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MAT 110

ASSIGNMENT 04

SET 7

Ans to the question no 01

Given function,

$f(x,y) = y \cdot e^{(xy+3x+2y^2)}$

Now,

$f_x = y \cdot e^{(xy+3x+2y^2)} \cdot (y+3) = (y^2+3y) \cdot e^{(xy+3x+2y^2)}$

$f_y = e^{(xy+3x+2y^2)} + y \cdot e^{(xy+3x+2y^2)} \cdot (x+4y)$

$\Rightarrow f_y = e^{(xy+3x+2y^2)}(1+xy+4y^2)$

$f_{xx} = e^{(xy+3x+2y^2)} \cdot (y+3) \cdot y(y+3) = e^{(xy+3x+2y^2)} \cdot y(y+3)^2$

$f_{yy} = e^{(xy+3x+2y^2)} \cdot (x+4y) \cdot (1+xy+4y^2) + e^{(xy+3x+2y^2)} \cdot (x+8y)$

$\Rightarrow f_{yy} = e^{(xy+3x+2y^2)} \cdot \{(x+4y)(1+xy+4y^2) + (x+8y)\}$

$f_{xy} = (2y+3) \cdot e^{(xy+3x+2y^2)} + (y^2+3y) \cdot e^{(xy+3x+2y^2)} \cdot (x+4y)$

$\Rightarrow f_{xy} = e^{(xy+3x+2y^2)}\{(2y+3) + (y^2+3y)(x+4y)\}$

Now, $f(0,0) = 0$

$f_x(0,0) = 0$

$f_y(0,0) = 1$

$f_{xx}(0,0) = 0$

$f_{yy}(0,0) = 0$

$f_{xy}(0,0) = 3$

The first degree Maclaurin polynomial approximation,

$\therefore L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$

$\Rightarrow L(x,y) = 0 + 0 + 1 \cdot y = y$

$\therefore L(x,y) = y$

The second degree Maclaurin polynomial approximation,

$Q(x,y) = L(x,y) + \frac{f_{xx}(0,0)}{2!} \cdot (x-0)^2 + \frac{f_{yy}(0,0)}{2!} \cdot (y-0)^2 + \frac{f_{xy}(0,0)}{1!} \cdot (x-0)(y-0)$

$\Rightarrow Q(x,y) = y + 0 + 0 + 3xy = y(3x+1)$

$\therefore Q(x,y) = y(3x+1)$

Ans to the question no 02

Given function,

$f(x, y) = f(x, y) = 4xye^{-x^2-y^2}$

Now,

$f_x = 4y \cdot \{e^{-x^2-y^2} + xe^{-x^2-y^2}(-2x)\}$

$\Rightarrow f_x = 4y \cdot e^{-x^2-y^2}(1 - 2x^2)$

$f_y = 4x \cdot \{e^{-x^2-y^2} + ye^{-x^2-y^2}(-2x)\}$

$\Rightarrow f_y = 4x \cdot e^{-x^2-y^2}(1 - 2y^2)$

For extreme values,

$f_x = 0$

$\Rightarrow 4y \cdot e^{-x^2-y^2}(1 - 2x^2) = 0$

$e^{-x^2-y^2}$ cannot be zero for any finite values of x or y

$\therefore y = 0$ or $x = \pm\frac{1}{\sqrt{2}}$

Similarly,

$f_y = 0$

$\Rightarrow 4x \cdot e^{-x^2-y^2}(1 - 2y^2) = 0$

$e^{-x^2-y^2}$ cannot be zero for any finite values of x or y

$\therefore x = 0$ or $y = \pm\frac{1}{\sqrt{2}}$

The critical points are: $(0, 0), (0, \frac{1}{\sqrt{2}}), (0, -\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, 0)$

$(-\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

Now,

$f_{xx} = 4y\{e^{-x^2-y^2}(-4x) + (1-2x^2)e^{-x^2-y^2}(-2x)\}$

$\Rightarrow f_{xx} = 4y\{2xe^{-x^2-y^2}(2x^2-3)\}$

$\Rightarrow f_{xx} = 8xy(2x^2-3)e^{-x^2-y^2}$

$f_{yy} = 4x\{e^{-x^2-y^2}(-4y) + (1-2y^2)e^{-x^2-y^2}(-2y)\}$

$\Rightarrow f_{xx} = 4x\{2xe^{-x^2-y^2}(2y^2-3)\}$

$\Rightarrow f_{xx} = 8xy(2y^2-3)e^{-x^2-y^2}$

$f_{xy} = 4\{e^{-x^2-y^2}(1-2x^2) + ye^{-x^2-y^2}(-2y)(1-2x^2)\}$

$\Rightarrow f_{xy} = 4e^{-x^2-y^2}(1-2x^2)(1-2y^2)$

For $(0, 0)$,

$A = f_{xx}(0, 0) = 0$

$B = f_{xy}(0, 0) = 4$

$C = f_{yy}(0, 0) = 0$

$D = AC - B^2 = 0 - 4^2 = -16$

Since $D < 0$, f has a saddle point at (0,0)

For $(0, \frac{1}{\sqrt{2}})$,

$A = f_{xx}(0, \frac{1}{\sqrt{2}}) = 0$

$B = f_{xy}(0, \frac{1}{\sqrt{2}}) = 0$

$C = f_{yy}(0, \frac{1}{\sqrt{2}}) = 0$

$D = AC - B^2 = 0$

Since $D = 0$, No conclusion can be done

For $(0, -\frac{1}{\sqrt{2}})$,

$A = f_{xx}(0, -\frac{1}{\sqrt{2}}) = 0$

$B = f_{xy}(0, -\frac{1}{\sqrt{2}}) = 0$

$C = f_{yy}(0, -\frac{1}{\sqrt{2}}) = 0$

$D = AC - B^2 = 0$
Since $D = 0$, No conclusion can be done

For $(\frac{1}{\sqrt{2}}, 0)$,

$A = f_{xx}(\frac{1}{\sqrt{2}}, 0) = 0$

$B = f_{xy}(\frac{1}{\sqrt{2}}, 0) = 0$

$C = f_{yy}(\frac{1}{\sqrt{2}}, 0) = 0$

$D = AC - B^2 = 0$
Since $D = 0$, No conclusion can be done

For $(-\frac{1}{\sqrt{2}}, 0)$,

$A = f_{xx}(-\frac{1}{\sqrt{2}}, 0) = 0$

$B = f_{xy}(-\frac{1}{\sqrt{2}}, 0) = 0$

$C = f_{yy}(-\frac{1}{\sqrt{2}}, 0) = 0$

$D = AC - B^2 = 0$
Since $D = 0$, No conclusion can be done

For $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$,

$A = f_{xx}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = -8e^{-1} = -2.943$

$B = f_{xy}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 0$

$C = f_{yy}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = -8e^{-1} = -2.943$

$D = AC - B^2 = 8.66$
Since $D > 0$ and $A < 0$, f has a relative maximum point at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

For $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$,

$A = f_{xx}(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 8e^{-1} = 2.943$

$B = f_{xy}(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 0$

$C = f_{yy}(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 8e^{-1} = 2.943$

$D = AC - B^2 = 8.66$
Since $D > 0$ and $A > 0$, f has a relative minimum point at $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

For $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$,

$A = f_{xx}(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 8e^{-1} = 2.943$

$B = f_{xy}(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 0$

$C = f_{yy}(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 8e^{-1} = 2.943$

$D = AC - B^2 = 8.66$
Since $D > 0$ and $A > 0$, f has a relative minimum point at $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

For $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$,

$A = f_{xx}(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -8e^{-1} = -2.943$

$B = f_{xy}(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 0$

$C = f_{yy}(-\frac{1}{-\sqrt{2}}, -\frac{1}{\sqrt{2}}) = -8e^{-1} = -2.943$

$D = AC - B^2 = 8.66$

Since $D > 0$ and $A < 0$, f has a relative maximum point at $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$\therefore$ Saddle point:(0,0)

Maxima:$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$,$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

Minima:$(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$,$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

<u>Ans to the question no 03</u>

Given function,

$x - 2y + z = 16 \Rightarrow z = 16 - x + 2y$

At origin (0,0),

$D = (Distance)^2 = (x-0)^2 + (y-0)^2 + (z-0)^2$

$= x^2 + y^2 + z^2$

$= x^2 + y^2 + (16 - x + 2y)^2$

Here,

$D_x = 2x + 2(16 - x + 2y)(-1)$

$= 2x - 32 + 2x - 4y$

$= 4x - 4y - 32$

When $D_x = 0$,

$4x - 4y - 32 = 0$

$\Rightarrow x - y - 8 = 0$

$\Rightarrow y = x - 8$

Again, $D_y = 2y + 2(16 - x + 2y)(2)$

$= 2y + 64 - 4x + 8y$

$= 10y - 4x + 64$

When $D_y = 0$

$10y - 4x + 64 = 0$

$\Rightarrow 5(x - 8) - 4x + 64 = 0$

$\Rightarrow 3x - 8 = 0$

$\Rightarrow x = \frac{8}{3}$

Substituting the value of x in the equation $y = x - 8$,

$y = \frac{-16}{3}$

$\therefore (x, y) = (\frac{8}{3}, \frac{-16}{3})$

Now, $A = D_{xx} = 4$

$C = D_{yy} = 10$

$B = D_{xy} = -4$

$D = AC - B^2 = 24$

Since $D > 0$ and $A > 0$, D has a minimum at point $(\frac{8}{3}, \frac{-16}{3})$

Now,

$D(\frac{8}{3}, \frac{-16}{3}) = (\frac{8}{3})^2 + (\frac{-16}{3})^2 + (16 - \frac{8}{3} - \frac{2 \cdot 16}{3})^2$

$= \frac{64}{9} + \frac{256}{9} + (\frac{40}{3} - \frac{32}{3})^2$

$= \frac{64}{9} + \frac{256}{9} + (\frac{8}{3})^2$

$= \frac{128}{9} + \frac{256}{9}$

$= \frac{128}{3}$

$\therefore Distance = \sqrt{\frac{128}{3}} = \frac{8\sqrt{6}}{3}$

$Since, x = \frac{8}{3}, y = \frac{-16}{3},$

$z = 16 - \frac{8}{3} + \frac{2 \cdot -16}{3} = \frac{8}{3}$

the point in the given plane that is closest to the origin is$(\frac{8}{3}, \frac{-16}{3}, \frac{8}{3})$

Ans to the question no 04

Given function,

$\vec{F} = -x^2(y-z)\hat{i} - \left(x^2+y^4\right)\hat{j} + \left(\frac{4z^2}{y^2}\right)\hat{k}$

$\Rightarrow \vec{F} = (-x^2y + -x^2z)\hat{i} + \left(-x^2 - y^4\right)\hat{j} + \left(4z^2y^{-2}\right)\hat{k}$

$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x^2y + x^2z & -x^2 - y^4 & 4z^2y^{-2} \end{vmatrix}$

$= \hat{i}\{\frac{\partial}{\partial y}(4z^2y^{-2}) - \frac{\partial}{\partial z}(-x^2-y^4)\} - \hat{j}\{\frac{\partial}{\partial x}(4z^2y^{-2}) - \frac{\partial}{\partial z}(-x^2y+ x^2z) + \hat{k}\{\frac{\partial}{\partial x}(-x^2-y^4) - \frac{\partial}{\partial y}(-x^2y+x^2z)\}$

$= (-8z^2y^{-3})\hat{i} - (0 - 0 + x^2)\hat{j} + (-2x + x^2)\hat{k}$

$= -8z^2y^{-3}\hat{i} - x^2\hat{j} - (2x + x^2)\hat{k}$

$\text{Div } \vec{F} = \Delta F$

$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

$= \frac{\partial}{\partial x}(-x^2y + -x^2z) + \frac{\partial}{\partial y}(-x^2 - y^4) + \frac{\partial}{\partial z}(4z^2y^{-2})$

$= -2x(y-z) - 4y^3 + \frac{8z}{y^2}$

$\therefore Curl\vec{F} = -8z^2y^{-3}\hat{i} - x^2\hat{j} - (2x + x^2)\hat{k}$

$\therefore \Delta F = -2x(y-z) - 4y^3 + \frac{8z}{y^2}$

Ans to the question no 05

Given function,

$\vec{F} = 2yz\hat{i} - x^2y\hat{j} + xz^2\hat{k}$

$\phi = 2x^2yz^3$

$\nabla\phi = (\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}) \cdot (2x^2yz^3)$

$= \frac{\partial}{\partial x}(2x^2yz^3)\hat{i} + \frac{\partial}{\partial y}(2x^2yz^3)\hat{j} + \frac{\partial}{\partial z}(2x^2yz^3)\hat{k}$

$= 4xyz^3\hat{i} + 2x^2z^3\hat{j} + 6x^2yz^2\hat{j}$

Now,

$\vec{F} \cdot (\nabla\phi) = (2yz\hat{i} - x^2y\hat{j} + xz^2\hat{k}) \cdot (4xyz^3\hat{i} + 2x^2z^3\hat{j} + 6x^2yz^2\hat{j})$

$= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$

$\therefore \vec{F} \cdot (\nabla\phi) = 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$

Ans to the question no 06

Given function,

$f(x,y,x) = x^2y + y^2z + xz^2$

$\nabla \cdot f(x,y,x) = (\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k})$

$= \frac{\partial}{\partial x}(x^2y+y^2z+xz^2)\hat{i} + \frac{\partial f}{\partial y}(x^2y+y^2z+xz^2)\hat{j} + \frac{\partial f}{\partial z}(x^2y+y^2z+xz^2)\hat{k})$

$= (2xy + z^2)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 + 2zx)\hat{k}$

At the point (2,4,5),

$\nabla \cdot f(2,4,5) = (2\cdot2\cdot4+5^2)\hat{i}+(2^2+2\cdot4\cdot5)\hat{j}+(4^2+2\cdot5\cdot2)\hat{k}$

$= 41\hat{i} + 44\hat{j} + 36\hat{k}$

unit vector along (1,-1,3),

$\hat{u} = \frac{\hat{i}-\hat{j}+3\hat{k}}{\sqrt{1^2+1^2+3^2}}$

$= \frac{\hat{i}-\hat{j}+3\hat{k}}{\sqrt{11}}$

Directional derivative of the function f(x,y,z) at the point (2,4,5) in the direction of the point (1,-1,3),

$(\nabla \cdot f(2,4,5)) \cdot \hat{u} = \frac{41-44+108}{\sqrt{11}}$

$= \frac{105}{\sqrt{11}} = 31.66$