MAT 216

Linear Algebra & Fourier Analysis

PART A

Lecture Note

Contents:

- ***Odd Even Functions***
- ***Fourier Series***

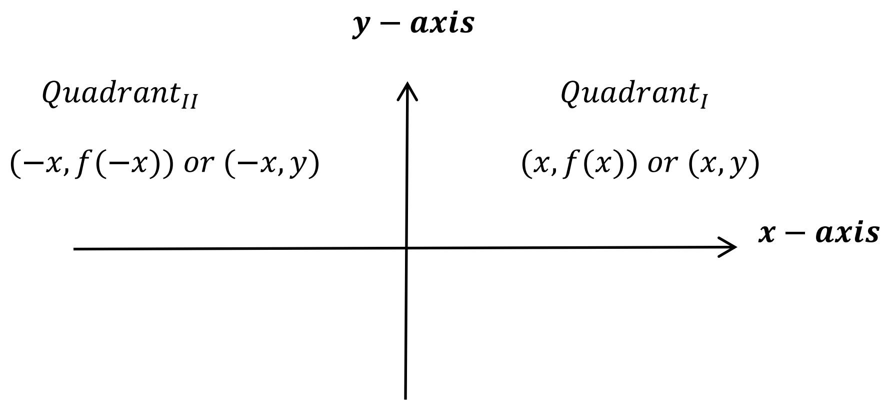
Reference Book:

Schaum's Outline Series Theory problems of Fourier Analysis – Murray R. Spiegel

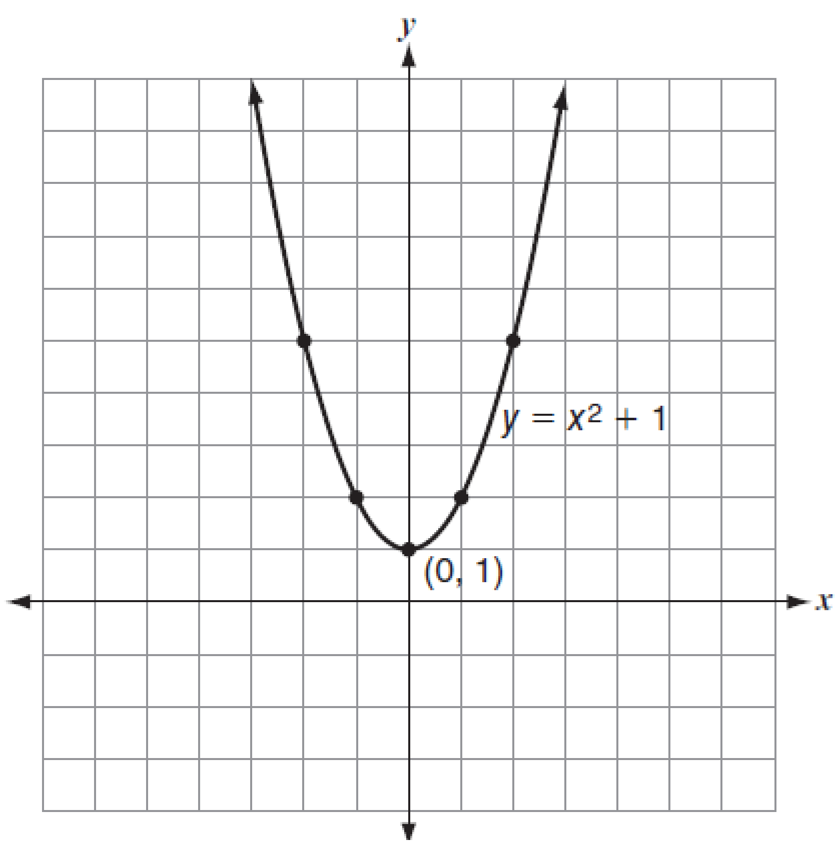
Odd Even Functions

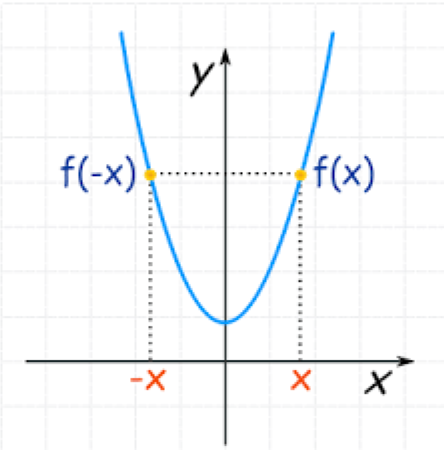
Even Function

A function is **even** when $f(x) = f(-x), \forall x$



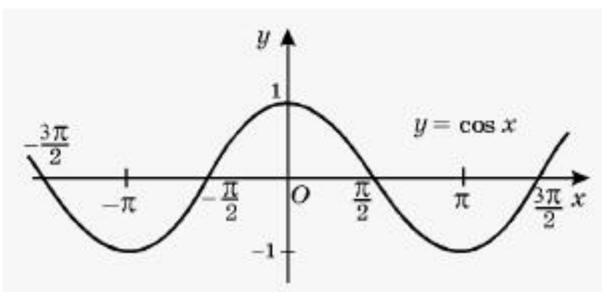
In other words they are symmetric about the $y - axis$ (like a reflection)

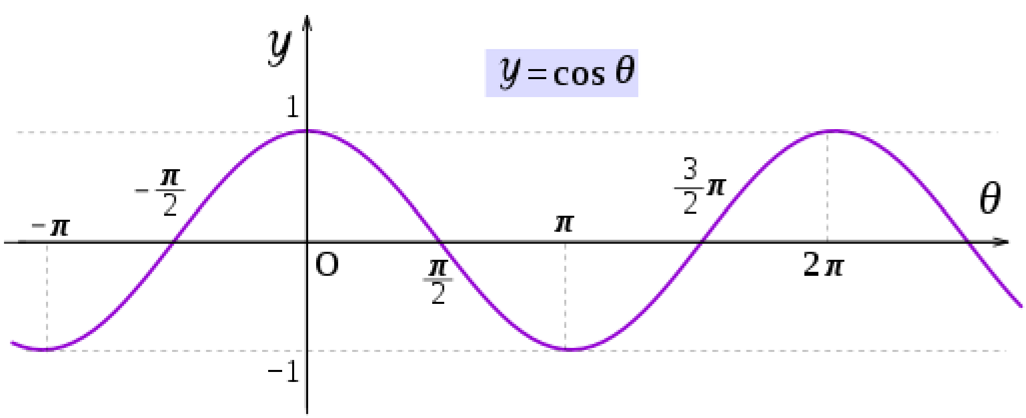




While we observe the curve $y = x^2 + 1$, it is vivid that distance from $x - axis$ to $f(x)$ and $f(-x)$ is equivalent.

Examples of **even** function: $x^2,\ x^4,\ x^6,\ x^8, \cos(x)\ etc$





An **even** exponent **does not** always make an **<u>even function</u>**.

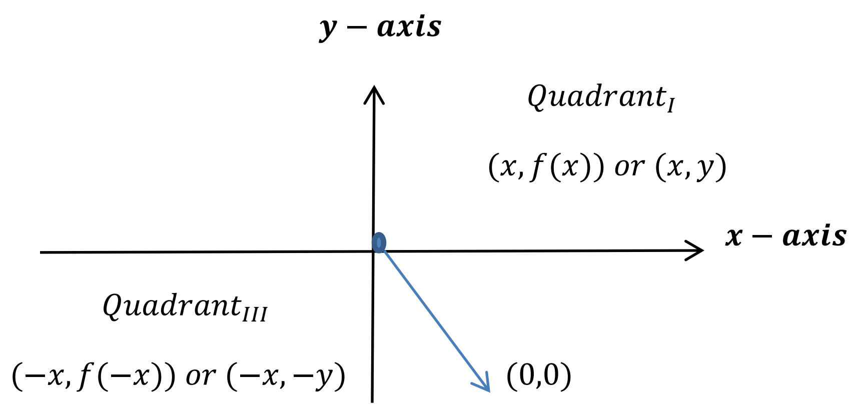
$y = (x+1)^2$ is not an even function.

$f(x) = (x+1)^2 = x^2 + 2x + 1$

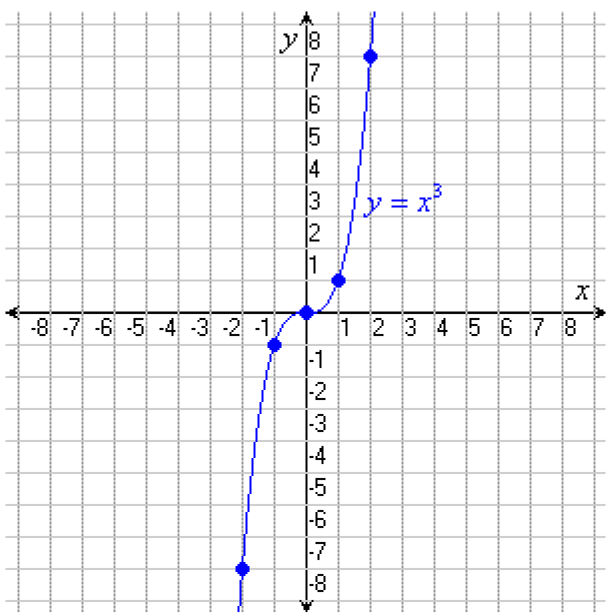
$f(-x) = (-x+1)^2 = x^2 - 2x + 1 \quad \therefore f(x) \neq f(-x)$

Odd Function:

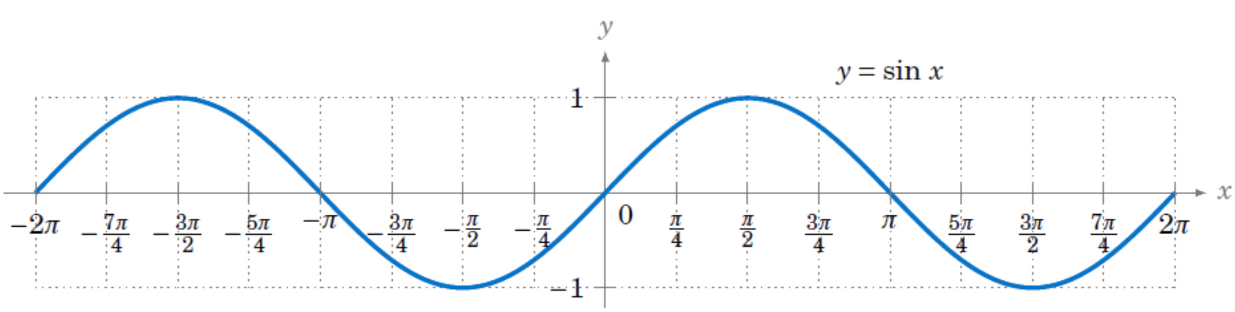
A function is **odd** when $f(x) = -f(-x), \forall x$



It is symmetric to the origin {i.e. (0,0)}



Examples of **odd** functions: $x, x^3, x^5, x^7, \sin(x)\ \ etc.$

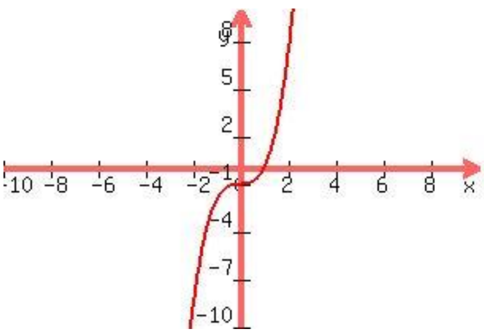


An **odd** exponent does not always make an odd function, for example $y = x^3 + 1$ is not an odd function.

$f(x) = x^3 + 1$

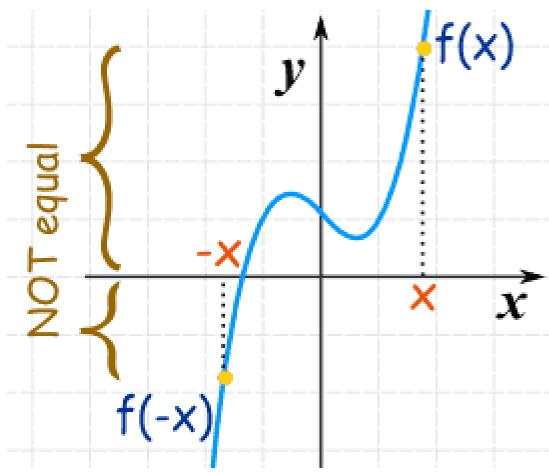
$-f(-x) = -[(-x)^3 + 1] = -[-x^3 + 1] = x^3 - 1$

$\therefore f(x) \neq -f(-x)$



Neither Odd nor Even Function:

Most functions are neither odd nor even. For example $y = x^3 - x + 1$ is **not an odd function**. It is **not an even function** either. It is neither odd nor even.



Example: Is $f(x) = \frac{x}{x^2-1}$ Even or Odd or neither?

$$f(-x) = \frac{(-x)}{[(-x)^2-1]} = \frac{-x}{x^2-1}$$

$\therefore f(x) \neq f(-x)$; Which implies it is not an even function.

$$-f(-x) = -\left(\frac{-x}{x^2-1}\right) = \frac{x}{x^2-1}$$

$\therefore f(x) = -f(-x)$; Which implies it is an odd function.

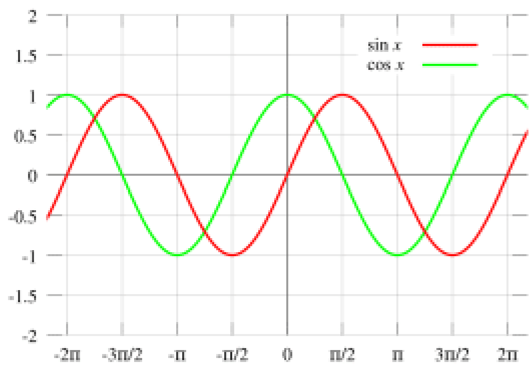
Even and Odd

The only function that is even and odd is $f(x) = 0$

Periodic Function

A periodic function is a function that repeats its values at regular intervals, for example, the trigonometric functions, which repeat at intervals of 2π.

Periodic functions are used throughout science to describe oscillations, waves, and other phenomena that exhibit periodicity.



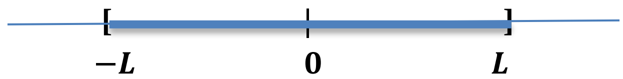
Fourier Series

A Fourier series is expansion of a periodic function f(x) in terms of an infinite sum of sine and cosine. It is useful to breakup an arbitrary periodic function into a set of simple terms that can be solved individually.

Fourier series defined by:

$$f(x) = \frac{a_0}{2} + \sum_{n-1}^{\infty} a_n cos\frac{n\pi}{L}x + \sum_{n=1}^{\infty} b_n sin\frac{n\pi}{L}x \quad \cdots\cdots\cdots \mathbf{(1)}$$

over the interval $(-L, L)$, while $f(x)$ has period $2L$.



The distance from $-L$ to L is $2L$.

- Even part of the function:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi}{L}x;\ a_0\ is\ the\ constant\ term\ of\ even\ part$$

- Odd part of the function:

$$\sum_{n=1}^{\infty} b_n sin\frac{n\pi}{L}x$$

Coefficients:

Even:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi}{L}x\,dx;\ \ n = 0,1,2,3\ \cdots$$

- ***Note:*** $\cos(n\pi) = \begin{cases} +1, & for\ \text{"n" even} \\ -1, & for\ \text{"n"}\ odd \end{cases}$
- ***Note:*** The constant term of **equation (1)** is $\frac{a_0}{2}$

$$\frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)cos\frac{(0)\pi}{L}x\,dx = \frac{1}{2L}\int_{-L}^{L} f(x)(1)\,dx = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

It is the average of $f(x)$ over a given period.

Odd:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) sin\left(\frac{n\pi}{L}\right) x\, dx\,; \quad n = 0, 1, 2, \cdots$$

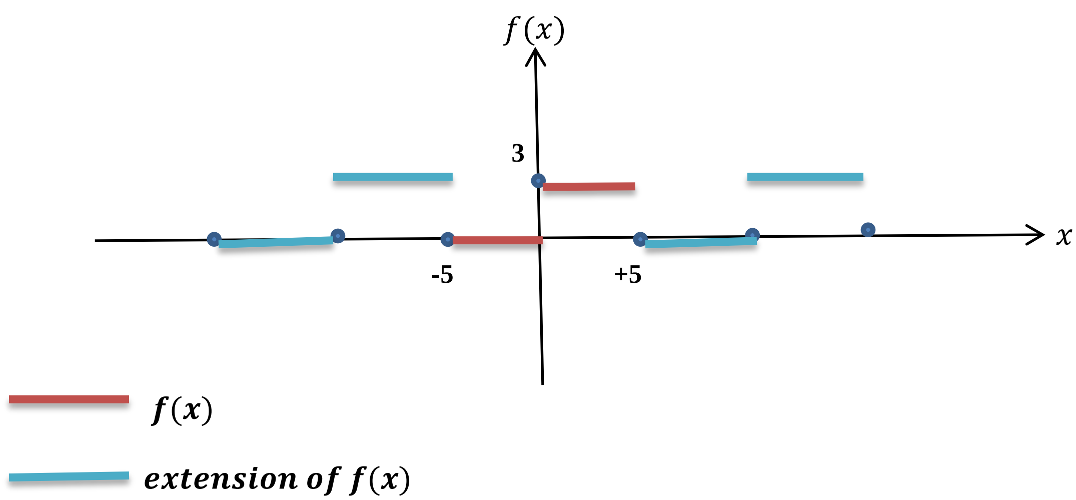
Note: $\sin(n\pi) = 0;\ \forall n$

Example

I. *Find Fourier Series coefficients corresponding to the function*

$$f(x) = \begin{cases} 0, -5 < x < 0 \\ 3, 0 < x < 5 \end{cases} ;Peroid = 10$$

II. *Write the corresponding Fourier Series*



I. $L = 5, 2L = 10$

$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x) cos\frac{n\pi}{L}(x)dx$$

$$= \frac{1}{5}\int_{-5}^{5} f(x) cos\,\frac{n\pi}{5}\,(x)dx$$

$$= \frac{1}{5}\left[\int_{-5}^{0}(0) cos\frac{n\pi}{5}(x)dx + \int_{0}^{5}(3) cos\frac{n\pi}{5}(x)dx\right]$$

$$= \frac{1}{5}\left[0 + \int_{0}^{5}(3) cos\frac{n\pi}{5}(x)dx\right]$$

$$= \frac{3}{5}\left[\frac{sin\frac{n\pi}{5}(x)}{\frac{n\pi}{5}}\right]_0^5 ; n \neq 0 \;\because \boldsymbol{n}\, is\ in\ the\ denominator$$

.

$$= \frac{3}{5}.\frac{5}{n\pi}\left[sin\frac{n\pi}{5}(5) - sin\frac{n\pi}{5}(0)\right] = \frac{3}{n\pi}[0-0] = 0 \quad \because \sin(n\pi) = 0$$

When $n = 0$ then a_n becomes a_0

$$a_0 = \frac{3}{5}\int_0^5 cos\frac{n\pi}{5}(x)dx = \frac{3}{5}\int_0^5 cos\frac{(0)\pi}{5}(x)dx = \frac{3}{5}\int_0^5 (1)dx = 3; \because \cos(0)^0 = 1$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi}{L}(x)dx$$

$$= \frac{1}{5}\int_{-5}^{5} f(x)sin\frac{n\pi}{5}(x)dx$$

$$= \frac{1}{5}\left[\int_{-5}^{0}(0)sin\frac{n\pi}{5}(x)dx + \int_0^5 (3)sin\frac{n\pi}{5}(x)dx\right]$$

$$= \frac{1}{5}\left[0 + \int_0^5 (3)sin\frac{n\pi}{5}(x)dx\right]$$

$$= \frac{3}{5}\left[\int_0^5 sin\frac{n\pi}{5}(x)dx\right]$$

$$= \frac{3}{5}\left[\frac{-cos\frac{n\pi}{5}(x)}{\frac{n\pi}{5}}\right]_0^5$$

$$= \frac{-3}{n\pi}[\cos(n\pi) - 1] \quad \because \cos(0)^0 = 1$$

II. Corresponding Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n cos \frac{n\pi}{L}(x) + b_n sin \frac{n\pi}{L}(x) \right)$$

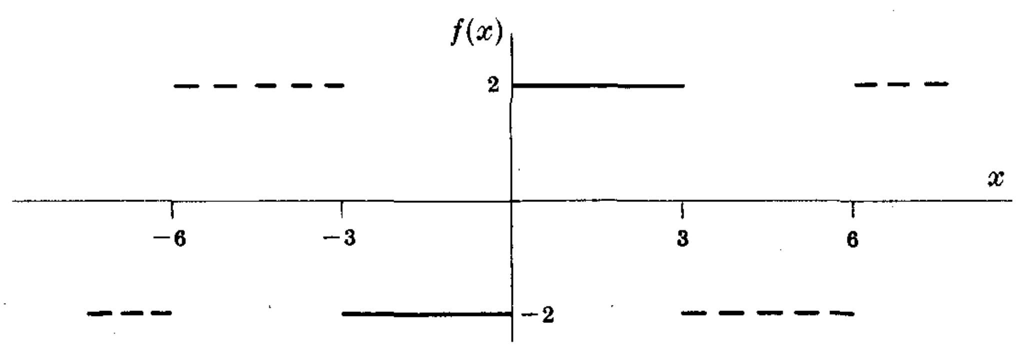
$$= \frac{3}{2} + \sum_{n=1}^{\infty} \left(0 + \frac{3(1-\cos(n\pi))}{n\pi} sin \frac{n\pi}{5}(x) \right)$$

Solved Examples from Tutorial:

Example 1:

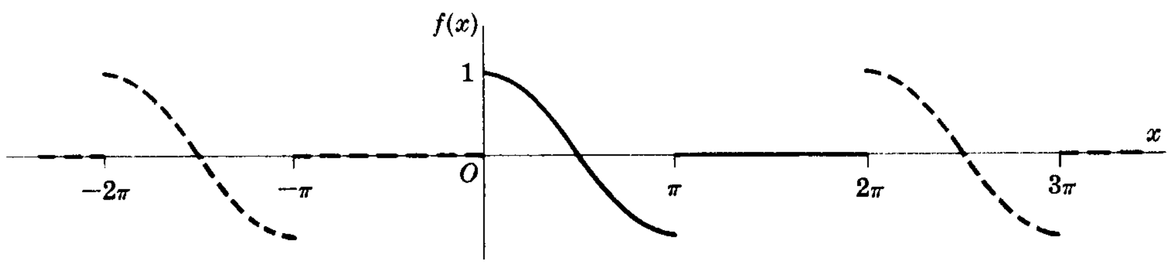
Classify each of the following functions according as they are even, odd, or neither even nor odd.

$$(a) \quad f(x) = \begin{cases} 2 & 0 < x < 3 \\ -2 & -3 < x < 0 \end{cases} \qquad \text{Period} = 6$$



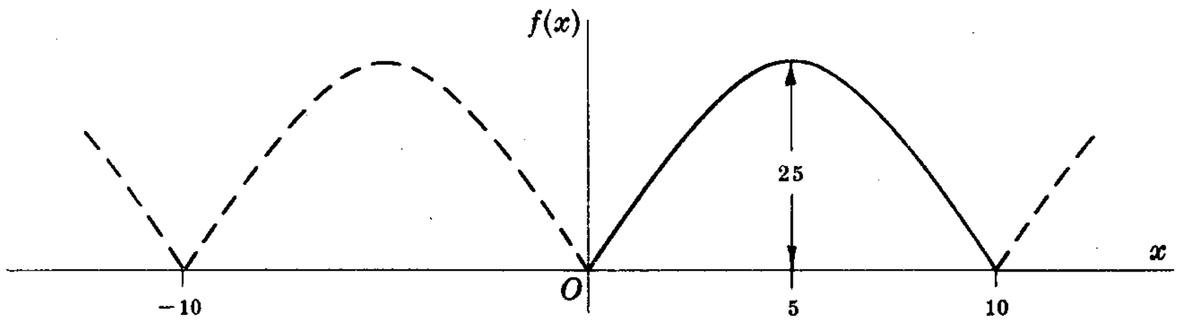
The graph of the function is seen to be odd.

$$(b) \quad f(x) = \begin{cases} \cos x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases} \qquad \text{Period} = 2\pi$$



The graph of the function is neither odd nor even.

(c) $f(x) = x(10 - x), \quad 0 < x < 10, \quad \text{Period} = 10.$



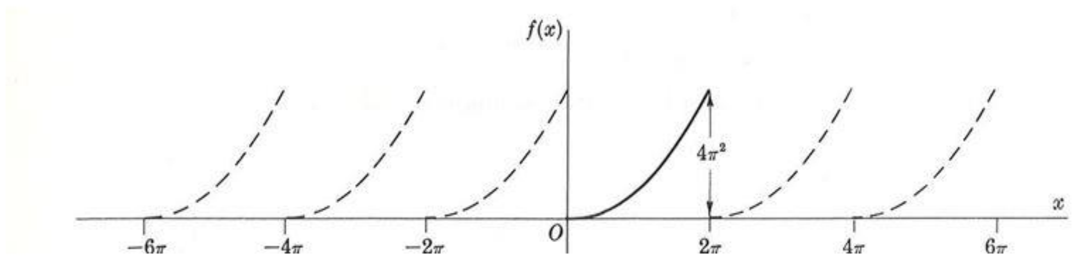
The graph of the function is seen to be even.

Example 2:

Note: the response from $-\pi$ to π is exactly the same as from 0 to 2 π so integrating over either is the same…..and the later is easier

Expand $f(x) = x^2, \ 0 < x < 2\pi$, in a Fourier series if the period is 2π.

The graph of $f(x)$ with period 2π is shown



$$Period = 2L = 2\pi \quad \therefore L = \pi$$

We know

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi}{L}(x) + \sum_{n=1}^{\infty} b_n sin\frac{n\pi}{L}(x)$$

Now we will evaluate the coefficients a_0, a_n, b_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}(x)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x^2 cos\frac{n\pi}{\pi}(x)dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x^2\cos(nx)\,dx$$

let $\boldsymbol{x^2 = u}$ ***and*** $\boldsymbol{\cos(nx) = v}$***, applying*** $\boldsymbol{\int (uv)dx = u\int v\,dx - \int\{\frac{d}{dx}(u)\int (v)dx\}dx}$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin(nx)}{n}\right) - \int 2x\left(\frac{\sin(nx)}{n}\right)dx\right]$$

$$let\ u = 2x, and\ v = \frac{\sin(nx)}{n}$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin(nx)}{n} - \left\{2x\left(-\frac{\cos(nx)}{n^2}\right) - \int 2\left(-\frac{\cos(nx)}{n^2}\right)dx\right\}\right]$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin(nx)}{n} + 2x\frac{\cos(nx)}{n^2} - \frac{2}{n^2}\int \cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[x^2\frac{\sin(nx)}{n} + 2x\frac{\cos(nx)}{n^2} - 2\frac{\sin(nx)}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[4\pi^2\frac{\sin(2\pi n)}{n} + 4\pi\frac{\cos(2\pi n)}{n^2} - 2\frac{\sin(2\pi n)}{n^3} - 0^2\frac{\sin(0)}{n} - 2(0)\frac{\cos(0)}{n^2} + 2\frac{\sin(0)}{n^3}\right]$$

Note:

- $\boldsymbol{\sin(n\pi) = 0, \forall n}$
- $\boldsymbol{\cos(n\pi) = \begin{cases} +1, & for \text{ "n" even} \\ -1, & for \text{ "n" } odd \end{cases}}$
- **In this case** $\boldsymbol{\cos(2\pi n) = +1,\ \because 2\times(\pi n)}$ **is an even number as we know 2 multiplied by any number is an even number.**

$$= \frac{1}{\pi}\left[0 + 4\pi\frac{1}{n^2} - 0 - 0 - 0 + 0\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right] = \frac{4}{n^2} \qquad while\ n \neq 0\ \because n\ is\ in\ the\ denominator$$

$$a_n : [n = 0] \xrightarrow{\text{yields}} a_0 = \frac{1}{L}\int_{-L}^{L} x^2 cos\frac{n\pi}{L}(x)dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x^2 cos\frac{0.\pi}{\pi}(x)dx;\ since\ n = 0$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x^2 \cos(0)\ dx,$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x^2(1)dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{8\pi^2}{3}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) sin\frac{n\pi}{L}(x)dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x^2 \sin(nx)\, dx$$

$$let\ u = x^2, \qquad v = \sin(nx)$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos(nx)}{n}\right) - \int 2x\left(-\frac{\cos(nx)}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos(nx)}{n}\right) + \int 2x\left(\frac{\cos(nx)}{n}\right)dx\right]$$

$$let\ u = 2x, \qquad v = \frac{\cos(nx)}{n}$$

$$= \frac{1}{\pi}\left[-x^2\frac{\cos(nx)}{n} + \left\{2x\left(\frac{\sin(nx)}{n^2}\right) - \int 2\frac{\sin(nx)}{n^2}dx\right\}\right]$$

$$= \frac{1}{\pi}\left[-x^2\frac{\cos(nx)}{n} + 2x\frac{\sin(nx)}{n^2} - 2\left(-\frac{\cos(nx)}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-x^2\frac{\cos(nx)}{n} + 2x\frac{\sin(nx)}{n^2} + 2\frac{\cos(nx)}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-4\pi^2\frac{\cos(2\pi n)}{n} + 4\pi\frac{\sin(2\pi n)}{n^2} + 2\frac{\cos(2\pi n)}{n^3} - \left\{-0^2\frac{\cos(0)}{n} + 2(0)\frac{\sin(0)}{n^2} + 2\frac{\cos(0)}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[-4\pi^2\frac{1}{n} + 4\pi(0) + 2\frac{1}{n^3} - \left\{-0 + 0 + 2\frac{1}{n^3}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{-4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right] = \frac{1}{\pi}\left[-\frac{4\pi^2}{n}\right] = -\frac{4\pi}{n}$$

Since

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos \frac{n\pi}{L}(x) + \sum_{n=1}^{\infty} b_n sin \frac{n\pi}{L}(x)$$

$$= \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2} cos \frac{n\pi}{\pi}(x) + \sum_{n=1}^{\infty} \left(\frac{-4\pi}{n}\right) sin \frac{n\pi}{\pi}(x)$$

$$= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} 4\left(\frac{1}{n^2}\cos(nx) - \frac{\pi}{n}\sin(nx)\right).$$

Exercise

1. Determine the Fourier series for:

$$f(x) = \begin{cases} -x; \ -4 \le x \le 0 \\ x; 0 \le x \le 4 \end{cases} \quad Period = 8$$

2. Graph each of the following functions and find its corresponding Fourier series:
   a) $f(x) = \begin{cases} 8, \ 0 < x < 2 \\ -8, \ 2 < x < 4 \end{cases}; Period = 4$
   b) $f(x) = \begin{cases} -x, -4 \le x \le 0 \\ x, \ 0 \le x \le 4 \end{cases}; Period = 8$
   c) $f(x) = 4x, \ 0 < x < 10, \ Period = 10$
   d) $f(x) = \begin{cases} 2x, \ 0 \le x \le 3 \\ x, -3 \le x \le 0 \end{cases}; \ Period = 6$

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