MAT 110

MIDTERM EXAM

SUMMER 21

Name : Shihab Muhtasim

ID : 21301610

section : 04 (FAB)

Ans to the qr no 1

Given, $\lim_{x \to 0} x \left[\frac{1}{x}\right]$

For $x \in \mathbb{R}$,

$$\frac{1}{x} + 1 > \left[\frac{1}{x}\right] \geq \frac{1}{x}$$

$$\Rightarrow x\left(\frac{1}{x} + 1\right) > x\left[\frac{1}{x}\right] \geq x\frac{1}{x}$$

[multiply by x]

$$\Rightarrow (1+x) > x\left[\frac{1}{x}\right] \geq 1$$

$$\lim_{x \to 0} (1+x) = 1$$

using squeeze theorem,

$$\lim_{x \to 0} x\left[\frac{1}{x}\right] = 1$$

Ans to the qr no 2

Given,

$$\lim_{x \to 0} \frac{\tan(4x)}{x}$$

$$= \lim_{x \to 0} \left\{ \frac{\frac{\sin(4x)}{\cos(4x)}}{\frac{x}{1}} \right\}$$

$$= \lim_{x \to 0} \frac{\sin(4x)}{\cos(4x)} \cdot \frac{1}{x}$$

$$= \lim_{x \to 0} \frac{1}{\cos(4x)} \cdot \frac{\sin(4x)}{x}$$

$$= \lim_{x \to 0} \frac{1}{\cos(4x)} \cdot 4 \cdot 1 \quad \left[\because \frac{\sin x}{x} = 1 \right]$$

$$= 4 \cdot \lim_{x \to 0} \frac{1}{\cos 4x}$$

$$= 4 \cdot \frac{1}{\cos(4 \cdot 0)}$$

$$4 \cdot 1 = 4$$

$$\therefore \lim_{x \to 0} \frac{\tan(4x)}{x} = 4$$

Ans to the qr no 3

$f(x) = \tan^{-1}(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3$

$f(0) = \pi$; $f'(x) = \frac{1}{1+x^2}$

$f'(0) = 1$, $f''(x) = (-1)(1+x^2)^{-2}(0+2x)$

$= \frac{-2x}{(1+x^2)^{-2}}$

$f''(0) = 0$

$f'''(x) = \frac{-2}{(1+x^2)^2} - \frac{6x^2}{(1+x^2)^3}$

$f'''(0) = -2$

$\tan^{-1} x = \pi + x - 2x^3$

setting $x = 1.2$

$\tan^{-1}(1.2) = \pi + 1.2 - 2 \cdot (1.2)^3$

$\pi = 0.88 + 2.26$

$= 3.14$

$\therefore$ Ans. $\pi = 3.1$

Ans to the qr no 4

Let,

x = length of box

y = height of box

x = width of box

Given, volume = 4.

$\Rightarrow x^2 \cdot y = 4$

$\Rightarrow y = \frac{4}{x^2}$ ...(1)

Now, $C = 5x^2 + 4xy \cdot 2$

$C = 5x^2 + 8xy$

Substituting $y = \frac{4}{x^2}$ we get,

$C(x) = 5x^2 + 8x\left(\frac{4}{x^2}\right)$

$C(x) = 5x^2 + \frac{32}{x}$

$\therefore \quad C'(x) = \frac{d}{dx}\left(5x^2 + \frac{32}{x}\right)$

$= 10x \left(-\frac{32}{x^2}\right)$

Now,

$C'(x) = 0$

$\Rightarrow 10x - \frac{32}{x^2} = 0$

$\Rightarrow 10x^3 - 32 = 0$

$\Rightarrow 10x^3 = 32$

$\Rightarrow x^3 = 3.2$

$\Rightarrow x = 1.47$

$\therefore \quad y = \frac{4}{x^2} = \frac{4}{(1.47)^2} = 1.85$

Now, $\quad C''(x) = \frac{d}{dx}\left(10x - \frac{32}{x^2}\right)$

$= 10 + \frac{64}{x^3}$

at $x = 1.47$,

$$c''(1.47) = 10 + \frac{64}{(1.47)^3}$$

$$= 30.14$$

since $c''(1.47) > 0$,

so, $x = 1.47$ is a minima point

Ans to the or no 5

Given,

$$f(x) = \begin{cases} 3ax & , x < 2 \\ ax^2 + bx + 1 & , x \geq 2 \end{cases}$$

We have to find if $f(x)$ is continuous at $x = 2$.

$$f(2) = a(2)^2 + 2b + 1$$
$$= 4a + 2b + 1$$

$$L.H.L = \lim_{x \to 2^-} f(x)$$
$$= \lim_{x \to 2^-} 3ax$$
$$= 3a(2)$$
$$= 6a$$

$$R.H.L = \lim_{x \to 2^+} f(x)$$

$$= \lim_{x \to 2^+} ax^2 + bx + 1$$

$$= 4a + 2b + 1$$

For $f(x)$ to be continuous at $x = 2$

L.H.L and R.H.L have to be equal.

$$L.H.L = R.H.L$$

$$6a = 4a + 2b + 1$$

$$\Rightarrow 2a - 2b = 1 \quad \ldots (i)$$

Now, using the first principal of differenciation,

$$L.H.D = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0^-} \frac{3a(x+h) - 3ax}{h}$$

$$= \lim_{h \to 0^-} \frac{3ax + 3ah - 3ax}{h}$$

$$= \lim_{h \to 0^-} \frac{3ah}{h} = 3a$$

$$R.H.D = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0^+} \frac{a(x+h)^2 + b(x+h) + 1 - ax^2 - bx - 1}{h}$$

$$= \lim_{h \to 0^+} \frac{a(x^2 + 2hx + h^2) - ax^2 + b(x+h) - bx}{h}$$

$$= \lim_{h \to 0^+} \frac{a(\cancel{x^2} + 2hx + h^2 - \cancel{x^2}) + b(x + h - \cancel{x})}{h}$$

$$= \lim_{h \to 0^+} \frac{2hax + ah^2 + bh}{h}$$

$$= \lim_{h \to 0^+} \frac{h(2ax + ah + b)}{h}$$

$$= \lim_{h \to 0^+} 2ax + ah + b$$

$$= 4a + b \quad [\because x = 2]$$

$$=$$

$\therefore$ L.H.D = R.H.D

$\Rightarrow$ $3a = 4a + b$

$\Rightarrow$ $a + b = 0$

$= \quad a = -b \quad \ldots (ii)$

solving (i) and (ii) we get,

$-2b - 2b = 1$

$\Rightarrow$ $-4b = 1$

$\Rightarrow$ $b = -\frac{1}{4}$

And, $2a - 2(-a) = 1$

$\Rightarrow$ $a = \frac{1}{4}$

$\therefore$ $a = \frac{1}{4}$, $b = -\frac{1}{4}$ ~~and~~ are the values for which f is differenciable at $x = 2$