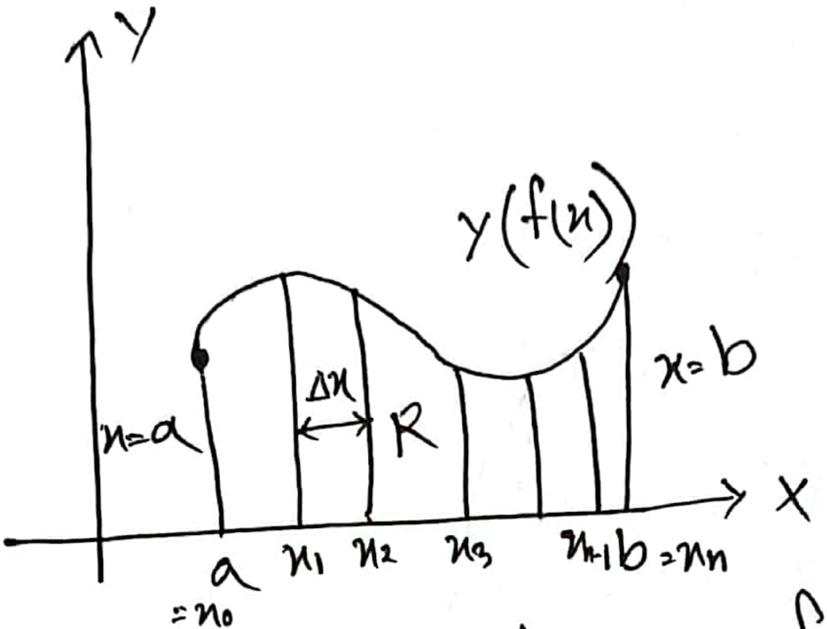
MAT120 01/06/2022

5.4

The Definition of area as limit:



Suppose $y=f(x)$ is a continuous function on the interval $[a,b]$ and R denote the region bounded below by x-axis, bounded on the sides by the vertical ~~time~~ lines $x=a$ and $x=b$, and bounded above by the curve $y=f(x)$.

Now, to find the area of the region R:

① Divide the interval $[a,b]$ into n equal subintervals and denote those parts as $x_1, x_2, x_3 ------ x_{n-1}$.

Each subinterval $\Delta x = \frac{b-a}{n}$

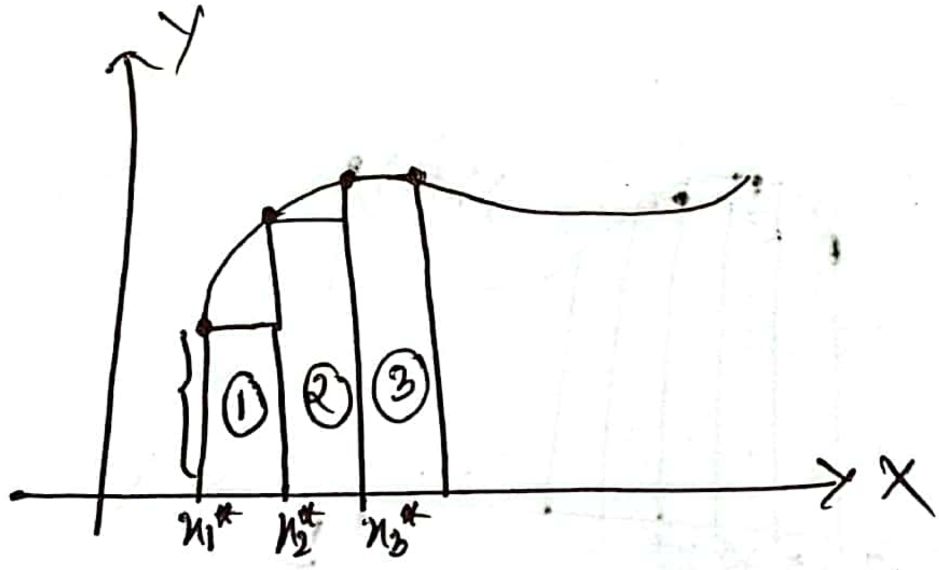
⑤ Repeat the process using more and more subdivisions then the area of the region R will be.

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$$

Height point

① Left end point approximation

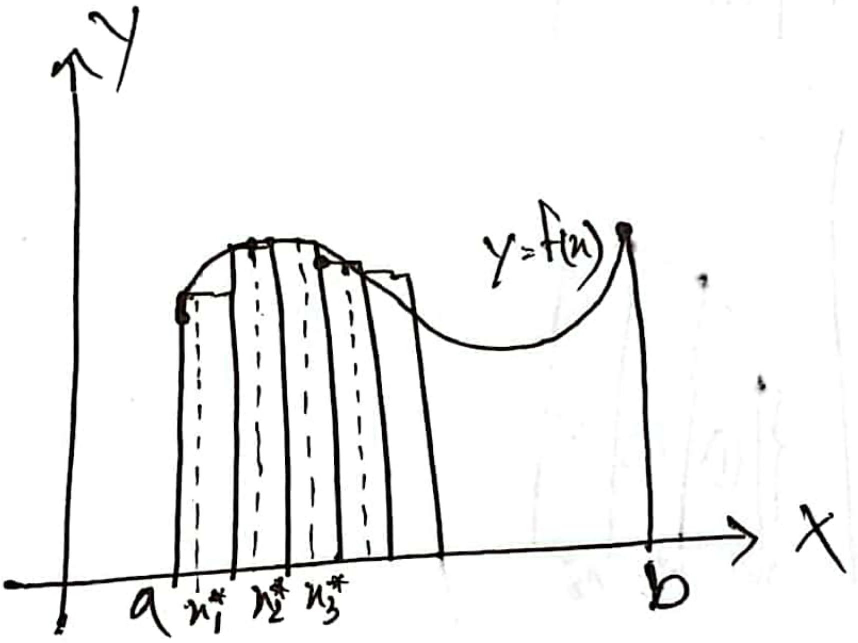
②



$$x_k^* = a + (k-1) \Delta x$$

(2) Over each subinterval construct a rectangle whose height is the value of 'f' at an arbitarily selected point. Thus if $x_1^*, x_2^*, \ldots, x_n^*$ denote the points of height then the length of height will be

$f(x_1^*), f(x_2^*) \ldots f(x_n^*)$



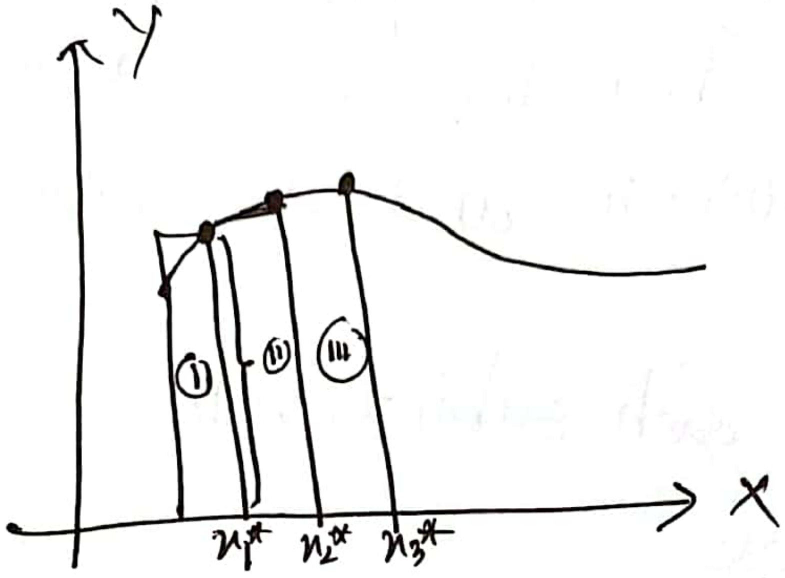
(3) So, the area of each rectangle could be

$f(x_1^*)\Delta x, \; f(x_2^*)\Delta x \ldots, \; f(x_n^*)\Delta x$

(4) So, the total area, $A \approx f(x_1^*)\Delta + f(x_2^*) + \ldots + f(x_n^*)\Delta x$

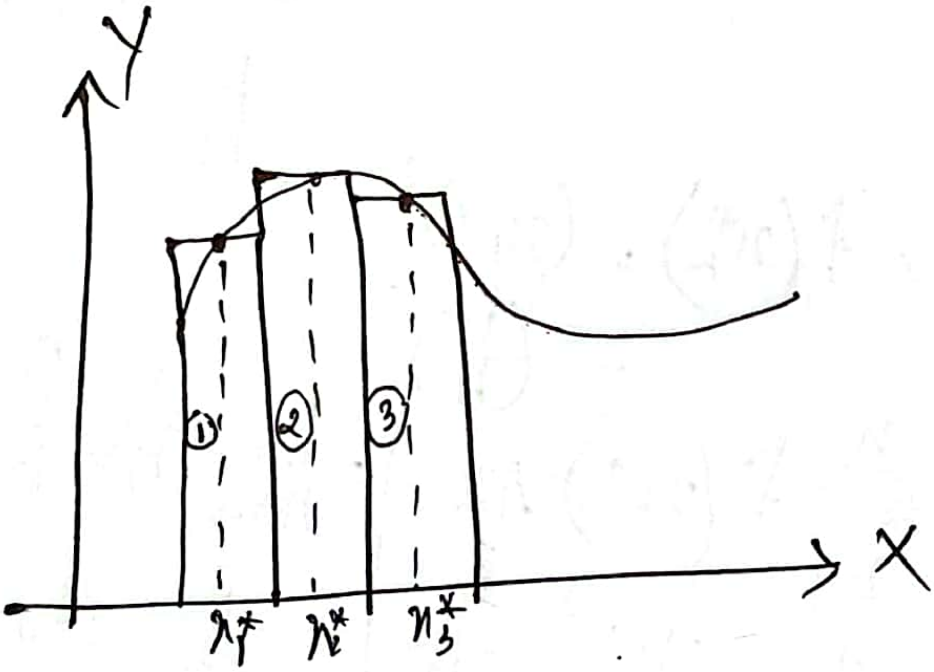
$$\approx \sum_{k=1}^{n} f(x_k^*)\Delta x$$

② Right end point approximation



$$x_k^* = a + k \cdot \Delta x$$

③ Mid point approximation



$$x_k^* = a + \left(k - \frac{1}{2}\right) \Delta x$$

use the definition of area as a limit with x_k^* as the right end point of each subinterval to find the area between the graph $f(x) = x^2$ and the interval [a,b] [0,1]

Solⁿ: Width of each subinterval,

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{1-0}{n} = \frac{1}{n}$$

Height point, $x_k^* = a + k(\Delta x)$

$$= 0 + k\left(\frac{1}{n}\right)$$

$$= \frac{k}{n}$$

Height, $f(x_k^*) = (x_k^*)^2$

$$= \left(\frac{k}{n}\right)^2$$

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \frac{1}{6}\left[\frac{n}{n}\,\frac{n+1}{n}\,\frac{2n+1}{n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{6}\left[1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= \frac{2}{6} = \frac{1}{3}$$

$$= \frac{1}{3} \text{ (Ans)}$$

(i) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1 = 1$

(ii) $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{2}$.

(iii) $\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{3}$.

(iv) $\lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \frac{1}{4}$

* $1+2+3+\dots+n$

$= \frac{n(n+1)}{n}$

* $1^2+2^2+3^2+\dots+n^2$

$= \frac{n(n+1)(2n+1)}{6}$

* $1^3+2^3+3^3+\dots+n^3$

$= \left\{\frac{n(n+1)}{2}\right\}^2$

$$\int_0^1 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1}{3} - 0\right)$$

$$= \frac{1}{3}$$

* width of each subinterval,

$$\Delta x = \frac{b-a}{n}$$

$$= \cancel{1-0}\ \frac{3-0}{n} = \frac{3}{n}$$

Height point, $x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$

$$= 0 + \left(k - \frac{1}{2}\right)\cdot\frac{3}{n}$$

$$= \left(\frac{2k-1}{2}\right)\frac{3}{n}$$

$$= \frac{6k-3}{2n}$$

Height, $f(x_k^*) = 9 - (x_k^*)^2$

$$= 9 - \left(\frac{6k-3}{2n}\right)^2$$

$$= 9 - \left(\frac{36k^2 - 36k + 9}{4n^2}\right)$$

$$= \frac{36n^2 - 36k^2 + 36k - 9}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{36n^2 - 36k^2 + 36k - 9}{4n^2} \times \frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{108n^2 - 108k^2 + 108k - \cancel{9}\,27}{4n^3}$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} \cancel{108}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\cancel{108}\,\frac{27}{n} - \frac{27k^2}{4n^3} + \frac{27k}{n^3} - \frac{27}{4n^3}\right]$$

$$= \lim_{n\to\infty} \sum \frac{27}{n} - \cancel{\sum} \lim_{n\to\infty} \sum \frac{27k^3}{n^3}$$

$$+ \lim_{n\to\infty} \sum \frac{27k}{n^3} + \lim_{n\to\infty} \sum \frac{27}{n^3}$$

$$= 27 - \frac{27}{3} + \lim_{n\to\infty} \cancel{\sum} 27 \cdot \underbrace{\frac{1}{n}}_{0} \cdot \boxed{\frac{1}{n^2} \sum k}$$

$$= 27 - \frac{27}{3} + 0 + 0$$

$$= 27 - 9 = 18$$

5.4

27 - 30

35 - 40