
$36 = b^2$	Subtract 64 from both sides of the equation.
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Standard equation of hyperbola with horizontal transverse axis.
$\frac{x^2}{8^2} - \frac{y^2}{36} = 1$	Substitute $a = 8$ and $36 = b^2$.
$\frac{x^2}{64} - \frac{y^2}{36} = 1$	Simplify.

To graph a hyperbola, graph the vertices, foci, and asymptotes, and then sketch the graph.

Example: Write the equation $16y^2 - 9x^2 - 144 = 0$ in standard form and sketch a graph of the hyperbola.

Solution: Transform the equation into standard form. Identify a^2 and b^2, and solve to find *a* and *b*. Use the equation $c^2 = a^2 + b^2$ to find *c*. Plot the foci, vertices, and asymptotes, and then sketch the graph.

$16y^2 - 9x^2 - 144 = 0$

$16y^2 - 9x^2 = 144$ Add 144 to both sides of the equation.

$\frac{y^2}{9} - \frac{x^2}{16} = 1$ Divide both sides of the equation by 144.

Since y^2 is not being subtracted, the hyperbola has a vertical transverse axis and the standard equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

$a^2 = 9$ $b^2 = 16$ $c^2 = a^2 + b^2$

$a = 3$ $b = 4$ $c^2 = 9 + 16$

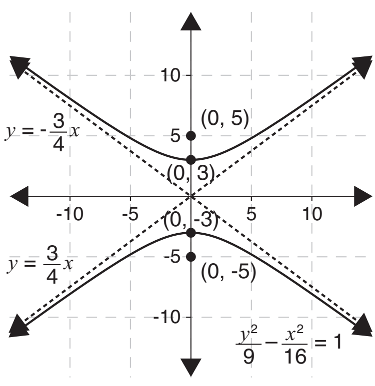
$c^2 = 25$

$c = 5$

Foci: $(0, -c), (0, c) \rightarrow (0, -5), (0, 5)$

Vertices: $(0, -a), (0, a) \rightarrow (0, -3), (0, 3)$

Asymptotes: $y = \frac{a}{b}x, y = -\frac{a}{b}x \rightarrow y = \frac{3}{4}x, y = -\frac{3}{4}x$



The graph of a hyperbola can be translated so that the center is (h, k). Just as you have seen with other translated graphs, $x - h$ and $y - k$ replace x and y.

Standard Form of the Equation of a Hyperbola Centered at (*h, k*)

Transverse Axis	**Horizontal**	**Vertical**
Equation	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$
Foci	$(h-c, k), (h+c, k)$	$(h, k-c), (h, k+c)$
a, b, c	$c^2=a^2+b^2$	$c^2=a^2+b^2$
Vertices	$(h-a, k), (h+a, k)$	$(h, k-a)$, (h, $k+a$)
Asymptotes	Lines through (h, k) with slopes $\pm\frac{b}{a}$	Lines through (h, k) with slopes $\pm\frac{a}{b}$

Example: Write the equation $9x^2+54x-4y^2+40y=55$ in standard form and sketch a graph of the hyperbola.

Solution: Transform the equation into standard form by completing the square for x and y and then divide both sides of the equation by a value that will make the equation equal to 1. Identify a^2 and b^2 and solve to find a and b. Use the equation $c^2=a^2+b^2$ to find c. Plot the foci, vertices, and asymptotes, and sketch the graph.

$$9x^2+54x-4y^2+40y=55$$

$$9(x^2+6x)-4(y^2-10y)=55 \qquad \text{Regroup and factor.}$$

$$9(x^2+6x+9)-4(y^2-10y+25)=55+81-100 \qquad 9\left(\frac{6}{2}\right)^2=81;\ -4\left(\frac{10}{2}\right)^2=-100$$

$$9(x+3)^2-4(y-5)^2=36 \qquad \text{Factor.}$$

$$\frac{(x+3)^2}{4}-\frac{(y-5)^2}{9}=1 \qquad \text{Divide both sides by 36.}$$

Since x^2 is not being subtracted, the ellipse has a horizontal transverse axis and the standard equation is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

The center of the hyperbola is (–3, 5).

$a^2 = 4$ $\quad b^2 = 9$ $\quad c^2 = a^2 + b^2$

$a = 2$ $\quad b = 3$ $\quad c^2 = 4 + 9$

$c^2 = 13$

$c \approx 3.61$

Foci: $(h - c, k), (h + c, k)$

$(-3 - 3.61, 5), (-3 + 3.61, 5)$

$(-6.61, 5), (0.61, 5)$

Vertices: $(h - a, k), (h + a, k)$

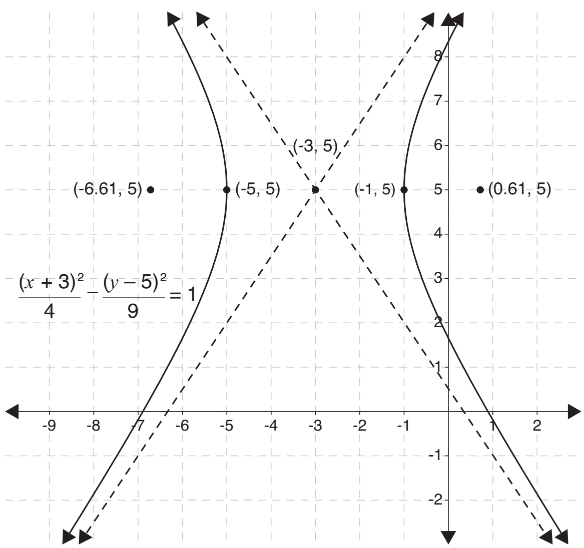
$(-3 - 2, 5), (-3 + 2, 5)$

$(-5, 5), (-1, 5)$

Asymptotes: Lines through (–3, 5) with slopes $\pm\frac{3}{2}$

Graph the center, foci, and vertices. Graph the asymptotes by starting at the center and graphing a slope of $\frac{3}{2}$ by going up 3 and to the right 2 and plotting another point.

Graph the other asymptote by starting at the center and graphing a slope of $-\frac{3}{2}$ by going down 3 and to the right 2 and plotting another point. Sketch the hyperbola.



Practice Exercises

11.13 Write the equation of the hyperbola if the coordinates of the foci are (0, 13), (0, –13) and the vertices are (0, 5), (0, –5).

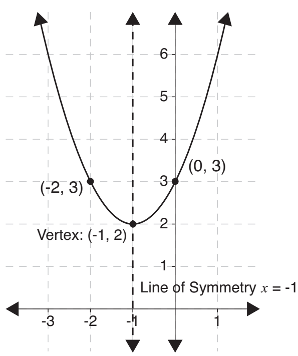
11.14 Write the equation $x^2 - 4y^2 = 4$ in standard form and sketch the graph of the hyperbola.

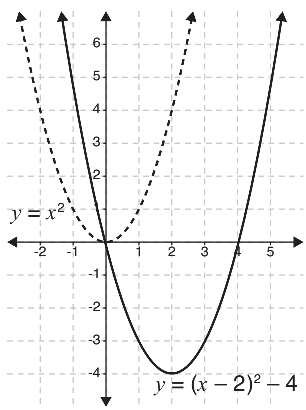
11.15 Write the equation $16y^2 - 9x^2 + 64y - 108x = 404$ in standard form and sketch the graph of the hyperbola.

11.5 Parabolas

Recall from Chapter 5 that the graph of a quadratic function is a parabola. In Chapter 5, we focused on parabolas that open vertically, either up or down, and were functions.

	General Form	**Vertex Form**
Example	$y = x^2 + 2x + 3$	$y = (x-2)^2 - 4$



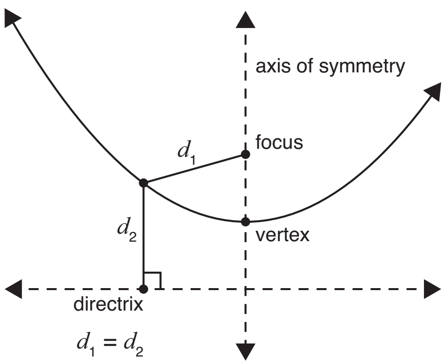


	$y = ax^2 + bx + c,\ a \neq 0$	$y = a(x-h)^2 + k,\ a \neq 0$
Line of symmetry	$x = \dfrac{-b}{2a}$	$x = h$
Vertex	x-coordinate = $\dfrac{-b}{2a}$ Substitute x and solve for y.	(h, k)
y-intercept	$(0, c)$	

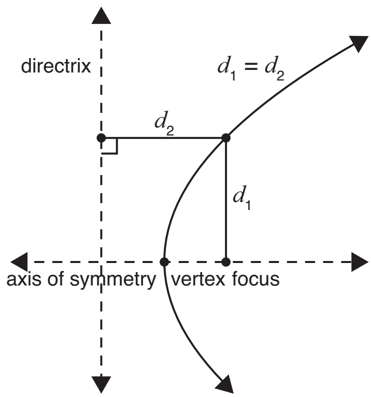
From a geometric perspective, a *parabola* can also be defined as the set of all points in a plane that are the same distance from a line called the *directrix* and a point not on the line called the *focus*. The midpoint of the line segment perpendicular to the directrix and passing through the focus is the *vertex* of the parabola. The line that passes through the vertex and the focus is the *axis of symmetry*. The axis of symmetry is perpendicular to the directrix.

In this study of parabolas, we will include parabolas that open vertically as well as parabolas that open horizontally.

Parabola that opens vertically



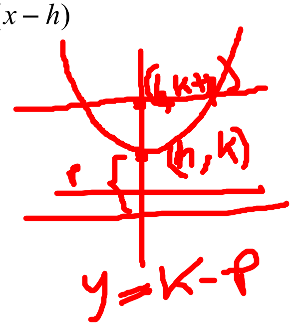
Parabola that opens horizontally



You can transform the equation of a parabola to a form that will help you readily identify the direction, focus, and directrix of the parabola.

Standard Equations of a Parabola with Vertex (*h*, *k*)

Direction	**Opens vertically (up or down)**	**Opens horizontally (left or right)**
Equation	$(x-h)^2 = 4p(y-k)$	$(y-k)^2 = 4p(x-h)$
Focus	$(h, k+p)$	$(h+p, k)$
Axis of symmetry	$x = h$	$y = k$
Directrix	$y = k - p$	$x = h - p$
If $p > 0$	opens up	opens right
If $p < 0$	opens down	opens left



y = k+p

Notice that an equation with an x that is squared will open vertically, while an equation with a y that is squared will open horizontally.

Example: Find the vertex, focus, and directrix of the parabola $-\frac{y^2}{8}=x$ and sketch a graph of the parabola.

Solution: Since y is the squared term, the parabola opens horizontally. Use the standard equation $(y-k)^2=4p(x-h)$. Write the equation in standard form and identify h, k, and p. Graph the vertex, focus, and directrix. Substitute a few values for x and find y to identify a few points on the graph. Sketch a graph of the parabola.

$(y-k)^2=4p(x-h)$	Standard form.
$-\frac{y^2}{8}=x$	Given equation.
$y^2=-8x$	Multiply both sides of the equation by –8.
$(y-0)^2=4(-2)(x-0)$	Transform to standard form.

$p=-2$; the parabola opens left

Vertex: $(h, k) \rightarrow (0, 0)$

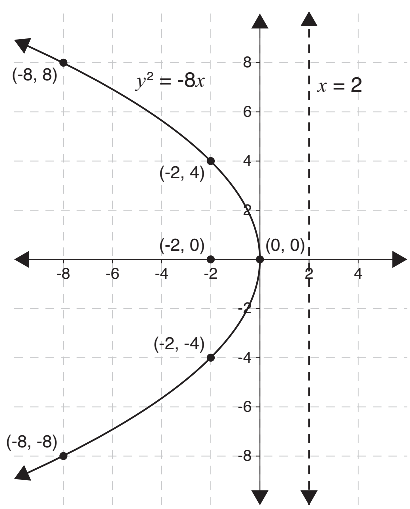
Focus: $(h+p, k) \rightarrow (0+(-2), 0) = (-2, 0)$

Directrix: $x = h - p \rightarrow x = 0 - (-2) \rightarrow x = 2$

$y=-2$

$|4p|=8$

y	$-\frac{y^2}{8}=x$	(x, y)
–8	–8	(–8, –8)
–4	–2	(–2, –4)
4	–2	(–2, 4)
8	–8	(–8, 8)



Notice from the graph to the right that the parabola is not a function because it fails the vertical line test.

Example: Find the vertex, focus, and directrix of the parabola

$y = \frac{1}{6}(x^2 - 4x + 22)$ and sketch a graph of the parabola.

Solution: Since x is the squared term, the parabola opens vertically. Use the standard equation $(x-h)^2 = 4p(y-k)$. Write the equation in the general form by completing the square and identify h, k. and p. Graph the vertex, focus, and directrix. Substitute a few values for y and find x to identify a few points on the graph. Sketch a graph of the parabola.

$(x-h)^2 = 4p(y-k)$	Standard equation.
$\frac{1}{6}(x^2 - 4x + 22) = y$	Given equation.
$x^2 - 4x + 22 = 6y$	Multiply both sides of the equation by 6.
$x^2 - 4x = 6y - 22$	Subtract 22 from both sides.
$(x^2 - 4x + 4) = 6y - 22 + 4$	$\left(\frac{4}{2}\right)^2 = 4$; add 4 to both sides.
$(x-2)^2 = 6y - 18$	Factor and simplify.
$(x-2)^2 = 6(y-3)$	Factor.
$(x-2)^2 = 4\left(\frac{3}{2}\right)(y-3)$	Transform to standard form.

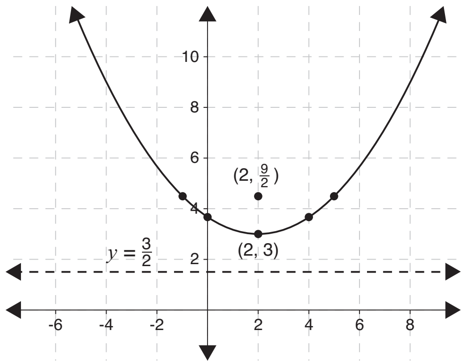
$p = \frac{3}{2}$; the parabola opens up

Vertex: $(h, k) \rightarrow (2, 3)$

Focus: $(h, k + p) \rightarrow \left(2, 3 + \frac{3}{2}\right) \rightarrow \left(2, \frac{9}{2}\right)$

Directrix: $y = k - p \rightarrow y = 3 - \frac{3}{2} \rightarrow y = \frac{3}{2}$

x	$\frac{1}{6}(x^2 - 4x + 22)$	y
-1	$\frac{1}{6}((-1)^2 - 4(-1) + 22)$	$\frac{9}{2}$
0	$\frac{1}{6}(0^2 - 4(0) + 22)$	$\frac{11}{3}$
4	$\frac{1}{6}(4^2 - 4(4) + 22)$	$\frac{11}{3}$
5	$\frac{1}{6}(5^2 - 4(5) + 22)$	$\frac{9}{2}$



You can use what you know about the vertex, focus, and directrix to write the equation of a parabola.

Example: The point (0, 7) is the focus of a parabola that has its vertex at (2, 7). Find the equation of the parabola.

Solution: The vertex is (2, 7), so $h = 2$ and $k = 7$. The focus and the vertex have a common value of 7, so the axis of symmetry is the horizontal line $y = 7$. The parabola opens horizontally and the focus is $(h + p, k)$. Use the focus to identify p and write the equation of the parabola using the general equation $(y-k)^2 = 4p(x-h)$.

Focus: $(h + p, k) = (0, 7)$

So $h + p = 0$ and $h = 2$.

$2 + p = 0$

$p = -2$

$$(y-k)^2 = 4p(x-h)$$

$$(y-7)^2 = 4(-2)(x-2)$$

$$(y-7)^2 = -8(x-2)$$

Practice Exercises

11.16 Find the vertex, focus, and directrix of the parabola $y = \frac{x^2}{4}$, and sketch a graph.

11.17 Find the vertex, focus, and directrix of the parabola

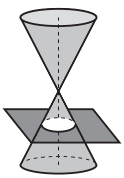
$$y^2 = 12x - 8y + 8.$$

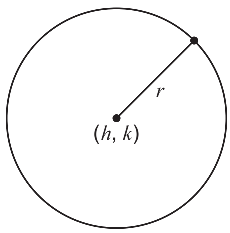
11.18 Find the equation of the parabola if the vertex is (6, 2) and the directrix is $x = 4$.

11.6 Identifying Conic Sections

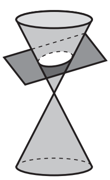
Conic sections are formed by a plane intersecting a cone. The equation for a conic section can be written in a form that identifies the center or vertex (h, k). Circles, ellipses, hyperbolas, and parabolas are all examples of conic sections.

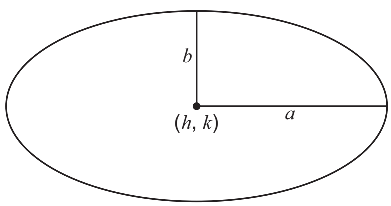
Circle $(x-h)^2 + (y-k)^2 = r^2$



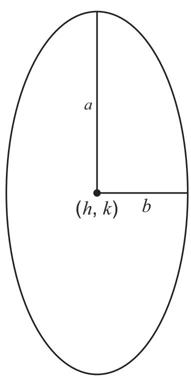


Ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,\ a > b$

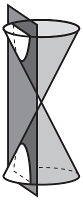


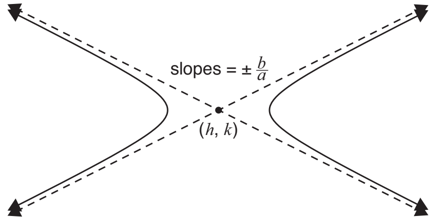


$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1,\ a > b$$

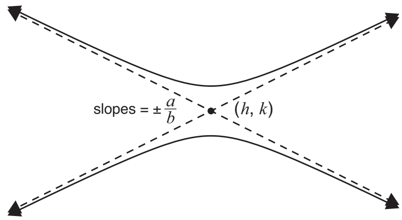


Hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

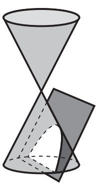


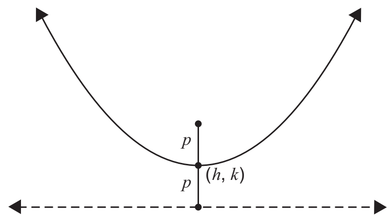


$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

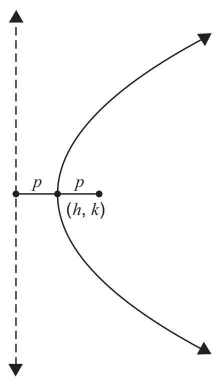


Parabola $(x-h)^2=4p(y-k)$





$(y-k)^2=4p(x-h)$



Each conic section is a special case of the general form of a second-degree equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, where at least one of the coefficients A, B, or C is non-zero. If you can transform a second-degree equation into one of the conic section equations above, you can identify and graph the conic section.

Example: Identify and describe the conic section given by the general equation $x^2 + y^2 - 4x - 10y + 13 = 0$.

Solution: Complete the square for x and y and transform the equation to match one of the preceding conic section equations. Identify and describe the conic section.

$x^2 + y^2 - 4x - 10y + 13 = 0$ Given equation.

$(x^2 - 4x) + (y^2 - 10y) + 13 = 0$ Regroup.

$(x^2 - 4x) + (y^2 - 10y) = -13$ Subtract 13 from both sides of equation.

$(x^2 - 4x + 4) + (y^2 - 10y + 25) = -13 + 4 + 25$ $\left(\frac{4}{2}\right)^2 = 4$; $\left(\frac{-10}{2}\right)^2 = 25$; Add.

$(x-2)^2 + (y-5)^2 = 16$ Factor.

The conic section is a circle with center (2, 5) and radius 4.

Note in the example that if you divided both sides of the equation by 16, you would have the equation $\frac{(x-2)^2}{16} + \frac{(y-5)^2}{16} = 1$, which would appear to be an ellipse. However, $a = b$, so the conic section would be a circle, or a special case of the ellipse.

Many second-degree equations are very difficult to factor into the conic section equations. However, you can use the values of the coefficients A, B, and C in the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ to determine the type of conic section that the equation represents.

Discriminant Theorem of Conic Sections

If at least one of the coefficients A, B, or C is non-zero, then the graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is a(n)

- ellipse if $B^2 - 4AC < 0$ and $A \neq C$
- circle if $B^2 - 4AC < 0$ and $A = C$
- hyperbola if $B^2 - 4AC > 0$
- parabola if $B^2 - 4AC = 0$

Recall that $b^2 - 4ac$ is the discriminant of a quadratic equation.

Example: Identify the conic section given by the equation

$$6x^2 + 5x + 3xy = -y^2 + 2y - 18.$$

Solution: Transform the equation into the general form of a second-degree equation and identify A, B, and C. Calculate $B^2 - 4AC$ and identify the conic section.

$6x^2 + 5x + 3xy = -y^2 + 2y - 18$	Given equation.
$6x^2 + 3xy + y^2 + 5x - 2y + 18 = 0$	Add $y^2 - 2y$ to both sides of the equation.

$A = 6$, $B = 3$, $C = 1$

$$B^2 - 4AC = 3^2 - 4(6)(1)$$
$$= -15$$

$B^2 - 4AC < 0$ and $A \neq C$

The conic section is an ellipse.

Example: Identify the conic section given by the equation

$$x^2 - 2y^2 = -5xy - 3x + 7y - 3.$$

Solution: Transform the equation into the general form of a second-degree equation and identify A, B, and C. Calculate $B^2 - 4AC$ and identify the conic section.

$x^2 - 2y^2 = -5xy - 3x + 7y - 3$	Given equation.
$x^2 + 5xy - 2y^2 + 3x - 7y + 3 = 0$	Add $5xy + 3x - 7y + 3$ to both sides.

$A = 1$, $B = 5$, $C = -2$

$$B^2 - 4AC = 5^2 - 4(1)(-2)$$
$$= 33$$

$B^2 - 4AC > 0$

The conic section is a hyperbola.

Practice Exercises

Identify each conic section.

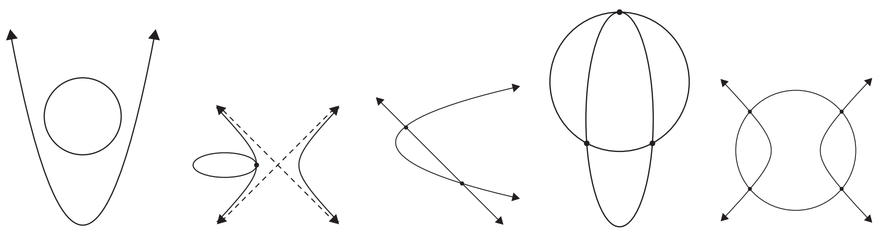
11.19 $4x^2 + y^2 + 4xy = 3x - 2y + 11$

11.20 $y^2 = -x^2 - 3x + 4y - 2$

11.7 Solve Quadratic Systems

A quadratic system is a system of equations where at least one equation is quadratic. Solutions to a system can be represented by the points of intersection of the graphs of the equations. An independent quadratic system can have zero or up to four solutions.

Consider the following systems:



0 Solutions 1 Solution 2 Solutions 3 Solutions 4 Solutions

Solve by Graphing

One way to solve a quadratic system is by graphing. Graph each equation and locate the point or points of intersection.

Example: Solve the system by graphing.

$$\begin{cases} x^2 + y^2 = 100 \\ 3x - y = 10 \end{cases}$$

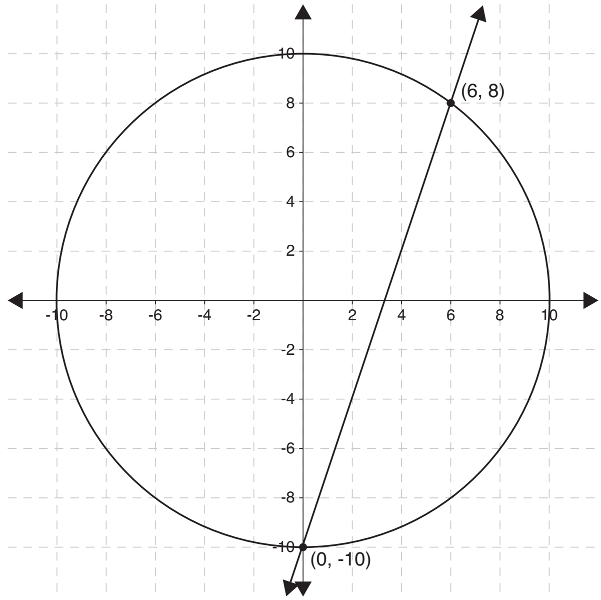
Solution: The first equation is a circle centered at the origin with radius 10. The second equation is a line. Solve the linear equation for y and graph the line using the slope and the y-intercept. Identify the points of intersection.

$3x - y = 10$

$-y = -3x + 10$ Subtract $3x$ from both sides of the equation.

$y = 3x - 10$ Multiply both sides of the equation by -1.

slope = 3, y-intercept = –10



The points of intersection on the graph are (6, 8), (0, –10).

Check: (6, 8)

$x^2 + y^2 = 100$	$3x - y = 10$
$6^2 + 8^2 = 100$	$3(6) - 8 = 10$
$36 + 64 = 100$ ✓	$18 - 8 = 10$ ✓

Check: (0, –10)

$x^2 + y^2 = 100$	$3x - y = 10$
$0^2 + (-10)^2 = 100$	$3(0) - (-10) = 10$
$0 + 100 = 100$ ✓	$0 + 10 = 10$ ✓

The solutions to the system are (6, 8), (0, –10).

Identifying solutions from a graph can be inefficient and difficult. The system may be challenging to transform into the standard form of a conic section, or the intersection points may not be integer values. Algebraic methods for solving a quadratic system can be more efficient. Two algebraic methods are substitution and combinations.

Solve by Substitution

Substitution is an effective method if one of the equations is linear. Solve the linear equation for x or y and substitute the value for the corresponding variable in the other equation to find the value of one variable. Substitute that value into one of the equations and solve for the remaining variable. Check your solutions by graphing or by substituting each solution into both of the original equations.

Example: Solve the system by substitution.

$$\begin{cases} (x-3)^2 = y+4 \\ x+y=-1 \end{cases}$$

Solution: Solve the linear equation for x or y and substitute the value for the corresponding variable in the quadratic equation. Solve for the remaining variable. Substitute the known value into one of the equations to find the remaining variable.

$x + y = -1$

$y = -x - 1$ Subtract x from both sides.

Substitute $-x - 1$ for y in the equation $(x-3)^2 = y+4$.

$(x-3)^2 = y+4$

$(x-3)^2 = -x-1+4$ Substitute $y = -x - 1$.

$x^2 - 6x + 9 = -x + 3$ Simplify.

$x^2 - 5x + 6 = 0$ Add x to and subtract 3 from both sides.

$(x - 3)(x - 2) = 0$ Factor.

$x = 3$ or 2 Zero Product Property.

Substitute to find y values.

If $x = 3$:	If $x = 2$:
$3 + y = -1$	$2 + y = -1$
$y = -4$	$y = -3$
$(3, -4)$	$(2, -3)$

Check: (3, –4)

$x + y = -1 \qquad (x-3)^2 = y+4$

$3 + -4 = -1$ ✓ $\qquad (3-3)^2 = -4+4$

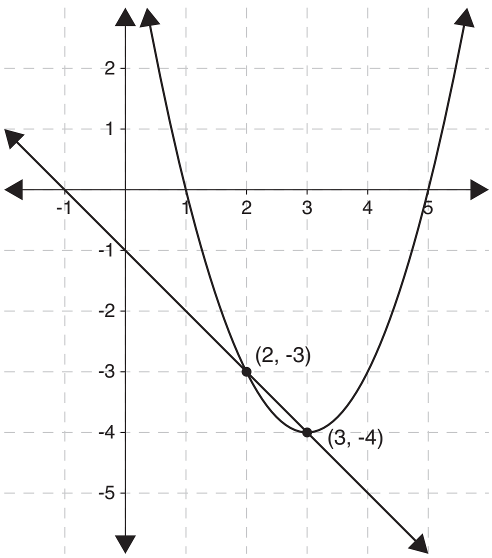
$0 = 0$ ✓

Check (2, –3):

$x + y = -1 \qquad (x-3)^2 = y+4$

$2 + -3 = -1$ ✓ $\qquad (2-3)^2 = -3+4$

$1 = 1$ ✓



The solutions to the system are (3, –4), (2, –3).

Solve by Combinations

If both equations in a quadratic system are quadratic, the best method to solve is usually by combinations. Write each equation in the general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Multiply one or both equations by numbers so that when you add the equations together, a variable will cancel out. Solve that equation to find the value of the variable.

Substitute that value back into one of the original equations to find the value of the other variable. Check your solutions.

Example: Solve the system by combinations.

$$\begin{cases} (x-4)^2+(y-1)^2=13 \\ x^2-8x-y^2+2y=-20 \end{cases}$$

Solution: Transform each equation into $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. Add the equations together to eliminate a variable. Solve the resulting equation to find the value(s) of one variable. Substitute each value back into one of the original equations to find the corresponding value of the other variable. Check every solution in both equations or check by graphing.

Eq. 1: $(x-4)^2+(y-1)^2=13$ Eq. 2: $x^2-8x-y^2+2y=-20$

$x^2-8x+16+y^2-2y+1=13$ $x^2-y^2-8x+2y+20=0$

$x^2+y^2-8x-2y+4=0$

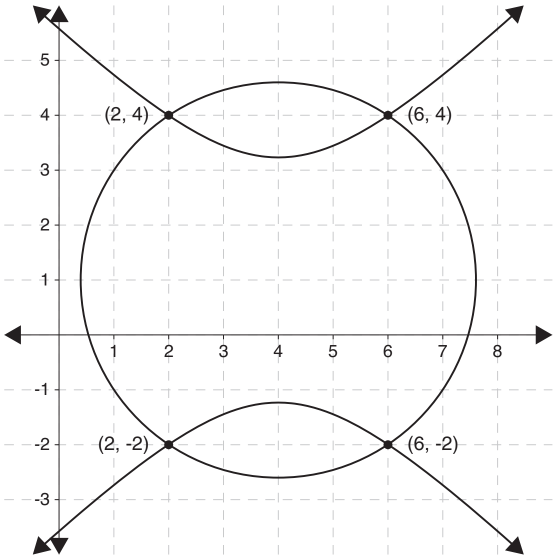
$x^2+y^2-8x-2y+4=0$	Equation 1.
$x^2-y^2-8x+2y+20=0$	Equation 2.
$2x^2-16x+24=0$	Add Equation 1 and Equation 2.
$x^2-8x+12=0$	Divide both sides of the equation by 2.
$(x-6)(x-2)=0$	Factor.
$x=6$ or 2	Zero Product Property.

Substitute to find y values.

If $x=6$:	If $x=2$:	
$6^2-8(6)-y^2+2y=-20$	$2^2-8(2)-y^2+2y=-20$	Substitute value of x.
$36-48-y^2+2y=-20$	$4-16-y^2+2y=-20$	Simplify.
$-y^2+2y+8=0$	$-y^2+2y+8=0$	Simplify.

$y^2 - 2y - 8 = 0$	$y^2 - 2y - 8 = 0$	Multiply equation by –1.
$(y - 4)(y + 2) = 0$	$(y - 4)(y + 2) = 0$	Factor.
$y = 4$ or $y = -2$	$y = 4$ or $y = -2$	Zero Product Property.
(6, 4), (6, –2)	(2, 4), (2, –2)	

You can check the solutions by substituting each solution into both equations. The check is left to you.



The solutions are (6, 4), (6, –2), (2, 4), (2, –2).

Practice Exercises

Use the following system for 11.21 and 11.22 $\begin{cases} y = x + 5 \\ y = x^2 + 3 \end{cases}$

11.21 Solve the system above by graphing.

11.22 Solve the system above by substitution.

11.23 Solve the system by combinations: $\begin{cases} y = x^2 + 3 \\ x^2 + (y - 4)^2 = 1 \end{cases}$

Chapter 11 Answers to Exercises

11.1 $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$; $M = \left(\frac{1 + (-5)}{2}, \frac{7 + 3}{2}\right) = (-2, 5)$

The midpoint of $\overline{AB}$ is (–2, 5).

11.2 $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$; $(-3,1) = \left(\frac{2 + x_2}{2}, \frac{5 + y_2}{2}\right)$; $-3 = \frac{2 + x_2}{2}$;

$-6 = 2 + x_2$; $-8 = x_2$; $1 = \frac{5 + y_2}{2}$; $2 = 5 + y_2$; $-3 = y_2$

The coordinates of D are (–8, –3).

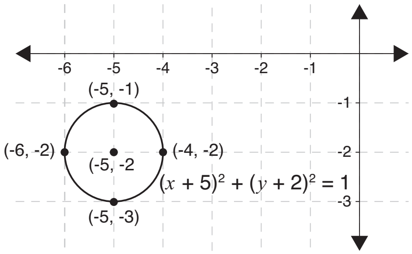
11.3 $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$; $d = \sqrt{(-2 - 8)^2 + (-5 - (-4))^2}$;

$d = \sqrt{(-10)^2 + (-1)^2}$; $d = \sqrt{101} \approx 10.05$

The length $\overline{AB}$ is approximately 10.05 units.

11.4 $(x - h)^2 + (y - k)^2 = r^2$; $(x + 5)^2 + (y + 2)^2 = 1$;

$(x - (-5))^2 + (y - (-2))^2 = 1^2$; Center (–5, –2), radius 1



11.5 $h = 2$, $k = 1$, $r = 9$; $(x - h)^2 + (y - k)^2 = r^2$; $(x - 2)^2 + (y - 1)^2 = 9^2$;

$(x - 2)^2 + (y - 1)^2 = 81$

11.6 $M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$; $(h,k)=\left(\frac{-1+3}{2},\frac{-4+-8}{2}\right)$; $(h, k) = (1, -6)$

The center of the circle is $(1, -6)$. $d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$;

$r=\sqrt{\left(1-3\right)^2+\left(-6-(-8)\right)^2}$; $r=\sqrt{4+4}$; $r=\sqrt{8}$. The radius of the circle is $\sqrt{8}$. $(x-h)^2+(y-k)^2=r^2$; $(x-1)^2+(y-(-6))^2=\sqrt{8}^2$;

$(x-1)^2+(y+6)^2=8$

11.7 $x^2+y^2+12x+18y=-106$; $(x^2+12x)+(y^2+18y)=-106$;

$(x^2+12x+36)+(y^2+18y+81)=-106+117$;

$(x+6)^2+(y+9)^2=11$; $(x+6)^2+(y+9)^2=\sqrt{11}^2$

11.8 The coordinates of the vertices are $(0, 8)$ and $(0, -8)$, so the length of the major axis is 16. $2a = 16$; $a = 8$; $a^2=64$; The foci are $(0, 5)$ and $(0, -5)$, so c is 5. $c^2=a^2-b^2$; $5^2=64-b^2$;

$25=64-b^2$; $-39=-b^2$; $39=b^2$; $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$; $\frac{x^2}{39}+\frac{y^2}{64}=1$

11.9 $2a = 10$; $a = 5$; $2b = 8$; $b = 4$; $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$; $\frac{x^2}{4^2}+\frac{y^2}{5^2}=1$; $\frac{x^2}{16}+\frac{y^2}{25}=1$

11.10 $x^2+4y^2=8$; $\frac{x^2}{8}+\frac{y^2}{2}=1$. Since the denominator of x^2 is larger, the ellipse has a horizontal major axis and the standard equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

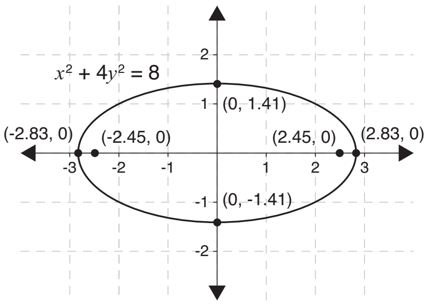
$a^2=8$; $a=\sqrt{8}$; $a \approx 2.83$; $b^2=2$; $b=\sqrt{2}$; $b \approx 1.41$; $c^2=a^2-b^2$;

$c^2=8-2$; $c^2=6$; $c=\sqrt{6}$; $c \approx 2.45$

Foci: $(-c, 0), (c, 0) \rightarrow (-2.45, 0), (2.45, 0)$

Vertices: $(-a, 0), (a, 0) \rightarrow (-2.83, 0), (2.83, 0)$

Co-Vertices: $(0, -b), (0, b) \rightarrow (0, -1.41), (0, 1.41)$



11.11 $25x^2 + 4y^2 + 50x - 32y = 11$; $25(x^2 + 2x) + 4(y^2 - 8y) = 11$;

$25(x^2 + 2x + 1) + 4(y^2 - 8y + 16) = 11 + 25 + 64$;

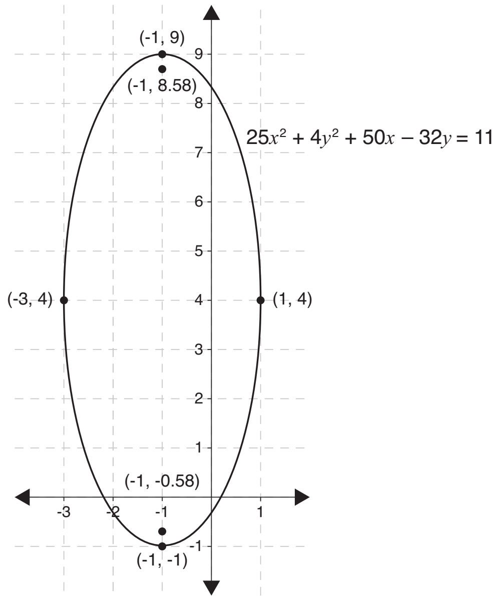
$25(x+1)^2 + 4(y-4)^2 = 100$; $\dfrac{(x+1)^2}{4} + \dfrac{(y-4)^2}{25} = 1$. Since the denominator of y^2 is larger, the ellipse has a vertical major axis and the standard equation is $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$. The center of the ellipse is $(-1, 4)$. $a^2 = 25$; $a = 5$; $b^2 = 4$; $b = 2$; $c^2 = a^2 - b^2$;

$c^2 = 25 - 4$; $c^2 = 21$; $c \approx 4.58$;

Foci: $(h, k - c) = (-1, 4 - 4.58) = (-1, -0.58)$;
$(h, k + c) = (-1, 4 + 4.58) = (-1, 8.58)$;

Vertices: $(h, k - a) = (-1, 4 - 5) = (-1, -1)$;
$(h, k + a) = (-1, 4 + 5) = (-1, 9)$;

Co-Vertices: $(h - b, k) = (-1 - 2, 4) = (-3, 4)$;
$(h + b, k) = (-1 + 2, 4) = (1, 4)$



11.12 $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$ $M = \left(\dfrac{-3+1}{2}, \dfrac{1+1}{2}\right)$; $M = (-1, 1)$; $h = -1$ and $k = 1$. Length of the horizontal axis = 1 – (–3) = 4; Length of the vertical axis = 2 – 0 = 2; $2a = 4$; $a = 2$; $2b = 2$; $b = 1$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1; \ \frac{(x+1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1; \ \frac{(x+1)^2}{4} + \frac{(y-1)^2}{1} = 1$$

11.13 Foci: (0, 13), (0, –13), so $c = 13$; Vertices: (0, 5), (0, –5), so $a = 5$; $c^2 = a^2 + b^2$; $13^2 = 5^2 + b^2$; $169 = 25 + b^2$; $144 = b^2$;

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1; \ \frac{y^2}{5^2} - \frac{x^2}{144} = 1; \ \frac{y^2}{25} - \frac{x^2}{144} = 1$$

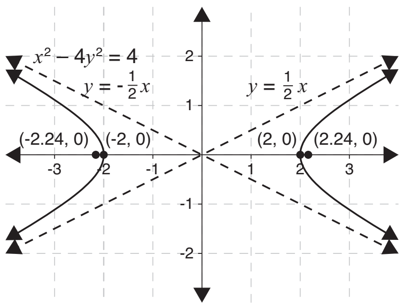
11.14 $x^2 - 4y^2 = 4$; $\dfrac{x^2}{4} - \dfrac{y^2}{1} = 1$; Since x^2 is not being subtracted, the hyperbola has a horizontal transverse axis and the standard equation is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. $a^2 = 4$; $a = 2$; $b^2 = 1$; $b = 1$

$c^2 = a^2 + b^2$; $c^2 = 4 + 1$; $c^2 = 5$; $c = \sqrt{5} \approx 2.24$

Foci: $(-c, 0), (c, 0) \rightarrow (-2.24, 0), (2.24, 0)$

Vertices: $(-a, 0), (a, 0) \rightarrow (-2, 0), (2, 0)$

Asymptotes: $y = \frac{b}{a}x, y = -\frac{b}{a}x \rightarrow y = \frac{1}{2}x, y = -\frac{1}{2}x$



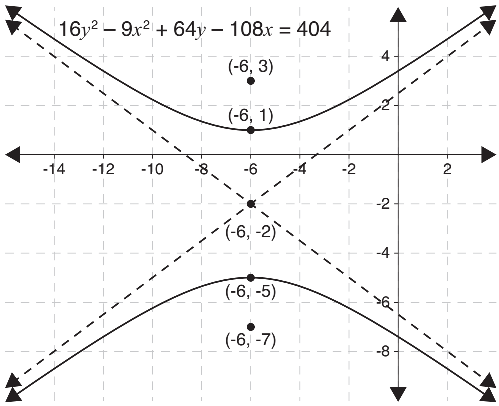
11.15 $16y^2 - 9x^2 + 64y - 108x = 404$; $16(y^2 + 4y) - 9(x^2 + 12x) = 404$;

$16(y^2 + 4y + 4) - 9(x^2 + 12x + 36) = 404 - 260$;

$16(y+2)^2 - 9(x+6)^2 = 144$;

$\frac{(y+2)^2}{9} - \frac{(x+6)^2}{16} = 1$; Since y^2 is not being subtracted, the ellipse has a vertical transverse axis and the standard equation is $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$. The center of the hyperbola is $(-6, -2)$.

$a^2 = 9$; $a = 3$; $b^2 = 16$; $b = 4$; $c^2 = a^2 + b^2$; $c^2 = 9 + 16$; $c^2 = 25$; $c = 5$; Foci: $(h, k - c) = (-6, -2 - 5) = (-6, -7)$; $(h, k + c) = (-6, -2 + 5) = (-6, 3)$; Vertices: $(h, k - a) = (-6, -2 - 3) = (-6, -5)$; $(h, k + a) = (-6, -2 + 3) = (-6, 1)$. Asymptotes: slope $\pm\frac{a}{b} = \pm\frac{3}{4}$.



11.16 Since x is the squared term, the parabola opens vertically.

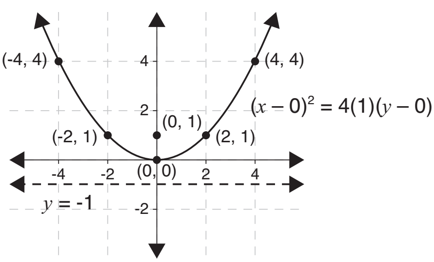
$$(x-h)^2 = 4p(y-k);\ y = \frac{x^2}{4};\ 4y = x^2;\ (x-0)^2 = 4(1)(y-0)$$

$p = 1$; the parabola opens up; Vertex: $(h, k) \to (0, 0)$

Focus: $(h, k+p) \to (0, 0+1) = (0, 1)$;

Directrix: $y = k - p \to y = 0 - 1 \to y = -1$

x	$\frac{x^2}{4}$	y	(x, y)
–4	$\frac{(-4)^2}{4}$	4	(–4, 4)
–2	$\frac{(-2)^2}{4}$	1	(–2, 1)
2	$\frac{(2)^2}{4}$	1	(2, 1)
4	$\frac{(4)^2}{4}$	4	(4, 4)



11.17 Since y is the squared term, the parabola opens horizontally.

$(y-k)^2 = 4p(x-h)$; $y^2 = 12x - 8y + 8$; $y^2 + 8y = 12x + 8$;

$y^2 + 8y + 16 = 12x + 8 + 16$; $(y+4)^2 = 12x + 24$; $(y+4)^2 = 12(x+2)$;

$(y-(-4))^2 = 4(3)(x-(-2))$; $p = 3$; the parabola opens to the right

Vertex: $(h, k) \rightarrow (-2, -4)$; Focus: $(h + p, k) \rightarrow (-2 + 3, -4) \rightarrow (1, -4)$; Directrix: $x = h - p \rightarrow x = -2 - 3 \rightarrow x = -5$

11.18 The vertex is (6, 2), and the directrix is $x = 4$, so the parabola opens horizontally and to the right. $h = 6$, $k = 2$; $x = h - p$; $4 = 6 - p$; $-2 = -p$; $2 = p$; $(y-k)^2 = 4p(x-h)$; $(y-2)^2 = 4(2)(x-6)$;

$(y-2)^2 = 8(x-6)$

11.19 $4x^2 + y^2 + 4xy = 3x - 2y + 11$; $4x^2 + 4xy + y^2 - 3x + 2y - 11 = 0$

$A = 4$, $B = 4$, $C = 1$; $B^2 - 4AC = 4^2 - 4(4)(1) = 0$; $B^2 - 4AC = 0$;

The conic section is a parabola.

11.20 $y^2 = -x^2 - 3x + 4y - 2$; $x^2 + y^2 + 3x - 4y + 2 = 0$;

$x^2 + 0xy + y^2 + 3x - 4y + 2 = 0$; $A = 1$, $B = 0$, $C = 1$

$B^2 - 4AC = 0^2 - 4(1)(1) = -4$; $B^2 - 4AC < 0$ and $A = C$.

The conic section is a circle.

11.21 The first equation is a line with a slope of 1 and a y-intercept of 5.

$y = x^2 + 3$; $y - 3 = x^2$; $(x-0)^2 = y - 3$; $(x-0)^2 = 4\left(\frac{1}{4}\right)(y-3)$

The second equation is a parabola with vertex (0, 3) that opens up.

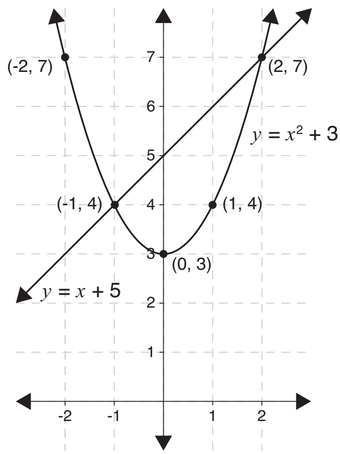
x	$x^2 + 3$	$f(x)$
–2	$(-2)^2 + 3$	7
–1	$(-1)^2 + 3$	4
1	$(1)^2 + 3$	4
2	$(2)^2 + 3$	7

The points of intersection appear to be (–1, 4) and (2, 7).

Check (–1, 4): $y = x + 5$; $4 = -1 + 5$ ✓; $y = x^2 + 3$; $4 = (-1)^2 + 3$ ✓

Check (2, 7): $y = x + 5$; $7 = 2 + 5$ ✓; $y = x^2 + 3$; $7 = 2^2 + 3$ ✓

The solutions to the system are (–1, 4) and (2, 7).



11.22 $y = x^2 + 3$; $x + 5 = x^2 + 3$; $0 = x^2 - x + 3 - 5$; $0 = x^2 - x - 2$;

$0 = (x-2)(x+1)$; $x = 2$ or $x = -1$. If $x = 2$: $y = 2 + 5$; $y = 7$; (2, 7). If $x = -1$: $y = -1 + 5$; $y = 4$; (–1, 4). The graph from 10.21 verifies the solutions (2, 7) and (–1, 4).

11.23 Equation 1: $y = x^2 + 3$; $x^2 - y + 3 = 0$; $-x^2 + y - 3 = 0$; Equation 2:

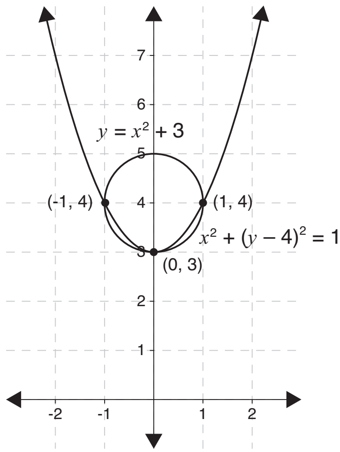
$x^2 + (y-4)^2 = 1$; $x^2 + y^2 - 8y + 16 = 1$; $x^2 + y^2 - 8y + 15 = 0$

$$\begin{array}{r} -x^2 + y - 3 = 0 \\ \underline{x^2 + y^2 - 8y + 15 = 0} \\ y^2 - 7y + 12 = 0 \end{array}$$

$(y - 3)(y - 4) = 0$; $y = 3$ or $y = 4$. If $y = 3$: $3 = x^2 + 3$; $0 = x^2$;

$0 = x$; (0, 3). If $y = 4$: $4 = x^2 + 3$; $1 = x^2$; $x = 1$ or $x = -1$;

(1, 4), (–1, 4). You can check the solutions by substituting each solution into both equations. The check is left to you.



The solutions are (0, 3), (1, 4), and (–1, 4).