Q1

1. For function $f(x) = 1/x$, at $x_0 = 2$, find the truncation error for backward difference ($h = 1, h = 0.1, h = 0.01, h = 0.001$) and figure out the relationship of error with the order of h. (5 marks)
ANS: Derivative of $1/x$ is $-1/x^2$. So $f'(x) = -1/x^2$. $f'(2) = -0.25$
Now using backward difference formula we find out derivative for each of the h value. And by subtracting the backward difference from the exact derivative at $x_0 = 2$ we find the truncation error.

h	Backward difference	Truncation error
1	-0.5	0.25
0.1	-0.2631578947368418	0.013157894736841813
0.01	-0.2512562814070307	0.0012562814070307127
0.001	-0.2501250625311924	0.00012506253119237698

We can see that the error is proportional to h^2.

2. A rocket has been launched, and its velocities at differet times are collected. From these data, the acceleration of the rocket, $a(t)$ at $t = 16sec$ is calculated numerically by using different methods (h=1) as shown in the table below:

Difference method	Forward	Backward	Central
a (t=20)	38.30383517015082	37.115963921706	37.70989954592841

Now, if the velocity of a rocket as function of time obey the equation below: where v is in m/s and t is in seconds,

$$v(t) = 1900 \ln(\frac{12 * 10^4}{12 * 10^4 - 2000t}) - 9.8t \tag{3}$$

find the truncation errors for the acceleration at t=20 sec for Forward, Backward and Central Difference methods. (4.5 marks)
ANS:

$$a(t) = 1900 * 2000 \frac{12 * 10^4 - 2000t}{(12 * 10^4 - 2000t)^2} - 9.8 \tag{4}$$

Following above equation, $a(t = 20) = 37.7$

Method	Truncation error
forward	-0.6038351701508162
backward	0.5840360782940053
central	-0.009899545928405473

Q2

② (a) $f(x) = 2x - \sec x \quad [-1, 1]$

$f(-1) = -2.8508$

$f(0) = -1$

$f(1) = 0.1492$

$f(0) \cdot f(1) < 0$

so, root lies between 0 & 1.

(b) $f(x) = x^3 - 5x^2 - 2x + 10$

$x^3 - 5x^2 - 2x + 10 = 0$

$\Rightarrow 2x = x^3 - 5x^2 + 10$

$$\boxed{\Rightarrow x = \frac{x^3 - 5x^2 + 10}{2} = g_1(x)}$$

Again, $x^3 - 5x^2 - 2x + 10 = 0$

$\Rightarrow 2x - 10 = x^3 - 5x^2$

$\Rightarrow x(x^2 - 5x) = 2x - 10$

$$\boxed{\Rightarrow x = \frac{2x - 10}{x^2 - 5x} = g_2(x)}$$

roots $\Rightarrow f(x) = 0$

$\Rightarrow x^3 - 5x^2 - 2x + 10 = 0$

$\Rightarrow x^2(x-5) - 2(x-5) = 0$

$\Rightarrow (x^2 - 2)(x - 5) = 0$

Monta

$x = \pm\sqrt{2}, 5$

$$\lambda_1 = |g_1'(x)| = \left|\frac{1}{2}(3x^2 - 10x)\right|$$

$$\lambda_1 = \begin{cases} +4.07107 > 1 : \quad \text{diverge} \\ 10.07107 > 1 : \text{diverge} \\ 12.5 > 1 ; \text{diverge} \end{cases}$$

$$\lambda_2 = |g_2'(x)| = \left|\frac{(x^2 - 5x)\times 2 - (2x-10)(2x-5)}{(x^2-5x)^2}\right|$$

$$\lambda_2 = \begin{cases} 0 \quad < 1 ; \text{ converge} \\ 1 \quad ; \text{diverge} \\ 1 \quad ; \text{diverge} \end{cases}$$

Q3

Set 2

A

Newton's Method

K	Xk	f(Xk)
0	0	1.0000000
1	1.0000000	-1.8314635
2	-0.9005241	-0.1210904
3	-0.7161246	0.1104702
4	-0.7789272	0.0092690
5	-0.7853177	0.0001139
6	-0.7853982	0.0000000
7	-0.7853982	0.0000000

B

Aitken Acceleration

K	Xk	f(Xk)
0	0	1.0000000
1	1.0000000	-1.8314635
2	-0.9005241	-0.1210904
2(^)	0.3447653	0.1731555
3	0.3939619	-0.0046692
4	0.3926997	-0.0000024
4(^)	0.3927313	-0.0001191
5	0.3926991	0.0000000

Set 2 Question 4

a) A = $\begin{pmatrix} 1 & -3 & 4 \\ 2 & -5 & 6 \\ -3 & 3 & 4 \end{pmatrix}$, det(A) =4$\neq$ 0. So, A is non-singular. Also, A is a square matrix. This system has unique solution.

b) Augmented matrix = $\left(\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 2 & -5 & 6 & 6 \\ -3 & 3 & 4 & 6 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -6 & 16 & 15 \end{array}\right)$ [R2=R2-2R1, R3=R3+3R1] = $\left(\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & 15 \end{array}\right)$ [R3=R3+6R2] = U

c) Back Substitution: 4 x3 = 15 => x3=15/4, x2 = 15/2, x1 = 45/2-12=21/2

x1 = 21/2, x2 = 15/2, x3 = 15/4

Q5

5 (a) Eqⁿ of straight line = $a_0 + a_1 x$

$$\underbrace{\begin{bmatrix} 1 & 500 \\ 1 & 700 \\ 1 & 990 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 120 \\ 180 \\ 200 \end{bmatrix}}_{b}$$

(b) $$\begin{bmatrix} 1 & 500 \\ 1 & 700 \\ 1 & 990 \end{bmatrix}$$

u_1 u_2

$$P_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_2 = u_2 - \frac{u_2 \cdot P_1}{P_1 \cdot P_1} P_1$$

$$= \begin{bmatrix} 500 \\ 700 \\ 990 \end{bmatrix} - \frac{(500\times1)+(700\times1)+(990\times1)}{(1\times1)+(1\times1)+(1\times1)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 500 \\ 700 \\ 990 \end{bmatrix} - 730 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} -230 \\ -30 \\ 260 \end{bmatrix}$$

$$q_1 = \frac{P_1}{|P_1|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{1}{348.425} \begin{bmatrix} -230 \\ -30 \\ 260 \end{bmatrix} = \begin{bmatrix} -0.660 \\ -0.086 \\ 0.746 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & -0.660 \\ 1/\sqrt{3} & -0.086 \\ 1/\sqrt{3} & 0.746 \end{bmatrix} = \begin{bmatrix} 0.577 & -0.660 \\ 0.577 & -0.086 \\ 0.577 & 0.746 \end{bmatrix}$$

$$R = Q^T A$$

$$= \begin{bmatrix} 0.577 & 0.577 & 0.577 \\ -0.660 & -0.086 & 0.746 \end{bmatrix} \begin{bmatrix} 1 & 500 \\ 1 & 700 \\ 1 & 990 \end{bmatrix}$$

$$R = \begin{bmatrix} 1.731 & 1263.63 \\ 0 & 348.34 \end{bmatrix}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} 1.731 & 1263.63 \\ 0 & 348.34 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0.577 & 0.577 & 0.577 \\ -0.660 & -0.086 & 0.746 \end{bmatrix} \begin{bmatrix} 120 \\ 180 \\ 200 \end{bmatrix}$$

$$\begin{bmatrix} 1.731 & 1263.63 \\ 0 & 348.34 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 288.5 \\ 54.52 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1.731 & 1263.63 \\ 0 & 348.34 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 288.5 \\ 54.52 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 52.412 \\ 0.156 \end{bmatrix}$$

$\therefore$ Eq^n of straight line = $52.412 + 0.156x$

(c) $f(x) = 52.412 + 0.156x$

$f(1800) = 52.412 + 0.156(1800)$

$= 333.2$

$\therefore \approx 334$ jerseys

Q6

* Find the integral of $f(x) = e^x + x$ in interval $[1, 3]$

(a)

$$\int_1^3 (e^x + x)\, dx = \int_1^3 e^x\, dx + \int_1^3 x\, dx$$

$$= [e^x]_1^3 + \left[\frac{x^2}{2}\right]_1^3$$

$$= [e^3 - e^1] + \left[\frac{9}{2} - \frac{1}{2}\right]$$

$$e^x + \frac{x^2}{2}$$

$$= e^3 + \frac{9}{2} - e^1 - \frac{1}{2}$$

$$= 21.3672$$

* Find the result for $m=3$ using Newton-cotes composite.

(b) $a = 1$, $b = 3$, $m = 3$, $h = \frac{3-1}{3} = \frac{2}{3}$

$x_0 = 1$ $\quad$ $x_2 = 7/3$

$x_1 = x_0 + h = \frac{5}{3}$ $\quad$ $x_3 = 3$

$$C_{1,3} = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$$

$$= \frac{1}{3}\left[e^0 + 2(e^{5/3} + 5/3) + 2(e^{7/3} + 7/3) + (e^3 + 3)\right]$$

$$= \cancel{21.09968}\ 21.0997$$

(c) Error:

$$\left|\frac{C_{1,3} - \text{Actual}}{C_{1,3}}\right| \times 100\%$$

$$= \left|\frac{21.0997 - 21.3672}{21.0997}\right| \times 100\%$$

$$= 1.27\%$$