LOWER BOUNDS ON THE FROBENIUS NORM OF THE INVERSE OF A NONNEGATIVE MATRIX

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Abstract. Motivated with a problem in spectroscopy, Sloane and Harwit conjectured in 1976 what is the minimal Frobenius norm of the inverse of a matrix having all entries from the interval $[0, 1]$. This is known as the S-matrix conjecture. In 1987, Cheng proved their conjecture in the case of odd dimensions, while for even dimensions he obtained a slightly weaker lower bound for the norm. In this report we discuss Frankel and Urschel's proof of the S-matrix conjecture for all even n larger than a small constant.

1. Introduction

Given a $n \times n$ nonnegative invertible matrix A with $\|A\|_{\max} \leq 1$ (i.e. all entries of A are ≤ 1), what is the minimum value of $\|A^{-1}\|_F$, the Frobenius norm of the inverse of A? This question is answered by the *S-matrix conjecture*.
Harwit and Sloane conjectured the following in 1976.

Conjecture 1 (S-matrix conjecture)**.** *For an invertible nonnegative $n \times n$ matrix A with $\|A\|_{max} \leq 1$,*

$$\left\|A^{-1}\right\|_F \geq \frac{2n}{n+1}.$$

Equality holds if and only if A is an S-matrix*.*

We say A is an S-matrix if $A_{ij} \in \{0, 1\}$ and

$$A^T A = \frac{n+1}{4}(I + \mathbf{1}\mathbf{1}^T),$$

where $\mathbf{1}$ is the all-ones vector. Since $\frac{n+1}{4}$ is an integer only when $n \equiv 3 \pmod 4$, equality can hold if and only if $n \equiv 3 \pmod 4$.
This question was motivated by a problem in spectroscopy, where we have n beams of light with different wavelengths, and we want to accurately measure the intensities of the different wavelengths using a detector that has some random error.

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The case where n is odd was resolved by Cheng, but when n was even, the proof they used gave the following, slightly worse, bound

$$\frac{2\sqrt{n^2-2n+2}}{n}. \tag{†}$$

Various other proofs of the same bounds were subsequently found, but no one was able to prove the even n case. Finally, in 2025, Frankel and Urschel [1] were able to prove the S-matrix conjecture for all even n larger than a small constant.

We will discuss the motivation for the S-matrix conjecture in Section 2, and then outline Frankel and Urschel's proof in Section 3.

All matrices discussed here are real.

2. Motivation

Suppose we have n objects of weights $w_1, \ldots, w_n$ and we want to measure them. We are allowed n measurements total, and we are given one of the following two scales:

Fig. 1

Fig. 2

However, the i'th meaurement will have some error e_i.

We assume that the e_i are independent random variables,

$$\mathbb{E}(e_i) = 0, \quad \mathbb{E}(e_i^2) = \sigma^2.$$

To measure the w_i, we can try the naïve approach:

$$\begin{aligned} m_1 &= w_1 + e_1 \\ m_2 &= w_2 + e_2 \\ m_3 &= w_3 + e_3 \\ m_4 &= w_4 + e_4. \end{aligned}$$

Using $\widehat{w}_i$ to denote our estimate for w_i, we have

$$\widehat{w}_i = m_i = w_i + e_i.$$

The mean square error for $\widehat{w}_i$ is thus σ^2. However, we can do much better than this. The guiding idea is as follows:

> *If we weigh an object n times and average the results, the error decreases by a factor of $\sqrt{n}$.*

Obviously, we could just take more measurements to increase the accuracy; but recall that we were only allowed n measurements.
Consider the following weighing scheme (corresponding to the scale in Fig 1):

$$\begin{aligned} m_1 &= w_1 + w_2 + w_3 + w_4 + e_1 \\ m_2 &= w_1 - w_2 + w_3 - w_4 + e_2 \\ m_3 &= w_1 + w_2 - w_3 - w_4 + e_3 \\ m_4 &= w_1 - w_2 - w_3 + w_4 + e_4 \end{aligned}$$

Solving the equations, we get

$$\widehat{w}_1 = \frac{m_1 + m_2 + m_3 + m_4}{4} = w_1 + \frac{e_1 + e_2 + e_3 + e_4}{4}, \quad \text{etc.}$$

Using basic statistics, the mean square error of $\widehat{w}_1$ is $\sigma^2/4$; similarly for the other $\widehat{w}_i$. Thus, we have decreased the mean square error by a factor of four!
For general n, we use a $n \times n$ matrix to describe the weighing setup:

$$\mathbf{m} = A\mathbf{w} + \mathbf{e},$$

where

$$\mathbf{m} = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}.$$

Then our estimate for $\mathbf{w}$ is

$$\widehat{\mathbf{w}} = A^{-1}\mathbf{m} = \mathbf{w} + A^{-1}\mathbf{e}.$$

The sum of the mean square errors of the $\widehat{w}_i$ is the sum of the variances of the entries of $A^{-1}\mathbf{e}$. Writing $A^{-1} = [a_{ij}^{-1}]$, it is easily checked that this is

$$\sigma^2 \left(\sum_{i,j=1}^{n} a_{ij}^2 \right) = \sigma^2 \left\| A^{-1} \right\|_F^2 .$$

Thus, the best possible weighing design corresponds to the matrices that minimize $\|A^{-1}\|_F$!
What are the constraints on A? If we are using the scale in Fig 1, then we must have $A_{ij} \in \{-1, 0, 1\}$, because 1 corresponds to the weight on the right pan, 0 corresponds to the weight not on either pan, -1 corresponds to the weight on the left pan. If we use the scale in Fig 2, then we must have $A_{ij} \in \{0, 1\}$. We will generalize and allow $A_{ij} \in [-1, 1]$ in the first case and $A_{ij} \in [0, 1]$ in

the second case. The reason for this is that an analogous problem exists in spectroscopy, where we use slits to allow certain wavelengths of light to be detected, but we can also use partially open slits, corresponding to A_{ij} not being an integer. Thus, our two optimization problems are to minimize $\|A^{-1}\|_F$ subject to:

(1) A is a invertible matrix with $\|A\|_{\max} \leq 1$
(2) A is a nonnegative invertible matrix with $\|A\|_{\max} \leq 1$

We can answer the first question quite easily. Let $\langle X, Y\rangle = \operatorname{tr}(Y^T X)$ be the Frobenius inner product. Then

$$\begin{aligned} n^2 &= \langle A^T, A^{-1}\rangle_F^2 \\ &\leq \|A\|_F^2 \left\|A^{-1}\right\|_F^2 \\ &\leq n^2 \|A\|_{\max}^2 \left\|A^{-1}\right\|_F^2 . \end{aligned}$$

So if $\|A\|_{\max} \leq 1$, then $\|A^{-1}\|_F \geq 1$.
Furthermore, if $A_{ij} \in \{\pm 1\}$ and $A^T = cA^{-1}$ then equality holds.

$$A^T = cA^{-1} \quad \Longleftrightarrow \quad A^T A = cI$$

Hence $c = n$ and $A^T A = nI$. Such matrices are called *Hadamard matrices* and correspond to the best possible weighing designs.

3. Proof

We now discuss the proof of the S-matrix conjecture for even n. First, we discuss one of the proofs of Cheng's bound (†), given by Drnovšek.
Let n be even, A invertible and nonnegative with $\|A\|_{\max} \leq 1$,

$$\begin{aligned} F(A) &= \begin{bmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & \sqrt{\frac{n}{n-2}}\frac{n}{2}A^{-1} \end{bmatrix}, \\ G(A) &= \begin{bmatrix} 0 & \mathbf{1}^T \\ \mathbf{1} & \sqrt{\frac{n-2}{2}}\left(\frac{2(n-1)}{n-2}I - \frac{2}{n}\mathbf{1}\mathbf{1}^T\right)A^T \end{bmatrix}, \\ H(A) &= \frac{\sqrt{n(n-2)}}{2(n-1)}\left[\sqrt{\frac{n}{n-2}}\frac{n}{2}A^{-1} - \sqrt{\frac{n-2}{2}}\left(\frac{2(n-1)}{n-2}I - \frac{2}{n}\mathbf{1}\mathbf{1}^T\right)A^T\right]. \end{aligned}$$

We can think of $H(A)$ as measuring how close $F(A)$ and $G(A)$ are to each other. By Cauchy-Schwartz,

$$\begin{aligned}\frac{n^2(n^2-2)^2}{(n-2)^2} &= \langle F(A), G(A)\rangle_F^2 \\ &\le \|F(A)\|_F^2 \, \|G(A)\|_F^2 \\ &= \left(2n + \frac{n^3}{4(n-2)}\left\|A^{-1}\right\|_F^2\right)(2n + h(A))\end{aligned}$$

where

$$\begin{aligned}h(A) &= \left\|\sqrt{\frac{n-2}{n}}\left(\frac{2n-2}{n-2}I - \frac{2}{n}\mathbf{1}\mathbf{1}^T\right)A^T\right\|_F^2 \\ &= \frac{4(n-1)^2}{n(n-2)}\sum_{i,j=1}^n A_{ij}^2 - \frac{4}{n}\sum_{i=1}^n\left(\sum_{j=1}^n A_{ij}\right)^2.\end{aligned}$$

By taking the derivative of $h(A)$ with respect to an entry A_{ij}, we see that it is never 0 for $A_{ij} \in (0,1)$. So $h(A)$ is maximized when $A_{ij} \in \{0,1\}$. In this case, it is easily checked that $h(A)$ is maximized when $A\mathbf{1} = \frac{n}{2}I$, in which case,

$$h(A) \le \frac{n(n^2-2n+2)}{n-2}.$$

Plugging in this to above, we get the bound

$$\left\|A^{-1}\right\|_F \ge \frac{2\sqrt{n^2-2n+2}}{n}.$$

For the main proof, suppose B is a counterexample to the S-matrix conjecture. Then $\|B\|_{\max} \le 1$ but $\|B^{-1}\|_F < \frac{2n}{n+1}$.
The argument above must be nearly tight, since

$$2 \cdot \frac{\sqrt{n^2-2n+2}}{n} \approx \frac{2n}{n+1}.$$

Hence all of the inequalities must be nearly equalities, which implies B must have some structure.
In the following lemma which is the core of the proof, this intuition is formalized, and after each statement we explain what it heuristically means.

Lemma 2. *Let $\boldsymbol{r} = B\mathbf{1} - \frac{(n-1)^2}{2(n-2)}\mathbf{1}$, $c = \frac{n(n^2-2n+2)}{n-2} - h(B)$. Then $0 \le c < 1$,*

(1) $\|\boldsymbol{r}\|_2^2 + \dfrac{(n-1)^2}{(n-2)}\displaystyle\sum_{i,j=1}^n B_{ij}(1-B_{ij}) = \frac{cn}{4} + \frac{n}{4(n-2)^2}$

i.e., B is almost a 01-matrix and its row sums are almost $\frac{n}{2}$

(2) $\|H(B)\|_F^2 \le \dfrac{n(n-2)}{4(n-1)^2}\left[\dfrac{n(n^2-2n-2)}{(n-2)(n+1)^2} - c\right]$

i.e., $F(B), G(B)$ are almost the same.

(3) $B(B^T + H(B)) = \dfrac{n^2}{4(n-1)} I + \dfrac{(n-1)^3}{4n(n-2)} \mathbf{1}\mathbf{1}^T + \dfrac{n-1}{2n}(\boldsymbol{r}\mathbf{1} + \mathbf{1}\boldsymbol{r}) + \dfrac{n-2}{n(n-1)} \boldsymbol{r}\boldsymbol{r}^T$

i.e., $B \cdot B^T$ plus a small perturbation has off-diagonal entries which are almost $\frac{1}{4}$ away from an integer

The rest of the proof is quite involved, using 3 more lemmas, but the main idea is that $H(B)$ is too small for $B(B^T + H(B))$ to have off-diagonal entries which are almost 1/4 away from an integer, resulting in a contradiction.

References

[1] Elsa Frankel and John Urschel (2025). *On the Frobenius norm of the inverse of a non-negative matrix*. Linear Algebra and its Applications **708**, 193–203.

[2] Martin Harwit and Neil J.A. Sloane (1979). *Hadamard Transform Optics*. Academic Press, Elsevier Inc.

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