Department of Mathematics and Natural Sciences

Semester: Summer 2022

Midterm Examination

Course Title: Integral Calculus and Differential Equations (Mathematics II)

Course Code: MAT 120

Section: 23

Total marks: 40

Times: 1 hour

Date: July 24, 2022

Answer any FOUR including question 1:

Q1. [4+3+3]

a. Solve the infinite integral $\int_{-\infty}^{+\infty} \frac{e^{-t}}{\sqrt{1+e^{-2t}}} dt$. See if it convergent or divergent.

b. Evaluate $\int_0^1 \frac{x^5}{\sqrt{1-x^2}} dx$.

c. With completing the partial fraction decomposition evaluate $\int \frac{-3}{(x+1)(2x-1)} dx$.

Q2. [5+5]

a. Use signed area theorem with x_k^* as the left end point of each subinterval to find the area under the curve $f(x) = x^3$ over the interval $[2,6]$.

b. Evaluate $\int_0^{\infty} \frac{x^3}{\sqrt{(1+x^2)^{\frac{9}{2}}}} dx$.

Q3. [5+5]

a. Find the arc length of the curve parameterized by the equations: $x = cost + t\sin t, y = sint - tcost; (0 \le t \le \pi)$.

b. Sketch the region enclosed by the curves and find its area, where $y = \cos 2x, y = 0, x = \frac{\pi}{4}, x = \frac{\pi}{2}$.

Q4.

[4+3+3]

Determine whether the statement is true or false. Explain your answer.

i. $\int \sin^4 x \cos^5 x \, dx = \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + c$

ii. If $f(x) = x^3$ is a smooth, nonnegative function on $[0,1]$, then the surface area generated by revolving the portion of the curve $y = f(x)$ between $x = 0$ and $x = 1$ about x-axis is $S = \int_0^1 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx \approx 3.56$.

iii. If C denote a constant of integration, the two formulas $\int \cos x \, dx = \sin x + C$ and $\int \cos x \, dx = (\sin x + \pi) + C$ are both correct equations.

Q5.

[5+5]

a. Find the area in terms of integral formula of the surface generated by revolving the curve $y = \sqrt{x}, 1 \le x \le 2$, about the x-axis.

b. Use cylindrical shells to find the solid (in terms of integral formula) generated when the region R under $y = x^2$ over the interval $[0, 2]$ is revolved about the line $y = -3$.

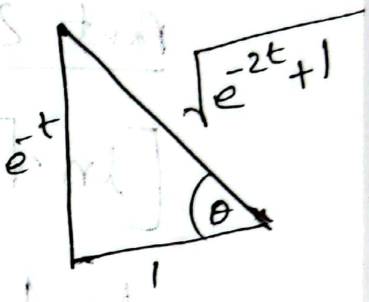
sec 23 Q

1 (a) $\int_{-\infty}^{\infty} \frac{e^{-t}}{\sqrt{1+e^{-2t}}}\, dx$ $\quad\Big|\quad e^{-t} = \tan\theta$

$\frac{d e^{-t}}{d\theta} = \sec^2\theta \, d\theta$

$= \int [I]_k^0 + [I]_0^p$

$I = \int \frac{\sec^2\theta \, d\theta}{\sqrt{1+\tan^2\theta}}$



$= \int \frac{\sec^2\theta}{\sec\theta} = \int \sec\theta$

$= \ln(\sec\theta + \tan\theta)$

$= \ln\left\{ \sec\theta\left(\sec^{-1}\frac{\sqrt{e^{-2t}+1}}{1}\right) + \tan\left(\tan^{-1} e^{-t}\right) \right\}$

$= \ln\left(\sqrt{e^{-2t}+1} + e^{-t}\right)$

$\therefore I_{Final} = \lim_{k \to -\infty} \left[\ln\left(\sqrt{e^{-2t}+1}\right) + e^{-t}\right]_k^0 + \lim_{p\to\infty} \left[\ln\left(\sqrt{e^{-2t}+1} + e^{-t}\right)\right]_0^p$

$= \cancel{\ln(\sqrt{2}+1) - \ln}$

Part 1

$$= \left[\ln \sqrt{\frac{1}{e^{2t}} + 1} + \frac{1}{e^t} \right]_K^0$$

$$= \{ \ln(\sqrt{2}+1) - 0 \}$$

Part 2

$$\left[\ln \sqrt{\frac{1}{e^{2t}} + 1} + \frac{1}{e^t} \right]_0^P$$

$$= \ln 1 - \ln(\sqrt{2}+1)$$

$$= 0 - \ln(\sqrt{2}+1)$$

final $I = \ln(\sqrt{2}+1) - \ln(\sqrt{2}+1)$

$= 0$ (ans)

(b) $u = (1-x)^2$ $u = x^5$

$= 2x$

$x = \tan\theta$

$$\int_0^1 \frac{\tan^5\theta}{\sec\theta} \sec^2\theta$$

$$\int_0^1 \tan^5\theta \sec^2\theta$$

$P = \tan\theta$

$dP = \sec^2\theta \, d\theta$

$$= \int_0^1 p^5 \, dp$$

$$= \int_0^1$$

$$\int_0^\infty \frac{x^3}{\sqrt{(1+x^2)^{9/2}}}\, dx$$

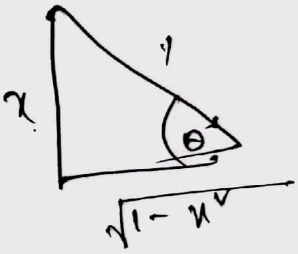
(6) $\int_0^1 \frac{x^5}{\sqrt{1-x^2}}$ $\quad\Big|\quad x = \sin\theta$, $\quad dx = \cos\theta \, d\theta$

$= \int_0^1 \frac{\sin^5\theta}{\cancel{\cos\theta}} \cancel{\cos\theta}\, d\theta$ $\qquad \theta = 0,\ x = 0$

$\qquad \theta = 1$

$= \int_0^1 \sin^5\theta \, d\theta$

$= \int_0^1 (1-\cos^2\theta)^2 \sin\theta \, d\theta$ $\quad\Big|\quad u = \cos\theta$, $\quad du = -\sin\theta \, d\theta$



$= -\int_0^1 (1-u^2)^2 \, du$

$= -\int_0^1 (1-2u^2+u^4)\, du$

$= -\int 1 \, du + 2\int u^2 \, du - \int u^4 \, du$

$= -[u] + \frac{2}{3}[u^3] - \frac{1}{5}[u^5]$

$= -[\cos\theta] + \frac{2}{3}[\cos^3\theta] - \frac{1}{5}[\cos^5\theta]$

$= -\left[\cos(\cos^{-1}\sqrt{1-x^2})\right]_0^1 + \frac{2}{3}\left((\sqrt{1-x^2})^3\right)_0^1 - \frac{1}{5}\left[(\sqrt{1-x^2})^5\right]_0^1$

$= -(0-1) + \frac{2}{3}(0-1) - \frac{1}{5}(0-1)$

$= +1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$

(c) $\int \frac{-3}{(x+1)(2x-1)}$

$$\frac{-3}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}$$

$$\Rightarrow \quad -3 = 2Ax - A + Bx + B$$

$$\therefore \quad 2A + B = 0 \quad \Big| \quad -A + B = -3$$

$$\Rightarrow 2A + A - 3 = 0 \qquad B = -2$$

$$\Rightarrow \quad A = 1$$

$u = 2x - 1$
$du = 2$

$$I = \int \frac{1}{x+1} - \frac{2}{2x-1}$$

$$= \ln(x+1) - \frac{\cancel{2}}{\cancel{2}} \ln(2x-1)$$

$$= \ln(x+1) - \ln(2x-1)$$

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(2) $f(x) = x^3 \quad [2, 6]$

$\Delta x = \frac{4}{n} \quad , \quad x_k^* = 2 + (k-1)\frac{4}{n}$

$= 2 + \frac{4k}{n} - \frac{4}{n}$

$f(x_k^*) = \left(\frac{2n+4k-4}{n}\right)^3 = \frac{2n+4k-4}{n}$

$=$

$\left[2 + (k-1)\frac{4}{n}\right]^3$

$= 2^8 + \frac{(k-1)^3 4^3}{n^3} + 3\cdot 4 \cdot \frac{(k-1)4}{n} + 3 \cdot 2 \cdot (k-1)^2 \frac{16}{n^2}$

$= 2^8 + \frac{(k^3 - 3k^2 + 3k + 1)4^3}{n^3} + \frac{48(k-1)}{n} + \frac{96(k^2 - 2k + 1)}{n^2}$

$2^8 + \frac{k^3}{n^3} - \frac{3k^2}{n^3} + \frac{3k}{n^3} + \frac{4^3}{n^3} + \frac{48k}{n} - \frac{48}{n} + \frac{96k^2}{n^2} - \frac{192k}{n^2} + \frac{96}{n^2}$

$f(x_k^*) \times \Delta x$

$= \frac{32^2}{n} + \frac{4k^3}{n^4} - \frac{12k^2}{n^4} + \frac{12k}{n^4} + \frac{4^4}{n^4} + \frac{192k}{n^2}$

$- \frac{192}{n^2} + \frac{384k^2}{n^3} - \frac{768k}{n^3} + \frac{384}{n^3}$

$\lim_{n \to \infty} \sum_{k=1}^{n}$

$\Rightarrow$

$= 32^2 + \frac{4 \cdot 1}{4} - 0 + 0 + 0 + \frac{192}{2} - 0 + \frac{384}{3}$

$- 0 + 0$

$= 1248$

(b) $$\int_0^\infty \frac{x^3}{\sqrt{(1+x^2)^{9/2}}}$$

$$= \lim_{k\to\infty} \cancel{\int} \left[I_1 \right]_0^k$$

$$I_1 = \int \frac{x^3}{\sqrt{(1+x^2)^{9/2}}}\, dx \qquad \begin{array}{|l} x = \cancel{a}\tan\theta \\ dx = \sec^2 d\theta \end{array}$$

$$= \int \frac{\tan^3\theta}{\sqrt{\sec^9\theta}} \times \sec^2\theta\, d\theta$$

$$= \int \tan^3\theta \, \sec^{2-\frac{9}{2}}\theta$$

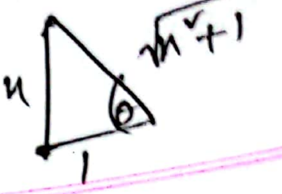
$$= \int \frac{\tan^3\theta}{\sec^{5/2}\theta} \qquad \begin{array}{|l} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{array}$$

$$= \int \frac{\sin^3\theta}{\cos^3\theta} \times \cancel{\frac{1}{6}} \cos^{5/2}\theta$$

$$= \int \frac{\sin^3\theta}{\sqrt{\cos\theta}} = -\int \frac{(1-\cos^2\theta)}{\sqrt{\cos\theta}} \sin\theta\, d\theta$$

$$= -\int \frac{1-u^2}{\sqrt{u}}\, du$$

$= u^{-\frac{1}{2}+1}$



$$= -\int \frac{1}{\sqrt{u}}\,du + \int u^{3/2}\,du$$

$$= -2\left[\sqrt{u}\right] + \frac{2}{5}\left[u^{5/2}\right]$$

$$= -2\left[\sqrt{\cos\theta}\right] + \frac{2}{5}\left[\cos^{5/2}\theta\right]$$

$\tan\theta = x$

$\cos\theta = \frac{1}{\sqrt{x^2+1}}$

$$= -2\left[\sqrt{\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)}\right] + \frac{2}{5}\left[\cos^{5/2}\left(\cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right]$$

$$= -2\left[\left(\frac{1}{\sqrt{x^2+1}}\right)^{1/2}\right]_0^k + \frac{2}{5}\left[\left(\frac{1}{\sqrt{x^2+1}}\right)^{5/2}\right]_0^k$$

Now, $\lim_{k\to\infty}$ $= -2\left\{\left(\frac{1}{\infty}\right)^{1/2} - 1\right\} + \frac{2}{5}\left\{0 - 1\right\}$

$= 2 - \frac{2}{5} = \frac{8}{5}$

③ ⓐ $\cancel{t}\ x = \cos t + t\sin t$ $[0, \pi]$

$y = \sin t - t\cos t$

$$\frac{dx}{dt} = -\cancel{\sin t} + t\cos t + \cancel{\sin t} = t\cos t$$

$$\frac{dy}{dt} = \cos t - (t(-\sin t) + \cos t)$$

$$= \cancel{\cos t} + t\sin t - \cancel{\cos t} = t\sin t$$

$$L = \int \sqrt{t^2\cos^2 t + t^2\sin^2 t}\ dt$$

$$= \int \sqrt{t^2(1)}\ dt = \int t\ dt = \frac{t^2}{2}$$

ⓑ

2?

(4) (1) $\int \sin^4 x \cos^5 x \, dx$

$\Rightarrow \int \sin^4 x \, (1-\sin^2 x)^2 \cos x \, dx$

$$\left| \begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right.$$

$\Rightarrow \int u^4 (1-u^2)^2 \, du$

$= \cancel{\int u^4 du - \int u^6}$

$= \int u^4 (1 - 2u^2 + u^4) \, du$

$= \int u^4 du - 2\int u^6 + \int u^8 du$

$= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}$

$= \frac{\sin^5 x}{5} - \frac{2\sin^7}{7} + \frac{\sin^9 x}{9} + C$

(11) $f(x) = x^3 \quad [0,1]$

$f'(x) = 3x^2$

$[f'(x)]^2 = (3x^2)^2$

$1+[f'(x)]^2 = 9x^4+1$

$S = 2\pi \int_0^1 x^3 \sqrt{9x^4+1}\, dx$ | $u = 9x^4+1$, $du = 36x^3\, dx$

$= \frac{2\pi}{36} \int \sqrt{u}\, du$

$= \frac{\pi}{\cancel{18}\,9} \times \frac{2}{3} \left[u^{3/2}\right]$

$= \frac{\pi}{27} \left[(9x^4+1)^{3/2}\right]_0^1$

$= \frac{\pi}{27} (31.62 - 1)$ True

$= 3.56 \ldots$

Q2 - (14)

(5) $f(x) = (x)^{1/2} \quad [1, 2]$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$1 + [f'(x)]^2 = \frac{1}{4x} + 1 = \frac{1+4x}{4x}$

$S = 2\pi \int \sqrt{x} \cdot \sqrt{\frac{1+4x}{4x}}$

$= \cancel{2}\pi \int \cancel{\sqrt{x}} \cdot \frac{\sqrt{1+4x}}{\cancel{2}\cancel{\sqrt{x}}}$

$= \pi \int \sqrt{1+4x} \; dx$

$u = 1+4x$
$du = 4\,dx$

$= \frac{\pi}{4} \int \sqrt{u}$

$= \frac{\pi}{\cancel{4}_2} \times \frac{\cancel{2}}{3} \times \left[u^{3/2}\right]$

$= \frac{\pi}{6} \left[1+4x\right]_1^2$

$= \frac{\pi}{6} (9 - 5) = \frac{2\cancel{4}\pi}{\cancel{6}_3} = \frac{2\pi}{3}$