Assignment 2

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Ans to or 1 (a)

Given, $f(x) = \tan x$

In taylor expansion:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0}{2!}(x-x_0)^2 + \cdots$$

Now, let $x_0 = 0$,

$f(x_0) = \tan 0 = 0$

$f'(x_0) = \sec^2 x_0 \cdot\ f'(x_0) = \sec^2 0 = 1$

$$f^2(x) = 2\sec x \cdot \frac{d}{dx}\sec x$$

$$= 2\sec^2 x \tan x$$

$$f^2(x_0) = 2\sec^2 0 \tan 0$$

$$= 2 \cdot 1 \cdot 0 = 0$$

$$f^3(x) = 2\left[\sec^2 x \cdot \sec^2 x + \tan x \cdot 2\sec^2 x \tan x\right]$$

$$= 2\sec^4 x + 4\sec^2 x \tan^2 x$$

$$f^3(x_0) = 2\sec^4 0 + 4\sec^2 0 \tan^2 0$$

$$= 2$$

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$$\therefore P_3(x) = 0 + 1.(x-x_0) + \frac{0.(x-x_0)^2}{2!} + \frac{2(x-x_0)^3}{3!}$$

$$= x - 0 + \frac{0(x-0)^2}{2!} + \frac{2(x-0)^3}{3!}$$

$$= x + \frac{2x^3}{3!}$$

$$\therefore P_3(x) = x + \frac{2x^3}{3!}$$

Ans to 1 (b).

At $x = \frac{\pi}{4}$,

$$f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$$

$$P_3\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + \frac{2\left(\frac{\pi}{4}\right)^3}{3!}$$

$$= 0.94689$$

$$\therefore \text{Relative error} = \left|f\left(\frac{\pi}{4}\right) - P_3\left(\frac{\pi}{4}\right)\right| / \left|f(\pi/4)\right|$$

$$= |1 - 0.94689| / |1|$$

$$= 0.05311$$

$\therefore$ 5.31% of relative error

Given

Ans to Q. 1(c)

Given, $f(x) = \tan(x)$

We know from (a),

$$f^3(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$$

$$f^4(x) = 8\sec^3 x \cdot \sec x \tan x + 4\left[\tan^2 x \cdot 2\sec^2 x \cdot \tan x + \sec^2 x \cdot 2\tan x \frac{d}{dx}\tan x\right]$$

$$= 8\sec^4 x \cdot \tan x + 8\tan^3 x \sec^2 x + 8\sec^4 x \tan x$$

Now, to maximize $f^4(\xi)$, Let $\xi = \frac{\pi}{4}$,

$$f^4\left(\frac{\pi}{4}\right) = 8\sec^4\left(\frac{\pi}{4}\right) \cdot \tan\frac{\pi}{4} + 8\tan^3\frac{\pi}{4}\sec^2\frac{\pi}{4} + 8\sec^4\frac{\pi}{4}\tan\frac{\pi}{4}$$

$$= 8 \times (\sqrt{2})^4 \times 1 + 8 \times 1 \times (\sqrt{2})^2 + 8 \times (\sqrt{2})^4 \times 1$$

$$= 8 \times 4 \times 1 + 8 \times 1 \times 2 + 8 \times 4 \times 1$$

$$= 80$$

Now,

$$|f(x) - P_3(x)| \leq \frac{f^4(\xi)}{4!} \times (x - x_0)^4$$

$$\leq \frac{f^4\left(\frac{\pi}{4}\right)}{4!} \times \cancel{x}\left(\frac{\pi}{4} - 0\right)^4$$

$$\leq \frac{10}{3} \times \left(\frac{\pi}{4}\right)^4$$

$$\leq 1 \cdot 2683$$

Ans to qr no 2

Given, $f(x) = e^{x} - e^{-x}$

Nodes, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

Ans to 2 (a)

Using vandermonde methode,

Given, $(2+1)$ nodes, degree $n = 2$

$$V = \begin{bmatrix} 1 & x_0^1 & x_0^2 \\ 1 & x_1^1 & x_1^2 \\ 1 & x_2^1 & x_2^2 \end{bmatrix} = \begin{bmatrix} 1 & (-1)^1 & (-1)^2 \\ 1 & 0^1 & 0^2 \\ 1 & 1^1 & 1^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

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Here,

$f(-1) = e^{-1} - e^{1} = e^{-1} - e$

$f(0) = e^{0} - e^{-0} = 0$

$f(1) = e^{1} - e^{-1} = e - e^{-1}$

$$\therefore b = \begin{bmatrix} e^{-1} - e \\ 0 \\ e - e^{-1} \end{bmatrix} = \begin{bmatrix} -2.3504 \\ 0 \\ 2.3504 \end{bmatrix}$$

Ans to Q 2 (b)

$$v = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|v| = -1(-1-1) = -1(-2) = 2$$

$\therefore$ Determinant of $v = 2$

Ans to Q 2(c)

We know,

$$a = v^{-1} b$$

Now,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} e^{-1} - e \\ 0 \\ e - e^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} -2.3504 \\ 0 \\ 2.3504 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2.3504 \\ 0 \end{bmatrix}$$

$\therefore\ a_0 = 0,\ a_2 = 0$

$a_1 = 2.3504$

Now, $P_2(x) = 0 \times x^0 + 2.3504\, x^1 + 0 \times x^2$

$= 2.3504\, x$

Ans to 2(d)

$f(x) = e^x - e^{-x}$

$f'(x) = e^x + e^{-x}$

$f^2(x) = e^x - e^{-x}$

$f^3(x) = e^x + e^{-x}$

Here,

$x = 2.1$

$x_0 = -2.1$

[$\because$ to maximize the error]

We know,

$$|f(x) - P_2(x)| \leq \left| \frac{f^3(\xi)}{3!} \times (x - x_0)^3 \right|$$

$$\leq \left| \frac{e^3 + e^{-3}}{3!} \times [2.1 - (-2.1)]^3 \right|$$

$$|f(x) - P_2(x)| \leq \left| \frac{e^{+2.1} - e^{-(+2.1)}}{3!} \times (2.1+2.1)^3 \right|$$

$$\leq \frac{8.2886}{3!} \times (4.2)^3 \quad \left[\begin{array}{l}\text{let,} \\ \xi = 2.1\end{array}\right]$$

$$\leq 1.38143 \times 74.088$$

$$\leq 102.347$$

$\therefore$ The upper bound of interpolation error for the given function for the interval $\xi \in [-2.1, 2.1]$ is 102.347

Ans to qr 3

Given, $f(x) = e^{x} + e^{-x}$

$x_0 = -1, \quad x_1 = 0, \quad x_2 = 1$

Ans to qr 3 (a)

Given, (2+1) nodes we have $n = 2$

Now, for $k = 0$,

$$l_0(x) = \frac{x - x_0}{(x_0 - x_0)} \times \frac{x - x_1}{x_0 - x_1} \times \frac{x - x_2}{x_0 - x_2}$$

$$= \frac{x+1}{-1+1} \times \frac{x-0}{-1-0} \times \frac{x-1}{-1-1}$$

$$= \frac{x}{-1} \times \frac{x-1}{-2}$$

$$= \frac{-x^2 + x}{-2} = \frac{x^2 - x}{2}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} \times \frac{x - x_1}{x_1 - x_1} \times \frac{x - x_2}{x_1 - x_2}$$

$$= \frac{x + 1}{0 - (-1)} \times \frac{x - 1}{0 - 1}$$

$$= (x+1) \times \frac{x - 1}{-1}$$

$$= \frac{x^2 - 1}{-1} = -x^2 + 1$$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1} \times \frac{x - x_2}{x_2 - x_2}$$

$$= \frac{x+1}{1 - (-1)} \times \frac{x - 0}{1 - 0} \times \frac{x - 1}{0}$$

$$= \frac{x+1}{2} \times x$$

$$= \frac{x^2 + x}{2}$$

Ans to Qr 3(b)

We know,

$$P_n(x) = \sum_{k=0}^{n} f(x_k)\, l_k(x)$$

Now,

$$\cancel{P_2(x) =}\ f(x_0) = e^{-1} + e^{1} = 3.0861$$

$$f(x_1) = e^{0} + e^{-0} = 2$$

$$f(x_2) = e^{1} + e^{-1} = 3.0861$$

Now,

$$P_2(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + f(x_2)\, l_2(x)$$

$$= 3.0861 \times \frac{x^2 - x}{2} + 2(-x^2 + 1) + 3.0861 \times \frac{x^2 + x}{2}$$

$$= 1.543x^2 - 1.543x - 2x^2 + 2 + 1.543x^2 + 1.543x$$

$$= 3.0861x^2 - 2x^2 + 2$$

$$= 1.0861x^2 + 2$$

$$= 2 + 1.0861x^2 \quad \text{[Natural basis]}$$

Now, $P_2(6) = 2 + 1.0861 \times 6^2$

$= 39.0996 + 2$

$= 41.0996$

Ans to Q 3 (c)

At $x = 1.5$,

$f(1.5) = e^{1.5} + e^{-1.5}$

$= 4.7048$

$P_2(1.5) = 2 + 1.0861 \times (1.5)^2$

$= 4.443725$

$\therefore$ Relative error $= |f(1.5) - P_2(1.5)|$

$= |4.7048 - 4.443725| / |4.70481|$

$= 0.0568$

$\therefore$ 5.686% of relative error.

Ans to Q 4 (a)

Given, $f(x) = e^x - e^{-x}$

$x_0 = -2$, $x_1 = 0$, $x_2 = 2$

$\therefore n = 2$

$f(x_0) = e^{-2} - e^{2} = -7.253$

$f(x_1) = e^{0} - e^{-0} = 0$

$f(x_2) = e^{2} - e^{-2} = +7.253$

We know,

$a_0 = f[x_0]$

$a_1 = f[x_0, x_1]$

$a_2 = f[x_0, x_1, x_2]$

$x_0 = -2$

$f[x_0] = -7.253$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = 3.626$$

$x_1 = 0$

$f[x_1] = 0$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = 3.626$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{0}{x_2 - x_0} = 0$$

$x_2 = 2$

$f[x_2] = 7.253$

$\therefore$ $a_0 = -7.253$

$a_1 = f[x_0, x_1] = 3.626$

$a_2 = f[x_0, x_1, x_2] = 0$

Ans to Q 4(b)

Newton's interpolation polynomial for the given function:

$$P_2(x) = a_0 n_0(x) + a_1 n_1(x) + a_2 n_2(x)$$

We know, $n_0(x) = 1$, $n_1(x) = x - x_0$

$n_2(x) = (x - x_0)(x - x_1)$

$$\therefore P_2(x) = -7.253 \times (1) + 3.626 \times (x - x_0) + 0 \times n_2(x)$$

$$= -7.253 + 3.626 \times (x+2)$$

$$= -7.253 + 7.253 + 3.626x$$

$$= 3.626x$$

$$\therefore P_2(6) = 3.626 \times 6 = 21.756$$

Ans to qr no 4(c)

At ; $x = 1.5$,

$P(1.5) = 3.626 \times 1.5 = 5.439$

$f(1.5) = e^{1.5} - e^{-1.5} = 4.258$

$$\text{Relative error} = \left| \frac{f(1.5) - P(1.5)}{f(1.5)} \right|$$

$$= \left| \frac{4.258 - 5.439}{4.258} \right|$$

$$= 0.27736$$

$\therefore$ 27.736 % of relative error at $x = 1.5$

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