Pattern Avoidance in Sequences

Shihan Kanungo
Mentor: Prof. Jesse Geneson (SJSU)

SJSU

PRIMES October Conference 2025

Outline

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1. Introduction
2. Saturation for Sequences
3. New Results

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- What is the largest possible group of people, such that for any set of k people, they are not all friends or not all strangers? (Ramsey Theory)
- What is the largest possible subset of $\{1, \ldots, n\}$ that does not contain a k-term arithmetic progression? (Szemeredi's Theorem)

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What is the SMALLEST possible structure that avoids a given forbidden substructure, BUT making it larger in any way induces a copy of the forbidden structure?

In other words: the *minimum* size of a *maximal* structure, rather than the *maximum* size.

Saturation for Graphs

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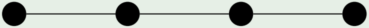
$H =$ 

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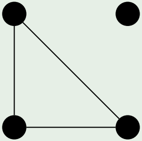

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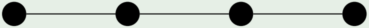
$H =$ 

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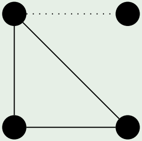

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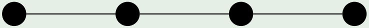
$H =$ 

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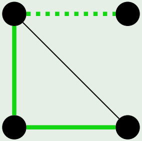

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Dichotomy

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In many other settings, it has been seen that the saturation function exhibits the same dichotomy.

Outline

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Sequence Saturation

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Definition

Let s, u be two sequences. We say that s contains a copy of u if s has a subsequence that can be turned into u by a *one-to-one* renaming of letters.

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The sequence $s = \mathbf{1}, \mathbf{2}, 3, 2, 3, \mathbf{1}, \mathbf{2}$ contains a copy of $u = abab$:

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The sequence $s = 1, 2, \mathbf{3}, \mathbf{2}, \mathbf{3}, 1, 2$ is 2-sparse, but not 3-sparse.

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If we dropped the r-sparsity condition, we would have arbitrarily long sequences like $1, 1, \cdots$ which avoid u.

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Similar checks for the other letters.

Conjecture

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This implies the dichotomy $\text{Sat}(n, u) = O(1)$ or $\Theta(n)$. They proved

Theorem (Anand-Geneson-Kaustav-Tsai, 2021)

We have $\text{Sat}(n, u) = O(n)$ *for all sequences* u *with two distinct letters.*

However, the cases for u having ≥ 3 distinct letters remained completely open.

Outline

Algorithm

Consider the following algorithm:

Input: Alphabet $A = \{1, \ldots, n\}$, forbidden sequence u
Output: u-saturated sequence
Initialize the sequence: $s \leftarrow 1, 2, \ldots, r-1$ ▷ Initial sequence avoids u
while it is possible to extend the sequence **do**
  for each letter $x \in A$ **do**
    if x can be *properly inserted* into s **then**
      Insert x appropriately into s to form s' ▷ Smallest x, leftmost position
      Update $s \leftarrow s'$ ▷ New sequence
      break ▷ Exit loop after the first valid insertion
    end if
  end for
end while
Return s ▷ Final sequence is u-saturated

The output of the algorithm:

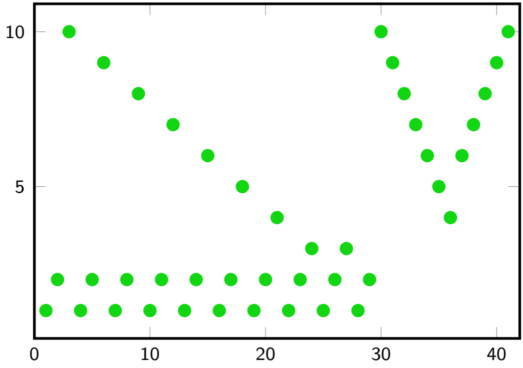

Figure: Algorithm on $u = abcacbc$.

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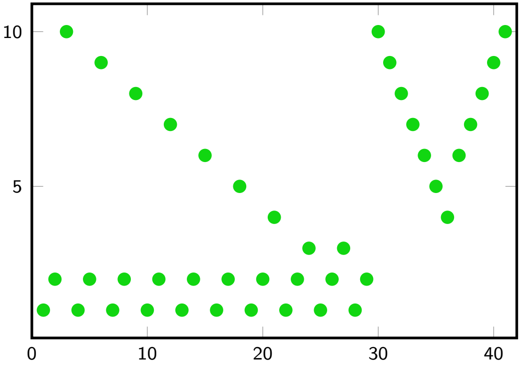


Figure: Algorithm on $u = abcacbc$.

Here, we represent $s = s_1 \cdots s_\ell$ by plotting the points (i, s_i). Using this pattern, we get $\mathrm{Sat}(n, abcacbc) = O(n)$!

Some more pictures:

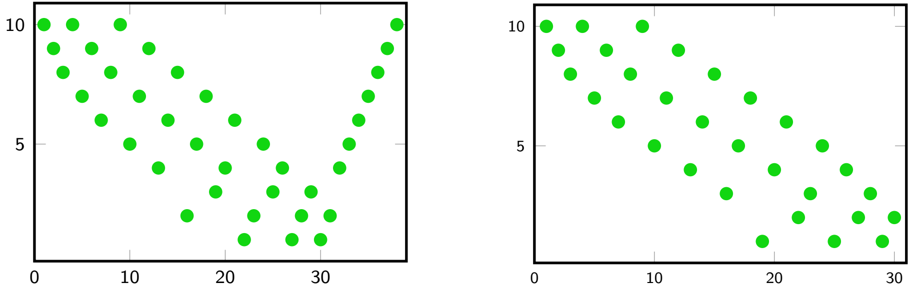
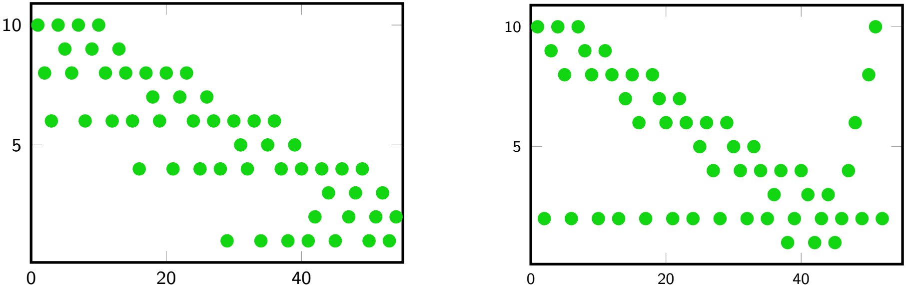


Figure: Algorithm on $u = abbacac$ (left) and $abcacba$ (right).

And even more:



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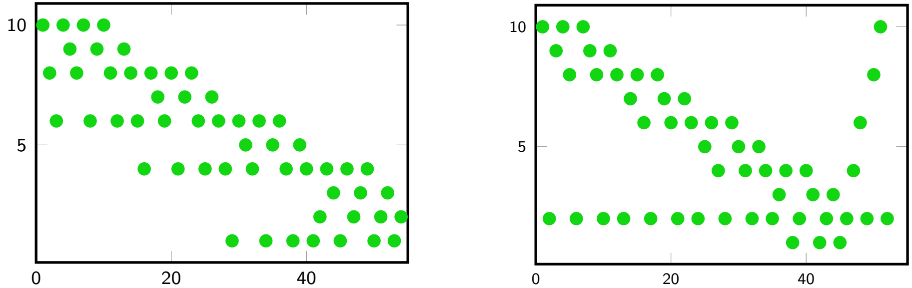

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This lets us resolve the conjecture for many specific sequences u.

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Theorem (Kanungo, 2025+)

If u is irreducible and of the form $aa \cdots bb$, then $\mathrm{Sat}(n, u) = O(n)$.

Theoretical Results

Suppose u is a sequence on 3 letters, and $u = abc \cdots xyz$ where a, b, c are distinct.

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$$f_0(u) = \#\{\text{ consecutive pairs of the form } ab, bc, ca\},$$
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Theorem (Kanungo, 2025+)

Let $u = abc \ldots xyz$ be a three-letter sequence with a, b, c distinct. Suppose

$$xyz \in \{abc, bca, cab\}, \qquad f_0(u) \geq f_1(u) + 5.$$

Then $\mathrm{Sat}(n, u) = O(n)$.

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Corollary

For any sequence u on 3 letters, $\mathrm{Sat}(n, (abc)u(abc)^t) = O(n)$ for large enough t.

Questions?

Thank You!